Monte Carlo Simulation for a Knock-In Bivariate Barrier Option

Introduction

This report presents a Monte Carlo simulation framework to price a knock-in bivariate option. We used Python to price the option with a Monte Carlo simulation in a base case. Then, we created a Monte Carlo pricing function and applied it to a series of different scenarios, analyzing how the price changes by varying monitoring frequency, volatilities, and correlation. We continued with a comparison to the digital barrier option and the computation of the Greeks.

Specifically, this case considers a Knock-In European Call option, whose payoff depends on two securities. The first, S_1, determines the activation of the barrier, while S_2 is the underlying of the call once the barrier is touched.

Option Description

Let $T = 1$ year and consider a barrier $b > S_1(0)$ on the first asset S_1, monitored at discrete times $t_m = \frac{mT}{M}$ for $m = 1, \ldots, M$. The barrier is activated if:

$$M_1(T) = \max\{S_1(t) : t \in \{t_1, \ldots, t_M\}\} \geq b.$$

We also consider a standard European call option on S_2 with maturity T and strike price K, which has the payoff:

$$c(T) = \max\{S_2(T) - K, 0\}.$$

The S_1-knock-in call option on S_2 has the payoff:

$$X(T) = c(T) \cdot \mathbb{I}\{M_1(T) \geq b\},$$

where $\mathbb{I}\{M_1(T) \geq b\}$ is the indicator function that takes the value 1 if the barrier b is breached and 0 otherwise.

Monte Carlo Simulation

The simulation uses the following parameters:

r	0.05
q1, q2	0.02
ρ	0.5
$S_1(0), S_2(0)$	100
B, K	120
$[\sigma_1, \sigma_2]$	[0.20, 0.25]
Monitoring dates	52
Simulations	10,000

Table 1: Simulation Parameters

We implemented stochastic differential equations (SDE) for S_1 and S_2 as discrete-time processes:

$$S_i(t + \Delta t) = S_i(t) \cdot e^{\left(r - q - \frac{1}{2}\sigma^2\right)\Delta t + \sigma_i \Delta W}$$

With:

$$\sigma_1 = [v_1, 0]$$

$$\sigma_2 = [\rho * v_2; \sqrt{(1 - \rho^2)} * v_2]$$

We divided the option's life (T=1 year) into m=52 steps for weekly monitoring. We then generated 10000 independent paths for each asset, which were initialized at $\mathrm{S}_1 = \mathrm{S}_2 = 100$ and evolved over time using the above equations.

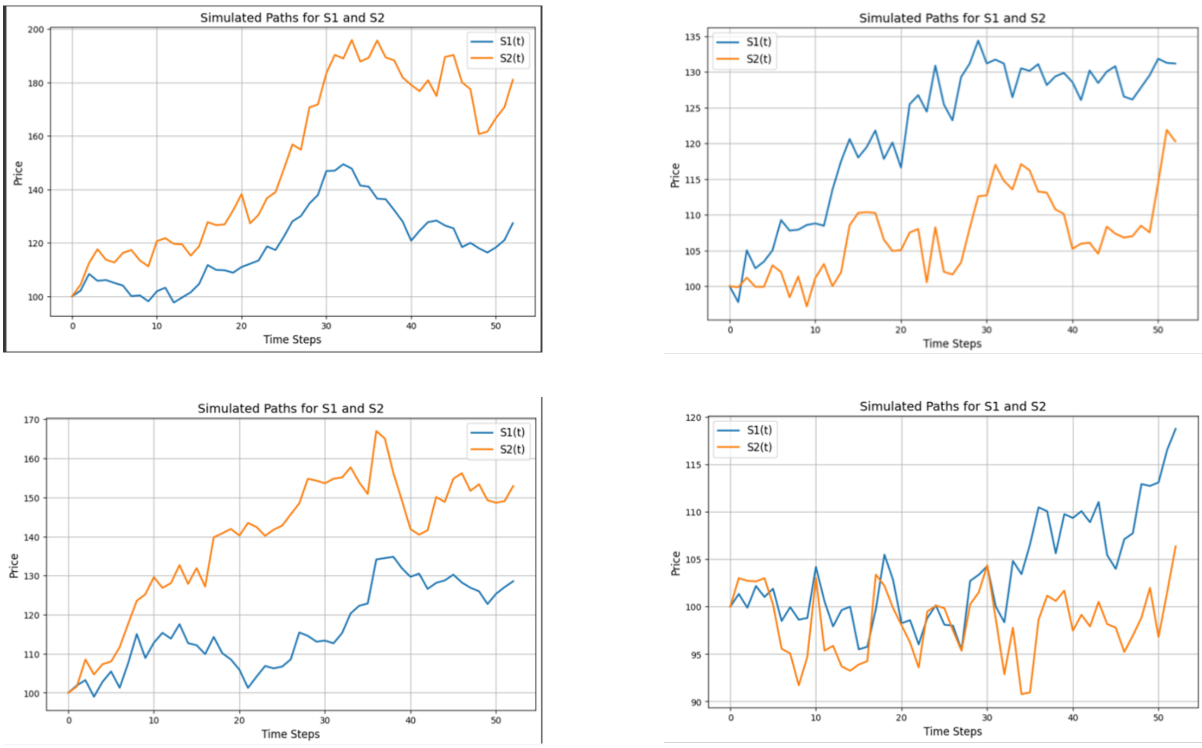


Figure 1: Sample paths from simulation of S_1 and S_2

After that, we calculated the price of the knock-in bivariate barrier option based on the Monte Carlo simulation previously implemented. For each simulation, we determined whether the first asset S_1 reached or exceeded the barrier b=120 at any time, by computing the maximum price of S_1 in all time steps for each simulation and created a binary indicator to record if the barrier was breached.

We calculated the payoff of a European call on S_2 at maturity and using the barrier indicator, we filtered out the payoffs for paths where the barrier was not breached.

Finally, we computed the discounted expectation on the knock-in payoffs and obtained the price given our initial inputs.

Price in Different Scenarios

We analyzed the impact of changes in volatilities and correlation on the price of the KI call option.

Impact of Volatility 1 and Correlation

Talking about the sensitivity of the option price with respect to volatility 1 (σ_1), there are several aspects to analyze. First of all, there is a clear positive relation between the option price and volatility 1 and the option price and the correlation. However, the most interesting thing is that a higher correlation implies a different sensitivity to the volatility, meaning that as the correlation increases, volatility 1 has a lower impact on the option price. The intuition is simple: for high correlations, the paths in which S_2 is ITM are the same in which S_1 touches the barrier. Therefore, σ_1 matters less after a certain threshold. This effect can be seen in the two graphs above with the flattening of the curve.

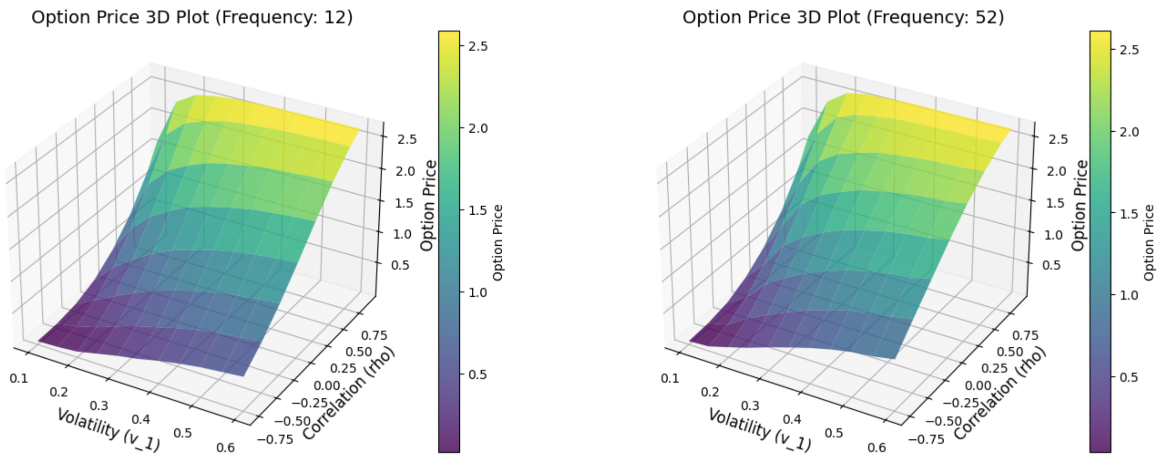


Figure 2: option price for varying v_1 and ρ

Impact of Volatility 2

Talking about the impact of σ_2, this is very different from the previous one. In particular, there is no flattening of the curve, meaning that there is not a bound where σ_2 stops influencing the price. In addition, a higher correlation implies a steepening of the curve, meaning that the sensitivity of the option price to changes in σ_2 increases. The differences between the impact of σ_1 and σ_2 are due to the different importance that S_1 and S_2 have for the option. In fact, while S_2 serves as an underlying asset for a standard call option, S_1 serves only to understand whether the barrier has been touched or not.

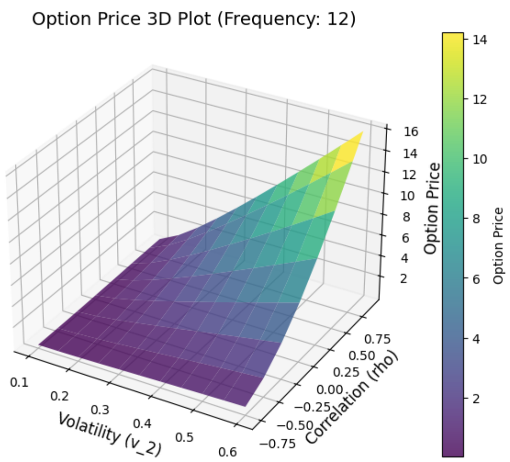


Figure 3: option price for varying v_2 and ρ

Analysis of Confidence Intervals

In this section our goal was to understand whether the frequency of monitoring dates had an impact on the precision of the Monte Carlo estimate i.e., if and how confidence interval of our price estimate changed by increasing the number of monitoring dates in a year. Since the call option activates when the barrier is touched, obviously increasing the monitoring dates will increase the option price as S_1 has more "chances" to touch the barrier. For this reason, we used standardized confidence intervals (width of the interval divided by the option price) and obtained comparable numbers. Increasing the number of monitoring dates improves precision, as shown by a decrease in standardized confidence intervals:

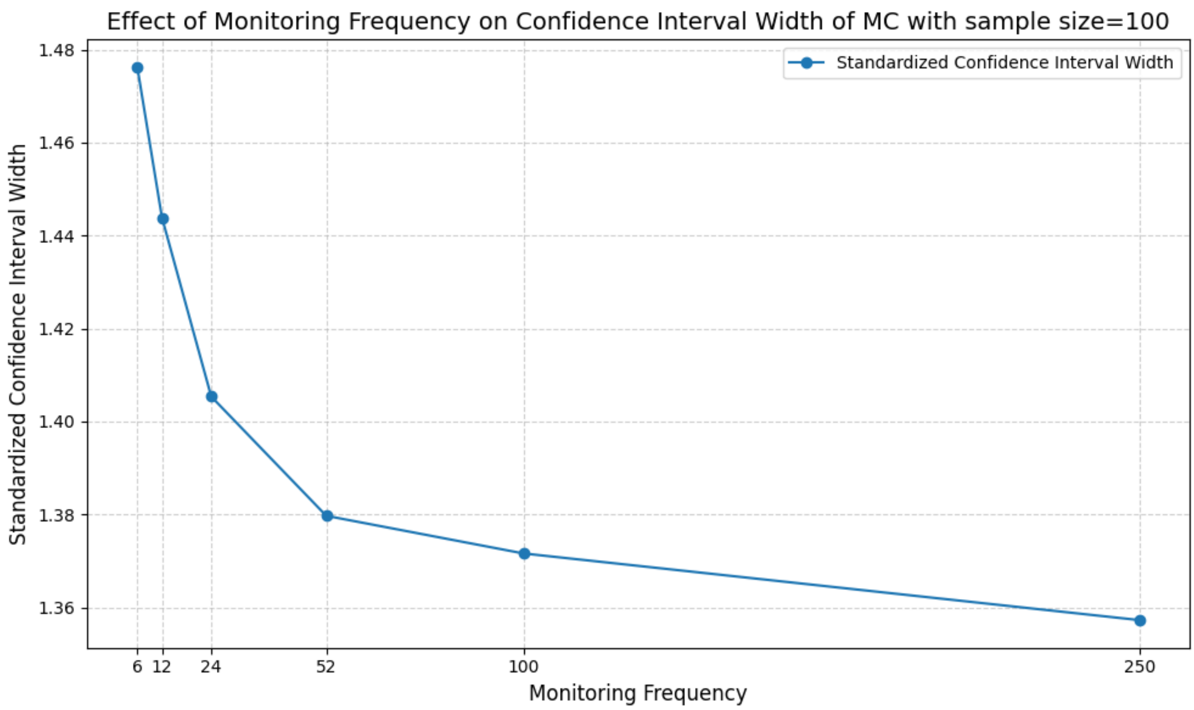
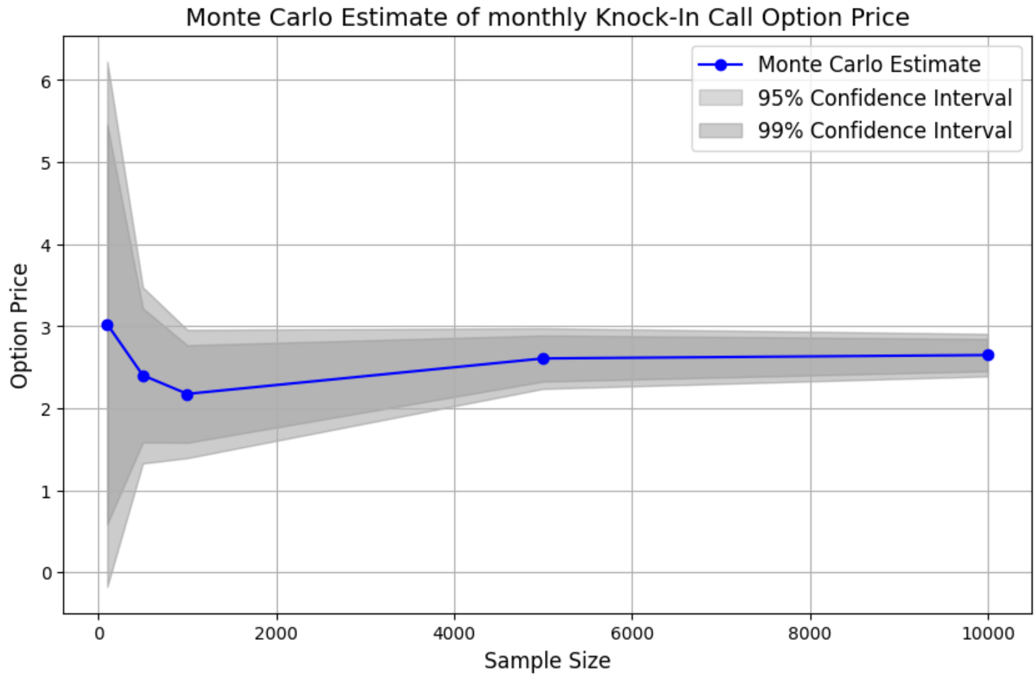


Figure 4: improving estimate precision by increasing monitoring dates

We also analyzed how the precision of the estimate increases with a higher number of simulations. Especially when the simulations are fewer, increasing them has a huge positive effect in reducing the confidence interval. From 5000 simulations, we keep to see improvements but not that much: the CI stabilizes. We could deduce that 5000-6000 is the perfect number of simulations since it decreases computational costs without losing much precision.



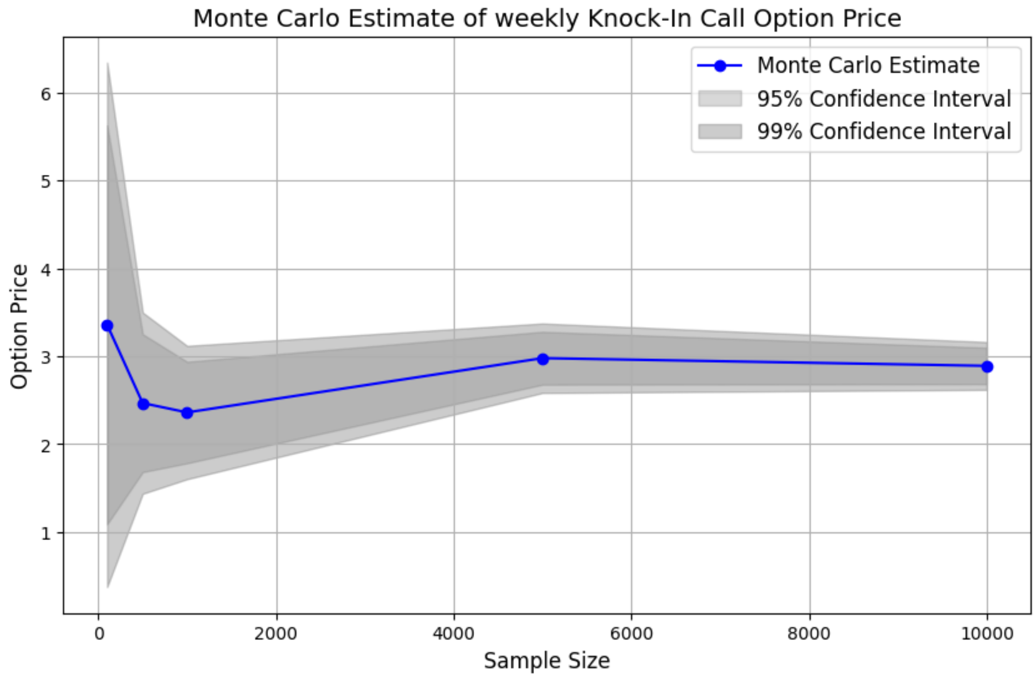


Figure 5: Confidence interval with larger sample sizes

Comparison with Digital Option

To compare the knock-in-call with the digital option we repeated the analysis of the option prices with respect to different volatilities and correlations.

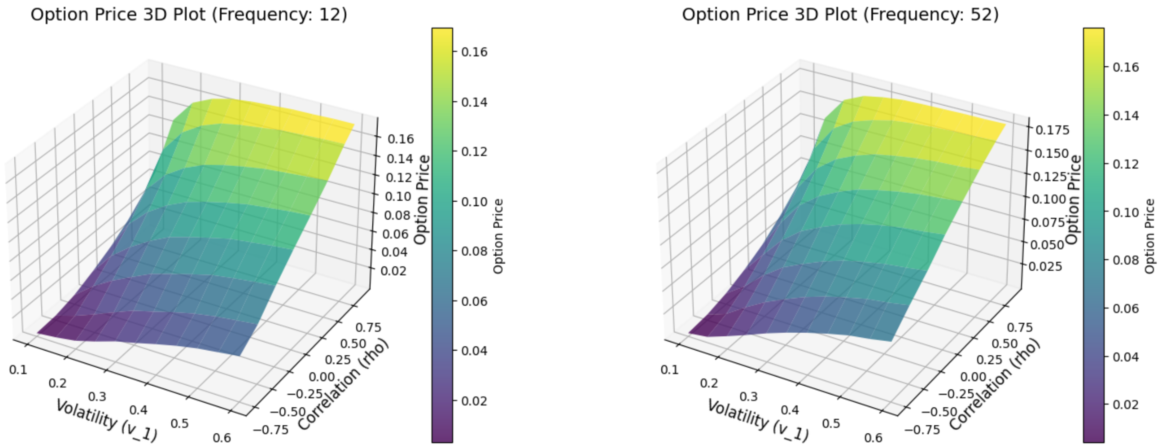


Figure 6: digital option price for varying v_1 and ρ

Both options exhibit similar behavior with respect to σ_1. In particular, as can be seen from the two images above, the main characteristic is that with higher correlation the bound where σ_1 stops influencing the price decreases.

Talking about the behavior related to σ_2, this is slightly different for the two options (the KI call and the KI digital). In fact, while in the case of a standard KI the increase in price with respect to the volatility of the second stock is almost constant, here we can see (from the two images below) that σ_1 stops influencing the price. This makes sense because now the reward is no longer influenced by the absolute level of the stock (since it is 0 or 1) but only by the fact that the strike price has been breached or not at T.

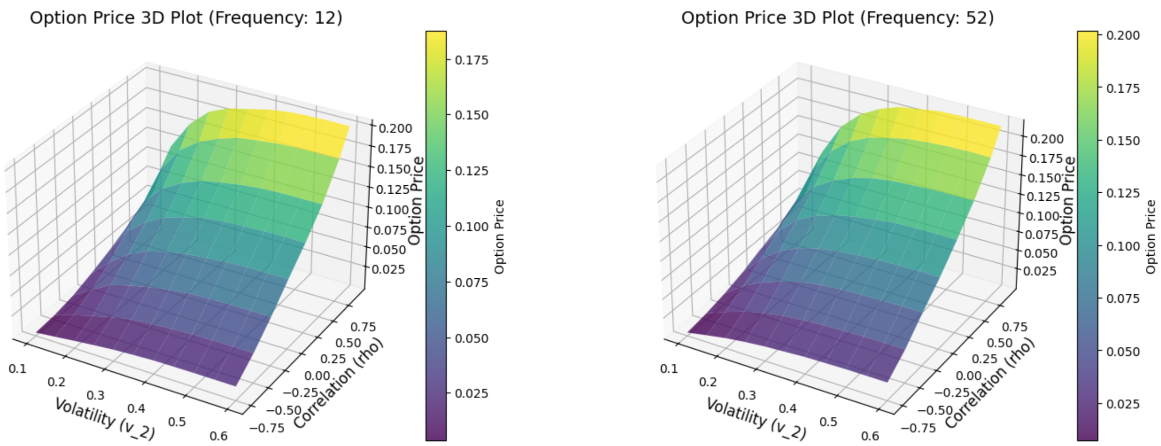


Figure 7: digital option price for varying v_2 and ρ

Computation of Greeks

We define the function `compute_greeks` to calculate **Delta**, **Gamma**, and **Vega** for each security at inception.

Delta quantifies the sensitivity of the option's price to changes in the underlying asset prices.

Gamma measures the rate of change of Delta for a given change in the underlying asset.

Vega represents the sensitivity to changes in volatility.

To estimate these sensitivities for exotic derivatives, we used the Central Difference Estimator (CDE) method for a large number of Monte Carlo simulations.

	Knock-In Call Option	**Knock-In Digital Call**
Delta (S_1)	0.1037	0.0057
Gamma (S_1)	-0.0040	-0.0002
Delta (S_2)	0.1847	0.0052
Gamma (S_2)	0.0062	0.0002
Vega (v_1)	7.6356	0.4411
Vega (v_2)	22.6026	0.2556

Table 2: Option Greeks for Standard Knock-In Call Option and Knock-In Digital Option

From the Greeks calculation we understand the following: for the KI call we are long both S_1 and S_2, but we are more sensitive to changes in S_2 since it directly influences the payoff of the option.

With regard to Gammas, the values are very small, but they will gain importance as we approach the barrier during the life of the option.

As we saw in Figures 2 and 3, and as can be seen from the Vegas, we are more sensitive to changes in σ_2 compared to σ_1.

Conclusion

This report demonstrated the use of Monte Carlo simulations to price knock-in bivariate barrier options under various scenarios. By analyzing sensitivity to volatilities and correlation, comparing with a digital option, and computing the Greeks, we gained valuable insights into the behavior and pricing of these derivatives.