$i := 1..n \quad nint := 1 \quad n := 3 \cdot nint \quad n = 3 \quad j := 1..n$

$$mx := \begin{array}{l} \text{for } i \in 1..n \\ \quad \left\| \begin{array}{l} mx_{1,i} \leftarrow 1 \\ mx \end{array} \right. \end{array} \qquad v := \begin{array}{l} \text{for } i \in 1..n \\ \quad \left\| \begin{array}{l} v_{1,i} \leftarrow 1 \\ v \end{array} \right. \end{array} \qquad \rho := \begin{array}{l} \text{for } i \in 1..n \\ \quad \left\| \begin{array}{l} \text{for } j \in 1..n \\ \quad \left\| \begin{array}{l} \text{if } i = j \\ \quad \left\| \begin{array}{l} \rho_{i,j} \leftarrow 1 \\ \rho \end{array} \right. \\ \text{else} \\ \quad \left\| \begin{array}{l} \rho_{i,j} \leftarrow 0.5 \\ \rho \end{array} \right. \end{array} \right. \end{array} \right. \end{array}$$

$mx = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$\rho = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$$

$\Sigma := \rho$

$$\Sigma = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \qquad A := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 5 & -1 \end{bmatrix}$$

$I := \begin{bmatrix} 1 & nint + 1 & nint \cdot 2 + 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

___ V[Y] and E[Y] ___

$V := A \cdot \Sigma \cdot A^{T} \qquad V = \begin{bmatrix} 15 & 7.5 \\ 7.5 & 33 \end{bmatrix} \qquad l := 1..2 \qquad a \leftarrow \sqrt{V}$

$mY := A \cdot mx^{T}$

$V_{1,1} + V_{2,2} = 48$

$mY = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

____ Conditional Σ and mcond(x) ____

$$\Sigma cond1 := \left\| \begin{array}{l} k \leftarrow 1 \\ \text{for } j \in (1..n) \\ \quad \left\| \begin{array}{l} \text{for } t \in (1..n) \\ \quad \left\| \; \Sigma\Sigma_{t,j} \leftarrow \Sigma_{t,j} - \dfrac{\Sigma_{t,k} \cdot \Sigma_{k,j}}{\Sigma_{k,k}} \right. \end{array} \right. \\ \Sigma cond1 \leftarrow (A) \cdot \Sigma\Sigma \cdot A^{T} \\ \Sigma cond1 \end{array} \right. \qquad \Sigma cond2 := \left\| \begin{array}{l} k \leftarrow 2 \\ \text{for } j \in (1..n) \\ \quad \left\| \begin{array}{l} \text{for } t \in (1..n) \\ \quad \left\| \; \Sigma\Sigma_{t,j} \leftarrow \Sigma_{t,j} - \dfrac{\Sigma_{t,k} \cdot \Sigma_{k,j}}{\Sigma_{k,k}} \right. \end{array} \right. \\ \Sigma cond2 \leftarrow (A) \cdot \Sigma\Sigma \cdot A^{T} \\ \Sigma cond2 \end{array} \right.$$

$$\Sigma cond3 := \left\| \begin{array}{l} k \leftarrow 3 \\ \text{for } j \in (1..n) \\ \quad \left\| \begin{array}{l} \text{for } t \in (1..n) \\ \quad \left\| \Sigma\Sigma_{t,j} \leftarrow \Sigma_{t,j} - \dfrac{\Sigma_{t,k} \cdot \Sigma_{k,j}}{\Sigma_{k,k}} \right. \end{array} \right. \\ \Sigma cond3 \leftarrow (A) \cdot \Sigma\Sigma \cdot A^{T} \\ \Sigma cond3 \end{array} \right.$$

$$\Sigma cond1 = \begin{bmatrix} 2.75 & -6.5 \\ -6.5 & 17 \end{bmatrix} \qquad \Sigma cond2 = \begin{bmatrix} 14.75 & 4.75 \\ 4.75 & 2.75 \end{bmatrix} \qquad \Sigma cond3 = \begin{bmatrix} 11 & 2.5 \\ 2.5 & 26.75 \end{bmatrix}$$

$$m11(x) := \sum_{j=1}^{n} \left(A_{1,j} \cdot \left(mx_{1,j} + (x - mx_{1,j}) \cdot \frac{\Sigma_{j,1}}{\Sigma_{1,1}} \right) \right) \quad m12(x) := \sum_{j=1}^{n} \left(A_{2,j} \cdot \left(mx_{1,j} + (x - mx_{1,j}) \cdot \frac{\Sigma_{j,1}}{\Sigma_{1,1}} \right) \right)$$

$$m22(x) := \sum_{j=1}^{n} \left(A_{2,j} \cdot \left(mx_{1,j} + (x - mx_{1,j}) \cdot \frac{\Sigma_{j,2}}{\Sigma_{2,2}} \right) \right) \quad m21(x) := \sum_{j=1}^{n} \left(A_{1,j} \cdot \left(mx_{1,j} + (x - mx_{1,j}) \cdot \frac{\Sigma_{j,2}}{\Sigma_{2,2}} \right) \right)$$

$$m31(x) := \sum_{j=1}^{n} \left(A_{1,j} \cdot \left(mx_{1,j} + (x - mx_{1,j}) \cdot \frac{\Sigma_{j,3}}{\Sigma_{3,3}} \right) \right) \quad m32(x) := \sum_{j=1}^{n} \left(A_{2,j} \cdot \left(mx_{1,j} + (x - mx_{1,j}) \cdot \frac{\Sigma_{j,3}}{\Sigma_{3,3}} \right) \right)$$

$$W1inn(x) := \sqrt{(mY_{1,1} - m11(x))^2 + (mY_{2,1} - m12(x))^2 + \boxed{trace}(V) + trace(\Sigma cond1) - 2 \cdot \left(trace\left(\left(\sqrt{M1} \cdot M2 \cdot \sqrt{M1} \right)^{\frac{1}{2}} \right) \right)}$$

From File matlab MultivariateGaussianAnalytical.m

$$SqrtVold := \begin{bmatrix} 3.7913 & 0.7910 \\ 0.7910 & 5.6 \end{bmatrix} \qquad SqrtSig1old := \begin{bmatrix} 0.9945 & -1.3270 \\ -1.3270 & 3.9037 \end{bmatrix}$$

$$SqrtSig2old := \begin{bmatrix} 3.7258 & 0.9318 \\ 0.9318 & 1.3718 \end{bmatrix}$$

$$lambda1aux := \text{eigenvals}(\Sigma cond1) \qquad lambda2aux := \text{eigenvals}(\Sigma cond2)$$

$$lambda1 := \text{for } i \in 1..2 \; \left\| \begin{array}{l} lambda1_i \leftarrow \max(lambda1aux_i, 0) \\ lambda1 \end{array} \right. \qquad lambda2 := \text{for } i \in 1..2 \; \left\| \begin{array}{l} lambda2_i \leftarrow \max(lambda2aux_i, 0) \\ lambda2 \end{array} \right.$$

$lambda3aux := \mathrm{eigenvals}(\Sigma cond3)$

$$lambda3 := \begin{array}{l} \text{for } i \in 1..2 \\ \quad lambda3_i \leftarrow \max\left(lambda3aux_i, 0\right) \\ lambda3 \end{array}$$

$eigvct1 := \mathrm{eigenvecs}(\Sigma cond1)$ $\quad sigma1 := \sqrt{lambda1}$ $\quad eigvct2 := \mathrm{eigenvecs}(\Sigma cond2)$

$sigma2 := \sqrt{lambda2}$ $\quad eigvct3 := \mathrm{eigenvecs}(\Sigma cond3)$ $\quad sigma3 := \sqrt{lambda3}$

$SqrtSig1 := eigvct1 \cdot \mathrm{diag}(sigma1) \cdot eigvct1^{\mathrm{T}}$ $\quad SqrtSig2 := eigvct2 \cdot \mathrm{diag}(sigma2) \cdot eigvct2^{\mathrm{T}}$

$SqrtSig3 := eigvct3 \cdot \mathrm{diag}(sigma3) \cdot eigvct3^{\mathrm{T}}$

$$SqrtSig1 = \begin{bmatrix} 0.995 & -1.327 \\ -1.327 & 3.904 \end{bmatrix} \quad SqrtSig2 = \begin{bmatrix} 3.726 & 0.932 \\ 0.932 & 1.372 \end{bmatrix} \quad SqrtSig3 = \begin{bmatrix} 3.303 & 0.295 \\ 0.295 & 5.164 \end{bmatrix}$$

$lambdaV := \mathrm{eigenvals}(V)$

$eigvctV := \mathrm{eigenvecs}(V)$ $\quad sigmaV := \sqrt{lambdaV}$

$SqrtSigV := eigvctV \cdot \mathrm{diag}(sigmaV) \cdot eigvctV^{\mathrm{T}}$

$$SqrtSigV = \begin{bmatrix} 3.791 & 0.791 \\ 0.791 & 5.69 \end{bmatrix}$$

$P1aux := SqrtSig1 \cdot V \cdot SqrtSig1$ $\quad P2aux := SqrtSig2 \cdot V \cdot SqrtSig2$ $\quad P3aux := SqrtSig3 \cdot V \cdot SqrtSig3$

$lambdaP1 := \mathrm{eigenvals}(P1aux)$ $\quad lambdaP2 := \mathrm{eigenvals}(P2aux)$ $\quad lambdaP3 := \mathrm{eigenvals}(P3aux)$

$eigvctP1 := \mathrm{eigenvecs}(P1aux)$ $\quad sigmaP1 := \sqrt{lambdaP1}$ $\quad eigvctP2 := \mathrm{eigenvecs}(P2aux)$

$sigmaP2 := \sqrt{lambdaP2}$ $\quad P2 := eigvctP2 \cdot \mathrm{diag}(sigmaP2) \cdot eigvctP2^{\mathrm{T}}$

$eigvctP3 := \mathrm{eigenvecs}(P3aux)$

$P1 := eigvctP1 \cdot \mathrm{diag}(sigmaP1) \cdot eigvctP1^{\mathrm{T}}$

$sigmaP3 := \sqrt{lambdaP3}$

$P3 := eigvctP3 \cdot \mathrm{diag}(sigmaP3) \cdot eigvctP3^{\mathrm{T}}$

Defining the Trace Function

$Tr(X) := X_{1,1} + X_{2,2}$ $\quad Tr(V) = 48$ $\quad TrP1old := 24.3643$ $\quad TrP2old := 23.6851$

$Tr(P1) = 24.364$ $\quad Tr(P2) = 23.685$ $\quad Tr(P3) = 42.382$ $\quad TrP3old := 42.3815$

$$W1inn(x) := \sqrt{\left(mY_{1,1} - m11(x)\right)^2 + \left(mY_{2,1} - m12(x)\right)^2 + Tr(V) + Tr(\Sigma cond1) - 2 \cdot Tr(P1)}$$

$$W12inn(x) := \left(mY_{1,1} - m11(x)\right)^2 + \left(mY_{2,1} - m12(x)\right)^2 + Tr(V) + Tr(\Sigma cond1) - 2 \cdot Tr(P1)$$

$$Adv1aux1(x) := \left(mY_{1,1} - m11(x)\right)^2 + \left(mY_{2,1} - m12(x)\right)^2$$

$$Adv1aux2 := \int_{-\infty}^{\infty} Adv1aux1(y) \cdot \left(\frac{1}{v_{1,1} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,1}\right)^2}{\left(v_{1,1}\right)^2}} \right) \mathrm{d}y$$

$$Adv1 := \frac{Adv1aux2}{2 \cdot \left(V_{1,1} + V_{2,2}\right)} \qquad Adv1 = 0.294$$

$$W1 := \int_{-\infty}^{\infty} W1inn(y) \cdot \frac{1}{v_{1,1} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,1}\right)^2}{\left(v_{1,1}\right)^2}} \mathrm{d}y$$

$$W12 := \int_{-\infty}^{\infty} W12inn(y) \cdot \frac{1}{v_{1,1} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,1}\right)^2}{\left(v_{1,1}\right)^2}} \mathrm{d}y$$

$$Adv1 + Diff1 = 0.492$$

$$W1 = 6.467$$

$$W1Normalized := \frac{W1^2}{2 \cdot \left(V_{1,1} + V_{2,2}\right)} \qquad W1Normalized = 0.436$$

$$W12 = 47.271 \qquad i1 := \frac{W12}{2 \cdot \left(V_{1,1} + V_{2,2}\right)} \qquad i1 = 0.492$$

$$WVB1 = 4.241$$

$$W2inn(x) := \sqrt{\left(mY_{1,1} - m21(x)\right)^2 + \left(mY_{2,1} - m22(x)\right)^2 + Tr(V) + Tr(\Sigma cond2) - 2 \cdot Tr(P2)}$$

$$W22inn(x) := \left(mY_{1,1} - m21(x)\right)^2 + \left(mY_{2,1} - m22(x)\right)^2 + Tr(V) + Tr(\Sigma cond2) - 2 \cdot Tr(P2)$$

$$Adv2aux1(x) := \left(mY_{1,1} - m21(x)\right)^2 + \left(mY_{2,1} - m22(x)\right)^2$$

$$Diff2Aux1(x) := Tr(V) + Tr(\Sigma cond2) - 2 \cdot Tr(P2)$$

$$W2 := \int_{-\infty}^{\infty} W2inn(y) \cdot \frac{1}{v_{1,2} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{(y - mx_{1,2})^2}{(v_{1,2})^2}} \, dy$$

$$W22 := \int_{-\infty}^{\infty} W22inn(y) \cdot \frac{1}{v_{1,2} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{(y - mx_{1,2})^2}{(v_{1,2})^2}} \, dy$$

$W2 = 6.519$

$W22 = 48.63$

$$i2 := \frac{W22}{2 \cdot (V_{1,1} + V_{2,2})} \qquad i2 = 0.507$$

$$W3inn(x) := \sqrt{(mY_{1,1} - m31(x))^2 + (mY_{2,1} - m32(x))^2 + Tr(V) + Tr(\Sigma cond3) - 2 \cdot Tr(P3)}$$

$$W32inn(x) := (mY_{1,1} - m31(x))^2 + (mY_{2,1} - m32(x))^2 + Tr(V) + Tr(\Sigma cond3) - 2 \cdot Tr(P3)$$

$$Adv3aux1(x) := (mY_{1,1} - m31(x))^2 + (mY_{2,1} - m32(x))^2$$

$$W3 := \int_{-\infty}^{\infty} W3inn(y) \cdot \frac{1}{v_{1,3} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{(y - mx_{1,3})^2}{(v_{1,3})^2}} \, dy$$

$$W32 := \int_{-\infty}^{\infty} W32inn(y) \cdot \frac{1}{v_{1,3} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{(y - mx_{1,3})^2}{(v_{1,3})^2}} \, dy$$

$W3 = 2.856$

$$W1inn(x) := \sqrt{(mY_{1,1} - m11(x))^2 + (mY_{2,1} - m12(x))^2 + Tr(V) + Tr(\Sigma cond1) - 2 \cdot Tr(P1)}$$

$W32 = 11.237$

$$i3 := \frac{W32}{2 \cdot \left(V_{1,1} + V_{2,2}\right)} \qquad i3 = 0.117$$

Sinkhorn

$$d := 2 \qquad Id := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad sig := 10$$

$$Dsigma1old := \begin{bmatrix} 8.1233 & -12.1498 \\ -12.1498 & 40.7404 \end{bmatrix}$$

$$Dsigma2old := \begin{bmatrix} 31.9256 & 11.7295 \\ 11.7295 & 15.5112 \end{bmatrix}$$

$$Dsigma3old := \begin{bmatrix} 25.3466 & 9.1295 \\ 9.1295 & 59.4462 \end{bmatrix}$$

$$AuxDsigma1 := 4 \cdot SqrtSig1 \cdot V \cdot SqrtSig1 + sig^4 \cdot \mathrm{identity}(2)$$

$$\mathrm{eigenvals}(AuxDsigma1) = \begin{bmatrix} 1.2 \cdot 10^4 \\ 1.002 \cdot 10^4 \end{bmatrix} \qquad lambdaDs := \mathrm{eigenvals}(AuxDsigma1)$$

$$sigmaDs := \sqrt{lambdaDs}$$

$$eigvctDs := \mathrm{eigenvecs}(AuxDsigma1)$$

$$Dsigma1 := eigvctDs \cdot \mathrm{diag}(sigmaDs) \cdot eigvctDs^{\mathrm{T}}$$

$$Dsigma1 = \begin{bmatrix} 101.018 & -2.832 \\ -2.832 & 108.62 \end{bmatrix}$$

$$AuxDsigma2 := 4 \cdot SqrtSig2 \cdot V \cdot SqrtSig2 + sig^4 \cdot \mathrm{identity}(2)$$

$$lambdaDs2 := \mathrm{eigenvals}(AuxDsigma2)$$

$$sigmaDs2 := \sqrt{lambdaDs2}$$

$$eigvctDs2 := \mathrm{eigenvecs}(AuxDsigma2)$$

$$Dsigma2 := eigvctDs2 \cdot \mathrm{diag}(sigmaDs2) \cdot eigvctDs2^{\mathrm{T}}$$

$$Dsigma2 = \begin{bmatrix} 105.587 & 2.683 \\ 2.683 & 101.833 \end{bmatrix}$$

$$AuxDsigma3 := 4 \cdot SqrtSig3 \cdot V \cdot SqrtSig3 + sig^4 \cdot \mathrm{identity}(2)$$

$$lambdaDs3 := \mathrm{eigenvals}(AuxDsigma3)$$

$$sigmaDs3 := \sqrt{lambdaDs3}$$

$$eigvctDs3 := \mathrm{eigenvecs}(AuxDsigma3)$$

$$Dsigma3 := eigvctDs3 \cdot \mathrm{diag}(sigmaDs3) \cdot eigvctDs3^{\mathrm{T}}$$

$$Dsigma3 = \begin{bmatrix} 103.501 & 3.517 \\ 3.517 & 116.635 \end{bmatrix}$$

$$OT1inn(x) := \left(mY_{1,1} - m11(x)\right)^2 + \left(mY_{2,1} - m12(x)\right)^2 + \left(Tr(V) + Tr(\Sigma cond1)\right) - Tr(Dsigma1) + \left(d \cdot sig^2 \cdot \left(1 - \ln\left(2 \cdot sig\right.\right.\right.$$

$$OT1 := \int_{-\infty}^{\infty} OT1inn(y) \cdot \frac{1}{v_{1,1} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,1}\right)^2}{\left(v_{1,1}\right)^2}} \, \mathrm{d}y = 91.07$$

$\sqrt{OT1inn(}$

$W1inn(q)$

$$SqrtOT1 := \int_{-\infty}^{\infty} \sqrt{OT1inn(y)} \cdot \frac{1}{v_{1,1} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,1}\right)^2}{\left(v_{1,1}\right)^2}} \, \mathrm{d}y = 9.376$$

$$OT2inn(x) := \left(mY_{1,1} - m21(x)\right)^2 + \left(mY_{2,1} - m22(x)\right)^2 + \left(Tr(V) + Tr(\Sigma cond2)\right) - Tr(Dsigma2) + \left(d \cdot sig^2 \cdot \left(1 - \ln\right.\right.$$

$$OT2 := \int_{-\infty}^{\infty} OT2inn(y) \cdot \frac{1}{v_{1,2} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,2}\right)^2}{\left(v_{1,2}\right)^2}} \, \mathrm{d}y = 92.23$$

$$\sqrt{OT2} = 9.604$$

$$SqrtOT2 := \int_{-\infty}^{\infty} \sqrt{OT2inn(y)} \cdot \frac{1}{v_{1,2} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,2}\right)^2}{\left(v_{1,2}\right)^2}} \, dy = 9.415$$

$$OT3inn(x) := \left(mY_{1,1} - m31(x)\right)^2 + \left(mY_{2,1} - m32(x)\right)^2 + \left(Tr(V) + Tr(\Sigma cond3)\right) - Tr(Dsigma3) + \left(d \cdot sig^2 \cdot (1 - \ln\right.$$

$$\left(mY_{1,1} - m31(4)\right)^2 + \left(mY_{2,1} - m32(4)\right)^2 = 92.25$$

$$\left(Tr(V) + Tr(\Sigma cond3)\right) - Tr(Dsigma3$$

$$\left(Tr(V) + Tr(\Sigma cond3)\right) - Tr(Dsig$$

$$OT3 := \int_{-\infty}^{\infty} OT3inn(y) \cdot \frac{1}{v_{1,3} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,3}\right)^2}{\left(v_{1,3}\right)^2}} \, dy = 85.561$$

$$Tr(V) + Tr(\Sigma cond3) = 85.75$$

$$\sqrt{OT3} = 9.25$$

$$OT3inn(x)$$

$$OT2inn(x)$$

$$OT1inn(x)$$

$$VY := V_{1,1} + V_{2,2}$$

$$SqrtOT3 := \int_{-\infty}^{\infty} \sqrt{OT3inn(y)} \cdot \frac{1}{v_{1,3} \cdot \sqrt{2 \cdot \pi}} \cdot e^{\frac{-1}{2} \cdot \frac{\left(y - mx_{1,3}\right)^2}{\left(v_{1,3}\right)^2}} \, dy = 9.222$$