5 A detailed study of the Friedmann equations

In the last lecture we introduced the Friedmann equations, which are just the Einstein equations that the FRW metric satisfies. The two equations read:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \end{aligned} \tag{5.1}$$

They reveal everything we need to describe the evolution of the Universe, that is they are differential equations for the scale factor $a(t)$. Now, given that the scale factor controls the dynamics of the Universe, these equations give an explicit overview of the various stages the Universe itself went through, i.e. if it has been expanding forever and at which rate.
Remember another fundamental equation, the continuity equation:

$$\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0 \tag{5.2}$$

through which we obtain the second Friedmann equation in (5.1) by differentiating with respect to time the first one.

5.1 Some properties

Before moving on to solve the equations themselves, it is interesting to see how a multitude of useful insights can be gained through some simple considerations that don't involve specific solutions.

5.1.1 The Newtonian perspective and the fate of the Universe

The main intuition behind the Friedmann equations is that they are simply a statement about conservation of energy in the Newtonian framework. To see this, consider a non-relativistic system containing a distribution of matter ($P = 0$) with density $\rho(t)$ and organized in a spherical fashion, where the sphere has radius $a(t)$. If a test particle is placed on the surface of the said sphere, by conservation of energy we have:

$$\frac{1}{2}m\dot{a}^2 - G\frac{mM}{a} = E \tag{5.3}$$

where E is the total energy and $M = 4/3\pi a^3(t)\rho(t)$ is the mass of the matter distribution. Rewriting this equation we have the following:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2E}{ma^2} \tag{5.4}$$

Clearly, this matches the the first Friedmann equation (5.1) on two conditions. First, we need to introduce in the Newtonian theory both radiation and dark energy, so that the density ρ becomes the sum of all these terms. Secondly, the identification $2E/m$ with -2κ needs to be made. This means that the total energy of the system is related to the spatial curvature k. Since the energy is related to the concept of escape velocity, this implies that the curvature actually determines the fate of the Universe. If, for example, the energy of the system is 0, or equivalently the space is flat, then the sphere will expand forever, with a velocity that tends to 0 (note that, just because we are talking about a spherical matter distribution, it doesn't have any implication for the spatial geometry, which can be flat, open or closed). On the other hand, we can reason in the opposite way, because it is the density $\rho(t)$ that specifies the curvature of space in the first place, so that the two concepts are tightly linked. To make even more explicitly the connection between the two, curvature and energy, we note that a flat Universe today ($t = t_0$) corresponds to the following *critical density*:

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \simeq 1.26 \times 10^{11} M_\odot \text{Mpc}^{-3} \tag{5.5}$$

Now, in cosmology it is convenient to scale all the densities to the critical density, and work with the adimensional parameters Ω_i defined as:

$$\Omega_i = \frac{\rho_{i,0}}{\rho_{\text{crit},0}} \quad i \in \{\text{Matter, Radiation, Dark Energy}\} \tag{5.6}$$

In the last lecture we calculated the dependence of the different densities $\rho(t)$ on time for all the species, finding:

$$\rho_m(t) = \rho_{m,0}a^{-3} \quad \rho_r(t) = \rho_{r,0}a^{-4} \quad \rho_\Lambda(t) = \rho_{\Lambda,0}a^0 \tag{5.7}$$

Using these relations, the first Friedmann equation can then be rewritten as:

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda \tag{5.8}$$

where we have defined the adimensional energy density parameter associated with curvature $\Omega_k = -kc^2/H_0^2$. Evaluating (5.8) at present time yields an important constraint:

$$1 = \sum_i \Omega_i \equiv \Omega_0 + \Omega_k \tag{5.9}$$

Where we have defined the total energy content $\Omega_0 \equiv \Omega_r + \Omega_m + \Omega_\Lambda$. This implies that the sign of $\Omega_0 - 1$ is related to that of Ω_k:

$$\Omega_0 - 1 = -\Omega_k = \frac{kc^2}{H_0^2} \tag{5.10}$$

In other words, if the energy content of the Universe happens to satisfy $\Omega - 1 = 0$, the spatial curvature will be zero, and the expansion will continue forever; this is the case of a *flat* Universe. The same line of reasoning can be carried out for the other cases, $\Omega - 1 < 0$ or $\Omega - 1 > 0$, which are the cases of *open* and *closed* Universes, where, respectively, the expansion will continue again forever or halt:

$$\begin{aligned} k = +1 &\longleftrightarrow \Omega_0 > 1 \qquad \text{Closed Universe} \\ k = 0 &\longleftrightarrow \Omega_0 = 1 \qquad \text{Flat Universe} \\ k = -1 &\longleftrightarrow \Omega_0 < 1 \qquad \text{Open Universe} \end{aligned} \tag{5.11}$$

Notice one final important consequence of the last form of the Friedmann equation (5.8). The different scalings of the various energy densities imply that the Universe underwent different phases in which only one of the species dominated. In particular, for very early times $a \ll 1$, the Universe underwent a period of *radiation domination* (RD), whereas only later and until close to present day was the evolution controlled by matter, in the period of *matter domination*

(MD). Finally, we can see that in the far future (actually, even today) the dynamics are specified by dark energy in the period of *dark energy domination* (ΛD). This last period is still subject to uncertainty, as we remarked when we defined the cosmological event horizon. Since the nature of dark energy is still subject to intense study and research, it is not known if, for instance, its equation of state is really a constant $w_\Lambda = -1$ or if it depends on time $w_\Lambda = w_\Lambda(a)$. In this last case, the dark energy domination period will need to be revised. Mathematically speaking, we have the following limits of (5.8):

$$\begin{aligned} \frac{H^2}{H_0^2} &\approx \Omega_r a^{-4} \longleftrightarrow \text{Early Universe (RD)} \\ \frac{H^2}{H_0^2} &\approx \Omega_m a^{-3} \longleftrightarrow \text{Late Universe (MD)} \\ \frac{H^2}{H_0^2} &\approx \Omega_\Lambda \longleftrightarrow \text{Today and the Future (ΛD)} \end{aligned} \tag{5.12}$$

5.1.2 Acceleration or deceleration?

During the study of distances in cosmology, we introduced the deceleration parameter q_0:

$$q_0 = -\frac{\ddot{a}}{aH^2}\bigg|_{t_0} \tag{5.13}$$

Not surprisingly, the Friedmann equations allow us to find an analytical expression for this parameter. Using the second equation in (5.1) and the definition of critical density (5.5), we find:

$$q_0 = \frac{1}{2}\sum_i \Omega_i(1+3w_i) \tag{5.14}$$

where, as in (5.6), the index i spans all species. Now, using the various equations of state, this equation simplifies to:

$$q_0 = \frac{1}{2}(\Omega_m + 2\Omega_r - 2\Omega_\Lambda) \tag{5.15}$$

Therefore, by measuring the adimensional density parameters Ω_i, we can determine if the Universe today is accelerating or decelerating. Obviously, this calculation can be done for every instant of cosmic time, providing us even more insight into the dynamics of the Universe.

5.1.3 The age of the Universe

A final, useful application of the Friedmann equations is the calculation of the age of the Universe. To estimate this quantity, the following relationship can in fact be used:

$$\frac{da}{dt}\frac{1}{a} = H \tag{5.16}$$

This implies:

$$\int_0^{t_0} dt = \int_0^1 \frac{da}{a}\frac{1}{H} \tag{5.17}$$

where, again, the extrapolation to the past is intended to be valid up until (approximately) $t_{100\ \mathrm{GeV}}$ and not $t = 0$. This is however just a nuisance (it techincally isn't) so we can go ahead and use (5.8) to find:

$$t_0 = \frac{1}{H_0}\int_0^1 da\ [\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^{-2} + \Omega_k]^{-1/2} \tag{5.18}$$

The major takeaway from these properties derived from the Friedmann equations is that the measurement of the adimensional density parameters Ω_i is of fundamental importance. These parameters specify uniquely the scale factor $a(t)$ through the first Friedmann equation. In addition, they offer an expression for the deceleration parameter, making possible the measurement of distances of different sources. Finally, these same parameters determine the age of the Universe.
Having analysed the equations, we now turn to their solutions.

5.2 Solutions

In this section we give exact solutions to the first Friedmann equation:

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_\Lambda a^2 + \Omega_k \right) \tag{5.19}$$

While a general analytical solution to this equation does not exist, we can find some relatively simple solutions if we restrict ourselves to special cases. For example, simple solutions exist for single-component Universes, which are a good approximation for periods in cosmic time where the equation of state was approximately constant, like the periods of matter, radiation and dark energy domination. Moreover, analytical solutions exist for two-component Universes, which are interesting because they incorporate the transition era between the two equations of state.

5.2.1 Matter Universes: $\Omega_m + \Omega_k = 1$

We first study the solutions to a Universe with no amount radiation and dark energy:

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_m}{a} + \Omega_k \right) \tag{5.20}$$

- Consider firstly a Universe where there is only matter, so that $\Omega_m = 1$ and $\Omega_k = 0$. In this case, the Friedmann equation simplifies down to the first in (5.12):

  $$\dot{a}^2 = H_0^2 \Omega_m a^{-1} \tag{5.21}$$

  Imposing the condition $a(t_0) = 1$ we have the following solution:

  $$a(t) = \left(\frac{t}{t_0} \right)^{2/3} \tag{5.22}$$

  This solution is historically interesting. It's called the *Einstein-de Sitter* (EdS) Universe, having been first derived by them. This connects back to the first lecture, where we hinted at the fact that a "matter-only" Universe (the one predicted by EdS) was having serious trouble explaining some observational facts, among which the predicted age of the Universe being shorter than some stars within it. Solving the integral (5.18), we find that the age of the Universe predicted by this model is:

  $$t_0 = \frac{2}{3} \frac{1}{H_0} \tag{5.23}$$

  Therefore, given an estimate of the Hubble constant H_0, one can easily calculate the age of the Universe. In the next lecture we will give an overview of the various measurements that were carried out in the last few decades to form the ΛCDM model, and we will return to this issue. Although this solution doesn't completely describe the evolution of the Universe (we know there is more than just matter, like dark energy and radiation), it is a good approximation for the period of matter domination, as we said above.

- Consider now the case of $k = -1$, i.e. an open Universe. The solution can be expressed in parametric form:

$$a(\theta) = \frac{1}{2}\frac{\Omega_m}{|1-\Omega_m|}(\cosh\theta - 1) \quad t(\theta) = \frac{1}{4c}\frac{\Omega_m}{|1-\Omega_m|}(\sinh\theta - \theta) \tag{5.24}$$

where $0 \leq \theta \leq \infty$. Note first that the solution reduces to $a(t) \propto t^{2/3}$ at early epochs $\theta \ll 1$: because of the difference in scaling between matter (a^{-3}) and curvature (a^{-2}), as long as $\Omega_m \neq 0$, there will always be an arbitrarily early time when matter dominates. More importantly, at later epochs $\theta \gg 1$ we have $a \propto t$, a phase of free expansion of the Universe, which will continue forever.

- Finally, we consider the solution for $k = +1$, given in parametric form by:

$$a(\theta) = \frac{1}{2}\frac{\Omega_m}{|1-\Omega_m|}(1 - \cos\theta) \quad t(\theta) = \frac{1}{4c}\frac{\Omega_m}{|1-\Omega_m|}(\theta - \sin\theta) \tag{5.25}$$

where $0 \leq \theta \leq 2\pi$. We can see that the Universe reaches a maximum size $a_{\rm max}$ at time $t_{\rm max}$, given by:

$$a_{\rm max} = \frac{\Omega_m}{|1-\Omega_m|} \quad t_{\rm max} = \frac{\pi}{4c}\frac{\Omega_m}{|1-\Omega_m|} \tag{5.26}$$

After reaching this maximum size, the Universe recollapses in a *big crunch* at $\theta = 2\pi$, where:

$$a_{\rm crunch} = 0 \quad t_{\rm crunch} = \frac{\pi}{2c}\frac{\Omega_m}{|1-\Omega_m|} \tag{5.27}$$

Again, note that at early epochs we have $a \propto t^{2/3}$, for the same reason as before.

The following plot summarises these three cases:

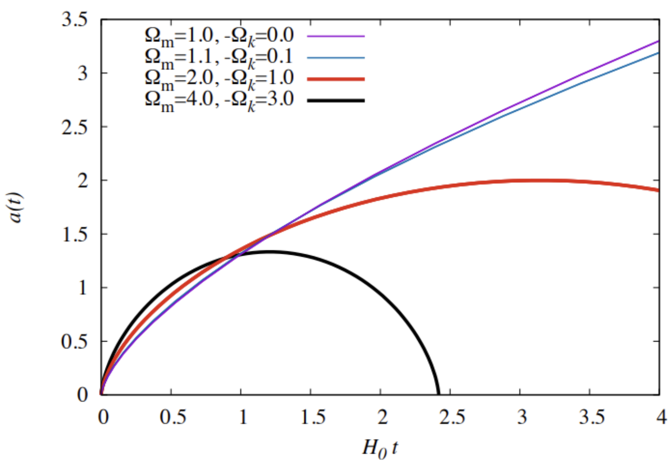

Figure 1: The three cosmologies where $\Omega_m + \Omega_k = 1$, corresponding to flat, open and closed Universes.

In the plot we can see the results we just derived. For a flat geometry ($\Omega_m = 1$ and $k = 0$), we get an ever expanding Universe with "escape velocity". On the other hand, an open Universe ($\Omega_m < 1$ and $k = -1$) still expands forever, whereas in a closed Universe ($\Omega_m > 1$ and $k = +1$) the matter density is high enough for it to recollapse on itself in a big crunch.

5.2.2 Radiation Universes: $\Omega_r + \Omega_k = 1$

Now we delve into the study of the solutions for a radiation Universe void of both matter and dark energy:

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_r}{a^2} + \Omega_k \right) \tag{5.28}$$

- As for the matter Universes, let's analyse first the case of zero spatial curvature $k = 0$:

  $$\dot{a}^2 = H_0^2 \Omega_r a^{-2} \tag{5.29}$$

  Again, imposing the condition $a(t_0) = 1$, we have the following solution:

  $$a(t) = \left(\frac{t}{t_0} \right)^{1/2} \tag{5.30}$$

  Calculating the age of the Universe with the integral (5.18), one gets:

  $$t_0 = \frac{1}{2} \frac{1}{H_0} \tag{5.31}$$

  This solution is particularly interesting for the very early Universe. As said above, in fact, as long as $\Omega_r \neq 0$, there will always be an arbitrarily early time such that radiation dominates, because of the scaling with a^{-4}.

- Next, we add some amount of curvature, such that $k = +1$, i.e. a closed Universe. This case is very similar to the closed Universe with matter, in the sense that it has a big crunch at some finite time in the future:

  $$a(\theta) = \sqrt{\frac{1}{2} \frac{\Omega_r}{|1 - \Omega_r|}} \sin\theta \quad t(\theta) = \frac{1}{2c} \sqrt{\frac{1}{2} \frac{\Omega_r}{|1 - \Omega_r|}} (1 - \cos\theta) \tag{5.32}$$

  Where, again, $0 \leq \theta \leq 2\pi$. Notice that the crunch time, and therefore the maximum, is reached at an earlier time compared to the matter Universe:

  $$a_{\text{curnch}} = 0 \quad t_{\text{crunch}} = \frac{1}{c} \sqrt{\frac{1}{2} \frac{\Omega_r}{|1 - \Omega_r|}} \tag{5.33}$$

  Basically, in both models, the Universe expands with a cycloidal evolution of the scale factor, until a maximum size is obtained at t_{max}, where the expansion is reversed. Then, the Universe starts to compress until it reaches the big crunch at t_{crunch}.

- Finally, consider a radiation dominated open Universe, with $k = -1$. In this case, the expansion will go on forever:

  $$a(\theta) = \sqrt{\frac{1}{2} \frac{\Omega_m}{|1 - \Omega_r|}} \sinh\theta \quad t(\theta) = \frac{1}{2c} \sqrt{\frac{1}{2} \frac{\Omega_r}{|1 - \Omega_r|}} (\cosh\theta - 1) \tag{5.34}$$

  where $0 \leq \theta \leq \infty$. The early Universe limit $\theta \ll 1$ correctly reproduces the zero curvature solution $a \propto t^{1/2}$, whereas the late Universe $\theta \gg 1$ sees an expansion $a \propto t$ that will continue forever, as we remarked.

5.2.3 Dark energy Universes: $\Omega_\Lambda + \Omega_k = 1$

Consider a Universe with a cosmological constant $\Lambda > 0$ and some curvature:

$$\dot{a}^2 = H_0^2 \left(\Omega_\Lambda a^2 + \Omega_k \right) \tag{5.35}$$

This kind of Universe is a good approximation to present and late times, where dark energy starts to dominate all other kinds of energies. The solutions to this equation are:

$$a(t) = \sqrt{\frac{3}{\Lambda}} \begin{cases} \cosh(\sqrt{\Lambda c^2/3}t) & k = +1 \\ \exp(\sqrt{\Lambda c^2/3}t) & k = 0 \\ \sinh(\sqrt{\Lambda c^2/3}t) & k = -1 \end{cases} \tag{5.36}$$

Note that the $k = +1$ solution doesn't have a singularity, whereas the scale factor vanishes at $t = -\infty$ and $t = 0$ for the $k = 0$ and $k = -1$ solutions respectively. These singularities are, however, only due to poor coordinate choices, since the three solutions (5.36) are only three different ways to slice the the same *de Sitter* space.

5.2.4 Early Universe: matter and radiation

As we will learn in the next lecture, a flat ($k = 0$) Universe containing matter and radiation is very close to resembling our own, at least at early stages. The Friedmann equation we have to solve is:

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} \right) \tag{5.37}$$

This model presents a transition from a period of radiation domination to a period of matter domination. The transition time, or scale factor a_{eq}, called *matter-radiation equality* is defined by:

$$\frac{\Omega_m}{a} = \frac{\Omega_r}{a^2} \implies a_{\mathrm{eq}} = \frac{\Omega_r}{\Omega_m} \tag{5.38}$$

Rewriting (5.37) using this new definition simplifies things:

$$\dot{a}^2 = \frac{H_0^2 \Omega_r}{a^2} \left(1 + \frac{a}{a_{\mathrm{eq}}} \right) \tag{5.39}$$

This implies the following differential equation:

$$H_0 dt = \frac{a da}{\Omega_r^{1/2}} \left(1 + \frac{a}{a_{\mathrm{eq}}} \right)^{-1/2} \tag{5.40}$$

Integrating it, we get the solution:

$$H_0 t = \frac{4a_{\mathrm{eq}}^2}{3\Omega_r^{1/2}} \left(1 - \left(1 - \frac{a}{2a_{\mathrm{eq}}} \right) \left(1 + \frac{a}{a_{\mathrm{eq}}} \right)^{1/2} \right) \tag{5.41}$$

As a sanity check, it is useful to compute the radiation and matter domination (RD and MD) limits and make sure they correspond to (5.30) and (5.22). First we impose the RD limit, obtained by $a/a_{\mathrm{eq}} \ll 1$, finding the following scaling:

$$a \simeq (3H_0 \Omega_r^{1/2} t)^{1/2} \propto t^{1/2} \tag{5.42}$$

which is indeed the right proportionality. Now, the MD limit $a/a_{\rm eq} \gg 1$ implies:

$$a = \left(\frac{3\Omega_r H_0 t}{4a_{\rm eq}^{1/2}}\right)^{2/3} \propto t^{2/3} \tag{5.43}$$

Again, this coincides with the Einstein-de Sitter solution.
As a final application, one can also calculate the time of matter-radiation equality $t_{\rm eq}$, which is found by setting $a = a_{\rm eq}$ in (5.41):

$$t_{\rm eq} = \frac{4a_{\rm eq}^2}{3\Omega_r^{1/2}}\left(1 - \frac{1}{\sqrt{2}}\right) \tag{5.44}$$

5.2.5 Late Universe: matter and dark energy

Consider now a flat Universe containing matter and a positive cosmological constant, such that $\Omega_m + \Omega_\Lambda = 1$. This model, as we will see in the next lecture, describes rather well our own Universe at present and future times. The Friedmann equation to solve is, in this case:

$$\dot{a}^2 = H_0^2\left(\frac{\Omega_m}{a} + \Omega_\Lambda a^2\right) \tag{5.45}$$

The time of interest is now the moment of *matter-dark energy equality* $a_{\Lambda\rm m}$, defined by:

$$\frac{\Omega_m}{a} = \Omega_\Lambda a^2 \implies a_{\Lambda\rm m} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \tag{5.46}$$

Rewriting the Friedmann equation in the following way:

$$\dot{a} = H_0 a \Omega_\Lambda^{1/2}\left(\frac{a_{\Lambda\rm m}^3}{a^3} + 1\right)^{1/2} \tag{5.47}$$

yields the differential equation:

$$H_0 dt = \frac{a da}{\Omega_\Lambda^{1/2}}\left(\frac{a_{\Lambda\rm m}^3}{a^3} + 1\right)^{-1/2} \tag{5.48}$$

The solution is the following:

$$H_0 t = \frac{2}{3\Omega_\Lambda^{1/2}} \log\left[\left(\frac{a}{a_{\Lambda\rm m}}\right)^{3/2} + \left(\frac{a^3}{a_{\Lambda\rm m}^3} + 1\right)^{1/2}\right] \tag{5.49}$$

Let's check whether this solution gives the right scalings at early (5.22) and late (5.36) times. First, the limit $a/a_{\Lambda\rm m} \ll 1$ in (5.49) yields:

$$a \simeq \left(\frac{3H_0 t a_{\Lambda\rm m}^{3/2}\Omega_\Lambda^{1/2}}{2} t\right)^{2/3} \tag{5.50}$$

which is again the right proportionality for a matter dominated Universe. The late limit $a/a_{\Lambda\rm m} \gg 1$ instead results in:

$$a \simeq \frac{a_{\Lambda\rm m}}{2^{2/3}} \exp\left(\frac{3H_0\Omega_\Lambda^{1/2} t}{2}\right) \tag{5.51}$$

which corresponds with the result of a flat de Sitter Universe.
As before, the time of matter-dark energy equality $t_{\Lambda\rm m}$ can be found by setting $a = a_{\Lambda\rm m}$ in (5.49):

$$t_{\Lambda\rm m} = \frac{2}{3H_0\Omega_\Lambda^{1/2}} \log(1 + \sqrt{2}) \tag{5.52}$$