4 The dynamics of the Universe

Up until now we were only concerned with the properties of the Universe at large scales. Making use of symmetry arguments we determined the metric:

$$\begin{aligned} ds^2 &= -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \\ &= ds^2 = a^2(\eta)(-c^2d\eta^2 + (d\chi^2 + S_k^2(\chi)d\Omega^2)) \end{aligned} \tag{4.1}$$

We have also explored all these kind of spacetimes, but the scale factor $a(t)$, which determines the *dynamics* of the Universe, was never specified. It is in fact necessary to solve the Einstein equations to find how $a(t)$ evolves:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{4.2}$$

4.1 Curvature

In order to solve the Einstein equations, we have to calculate all the necessary tensors. We start with the left hand side first of (4.2). Unfortunately, although being quite straightforward, the calculation of the Ricci tensor and scalar is very tedious. Let's start by recalling the Christoffel symbols:

$$\begin{aligned} &\Gamma^{\mu}_{00} = \Gamma^{0}_{\alpha 0} = 0 \\ &\Gamma^{0}_{ij} = c^{-1}a\dot{a}\gamma_{ij} = c^{-1}\frac{\dot{a}}{a}g_{ij} \\ &\Gamma^{i}_{0j} = c^{-1}\frac{\dot{a}}{a}\delta^i_j \\ &\Gamma^{i}_{jk} = \frac{1}{2}\gamma^{il}(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}) \end{aligned} \tag{4.3}$$

Now we have, by definition:

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \tag{4.4}$$

Setting $\mu = \nu = 0$, we can calculate R_{00}:

$$R_{00} = \partial_\lambda\Gamma^\lambda_{00} - \partial_0\Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{00} - \Gamma^\rho_{0\lambda}\Gamma^\lambda_{0\rho} \tag{4.5}$$

From (4.3) we can see that all the Christoffel's with two time indices vanish, so that we get:

$$R_{00} = -\partial_0\Gamma^i_{0i} - \Gamma^i_{0j}\Gamma^j_{0i} \tag{4.6}$$

But again from (4.3) we have $\Gamma^i_{0j} = c^{-1}\frac{\dot{a}}{a}\delta^i_j$, so we find:

$$R_{00} = -\frac{1}{c^2}\frac{d}{dt}\left(3\frac{\dot{a}}{a}\right) - \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = -\frac{3}{c^2}\frac{\ddot{a}}{a} \tag{4.7}$$

The other components of the Ricci tensor are only shown here below, because the necessary calculations are simply too long and uninteresting to be included:

$$\begin{aligned} &R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a} \\ &R_{0i} = 0 \\ &R_{ij} = \frac{1}{c^2}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{kc^2}{a^2}\right) \end{aligned} \tag{4.8}$$

Given the Ricci tensor, it is straightforward to compute the Ricci scalar:

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
&= -R_{00} + \frac{1}{a^2}\gamma^{ij} R_{ij} = \frac{3}{c^2}\frac{\ddot{a}}{a} + \frac{\delta^i_i}{c^2}\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{kc^2}{a^2}\right) \\
&= \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right)
\end{aligned}
\tag{4.9}
$$

Finally, we state the components of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2 R g_{\mu\nu}$:

$$
\begin{aligned}
G^0_0 &= -\frac{3}{c^2}\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right) \\
G^0_i &= 0 \\
G^i_j &= -\frac{1}{c^2}\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right)\delta^i_j
\end{aligned}
\tag{4.10}
$$

4.2 The energy content

To complete the Einstein equations, we have to specify the form of the energy-momentum tensor $T_{\mu\nu}$ for all the fields (matter, radiation, dark energy) present in the Universe. Let us first recall the properties of the energy-momentum tensor itself and only later describe how it differs between the various species.

4.2.1 The energy-momentum tensor

The energy-momentum tensor (EMT) describes the presence and the dynamics of all the matter contained in spacetime. Let's start from scratch to understand what form of the EMT is necessary for our case, i.e. for a homogeneous and isotropic distribution of matter at large scales.
In Newtonian dynamics, many gravitational problems "only" present the interaction between a few particles or objects, such as the two body problem, from which we can deduce, for example, the trajectory of the Earth around the Sun. However, in many cases, where the number of interacting particles is especially large, describing each and every interaction becomes inappropriate or straight impossible. This is where the *fluid* approximation comes in handy. This approximation treats the system as a smooth continuum, which is now described only by averaged quantities, or fields (such as velocity, density, temperature, pressure etc...). That is, we assume that, in a sufficiently small neighborhood of every point $\vec{x}$, the system looks homogeneous, so that it can be specified by just a few variables. Basically, since the particles of the fluid will be moving randomly, this will give rise to a *pressure field*, a certain amount of *heat conduction* and *viscous forces* between neighboring fluid elements. In addition, viscous forces can result in *shearing* of the fluid.
One of the simplest type of fluids is the *perfect fluid*, and it turns out to be the one we are looking for in our description of the Universe at large scales. A fluid is said to be perfect if there exists a frame, the *comoving frame*, where there are no shear stresses nor heat conduction. As a consequence, in this reference frame, the local properties in the neighborhood of any point $\vec{x}$ are isotropic.
To express the fluid in this comoving frame, in special relativity, we need the energy-momentum tensor $T^{\mu\nu}$. Let us first recall what the various components mean (in a general reference frame):

- $T^{00}(x)$ is the energy density of the fluid element
- $cT^{0j}(x)$ is the energy flux of the fluid element in the j-th direction
- $\frac{1}{c}T^{i0}(x)$ is the i-th component of the momentum density of the fluid element
- $T^{ij}(x)$ is the flux of the i-th component of the momentum in the j-th direction. The momentum flux across two fluid elements indicates an exertion of force. If these forces are perpendicular to the interface between the fluid elements, then such forces are represented by a diagonal T^{ij}. If, on the other hand, the forces are parallel at the interface between the fluid elements, then these are represented by off-diagonal terms in T^{ij}.

Before moving on, note that the energy-momentum tensor is a symmetric quantity, i.e. $T^{\mu\nu} = T^{\nu\mu}$. This is a non-trivial statement. For instance, this implies that $T^{i0} = T^{0i}$, so (neglecting the c factors) the energy flux in the i-th direction is equal to component of momentum density in that same direction.
Now, if we want to describe the EMT of a perfect fluid, we can reason in the following way for the comoving frame, where there is no shear stress nor conductivity. First, since the conductivity is zero, we have $T^{0j} = 0 = T^{i0}$, since those components measure a flux (we could equally have said that these vectors have to be zero because of isotropy, as we did when arguing that $g_{0i} = 0$). Then, we have $T^{ij} = 0$ for $i \neq j$ because these terms correspond to viscous forces at the interfact between fluid elements. But T^{ij}, being a 3-tensor, must be diagonal in all reference frames connected to one another by a rotation. This necessarily implies that $T^{ij} \propto \delta^{ij}$. The proportionality constant is in this case the pressure $T^{ij} = P(x)\delta^{ij}$. Basically, we have found that in the comoving frame, the energy-momentum tensor reads:

$$T^{\mu\nu}(x) = \begin{pmatrix} \rho(x)c^2 & & & \\ & P(x) & & \\ & & P(x) & \\ & & & P(x) \end{pmatrix} \tag{4.11}$$

Finally, we turn to general relativity. Clearly, a good model for energy-matter content of the Universe at large scales is the perfect fluid, because it incorporates isotropy by itself. The requirement of homogeneity then only takes out the $\vec{x}$ dependence, as seen below. We also have to notice that the metric is not $\eta^{\mu\nu}$, but the FRW one. For this reason, the convention is that, in the comoving frame, which in our case is just the one specified by the comoving coordinates, we have:

$$T^{\mu}_{\nu}(x) = \begin{pmatrix} -\rho(t)c^2 & & & \\ & P(t) & & \\ & & P(t) & \\ & & & P(t) \end{pmatrix} \tag{4.12}$$

where now the energy ρ and pressure P fields are dependent only on time, because of homogeneity. The energy-momentum tensor is then extended to the explicitly covariant form:

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu} \tag{4.13}$$

where u^μ is the relative 4-velocity between the fluid and a comoving observer, i.e. the fluid's peculiar velocity. Notice that, as expected, in the comoving frame, with $u^\mu = (c, 0, 0, 0)$, we return to (4.11).

As a final note about the energy-momentum tensor, let's analyse the conservation equation it obeys:

$$\nabla_\mu T^\mu_\nu = 0 \tag{4.14}$$

Expanding it we have:

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda = 0 \tag{4.15}$$

These are four separate equations, one for every value of ν. The spatial components:

$$\nabla_\mu T^\mu_i = 0 \tag{4.16}$$

are identically satisfied. We actually expected this, as $\nabla_\mu T^\mu_i$ is a vector which, again because of isotropy, has to vanish. The time component of the conservation equation is instead more interesting:

$$\partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda = 0 \tag{4.17}$$

This reduces to:

$$\frac{1}{c}\frac{d(\rho c^2)}{dt} + \Gamma^\mu_{\mu 0}(\rho c^2) - \Gamma^\lambda_{\mu 0} T^\mu_\lambda = 0 \tag{4.18}$$

From the Christoffel's (4.3) we see that $\Gamma^\lambda_{\mu 0}$ vanishes unless the indices are both spatial and equal to each other, in which case $\Gamma^i_{i0} = 3c^{-1}\dot{a}/a$. We then finally get:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = \dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0 \tag{4.19}$$

This equation is the continuity equation in an expanding spacetime. In order to solve (4.19), we need the last piece of the puzzle, the *equation of state*, which relates pressure to energy density:

$$P = P(\rho) \tag{4.20}$$

In general, most cosmological fluids are well described by a constant equation of state:

$$P = w\rho c^2 \tag{4.21}$$

With this in mind, the continuity equation reads:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{4.22}$$

To which the solution obviously is:

$$\rho \propto a^{-3(1+w)} \tag{4.23}$$

In other words, the dilution of energy of a fluid in an expanding spacetime depends on the equation of state of the fluid itself.
In the first lecture we have described the types of fluids present in the Universe: matter (dark and baryonic), radiation and dark energy. What is the equation of state, i.e. the w, for each of these?

4.2.2 Equations of state

Let's think about what we have done so far. We have written down the Einstein equations (4.2) and calculated the left hand side, the "curvature part". To complete the equations, however, we also need to know the energy-momentum tensor of every species of the Universe. As we have argued in the above, each cosmological fluid, being isotropic and homogeneous (at large scales), is rather well described by a perfect fluid, whose energy-momentum tensor (in the comoving frame) is of the form (4.12). Then, solving the continuity equation using a constant equation of state, we have found the scaling of the energy density with the scale factor $a(t)$ (4.23). For this reason, to complete the Einstein equations, we need to find how $\rho(t)$ evolves (or $P(t)$, since they are proportional), which means we need to find w for matter, radiation and dark energy.

In order to do this, we need to turn to statistical mechanics. Consider a gas of particles that are at thermodynamic equilibrium with temperature T, inside a box of volume V. Then, the number density of particles with momentum (the absolute value) in the interval $(p, p+dp)$ is given by:

$$dn(p) = \frac{g}{(2\pi\hbar)^3} f(p,T) 4\pi p^2 dp \tag{4.24}$$

where $f(p,T)$ is the probability distribution function, i.e. the probability to find a randomly chosen particle with momentum p. For a classical gas, this is the Maxwell-Boltzmann distribution, whereas for a quantum gas, it's either the Fermi-Dirac distribution (if the particles are fermions), or the Bose-Einstein distribution (if the particles are bosons). The g factor in (4.24) is the spin degeneracy of the particles. This probability distribution is derived using basic quantum mechanics, but we will not do so here. Once we have $dn(p)$, we can calculate the thermodynamic variables:

$$\begin{aligned}
\frac{N}{V} &= \int_0^\infty dn(p) \\
\rho c^2 &= \frac{E}{V} = \int_0^\infty dn(p) E(p) \\
\frac{P}{V} &= \frac{1}{3}\int_0^\infty dn(p) p v(p) = \frac{1}{3}\frac{\int_0^\infty dn(p) p v(p)}{\int_0^\infty dn(p)} n = \frac{1}{3} n \langle p v(p) \rangle
\end{aligned} \tag{4.25}$$

where $E(p) = \sqrt{m^2c^4 + p^2c^2}$ is the particles' energy.

Now, let's evaluate P for every species that we know exists in the Universe. We start with matter, which is basically defined to be the opposite of radiation, that is:

$$E(p) = \sqrt{m^2c^4 + p^2c^2} \simeq mc^2 \qquad v(p) = \frac{p}{m} \tag{4.26}$$

This is the non-relativistic limit, and it applies to both dark energy and baryonic matter. The pressure then is:

$$P = \frac{1}{3} n \langle p v(p) \rangle = \frac{1}{3} n m \langle v(p)^2 \rangle = \frac{1}{3} n \frac{E}{c^2} \langle v(p)^2 \rangle \simeq 0 \tag{4.27}$$

where the last approximation is due to the non-relativistic nature of the particles, $\langle v^2 \rangle / c^2 \simeq 0$. For this reason we have:

$$P_{\text{matter}} = 0 \tag{4.28}$$

which implies $w_{\text{matter}} = 0$. The dilution of the energy density is then, from (4.23):

$$\rho_{\text{matter}} \propto a^{-3} \tag{4.29}$$

That is, the matter energy density scales like the inverse of the volume, with size a^{-3}.

Instead, for radiation (remember that by radiation we mean any ultra-relativistic species. At present time the only ones are photons and neutrinos, but every particle was once ultra-relativistic, as the Universe was very hot) we have the following properties:

$$E(p) = \sqrt{m^2c^4 + p^2c^2} \simeq pc \qquad v(p) = c \tag{4.30}$$

The pressure then becomes:

$$P = \frac{1}{3}n\langle pv(p)\rangle = \frac{1}{3}n\langle pc\rangle \tag{4.31}$$

whereas the energy density:

$$\rho c^2 = \frac{E}{V} = \int_0^\infty dn(p)pc = \frac{\int_0^\infty dn(p)pc}{\int_0^\infty dn(p)}n = n\langle pc\rangle \tag{4.32}$$

Putting (4.32) and (4.31) together, we find:

$$P_{\text{radiation}} = \frac{1}{3}\rho \tag{4.33}$$

which means $w_{\text{radiation}} = 1/3$, leading to an energy dilution of the form:

$$\rho_{\text{radiation}} \propto a^{-4} \tag{4.34}$$

Again, we find the scaling of the volume a^{-3}, but this time enhanced by another a^{-1}, which comes from the redshift.

The last component we have to analyse is dark energy. As we have seen in the first lectures, it was clear that matter and radiation only could not describe our Universe. Because of this shortcoming, the concept of dark energy was introduced. To the best of our knowledge, this entity satisfies an equation of state of the following form:

$$P_\Lambda \simeq -\rho c^2 \tag{4.35}$$

that is $w_\Lambda = -1$, and its energy density stays constant:

$$\rho_\Lambda \propto a^0 \tag{4.36}$$

Since the Universe is expanding, energy has to be created continuously. A natural candidate that satisfies this behavior is *vacuum energy* because, as the Universe expands, more space is being created and therefore the energy increases in proportion to the volume, leading to a constant energy density. In quantum field theory, this vacuum energy is actually predicted, leading to an energy-momentum tensor of the form:

$$T_{\mu\nu}^{\text{vac}} = -\rho c^2 g_{\mu\nu} \tag{4.37}$$

Comparing this with (4.13), we see that this form implies $P = -\rho c^2$. This type of vacuum energy is actually also predicted by general relativity, through the cosmological constant. In fact, we could rewrite the Einstein equations (4.2) with a modified energy-momentum tensor:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} + T_{\mu\nu}^\Lambda) \tag{4.38}$$

where:

$$T_{\mu\nu}^\Lambda = -\frac{\Lambda c^4}{8\pi G}g_{\mu\nu} = -\rho_\Lambda c^2 g_{\mu\nu} \tag{4.39}$$

This clearly has the same form of (4.37). In general, the terms "vacuum energy" and "dark energy" are used intercheangably, to mean a certain entity that satisfies an equation of state with $w = -1$.

4.3 The Friedmann equations

We finally have all the pieces to solve, or at least rewrite, the Einstein equations. Let's summarise all the different items here. The non-zero components of the Einstein tensor $G_{\mu\nu}$ are:

$$
\begin{aligned}
G^0_0 &= -\frac{3}{c^2}\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right) \\
G^0_i &= 0 \\
G^i_j &= -\frac{1}{c^2}\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right)\delta^i_j
\end{aligned}
\tag{4.40}
$$

Then, the form of the energy-momentum tensor is that of a perfect fluid, which in the rest (comoving) frame reads:

$$
T^\mu_\nu(x) = \begin{pmatrix} -\rho(t)c^2 & & & \\ & P(t) & & \\ & & P(t) & \\ & & & P(t) \end{pmatrix} \tag{4.41}
$$

This has to satisfy the conservation equation, which yields a scaling of the energy density of:

$$
\rho \propto a^{-3(1+w)} \tag{4.42}
$$

The total energy density is the sum of the energy density of every component in the Universe:

$$
\rho = \rho_{\text{matter}} + \rho_{\text{radiation}} + \rho_\Lambda \tag{4.43}
$$

which scale as:

$$
\rho_{\text{matter}} \propto a^{-3} \quad \rho_{\text{radiation}} \propto a^{-4} \quad \rho_\Lambda \propto a^0 \tag{4.44}
$$

Now, let's pull everything together. There are only two Einstein equations, namely the 00 one, and any one of the spatial ones ij, as the G^i_j is proportional to δ^i_j, because of isotropy. The 00 equation reads:

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{4.45}
$$

This equation is called the *Friedmann equation* and is particularly important. Given initial conditions, from this equation it is rather straightforward to calculate $a(t)$, which controls the dynamics of the whole Universe. The ρ present in this equation is to be understood as a sum, namely the one found in (4.43). The spatial part of the Einstein equations yields the (second) Friedmann equation:

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \tag{4.46}
$$

This equation actually follows from taking the time derivative of (4.45) and using the continuity equation (4.19). This shouldn't surprise us, because the Einstein equation implies the conservation equation for the energy-momentum tensor via the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$.

These two equations (4.45) and (4.46) control the large-scale dynamics of our Universe. In the next lecture we will explore them and analyse their fundamental implications.