Lecture notes from the course ”Gravità e Superstringhe 1” given by Silke Klemm during the academic year 2021/2022

Emanuele Besana

Contents

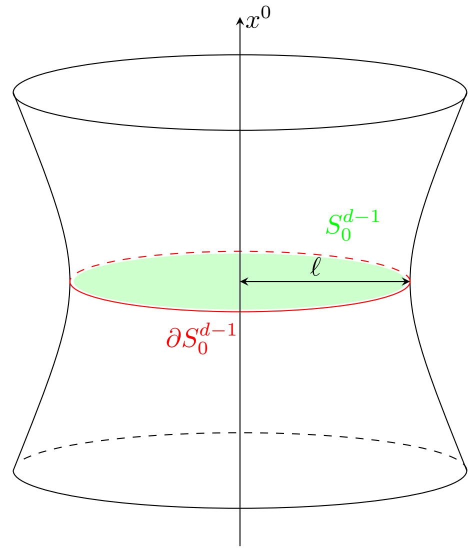


Figure 1: Sketch of anti-de Sitter space.

1 De Sitter spaces

De Sitter spacetime is the maximally symmetric spacetime of constant positive curvature; it is a solution of the vacuum Einstein field equations with positive cosmological constant. This spacetime is directly relevant for observation primarily because there is evidence that the early universe underwent a period of rapid expansion, generally called *inflation*, which turns out to be well approximated by the de-Sitter model. Consider a (d+1) dimensional Minkowski space: R_1^{d+1}, where the upper indexes refer to the total number of dimensions and the lower ones refer to the metric signature (i.e. the number of negative eigenvalues of the metric.). We thus take x^α to be the coordinates with $\alpha = 0, 1, ..., d$ and $\eta = (-1, 1, ..., 1)$ to be the metric.

A d-dimensional *de Sitter space* dS_d can be represented via a hypersurface satisfying:

$$\eta_{\alpha\beta} x^\alpha x^\beta = \ell^2 \tag{1.1}$$

where ℓ is a constant.

From construction it is clear that the isometry group of dS_d is $O(d, 1)$, whose Lie Algebra is $1/2d(d+1)$ dimensional, the same as the Poincaré group in d dimensions. This means that the space is maximally symmetric, since the number of isometries is the number of linearly independent Killing vectors. It is also known that a maximally symmetric space has constant (scalar) curvature, heuristically speaking because it should "look the same" everywhere (as expressed by rotation-like isometries) and the same in every direction (as expressed by rotation like isometries). This implies that:

$$R_{\mu\nu\rho\sigma} = \frac{1}{\ell^2}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}) \tag{1.2}$$

and thus dS_d is, also, an Einstein space, i.e.:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1.3}$$

where $G_{\mu\nu}$ is the Einstein tensor and Λ is the cosmological constant.
One can also show:

$$\Lambda = \frac{(d-2)(d-1)}{2\ell^2} \tag{1.4}$$

A list of coordinate systems:

i) Global coordinates:

$$x^0 = \ell \sinh \frac{\tau}{\ell} \tag{1.5}$$

which takes us to:

$$(x^1)^2 + (x^2)^2 + ... + (x^d)^2 = \ell^2 \cosh^2 \frac{\tau}{\ell} \tag{1.6}$$

We introduce polar coordinate in $\mathbb{R}^d$:

$$\begin{aligned}(dx^1)^2 + (dx^2)^2 + ... + (dx^d)^2 &= dr^2 + r^2 d\Omega_{d-1}^2 \\ &= \sinh^2 \frac{\tau}{\ell} d\tau^2 + \ell^2 \cosh^2 \left(\frac{\tau}{\ell}\right) d\Omega_{d-1}^2\end{aligned} \tag{1.7}$$

Thus the induced metric on dS_d:

$$\begin{aligned}ds^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + ... + (dx^d)^2 \\ &= -d\tau^2 + \ell^2 \cosh^2 \left(\frac{\tau}{\ell}\right) d\Omega_{d-1}^2\end{aligned} \tag{1.8}$$

where in the last equality we have used equation 1.5 and the useful identity:

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.9}$$

This is the $K = 1$ $\Lambda > 0$ solution of the Friedmann equations (*FLRW* model with curvature and cosmological constant. Blau, page 847) We notice that by differentiating two times we find cosh which is positive and thus we have defined an accelerated spacetime. The sphere is zero at $\tau = 0$ and then expands. Equation (1.8) tells us that constant time hypersurfaces are S^{d-1}. In our universe these hypersurfaces are flat! This is the reason we introduce the next coordinates.

ii) Inflationary Coordinates: we introduce coordinates (t, x^i) with $i = 1, ..., d-1$ defined as follows:

$$\begin{aligned} x^0 &= \ell \sinh \frac{t}{\ell} + \frac{\mathbf{x}^2}{2\ell} e^{t/\ell} \\ x^i &= x^i e^{t/\ell} \\ x^d &= \ell \cosh \frac{t}{\ell} - \frac{\mathbf{x}^2}{2\ell} e^{t/\ell} \end{aligned}$$

with:

$$\mathbf{x}^2 = \sum_{i=1}^{d-1} (x^i)^2 \tag{1.10}$$

The induced metric is thus:

$$ds^2 = -dt^2 + e^{2t/\ell} d\mathbf{x}^2 \tag{1.11}$$

where $d\mathbf{x}^2$ is the line element for flat spacetime in $d-1$ dimensions.

This is the $K = 0$ $\Lambda > 0$ solution of the Friedmann equations. Notice also that these coordinates only cover half of the de Sitter space.

iii) Static coordinates: We define r from:

$$(x^0)^2 - (x^d)^2 = r^2 - l^2 \tag{1.12}$$

inserting this in equation (1.2) we find:

$$(x^1)^2 + ... + (x^{d-1})^2 = r^2 \tag{1.13}$$

which defines S^{d-2} with radius r.

We parametrise:

$$\begin{aligned} x^0 &= \sqrt{\ell^2 - r^2} \sinh \frac{t}{\ell} \\ x^d &= \sqrt{\ell^2 - r^2} \cosh \frac{t}{\ell} \end{aligned}$$

which differentiated give us:

$$dx^0 = -\frac{rdr}{\sqrt{\ell^2 - r^2}} \sinh \frac{t}{\ell} + \frac{\sqrt{\ell^2 - r^2}}{\ell} \cosh \frac{t}{\ell} dt$$
$$dx^d = -\frac{rdr}{\sqrt{\ell^2 - r^2}} \cosh \frac{t}{\ell} + \frac{\sqrt{\ell^2 - r^2}}{\ell} \sinh \frac{t}{\ell} dt$$

Introducing spherical coordinates:

$$(dx^1)^2 + ... + (dx^{d-1})^2 = dr^2 + r^2 d\Omega_{d-2}^2 \tag{1.14}$$

we find the induced metric to be:

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \tag{1.15}$$

Where $r = \ell$ we have a *cosmological horizon*. In this coordinate system the metric is time-independent, i.e. ∂_t is a Killing vector.

The *Schwarzschild-de Sitter metric* describes a black hole in dS_d:

$$ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2 \tag{1.16}$$

with:

$$V(r) = 1 - \frac{2M}{r^{d-3}} - \frac{r^2}{\ell^2} \tag{1.17}$$

If $M = 0$ we find dS_d. If $\ell \mapsto \infty$ we find Schwarzschild in d dimensions, where the curvature singularity is in $r = 0$.

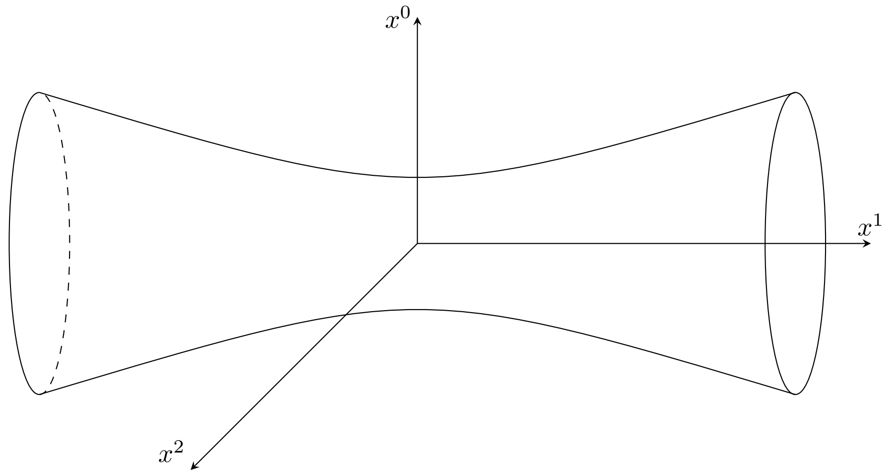


Figure 2: Anti-de Sitter space in two dimensions.

2 Anti-de Sitter spaces

The anti-de Sitter spacetime is the maximally symmetric spacetime of constant negative curvature; it is a solution of the vacuum Einstein field equations with negative cosmological constant. Anti-de Sitter spaces have a wider application in String Theory.

What we consider is $\mathbb{R}^{d+1}_2$. Thus we have two "time coordinates", i.e. a metric with signature -2.

The flat metric we consider is:

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, ..., 1, -1) \tag{2.1}$$

The d-dimensional anti-de Sitter spacetime AdS_d is the embedded hypersurface satisfying:

$$\eta_{\alpha\beta} x^\alpha x^\beta = -\ell^2 \tag{2.2}$$

The isometry group is $O(d-1, 2)$, which means that the space is maximally symmetric, analogously to its de Sitter counterpart.

Example 1. We take anti-de Sitter: AdS_2.

This gives us:

$$-(x^0)^2 + (x^1)^2 - (x^2)^2 = -\ell^2 \tag{2.3}$$

Fixing x^1 means finding a circumference: $(x^0)^2 + (x^2)^2 = \ell^2 + (x^1)^2$, which is a S^1.

Getting back to our main discussion, with the induced metric, AdS_d is a spacetime with constant negative curvature:

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}) \tag{2.4}$$

As said before, it is also solution to the Einstein equations with:

$$\Lambda = -\frac{(d-2)(d-1)}{2\ell^2} \tag{2.5}$$

Some coordinate systems:

i) Global coordinates: we define r with the following parametrisation:

$$(x^0)^2 + (x^d)^2 = r^2 + \ell^2 \tag{2.6}$$

This gives us:

$$(x^1)^2 + ... + (x^{d-1})^2 = r^2 \tag{2.7}$$

and thus a S^{d-2} with radius r. We then parametrise:

$$x^0 = \sqrt{\ell^2 + r^2}\sin\frac{t}{\ell}$$
$$x^d = \sqrt{\ell^2 + r^2}\cos\frac{t}{\ell}$$

Differentiating we find the induced metric:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\Omega^2_{d-2} \tag{2.8}$$

We notice that we can obtain these global coordinates by substituting $\ell \mapsto i\ell$ in the de Sitter static coordinates; this is in fact the negative curvature counterpart of the static de Sitter metric introduced before. Notice also that this metric is static.

With a black hole:

$$ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 d\Omega^2_{d-2} \tag{2.9}$$

where:

$$V(r) = 1 - \frac{2M}{r^{d-3}} + \frac{r^2}{\ell^2} \tag{2.10}$$

where if:

$$\begin{cases} M = 0 \text{ we find d-dimensional anti-deSitter } AdS_d \\ \ell \mapsto +\infty \text{ gives us d-dimesnional Schwarzschild} \end{cases} \tag{2.11}$$

ii) Poincaré coordinates: (z, x^i, t) with $i = 1, ..., d-2$ and $0 < z < +\infty$. We parametrise:

$$x^0 = \frac{z^2 + \ell^2 + \mathbf{x}^2 - t^2}{2z}$$
$$x^i = \frac{\ell}{z} x^i$$
$$x^{d-1} = -\frac{z^2 + \ell^2 + \mathbf{x}^2 - t^2}{2z}$$

and thus:

$$ds^2 = \frac{\ell^2}{z^2}(-dt^2 + dz^2 + d\mathbf{x}^2) \tag{2.12}$$

which is conformally flat. This shouldn't surprise us though, since a well known differential geometry theorem states that every space with constant curvature is conformally flat.

iii) FLRW form (hyperbolic slicing):

$$x^0 = \ell \sin \frac{\tau}{\ell}$$
$$x^i = \ell \cos \frac{\tau}{\ell} (\sinh \rho)$$
$$x^d = \ell \cos \frac{\tau}{\ell} (\cosh \rho)$$

The induced metric is thus:

$$ds^2 = -d\tau^2 + \ell^2 \cos^2 \left(\frac{\tau}{\ell}\right) [d\rho^2 + (\sinh \rho)^2 d\Omega_{d-2}^2] \tag{2.13}$$

where:

$$d\tilde{s}^2 = d\rho^2 + (\sinh \rho)^2 d\Omega_{d-2}^2 \tag{2.14}$$

is defined as the metric of hyperbolic space H^{d-1}. Note that this is the $K = -1$ $\Lambda < 0$ solution to the Friedmann equations.

Some facts on Einstein manifolds:

1. In two dimensions any manifold is an Einstein manifold:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \Rightarrow G_{\mu\nu} = 0 \Rightarrow \Lambda = 0 \tag{2.15}$$

(consequence of Gauss-Bonnet theorem).

2. In $d \geq 3$ we have $\Lambda = const$ because of the 2^{nd} Bianchi identity.

3. Above we have used the fact that a maximally symmetric space has constant curvature and is therefore an Einstein space, actually though the viceversa is not always true. However one can show that in $d = 3$ it holds: an Einstein space exhibits constant curvature.

2.1 The Weyl tensor

In n dimensions, the *Weyl tensor* is defined as:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{n-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu} \quad (2.16)$$

which implies:

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + 2(g_{\mu[\rho}L_{\sigma]\nu} - g_{\nu[\sigma}L_{\sigma[\mu}) \quad (2.17)$$

where:

$$L_{\mu\nu} = \frac{1}{n-2}(R_{\mu\nu} - \frac{R}{2(n-1)}g_{\mu\nu}) \quad (2.18)$$

is the *Schouten tensor*. This way (2.17) corresponds to the decomposition of the Riemann curvature tensor into a traceless-symmetric part (the Weyl tensor) and a non-zero trace part (the Schouten tensor) in the following sense: the Riemann tensor, due to its symmetries ($R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$), defines a symmetric endomorphism on the space of 2-forms, and every symmetric matrix can be decomposed into a symmetric-traceless part and a non-zero trace part.

Notice that in $n = 3$ $C_{\mu\nu\rho\sigma} = 0$, meaning that the Riemann tensor is completely determined by the Schouten tensor that, in an Einstein space, is solely determined by $g_{\mu\nu}$. Two interesting facts about the Weyl tensor[1]:

- The Weyl tensor is strictly connected to gravitational waves, and the fact that it vanishes in 3 dimensions (as we just said) reflects in the fact that in $d = 3 = (2+1)$ there are no gravitational waves, since the number of diffeomorphisms is just enough to set every element of the polarization tensor to zero. To see this, remember the solution to the linearized Einstein equations:
$$\bar{h}_{\mu\nu} = e_{\mu\nu}e^{ik_\rho x^\rho} \quad (2.19)$$
In 3d the $e_{\mu\nu}$ tensor has 6 free components, however the Lorenz gauge implies that $e_{\mu\nu}k^\mu = 0$, thus 3 components are fixed and only 3 free components remain. There are as usual some residual gauge transformations:
$$x^\mu \to x^\mu + \xi^\mu(x) \quad (2.20)$$
that preserve the Lorenz gauge ($\partial^\mu \bar{h}_{\mu\nu} = 0$): imposing this condition one finds $\xi_\mu(x) = b_\mu e^{ik_\rho x^\rho}$, which means that ξ has 3 free components. There remain 0 free components for $e_{\mu\nu}$ in the end.

- The Weyl tensor is conformally invariant, i.e. if $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\Omega(x)}g_{\mu\nu}$ then $\tilde{C}^\mu_{\nu\rho\sigma} = C^\mu_{\nu\rho\sigma}$

[1] For the physical interpretations of the Weyl tensor: http://math.ucr.edu/home/baez/gr/ricci.weyl.html

3 Mauer-Cartan structure equations

[2] Consider the usual connections and curvature formalism, although this time using a non-coordinate basis for the tangent space (i.e. a basis that's not derived from a coordinate system.). It turns out that this slight change in emphasis reveals a different point of view on the connection and curvature, one in which the relationship to gauge theories of particle physics is more transparent.
Up until now the basis vectors for the tangent space $T_p(M)$ was $\hat{e}_{(\mu)} = \partial_\mu$ and $\hat{\theta}^\mu = dx^\mu$ for the dual $T^*_p(M)$, however we could have chosen an arbitrary basis. Let us therefore imagine that at each point in the manifold we introduce a set of basis vectors $\hat{e}_{(a)}$ (indexed by a Latin letter rather than Greek, to remind us that they are not related to any coordinate system) with $a = 0, 1, ..., 1, n-1$ that we take to be orthonormal, in a sense that is appropriate to the signature of the manifold on which we are working. That is, if the canonical form of the metric is written η_{ab}, we demand:

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab} \tag{3.1}$$

where $g(,)$ is the usual metric tensor. Thus, in a Lorentzian spacetime η_{ab} represents the Minkowski metric (while in a space with positive definite metric it would represent the Euclidian metric). As a reference, this new basis $\hat{e}_{(a)}$ is called *tetrad* or *vierben* in 4 dimensions, *dreibein* in 3 dimensions and *vielbein* more generally for any number of dimensions.
Let us expand the vierbein on the coordinate basis:

$$\hat{e}_{(a)} = e^\mu{}_a \hat{e}_{(\mu)} \tag{3.2}$$

where $e^\mu{}_a \in GL(n, \mathbb{R})$ and we denote their inverse by switching idices to obtain $e_\mu{}^a$ which satisfy:

$$e_\mu{}^a e^\mu{}_b = \delta^a_b \quad \text{and} \quad e_a{}^\mu e_\nu{}^a = \delta^\mu_\nu \tag{3.3}$$

With the use of (3.2) equation (3.1) becomes:

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab} \quad \text{or} \quad g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{3.4}$$

This last equation sometimes leads people to say that the vielbeins are the "square root" of the metric. Also it's important to note that the distinction between the vielbeins and their components is blurred, so that we will refer to $e_\mu{}^a$ as the vielbein. We could repeat the steps for $T^*_p(M)$ and define a new non-coordinate basis $\hat{\theta}^{(a)}$ that naturally satisfies:

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta^a_b \tag{3.5}$$

This implies that we can use the same matrix in (3.2) to change the basis for the dual:

[2] This chapter is basically a transcription of appendix J in Carroll's book *Spacetime and Geometry*

$$\hat{\theta}^{(a)} = e_{\mu}{}^{a}\hat{\theta}^{(\mu)} \text{ and } \hat{\theta}^{(\mu)} = e^{\mu}{}_{a}\hat{\theta}^{(a)} \tag{3.6}$$

Example 2. The Schwarzschild metric is known to be:

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2(d\theta^2 + sen^2(\theta)d\varphi^2) \tag{3.7}$$

where $V(r) = 1 - \frac{2m}{r}$. Let's choose the new basis to be:

$$\hat{\theta}^{(0)} = \sqrt{V(r)}dt \tag{3.8}$$

$$\hat{\theta}^{(1)} = \frac{dr}{\sqrt{V(r)}} \tag{3.9}$$

$$\hat{\theta}^{(2)} = rd\theta \tag{3.10}$$

$$\hat{\theta}^{(3)} = rsin(\theta)d\varphi \tag{3.11}$$

now the line element becomes:

$$ds^2 = \eta_{ab}\hat{\theta}^{(a)}\hat{\theta}^{(b)} \tag{3.12}$$

therefore it is easy to see that the only non-zero vielbeins are:

$$e_t{}^0 = \sqrt{V(r)} \tag{3.13}$$

$$e_r{}^1 = \frac{1}{\sqrt{V(r)}} \tag{3.14}$$

$$e_{\varphi}{}^3 = rsen(\theta) \tag{3.15}$$

$$e^2{}_{\theta} = r \tag{3.16}$$

Any other vector can be expressed in terms of its components in the orthonormal basis. If a vector V is written in the coordinate basis as $V^{\mu}\hat{e}_{(\mu)}$ and in the orthonormal non-coordinate basis as $V^a\hat{e}_{(a)}$, the sets of components will be related by:

$$V^a = e_{\mu}{}^{a}V^{\mu} \tag{3.17}$$

therefore the vielbeins allow us to "switch from Latin to Greek indices and back". Now looking back at the first equation of (3.4), we see that the components of the metric tensor in the orthonormal basis are just those of the flat metric, η_{ab}.(In fact we can go so far as to raise and lower the Latin indices using the flat metric and its inverse η^{ab}).
Now that we have non-coordinate bases, these bases can be changed independently of the coordinates. The only restriction is that the orthonormality property (3.5) be preserved; but we know what kind of transformations preserve the flat metric: they are the Lorentz transformations. We therefore consider changes of basis of the form:

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda^{a}{}_{a'}(x)\hat{e}_{(a)} \tag{3.18}$$

where the matrices $\Lambda^{a}{}_{a'}(x)$ represent position-dependent transformations which (at each point) leave the form of the metric unchanged:

$$\Lambda^{a}{}_{a'}\Lambda^{b}{}_{b'}\eta_{ab} = \eta_{a'b'} \tag{3.19}$$

So we now have the freedom to perform a Lorentz transformation at every point in spacetime. These transformations are therefore called *local Lorentz transformations* (LLT). We still have our usual freedom to make changes in coordinates, which are called *general coordinate transformations* (GCT). Both can happen at the same time, resulting in a mixed tensor transformation law:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_{a}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\Lambda^{b}{}_{b'}\frac{\partial x^{\nu}}{\partial x^{\nu'}}T^{a\mu}{}_{b\nu} \tag{3.20}$$

The important stuff happens when we try to differentiate: in our usual formalism, the covariant derivative of a tensor is given by its partial derivative plus correction coefficients that guarantee the right transformation law under GCTs.
The same procedure will continue to be true for the non-coordinate basis, but we replace the Christoffel symbols with the so called *spin connection* denoted $\omega_{\mu}{}^{a}{}_{b}$. Basically every index gives rise to a correction term that contains this new connection:

$$\nabla_{\mu}X^{a}{}_{b} = \partial_{\mu}X^{a}{}_{b} + \omega_{\mu}{}^{a}{}_{c}X^{c}{}_{b} - \omega_{\mu}{}^{c}{}_{b}X^{a}{}_{c} \tag{3.21}$$

this now transforms correctly as (3.20). (The name ”spin connection” comes from the fact that this can be used to take covariant derivatives of spinors, which is actually impossible using the conventional connection coefficients.)

In order to find a relationship between the spin connection and the Christoffels, we use the fact that a tensor needs to be independent of the way it is written. Consider on the one hand:

$$\begin{aligned}\nabla X =&(\nabla_{\mu}X^{\nu})dx^{\mu}\otimes\partial_{\nu}\\ =&(\partial_{\mu}X^{\nu} + \Gamma^{\nu}_{\mu\lambda})dx^{\mu}\otimes\partial_{\nu}\end{aligned} \tag{3.22}$$

now the same object can be rewritten as:

$$\begin{aligned}\nabla X =&(\nabla_{\mu}X^{a})dx^{\mu}\otimes\hat{e}_{(a)}\\ =&(\partial_{\mu}X^{a} + \omega_{\mu}{}^{a}{}_{b}X^{b})dx^{\mu}\otimes\hat{e}_{(a)}\\ =&(\partial_{\mu}(e_{\nu}{}^{a}X^{\nu}) + \omega_{\mu}{}^{a}{}_{b}e_{\lambda}{}^{b}X^{\lambda})dx^{\mu}\otimes(e^{\sigma}{}_{a}\partial_{\sigma})\\ =&e^{\sigma}{}_{a}(e_{\nu}{}^{a}\partial_{\mu}X^{\nu} + X^{\nu}\partial_{\mu}e_{\nu}{}^{a} + \omega_{\mu}{}^{a}{}_{b}e_{\lambda}{}^{b}X^{\lambda})dx^{\mu}\otimes\partial_{\sigma}\\ =&(\partial_{\mu}X^{\nu} + e^{\nu}{}_{\lambda}\partial_{\mu}e_{\lambda}{}^{a}X^{\lambda} + e^{\nu}{}_{a}e_{\lambda}{}^{b}\omega_{\mu}{}^{a}{}_{b}X^{\lambda})dx^{\mu}\otimes\partial_{\nu}\end{aligned} \tag{3.23}$$

now comparing (3.22) to (3.23) we can easily find that:

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a \tag{3.24}$$

We can actually rewrite equation (3.24) in a more compact way. Notice in fact that:

$$\nabla_\mu e^a{}_\nu = 0 \Longleftrightarrow \partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu} e_\rho{}^a + \omega_\mu{}^a{}_b e^b_\nu = 0 \tag{3.25}$$

which, contracting with $e^\nu{}_c$, becomes:

$$e^\nu{}_c \partial_\mu e_\nu{}^a - e^\nu{}_c \Gamma^\rho_{\mu\nu} e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b e^\nu{}_c = 0 \tag{3.26}$$

and remembering that $e_\nu{}^b e^\nu{}_c = \delta^b_c$ gives us (3.24), thus the relationship between the Christoffels and the spin connection is summarized by:

$$\nabla_\mu e^a{}_\nu = 0 \tag{3.27}$$

The application of all this mathematical jargon comes from a slight change in viewpoint, where we think of various tensors as tensor-valued differential forms. For example, an object like $X_\mu{}^a$, which we think as a (1,1) tensor, can also be thought of as a "vector-valued one form". It has one lower Greek index, so we can think of it as a one form, but for each value of the lower index it is a vector. Similarly a tensor $A_{\mu\nu}{}^a{}_b$, antisymmetric in μ and ν, can be thought of as a "(1,1)-tensor-valued two-form".
The usefulness of this viewpoint comes from considering exterior derivatives[3]:

$$(dX)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a \tag{3.29}$$

where we have considered $X_\mu{}^a$ as a one form, i.e. $X_\mu{}^a dx^\mu$. This object transforms as a two-form under CGTs, but not as a vector under LLTs. However the object:

$$(dX)_{\mu\nu}{}^a + (\omega \wedge X)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a + \omega_\mu{}^a{}_b X_\nu{}^b - \omega_\nu{}^a{}_b X_\mu{}^b \tag{3.30}$$

transforms correctly both under CGTs and LLTs. One can think of the l.h.s. of the equation above as a "covariant exterior derivative".
An immediate application of this formalism is to the expressions for the torsion and the curvature. The torsion, since it is antisymmetric in its two lower indices, can be thought of as a vector-valued two form $T_{\mu\nu}{}^a$. The curvature, which is always antisymmetric in its last two indices, is a (1-1)-tensor-valued two-form, $R^a{}_{b\mu\nu}$. Expressing these in the basis one-forms $e^a = e_\mu{}^a dx^\mu$ and the

[3]Recall that for a 1-form $\omega = A_j dx^j$, its exterior derivative is:

$$d\omega = \frac{\partial A_j}{\partial x^k} dx^k \wedge dx^j \tag{3.28}$$

spin-connection one forms $\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu$. The defining relations for the torsion and curvature are then:

$$T^a = de^a + \omega^a{}_b \wedge e^b \tag{3.31}$$

and

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{3.32}$$

equations (3.30) and (3.31) are known as the *Mauer-Cartan structure equations*. These are equivalent to the usual definitions of course, and we can actually check for the torsion very rapidly:

$$\begin{aligned} T_{\mu\nu}{}^\lambda &= e^\lambda{}_a T_{\mu\nu}{}^a \\ &= e^\lambda{}_a(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b) \\ &= \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \end{aligned} \tag{3.33}$$

So far our equations have been true for general connections; let's see what we get for the Christoffel connection. Metric compatibility is expressed as the vanishing of the covariant derivative of the metric : $\nabla g = 0$, which, when expressed in orthonormal coordinates, becomes:

$$\begin{aligned} \nabla_\mu \eta_{ab} &= \partial_\mu \eta_{ab} - \omega_\nu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} \\ &= -\omega_{\mu ab} - \omega_{\mu ba} \end{aligned} \tag{3.34}$$

thus setting it equal to zero gives:

$$\omega_{\mu ab} = -\omega_{\mu ba} \tag{3.35}$$

The last application of the vielbeins comes from taking the exterior derivative of the structure equations:

$$\begin{aligned} dT^a &= dde^a + d(\omega^a{}_b \wedge e^b) \\ &= d\omega^a{}_b \wedge e^b - \omega^a{}_b \wedge de^b \end{aligned} \tag{3.36}$$

now using the Mauer-Cartan equations one finds this becomes:

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \tag{3.37}$$

Analogously for (3.32):

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \tag{3.38}$$

Now equation (3.36) is the first Bianchi identity for a torsion-free spacetime ($T^a = 0$):

$$
\begin{aligned}
R^a{}_b \wedge e^b \; &\Longrightarrow R^a{}_{b\mu\nu} e_\rho{}^b dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \\
&\Longrightarrow R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \\
&\Longrightarrow R^a{}_{[\rho\mu\nu]} = 0
\end{aligned} \tag{3.39}
$$

whereas (3.37) is the Bianchi identity $\nabla_{[\lambda|} R^\rho{}_{\sigma|\mu\nu]}$.

Example 3. We want to calculate the curvature for the metric:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + sen^2(\theta)d\varphi^2) \tag{3.40}$$

Let's choose the new basis to be:

$$\hat{\theta}^{(0)} = \sqrt{f(r)}dt \tag{3.41}$$

$$\hat{\theta}^{(1)} = \sqrt{h(r)}dt \tag{3.42}$$

$$\hat{\theta}^{(2)} = rd\theta \tag{3.43}$$

$$\hat{\theta}^{(3)} = rsin(\theta)d\varphi \tag{3.44}$$

Change of notation: $\hat{\theta}^{(b)} \equiv e^a$:

$$de^0 = -\frac{f'}{2\sqrt{f}}\frac{e^0 \wedge e^1}{\sqrt{hf}} \tag{3.45}$$

Comparing this with (3.31) with a=0 and in a torsion-free spacetime:

$$de^0 + \omega^0{}_b \wedge e^b = 0 \tag{3.46}$$

we find that:

$$\omega^0{}_1 = \frac{f'}{2f\sqrt{h}}e^0 \tag{3.47}$$

Doing the same for the other indices we find that the only non zero terms of the spin connection are:

$$\omega^2{}_1 = \frac{1}{r\sqrt{h}}e^2 \tag{3.48}$$

$$\omega^3{}_1 = \frac{1}{r\sqrt{h}}e^3 \tag{3.49}$$

$$\omega^3{}_2 = \frac{cot(\theta)}{r}e^3 \tag{3.50}$$

$$\tag{3.51}$$

inserting in (3.32) gives the components of the Riemann tensor.

4 Lagrange Formulation of General Relativity

Spacetime is a manifold M with a metric of Lorentzian signature. An action is an integral over a region $\mathcal{V}$ of the manifold. A metric defined on the manifold provides us with a canonical volume form thus with a way of integrating a scalar function over a region of spacetime. The simplest non-trivial scalar function coming from the metric is the Ricci scalar. This motivates us to introduces the *Einstein-Hilbert action*:

$$S_{EH}[g_{\alpha\beta}] = \frac{1}{16\pi}\int_{\mathcal{V}} \sqrt{-g}d^4xR \tag{4.1}$$

To ensure the well-posedness of the action functional, as we will see in section 4.2, it turns out that we need the addition of the so called *Gibbons-Hawking-York boundary term*:

$$S_{GHY}[g_{\alpha\beta}] = \frac{1}{8\pi}\oint_{\partial\mathcal{V}} \epsilon K\sqrt{h}d^3y \tag{4.2}$$

where K is the trace of the extrinsic curvature of $\partial\mathcal{V}$, ϵ is +1 or -1 where the boundary is respectively timelike or spacelike and h is the determinant of the induced metric on $\partial\mathcal{V}$.

To define a scale of values for the action with some sense, as we'll see in section 4.4, it is useful to consider everything being embedded in flat spacetime. This translates to adding a non-dynamical term to the action:

$$S_0[g_{\alpha\beta}] = \frac{1}{8\pi}\oint_{\partial\mathcal{V}} \epsilon K_0\sqrt{h}d^3y \tag{4.3}$$

where K_0 is the extrinsic curvature of $\partial\mathcal{V}$ embedded in flat spacetime.

Thus, the gravitational action $S_G[g]$ comprises three terms:

$$S_G[g_{\alpha\beta}] = S_{EH}[g_{\alpha\beta}] + S_{GHY}[g_{\alpha\beta}] - S_0 \tag{4.4}$$

If matter enters the scene one adds a matter interaction term:

$$S_M[\psi, g_{\alpha\beta}] = \int_{\mathcal{V}} \mathcal{L}(\psi, \psi_{,\alpha}, g_{\alpha\beta})\sqrt{-g}d^4x \tag{4.5}$$

where $L(\psi, \psi_{,\alpha}, g_{\alpha\beta})$ is some Lagrangian describing the interaction and ψ is the matter field.

4.1 Variation of the Einstein-Hilbert Action

Upon a variation of the metric:

$$g_{\alpha\beta}(x) \longmapsto g_{\alpha\beta}(x) + \delta g_{\alpha\beta} \tag{4.6}$$

the Einstein-Hilbert action varies as follows:

$$(16\pi)\,\delta S_{EH} = \int_{\mathcal{V}} \left[\delta(\sqrt{-g})R + \sqrt{-g}\delta(g^{\alpha\beta})R_{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta(R_{\alpha\beta})\right] d^4x \tag{4.7}$$

Using the following identities:

$$g_{\gamma\beta}g^{\alpha\gamma} = \delta^{\alpha}_{\beta} \Rightarrow \delta(g_{\gamma\beta})g^{\alpha\gamma} + g_{\gamma\beta}\delta(g^{\alpha\gamma}) = 0 \Rightarrow \delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\nu\beta}\delta g_{\mu\nu} \tag{4.8}$$

and the following result:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{4.9}$$

whose proof is found in Appendix **??**, one can find the total variation to be the following:

$$(16\pi)\,\delta S_{EH} = \int_{\mathcal{V}} \sqrt{-g}d^4x(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta})\delta g^{\alpha\beta} + \int_{\mathcal{V}} \sqrt{-g}d^4x\; g^{\alpha\beta}\delta R_{\alpha\beta} \tag{4.10}$$

There is no particularly elegant way of showing that the second integral is one of a total derivative.

We start by writing down the explicit expression of the Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} \tag{4.11}$$

Its variations will be:

$$\delta R_{\mu\nu} = \partial_\lambda\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu\delta\Gamma^\lambda_{\mu\lambda} + \delta(\Gamma^\lambda_{\lambda\rho})\Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\lambda\rho}\delta(\Gamma^\rho_{\mu\nu}) - \delta(\Gamma^\lambda_{\nu\rho})\Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\nu\rho}\delta(\Gamma^\rho_{\mu\lambda}) \tag{4.12}$$

The Christoffel symbols themselves are not tensors but their variation turn out to be tensors. In normal coordinates one can easily derive the following:

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}) \tag{4.13}$$

This is because in normal coordinates the Christoffel symbols are null, thus we can replace partial derivatives with covariant derivatives. Due to both terms being tensors in the last equation, this equality holds in any coordinate system.

All of this means that for equation 4.12 one simply drops terms involving undifferentiated Christoffel symbols and replaces partial derivatives with covariant derivatives. Thus we find:

$$\delta R_{\mu\nu} = \nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda} \tag{4.14}$$

and knowing that $\nabla_X = 0$, we find:

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\Gamma^\lambda_{\mu\lambda}) \\ &= \nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\Gamma^\nu_{\mu\nu}) \end{aligned} \tag{4.15}$$

Using the given explicit expression for the variation of the Christoffel symbols:

$$\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\delta g_{\mu\nu} \\
&= (g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\mu\nabla_\nu\delta g_{\alpha\beta} \\
&= \nabla_\lambda\left((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta}\right)
\end{aligned} \tag{4.16}$$

Thus we have almost concluded showing that the last term in the variation of the Einstein-Hilbert action turns out to be a total derivative and thus a boundary term:

$$\begin{aligned}
\int_{\mathcal{V}} \sqrt{-g}d^4x\ \nabla_\lambda\Delta B^\lambda &= \epsilon\oint_{\partial\mathcal{V}} \sqrt{h}d^3yN_\lambda(g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta} \\
&= \epsilon\oint_{\partial\mathcal{V}} \sqrt{h}d^3yN_\lambda(g^{\lambda\alpha}\partial^\beta\delta g_{\alpha\beta} - g^{\alpha\beta}\partial^\lambda\delta g_{\alpha\beta}) \\
&= -\epsilon\oint_{\partial\mathcal{V}} \sqrt{h}d^3yg^{\alpha\beta}\partial_\lambda(\delta g_{\alpha\beta})N^\lambda \\
&= -\epsilon\oint_{\partial\mathcal{V}} \sqrt{h}d^3yh^{\alpha\beta}\partial_\lambda(\delta g_{\alpha\beta})N^\lambda
\end{aligned} \tag{4.17}$$

where ΔB^λ is the argument of the total derivative and ϵ is the modulus of the normal vector to the hypersurface.

4.2 Variation of the Gibbons-Hawking-York Boundary Term

For the well-posedness of the action principle we want an action whose variation doesn't comprise boundary terms, thus giving us the necessity of adding the Gibbons-Hawking-York boundary term to the gravitational action. We recall the boundary term:

$$S_{GHY}[g_{\alpha\beta}] = \frac{1}{8\pi}\oint_{\partial\mathcal{V}} \epsilon K\sqrt{h}d^3y \tag{4.18}$$

Because the induced metric is fixed on the boundary itself , a variation of the Gibbons-Hawking-York term yields:

$$(16\pi)\delta S_{GHY} = 2\oint_{\partial\mathcal{V}} \epsilon\delta K\sqrt{h}d^3y \tag{4.19}$$

The trace of the extrinsic curvature K is the following:

$$\begin{aligned}
K &= g^{\alpha\beta}K_{\alpha\beta} \\
&= (h^{\alpha\beta} + \epsilon N^\alpha N^\beta)\nabla_\alpha N_\beta \\
&= h^{\alpha\beta}\nabla_\alpha N_\beta \\
&= h^{\alpha\beta}(\partial_\alpha N_\beta - \Gamma^\lambda_{\alpha\beta}N_\lambda)
\end{aligned} \tag{4.20}$$

Thus a variation of the extrinsic curvature yields the following:

$$
\begin{aligned}
\delta K &= -h^{\alpha\beta}\delta\Gamma^{\lambda}_{\alpha\beta}N_{\lambda} \\
&= -\frac{1}{2}h^{\alpha\beta}(\partial_{\alpha}\delta g_{\sigma\beta} + \partial_{\beta}\delta g_{\alpha\sigma} - \partial_{\sigma}\delta g_{\alpha\beta})N^{\sigma} \\
&= \frac{1}{2}h^{\alpha\beta}\partial_{\lambda}(\delta g_{\alpha\beta})N^{\lambda}
\end{aligned}
\tag{4.21}
$$

where we have used the fact that the variation of the induced metric on the boundary is null and the same goes for the variation of the normal vector and that the tangential derivatives of $\delta g_{\alpha\beta}$ vanish on $\partial\mathcal{V}$.

Thus:

$$
(16\pi)\delta S_{GHY} = \epsilon \oint_{\partial\mathcal{V}} \sqrt{h}d^3y h^{\alpha\beta}\partial_{\lambda}(\delta g_{\alpha\beta})N^{\lambda} \tag{4.22}
$$

which is opposite to the boundary term arising from the variation of the Einstein-Hilbert action.

Thus the variation of the gravitational action is the following:

$$
(16\pi)\delta S_G = \int_{\mathcal{V}} \sqrt{-g}d^4x G_{\alpha\beta}\delta g_{\alpha\beta} \tag{4.23}
$$

where:

$$
G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \tag{4.24}
$$

is the Einstein tensor. Thus asking $\delta S_G = 0$ gives us:

$$
R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 0 \tag{4.25}
$$

which are the *Einstein field equations* in a vacuum.

4.3 Coupling with Matter

If matter is involved, as already previously mentioned, we need to add a further term S_M to the action, whose variation is the following:

$$
\begin{aligned}
\delta S_M &= \int_{\mathcal{V}} \left(\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g} + \mathcal{L}\delta\sqrt{-g} \right) d^4x \\
&= \int_{\mathcal{V}} \left(\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2}\mathcal{L}g_{\alpha\beta} \right) \delta g^{\alpha\beta}\sqrt{-g}d^4x
\end{aligned}
\tag{4.26}
$$

We define the *stress-energy tensor* to be the following:

$$
T_{\alpha\beta} = -2\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}} + \mathcal{L}g_{\alpha\beta} \tag{4.27}
$$

Then we have:

$$\delta S_M = -\frac{1}{2}\int_{\mathcal{V}} T_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x \tag{4.28}$$

and thus the Einstein-Field equations in the presence of matter, obtained by asking $\delta(S_G + S_M) = 0$, are the following:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{4.29}$$

Thus the Einstein-Field equations have been obtained from a variational principle.

4.4 Non-Dynamical Term

The need for a non-dynamical term can be explained by calculating the action in a vacuum and in flat spacetime, i.e. R=0.

$$S_G[g_{\alpha\beta}] = S_{GHY}[g_{\alpha\beta}] = \frac{1}{8\pi}\oint_{\partial\mathcal{V}} \epsilon K\sqrt{h}d^3y \tag{4.30}$$

We can choose the hypersurface $\partial\mathcal{V}$, boundary to the region of integration, to be composed by two hypersurfaces of constant time t_0 and $t_1 > t_0$ and a three-cylinder with $r = R$. Thus the induced metric on the hypersurface is the following:

$$ds^2 = -dt^2 + R^2(d\theta^2 + sin^2\theta d\phi^2) \tag{4.31}$$

which gives us $\sqrt{h} = R^2 sin\theta$. As found in example **??**, $K = 0$ on the constant time spacelike ($\epsilon = -1$) hypersurfaces and $K = 2/R$ on the timelike ($\epsilon = +1$) hypersurface of constant $r = R$.

Thus:

$$S_G = \frac{1}{8\pi}\oint_{\partial\mathcal{V}} \epsilon K\sqrt{h}d^3y = 8\pi R(t_1 - t_0) \tag{4.32}$$

which diverges when considering the whole of spacetime ($R \to +\infty$).

This would then give us a non-well-posed action principle, because for asymptotically flat spacetimes the action diverges and thus is not a well defined quantity.

This problem is remedied by the non-dynamical S_0 term in the gravitational action by asking K_0 to be the extrinsic curvature of the boundary $\partial\mathcal{V}$ embedded in flat spacetime. With this definition $S_{GHY} - S_0$ is asymptotically well defined and, in particular, $S_G = 0$ for flat spacetime.

5 Black Holes in anti-de Sitter space

A black hole in d-dimensional anti-de Sitter is the following:

$$ds^2 = -\left(k - \frac{m}{r^{d-3}} + \frac{r^2}{\ell^2}\right) dt^2 + \left(k - \frac{m}{r^{d-3}} + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_k^2 \tag{5.1}$$

where:

$$d\Omega_k^2 = \begin{cases} d\psi^2 + \sin^2\psi d\sigma_{d-3}^2 \qquad k=1 \qquad \Rightarrow \qquad S^{d-2} \\ d\psi^2 + \sinh^2\psi d\sigma_{d-3}^2 \qquad k=-1 \qquad \Rightarrow \qquad H^{d-2} \\ (dx^1)^2 + ... + (dx^{d-2})^2 \qquad k=0 \qquad \Rightarrow \qquad E^{d-2} \end{cases} \tag{5.2}$$

where $d\sigma_{d-3}^2$ is the standard metric on a $d-3$-dimensional sphere, thus:

- S^{d-2} is a $d-2$-dimensional sphere;
- H^{d-2} is a $d-2$-dimensional hyperbolic space;
- E^{d-2} is a $d-2$-dimensional euclidean space.

As a sidenote, equation (5.1) is the most general spherically-symmetric solution of the Einstein equations with negative cosmological constant, with $T_t^t = T_r^r = -\frac{\Lambda}{8\pi G}$, i.e. a black hole in an AdS spacetime. (see Blau p. 883). We concentrate on $d = 4$-dimensional cases:

1. $k = 0$ case:
   We start with an example of compactification of the $(x - y)$-plane as in Figure 3, where identifying alternate parallel segments with each other we obtain a torus known as the *Clifford torus*.

   Due to the $x - y$ plane being flat, the torus also is "flat" because induced metric on the torus is the same as 2-dimensional euclidean space. We cannot embed this torus in E^3, but we can do so in S^3 with the following reasoning. We start by writing the S^3 metric:

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \tag{5.3}$$

   where $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq 2\pi$??? and $0 \leq \psi \leq 2\pi$ are the *Hopf coordinates* for the S^3 sphere.

   The S^3 sphere can be embedded in E^4 by the following:

$$\begin{aligned} x^1 &= \cos\phi \sin\theta \\ x^2 &= \sin\phi \sin\theta \\ x^3 &= \cos\psi \cos\theta \\ x^4 &= \sin\psi \cos\theta \end{aligned} \tag{5.4}$$

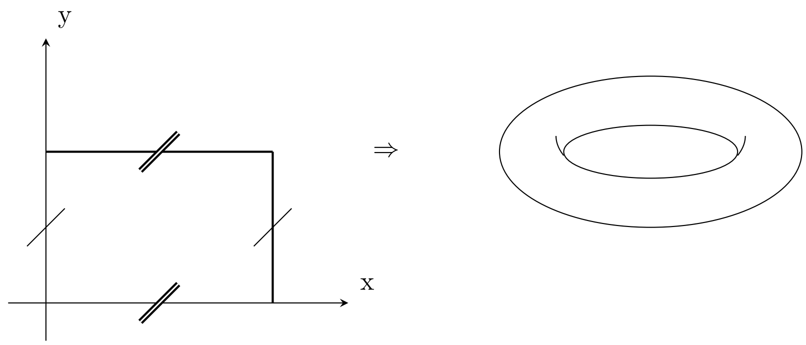


Figure 3: Compactification of the $x - y$ plane to a torus.

and by asking $\theta = const$ we find a Clifford torus. This is because it provides us with $d\theta = 0$ which plugged in the above metric of S^3 gives us the 2-dimensional euclidean space metric, which, as mentioned previously, is the induced metric on the torus.

Thus one finds that black holes can be of a torus shape.

N.B. In general a torus is characterised by a complex number τ, with $Im\tau > 0$, known as the *Teichmüller parameter* which tells us that the torus can also be obtained by identifying in pairs the opposite sides of a parallelepiped.
This is the way one can compactify the black hole horizon in AdS spacetime (why was it not compact??)

2. $k = -1$ case:
We start in two-dimensional hyperbolic space H^2. The standard metric on the two-sphere thus takes the form:

$$d\Omega_{-1}^2 = d\psi^2 + \sinh^2 \psi d\phi^2 \tag{5.5}$$

where ψ is thus seen as some sort of radial coordinate.

We want to compactify H^2, the simplest way is taking a geodetic octagon as in Figure 4, this way obtaining a compact Riemann surface of genus $g = 2$. In order to obtain a Riemann surface with $g > 2$ we would need to start with a more compact (?) polygon. Why was the simplest polygon an octagon? Couldn't have we started with a 4 side polygon? Generally the answer is no, because of the Gauss-Bonnet theorem.

We have thus found that black holes can have an arbitrary number of "holes".

Questions:

- "Can the holes communicate?"

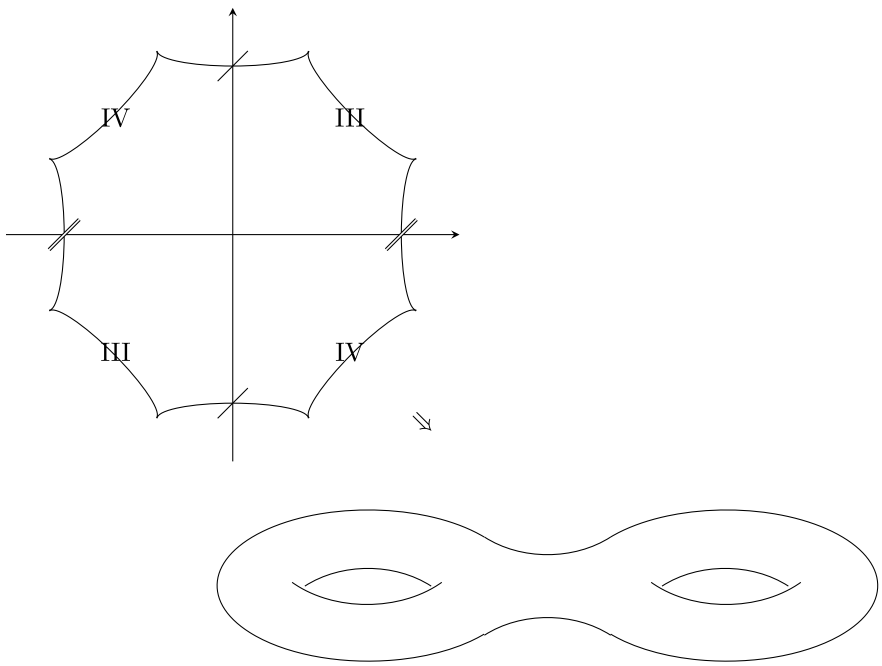


Figure 4: Compactification of H^2.

- ”What is the black hole, is it in the two holes or is it the thickness?”
- lolol

6 Black Holes Axiomatics

Here we consider $(M, <,>)$ to be a pseudo-Riemannian manifold with $<,>$ a scalar product of signature $(-,+,...,+)$.

We start by noting that for any point $\underline{x} \in M$ we are provided with the space tangent to the manifold in $\underline{x} \in M$ denoted as $T_{\underline{x}}$. Thus we define the light-cone in $\underline{x} \in M$ as:

$$C_{\underline{x}} = \{v \in T_{\underline{x}} : v_\mu v^\mu = 0\}. \tag{6.1}$$

The future semi-cone is thus defined as:

$$C^+_{\underline{x}} = \{v \in T_{\underline{x}} : v_\mu v^\mu = 0 \wedge v^0 > 0\} \tag{6.2}$$

and the past semi-cone as:

$$C^-_{\underline{x}} = \{v \in T_{\underline{x}} : v_\mu v^\mu = 0 \wedge v^0 < 0\}. \tag{6.3}$$

By choosing one of the two semi-cones to be the future, for instance, we say $T_{\underline{x}}$ is time-oriented.

Definition 6.1. (*Time-Orientable Manifold*)
If $T_{\underline{x}}$ is continuously time-oriented $\forall x \in M$, then M is called a *Time-Orientable manifold.*

Basically, at each $p \in M$ we are designating half of the light cone as "future" and the other half as "past". If such a separation can be made continuously over the manifold, then it is said to be time orientable. Non-time orientable manifolds have thus the phyisically pathological property that we cannot consistently distinguish between the notions of "going forward in time" as opposed to "going backward in time". To rigorously define a black hole we need a precise notion of infinity.

Definition 6.2. (*Chronological future* (*or past*))
The chronological future in a point $p \in M$, denoted as $I^+(p)$, is the set of points approachable from p with a time-like curve, i.e.:

$$I^+(p) := \{q \in M |\ \exists\, c : [a,b] \to M;\ c(a) = p;\ c(b) = q;\ <\dot{c}, \dot{c}>< 0\}.$$

As a nomenclature, one also says that in the case above, $\dot{c}$ is *future directed.* The chronological past $I^-(p)$ is defined analogously.

With $I^-(p)$ we have thus identified the set of points that can influence materially p and with $I^+(p)$ we have identified the set of point that can be influenced materially by p and thus also by $I^-(p)$. We specify materially because the curve c is required to satisfy $<\dot{c}, \dot{c}>< 0$, thus information that arrives in p via null curves, i.e. light signals, is not contemplated.

Definition 6.3. (*Causal future*) The causal future in a point $p \in M$, denoted as $J^+(p)$, is the set of points approachable from p with a time-like or light-like curve, i.e.:

$$J^+(p) := \{q \in M | \ \exists \ c : [a,b] \to M; \ c(a) = p; \ c(b) = q; \ < \dot{c}, \dot{c} > \leq 0\}.$$

The causal past $J^-(p)$ is defined analogously.

Now any point in $J^-(p)$ can influence p and thus $J^+(p)$ as well.

Definition 6.4. (*Strong Causality*) In $p \in M$ strong causality is satisfied if any neighbourhood of p contains a neighbourhood of p that is not intersected more than once by any causal curve, i.e. $< \dot{c}, \dot{c} > \leq 0$.

A manifold M is strongly causal if the above condition satisfied $\forall p \in M$.

In a strongly causal manifold M there can be no closed light-like curves: the future cannot influence the past. There can be no snake that bites its own tail.

Nietzsche would be really happy in *Gödel's universe*, where strong causality is not satisfied.

Definition 6.5. (*Asymptotic Simplicity*)
A manifold M is asymptotically simple and empty if there exists an embedding $f : (M,g) \mapsto (\tilde{M}, \tilde{g})$ in a strongly causal manifold $\tilde{M}$ with a boundary I satisfying:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \text{in} \quad \tilde{M} \cap M.$$

(More precisely if $\Omega^2 g = f_*(\tilde{g})$ on M) where Ω is a differentiable function and:

i) $\Omega = 0$ and $d\Omega \neq 0$ on I;

ii) Any null geodesic on M has two extremes in I;

iii) $R_{\mu\nu} = 0$ in an open neighbourhood of I on $\tilde{M}$.

N.B.

- What is the reason for condition *i*) ? Two affine parameters λ and $\tilde{\lambda}$ of the two metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are related by a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ as follows:
$$\frac{d\lambda}{d\tilde{\lambda}} = \frac{1}{\Omega^2} \tag{6.4}$$
therefore if $\Omega = 0$ on I, $\int d\lambda$ diverges, which is what we want, since I is the null future infinity.

- A brief comment on condition *ii*) is also needed. Under a conformal transformation of the metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, the scalar curvature transforms as (Wald appendix p.446)(in $d = 4$):
$$\tilde{R} = \Omega^{-2} R - 6\Omega^{-1}(\Omega)|_{\mu\nu}\tilde{g}^{\mu\nu} + 3(\Omega^{-2})|_{\mu}(\Omega)|_{\nu}\tilde{g}^{\mu\nu} \tag{6.5}$$

where this notation is used:

$$(\cdot)|_\mu = \nabla_\mu(\cdot) \tag{6.6}$$

Now let Ω be at least C^3 on $\tilde{M}$, this then implies that $\tilde{g}$ is C^3, which also implies that $\tilde{R}$ is C^1 even on I, where $\Omega = 0$. Now multiply (6.5) by Ω^2:

$$\underbrace{\Omega^2 \tilde{R}}_{=0 \text{ on } I} = R \underbrace{-6\Omega(\Omega)|_{\mu\nu}\tilde{g}^{\mu\nu}}_{=0 \text{ on } I} + 3(\Omega)|_\mu(\Omega)|_\nu\tilde{g}^{\mu\nu} \tag{6.7}$$

which implies:

$$(\Omega)|_\mu(\Omega)|_\nu\tilde{g}^{\mu\nu} = 0 \qquad \text{on } I \tag{6.8}$$

therefore I is a null hypersurface.

We follow this definition with an example in two-dimensional Minkowski space, to elucidate a useful tool: *Penrose diagrams*.

Example 4. Two-dimensional Minkowski space is described by the metric:

$$ds^2 = -dt^2 + dx^2. \tag{6.9}$$

In null coordinates $u = t - x$ and $v = t + x$, the metric is rewritten as:

$$ds^2 = -du dv \tag{6.10}$$

And in coordinates $u =: tan(U/2)$ and $v =: tan(V/2)$ we find:

$$ds^2 = -\frac{1}{4\cos^2\frac{U}{2}\cos^2\frac{V}{2}} dU dV \tag{6.11}$$

with $-\pi \leq U, V \leq \pi$.

Thus we have shown:

$$\Omega^{-1} = \frac{1}{2\cos(U/2)\cos(V/2)} \tag{6.12}$$

which tells us that the Minkowski metric is conformal to $d\tilde{s}^2 = -dUdV$. Indeed, we have that the two metrics are equivalent with (U, V) being compact coordinates.

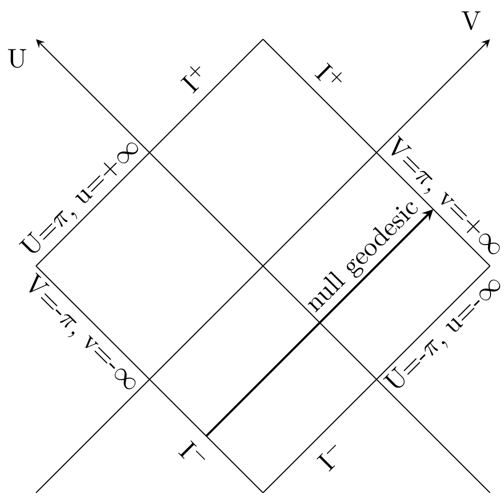


Figure 5: Penrose diagram of two-dimensional Minkowski space.

Due to the coordinate transformation being conformal, orientation and angles are preserved. This tells us that we can represent Minkowski space as in Figure 5, where time t is still the vertical axis and space x is the horizontal one. The 45^o rotated square represents the whole of Minkowski space and its border represents infinity and light-like curves remain 45^o inclined.

It is easy to check that the requirements for asymptotic simplicity are satisfied by Minkowski space-time.

Theorem 6.1. *($d = 4$)*
If M is asymptotically simple and empty, the following are satisfied:

a) Any null geodesic on M that reaches I has an unlimited affine parameter and $I = I^+ \cup I^-$, where $I^\pm$ is a null connected hypersurface with topology $\mathbb{R} \times S^2$ and is denoted as infinite null future (+) or null past (-).

b) (M,g) is globally hyperbolic.
Globally hyperbolic means that (M,g) comprises a Cauchy hypersurface Σ, which means $D(\Sigma) = M$, with $D(\Sigma) := D^+(\Sigma) \cup D^-(\Sigma)$, where:

$$D^+(\Sigma) := \{p \in M | \textit{any causal curve extendable in the past containing } p \textit{ intersects } \Sigma\}$$

and $D^-(\Sigma)$ is defined analogously. (A sketch of a Cauchy hypersurface Σ is found in Figure 6)

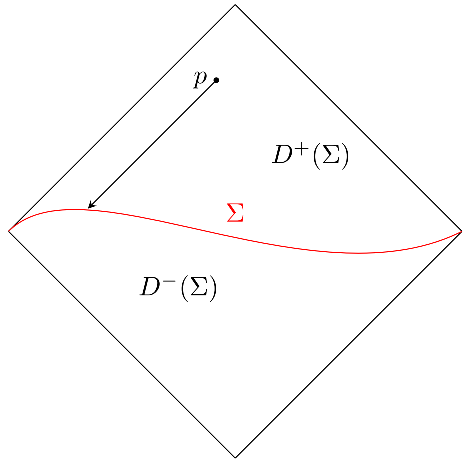


Figure 6: Example of a Cauchy hypersurface Σ in Minkowski space.

c) $g \to \eta$ as we get close to I, which means M is asymptotically flat at null infinity I^+.

N.B.

- The a) condition in the above theorem implies that I represents null infinity for M with I^+ being the set of all future extremes of null geodesics (see Figure 5. Analogously for I^-).
- We iused in b) the concept of *extendability*. We need to distinguish clearly between the possibilities that a curve "runs off to infinity" or "runs into a singularity" as opposed the possibility that a curve "stops" somewhere simply because one did not define it to go further. This distinction can be made precise via the notion of an *endpoint* of a curve. Let $\lambda(t)$ be a future directed causal curve. We say that $p \in M$ is a *future endpoint* of λ if for every neighbourhood O of p there exists a t_0 such that $\lambda(t) \in O$ for all $t > t_0$ (the curve "stays there"). The curve $\lambda(t)$ is said to be *future inextendable* if it has no future endpoint. Past inextendability is defined similarly.
- Condition ii) in definition 6.5 is too strong. It excludes the possibility of having black holes. This is seen by selecting a light-like curve approaching the singularity. This is why we introduce the concept of *weakly asymptotically simple manifold*, where condition ii) becomes iv): there exists an open neighbourhood of I isometric to an open neighbourhood of I' of an asymptotically simple manifold M'. (Basically we require that only a part of M is asymptotically simple, i think).

Definition 6.6. (*Null Hypersurfaces*)
Let $S(x)$ be a smooth function of some spacetime coordinates x^μ. Consider a

family of hypersurfaces satisfying $S = const$. The vector field normal to the hypersurfaces is:

$$\ell = \tilde{f}(x) g^{\mu\nu}(\partial_\nu S)\partial_\mu,$$

where $\tilde{f}$ is a non-zero arbitrary function.
If $\ell^2 = 0$ for a particular hypersurface N of the family, then N is called null hypersurface.

In the case at hand, we asked $\Omega = 0$ and showed that $\tilde{g}^{\mu\nu}\partial_\nu\Omega$ is a null vector.

Definition 6.7. (*Asymptotic Predictability in the Future*)
A weakly asymptotically simple and empty space-time is said to be asymptotically predictable in the future if there exists a partial Cauchy surface Σ which satisfies:

$$I^+ \subseteq \overline{D^+(\Sigma)}$$

the bar indicates the closure of $\tilde{M}$ and $D^+(\Sigma)$ has previously been defined.

We are now ready to define what we mean by black hole.

Definition 6.8. (*Black Hole*)
Space-time contains a black hole, if:

$$B := M \backslash J^-(I^+) \neq \emptyset.$$

Also, we denote with H the boundary of $J^-(I^+)$ and call it event horizon.

One can easily check, by looking at Figure 5, that $B = 0$ for Minowski space. Thus Minkowski space cannot host a black hole.

Physically this definition of black hole means that a spacetime hosts a black hole if there are some points of spacetime that cannot reach the chronological future, thus are trapped in the black hole region. Thus the event horizon is the the border of the region of spacetime where particles can't reach infinity.

In the following we consider the Schwarzschild metric to see how the Penrose diagram of Minkowski space changes in the presence of a spherically symmetric black hole.

Example 5. The Schwarzschild metric is the following:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.13}$$

Because of the spherical symmetry we consider only the (t, r) coordinates. We, thus, define the change of coordinates as:

$$\left(1 - \frac{2m}{r}\right)^{-2} dr^2 =: dr_*^2 \tag{6.14}$$

which gives us:

$$r_* = \int \frac{dr}{1 - \frac{2m}{r}} = r + 2m \ln\left(\frac{r}{2m} - 1\right) \tag{6.15}$$

where $2m < r < +\infty \ \Rightarrow \ -\infty < r_* < +\infty$. We introduce null coordinates: $u = t - r_*$ and $v = t + r_*$. This takes us to the following metric:

$$ds^2 = -\left(1 - \frac{2m}{r(u,v)}\right) dudv = -\frac{2m}{r} e^{-r/2m} e^{(u-v)/4m} dudv \tag{6.16}$$

We introduce the *Kruskal coordinates*:

$$\begin{aligned} U &:= -e^{-u/4m} \\ V &:= e^{-v/4m} \end{aligned} \tag{6.17}$$

which takes us to the following metric:

$$ds^2 = -\frac{32m^3}{r} e^{-r/2m} dU dV. \tag{6.18}$$

Thus we have the following identification of regions:

$$r \geq 2m, \ -\infty < t < +\infty \qquad \Rightarrow \qquad -\infty < U < 0, \ 0 < V < +\infty. \tag{6.19}$$

which is regular in $r = 2m$ and can be extended to the region $U \geq 0 \wedge V \leq 0$.

We finally introduce the coordinates:

$$\begin{aligned} U &=: \tan\left(\frac{\eta - \zeta}{2}\right) \\ V &=: \tan\left(\frac{\eta + \zeta}{2}\right) \end{aligned} \tag{6.20}$$

thus obtaining:

$$ds^2 = \frac{32m^3}{r(\eta,\zeta)} \frac{e^{-r/2m}}{4\cos^2\left(\frac{\eta+\zeta}{2}\right)\cos^2\left(\frac{\eta-\zeta}{2}\right)}(-d\eta^2 + d\zeta^2) =: \Omega^{-2}(-d\eta^2 + d\zeta^2) \tag{6.21}$$

. And thus we've shown the Schwarzschild metric is conformal to the flat metric. The Schwarzschild metric is thus the region:

$$-\pi/2 \leq \frac{\eta \pm \zeta}{2} \leq \pi/2 \qquad \wedge \qquad r > 0 \tag{6.22}$$

which is represented by the Penrose diagram in Figure 7. We find that the two top and bottom triangles over the wavy lines are not taken because of the condition $r > 0$. The wavy line is there to highlight the fact that $r = 0$ is a coordinate singularity.

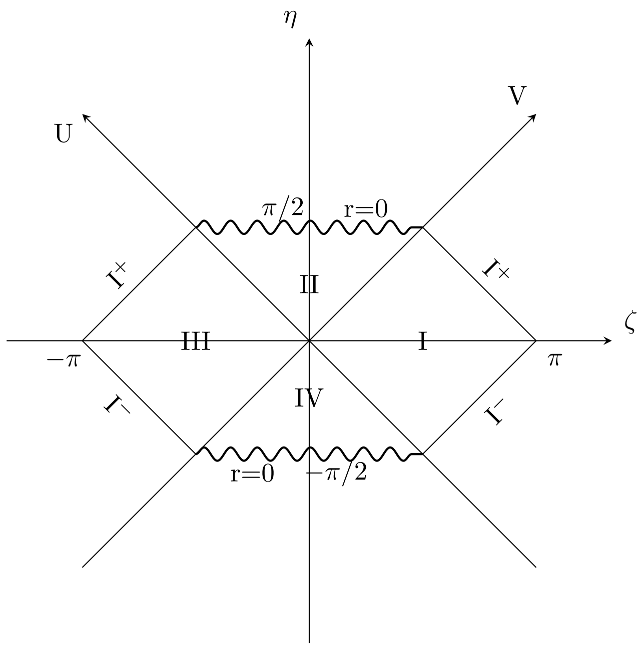


Figure 7: Penrose Diagram of Schwarzschild spacetime.

- First note that the diagram in Figure 7, if one just takes equations (6.20) as a reference, is supposed to be a diamond, however the further condition $r > 0$ imposes a cutoff at $\eta = \pi/2$, thus yielding Figure 7.
- We call H^+ the coordinate line $U = 0$, which is the "future horizon", and H^- the line $V = 0$, the past horizon.
- In order to check if the Schwarzschild spacetime contains a black hole, we use Definition 6.8, which tells us that the region $J^-(I^+)$ is in fact region I+III+IV, thus $M - J^-(I^+)$ it is not the empty set, and the spacetime contains a region of no return.
- Notice that the Kruskal coordinates U, V as defined, only cover region I, which is clear from (6.19), and by switching to the new coordinates η, ζ, we were able to cover all of spacetime.
  However, at $r = 2m$ the Kruskal coordinates are well behaved (as said above), and therefore there is no reason not to extend these coordinates beyond the horizon; in fact one can choose the following coordinates:

$$\begin{aligned} U &:= e^{-u/4m} \\ V &:= e^{-v/4m} \end{aligned} \tag{6.23}$$

  so now U can also take values greater than zero.

- As is known from more elementary courses, the maximally extended Schwarzschild spacetime contains region IV, a *white hole*, and region III, a *mirror universe*, where the latter can communicate with our universe inside the horizon. A clarification is called for: the Schwarzschild metric describes a black hole that was always there from the beginning of time, and will be there for an infinite amount of time. Later on we will show that the metric for *astrophysical black holes* (i.e. the ones generated from a stellar collapse, like for a Type II Supernova progenitor that's enough massive) is actually devoid of regions III and IV.

We could do the same analysis on the Schwrazschild metric (6.13), although this time imposing $m < 0$, since there would seem to be no a priori restriction on the values of m. Now the pathological term $(1 - 2m/r)$ is always positive and there are no singularities in the metric except for $r = 0$. In order to obtain the Penrose diagram, start by doing the same transformation as in (6.14), now yielding:

$$r_* = r + 2m \ln(1 - \frac{r}{2m}) \tag{6.24}$$

and introduce analogously the coordinates u, v, so that the metric becomes (6.16). Introduce now these new coordinates:

$$\begin{aligned} u &:= tan(\frac{U}{2}) \\ v &:= tan(\frac{V}{2}) \end{aligned} \tag{6.25}$$

so that:

$$ds^2 = -(1 - \frac{2m}{r})\frac{1}{4cos^2(U/2)cos^2(V/2)}dUdV = \Omega^{-2}(-dUdV) \tag{6.26}$$

with: $-\pi \leq U, V \leq \pi$ and $r > 0$ and $r = 0 \implies v = u \implies V = U$. The only constraint is: $r > 0 \implies r_* > 0 \implies v > u \implies arctan(v) > arctan(u) \implies V > U$. The Penrose diagram is shown in Figure 8. Notice that this singularity is different from the black hole singularity: it can be seen from I^+. Conversely, the singularity inside the black hole is hidden behind the horizon $r = 2m$. A singularity that can be seen from I^+ is known as a *naked singularity*. Remember also that the time reversal of a black hole solution, a white hole, has a naked singularity too.

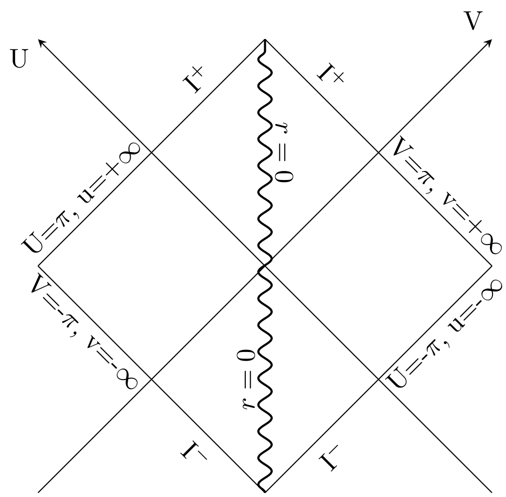


Figure 8: Naked singularity from Schwarzschild metric with negative mass.

6.1 Surface Gravity and Black Hole Temperature

Definition 6.9. (*Killing Horizons*)
A null hypersurface N is called a Killing horizon of a Killing vector field ξ if ξ is orthogonal to N.

It is useful now to remember a fundamental property of null hypersurfaces: let ℓ be the normal vector to the null hypersurface N, then this of course implies that $<\ell, t>=0$ for any t tangent to N; since $<\ell,\ell>=0$, it is also clear that ℓ is also tangent to N in addition to being orthogonal to it. For nomenclature purposes call $\ell^\mu = \frac{dx(\lambda)^\mu}{d\lambda}$, where $x(\lambda)^\mu$ is a path on N.

Lemma 6.2. *The curves $x(\lambda)^\mu$ are geodesics*

Proof. Let N be a null hypersurface defined by $S = const$, then the normal vector to it is naturally defined as :

$$\ell^\mu = \tilde{f} g^{\mu\nu}\partial_\nu S \tag{6.27}$$

Now calculate $\nabla_\ell \ell$:

$$\ell^\rho \nabla_\rho \ell^\mu = \left(\ell^\rho \partial_\rho \tilde{f}\right) \underbrace{g^{\mu\nu}\partial_\nu S}_{=\ell^\mu/\tilde{f}} + \tilde{f} g^{\mu\nu} \ell^\rho \nabla_\rho \partial_\nu S \tag{6.28}$$

where we can manipulate the last term:

$$\nabla_\rho \partial_\nu S = \partial_\rho \partial_\nu S - \Gamma^\lambda_{\rho\nu}\partial_\lambda S = \partial_\nu \partial_\rho S - \Gamma^\lambda_{\nu\rho}\partial_\lambda S = \nabla_\nu \partial_\rho S \tag{6.29}$$

so (6.28) becomes:

$$\begin{aligned}\ell^\rho \nabla_\rho \ell^\mu &= \left(\frac{dx(\lambda)^\rho}{d\lambda}\partial_\rho log\tilde{f}\right)\ell^\mu + \tilde{f}g^{\mu\nu}\ell^\rho\nabla_\nu\partial_\rho S \\ &= \left(\frac{d}{d\lambda}log\tilde{f}\right)\ell^\mu + \ell^\rho\tilde{f}\nabla^\mu\partial_\rho S\end{aligned} \tag{6.30}$$

again let us look at the last term:

$$\nabla^\mu\partial_\rho S = \nabla^\mu\tilde{f}^{-1}\ell_\rho = \tilde{f}^{-1}\nabla^\mu\ell_\rho + (\partial^\mu\tilde{f}^{-1})\ell_\rho \tag{6.31}$$

which means (6.30) is:

$$\nabla_\ell\ell^\mu = \left(\frac{d}{d\lambda}log\tilde{f}\right)\ell^\mu + \frac{1}{2}\partial^\mu(\ell^2) - (\partial^\mu log\tilde{f})\ell^2 \tag{6.32}$$

Now use these two facts: that $\ell^2 = 0$ since ℓ is a null vector, and that $\partial^\mu\ell^2 \neq 0$ but since $\ell^2 = 0$ everywhere on N, ℓ^2 is constant along directions tangent to N, therefore $t^\mu\partial_\mu\ell^2 = 0$ for every tangent vector t^μ, which means $\partial^\mu\ell^2|_N \propto \ell^\mu$. We thus find from (6.32) that:

$$\nabla_\ell\ell^\mu \propto \ell^\mu \tag{6.33}$$

which is the geodesic equation in a non-affine parametrization. Usually one can choose $\tilde{f}$ conveniently to find $\nabla_\ell\ell^\mu = 0$ □

Definition 6.10. (*Null Generators*)
The null geodesics $x(\lambda)^\mu$ with affine parameter λ and tangent vector $\frac{x(\lambda)^\mu}{d\lambda}$ orthogonal to a null hypersurface N, are called the null generators of N.

Example 6. Kruskal spacetime
Take the Schwarzschild metric written under Krukal coordinates, and look back to its Penrose diagram in Figure 7, if we take the null hypersurface N to be $U = 0$, then:

$$\ell = \tilde{f}g^{\mu\nu}\partial_\nu S\partial_\mu = \tilde{f}g^{VU}\partial_U S\partial_V = -\frac{\tilde{f}re^{\frac{r}{2m}}}{16m^3}\partial_V \tag{6.34}$$

Of course it is easy to check that ℓ is a null vector, since ∂_V is one. Notice that in this case $\partial_\mu\ell^2 = 0$ on N, which means (if $\tilde{f}$ is constant) that $\nabla_\ell\ell^\mu = 0$. Since $U = 0 \implies r = 2m$:

$$\ell|_N = \frac{-\tilde{f}e}{8m^2}\partial_V = \partial_V \tag{6.35}$$

where we have chosen the appropriate $\tilde{f}$, in order to obtain that V is an affine parameter.

Definition 6.11. (*Surface gravity*)
Let ℓ be orthogonal to N such that $\nabla_\ell \ell = 0$. Since $\xi = f\ell$ on N for some function f, it follows that:

$$\nabla_\xi \xi = \nabla_{(f\ell)} f\ell = f(\nabla_\ell f\ell) = f(f\nabla_\ell \ell + \ell(f) \cdot \ell) \tag{6.36}$$

which implies, using the fact that we are employing an affine parametrization, that:

$$\nabla_\xi \xi = \ell(f)\xi \equiv \kappa\xi \tag{6.37}$$

where:

$$\kappa = \ell(f) = \ell^\mu \partial_\mu f = f^{-1}\xi^\mu \partial_\mu f = \xi^\mu \partial_\mu \log(f) \tag{6.38}$$

We call κ surface gravity.

It is possible to obtain an expression for κ. First it is necessary to use the dual formulation of Frobenius' theorem (see Wald p. 436), which says that ξ is orthogonal to the hypersurface N iff:

$$\xi_{[\mu} \nabla_\nu \xi_{\rho]}|_N = 0 \tag{6.39}$$

which translates into:

$$\xi_\rho \nabla_\mu \xi_\nu|_N + (\xi_\mu \nabla_\nu \xi_\rho - \xi_\nu \nabla_\mu \xi_\rho)|_N = 0 \tag{6.40}$$

where we have used the fact that ξ is a Killing vector, therefore $\nabla_{[\mu}\xi_{\nu]} = 0$. Multiplying equation (6.40) by $\nabla^\mu \xi^\nu$ gives:

$$\begin{aligned}\xi_\rho (\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_N &= -2(\nabla^\mu \xi^\nu)\xi_\mu(\nabla_\nu \xi_\rho)|_N \\ &= -2(\nabla_\xi \xi^\nu)\nabla_\nu \xi_\rho|_N \\ &= -2\kappa \xi^\nu \nabla_\nu \xi_\rho|_N = -2\kappa^2 \xi_\rho|_N\end{aligned} \tag{6.41}$$

which means:

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_N \tag{6.42}$$

which is the expression for surface gravity.

Lemma 6.3. *For a killing vector ξ it is true that $\nabla_\rho \nabla_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma}\xi^\sigma$*

Proof. By definition of the Riemann tensor we have:

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = R_{abc}{}^d \xi_d \tag{6.43}$$

On the other hand, by Killing's equation, we can rewrite this as:

$$\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a = R_{abc}{}^d \xi_d \tag{6.44}$$

If we write down the same equation with cyclic permutations of the indices (abc), and then add the (abc) equation to the (bca) equation and subtract the (cab) equation, we obtain:

$$\begin{aligned} 2\nabla_b \nabla_c \xi_a &= \left(R_{abc}{}^{d} + R_{bca}{}^{d} - R_{cab}{}^{d}\right) \xi_d \\ &= -2R_{cab}{}^{d}\xi_d \end{aligned} \tag{6.45}$$

where the symmetry properties of the Riemann tensor were used in the last equality. Thus, for any Killing field ξ^a, we obtain the formula:

$$\nabla_a \nabla_b \xi_c = -R_{bca}{}^{d}\xi_d \tag{6.46}$$

which is equivalent to what we wanted to prove. □

Lemma 6.4. *κ is constant on ξ 's orbits (on N)*

Proof. Let t^μ be the tangent vector field to N. Since (6.42) is valid over all of N, we get:

$$\begin{aligned} \nabla_t \kappa^2 &= -(\nabla^\mu \xi^\nu)\nabla_t(\nabla_\mu \xi_\nu) \\ &= -(\nabla^\mu \xi^\nu) t^\rho \nabla_\rho (\nabla_\mu \xi_\nu) \\ &= -(\nabla^\mu \xi^\nu) t^\rho R_{\nu\mu\rho\sigma}\xi^\sigma \end{aligned} \tag{6.47}$$

Now since ξ is also tangent to N, we can take it to be the tangent vector t^μ:

$$\nabla_\xi \kappa^2 = -(\nabla^\mu \xi^\nu)\xi^\rho R_{\nu\mu\rho\sigma}\xi^\sigma = 0 \tag{6.48}$$

due to the fact that the Riemann tensor is antisymmetric in its last two indices. □

Definition 6.12. (*Non-degenerate Killing horizons*)
Suppose $\kappa \neq 0$ along an orbit of ξ on N. This orbit coincides only with a part of a null generator of N. To see this choose some coordinates on N such that $\xi = \frac{\partial}{\partial \alpha}$. If $\alpha = \alpha(\lambda)$ along an orbit of ξ with affine parameter λ, we get:

$$\xi|_{\text{orbit}} = \frac{d\lambda}{d\alpha}\frac{d}{d\lambda} \equiv f\ell \tag{6.49}$$

where we have defined $f \equiv \frac{d\lambda}{d\alpha}$, $\ell \equiv \frac{d}{d\lambda} = \frac{dx(\lambda)^\mu}{d\lambda}\partial_\mu$. We know that the surface gravity is:

$$\kappa = \xi^\mu \partial_\mu \log(f) = \frac{\partial}{\partial \alpha}\log(f) \tag{6.50}$$

where we have just found that κ is constant on N, which means that the above equation is a trivial differential equation on N that is easily solved:

$$f = f_0 e^{\kappa\alpha} \tag{6.51}$$

for some arbitrary constant f_0. We make use of our freedom to shift α with a constant, in order to select $f_0 = \pm\kappa$, so that (6.51) becomes:

$$\frac{d\lambda}{d\alpha} = \pm\kappa e^{\kappa\alpha} \implies \lambda = \pm e^{\kappa\alpha} + \text{const} \tag{6.52}$$

We see that for $-\infty < \alpha < +\infty$ we can cover the $\lambda > 0$ or $\lambda < 0$ parts of the generator of N. The bifurcation point $\lambda = 0$ is a fixed point of ξ. One can show that at this point κ changes sign and it is topologically a two-sphere, called the *bifurcation 2-sphere*.

Example 7. Kruskal spacetime
Choose N to be $U = 0 \cup V = 0$ which are the future and the past horizon respectively, and ξ to be ∂_t, that when written in Kruskal coordinates becomes $\frac{1}{4m}(V\partial_V - U\partial_U)$. Explicitly:

$$\xi = \begin{cases} \frac{1}{4m}V\partial_V & \text{at } U = 0 \\ -\frac{1}{4m}U\partial_U & \text{at } V = 0 \end{cases} = f\ell \tag{6.53}$$

however, in the previous example, we found that $\ell|_{N\to U=0} = \partial_V$ and $\ell|_{N\to V=0} = \partial_U$, so that:

$$f = \begin{cases} \frac{1}{4m}V & \text{at } U = 0 \\ -\frac{1}{4m}U & \text{at } V = 0 \end{cases} \tag{6.54}$$

however since ℓ is orthogonal to N, we have just found that N is a Killing horizon of $\xi = \partial_t$. Let's clarify the bifurcation 2-sphere business as well: we know that

$$\kappa = \xi^\mu\partial_\mu \log(f) = \begin{cases} \frac{1}{4m}V\partial_V(\log(V)) & \text{at } U = 0 \\ -\frac{1}{4m}U\partial_U(\log(U)) & \text{at } V = 0 \end{cases} = \begin{cases} \frac{1}{4m} & \text{at } U = 0 \\ -\frac{1}{4m} & \text{at } V = 0 \end{cases} \tag{6.55}$$

thus we see that κ changes sign at $U = V = 0$, which is in this case the famous bifurcation 2-sphere.

We are now ready to find the link between surface gravity and the temperature of a black hole. Let's define a *Wick rotation* as:

$$t \to -i\tau \tag{6.56}$$

where τ is the "Euclidian time". If we apply (6.56) to the Schwarzschild solution we get an euclidian signature metric (which is the reason for the name given to τ):

$$ds_E^2 = \left(1 - \frac{2m}{r}\right)d\tau^2 + dr^2\left(1 - \frac{2m}{r}\right)^{-1} + r^2 d\Omega_2^2 \tag{6.57}$$

In order to study the region near the "horizon" $r = 2m$, define the new coordinate x by:

$$r - 2m = \frac{x^2}{8m} \tag{6.58}$$

and expand the solution around $x = 0$

$$ds_E^2 = (\kappa x)^2 d\tau^2 + dx^2 + \frac{1}{4\kappa^2} d\Omega^2 \tag{6.59}$$

where we have used $\kappa = 1/4m$. Notice that the part (x, τ) of the metric is the Euclidian plane in polar coordinates, therefore we need to impose τ to be $\frac{2\pi}{\kappa}$ periodic in order to avoid conic singularities,[4] which means:

$$\tau \backsim \tau + \frac{2\pi}{\kappa} \tag{6.61}$$

thus we see that the periodic nature of the euclidian time is less than 2π, which corresponds physically to cut a slice of the 2d circle and identify the two cuts as being the same, as we see in the figure below.

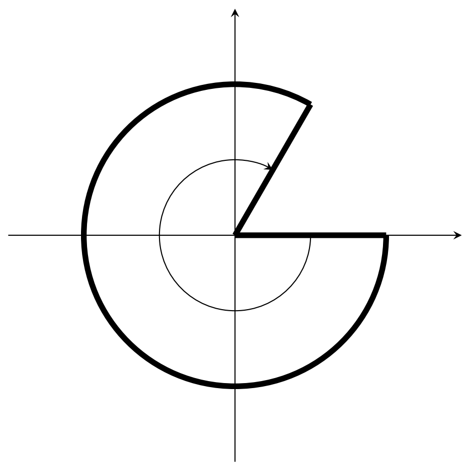


Figure 9: Conic singularities.

The important feature of this so called euclidian time is that its periodicity is related to the temperature of the black hole, therefore the surface gravity is related to it as well. To see this, remember that in the path integral approach to the quantization of the scalar field, the amplitude of going from configuration $|\phi_1, t_1>$ to $|\phi_2, t_2>$ is:

$$<\phi_2, t_2|\phi_1, t_1> = \int [\mathcal{D}\phi] e^{\frac{iS[\phi]}{\hbar}} \tag{6.62}$$

where the path integral is calculated over all of the configurations of ϕ that take the correct boundary values ϕ_1 and ϕ_2. We also know that:

$$<\phi_2, t_2|\phi_1, t_1> = <\phi_2|e^{-\frac{i}{\hbar}H(t_2 - t_1)}|\phi_1> \tag{6.63}$$

[4] Remember that in standard polar coordinates (r, θ), the euclidian plane's metric is:

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{6.60}$$

and in order to avoid conic singularities, we need to impose that the periodicity of θ is 2π.

where H is the Hamiltonian. Now switch to euclidian time through a Wick rotation and impose $\phi_1 = \phi_2$, then summing over all of the possible ϕ_1 we get:

$$\mathrm{Tr}(e^{-\Delta\tau H}) = \int [\mathcal{D}\phi] e^{\frac{iS[\phi]}{\hbar}} \tag{6.64}$$

where now the integral, since we have imposed $\phi_1 = \phi_2$, is calculated through all of fields periodic in euclidian time with period $\Delta\tau$. However, we also know that the integral above is $\mathrm{Tr}(e^{-\beta H})$, since it is the partition function for the canonical ensemble of a field ϕ at temperature $T = \beta^{-1}$. Putting it all together, and remembering (6.61), we get:

$$\frac{2\pi}{\kappa} = \beta = \frac{1}{k_B T} \tag{6.65}$$

thus yielding the black hole temperature:

$$T_{\mathrm{BH}} = \frac{\kappa}{2\pi k_B} \tag{6.66}$$

6.2 Black Holes from Gravitational Collapse

We have said previously that the Schwarzschild metric represents an "eternal" black hole that was there from the beginning of time, and will be there until the end of times.
In this section we seek to introduce a simple model of astrophysical black holes, those formed after a star collapses in on itself.
The simplest model of gravitational collapse to a black hole is provided by a collapsing infinitely thin spherical shell of null matter (radiation) in Minkowksi space time. I order to describe such incoming radiation, it is natural to work in ingoing coordinates that are adapted to such null geodesics ($v = t + r, r$), such that the Minkowski metric takes the form:

$$ds^2 = -dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6.67}$$

and the Schwarzschild metric:

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6.68}$$

with $f(r) = 1 - 2m/r$. In both metrics, ingoing lightrays are described by lines of constant v. The relevance of these two metrics for the problem at hand arises from the fact that in the two vacuum regions inside and outside the shell one will have Minkowski spacetime and Schwarzschild spacetime respectively. We can suppose that the shell moves along the igoing null trajectory $v = v_0$, in both metrics.
The metric that covers the whole of spacetime is known as the *Vaidya metric*:

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2 d\Omega^2 \qquad f(v,r) = 1 - \frac{2m(v)}{r} \tag{6.69}$$

in this particular case with the distributional mass function (*shock wave* model):

$$m(v) = m_f \Theta(v - v_0) \tag{6.70}$$

Calculating the Einstein tensor of this metric one finds that this is a solution of the Einstein equations with an energy-momentum tensor whose only non vanishing component is:

$$T_{vv} = \frac{m_f \delta(v - v_0)}{4\pi r^2} \tag{6.71}$$

It is clear that at some point the radius of the shell will reach and then cross its Schwarzschild radius. Once that has happened, the exterior Schwarzschild geometry describes a black hole with a future event horizon. However, there is no trace here of either region III or IV from the orignal Penrose diagram for the eternal black hole. The correct diagram to picture this metric is naturally obtained by gluing together the two Penrose diagrams of Figure 5 and Figure 7 along an ingoing worldline of the shell, yielding the diagram represented in the figure below:

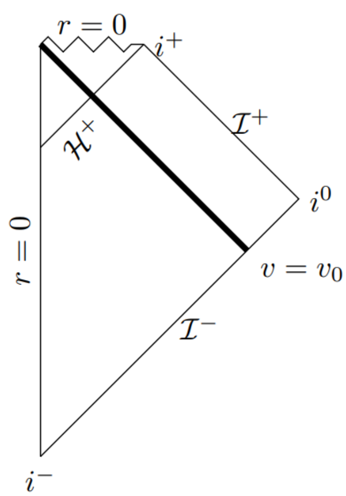


Figure 10: : Penrose diagram for ”shockwave” gravitational collapse

Notice that, as expected, region III and IV do not appear. Also we see that in the region $v < v_0$ inside the shell, the geometry is that of Minkowski space, whereas the geometry outside is Schwarzschild. Formation of the black hole occurs when the shell crosses the event horizon H^+.

7 Charged and Rotating Black Holes

We begin by defining certain things that will come up later on in the chapter.

Definition 7.1. (*Stationary spacetime*)
An asymptotically flat spacetime is said to be stationary iff there exist a timelike Killing vector field near spatial infinity.

Basically outside of a potential horizon we can choose a temporal coordinate such that $K = \partial_t$. The most general stationary metric that allows for these coordinates is:

$$ds^2 = g_{00}(\vec{x})dt^2 + 2g_{0i}(\vec{x})dtdx^i + g_{ij}(\vec{x})dx^i dx^j \tag{7.1}$$

where $i = 1, ..., d-1$. A stationary spacetime is called *static* if it is invariant under time reversals, i.e. $t \to -t$. Note that this conditions holds iff $g_{0i} = 0$.

Definition 7.2. (*Axisymmetric spacetime*)
An asymptotically flat spacetime is called axisymmetric if there exists a spacelike Killing vector field m near spatial infinity, such that all of its orbits are closed. We can generally choose coordinates such that $m = \partial_\varphi$ with φ being modulo 2π.

Theorem 7.1. *(Birkhoff)*
Every solution to the vacuum Einstein equations that is spherically symmetric, is also static. This means that the solution is the Schwarzschild metric.

Theorem 7.2. *(Carter Robinson)*
Let (M,g) be a axisymmetric stationary asymptotically flat spacetime with $R_{\mu\nu} = 0$ *that is nonsingular on and outside of a potential horizon, then (M,g) is a family member of the Kerr black holes that are completely specified by two parameters: their mass and angular momentum.*

The above theorems can be generalized by the following:

Theorem 7.3. *(No-hair theorem)*
Stationary, asymptotically flat black hole solutions to general relativity coupled to electromagnetism that are nonsingular outside or on event horizons, are fully characterized by the parameters of mass, electric and magnetic charge, and angular momentum.

This means that we can solve the *Einstein-Maxwell* equations:

$$S_{EM} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}] \tag{7.2}$$

from which we obtain the following equation by varying with respect to the metric:

$$G_{\mu\nu} = \frac{1}{2}(F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \tag{7.3}$$

and the curved spacetime generalization of the Maxwell equations by varying with respect to A_μ:

$$\nabla_\mu F^{\mu\nu} = 0 \tag{7.4}$$

These are now two coupled differential equations that we can solve for $g_{\mu\nu}$ and A_μ. If we also require that the spacetime be asymptotically flat and stationary we get a $Kerr - Newman$ black hole, which is completely characterized by mass, charge and angular momentum, just as the theorem suggested. Notice that we did not ask for axisymmetry, however Hawking and Wald have proven that stationary implies axysimmetric, therefore our solutions are also axisymmetric by construction.
The solutions to these equations are the following:

$$\begin{aligned} ds^2 = &-\frac{(\Delta - a^2\sin^2(\theta))}{\rho^2}dt^2 - 2a\sin^2(\theta)\frac{(r^2+a^2-\Delta)}{\rho^2}dtd\varphi \\ &+ \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2(\theta)}{\rho^2}\sin^2(\theta)d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{7.5}$$

with:

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2(\theta) \\ \Delta &= r^2 - 2mr + a^2 + e^2 \end{aligned} \tag{7.6}$$

We can see that the three parameters mentioned above are m, a, e, where one can prove that a is related to the angular momentum by $J = ma$, and $e^2 = Q^2 + P^2$, where Q is the electric charge and P is the magnetic charge. The solution for the Maxwell one form (the generalization of the classical potential A_μ) is:

$$A = \frac{Qr(dt - a\sin^2(\theta)d\varphi) - P\cos(\theta)(adt - (r^2+a^2)d\varphi)}{\rho^2} \tag{7.7}$$

from which we can recover the familiar cases for instance by letting $P = a = 0$, which returns $A^0 = \frac{Q}{r}$, the electrostatic potential. The coordinates (t, r, θ, φ) that appear in (7.7) are called the *Boyer-Lindquist* coordinates. Notice also that:

i) $a = e = 0$ is the Schwarzschild metric.

ii) $a = 0$ is the *Reissner-Nordstrom* metric, which is a charged Schwarzschild black hole.

iii) $e = 0$ is a Kerr black hole, which is an uncharged spinning black hole.

We will focus on the third case Kerr black holes.

7.1 Kerr Black Holes

From (7.6) by letting $e = 0$ we get:

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2(\theta) \\ \Delta &= r^2 - 2mr + a^2 \end{aligned} \tag{7.8}$$

The Kerr metric has an horizon (a coordinate singularity) in $\Delta = 0$ [5], which means:

$$r_\pm = m \pm \sqrt{m^2 - a^2} \tag{7.9}$$

Therefore we can split the discussion for the three cases:

i) $m^2 < a^2$
This means that Δ is always greater than zero, thus we have no coordinate singularities for this case. We do however find that there is a real singularity at $\rho^2 = 0$ ($r = 0$ and $\theta = \pi/2$) which one can verify by calculating the scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and checking that it diverges for the said parameters. We can study this singularity by noting that the Boyer-Lindquist coordinates are oblate spheroidal coordinates, which are related to the usual cartesian coordinates by (the following are called *Kerr-Shild* coordinates):

$$\begin{aligned} x + iy &= (r + ia)exp\left(i\int(d\varphi + \frac{a}{\Delta}dr)\right) \\ z &= r\cos(\theta) \\ \tilde{t} &= \int(dt + \frac{r^2 + a^2}{\Delta}dr) - r \end{aligned} \tag{7.10}$$

and $r(x, y, z)$ is implicitly defined by:

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2z^2 = 0 \tag{7.11}$$

With (7.10) the metric becomes:

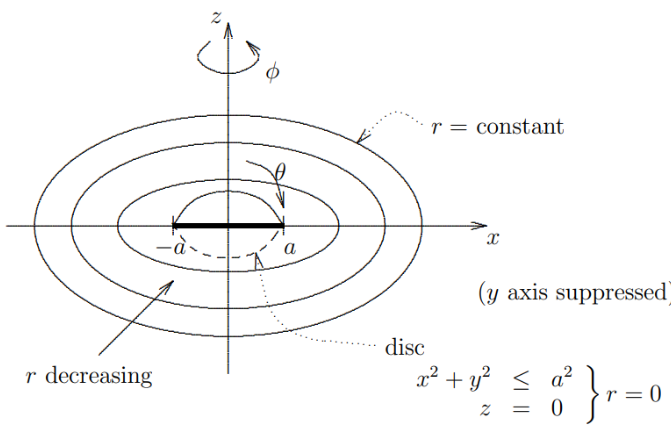
$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2mr^3}{r^4 + a^2z^2}\left(\frac{r(xdx + ydy)}{r^2 + a^2} + \frac{zdz}{r} + d\tilde{t}\right)^2 \tag{7.12}$$

We can see that the $r = const$,$\tilde{t} = const$ are confocal ellipsoids that degenerate to a disk for $r = 0$ (remember that r is a coordinate in oblate spheroidal coordinates, therefore it is NOT a radius) of equation:

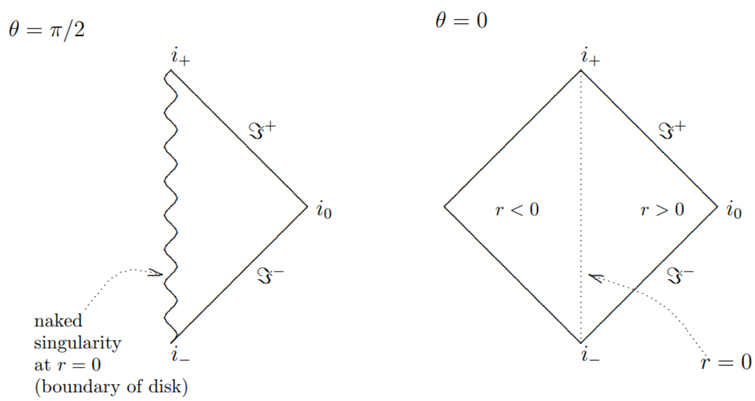
$$r^2 + a^2 \leq a^2 \tag{7.13}$$

[5] See Carrol p.241 for instance to see how we can find horizons for a given metric.

which follows from (7.10). In the plane $z = 0$ ($\theta = \pi/2$) we get the singularity, which is therefore a *ring singularity*: the rotation has "softened" the Schwarzschild singularity, spreading it out over a ring:



Now, there is no reason that $r \geq 0$, thus we can analytically continue the spacetime to the region inside the ring, which is another asymptotically flat spacetime. This new spacetime is not an identical copy of the one you came from, but it is described by the Kerr metric with $r < 0$, which means that Δ is never zero and as a consequence there are no horizons.
In order to study this spacetime's causal structure, we would need a 3D Penrose diagram, since we have don't have spherical symmetry and we can't therefore suppress two dimensions. However we can use the submanifolds $\theta = 0, \pi/2$ which are *totally-geodesic*, i.e. a geodesic that is initially tangent to the submanifold remains tangent to it, so we can draw 2D diagrams for them.



For $\theta = \pi/2$ each point in the diagram represents a circle ($0 \leq \varphi \leq 2\pi$), each ingoing radial geodesic hits the ring singularity at $r = 0$, which is clearly naked. For $\theta = 0$ we are considering only geodesics on the axis of symmetry, therefore ingoing radial null geodesics pass through the disc at $r = 0$ into the other region with $r < 0$. In addition to having a naked singularity, this spacetime is unphysical also for the following reason: consider the norm of the Killing vector field $m = \partial_\varphi$:

$$m^2 = g_{\varphi\varphi} = a^2 \sin^2(\theta) \left(1 + \frac{r^2}{a^2}\right) + \frac{ma^2}{r} \left(\frac{2\sin^4(\theta)}{1 + \frac{a^2}{r^2}\cos^2(\theta)}\right) \tag{7.14}$$

Let $r/a = \delta$ and consider $\theta = \pi/2 + \delta$. Then:

$$m^2 = a^2 + \frac{ma}{\delta} + \dots \tag{7.15}$$

which for sufficiently small negative δ becomes negative. But we starded with an axysimmetric spacetime which constitutes a vector field ∂_φ that has cloed orbits, which means that inside the ring $r < 0$ there exist close timelike curves, which obviously violate strong causality.

ii) $m^2 > a^2$
In this case the ring singularity is still there, but in addition we have two horizons at $r_\pm$ (in Boyer-Lindquist coordinates). This points are in fact coordinate singularities, not real ones. To see this introduce the *Kerr coordinates* (v, χ):

$$\begin{aligned} dv &= dt + \frac{r^2 - a^2}{\Delta} dr \\ d\chi &= d\varphi + \frac{a}{\Delta} dr \end{aligned} \tag{7.16}$$

so the metric becomes:

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2(\theta)}{\rho^2} dv^2 + 2dvdr - \frac{2a\sin^2(\theta)(r^2 + a^2 - \Delta)}{\rho^2} dvd\chi \\ &- 2a\sin^2(\theta) d\chi dr + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2(\theta)}{\rho^2} \sin^2(\theta) d\chi^2 + \rho^2 d\theta^2 \end{aligned} \tag{7.17}$$

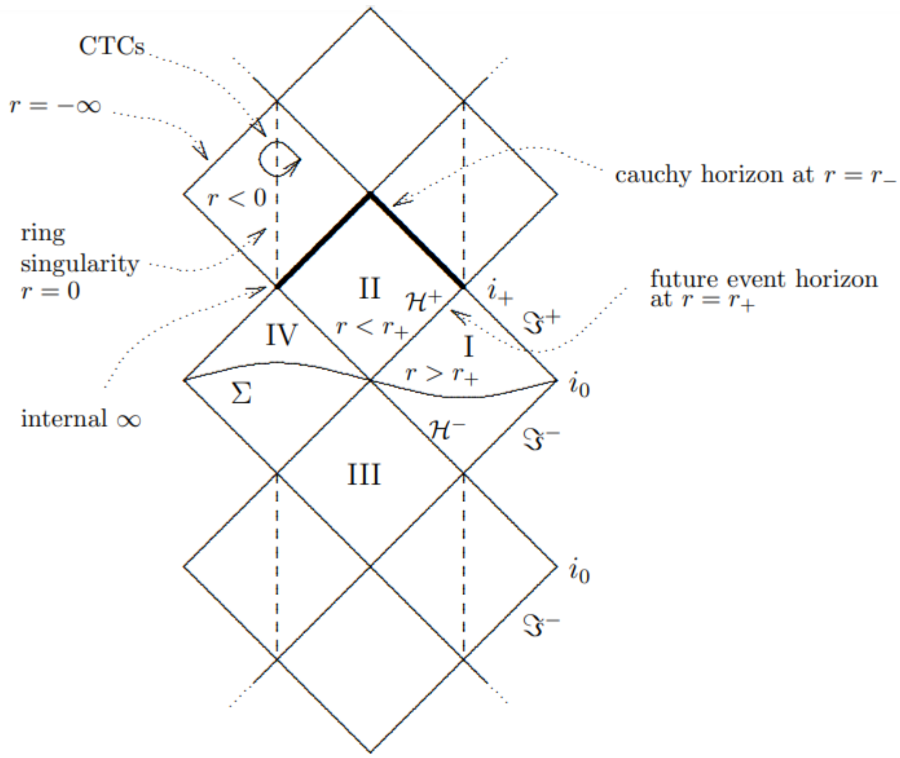
which is manifestly non-singular at $\Delta = 0$. In fact one can show that these hypersurfaces $r = r_\pm$ are Killing horizons of the following Killing vectors:

$$\xi_\pm = \partial_t + \frac{a}{r_\pm^2 + a^2} \partial_\varphi \tag{7.18}$$

with surface gravity:

$$\kappa_{\pm} = \frac{r_{\pm} - r_{\mp}}{2(r_{\pm}^2 + a^2)} \tag{7.19}$$

To see a proof of this claim see Townsend p.83. The following is the Penrose diagram for the Kerr solution:



The surfaces defined by $r = r_{\pm}$ are both null, and they are both event horizons as we just said. If you are an observer falling into the black hole from far away, r_+ is just like $2Gm$ in the Schwarzschild metric; at this radius r switches from being a spacelike coordinate to a timelike coordinate, and you necessarily move in the direction of decreasing r. Witnesses outside the black hole also see the same phenomena that they would outside an uncharged hole: the infalling observer is seen to move more and more slowly, and is increasingly redshifted.

But the inevitable fall from r_+ to ever decreasing radii only lasts until you reach the null surface r_-, where r switches back to being a spacelike coordinate and the motion in the direction of decreasing r can be arrested (if the observer is falling at $\theta = \pi/2$, he can reach the ring singularity at $r = 0$, where the laws of physics break down, whereas if he is falling at

$\theta \neq 0$, he will cross $r = 0$ and enter another universe, as was said above). Therefore you do not have to hit the singularity at $r = 0$ (if $\theta = \pi/2$); in fact you can choose either to continue on to $r = 0$, or begin to move in the direction of increasing r back through the null surface at r_-. Then once again r will become a timelike coordinate, but with reversed orientation: you are forced to move in the direction of increasing r. You will eventually be spit out past r_+ once more, which is like emerging from a white hole into the rest of the universe. From here you can choose to go back into the black hole, this time a different black hole than the one you entered in the first place, and repeat the voyage as many times as you like.

Note a final insight about this solution $m^2 > a^2$. We know that r_+ is a Killing horizon of a Killing field already mentioned in (7.18):

$$\xi_+ = \partial_t + \Omega_H \partial_\varphi \tag{7.20}$$

where we have defined:

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{7.21}$$

It is interesting to see that:

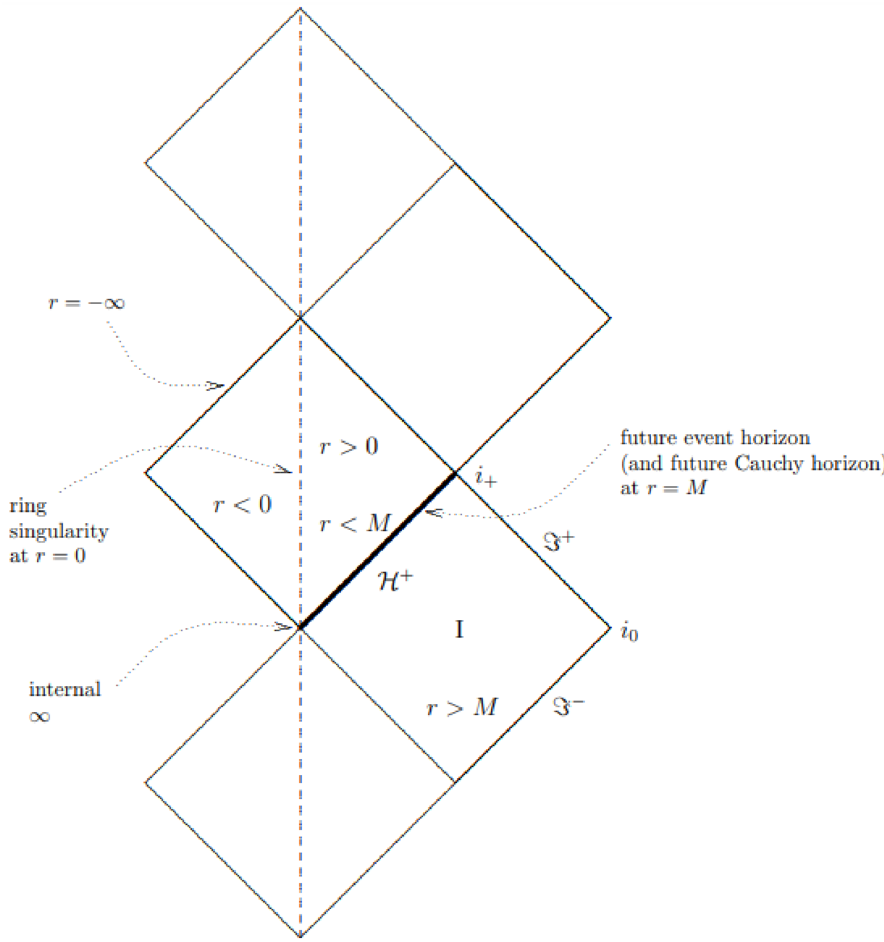
$$\xi_+(\varphi - \Omega_H t) = (\partial_t + \Omega_H \partial_\varphi)(\phi - \Omega_H t) = -\Omega_H + \Omega_H = 0 \tag{7.22}$$

This means that:

$$\phi = \Omega_H t + \text{const} \tag{7.23}$$

on the orbits of ξ, whereas naturally φ is constant on the orbits of ∂_t. The interpretation of this results is that particles on the orbits of ξ rotate with angular velocity Ω_H relative to those static particles, those on orbits of ∂_t, and hence relative to a stationary frame at ∞. However, since the null generators of the horizon are the orbits of ξ themselves, the black hole is rotating with angular velocity Ω_H.

iii) $m^2 = a^2$, i.e. an *Extreme Kerr Black Hole*
This solution features only one horizon $r_+ = r_- = m$ with surface gravity $\kappa = 0$, which implies that the temperature of the black hole is zero. In this case the Killing horizon(s) is degenerate (i.e. $\kappa = 0$) and $\Omega_H = \frac{a}{2m}$. The Penrose diagram is the following:



7.1.1 The Penrose Process

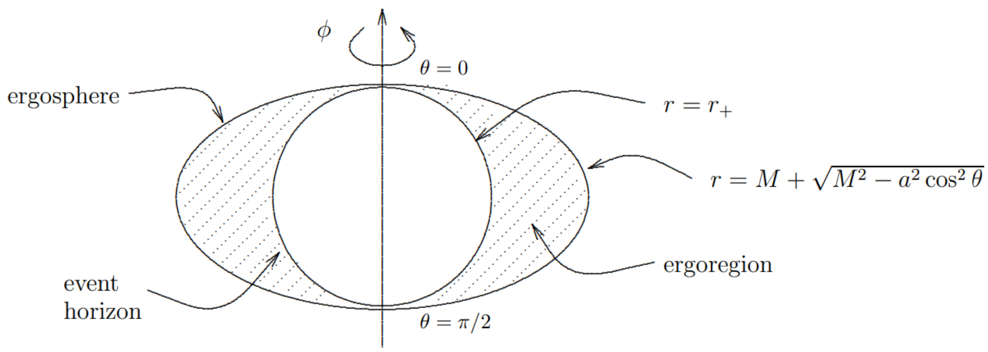
Consider the case of $m^2 > a^2$ and:

$$< \partial_t, \partial_t >= g_{tt} = -\frac{\Delta - a^2 \sin^2(\theta)}{\rho^2} = -\left(1 - \frac{2mr}{r^2 + a^2 \cos^2(\theta)}\right) \tag{7.24}$$

We clearly see that ∂_t is timelike for:

$$r^2 + a^2 \cos^2(\theta) - 2mr > 0 \implies r > m + \sqrt{m^2 - a^2 \cos^2(\theta)} \tag{7.25}$$

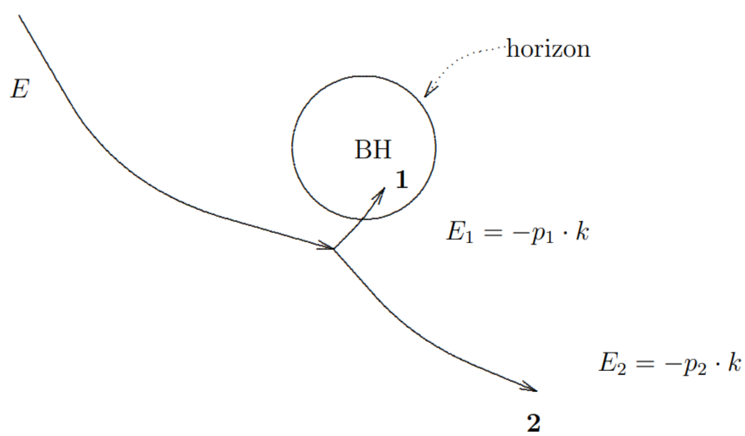
The edge of this region, $r = m + \sqrt{m^2 - a^2 \cos^2(\theta)}$ is called the *ergosphere*. Recalling the definition of $r_+ = m + \sqrt{m^2 - a^2}$, we can see that the ergosphere touches the event horizons at $\theta = 0, \pi$ and resides outside (at a greater r) of them for all the other values of θ. This means that the vector ∂_t can become spacelike in a region outside of r_+, this region is called the *ergoregion*. The following is a schematization of the various important quantities:



The existence of the ergoregion has some interesting physical consequences, like the *Penrose process*. Suppose that a particle draws near the (Kerr) black hole along a geodesic. If p is its 4-momentum, we can define a quantity that is conserved along its trajectory:

$$E = -p_\mu K^\mu \tag{7.26}$$

where $K = \partial_t$. This quantity is the energy of the particle. Now imagine that the said particle decays in two other particles, one of which falls into the black hole and the other runs off to ∞.



The conservation of energy requires that $E_2 = E - E_1$. Under normal circumstances we would have $E_1 > 0$ and therefore $E_2 < E$. In this case E_1 is not necessarily positive in the ergoregion, since K becomes spacelike. If the decay happens in the ergoregion, it is thus possible to extract energy from a black hole.
There is however a limit to how much energy one can extract from a black hole, since naturally the energy you gained is in spite of something else that has lost it. To see this more clearly notice the fact that for the particle that crosses the

horizon r_+, we get that $-(p_1)_\mu K^\mu \geq 0$, (where ξ is defined as (7.20)) since ξ is future directed and null on the horizon, and p_1 is future directed and timelike or null (massive or massless particle, i think). We can thus write:

$$-<p_1, K> -\Omega_H \underbrace{<p_1, \partial_\varphi>}_{L_1} \geq 0 \tag{7.27}$$

where L_1 is the infalling particle's angular momentum. This equation is in fact the following:

$$E_1 - \Omega_H L_1 \geq 0 \tag{7.28}$$

which means that $L_1 \leq \frac{E_1}{\Omega_H}$, therefore if $E_1 < 0$ it follows that $L_1 < 0$. We thus have found that the black hole's angular momentum has decreased as a result of the Penrose process, which means that the energy we gained was in spite of the hole's rotational energy.
Now the mass and angular momentum of the black hole are what they used to be plus the negative contributions of the particle that has fallen in:

$$\begin{aligned} \delta m &= E_1 \\ \delta J &= L_1 \end{aligned} \tag{7.29}$$

Thus:

$$\delta J = \frac{\delta m}{\Omega_H} = \frac{\delta m}{a/(r_+^2 + a^2)} \tag{7.30}$$

Using the fact that $r_+ = m + \sqrt{m^2 - a^2} = m + \sqrt{m^2 + J^2/m^2}$ we get:

$$\begin{aligned} \delta J \leq \frac{(r_+^2 + a^2)\delta m}{a} &= \frac{m^2 + 2m\sqrt{m^2 - J^2/m^2} + m^2 - J^2/m^2 + J^2/m^2}{J/m}\delta m \\ &= \frac{2m(m^2 + \sqrt{m^4 - J^2})}{J}\delta m \end{aligned} \tag{7.31}$$

which implies that:

$$\delta(m^2 + \sqrt{m^4 - J^2}) \geq 0 \tag{7.32}$$

therefore both the angular momentum and the mass of the black hole can decrease, but their combination must satisfy this inequality. This quantity is related in fact to the area of the black hole's horizon. To see this, first we let $t = const$ and $r = r_+$ in the Boyer-Lindquist metric (or $v = const$ and $r = r_+$ in Kerr coordinates) and find the induced metric to be:

$$d\sigma^2 = \frac{r_+^2 + a^2}{r_+^2 + a^2\cos^2(\theta)}\sin^2(\theta)d\chi^2 + (r_+^2 + a^2\cos^2(\theta))d\theta^2 \tag{7.33}$$

whose determinant is:

$$\sqrt{\sigma} = (r_+^2 + a^2)\sin^2(\theta) \tag{7.34}$$

The area is then the integral of the induced volume element:

$$A = \int \sqrt{\sigma} d\chi d\theta = \int_0^{2\pi} d\chi \int_0^{\pi} (r_+^2 + a^2) \sin^2(\theta) d\theta = 8\pi(m^2 + \sqrt{m^4 - J^2}) \tag{7.35}$$

which is what we found in (7.32), thus the interpretation of the previous limit is that both the mass and the angular momentum decrease, provided that the area of the horizon increases after the process has taken place. This is a special case of the second law of black holes mechanics, which we'll see more thoroughly below.

To see more precisely the amount of energy we can extract with the Penrose process, we define the *irreducible mass*:

$$m_{irr}^2 = \frac{1}{2}(m^2 + \sqrt{m^4 - J^2}) \tag{7.36}$$

from which:

$$m^2 = m_{irr}^2 + \frac{J^2}{4m_{irr}^2} \geq m_{irr}^2 \tag{7.37}$$

Let's begin the Penrose process with a black hole with mass m_0 and angular momentum J_0. After the process the variables change to m and J. The following inequalities hold:

$$m^2 \geq m_{irr}^2 \geq m_{0irr} \tag{7.38}$$

where the last inequality is true since $\delta m_{irr} \geq 0$. From (7.38) we can see that:

$$m_0 - m \leq m_0 - m_{irr} \leq m_0 - m_{0irr} \tag{7.39}$$

where the first term is the energy extracted from the black hole, which we therefore find to have an upper limit given by $m_0 - m_{0irr}$. In the case scenario we have $\delta m_{irr} = 0$ and thus $J = 0$, i.e. the black hole stops spinning.

7.1.2 Gravitational Dragging

Another interesting peculiarity of the Kerr solution is the gravitational dragging of intertial frames of reference. To see this phenomenon first note that we can rewrite the Kerr metric ((7.5) with $e = 0$) in an *ADM canonical form*:

$$ds^2 = -N^2 dt^2 + \sigma(d\varphi - \omega dt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \tag{7.40}$$

with a proper identification of the various constants:

$$\begin{aligned} N^2 &= \frac{\rho^2 \delta}{\Sigma^2} \\ \Sigma^2 &= (r^2 + a^2)^2 - a^2 \Delta \sin^2(\theta) \\ \sigma &= \frac{\Sigma^2}{\rho^2} \sin^2(\theta) \\ \omega &= \frac{2mar}{\Sigma^2} \end{aligned} \tag{7.41}$$

Consider next an observer with 4-velocity:

$$u = \frac{1}{N}(\partial_t + \omega\partial_\varphi) \tag{7.42}$$

with $< u, u >= -1$ of course. We also easily find that $< u, \partial_\varphi >=< u, \partial_\theta >=< u, \partial_r >= 0$, so the observer does not move with respect to the $t = const$ hypersurfaces. Also note that the fact that $< u, \partial_\varphi >$ implies there is no angular momentum, i.e. the observer is (at least locally) non-rotating. However we can see that the same observer rotates with respect to infinity with angular velocity ω, in fact:

$$\frac{d\varphi(\tau)}{dt} = \frac{\frac{d\varphi}{d\tau}}{\frac{dt}{d\tau}} = \frac{u^\varphi}{u^t} = \omega \tag{7.43}$$

Therefore the observer is rotating as seen from infinity, in spite of the fact that he locally isn't. This phenomenon is called *gravitational dragging*. To stay at a fixed φ outside a Kerr black hole, an observer would need to move with a certain speed in the opposite direction in which the hole is spinning; this required velocity increases as we approach the ergosphere and becomes the speed of light at the ergosphere. Thus in the ergoregion there is no sitting still: the velocity we would need to rotate at is greater than the speed of light. To prove this last fact let's first consider a timelike curve tangent to the vector field u defined in (7.42), then:

$$u^2 < 0 \implies \underbrace{g_{tt}u^t u^\varphi + g_{\varphi\varphi}u^\varphi u^\varphi + g_{\theta\theta}u^\theta u^\theta}_{> 0 \text{ in the ergoregion}} + 2g_{t\varphi}u^t u^\varphi \leq 0 \tag{7.44}$$

thus:

$$\underbrace{g_{t\varphi}}_{= -\omega\sigma < 0} \quad u^t u^\varphi < 0 \tag{7.45}$$

Let's prove that $u^t > 0$. Consider the vector field $\xi = \partial_t + \omega\partial_\varphi$, which is timelike as see from its norm $< \xi, \xi >= -N^2 < 0$ outside the horizon; in addition ξ is future directed (at infinity $\xi = \partial_t$ which is obviously future directed). But u is timelike and future directed as well, which means that $< u, \xi >= -N^2 u^t < 0$ so that $u^t > 0$.
As a consequence of (7.45) we then find that $u^\varphi = \frac{d\phi}{d\tau} > 0$, so that every observer that resides in the ergoregion have to rotate in the same direction of the black hole's spin. This is the extreme case of gravitational dragging, where the motion cannot be reversed.

7.1.3 Kerr Black Hole Temperature

The method for obtaining the temperature is straight forward. Let's Wick rotate the ADM metric (7.40):

$$ds^2 = N^2 d\tau^2 + \sigma(d\varphi + i\omega d\tau)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{7.46}$$

and expand the various quantities around the horizon r_+:

$$\begin{aligned}\Delta = \underbrace{\Delta(r_+)} = 0 + \Delta'(r_+)(r-r_+) + \dots \\ = 2(r_+ - m)(r - r_+) + \dots\end{aligned} \tag{7.47}$$

and:

$$\begin{aligned}N^2 = \underbrace{N^2(r_+)}_{=0} + \frac{\partial N^2}{\partial_r}|_{r_+}(r-r_+) \\ = \frac{r_+^2 + a^2 cos^2(\theta)}{(r_+^2+a^2)^2} 2(r_+ - m)(r-r_+)\end{aligned} \tag{7.48}$$

therefore the metric becomes:

$$\begin{aligned}ds^2 \cong \frac{r_+^2 + a^2 cos^2(\theta)}{(r_+^2+a^2)^2} 2(r_+ - m)(r-r_+) d\tau^2 \\ + \frac{r_+^2 + a^2 cos^2(\theta)}{2(r_+ - m)(r-r_+)} dr^2 + \sigma_+(d\varphi + i\omega_+ d\tau)^2 + \varphi_+^2 d\theta^2\end{aligned} \tag{7.49}$$

and if we rename $r - r_+ \equiv R^2$:

$$ds^2 = \frac{2(r_+^2 + a^2\cos^2(\theta))}{r_+ - m}\left(\underbrace{\frac{(r_+ - m)^2}{(r_+ + a^2)^2} R^2 d\tau^2}_{\equiv R^2 d\Phi^2} + dr^2\right) \tag{7.50}$$

From (7.50) we see that we have to identify Φ with itself after a period of $\tau \frac{r_+ - m}{r_+^2 + a^2}$ in order not to have singularities, so that $\tau \sim \tau + 2\pi \frac{r_+^2 + a^2}{r_+ - m}$. The period of the euclidian time is related to the Hawking temperature by $\beta = \frac{1}{T}$, which implies that:

$$T_{Kerr} = \frac{r_+ - m}{2\pi(r_+^2 + a^2)} = \frac{r_+ - m}{4\pi m r_+} \tag{7.51}$$

In the case of extreme Kerr black holes we know that $a = m \implies r_+ = m$ so the temperature is zero, meaning that this case is sort of a *ground state* of spinning black holes.
As a curiosity, we could ask ourselves if these maximally spinning black holes do in fact exist. Observation has found two quasi-extreme cases:

- *GRS1915+ 105*: $\frac{J}{m^2} \geq 0.98$
- *SMBH at the center of MCG-6-30-15*: $\frac{J}{m^2} \geq 0.99$

7.1.4 Killing-Yano tensors

A solution of the Einstein equations admits a *Killing-Yano tensor* $f_{\mu\nu}$ if it satisfies:

$$\nabla_{(\mu} f_{\mu)\nu} = 0 \qquad , f_{\mu\nu} = f_{\nu\mu} \tag{7.52}$$

This is a generalization of a Killing vector field. We can go further and define:

$$K\mu\nu = f_{\mu}{}^{\lambda} f_{\lambda\nu} = K_{\nu\mu} \tag{7.53}$$

from which it follows, using (7.52), that:

$$\nabla_{(\mu} K_{\nu\rho)} = 0 \tag{7.54}$$

This formalism is useful since one can prove that

$$Q = K_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \tag{7.55}$$

is a constant of motion along a geodesic, i.e. a conserved charge. For example for the Kerr-Newman solution, the coserved charges found are obviously the electric/magnetic charges, energy and angular momentum.
Question: is Q related to some continuous spacetime simmetry? [6]

7.1.5 Kerr solution in higher dimensions

It is straightforward to extend Birkhoff's theorem to higher dimensions. That is, one can solve Einstein's vacuum equations in any $D \geq 4$ with the assumption that the geometry is asymptotically flat and spherically symmetric, i.e. the solution has an $SO(D-1)$ isometry.
The features of these higher dimensional solutions are essentially unchanged from those of the $D = 4$ Schwarzschild metric.
The generalization of the Kerr solution to higher dimensions is more interesting, and in $D = 5(= 4{+}1)$ is called the *Myers-Perry solution.* [7] The group $SO(D{-}1)$ has $\lfloor \frac{D-1}{2} \rfloor$ Casimir operators, which means that in five dimensions there are two possible angular momentums, i.e. two planes that the hole can spin in. For simplicity coinsider a *single-spinning* Myers-Perry:

$$\begin{aligned} ds^2 = -dt^2 + \frac{M}{r^{D-5}\rho^2}(dt^2 + a\sin^2(\theta)d\varphi)^2 + \frac{\rho^2}{\Delta}dr^2 \\ + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2(\theta)d\varphi^2 + r^2\cos^2(\theta)d\Omega^2_{D-4} \end{aligned} \tag{7.56}$$

where:

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2(\theta) \\ \Delta &= r^2 + a^2 - \frac{M}{r^{D-5}} \end{aligned} \tag{7.57}$$

[6] See I. Bars - *Two-time physics* (1998)
[7] See https://arxiv.org/abs/1111.1903

and $d\Omega^2_{D-4}$ is the line element on S^{D-4}. If we put $D > 5$, the solution is similar to the Kerr one, although it has other interesting properties. In fact the solution has horizons for $\Delta = 0$, i.e.:

$$r^2 + a^2 - \frac{M}{r^{D-5}} = 0 \tag{7.58}$$

which has one solution for $D = 5$ with the constraint $M^2 < a^2$. If $D > 5$, this equation always has a solution, thus it doesn't have an upper bound on the spinning parameter a. Why does this happen physically? Let's look at the competition between the gravitational attraction of the hole and the centrifugal force:

$$\frac{\Delta}{r^2} - 1 = -\frac{M}{r^{D-3}} + \frac{a^2}{r^2} \tag{7.59}$$

where the first term is the attraction due to gravity which naturally depends on the number of dimensions D at play, whereas the second term decreases as r^2 in any number of dimensions, since the rotation takes place on a plane independently of D. Notice that in $D = 4$ the first term wins, therefore the solution will have an upper bound on the angular momentum. For $D > 5$ the centrifugal term wins out, and as a consequence the hole can spin indefinitely fast: the limit $a \to \infty$ is called the *ultra-spinning Myers-Perry* which leads to *blackfolds.* [8]

7.2 The Reissner-Nordstrom and Majumdar-Papapetrou solutions

The Reissner-Nordstrom solution is obtained by letting $a = 0$ in (7.5). This describes a static charged black hole:

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)} \\ &+ r^2(d\theta^2 + \sin^2(\theta) d\varphi^2) \end{aligned} \tag{7.60}$$

with the Maxwell one-form:

$$A = \frac{Q}{r} dt + P\cos(\theta) d\varphi \qquad , Q^2 + P^2 = e^2 \tag{7.61}$$

with Q and P being the electric and magnetic charge respectively.
This solution has horizons for $r^2 - 2mr + e^2 = 0$ which means:

$$r = r_\pm = m \pm \sqrt{m^2 - e^2} \tag{7.62}$$

Three cases arise like for the Kerr solution: the $e^2 > m^2$ case contains a naked singularity and is therefore unphysical, the "normal" case is $m^2 > e^2$ and the conformal diagram is similar to the $m^2 > a^2$ case for the Kerr black hole,

[8] See https://arxiv.org/abs/1106.2021

however for the charged one you cannot travel past the $r = 0$ singularity into another asymptotically flat universe.
The extreme case is $e = m$ and the metric becomes:

$$ds^2 = -\left(1 - \frac{e}{r}\right)^2 dt^2 + \left(1 - \frac{e}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \tag{7.63}$$

Now change the coordinates to a new isotropic coordinate $\rho = r - e$, thus the horizon at $r = e = m$ is at $\rho = 0$:

$$ds^2 = -\left(1 - \frac{e}{\rho}\right)^{-2} dt^2 + \left(1 - \frac{e}{\rho}\right)^2 (d\rho^2 + \rho^2 d\Omega^2) \tag{7.64}$$

where we can see that the last part of the metric is the euclidian space one, that's the reason for the name "isotropic" assigned to ρ. Also note that $1 + e/\rho$ is a harmonic function on $\mathbb{R}^3$, i.e. :

$$\nabla^2_{\mathbb{R}^3}\left(1 + \frac{e}{\rho}\right) = 0 \tag{7.65}$$

A fascinating property of extremal black holes is that the mass is in some sense balanced by the charge. More specifically, two extremal black holes with same sign charges will attract each other gravitationally, but repel each other electromagnetically, and it turns out that these effects precisely cancel. Indeed, we can find exact solutions for the coupled Einstein-Maxwell equations representing any number of such black holes in a stationary configuration. To see this, note that since the far right side of (7.64) is just the flat space metric, we can write it as:

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}^2 \tag{7.66}$$

where H can be written as:

$$H = 1 + \frac{m}{|\vec{x}|} \tag{7.67}$$

and the electric potential (setting $P = 0$ for simplicity) is:

$$A = \frac{dt}{H(\vec{x})} \tag{7.68}$$

But now let's forget that we know that H obeys (7.67) and simply plug the metric (7.66) and the electrostatic potential (7.68) into Einstein's equations and Maxwell's equations, only imagining that $\partial_0 H = 0$, i.e. H is time-independent. We can straightforwardly show that they can be simultaneously solved by any time-independent H that obeys:

$$\nabla^2_{\mathbb{R}^3} H = 0 \tag{7.69}$$

We know all the solutions to this equation is though (that are well behaved at infinity):

$$H(\vec{x}) = 1 + \sum_{j=1}^{N} \frac{m_j}{|\vec{x} - \vec{x}_j|} \tag{7.70}$$

for some set of N spatial points defined by $\vec{x}_j$. This static configuration is that of N extreme Reissner-Nordstrom black holes with masses m_j and at positions $\vec{x}_j$ (note that the configuration is static!). This solution is better known as the *Majumdar-Papapetrou solution.* Notice that we have just created a superposition of black hole solutions, despite the Einstein-Maxwell equations being highly non-linear, this means that the holes repel each other, creating a static solution!

We can also study the near-horizon geometry $\rho \to 0$ of the extremal charged black hole, taking (7.64) and expanding it:

$$\begin{aligned} ds^2 \to &-\frac{e^2}{\rho^2}dt^2 + \frac{e^2}{\rho^2}(d\rho^2 + \rho^2 d\Omega^2) \\ &= -\frac{e^2}{\rho^2}dt^2 + \frac{e^2}{\rho^2}d\rho^2 + e^2 d\Omega^2 \end{aligned} \tag{7.71}$$

which is the product space metric for $AdS_2 \times S^2$ (*Bertotti-Robinson*), and the isometry group is $SO(2,1) \times SO(3)$. We could compare this result with the near-horizon metric for the extremal Kerr black hole $a = m$, $r_+ = r_- = m$. We recall that the metric is:

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}(d\hat{t} - a\sin^2(\theta)d\hat{\varphi})^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 \\ &+ \frac{\sin^2(\theta)}{\rho^2}((\hat{r}^2 + a^2)d\hat{\varphi} - ad\hat{t})^2 + \rho^2 d\theta^2 \end{aligned} \tag{7.72}$$

with $\Delta = (\hat{r} - a)^2$ and $\rho = \hat{r}^2 + a\cos^2(\theta)$. Now we take the limit near the horizon $r_+ = m$ by introducing a scaling parameter λ and afterwards letting it go to zero:

$$\begin{aligned} r &= \frac{\hat{r} - m}{\lambda m} \\ t &= \frac{\lambda \hat{t}}{2m} \\ \varphi &= \hat{\varphi} - \frac{\hat{t}}{2m} \end{aligned} \tag{7.73}$$

Notice that if $\hat{r} \to m$ and $\lambda \to 0$ the quotient remains finite. The metric (7.72) becomes:

$$ds^2 = 2\Omega^2 m^2 \left(\frac{dr^2}{r^2} + d\theta^2 - r^2 dt^2 + \Lambda(d\varphi + dt)^2\right) \tag{7.74}$$

where $\Omega^2 = \frac{1+\cos^2(\theta)}{2}$ and $\Lambda = \frac{2\sin(\theta)}{1+\cos^2(\theta)}$. (7.74) is known as the *near horizon extremal Kerr geometry* and it's not a product space this time, since the dt term is multiplied by the factor Ω which contains the variable θ.

8 Conserved Charges

Since we have claimed above that the most general stationary black hole solution to general relativity is characterized by mass, charge and spin, we should consider how these quantities are defined. There exist several ways to accomplish this, and one of them is through the so called *Komar integrals*.
Let Σ be a spacelike hypersurface, $V \subset \Sigma$ and ∂V the border of V. To every Killng vector field ξ we associate a Komar integral:

$$Q_\xi(V) = \frac{c}{16\pi G} \oint_{\partial V} dS_{\mu\nu} \nabla^\mu \xi^\nu \tag{8.1}$$

(c is just a constant, not the speed of light) which can be equally rewritten with the help of Gauss's theorem:

$$Q_\xi(V) = \frac{c}{8\pi G} \int_V dS_\mu \nabla_\nu \nabla^\mu \xi^\nu \tag{8.2}$$

In order to see that this is a conserved charge, make use of the following:

Lemma 8.1. $\nabla_\nu \nabla_\mu \xi^\nu = R_{\mu\nu} \xi^\nu$ *for every Killing vector field.*

This can be easily obtained from Lemma 6.3 by taking the trace on both sides of the equation. Plugging the above lemma in (8.2) gives:

$$Q_\xi(V) = \frac{c}{8\pi G} \int_V dS_\mu R^\mu_\nu \xi^\nu \tag{8.3}$$

We know however that $R_{\mu\nu}$ is related to the energy momentum tensor $T_{\mu\nu}$ via the Einstein equations in the following way

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \tag{8.4}$$

where T is the trace of $T_{\mu\nu}$,thus:

$$\begin{aligned} Q_\xi(V) &= c \int_V dS_\mu (T^\mu{}_\nu \xi^\nu - \frac{1}{2} T \xi^\mu) \\ &\equiv \int_V dS_\mu J^\mu(\xi) \end{aligned} \tag{8.5}$$

where we have identified:

$$J^\mu(\xi) \equiv c(T^\mu{}_\nu \xi^\nu - \frac{1}{2} T \xi^\mu) \tag{8.6}$$

End with the following:

Lemma 8.2. $\nabla_\mu J^\mu(\xi) = 0$

Proof. Using the fact that $\nabla_\mu T^{\mu\nu} = 0$ we find:

$$\nabla_\mu J^\mu = c\left(T^{\mu\nu}\nabla_\mu \xi_\nu - \frac{1}{2}T\nabla_\mu \xi^\mu\right) - \frac{c}{2}\xi^\mu \nabla_\mu T \tag{8.7}$$

The term in parenthesis is clearly zero since ξ is a Killing vector field. Let's analyze in more detail the last term:

$$\xi^\mu \nabla_\mu T = \xi^\mu \partial_\mu T = -\frac{1}{8\pi G}\xi^\mu \partial_\mu R \tag{8.8}$$

Now if we take a reference system where $\xi^\mu \partial_\mu = \partial_\alpha$ for simplicity, the metric and therefore R as well are independent from α, thus this term goes to zero. □

This lemma tells us that J^μ is in fact a covariantly-conserved 4-current. This is very interesting since we can show that the energy is the conserved charge associated to the time translation Killing field $K = \partial_t$:

$$E(V) = -\frac{1}{8\pi G}\oint_{\partial V} dS_{\mu\nu}\nabla^\mu K^\nu \tag{8.9}$$

where we have set $c = -2$.
In fact, consider a metric $g_{\mu\nu}$ that resembles the Minkowski one $\eta_{\mu\nu}$ at $r \to \infty$ such that $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}\left(\frac{1}{r}\right)$. Linearize the Einstein equations at $r = \infty$ and consider a perfect fluid $T_{\mu\nu} = \mathrm{diag}(\rho, 0, 0, 0)$, then:

$$E(v) = \int dV T^{00} = -\frac{1}{8\pi G}\oint_{\partial V} dS_{\mu\nu}\nabla^\mu K^\nu \tag{8.10}$$

where for the second equality we have used Einstein equations and Gauss's theorem.

Example 8. We can show that for the metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{8.11}$$

the energy E is equal to the mass of the black hole m. To do this, define $N = \sqrt{\left(1 - \frac{2m}{r}\right)}$ (the *lapse function*), and take as the surface over which to integrate ∂V as a sphere outside the horizon, i.e. a $t = \mathrm{const}$ and $r = \mathrm{const} > 2m$ hypersurface.

- The vector orthogonal to the constant time hypersurface is $u = N^{-1}\partial_t$, in fact $< u, u >= -1$ and $< u, \partial_i >= 0$ for $i = r, \theta, \varphi$.
- The vector perpendicular to a sphere is naturally a radial vector $v = N\partial_r$, since $< v, v >= 1$ and $< v, \partial_\theta >=< v, \partial_\varphi >= 0$.

The measure we are looking for is:

$$dS_{\mu\nu} = \underbrace{(v^\mu u^\nu - u^\mu v^\nu)}_{\text{binormal to } \partial V}\sqrt{\sigma}d\theta d\varphi \tag{8.12}$$

with σ being the determinant of the induced metric on ∂V:

$$d\sigma^2 = r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{8.13}$$

The only non-zero term of the measure is:

$$dS^{rt} = (v^r u^t - \underbrace{u^r}_{=0} v^t)\sqrt{\sigma}d\theta d\varphi = \sqrt{\sigma}d\theta d\varphi = -dS^{tr} \tag{8.14}$$

Now let's calculate Komar's integral for the energy of the system:

$$E = -\frac{1}{8\pi G}\int_0^\pi d\theta \int_0^{2\pi} d\varphi\sqrt{\sigma}\nabla_r K_t \cdot 2 \tag{8.15}$$

where the 2 comes from the fact that K is a Killing field. Calculate the integrand:

$$\nabla_r K_t = \partial_r K_t - \Gamma^\rho_{rt}K_\rho = \partial_r K_t - \Gamma_{\rho rt}K^\rho = \partial_r K_t - \Gamma_{trt} \tag{8.16}$$

where:

$$\begin{aligned} \partial_r K_t &= \partial_r(g_{tt}K^t) = \partial_r g_{tt} \\ \Gamma_{trt} &= \frac{1}{2}(g_{tt,r} + gtr,t - g_{rt,t}) = \frac{1}{2}\partial_r g_{tt} \end{aligned} \tag{8.17}$$

This implies that:

$$\nabla_r K_t = \frac{1}{2}\partial_r g_{tt} = -\frac{m}{r^2} \tag{8.18}$$

Thus:

$$E = -\frac{1}{4\pi G}\int_0^\pi d\theta \int_0^{2\pi} d\varphi r^2 \sin(\theta)\left(-\frac{m}{r^2}\right) = \frac{m}{G} \tag{8.19}$$

Notice that the result is independent of where we place the sphere around the black hole that we are integrating over, provided that $r > 2m$. This was only possible because on the region $r > 2m$ we have $T_{\mu\nu} = 0$, i.e. there's a vacuum. If we were to do this for an AdS black hole, the integral would diverge if $r \to \infty$ since the energy density of the vacuum is a constant (Λ), therefore if we wanted to find the mass of the hole we would need to subtract the background.

Along with energy, another quantity of interest is angular momentum. Consider the Komar integral for the conserved charge with $\xi = m = \partial_\varphi$ and $c = 1$:

$$J(V) = \frac{1}{16\pi G}\oint_{\partial V} dS_{\mu\nu}\nabla^\mu m^\nu \tag{8.20}$$

We can show that this is in fact the angular momentum. First rewrite the integral using Gauss's theorem:

$$\begin{aligned} J &= \frac{1}{8\pi G}\int_V dS_\mu \nabla_\nu \nabla^\mu m^\nu = \frac{1}{8\pi G}\int_V dS_\mu R^\mu{}_\nu m^\nu \\ &= \int_V dS_\mu (T^\mu{}_\nu - \frac{1}{2}T\delta^\mu{}_\nu)m^\nu = \int_V dS_\mu J^\mu \end{aligned} \tag{8.21}$$

where $J^\mu = T^\mu{}_\nu m^\nu - \frac{1}{2}Tm^\mu$. Choosing the $t = \text{const}$ hypersurface $V = \Sigma_t$, this implies $dS_\mu = (dV, \vec{0})$ and $dS_\mu m^\mu = 0$. Therefore:

$$J = \int_V dV T^0{}_\nu m^\nu = \int_V dV (T^0{}_2 x^1 - T^0{}_1 x^2) \tag{8.22}$$

where we have used the fact that in cartesian coordinates $m = \partial_\varphi = x^1\partial_2 - x^2\partial_1$. If we assume a weak source $g_{\mu\nu} \simeq \eta_{\mu\nu}$ then:

$$J = \epsilon_{3jk}\int dV x^j T^{k0} \tag{8.23}$$

which is the third component of the angular momentum.

Example 9. Once again we can calculate the integral (8.20) for the Kerr metric:

$$ds^2 = -N^2dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma\sin^2(\theta)}{\rho^2}(d\varphi - \omega dt)^2 \tag{8.24}$$

with the usual:

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2(\theta) \\ \Delta &= r^2 - 2mr + a^2 \\ \omega &= \frac{2mar}{\Sigma^2} \\ \Sigma^2 &= (r^2+a^2)^2 - a^2\Delta\sin^2(\theta) \\ N^2 &= \frac{\rho^2\Delta}{\Sigma^2} \end{aligned} \tag{8.25}$$

Take as before the hypersurface of choice to be $t = \text{const}$ and $r = \text{const} > r_+$. The normal vectors are:

- $u = N^{-1}(dt + \omega d\varphi)$ with $< u, u >= -1$ and $< u, \partial_i >= 0$ with $i = r, \theta, \varphi$.
- $v = \frac{\sqrt{\Delta}}{\rho}\partial_r$ with $< v, v >= 1$ and $< v, \partial_\theta >=< v, \partial_\varphi >= 0$.

The measure of integration is:

$$dS^{\mu\nu} = (v^\mu u^\nu - u^\mu v^\nu)\sqrt{\sigma}d\theta d\varphi \tag{8.26}$$

and the induced metric on the hypersurface is:

$$d\sigma^2 = \rho^2 d\theta^2 + \frac{\Sigma\sin^2(\theta)}{\rho^2}d\varphi^2 \tag{8.27}$$

Therefore:

$$(v^\mu u^\nu - u^\mu v^\nu)\nabla_\mu m_\nu = v^\mu u^\nu \nabla_\mu m_\nu - u^\nu v^\mu \nabla_\mu m_\nu = -2u^\mu v^\nu \nabla_\mu m_\nu \tag{8.28}$$

The Komar integral thus is:

$$J = \frac{-2}{16\pi G}\int_0^\pi d\theta \int_0^{2\pi} d\varphi \Sigma \sin(\theta) u^\mu v^\nu \underbrace{\nabla_\mu m_\nu}_{= g_{\nu\rho}\nabla_\mu m^\rho} \tag{8.29}$$

but:

$$\nabla_\mu m^\rho = \partial_\mu m^\rho + \Gamma^\rho_{\mu\sigma} m^\sigma = \Gamma^\rho_{\mu\varphi} \tag{8.30}$$

where we have used the fact that $m^\varphi = 1$ is the only non zero component of the field m. So (8.29) becomes:

$$\begin{aligned} &= -\frac{1}{4G}\int_0^\pi d\theta \Sigma \sin(\theta) u^\mu v^\nu \Gamma_{\nu\mu\varphi} = -\frac{1}{4G}\int_0^\pi d\theta \Sigma \sin(\theta)(u^t v^r \Gamma_{rt\varphi} + u^\varphi v^r \Gamma_{r\varphi\varphi}) \\ &= -\frac{1}{4G}\int_0^\pi d\theta \sin(\theta)\Sigma \left(\frac{1}{N}\frac{\sqrt{\Delta}}{\rho}\Gamma_{rt\varphi} + \frac{\omega}{N}\frac{\sqrt{\Delta}}{\rho}\Gamma_{r\varphi\varphi}\right) \end{aligned} \tag{8.31}$$

and the Christoffels are given by:

$$\begin{aligned} \Gamma_{rt\varphi} &= \cdots = -\frac{1}{2}\partial_r\left(\frac{-\omega\Sigma^2\sin^2(\theta)}{\rho^2}\right) \\ \Gamma_{r\varphi\varphi} &= \cdots = -\frac{1}{2}\partial_r\left(\frac{\Sigma^2\sin^2(\theta)}{\rho^2}\right) \end{aligned} \tag{8.32}$$

The end result does not depend on r of course, and since the integral is very complicated we use this fact to our advantage and expand around $r \to \infty$ the term in parenthesis in (8.31):

$$(\dots) \simeq -3ma\sin^2(\theta) + \mathcal{O}(\frac{1}{r}) \tag{8.33}$$

which finally implies:

$$J = \frac{3ma}{4G}\int_0^\pi d\theta \sin^3(\theta) = \frac{ma}{G} \tag{8.34}$$

9 Black Hole thermodynamics

We have seen previously that for every physical process involving a black hole it is true that:

$$\delta A \geq 0 \tag{9.1}$$

which means that the area of the black hole must in fact increase. This is very reminiscent of the second law of thermodynamics, which says that the total entropy of an isolated system can never decrease:

$$\delta S \geq 0 \tag{9.2}$$

This analogy is not a coincidence, and in fact we can extend this to every law of thermodynamics:

Law	Thermodynamics	Black Holes
0	T is constant at thermal equilibrium	κ is constant on the horizon
1	$dE = TdS +$ work terms	$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ$
2	$\delta S \geq 0$	$\delta A \geq 0$
3	$T = 0$ is not achievable with a finite number of physical processes	$\kappa = 0$ is not achievable with a finite number of physical processes

These 4 laws (for black holes) are known as the *4 laws of black hole mechanics* and were found by Bardeen-Hawking-Carter (1973). As was said above, this is not just a mathematical analogy, since in 1974 Hawking found that black holes emit radiation (evaporate) as a consequence of quantum effects in curved spacetime, and the radiation follows a black-body with temperature:

$$T = \frac{\hbar\kappa}{2\pi} \tag{9.3}$$

therefore superficial gravity is related to temperature in a fundamental way, which is what we found in (6.65). Basically the 4-laws are just black-hole-applied thermodynamics.
This means that we can also associate an entropy to the black-hole:

$$S = \frac{A}{4G\hbar} \tag{9.4}$$

We also know form boltzmann's formula that:

$$S = klogW \tag{9.5}$$

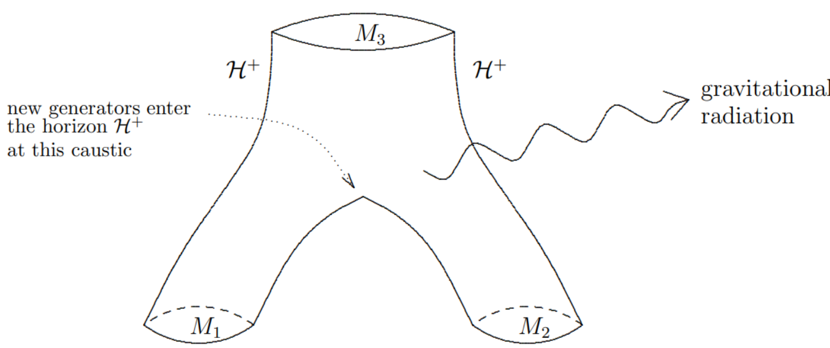
where W is the number of microstates of the system. Where are the microstates of the black hole? We already know the macrostate is characterized by three parameters: the mass, the angular momentum and the charge. What is the microstate of the system?

The second law has very interesting consequences. Consider first the metric:

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\Omega^2) \tag{9.6}$$

with the usual definition of $V(r)$, so that there is an horizon at $r_h = 2Gm$ so that the area of the horizon is $A_H = 4\pi r_H^2 \propto m^2$. Now the consequences are:

i) The second law limits to efficiency of mass/energy conversion in black hole collisions. Consider the Finkelstein diagram of two coalescing black holes:



We know that $\delta A \geq 0$, thus:

$$A_3 \geq A_1 + A_2 \implies m_3^2 \geq m_1^2 + m_2^2 \tag{9.7}$$

which means:

$$E_{rad} = m_1 + m_2 - m_3 \leq m_1 + m_2 - \sqrt{m_1^2 + m_2^2} \tag{9.8}$$

The efficiency is thus:

$$\eta = \frac{E_{rad}}{m_1 + m_2} \leq 1 - \frac{1}{\sqrt{2}} \tag{9.9}$$

ii) The second law also implies that black holes cannot bifurcate.
We know that:

$$A_3 \leq A_1 + A_2 \implies m_1^2 + m_2^2 \geq m_3^2 \tag{9.10}$$

The conservation of energy requires:

$$m_1 + m_2 = m_3 - E_{rad} \leq m_3 \implies m_3^2 \geq (m_1 + m_2)^2 \tag{9.11}$$

so that:

$$m_1^2 + m_2^2 \geq m_3^2 \geq (m_1 + m_2)^2 \implies 0 \leq 2m_1 m_2 \tag{9.12}$$

which is of course a contradiction.

9.1 The Kerr black hole as a thermodynamic system

Let's use the formalism we just introduced to a rotating black hole. Recall first some properties:

- $E = m$
- $J = ma$
- $S = \frac{A_H}{4} = \pi(r_+^2 + a^2)$
- $T = \frac{\kappa}{2\pi} = \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)}$
- $\Omega_H = \frac{a}{r_+^2 + a^2}$

and one can also prove that:

$$E = E(S, J) = \sqrt{\frac{S}{4\pi} + \frac{\pi}{S}J^2} \tag{9.13}$$

which is called the *fundamental relation*. This name comes from thermodynamics where we relate the energy to the extensive thermodynamics quantities (S and J in this case). We can verify that:

$$\begin{aligned} \left(\frac{\partial E}{\partial S}\right)|_J &= \dots = \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)} = T \\ \left(\frac{\partial E}{\partial J}\right)|_S &= \dots = \Omega_H \end{aligned} \tag{9.14}$$

These two are equations of state, i.e. they express the intensive parameters T and Ω_H as a function of the extensive parameters. Therefore:

$$dE = \left(\frac{\partial E}{\partial S}\right) dS + \left(\frac{\partial E}{\partial J}\right) dJ = TdS + \Omega_H dJ \tag{9.15}$$

which is the first law of black hole mechanics. (9.13) tells us that:

$$E(\lambda S, \lambda J) = \lambda^{1/2} E(S, J) \tag{9.16}$$

So we can say that:

$$\begin{aligned} \frac{\partial E(\lambda S, \lambda J)}{\partial \lambda} &= \frac{\partial E}{\partial(\lambda S)}\frac{\partial(\lambda S)}{\partial \lambda} + \frac{\partial E}{\partial(\lambda J)}\frac{\partial(\lambda J)}{\partial \lambda} \\ \implies \frac{1}{2}\lambda^{-1/2}E(S, J) &= \frac{1}{\lambda}\frac{\partial E}{\partial S}S + \frac{1}{\lambda}\frac{\partial E}{\partial J}J \end{aligned} \tag{9.17}$$

Now setting $\lambda = 1$ we find:

$$\frac{1}{2}E = TS + \Omega_H J \tag{9.18}$$

this equation is due to Smarr (1973). From this we can find the specific heat as usual:

$$c_J = T\left(\frac{\partial S}{\partial T}\right)|_J \tag{9.19}$$

To find this we have, from (9.14) $T = T(S, J)$ so that:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial S}\right)|_J dS + \left(\frac{\partial T}{\partial J}\right)|_S dJ \\ &= \left(\frac{\partial T}{\partial S}\right)|_J \left(\left(\frac{\partial S}{\partial T}\right)|_J dT + \left(\frac{\partial S}{\partial T}\right)|_S dJ\right) + \left(\frac{\partial T}{\partial J}\right)|_S \end{aligned} \tag{9.20}$$

From this, by fixing J ($\partial_J = 0$ and $dJ = 0$):

$$1 = \left(\frac{\partial T}{\partial S}\right)|_J \left(\frac{\partial S}{\partial T}\right)|_J \tag{9.21}$$

Which implies:

$$c_J = \frac{T}{\left(\frac{\partial}{\partial S}\right)|_J} = \frac{ETS^3}{\pi J^2 - T^2 S^3} \tag{9.22}$$

where we have used the first equation in (9.14). In the Schwarzschild geometry we know that $J = 0$, so:

$$C = -\frac{E}{T} = -\frac{1}{8\pi T^2} < 0 \tag{9.23}$$

therefore the black hole shrinks as it radiates. Notice that (9.22) is singular at $\pi J^2 - T^2 S^3 = 0 \Longleftrightarrow r_+^2/a^2 = 3 + 2\sqrt{3}$.
It is interesting to note that we can write C_J as a function of J and E, using $S = S(E, J) = 2\pi(E^2 + \sqrt{E^4 - J^2})$ and the first of (9.14). If we then fix E and then plot C_J as J varies we get: (disegno)

where for an extreme black hole $T = 0$, so that $S = 2\pi J$ and $E = \sqrt{J}$ (we have used (9.13).) C_J has a singularity as is evident from the graph. Consider the ratio:

$$\frac{J}{E^2} = \frac{2ar_+}{r_+^2 + a^2} = \frac{2\frac{r_+}{a}}{\frac{r_+^2}{a^2} + 1} = 2\sqrt{3 + 2\sqrt{3}} \tag{9.24}$$

This implies that:

$$\frac{J^2}{E^4} = -3 + 2\sqrt{3} \simeq 0.46 \tag{9.25}$$

Next we can find the Gibbs free energy:

$$G = E - TS - \Omega_H J = G(T, \Omega_H) = \frac{1}{2}E \tag{9.26}$$

One can verify that G is continuous at the singularity of C_J and has continuous gradients $\frac{\partial G}{\partial T}|_{\Omega_H}$ and $\frac{\partial G}{\partial \Omega_H}|_T$, so we can classify the phenomenon that happens

at the pole $J^2/E^4 \simeq 0.464$ as a 2^{nd} kind phase transition. (i.e. ∂ and ∂^2 are continuous).
The phase diagrams:

$$\frac{\Omega_H}{T} = \frac{\pi J}{S\left(\frac{1}{8\pi} - \frac{\pi J^2}{2S^2}\right)} \tag{9.27}$$

and if we use $S = 2\pi(E^2 + \sqrt{E^4 - J^2})$ and $J \simeq 0.681E^2$ at the singularity, we get:

$$\frac{\Omega_H}{T} \simeq 5.843 \tag{9.28}$$

(disegno)

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10 The BTZ Black Hole

[9] Take the vacuum Einstein equations with $\Lambda = -\frac{1}{\ell^2} < 0$ in $(2+1) - dim$ with the units $8G = 1$:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{10.1}$$

These equations admit a black hole solution:

$$ds^2 = -N^2 dt^2 + f^{-2} dr^2 + r^2 (d\varphi + N^\varphi dt)^2 \tag{10.2}$$

with the following:

$$\begin{aligned} N = f &= (-8M + \frac{r^2}{\ell^2} + \frac{16J^2}{r^2})^{1/2} \\ N^\varphi &= -\frac{4J}{r^2} \end{aligned} \tag{10.3}$$

N is again called the lapse function. The metric (1.1) is stationary and axisymmetric, with Killing vectors ∂_t and ∂_φ, and generically has no other symmetries.Note from (10.2) that there is a bound on the J parameter in order to avoid conical singularities: $J \leq M\ell^2$. Let M, J be the mass and the angular momentum respectively, then the metric is singular when:

$$r = r_\pm = 4M\ell^2 \left(1 \pm \left(1 - \left(\frac{J}{M\ell} \right)^2 \right)^{1/2} \right) \tag{10.4}$$

but these are merely coordinate singularities, analogous to the Kerr case that we studied earlier in $(3+1) - dim$. These two equations imply that:

$$\begin{aligned} M &= \frac{r_+^2 + r_-^2}{\ell^2} \\ J &= \frac{2r_+ r_-}{\ell} \end{aligned} \tag{10.5}$$

As a consequence, one can show that the following is the fundamental thermodynamic relation:

$$M(S,J) = \frac{1}{2\ell^2} \left(\left(\frac{S}{\pi} \right)^2 + \left(\frac{\ell\pi}{S}^2 \right)^2 \right) \tag{10.6}$$

from which one can recover the temperature of the horizon and its rotational speed:

$$\left(\frac{\partial M}{\partial S} \right) |_J = T \qquad \left(\frac{\partial M}{\partial J} \right) |_S \tag{10.7}$$

This means that the BTZ black hole satisfies the 1st law of black hole mechanics. Smarr's law (the 2nd law) follows from the scaling properties $M(\lambda S, \lambda^2 J) =$

[9] This section is mostly a summary of https://arxiv.org/abs/gr-qc/9506079

$\lambda^2 M(S,J)$ by using Euler's theorem.
The time component g_{tt} of the metric vanishes at $r = r_{\text{erg}}$:

$$r = r_{\text{erg}} = (M)^{1/2}\ell = (r_+^2 + r_-^2)^{1/2} \tag{10.8}$$

As in the Kerr solution in (3+1) dimensions, $r < r_{\text{erg}}$ determines an ergosphere: timelike curves in this region necessarily have $\frac{\partial\varphi}{\partial\tau} > 0$ (when $J > 0$), so all observers are dragged along by the rotation of the black hole. Note that the $r_\pm$ become complex if $|J| > M\ell$, and the horizons disappear, leaving a metric that has a naked conical singularity at $r = 0$. The $8M = -1, J = 0$ metric may be recognized as that of ordinary AdS_3 space:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2}\right)} + r^2 d\varphi^2 \tag{10.9}$$

The fact that behind the BTZ metric hides a black hole is best seen by switching to a set of Eddington-Finkelstein coordinates:

$$\begin{aligned} dv &= dt + \frac{dr}{N^2} \\ d\tilde{\varphi} &= d\varphi - \frac{N^\varphi}{N^2}dr \end{aligned} \tag{10.10}$$

The metric (10.2) becomes:

$$ds^2 = -N^2 dv^2 + 2dvdr + r^2(d\tilde{\varphi} + N^\varphi dv)^2 \tag{10.11}$$

It is now easy to see that the horizon $r = r_+$, where N vanishes, has tangent vectors ∂_v,$\partial_{\tilde{\varphi}}$ and has normal vector $\chi = \partial_v - N^\varphi(r_+)\partial_{\tilde{\varphi}}$. Since $<\chi, \chi>= 0$ and is a Killing vector, $r = r_+$ is a null surface and therefore a Killing horizon. The surface gravity is calculated using:

$$\kappa = \left(-\frac{1}{2}\nabla_\mu\chi_\nu\nabla^\nu\chi^\nu\right)^{1/2} = \frac{r_+^2 - r_-^2}{\ell^2 r_+} \tag{10.12}$$

so that the Hawking temperature:

$$T = \frac{\kappa}{2\pi}\frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+} \tag{10.13}$$

which corresponds to the one found with (10.7). To find the entropy of the black hole we need the area of its horizon, which is just the length of a circumference since we are in 2 spatial dimensions. Take thus the hypersurface $r = \text{const}, t = \text{const}$ and find the induced metric:

$$d\sigma^2 = r_+^2 d\varphi^2 \tag{10.14}$$

The area is thus:

$$A_{\text{hor}} = \int_0^{2\pi} \sqrt{\sigma} d\varphi = \int_0^{2\pi} r_+ d\varphi = 2\pi r_+ \tag{10.15}$$

The entropy is then:

$$S = \frac{A_{\text{hor}}}{4} = \frac{\pi r_+}{2} \tag{10.16}$$

10.1 Global Geometry

The BTZ black hole is a $3d$ Einstein space, thus it has constant curvature and it is locally AdS_3. This means that globally it has to be a quotient space of AdS_3. To see this remember that AdS_3 can be obtained by $\mathbb{R}^4_2$ with coordinates (X^1, X^2, T^1, T^2) and metric:

$$ds^2 = dX^{1^2} + dX^{2^2} - dT^{1^2} - dT^{2^2} \tag{10.17}$$

as a imbedded hypersurface given by the following equation:

$$X^{1^2} + X^{2^2} - T^{1^2} - T^{2^2} = -\ell^2 \tag{10.18}$$

It is clear from this that the isometry group of AdS_3 is $SO(2,2)$. We could equivalently combine the coordinates (X^1, X^2, T^1, T^2) into a 2×2 matrix:

$$X = \frac{1}{\ell}\begin{pmatrix} T^1 + X^1 & T^2 + X^2 \\ -T^2 + X^2 & T^1 - X^1 \end{pmatrix} \tag{10.19}$$

Equation (10.18) implies that $det(X) = 1$, i.e. $X \in SL(2,\mathbb{R})$. Isometries may now be represented as elements of the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2 \approx SO(2,2)$; the two copies of $SL(2,\mathbb{R})$ act by left and right multiplication, $X \to \rho_L X \rho_R$, with $(\rho_L, \rho_R) \backsim (-\rho_L, \rho_R)$.
The relevant region of the universal covering space of anti-de Sitter space may be covered by:

$$\begin{aligned} X^1 &= \ell\sqrt{\alpha}\sinh\left(\frac{r_+}{\ell}\varphi - \frac{r_-}{\ell^2}t\right) \\ X^2 &= \ell\sqrt{\alpha - 1}\cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\varphi\right) \\ T^1 &= \ell\sqrt{\alpha}\cosh\left(\frac{r_+}{\ell}\varphi - \frac{r_-}{\ell^2}t\right) \\ T^2 &= \ell\sqrt{\alpha - 1}\sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\varphi\right) \end{aligned} \tag{10.20}$$

with:

$$\begin{gathered} \alpha = \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \\ -\infty < \varphi, t < +\infty \end{gathered} \tag{10.21}$$

This parametrization is valid only up to $r \geq r_+$ and for the different patches $r_- \leq r \leq r_+$, $0 \leq r \leq r_-$ one must use different parametrizations.
It is straightforward to show that the standard AdS metric then transforms to the BTZ metric (10.2) in each patch. The "angle" φ in equation (10.21) has infinite range, however; to make it into a true angular variable, we must identify φ with $\varphi + 2\pi$. This identification is an isometry of anti-de Sitter space: it is a boost in the $X_1 - T_1$ and the $X_2 - T_2$ planes and corresponds to an element

$\rho_L X \rho_R$ of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$ with:

$$\rho_L = \begin{pmatrix} e^{(\frac{\pi(r_+ - r_-)}{\ell})} & 0 \\ 0 & e^{(-\frac{\pi(r_+ - r_-)}{\ell})} \end{pmatrix} \qquad \rho_R = \begin{pmatrix} e^{(\frac{\pi(r_+ + r_-)}{\ell})} & 0 \\ 0 & e^{(-\frac{\pi(r_+ + r_-)}{\ell})} \end{pmatrix} \tag{10.22}$$

The BTZ black hole may thus be viewed as a quotient space $AdS_3/ < (\rho_L, \rho_R) >$, where $< (\rho_L, \rho_R) >$ denotes the group generated by $< (\rho_L, \rho_R >$. This is an extraordinary result: anti-de Sitter space is an extremely simple, virtually structureless manifold, but appropriate identifications nevertheless convert it into a spacetime very much like the (3+1)-dimensional Kerr black hole. Note that in this context $r = 0$ is not a singularity since the curvature is constant everywhere. One could analytically continue the solution to $r < 0$ but $g_{\varphi\varphi}$ becomes negative, which implies the existence of closed timelike curves.

10.2 Penrose Diagram of AdS_n

Remember that in global coordinates we have anti-de Sitter space:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2}\right)} + r^2 d\Omega_{n-2}^2 \tag{10.23}$$

Now do the following coordinate swap:

$$r/\ell = \sinh(\rho) \implies dr = \ell \cosh(\rho) d\rho \tag{10.24}$$

so that $0 < r < \infty$ implies $0 < \rho < \infty$. The metric becomes:

$$ds^2 = -\cosh^2(\rho) dt^2 + \rho^2 d\rho^2 + \ell^2 \sinh^2(\rho) d\Omega_{n-2}^2 \tag{10.25}$$

Now define χ through $\tan\chi = \sinh\rho$, therefore $\chi \in [0, \frac{\pi}{2})$. The following equalities are true:

$$\frac{1}{\cos^2\chi} d\chi = \cosh\rho \; d\rho = (1 + \tan^2\chi)^{1/2} d\rho = \frac{1}{\cos\chi} d\rho \implies d\rho = \frac{d\chi}{\cos\chi} \tag{10.26}$$

The metric thus changes again into:

$$ds^2 = \frac{1}{\cos^2\chi}(-dt^2 + \ell^2 d\chi^2 + \ell^2 \sin^2\chi \; d\Omega_{n-2}^2) \equiv \Omega^{-2} d\tilde{s}^2 \tag{10.27}$$

The compactified metric $d\tilde{s}^2$ differs from the physical metric just by a conformal factor $\Omega^{-2} = 1/\cos^2\chi$ so it allows to include infinity, i.e. $\chi = \pi/2$. Notice that in $\chi = \pi/2$ the compactified metric:

$$d\tilde{s}^2|_{\chi=\pi/2} = -dt^2 + \ell^2 d\Omega_{n-2}^2 \tag{10.28}$$

is an $(n-1)$ dimensional cylinder. This means that the Penrose diagram of AdS_n is a stuffed cylinder(disegno)

11 Chern-Simons Theories

The *Chern-Simons* lagrangian is:

$$\mathcal{L}_{\text{CS}} = K\epsilon^{\mu\nu\rho}\text{Tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) \tag{11.1}$$

where the gauge field A_μ lives in a semisimple lie algebra $G = \mathfrak{su}(N)$. If G is non-abelian (non-commutative), the interaction term $\epsilon^{\mu\nu\rho}\text{Tr}(A_\mu \partial_\nu A_\rho)$ is not equal to zero. We could then expand the gauge field on the basis generators $A_\mu = A_\mu^a T^a$ where T^a are the $\mathfrak{su}(N)$ generators $a = 1, \dots, N$ that satisfy the usual commutation relations $[T^a, T^b] = f^{abc}T^c$.
Let $< \cdot >$ be an Ad-invariant non-degenerate bilinear form on the Lie algebra G. If G is semisimple then such a form is inherited from the Killing form $< x \cdot y >= \text{Tr}(Ad(x)Ad(y))$.On the other hand if G is not semisimple, the Killing form is not necessarily non-degenerate.
One could prove that the variation of (11.1) w.r.t. A_μ is:

$$\delta\mathcal{L}_{\text{CS}} = K\epsilon^{\mu\nu\rho} < \delta A_\mu \cdot F_{\nu\rho} > \tag{11.2}$$

with the non-abelian field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{11.3}$$

From (11.2) we can see that the equations of motion are $F_{\mu\nu} = 0$, thus $A_m u$ is of pure gauge (flat connection): $A_\mu = g^{-1}\partial_\mu g$ with g an element of the group. One can also show that under a gauge transformation:

$$A_\mu \to A_m u^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g \tag{11.4}$$

the CS lagrangian transforms as:

$$\begin{aligned} \mathcal{L}_{\text{CS}} \to &\mathcal{L}_{\text{CS}} - K\epsilon^{\mu\nu\rho}\partial_\mu < \partial_\nu g \cdot g^{-1} A_\rho > \\ &- \frac{K}{3}\epsilon^{\mu\nu\rho} < g^{-1}\partial_\mu g g^{-1} \cdot \partial_\nu g g^{-1} \partial_\rho g > \end{aligned} \tag{11.5}$$

Therefore the lagrangian does not seem to be gauge invariant. However the second term in (11.5) is a total derivative, thus it goes to zero once we calculate the action with proper boundary conditions. The third term appears only in the non-abelian case and it is called *winding number density*:

$$W(g) = \frac{1}{24\pi^2} < g^{-1}\partial_\mu g g^{-1} \cdot \partial_\nu g g^{-1} \partial_\rho g > \tag{11.6}$$

The integral of W with suitable boundary conditions is a whole number N. Therefore the CS action changes only by a constant under a gauge transformation:

$$S_{\text{CS}} \to S_{\text{CS}} - 8\pi^2 KN \tag{11.7}$$

In the calculation of path integrals the term $exp(iS_{\text{CS}})$ appears, and therefore it is this one that we want to render gauge invariant; one can show that this factor is in fact invariant iff the coupling constant K is quantized: $K = \frac{n}{4\pi},\ n \in \mathbb{Z}$.

The origin of the name "Chern-Simons" comes from the fact that the mathematicians Chern and Simons were studying the so called *Pontryagin density* defined in $4d$ by:

$$\epsilon^{\mu\nu\rho\sigma} < F_{\mu\nu} \cdot F_{\rho\sigma} > \tag{11.8}$$

They found that this could be expressed as a total derivative:

$$\epsilon^{\mu\nu\rho\sigma} < F_{\mu\nu} \cdot F_{\rho\sigma} >= 4\partial_\sigma[\epsilon^{\mu\nu\rho\sigma} < A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho >] \tag{11.9}$$

We can see from the equation above that the integral over the Pontryagin density is a boundary term which is exactly the CS action. Note that the Chern-Simons theory can be written in every dimension:

$$\mathcal{L}_{\text{CS}} = \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda \tag{11.10}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

11.1 $3d$ Gravity

The goal of this section is to show that the gravitational theory in $(2 + 1)$ dimensions with $\Lambda < 0$ can be obtained from a Chern-Simons theory for the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (remember that the 3 dimensional anti de Sitter solution has the isometry group $SO(2, 2)$, which is isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$). We in fact could show this even for $\Lambda > 0$, however we would need a different group.
In a first order formalism the fundamental variables are the triad and the spin connection:

$$\begin{aligned} e &= e^a_\mu dx^\mu \\ \omega^{ab} &= \omega^{ab}_\mu dx^\mu \end{aligned} \tag{11.11}$$

"First order formalism" means that we are varying the action with respect to ω^{ab} as well, which means that we are not taking it as an input as we did in the case of the Einstein-Hilbert action, where we assumed that the connection was given by the Christoffel symbols.
Remember that the latin indices a, b, c are raised and lowered with the Minkowski metric:

$$\eta^{ab} = \text{diag}(-1, 1, 1) \tag{11.12}$$

Now define:

$$\begin{aligned} \omega^a &\equiv \frac{1}{2}\epsilon^{abc}\omega_{bc} \\ A^{(\pm)a} &= \omega^a \pm \frac{e^a}{\ell} \end{aligned} \tag{11.13}$$

where ℓ has units of a distance. The generators of $SL(2,\mathbb{R})$ in the fundamental representation are:

$$\tau_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.14}$$

These satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra: $[\tau_a, \tau_b] = \epsilon^c_{ab}\tau_c$; note also that $\mathrm{Tr}(\tau_a\tau_b) = \frac{1}{2}\eta_{ab}$. The connections $A^{(\pm)} = A^{(\pm)a}\tau_a$ live in the algebra $\mathfrak{sl}(2,\mathbb{R})$. We posit that the Einstein Hilbert action can be written as:

$$\begin{aligned} S_{\mathrm{CS}} = K \int \mathrm{Tr}(A^+ \wedge dA^+ + \frac{2}{3}A^+ \wedge A^+ \wedge A^+) \\ - K \int \mathrm{Tr}(A^- \wedge dA^- + \frac{2}{3}A^- \wedge A^- \wedge A^-) \end{aligned} \tag{11.15}$$

using the expression for the connections:

$$\begin{aligned} S_{\mathrm{CS}} = K \int A^{(+)a} \wedge dA^{(+)b}\mathrm{Tr}(\tau_a\tau_b) + \frac{2}{3}A^{(+)a} \wedge A^{(+)b} \wedge A^{(+)c}\mathrm{Tr}(\tau_a\tau_b\tau_c) \\ - K \int A^{(-)a} \wedge dA^{(-)b}\mathrm{Tr}(\tau_a\tau_b) + \frac{2}{3}A^{(-)a} \wedge A^{(-)b} \wedge A^{(-)c}\mathrm{Tr}(\tau_a\tau_b\tau_c) \end{aligned} \tag{11.16}$$

To see that this leads in fact to the EH action, first note that:

$$A^{(\pm)a} \wedge A^{(\pm)b} = -A^{(\pm)b} \wedge A^{(\pm)a} \tag{11.17}$$

then:

$$\begin{aligned} &A^{(-)a} \wedge A^{(-)b} \wedge A^{(-)c}\mathrm{Tr}(\tau_a\tau_b\tau_c) \\ &= A^{(-)a} \wedge A^{(-)b} \wedge A^{(-)c}\mathrm{Tr}(\tau_{[a}\tau_{b]}\tau_c) \end{aligned} \tag{11.18}$$

However:

$$\begin{aligned} \mathrm{Tr}(\tau_{[a}\tau_{b]}\tau_c) &= \frac{1}{2}\mathrm{Tr}([\tau_a\tau_b]\tau_c) \\ &= \frac{1}{2}\mathrm{Tr}(\epsilon^d_{ab}\tau_d\tau_c) \end{aligned} \tag{11.19}$$

Therefore we can rewrite (11.16) recalling that $\mathrm{Tr}(\tau_a\tau_b) = \frac{1}{2}\eta_{ab}$:

$$\begin{aligned} S_{\mathrm{CS}} = K \int A^{(+)a} \wedge dA^{(+)}_a + \frac{2}{3}A^{(+)a} \wedge A^{(+)b} \wedge A^{(+)c}\frac{1}{2}\epsilon_{abc} \\ - K \int A^{(-)a} \wedge dA^{(-)}_a + \frac{2}{3}A^{(-)a} \wedge A^{(-)b} \wedge A^{(-)c}\frac{1}{2}\epsilon_{abc} \end{aligned} \tag{11.20}$$

Now use (11.13):

$$\begin{aligned}
S_{\mathrm{CS}} &= \frac{K}{2}\int(\omega^a + \frac{e^a}{\ell})(d\omega_a - \frac{de_a}{\ell}) + \frac{1}{3}\epsilon_{abc}(\omega^a + \frac{e^a}{\ell}) \wedge (\omega^b + \frac{e^b}{\ell}) \wedge (\omega^c + \frac{e^c}{\ell}) \\
&- \frac{K}{2}\int(\omega^a - \frac{e^a}{\ell})(d\omega_a - \frac{de_a}{\ell}) + \frac{1}{3}\epsilon_{abc}(\omega^a - \frac{e^a}{\ell}) \wedge (\omega^b - \frac{e^b}{\ell}) \wedge (\omega^c - \frac{e^c}{\ell}) \\
&= \frac{K}{2}\int \underbrace{(2\omega^a \wedge \frac{de^a}{\ell}}_{\text{Term A}} + \underbrace{2\frac{e^a}{\ell} \wedge d\omega_a}_{\text{Term B}} + \frac{1}{3}\epsilon_{abc}(2\omega^a \wedge \omega^b \wedge \frac{e^c}{\ell} \\
&+ \omega^a \wedge \frac{e^b}{\ell} \wedge \omega^c) + 2\frac{e^a}{\ell} \wedge \omega^b \wedge \omega^c + 2\frac{e^a}{\ell} \wedge \frac{e^b}{\ell} \wedge \frac{e^c}{\ell})
\end{aligned} \tag{11.21}$$

A bit of mathematical sleight of hands saves us:

$$\begin{aligned}
\epsilon_{abc}\omega^a \wedge \frac{e^b}{\ell} \wedge \omega^c = -\epsilon_{abc}\frac{e^b}{\ell}\omega^a \wedge \omega^c = -\epsilon_{bac}\frac{e^a}{\ell}\omega^b \wedge \omega^c \\
= \epsilon_{abc}\frac{e^a}{\ell}\omega^b \wedge \omega^c
\end{aligned} \tag{11.22}$$

and:

$$\begin{aligned}
\epsilon_{abc}\omega^a \wedge \omega^b \wedge \frac{e^c}{\ell} = \epsilon_{abc}\frac{e^c}{\ell} \wedge \omega^a \wedge \omega^b = \epsilon_{bca}\frac{e^a}{\ell} \wedge \omega^b \wedge \omega^c \\
= \epsilon_{abc}\frac{e^a}{\ell} \wedge \omega^a \wedge \omega^b
\end{aligned} \tag{11.23}$$

Furthermore:

$$\begin{aligned}
d(\frac{e^a}{\ell} \wedge \omega_a) &= \frac{de^a}{\ell} \wedge \omega_a - \frac{e^a}{\ell} \wedge \omega_a \\
&= \omega_a \wedge \frac{de^a}{\ell} - \frac{e^a}{\ell} \wedge d\omega_a
\end{aligned} \tag{11.24}$$

which implies that:

$$\omega_a \wedge \frac{de^a}{\ell} = d(\frac{e^a}{\ell} \wedge \omega_a) + \frac{e^a}{\ell} \wedge d\omega_a \tag{11.25}$$

This is very useful now since a boundary term appears, and we know we can neglect it when we integrate, i.e. when we put it into the CS action in our case. Following this reasoning it is clear that the terms A and B highlighted in equation (11.13) are the same except for a boundary term, which we can neglect anyway. The CS action therefore becomes:

$$\begin{aligned}
S_{\mathrm{CS}} &= \frac{K}{2}\int(4\frac{e^a}{\ell} \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}(6\frac{e^a}{\ell} \wedge \omega^b \wedge \omega^c) + \frac{2}{\ell^3}(e^a \wedge e^b \wedge e^c)) \\
&= \frac{K}{\ell}\int(e^a \wedge (2d\omega_a + \epsilon_{abc}\omega^b\omega^c) + \frac{1}{3\ell^2}\epsilon_{abc}e^a \wedge e^b \wedge e^c)
\end{aligned} \tag{11.26}$$

In order to make the Einstein equations explicit, notice the following things:

$$\epsilon_{abc}R^{bc} = \epsilon_{abc}(d\omega^{bc} + \omega^b_d \wedge \omega^{dc}) \tag{11.27}$$

however we know that $\omega^a = 1/2\epsilon^{abc}\omega_{bc}$:

$$\epsilon_{dea}\omega^a = \frac{1}{2} \underbrace{\epsilon_{dea}\epsilon^{abc}}_{= \delta^c_d \delta^b_l - \delta^b_d \delta^c_e} \omega_{bc} = \frac{1}{2}\omega_{ed} - \frac{1}{2}\omega_{de} = -\omega_{de} \tag{11.28}$$

which implies that $\omega_{de} = -\epsilon_{dea}\omega^a$. Thus:

$$\epsilon_{abc}R^{bc} = \epsilon_{abc}(-\epsilon^{bc}_d d\omega^d + \epsilon^b_{de}\omega^e \wedge \epsilon^{dc}_f \omega^f) \tag{11.29}$$

Now remember the identities :

$$\begin{aligned} \epsilon_{abc}\epsilon^{bc}_d &= -2\eta_{ad} \\ \epsilon^{bc}_f &= \eta^{bc}\eta_{ef} - \delta^b_f \delta^c_e \end{aligned} \tag{11.30}$$

(WTF are those? Indices??). Therefore we can rewrite (11.29) as:

$$\begin{aligned} \epsilon_{abc}R^{bc} &= 2\eta_{ad}d\omega^d + \epsilon_{abc}(\eta^{bc}\eta_{ef} - \delta^b_f \delta^c_e)\omega^e \wedge \omega^d \\ &= 2d\omega_a + \epsilon_{abc}(-\omega^c \wedge \omega^b) = 2d\omega_a + \epsilon_{abc}\omega^b \wedge \omega^c \end{aligned} \tag{11.31}$$

The CS action splits into two terms:

$$S_{\text{CS}} = \frac{K}{\ell} \int \underbrace{e^a \wedge \epsilon_{abc}R^{bc}}_{\text{Term 2: Ricci Scalar}} + \underbrace{\frac{1}{3\ell^2}\epsilon_{abc}e^a \wedge e^b \wedge e^c}_{\text{Term 1: Cosmological Constant}} \tag{11.32}$$

Let us analyze both of these terms individually:

- Term 1:

$$\epsilon_{abc}e^a \wedge e^b \wedge e^c = \epsilon_{abc}e^a_\mu \wedge e^b_\nu \wedge e^c_\rho \underbrace{dx^\mu \wedge dx^\nu \wedge dx^\rho}_{= \varepsilon^{\mu\nu\rho}d^3x} \tag{11.33}$$

Choose $dt \wedge dx \wedge dy$ as having positive orientation, i.e. $\varepsilon^{txy} = -1$. This means that:

$$= -\epsilon_{abc}\varepsilon^{\mu\nu\rho}e^a_\mu e^b_\nu e^c_\rho d^3x \tag{11.34}$$

Notice that:

$$\begin{aligned} \varepsilon^{\mu\nu\rho}e^a_\mu e^b_\nu e^c_\rho = e^a_t e^b_x e^c_y - e^a_x e^b_y e^c_t - e^a_y e^b_t e^c_x \\ + e^a_t e^b_y e^c_x + e^a_y e^b_x e^c_t + e^a_x e^b_t e^c_y \end{aligned} \tag{11.35}$$

and consider one of these terms specifically:

$$\epsilon_{abc}e^a_x e^b_y e^c_t = \epsilon_{bca}e^b_x e^c_y e^a_t = \epsilon_{abc}e^a_t e^b_x e^c_y \tag{11.36}$$

Proceeding analogously for every term we find that they are in fact all equal to eachother:

$$\begin{aligned}\epsilon_{abc}e^a \wedge e^b \wedge e^c &= 3! \underbrace{\epsilon_{abc}e^a_t e^b_x e^c_y}_{=(det e^a_\mu)^2 \equiv e} d^3x \\ &= 3! e d^3x = 3!\sqrt{-g}d^3x\end{aligned} \tag{11.37}$$

where in the last equality we have used:

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu} \to (det\ e^c_\mu)^2 det\ \eta_{ab} = det\ g_{\mu\nu} \implies -e^2 = g \tag{11.38}$$

- Term 2:

$$\begin{aligned}\epsilon_{abc}e^a \wedge R^{bc} &= \epsilon_{abc}e^a \wedge \frac{1}{2}R^{bc}{}_{de}e^d \wedge e^e \\ &= \frac{1}{2}\epsilon_{abc}R^{bc}{}_{de}e^a \wedge e^d \wedge e^e = -\epsilon^{ade}\frac{1}{3!}\epsilon_{fgh}e^f \wedge e^g \wedge e^h \\ &= -\epsilon^{ade}\frac{1}{3!}3!\sqrt{-g}d^3x\end{aligned} \tag{11.39}$$

The third equality comes from the fact that:

$$e^a \wedge e^d \wedge e^e = -\epsilon^{ade}e^0 \wedge e^1 \wedge e^2 \tag{11.40}$$

This implies that:

$$\begin{aligned}\epsilon_{abc}e^a \wedge R^{bc} &= \frac{1}{2}\epsilon_{abc}R^{bc}{}_{de}(-\epsilon^{ade}\sqrt{-g}d^3x) \\ &= \frac{1}{2}(\delta^d_b\delta^e_c - \delta^e_b\delta^d_c)R^{bc}{}_{de}\sqrt{-g}d^3x = \frac{1}{2}2R\sqrt{-g}d^3x\end{aligned} \tag{11.41}$$

The CS action thus finally resembles the EH action:

$$S_{\text{CS}} = \frac{K}{\ell}\int(R + \frac{2}{\ell^2}\sqrt{-g})d^3x \tag{11.42}$$

By identifying $\Lambda \equiv -\frac{1}{\ell^2}$ we get exactly the Einstein-Hilbert action in three dimensions.
If we wanted to repeat the same procedure for a positive cosmological constant, we would need to make the following substitution:

$$\ell \to i\ell \tag{11.43}$$

and also change the group of the Chern-Simons theory:

$$SL(2,\mathbb{R}) \to SL(2,\mathbb{C}) \tag{11.44}$$

and:

$$A^{-a} = A^{+a} \tag{11.45}$$

Notice the isomorphism $SL(2,\mathbb{C}) \simeq SO(3,1)$.
There also exists the so called *gravitational Chern-Simons*:

$$I \backsim \int \mathrm{Tr}(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge) \tag{11.46}$$

which is now a second order formalism, in the sense that we have assumed the connection to be precisely the spin connection, i.e. the torsion is:

$$T^a = de^a + \omega^a_b \wedge e^b = 0 \tag{11.47}$$

The equations of motion are now found by varying the action only w.r.t. e^a_μ, which are third order equations in $g_{\mu\nu}$ (or equivalently e^a_μ):

$$C_{\mu\nu\rho} = 0 \tag{11.48}$$

where C is the *Cotton-York tensor*, given by:

$$C_{\mu\nu\rho} = \nabla_\mu L_{\nu\rho} - \nabla_\nu L_{\mu\rho} \tag{11.49}$$

with L being the Schouten tensor introduced above. Notice that in 3 dimensions it is true that:

$$C_{\mu\nu\rho} = 0 \Leftrightarrow \text{The manifold is conformally flat} \tag{11.50}$$

12 Stellar interiors

We have already seen that an astrophysical black hole, one obtained from the gravitational collapse of a self-gravitating object like a star in a SN-like process, shares the singularity and future horizon with the maximally extended Schwarzschild solution, without any white hole, past horizon, or a separate asymptotically flat region (a mirror universe). There is no need in fact for the latter things to take form, since the past of the final Schwarzschild spacetime does not resemble at all the Schwarzschild solution itself. The reason for this is that a static spherical object, like a star, with radius larger than $2Gm$, does not have a singularity, since for $r < 2Gm$ the presence of matter (an energy momentum tensor) alters the solution to the Einstein equation for a spherically symmetric object. The goal of the next few calculations is to obtain a solution to the Einstein equations for stellar interiors, which we will model as a perfect fluid.

We already know from Birkhoff's theorem that the most general spherically symmetric solution is of the form:

$$ds^2 = -e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + d\Omega_2^2 \tag{12.1}$$

We have just said that the star is a perfect fluid:

$$T_{ab} = \mathrm{diag}(\rho, P, P, P) \tag{12.2}$$

where ρ is the matter density and P is the pressure. Now choose the following orthonormal basis:

$$\begin{aligned} e^0 &= e^{a(r)}dt \\ e^1 &= e^{b(r)}dr \\ e^2 &= r d\theta \\ e^3 &= r\sin(\theta)d\varphi \end{aligned} \tag{12.3}$$

In this new basis the only non-zero components of the Einstein tensor are:

$$\begin{aligned} G_{00} &= \frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) \\ G_{11} &= -\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) \\ G_{22} = G_{33} &= e^{-2b}\left(a'^2 - a'b' + a'' + \frac{a'-b'}{r}\right) \end{aligned} \tag{12.4}$$

The Einstein equations $G_{ab} = 8\pi G T_{ab}$ with (12.2) imply:

$$\begin{aligned} &\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = 8\pi G\rho \\ &-\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = 8\pi G P \\ &e^{-2b}\left(a'^2 - a'b' + a'' + \frac{a'-b'}{r}\right) = 8\pi G P \end{aligned} \tag{12.5}$$

The variables in these equations are P, ρ, a, b, which means that we can find equations for a and b as functions of ρ and P, and plugging these in a third equation will give us the relationship between P and ρ, the hydrostatic equilibrium equations. Notice that in order to close the system we would need a fourth equation, which is exactly the equation of state $P = P(\rho)$ that depends on the type of matter that we are considering, e.g. $P = \kappa\rho^{1+\frac{1}{n}}$ for polytropes, where n is the *polytropic index* etc...
Continue by defining $u \equiv re^{-2b(r)}$, therefore:

$$\frac{u'}{r^2} = e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) \tag{12.6}$$

Now plug (12.6) in the first of the (12.5) to find:

$$u(r) = r - 2GM(r), \qquad M(r) = \int_0^r dr' 4\pi\rho(r')r'^2 dr' \tag{12.7}$$

Comparing (12.7) with the definition of $u(r)$ gives:

$$e^{-2b} = \frac{u(r)}{r} = 1 - \frac{2GM(r)}{r} \tag{12.8}$$

This is the equation for $b(r)$. Let's find the equation defining a. Begin by summing the first two of (12.5):

$$\frac{2}{r}e^{-2b}(a' + b') = 8\pi G(\rho + P) \tag{12.9}$$

Isolate a' and then integrate:

$$a(r) = -b + \int_\infty^r 4\pi e^{2b} r'(\rho + P)dr' \tag{12.10}$$

with the boundary conditions $a(\infty) = -b(\infty)$ since the metric outside the star is given by the Schwarzschild solution.
Now solve the second of (12.5) for a' and then derive it:

$$a'' = 2b'e^{2b}\left(4\pi GPr + \frac{1}{2r}\right) + e^{2b}\left(4\pi G(P'r + P) - \frac{1}{2r^2}\right) + \frac{1}{2r^2} \tag{12.11}$$

Plug (12.11) in the third equation in (12.5):

$$\begin{aligned} &e^{-2b}\left(a'^2 - a'b' + \frac{1}{2r^2} + \frac{a' - b'}{r}\right) + 2b'\left(4\pi GPr + \frac{1}{2r}\right) \\ &+ 4\pi GP'r - \frac{1}{2r^2} = 4\pi GP \end{aligned} \tag{12.12}$$

Using again the first two of (12.5) in order to eliminate some terms in (12.12) we get:

$$e^{-2b}(a' + b')(a' + \frac{1}{r}) + 4\pi GP'r = 4\pi G(P + \rho) \tag{12.13}$$

From (12.10) find the quantity $e^{-2b}(a' + b')$ as a function of $(\rho + P)r$, and find:

$$a' = -\frac{P'}{\rho + P} \tag{12.14}$$

Which is the equation for a.
Note that we could have gotten this equation simply by the Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$. On the other hand from the second equation in (12.5) and from (12.8) we find:

$$a' = \frac{1}{1 - \frac{2GM(r)}{r}} \left(4\pi G P r + \frac{1}{2r} \right) - \frac{1}{2r} \tag{12.15}$$

Now simpy compare it with (12.14):

$$-P' = \frac{G(\rho + P)(M(r) + 4\pi r^3 P)}{r^2 \left(1 - \frac{2GM(r)}{r}\right)} \tag{12.16}$$

This is the *Tolman-Oppenheimer-Volkoff equation*, or simply the *TOV equation*, which is the equation of hydrostatic equilibrium in general relativity. An important boundary condition to keep in mind is that of the pressure at the star's radius: $P(R) = 0$. It is interesting to compare the TOV equation to the hydrostatic equilibrium one in newtonian physics, also known as *Euler's equation*:

$$-P' = \frac{G\rho M(r)}{r^2} \tag{12.17}$$

The follwing are some of the differences:

- From (12.16) it is clear that pressure itself is a source for the gravitational field. This is missing in the classical picture.
- The density ρ is substituted in favour of $\rho + P$, to accommodate for the additional pressure term.
- The gravitational force is steeper than the classical $1/r^2$, we have instead:

$$r^{-2} \left(1 - \frac{2GM(r)}{r}\right)^{-1} \tag{12.18}$$

These adjustments to the classical theory imply that an arbitrarily massive neutron star cannot exist. To see this, we could construct a simple and semi-realistic model of a star assuming that the fluid is incompressible: the density is constant out to the surface of the star, after which it vanishes. The continuity equation to the right of (12.7) implies that:

$$M(r) = \frac{4}{3}\pi\rho r^3 \tag{12.19}$$

By plugging this in (12.16) one finds:

$$-P' = \frac{4\pi G r(P+\rho)(\frac{\rho}{3}+P)}{1-\frac{8\pi G\rho r^2}{3}} \tag{12.20}$$

which is a differential equation for the function $P(r)$, solvable by separation of variables:

$$\frac{-dP}{(\rho+P)(\frac{\rho}{3}+P)} = \frac{3}{2\rho}\left(\frac{dP}{\rho+P} - \frac{dP}{\rho/3+P}\right) = \frac{4\pi G r dr}{1-\frac{8\pi G r^2}{3}} \tag{12.21}$$

The solution is:

$$P(r) = \rho\frac{c/3-(1-8\pi G\rho r^2/3)^{1/2}}{(1-8\pi G\rho r^2/3)^{1/2}-c} \tag{12.22}$$

where c is just a constant, which is fixed by asking that the pressure vanishes at the surface of the star $P(R)=0$:

$$P(r) = \rho\frac{(1-8\pi G\rho R^2/3)^{1/2}-(1-8\pi G\rho r^2/3)^{1/2}}{(1-8\pi G\rho r^2/3)^{1/2}-3(1-8\pi G\rho R^2/3)^{1/2}} \tag{12.23}$$

This diverges at $r^2 = -3/(\pi G\rho)+9R^2$; this pole is contained in the star ($0 << r < R$) iff:

$$\frac{1}{3\pi G\rho} < R^2 < \frac{3}{8\pi G\rho} \tag{12.24}$$

This represents an unphysical situation of course. The divergence appears at $r \leq 0$ for:

$$R^2 < \frac{1}{3\pi G\rho} \tag{12.25}$$

which we thus require in order to have physical significance. Use (12.19) evaluated at R and analyze the limit case:

$$R^2 = \frac{1}{3\pi G\rho} \implies M(R) = \frac{4}{3}\pi\rho\frac{1}{3\pi G\rho}R \implies 2GM = \frac{8}{9}R \tag{12.26}$$

so the limit radius is:

$$R = \frac{9}{8}R_S \tag{12.27}$$

where R_S is the Schwarzschild radius. We could also see this as a limit for the mass of the star:

$$M = \frac{4\pi\rho R^3}{3} < \frac{4\pi\rho}{3}\left(\frac{1}{3\pi G\rho}\right)^{3/2} \tag{12.28}$$

Rewrite this compactly as:

$$M(R) < \frac{4}{9}(3\pi G^3\rho)^{-1/2} \tag{12.29}$$

Basically if the star has some mass of density ρ and it accretes enough mass, it eventually shrinks to a black hole. We derived this result from the rather strong assumption that the density is constant, but continues to hold when that assumption is considerably weakened; *Buchdal's theorem* states that any reasonable static, spherically symmetric interior solution satisfies the limits to the mass and radius that we just found.

13 The Kaluza-Klein mechanism

[10] The best shot at an all-encompassing theory of nature is supposed to come from the string program, whose theory only works in 10 dimensions if one allows for supersymmetry, otherwise the number of dimensions skyrockets to 26. However, since the observable number is in fact 4, we should think at how one could get rid of all these extra dimensions.
The mechanism by which this is done is the so called *Kaluza-Klein mechanism.* In order to illustrate this, let's consider a dimensional reduction where the dimension of spacetime is reduced just by one.
Let us assume that we are starting with Einstein gravity in $(D+1)$ dimensions, described by the Einstein-Hilber lagrangian:

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} \tag{13.1}$$

where we put hats on the fields to signify that they are in $(D+1)$ dimensions. Now suppose that we wish to reduce the theory to D dimensions, by compactifying one of the coordinates on a circle S^1 of radius L. Let this coordinate be called z. In principle, we could simply now expand all of the components of the $(D+1)$ dimensional metric tensor as a Fourier series of the form:

$$\hat{g}_{MN}(x,z) = \sum_n g_{MN}^{(n)}(x)e^{\frac{inz}{L}} \tag{13.2}$$

We write x to denote collectively the D coordinates of the lower dimensional spacetime. This decomposition implies an infinite number of fields in D dimensions, labelled by the Fourier n modes. The notation we are using is:

- The M, N indices go from 0 to D. They are $(D+1)$ dimensional
- The μ, ν indices go from 0 to $D-1$. They are D dimensional

Note that by letting $\hat{g}_{MN}$ be a real field, we get from (13.2):

$$g_{MN}^{*\,(-n)} = g_{MN}^{(n)} \tag{13.3}$$

It turns out that the modes with $n \neq 0$ are associated with massive fields, while those with $n = 0$ are massless. The basic reason for this can be seen by considering a simple toy example of a massless scalar field $\hat{\phi}$ in flat $(D+1)$ dimensional space, where it satisfies thus the KG equation:

$$\hat{\Box}\hat{\phi} = 0, \qquad \hat{\Box} = \Box + \frac{\partial^2}{\partial z^2} \tag{13.4}$$

Now compactify the coordinate z and Fourier expand the field as we did earlier for the metric tensor:

$$\hat{\phi}(x,z) = \sum_n \phi_n(x)e^{\frac{inz}{L}} \tag{13.5}$$

[10] The following chapter is heavily based around :http://people.tamu.edu/ c-pope/ihplec.pdf

By plugging the above equation into (13.4) we get the massive/massless KG equation in 3 dimensions for the modes ϕ_n:

$$\Box\phi_n - \frac{n^2}{L^2}\phi_n = 0 \tag{13.6}$$

where the mass of the field is therefore $m = |n|/L$. The usual Kaluza-Klein philosophy is to assume that the radius L of the extra dimension is very small (otherwise we would see it!), in which case the masses of the non-zero modes will be enormous. (By small, we mean that L is roughly speaking of the order of the Planck length, 10^{-33} cm, so that the non-zero modes will have masses of the order of the Planck mass, 10^{-5} g, which is of course enormous). Thus unless we were working with accelerators way beyond even intergalactig scales, the energies of the particles we will ever see would be way below the scales of the masses of the KK massive modes, and they can safely be neglected. Thus usually, when one speaks of Kaluza-Klein reduction, one has in mind a compactification together with a truncation to the massless sector.At least in a case such as our compactification on S^1, this truncation is consistent, in a manner that will be explained later.

Our KK reduction ansatz, then, will be to be to take $\hat{g}_{MN}$ to be independent of z. The main point now is that from this D dimensional point of view, the index M, which runs over the $(D+1)$ values of the higher dimensions, splits into a range lying in the D lower dimensions, or it takes the value associated with the compactified dimension z. Thus the component of the metric will therefore split into a D-dimensional tensor, a D-dimensional vector and a scalar:

$$g_{\mu\nu} = \hat{g}_{\mu\nu}, \qquad A_\mu = \hat{g}_{\mu z}, \qquad \phi = \hat{g}_{zz} \tag{13.7}$$

This simple looking parametrization is actually very unnatural, and the reason is that it doesn't pay attention to the underlying symmetries of the theory, in the sense that $\hat{g}_{\mu\nu}$ does not transform as a second rank tensor under coordinate transformations, A_μ does not transform as a vector and ϕ does not transform as a scalar. A much better parametrization is to write the $(D+1)$ dimensional metric in terms of D dimensional fields $g_{\mu\nu}$, A_μ and ϕ:

$$d\hat{s}^2 = e^{2\alpha\phi}ds^2 + e^{2\beta\phi}(dz + A)^2 \tag{13.8}$$

where α and β are constant that we will choose conveniently in a moment, and $A = A_\mu dx^\mu$ of course. (13.8) is known as the *Kaluza-Klein* ansatz. All of the fields on the right hand side are independent of the extra dimension z. This form for the metric implies that:

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{2\phi\alpha}g_{\mu\nu} + e^{2\beta\phi}A_\mu A_\nu \\ \hat{g}_{\mu z} &= e^{2\beta\phi}A_\mu \\ \hat{g}_{zz} &= e^{2\beta\phi} \end{aligned} \tag{13.9}$$

Thus, as long as we choose $\beta \neq 0$, we will adequately parametrize the higher-dimensional metric.

We now need to express the $(D+1)$ dimensional Ricci scalar in (13.1) as a fucntion of the D dimensional fields. To do this, we make a convenient choice of vielbein basis, namely:

$$\hat{e}^a = e^{\alpha\phi} e^a, \qquad \hat{e}^d = e^{\beta\phi}(dz + A) \tag{13.10}$$

where $a = 0, \dots, d-1$ is the D dimensional Lorentz index.(the superscript of the second vielbein is $d = D$ for layout purposes.) After long and tedious calculations we find for the spin connection:

$$\begin{aligned}
\hat{\omega}^{ab} &= \omega^{ab} + \alpha e^{-\alpha\phi}(\nabla^b \phi \hat{e}^a - \nabla^a \phi \hat{e}^b) - \frac{1}{2} F^{ab} e^{(\beta - 2\alpha)\phi} \hat{e}^d \\
\hat{\omega}^{ad} &= -\beta e^{-\alpha\phi} \nabla^a \phi \hat{e}^d - \frac{1}{2} F^a{}_b e^{(\beta-2\alpha)\phi} \hat{e}^b
\end{aligned} \tag{13.11}$$

with $F = dA$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) and ∇_a is the D dimensional covariant derivative, for example $\nabla^a = e^a{}_\mu \nabla^\mu$. We should now exploit our freedom of choice for the constants α and β. There are two things we would like to achieve, one of which is to ensure that the dimensionally-reduced lagrangian is of the Einstein-Hilbert form $\mathcal{L} = \sqrt{-g}R + \dots$; if we calculate the first term of (13.1) by first calculating the form for $\hat{R}$ letting α and β be arbitrary, we get:

$$\mathcal{L} = e^{(\beta + (D-2)\alpha)\phi} \sqrt{-g} R \tag{13.12}$$

where the term $e^{(\beta + D\alpha)\phi}$ comes from $\sqrt{-\hat{g}}$ and $e^{2\alpha\phi}$ from $\hat{R}$. Thus we immediately see that we should choose:

$$\beta = -(D-2)\alpha \tag{13.13}$$

and provided that we are not reducing down to $D = 2$ dimensions, this will not present any problem. The other thing that we would like to ensure is that the scalar field ϕ acquires a kinetic term with the canonical normalisation, meaning a term of the form $-1/2\sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. It turns out that to get this, we should take:

$$\alpha^2 = \frac{1}{2(D-1)(D-2)} \tag{13.14}$$

Thus (13.13) and (13.14) are our choice for the constants. In this manner we obtain for the Ricci tensor:

$$\begin{aligned}
\hat{R}_{ab} &= e^{-2\alpha\phi}(R_{ab} - \frac{1}{2}\nabla_a \phi \nabla_b \phi - \alpha \eta_{ab} \Box \phi) - \frac{1}{2} e^{-2\alpha d \phi} F^c{}_a F_{bc} \\
\hat{R}_{ad} &= \frac{1}{2} e^{(D-3)\alpha\phi} \nabla^b (e^{-2(D-1)\alpha\phi} F_{ab}) \\
R_{dd} &= (D-2)\alpha e^{-2\alpha\phi} \Box \phi + \frac{1}{4} e^{-2D\alpha\phi F^2}
\end{aligned} \tag{13.15}$$

where R_{ab} is the Ricci tensor calculated with $g_{\mu\nu}$. (Recall that the d index is the last one for the $D+1$ dimensional indices, meaning $0, 1, \dots, D-1, D$ where now $D = d$) The Ricci scalar is then:

$$\hat{R} = e^{-2\alpha\phi}(R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi) - \frac{1}{4} e^{-2D\alpha\phi} F^2 \tag{13.16}$$

with $F^2 = F^{ab}F_{ab}$ of course. Also:

$$\sqrt{-\hat{g}} = e^{(\beta+D\alpha)\phi}\sqrt{-g} = e^{2\alpha\phi}\sqrt{-g} \tag{13.17}$$

The $(D+1)$ lagrangian is therefore completely split in the three D dimensional lagrangians:

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} = \sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2) \tag{13.18}$$

where we have dropped the $\sim \Box\phi$ in (13.16) since it is a total derivative, which therefore does not contribute to the field equations. In modern parlance, the scalar field ϕ is called a dilaton. If the field ϕ could be set to a constant, it would appear that we have unified General Relativity and Electromagnetism, and the latter would be seen as a consequence of gravity in five dimensions. The price we have paid for this is of course an extra dimension. However, it is not actually allowed to set ϕ to a constant, since it would be in conflict with the equations of motion for ϕ itself. To see this, let's work out the equations of motion coming from (13.16): varying w.r.t. $g_{\mu\nu}$ will give us some sort of Einstein equations, with A_μ the Maxwell equations and ϕ will give the KG equation. This is equivalent to setting all of the Ricci tensors (13.15) to zero:

$$\underbrace{\hat{R}_{ab}}_{\text{Gravity}}, \qquad \underbrace{\hat{R}_{ad}}_{\text{Maxwell}}, \qquad \underbrace{\hat{R}_{dd}}_{\text{Klein-Gordon}} \qquad = 0 \tag{13.19}$$

Let us take a look at the Klein-Gordon equation:

$$\Box\phi = -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi}F^2 \tag{13.20}$$

If ϕ were equal to a constant, this would result in $F^2 = 0$, which is not true in general as we know, and it holds in a vacuum (i.e. this is the equation of motion for electromagnetic waves in the absence of sources). In other words, the details of the interactions between the lower dimensional fields prevent the "truncation" of ϕ (letting it be a constant). This "parametrization" is appropriate for the symmetries at play here: there is a gauge symmetry that involves the higher dimension, which is the translation along S^1.

13.1 Lower Dimensional Symmetries from the S^1 Reduction

The original $(D+1)$-dimensional Einstein theory is invariant under general coordinate transformations, which can be written in infinitesimal form as:

$$\delta\hat{x}^M = -\hat{\xi}^M(x^\nu, z), \qquad \delta\hat{g}_{MN} = \hat{\xi}^P\partial_P\hat{g}_{MN} + \hat{g}_{PN}\partial_M\hat{\xi}^P + \hat{g}_{MP}\partial_N\hat{\xi}^P \tag{13.21}$$

Now, the form of the Kaluza-Klein ansatz (13.8) will not in general be preserved by such transformations. In fact, the most general allowed form for transformations that preserve (13.8) will be:

$$\hat{\xi}^\mu = \xi^\mu(x), \qquad \hat{\xi}^z = cz + \lambda(x) \tag{13.22}$$

where the parameter c is a constant. In general this parameter implies a scale transformation, which means that it multiplies the Lagrangian by a constant, and therefore this does not change the equations of motion. Let us thus look at the local transformations, namely those parametrized by $\xi^\mu(x)$ and $\lambda(x)$. First of all we see from (13.21) that:

$$\delta\hat{g}_{zz} = \xi^\rho \partial_\rho \hat{g}_{zz} \tag{13.23}$$

since ξ depends solely on the D-dimensional coordinates x. From (13.9) we thus deduce that:

$$\delta\phi = \xi^\rho \partial_\rho \phi \tag{13.24}$$

implying that ϕ is indeed transforming as a scalar under the D-dimensional general coordinate transformations parametrized by ξ^μ. On the other hand, for the vectorial part of (13.21), we get:

$$\delta\hat{g}_{\mu z} = \xi^\rho \partial_\rho \hat{g}_{\mu z} + \hat{g}_{\rho z}\partial_\mu \xi^\rho + \hat{g}_{zz}\partial_\mu \hat{\xi}^z \tag{13.25}$$

Again, using this equation with (13.9) we get the transformation laws for A_μ:

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda \tag{13.26}$$

This shows that A_μ transforms properly as a vector under general coordinate transformations ξ^μ, and that it has the usual gauge transformation of a $U(1)$ gauge field, under the parameter λ. Basically a gauge invariance in D-dimensions can be seen as a general coordinate transformation covariance in D dimensions along the compactified dimension!
The last transformation law is that of the metric:

$$\delta\hat{g}_{\mu\nu} = \xi^\rho \partial_\rho \hat{g}_{\mu\nu} + \hat{g}_{\rho\nu}\partial_\mu \xi^\rho + \hat{g}_{\mu\rho}\partial_\nu \xi^\rho + \hat{g}_{z\nu}\partial_\mu \hat{\xi}^z + \hat{g}_{\mu z}\partial_\nu \hat{\xi}^z \tag{13.27}$$

Using what we have now learned about the transformation rules for ϕ and A_μ, we find, after substituting from (13.9), that:

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho \tag{13.28}$$

showing that the D-dimensional metric indeed has the proper transformation properties under general coordinate transformations ξ^ρ.

13.2 The Kaluza-Klein Black-Hole

We will work in $D = 4 + 1$ dimensions. The solutions to the vacuum Einstein equations $\hat{R}_{\mu\nu} = 0$ allow for:

$$d\hat{s}^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2(d\theta^2 + \sin^2(\theta) d\varphi^2) + dz^2 \tag{13.29}$$

which is the so called *black string* solution, whose topology is:

$$\text{Schwarzschild} \times S^1 \tag{13.30}$$

This is in fact a solution since the Ricci scalar factorizes in a product manifold: one part is the Schwarzschild geometry which we know satisfies the four dimensional Einstein equations, and the other part is dz^2, but every one dimensional manifold is flat.
If we apply a boost in the z direction:

$$\begin{pmatrix} t \\ z \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \tag{13.31}$$

we get mixed components in the metric, like $dtdz$. Thus if we try writing the resulting metric in a KK ansatz-like form (13.8), the fields ϕ and A_μ will be highly non-trivial. If we then reduce the dimensions to $D = 4$, we get a charged black hole solution with a dilaton, the so called *Einstein-Maxwell dilaton black hole*.
We could actually proceed in reverse: start with an Einstein-Maxwell theory in $D + 1 = 4$, therefore we have two fields at first:

$$\underbrace{\hat{g}_{MN}}_{\text{metric}}, \qquad \underbrace{\hat{B}_M}_{\text{Maxwell field}} \tag{13.32}$$

If we make a dimensional reduction to $D = 3$ we get:

- $\hat{g}_{MN} \to g_{\mu\nu}, A_\mu, \phi$ as before
- $\hat{B}_M \to B_\mu, \chi = B_z$

where $\mu = 0, 1, 2$. Reduce again to $D = 2$, where $i = 0, 1$:

- $g_{\mu\nu} \to g_{ij}, C_i, \psi$
- $A_\mu \to A_i, \Xi$
- $\phi \to \phi$
- $B_\mu \to B_i, \zeta$
- $\chi \to \chi$

As a whole we get 5 scalar fields in $D = 2$ dimensions ϕ^I with $I = 1, \ldots, 5$, which define the non-linear sigma model, i.e. their kinetic term has the form of:

$$G_{IJ}(\phi^K)\partial_i\phi^I\partial^i\phi^J \tag{13.33}$$

where G_{IJ} is the *target space* metric. This metric has some symmetries that could be used to generate new solutions from a given "seed solution", and the numer of dimensions can be increased to 5.

14 Sources of Gravitational Waves

We know that by linearizing the Einstein field equations we get a wave-like equation:

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{14.1}$$

The solution of this equation is known to be:

$$\bar{h}_{\mu\nu}(x) = \int d^4x' G^{\text{rit}}_{\mu\rho}(x - x') T^{\rho}{}_{\nu}(x') \tag{14.2}$$

where we introduced the retarded Green's function:

$$G^{\text{rit}}_{\mu\rho}(r - r', t - t') = \begin{cases} 0, & t - t' < 0 \\ \frac{\delta(t-t'-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} 4G\eta_{\mu\rho} & t - t' > 0 \end{cases} \tag{14.3}$$

Therefore:

$$\bar{h}_{\mu\nu}(\vec{r}, t) = 4G \int d^3\vec{r}' \frac{T_{\mu\nu}(\vec{r}', t' = t - |\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} \tag{14.4}$$

where $\vec{r}$ is the distance of the observer from the source, the integration variable $\vec{r}'$ extends over the source, whose dimension is roughly d. We are interested in the *radiation zone*, where $|\vec{r}| \gg d$. We thus use the following expansion:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r}\left(1 + \frac{\vec{r} \cdot \vec{r}'}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \sim \frac{1}{r} \tag{14.5}$$

If we plug this in (14.4) we get:

$$\bar{h}_{\mu\nu}(\vec{r}, t) = \frac{4G}{r} \int d^3\vec{r}' T_{\mu\nu}(\vec{r}', t - |\vec{r} - \vec{r}'|) \tag{14.6}$$

Now Fourier transform the solution:

$$\tilde{\bar{h}}_{\mu\nu}(\vec{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \bar{h}_{\mu\nu}(\vec{r}, t) e^{i\omega t} dt \tag{14.7}$$

and the energy-momentum tensor:

$$\tilde{T}_{\mu\nu}(\vec{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} T_{\mu\nu}(\vec{r}, t) e^{i\omega t} dt \tag{14.8}$$

Putting these together:

$$\begin{aligned} \tilde{\bar{h}}_{\mu\nu}(\vec{r}, \omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \frac{4G}{r} \int d^3\vec{r}' T_{\mu\nu}(\vec{r}', t - |\vec{r} - \vec{r}'|) e^{i\omega(t - |\vec{r}-\vec{r}'|)} e^{i\omega|\vec{r}-\vec{r}'|} \\ &= \frac{4G}{r} \int d^3\vec{r}' e^{i\omega|\vec{r}-\vec{r}'|} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt T_{\mu\nu}(\vec{r}', t - |\vec{r} - \vec{r}'|) e^{i\omega(t-|\vec{r}-\vec{r}'|)} \end{aligned} \tag{14.9}$$

if we see that $dt = -d(t - |\vec{r} - \vec{r}'|)$, then:

$$\tilde{\bar{h}}_{\mu\nu}(\vec{r}, \omega) = \frac{4G}{r} \int d^3\vec{r}' e^{i\omega|\vec{r}-\vec{r}'|} \tilde{T}_{\mu\nu}(\vec{r}', \omega) \tag{14.10}$$

We now would like to solve (14.10) for the spatial components $\tilde{\bar{h}}_{ij}(\vec{r}, \omega)$. These are in fact all the components we need to solve the problem completely, since all the other components can be directly obtained from the spatial ones through the Lorenz gauge:

$$\begin{aligned} \partial_\mu \bar{h}^{\mu\nu}(\vec{r}, t) &= \partial_0 \bar{h}^{0\nu}(\vec{r}, t) + \partial_i \bar{h}^{i\nu}(\vec{r}, t) \\ &= \partial_0 \frac{1}{\sqrt{2pi}} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \tilde{\bar{h}}^{0\nu}(\vec{r}, \omega)(-i\omega) + \partial_i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \tilde{\bar{h}}^{i\nu}(\vec{r}, \omega) \end{aligned} \tag{14.11}$$

This means that:

$$i\omega \ \tilde{\bar{h}}^{0\nu}(\vec{r}, \omega) = \partial_i \tilde{\bar{h}}^{i\nu}(\vec{r}, \omega) \tag{14.12}$$

Therefore:

- Let $\nu = j$, then we can get $\tilde{\bar{h}}^{0j}$ from $\tilde{\bar{h}}^{ij}$
- Let $\nu = 0$, then we can get $\tilde{\bar{h}}^{00}$ from $\tilde{\bar{h}}^{i0}$

We can now proceed to find the spatial components of h using the fact that:

$$|\vec{r} - \vec{r}'| = r \left(1 - \frac{\vec{r} \cdot \vec{r}'}{r^2} + \mathcal{O}\left(\frac{1}{r^2} \right) \right) \sim r \tag{14.13}$$

Then:

$$\tilde{\bar{h}}(\vec{r}, \omega) \approx \frac{4G}{r} e^{i\omega r} \int d^3\vec{r}' \tilde{T}_{ij}(\vec{r}', \omega) \tag{14.14}$$

Notice now that:

$$\int d^3\vec{r}' \tilde{T}_{ij}(\vec{r}', \omega) \underbrace{=}_{\partial'_k x'^i = \delta^i{}_k} \int d^3\vec{r}' \left(\partial'_k (\tilde{T}_{kj} x'^i - \partial'_k \tilde{T}_{kj}) x'^i \right) \tag{14.15}$$

and use the conservation of T: $\partial^\mu T_{\mu\nu}(\vec{r}, t) = 0$:

$$\partial_k \tilde{T}_{kj} = i\omega \tilde{T}^{0j} \tag{14.16}$$

Thus:

$$
\begin{aligned}
&\int d^3\vec{r}'\tilde{T}_{ij}(\vec{r}',\omega) = -i\omega\int d^3\vec{r}'\tilde{T}^{oj}{x'}^{i} = -i\omega\int d^3\vec{r}'\tilde{T}^{o(j}{x'}^{i)} \\
&= \frac{-i\omega}{2}\int d^3\vec{r}'(\tilde{T}^{0j}{x'}^{i}+\tilde{T}^{0i}{x'}^{j}) \\
&= \frac{-i\omega}{2}\int d^3\vec{r}'\left(\underbrace{\partial'_k(\tilde{T}^{0k}{x'}^{i}{x'}^{j})}_{\text{border term}} - \underbrace{(\partial'_k\tilde{T}^{0k}{x'}^{i}{x'}^{j})}_{=i\omega\tilde{T}^{00}}\right) \\
&= \frac{-\omega^2}{2}\underbrace{\int d^3\vec{r}'\tilde{T}^{00}(\vec{r},\omega){x'}^{i}{x'}^{j}}_{\frac{1}{3}\tilde{q}_{ij}(\omega)}
\end{aligned}
\tag{14.17}
$$

where $\tilde{q}$ is the Fourier transform of the quadrupole moment tensor. Finally:

$$
\tilde{\bar{h}}_{ij}(\vec{r},\omega) = \frac{4G}{r}e^{i\omega r}\left(\frac{\omega^2}{6}\right)\tilde{q}_{ij}(\omega) \tag{14.18}
$$

If we transform it back:

$$
\begin{aligned}
\bar{h}_{ij} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega e^{-i\omega t}\frac{4G}{r}e^{i\omega r}\left(\frac{\omega^2}{6}\right)\tilde{q}_{ij}(\omega) \\
&= \frac{1}{\sqrt{2\pi}}\frac{d^2}{dt^2}\int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-r)}\frac{2G}{3r}\tilde{q}_{ij}(\omega)
\end{aligned}
\tag{14.19}
$$

So that in the end:

$$
\bar{h}_{ij}(\vec{r},t) = \frac{2G}{3r}\frac{d^2}{dt^2}q_{ij}|_{\text{rit}} \tag{14.20}
$$

where $|_{\text{rit}}$ means that $t \to t-r$. (14.20) clearly tells us that the dominant radiation is the one from the quadrupole contribution. The absence of the dipole radiation is a consequence of the conservation of momentum, which does not allow for a mass dipole moment that is varying in time.

Example 10. Spiraling binary system of equal masses.
In the Newtonian approximation:

$$
\frac{GM^2}{(2r^2)} = \frac{Mv^2}{r} \implies \left(\frac{GM}{4r}\right)^{1/2} \tag{14.21}
$$

and the period of rotation is:

$$
T = \frac{2\pi r}{v} \implies \Omega = \frac{2\pi}{T} = \cdots = \left(\frac{GM}{4r^3}\right)^{1/2} \tag{14.22}
$$

The coordinates of the first star are:

$$
\begin{aligned}
x_a^1 &= r\cos(\Omega t) \\
x_a^2 &= r\sin(\Omega t)
\end{aligned}
\tag{14.23}
$$

For the second one:

$$\begin{aligned} x_a^1 &= -r\cos(\Omega t) \\ x_a^2 &= -r\sin(\Omega t) \end{aligned} \tag{14.24}$$

The energy density is:

$$T^{00}(\vec{r},t) = M\delta(x^3)(\delta(x^1 - r\cos(\Omega t))\delta(x^2 - r\sin(\Omega t)) + \delta(x^1 + r\cos(\Omega t))\delta(x^2 + r\sin(\Omega t))) \tag{14.25}$$

So that:

$$\begin{aligned} q_{11} &= 6Mr^2\cos^2(\Omega t) = 3Mr^2(1+\cos^2(\Omega t)) \\ q_{22} &= 6Mr^2\sin^2(\Omega t) = 3Mr^2(1-\cos(2\Omega t)) \\ q_{12} &= q_{21} = GMr^2\cos(\Omega t)\sin(\Omega t) = 3Mr^2\sin(2\Omega t) \\ q_{i3} &= 0 \end{aligned} \tag{14.26}$$

Finally from (14.20):

$$\bar{h}_{ij}(\vec{r},t) = \frac{8GM}{R}\Omega^2 r^2 \begin{pmatrix} -\cos(2\Omega t_r) & -\sin(2\Omega t_r) & 0 \\ -\sin(2\Omega t_r) & \cos(2\Omega t_r) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14.27}$$

14.1 Non-linear Gravitational Waves ("pp-waves")

Take the solution to the vacuum Einstein equations:

$$ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 \tag{14.28}$$

and switch to null coordinates $u = t - z$, $v = t + z$:

$$ds^2 = -dudv + dx^2 + dy^2 \tag{14.29}$$

Now we can manually put in the system a gravitational wave, i.e. a perturbation:

$$ds^2 = -dudv + G(u,x,y)du^2 + dx^2 + dy^2 \tag{14.30}$$

which solves the vacuum Einstein equations if G, the *wave outline*, satisfies the two dimensional Laplace equation:

$$(\partial_x^2 + \partial_y^2)G = 0 \tag{14.31}$$

This wave travels in the z direction, and it is called *pp-wave*, i.e. plane-fronted and parallel wave. Plane-fronted means that it is a plane wave, whereas parallel means that there exist a parallel vector, i.e. covariantly constant ($K = \partial_\nu$, $\nabla_\mu K_\nu = 0$).