3 The Properties of the FRW Metric

Remember that in last lecture we introduced the *FRW metric*, which is supposed to represent the Universe, at least at large enough scales where it is homogeneous and isotropic:

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right) \tag{3.1}$$

where we recall that $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ is the 2-sphere metric, and k controls the curvature(geometry) of the spatial slices:

$$k = \begin{cases} 0 & \text{Euclidian} \\ +1 & \text{Spherical} \\ -1 & \text{Hyperbolical} \end{cases} \tag{3.2}$$

This metric (3.1) is the crux of cosmology, and the goal of this lecture and the following is to understand it better.

3.1 What do the coordinates stand for?

As we have said, it is in general not obvious how to give meaning to the coordinates in General Relativity. However, in this case, the coordinates (r, θ, ϕ) are called *comoving coordinates*, and we can actually try to understand what that means and the reason for that name with the following example.

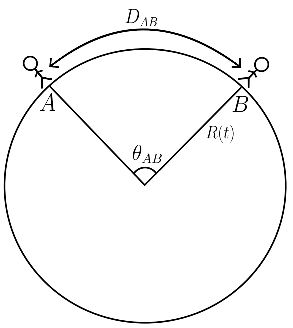

Figure 1: Two observers sitting on a circle

Consider a circle that is expanding, with a radius $R(t)$, and two observers fixed at positions A and B as in the figure on the left. These observers, then, are at a fixed *comoving* distance θ_{AB}, but their *physical* distance D_{AB} is instead increasing. The two are related by:

$$D_{AB}(t) = R(t) \times \theta_{AB} \tag{3.3}$$

We can actually go a bit further and ask what the *receding velocity* is, i.e. how fast A sees B going away from him or viceversa. Clearly, we just have to take a derivative:

$$\dot{D}_{AB}(t) = \dot{R}(t) \times \theta_{AB} = \frac{\dot{R}(t)}{R(t)} R(t) \times \theta_{AB} \equiv H(t) \times D_{AB} \tag{3.4}$$

We recognize this as Hubble's law! That is, the two observers see each other receding with a velocity that is proportional to their distance. We also have introduced the *Hubble constant* (even though it is not a constant, history has decided on this label) which, if evaluated at present time t_0, becomes H_0, the parameter we have introduced in the first lecture when we were talking about Hubble's discoveries.
Now, in this simplified model of the Universe, we can also understand how every observer sees everybody else receding with a velocity according to Hubble's law, like in the figure on the right below.

Clearly, A sees both observers B and C receding, but the same can be said from the point of view of observer C. This resembles a lot the picture of our Universe we have laid out in the first lecture, where every galaxy seems to be moving away from us independently of where we look. Also, homogeneity implies that this must be true for every galaxy, not just for us. We know, however, that our Universe is neither 2 dimensional nor a circle, so we need to generalize the discussion for the geometries that appear in the FRW metric. Consider a galaxy at a coordinate distance $\vec{r}$ and a scale factor $a(t)$, then its physical distance is $\vec{r}_{\text{phys}} = a(t) \times \vec{r}$ and its velocity is:

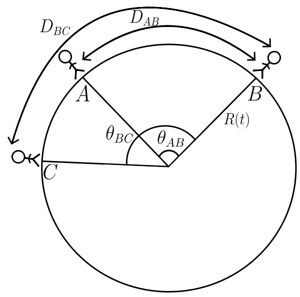

Figure 2: Multiple observers sitting on a circle

$$\vec{v}_{\text{phys}} = \dot{a}(t)\vec{r} + a(t)\dot{\vec{r}} \equiv H(t)\vec{r}_{\text{phys}} + \vec{v}_{\text{pec}} \tag{3.5}$$

We can see that the velocity has two contributions. First is the *Hubble flow* which, as we said before, results simply from the expansion of the Universe, and second is the *peculiar velocity*, because the galaxy might have its own local motion that may be due, for example, to the gravitational attraction of nearby galaxies.

We have gone through this example because it is applicable to our case, since the proper distance between two objects that are spatially separated by $d\sigma$ can be read from (3.1) by putting $dt = 0$:

$$d\ell(t) = a(t)d\sigma \implies d\dot{\ell}(t) = H(t)d\sigma \tag{3.6}$$

which is just the Hubble law, as before.

Basically, we have learned that the coordinates (r, θ, ϕ) are comoving, i.e. they are useful for describing observers moving with the Hubble flow. Moreover, this tells us that even the time t is the proper time of an observer moving with the Hubble flow itself, because if we place him at fixed comoving coordinates $(dr = d\theta = d\phi = 0)$ we get from the metric:

$$-c^2 d\tau^2 = -c^2 dt^2 \tag{3.7}$$

Finally, let us think about the scale factor itself. We have seen that it is basically a conversion factor between physical and coordinate distances at a certain time t. Intuitively, it doesn't really matter what the actual values of $a(t)$ are, since the only important things are ratios, like $a(t_1)/a(t_2)$ at two different times. This is reflected in the fact that the metric (3.1) has a rescaling symmetry (as we have already seen with the overall factor A in the previous lecture):

$$r \to r/\lambda, \quad a \to \lambda a \tag{3.8}$$

This freedom allows us to set the scale factor equal to one at present time $a(t_0) = 1$, which is just a convenient value.

3.2 New coordinates and conformal time

In general, it can be a bit inconvenient to have a complicated g_{rr} component in the FRW metric (3.1), so that we define a new radial coordinate to get rid of it:

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}} \tag{3.9}$$

With this change of coordinates, the FRW metric becomes:

$$ds^2 = -c^2dt^2 + a^2(t)(d\chi^2 + S_k^2(\chi)d\Omega^2) \tag{3.10}$$

where:

$$S_k(\chi) \equiv \begin{cases} \sinh\chi & k = -1 \\ \chi & k = 0 \\ \sin\chi & k = +1 \end{cases} \tag{3.11}$$

Note how for a flat Universe ($k = 0$), there is no distinction between r and χ.
Finally, it is very convenient to introduce the notion of *conformal time*, which is just another time coordinate defined by:

$$d\eta = \frac{dt}{a(t)} \tag{3.12}$$

With this new time the metric (3.10) becomes:

$$ds^2 = a^2(\eta)(-c^2d\eta^2 + (d\chi^2 + S_k^2(\chi)d\Omega^2)) \tag{3.13}$$

The obvious advantage of conformal time is that light rays ($ds^2 = 0$) now move at 45° in a spacetime diagram. However, it is a rather important concept altogether, as we will discover a bit later on in the lectures.

3.3 Geodesics, or the motion of objects

In the previous lecture we have seen how, given a certain metric, we can calculate how particles move by solving the geodesics equation. In this section we will do so for both massive and massless particles.
But first, since the geodesics equation requires the Christoffel's symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}), \quad \partial_\alpha \equiv \partial/\partial x^\alpha \tag{3.14}$$

we might as well calculate them now. Let's write the FRW metric as:

$$ds^2 = -c^2dt^2 + g_{ij}dx^idx^j = -c^2dt^2 + a^2(t)\gamma_{ij}dx^idx^j \tag{3.15}$$

Here we will calculate the Christoffel symbol $\Gamma^0_{\alpha\beta}$ and just list the other ones, since they are only a matter of calculations. Then:

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}g^{0\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) \tag{3.16}$$

Now, since $g^{0\lambda}$ is only nonzero if $\lambda = 0$ and $g_{00} = -1$, we have:

$$\Gamma^0_{\alpha\beta} = -\frac{1}{2}(\partial_\alpha g_{\beta 0} + \partial_\beta g_{\alpha 0} - \partial_0 g_{\alpha\beta}) \tag{3.17}$$

But $g_{00} = \text{const}$ and $g_{0i} = 0$, so that the first two terms vanish, leaving us with:

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\partial_0 g_{\alpha\beta} \tag{3.18}$$

clearly the temporal derivative is only nonzero for the spatial part of the metric, which contains the scale factor $a(t)$:

$$\Gamma^0_{ij} = c^{-1}a\dot{a}\gamma_{ij} \tag{3.19}$$

In all, the Christoffel's read:

$$\begin{aligned}
\Gamma^{\mu}_{00} &= \Gamma^{0}_{\alpha 0} = \Gamma^{\mu}_{\alpha\beta} = 0 \\
\Gamma^{0}_{ij} &= c^{-1} a\dot{a}\gamma_{ij} = c^{-1}\frac{\dot{a}}{a} g_{ij} \\
\Gamma^{i}_{0j} &= c^{-1}\frac{\dot{a}}{a}\delta^{i}_{j} \\
\Gamma^{i}_{jk} &= \frac{1}{2}\gamma^{il}(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk})
\end{aligned} \tag{3.20}$$

All the other ones are related to these by symmetries, because $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$ by virtue of the symmetry of the metric tensor $g_{\alpha\beta} = g_{\beta\alpha}$.
Having calculated them, we note one final thing. Consider the geodesics equation:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 0 \tag{3.21}$$

Using the four momentum of the particle we are considering $P^{\mu} = m\ dx^{\mu}/d\tau$, and the fact that:

$$\frac{d}{d\tau}P^{\mu}(x^{\alpha}(\tau)) = \frac{dx^{\alpha}}{d\tau}\frac{\partial P^{\mu}}{\partial x^{\alpha}} = \frac{P^{\mu}}{m}\frac{\partial P^{\mu}}{\partial x^{\alpha}} \tag{3.22}$$

we can rewrite the geodesics equation as:

$$P^{\alpha}\frac{\partial P^{\mu}}{\partial x^{\alpha}} + \Gamma^{\mu}_{\alpha\beta}P^{\alpha}P^{\beta} = 0 \tag{3.23}$$

This equation is the one we'll use below, since it is pretty handy. However, there is one subtleness that we decided to just gloss over. In fact, for the whole discussion we have used proper time τ as a parameter, but this is exactly zero for massless particles, so it really can't be that useful for them. It turns out that we can just use (3.23) even for photons in the following way. Technically, no one stops us from using a parameter λ to parametrise their path in spacetime, and once we have chosen such a parameter, there is only one continuation of it such that $x(\lambda)$ satisfies the geodesics equation, provided that we interpret $P^{\mu} \equiv dx^{\mu}/d\lambda$ as the four momentum of the massless particle.
At last, we can move on to calculate how massive and massless particles move in spacetime. We work for the moment with $c = 1$.

- Massive particles: let's consider the $\mu = 0$ component of (3.23):

  $$P^0\frac{dP^0}{dt} + \frac{\dot{a}}{a}g_{ij}P^iP^j = 0 \tag{3.24}$$

  where we have used the Christoffel's symbols and $\partial_i P^0 = 0$ because of homogeneity. We also introduce the *physical momentum* $p^2 \equiv g_{ij}P^iP^j$ which obeys the on-shell relation:

  $$-(P^0)^2 + p^2 = -m^2 \tag{3.25}$$

  Differentiating (3.25) we get $c^2P^0dP^0 = pdp$, which, substituted in (3.24) gives:

  $$p\frac{dp}{dt} + \frac{\dot{a}}{a}p^2 = 0 \tag{3.26}$$

  Finally, rearranging we have:

  $$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a} \tag{3.27}$$

This differential equation has a solution $p \sim a^{-1}$, meaning that the particles experience a *drag* given by the expansion of the Universe. This also reveals to us why the comoving coordinates are a natural reference frame, since we have:

$$p = \frac{mv}{\sqrt{1 - v^2}} \sim \frac{1}{a} \tag{3.28}$$

where $v^i = dx^i/dt$ is the peculiar velocity of the particles (remember that the peculiar velocity of an object is the velocity relative to the comoving frame, i.e. how the object deviates from perfect expansion proportional to $a(t)$) and $v^2 = g_{ij}v^i v^j$ as with the momentum. The first equality in (3.28) is just the usual special relativity relationship $dx^\mu/d\tau = U^\mu = (U^0, U^i) = (\gamma, \gamma v^i)$ with γ the local Lorentz factor. Now, consider a particle with initial physical (peculiar) velocity $v(t_1) \equiv v_1$. At a later time t_2 it will have a velocity:

$$v_2 = v_1 \frac{a(t_1)}{a(t_2)} \tag{3.29}$$

Since the Universe is expanding, $a(t)$ is growing, so that $v_2 < v_1$, i.e. even if an observer has a nonzero initial velocity, he will come to rest in the comoving frame. This, after a moment's thought, should be intuitive. After a while the Universe expands so much (if it will do so for a long time), that the observer's velocity will be negligible.

- Massless particles: we can take (3.24) with the new constraint $m = 0$:

  $$-(P^0)^2 + p^2 = 0 \tag{3.30}$$

  Using $P^0 = E$, we find:

  $$\frac{\dot{E}}{E} = \frac{\dot{a}}{a} \tag{3.31}$$

  which again implies that the energy scales as $E \sim a^{-1}$ for photons. This has the physical interpretations that their wavelength gets stretch proportionally to their scale factor, giving rise to such a decrease in energy (given that $E \sim \lambda^{-1}$). Interestingly, this is also the phenomenon at the base of cosmological redshift, which we will touch on in the next section

3.4 Cosmological redshift

An important concept in cosmology is the *cosmological redshift*, since what we know of the properties of the Universe is a result of the observation of distant objects. In order to analyze these correctly, we need to know how light propagates as the Universe evolves in time. Consider the scenario in which a galaxy that sends light signals to us positioned at (r_e, θ_e, ϕ_e). Since the Universe is isotropic and homogeneous we can freely choose the propagation of light to follow a straight line $\theta = \phi =$ const along null geodesics $ds^2 = 0$ in the parametrization (3.10):

$$0 = a(\eta)(-c^2 d\eta^2 + d\chi^2) \rightarrow \Delta\chi(\eta) = \pm c\Delta\eta \tag{3.32}$$

where the $\pm$ corresponds to outgoing and ingoing photons. If a wave crest is emitted at time t_e from the galaxy, the time t_0 when it reaches us is given by:

$$c^2(\eta(t_0) - \eta(t_e)) = \chi(r_e) - \chi(0) = \chi(r_e) \tag{3.33}$$

Since the comoving distance of the galaxy does not change with time, a successive wave crest emitted shortly after, at $t_e + \delta t_e$, reaches the origin at time $t_0 + \delta t_0$:

$$c^2(\eta(t_0 + \delta t_0) - \eta(t_e + \delta t_e)) = \chi(r_e) \tag{3.34}$$

Combining these two equations and noticing that for real applications we can approximate $\delta t_e \ll t_e$ and $\delta t_0 \ll t_0$, we find:

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)} \tag{3.35}$$

where we have used the definition of conformal time. Thus the period of the wave, hence its wavelength, increases proportionally to the scale factor:

$$\frac{\lambda_0}{\lambda_e} = \frac{\nu_e}{\nu_0} = \frac{\delta t_0}{\delta t_e} = \frac{1}{a(t_e)} \tag{3.36}$$

where we remember that we have the freedom to choose $a(t_0) = 1$. Clearly, this is the same thing we found before with $E \sim a^{-1}$, but now in a more concise way. Defining the relative change of wavelength as the *redshift* z, we have:

$$a(t_e) = \frac{1}{1+z} \tag{3.37}$$

A galaxy at redshift $z = 1$ thus emitted the observed light when the Universe was half its current size, a galaxy at redshift 2 when the Universe was one third its size, and so on.

Interestingly, this ties to the observation Hubble was making in the 1920s. To see this, consider the expansion of the wave factor for nearby sources ($z \ll 1$) around $t = t_0$:

$$a(t_e) = 1 + (t_e - t_0)H_0 + ... \tag{3.38}$$

where $t_e - t_0$ is the *look-back time*. Using our new definition of the redshift (3.37) we have $z = H_0(t_0 - t_e) + ...$, but for close objects we can just use $d/c = t_0 - t_e$. Crucially, we also know that the non-relativistic limit of the formula for a longitudinal Doppler shift is:

$$z = \frac{\lambda_0}{\lambda_e} - 1 = \sqrt{\frac{1+\beta}{1-\beta}} - 1 = \sqrt{\frac{1+v/c}{1-v/c}} - 1 \approx \frac{v}{c} \tag{3.39}$$

Putting all these thigs together, we find Hubble law again:

$$v \approx cz \approx H_0 d \tag{3.40}$$

Basically, what Hubble found was a cosmological redshift that masqueraded as a Doppler redshift, given that he was investigating close-by sources. For sources that are very distant, we actually need to pay more attention to how we even define distances, as we will see in the next lecture.

Let's wrap up this discussion now. First of all, the cosmological redshift has nothing to do with the Doppler redshift, for a number of reasons. Note, in fact, that even galaxies that are moving with the Hubble flow will experience a redshift, so velocity has nothing to do with it. Moreover, even if the galaxies had some peculiar velocity, in General Relativity there is no notion of comparing velocities with each other anyway. Having said this, the two notions can be used interchangeably for close sources that are moving "slowly", which is what happened with the Hubble law. Finally, today astronomers use redshift z to measure distances, instead of parsecs. To convert from a redshift to a distance we actually have to be careful, as distances in cosmology are a subtle concept, but for close enough sources we can just use, as we have said, $d = cz/H_0$.