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THE JACOBIAN VARIETY OF AN ALGEBRAIC CURVE.* †

By Wei-Liang Chow.

1. Preliminary remarks. We begin with a few remarks concerning terminology and notations. We shall follow in general the terminology and notations of A. Weil, as developed in his books [8], [9], with the modification that we shall operate throughout with points and (complete) varieties in projective spaces only. The projective space of dimension m will be denoted by $\mathfrak{S}_m$. If $(x) = (x_0, x_1, \cdots, x_m)$ is a point in $\mathfrak{S}_m$, the x_i being its homogeneous coordinates, and if K is a field, we shall denote by $K((x))$ the field generated over K by the adjunction of the mutual ratios of the x_i; similarly, if $(x), (y), \cdots$, are points in projective spaces, we denote by $K((x), (y), \cdots)$ the field generated over K by the adjunction of the mutual ratios of the homogeneous coordinates of every one of the points. The dimension of a point (x) over K is then the degree of transcendency of $K((x))$ over K, and (x) is said to be algebraic or rational over K if $K((x))$ is algebraic over K or coincides with K respectively; we shall also say that a homogeneous polynomial or form is rational over K, if the mutual ratios of its coefficients are all in K. We shall use the capital letters U, V, W, X, Y, Z (and these only) to denote indeterminates; for convenience, a system consisting of $m+1$ indeterminates $(X) = (X_0, X_1, \cdots, X_m)$ will be called a system of indeterminates in $\mathfrak{S}_m$.

We shall assume that the reader is familiar with the theory of associated forms of positive cycles in a projective space, as developed in Chow-van der Waerden [4]. For a positive cycle $\mathfrak{Z}$ of dimension r and degree d in a projective space $\mathfrak{S}_m$, the associated form is a form $f(V^0, \cdots, V^r)$ of degree d in each system of indeterminates $(V^i) = (V^i{}_0, V^i{}_1, \cdots, V^i{}_m)$, $i = 0, 1, \cdots, r$; this form is characterized by the following property: For any set of r hyperplanes in $\mathfrak{S}_m$ given by the equations $\sum_{j=0}^{m} v^i{}_j X_j = 0$, $i = 1, \cdots, r$, the form

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† This paper was written in 1948; a brief summary of the results contained in it appeared under the same title in the Abstract of the addresses given at the Conference on Algebraic Geometry and Algebraic Number Theory, University of Chicago, 1949, pp. 25-27. It is published here with some minor revisions, mainly in the first two sections.

$f(V^0, v^1, \cdots, v^r)$ does not vanish identically if and only if $\mathfrak{Z}$ has a finite number of common intersections with these r hyperplanes, and when such is the case, the form $f(V^0, v^1, \cdots, v^r)$ dissolves into a product of d linear forms, giving all the common intersections with the proper multiplicities. If we regard the set of coefficients (f) of the form $f(V^0, \cdots, V^r)$, arranged in an arbitrary but fixed order, as a point in a projective space $\mathfrak{S}_t$, and if (f') is a specialization of (f) over any field K, then it follows from the main theorem of associated form ([4], Satz 2) that the point (f') corresponds also to the associated form $f'(V^0, \cdots, V^r)$ of a positive cycle $\mathfrak{Z}'$ of dimension r and degree d in $\mathfrak{S}_m$; this positive cycle $\mathfrak{Z}'$ is said to be a specialization of $\mathfrak{Z}$ over K. A property or relation concerning a point (x) in $\mathfrak{S}_m$ and a point (y) in $\mathfrak{S}_n$ is said to be algebraic over a field K, if there is a set of forms $F_j(X, Y)$ in $K[X, Y]$, homogeneous in each of the systems of indeterminates $(X) = (X_0, X_1, \cdots, X_m)$ and $(Y) = (Y_0, Y_1, \cdots, Y_n)$, such that the property holds for any two points (x) and (y) in $\mathfrak{S}_m$ and $\mathfrak{S}_n$ respectively if and only if the equations $F_j(x, y) = 0$ hold; we then say that the property in question can be expressed as an algebraic condition (over K) in terms of the points (x) and (y). In case the set $F_j(X, Y)$ contains forms in (X) or (Y) alone, the point (x) or (y) can be restricted to the points of a certain bunch of varieties in $\mathfrak{S}_m$ or $\mathfrak{S}_n$ respectively; in such a case the property in question is often defined only for the points of certain bunches of varieties in $\mathfrak{S}_m$ and $\mathfrak{S}_n$, so that the equations defining these bunches of varieties are already implicitly contained in the definition of the property. The following fact is important for us later: The property that a given algebraic relation over K holds for a point (y) in $\mathfrak{S}_n$ and *every* point (x) in a positive cycle $\mathfrak{Z}$ in $\mathfrak{S}_m$ can itself be expressed as an algebraic condition over K in terms of (y) and the point (f) in $\mathfrak{S}_t$. To show this, we refer to the proof of Satz 2 in [4] and consider the equations (2), (3), (5) listed there; in our present notations these can be writen as a set of equations

$$G_j(V^1, \cdots, V^r, X^1, \cdots, X^d, Z) = 0,$$

where $(X^1), \cdots, (X^d)$ are d distinct systems of indeterminates in $\mathfrak{S}_m$ and (Z) is a system of indeterminates in $\mathfrak{S}_t$. Let $R_j(V^1, \cdots, V^r, Y, Z) = 0$ be the set of equations which we obtain by eliminating the indeterminates $(X^1), \cdots, (X^d)$ from the set of equations $F_j(X^1, Y) = 0, \cdots, F_j(X^d, Y) = 0$, $G_j(V^1, \cdots, V^r, X^1, \cdots, X^d, Z) = 0$, and let $A_j(Y, Z)$ be the set of all coefficients of the forms $R_j(V^1, \cdots, V^r, Y, Z)$, considered as forms in $(V^1), \cdots, (V^r)$ only; then it is easily seen from the cited proof in [4] that the property in question holds for (y) and $\mathfrak{Z}$ if and only if $A_j(y, f) = 0$. It is clear that this result remains true if more than one point (y) or more than one cycle $\mathfrak{Z}$ are involved, and it is also clear that it remains true even if the original algebraic property does not actually involve the point (y) at all, so that it is simply an algebraic property for the points of the positive cycle or cycles. For later reference we shall state our result as a lemma:

LEMMA 1. *Let $\mathfrak{Z}^1, \cdots, \mathfrak{Z}^s$ be positive cycles in projective spaces and let $(f^1), \cdots, (f^s)$ be the systems of coefficients of the associated forms of $\mathfrak{Z}^1, \cdots, \mathfrak{Z}^s$ respectively; let $(y^1), \cdots, (y^h)$ be points in projective spaces. Then the property that a given algebraic relation over a field K holds for the system of points $(y^1), \cdots, (y^h)$ and every system of points $(x^1), \cdots, (x^s)$, where each (x^i) runs through all the points in $\mathfrak{Z}^i$, can be expressed as an algebraic condition over K in terms of $(y^1), \cdots, (y^h)$ and $(f^1), \cdots, (f^s)$.*

It is easily seen from the definition of the associated form that if the positive cycle $\mathfrak{Z}$ is rational over a field K, then its associated form $f(V^0, \cdots, V^r)$ is also rational over K, i. e. $K((f)) = K$. We shall now show that the converse of this statement is also true, provided that $\mathfrak{Z}$ contains no component with a multiplicity which is divisible by the characteristic of the universal domain; more precisely, we shall now prove the following lemma:

LEMMA 2. *Let $\mathfrak{Z}$ be a positive cycle in a projective space and let (f) be the set of coefficients of the associated form of $\mathfrak{Z}$; if $\mathfrak{Z}$ contains no component with a multiplicity which is divisible by the characteristic of the field, then $\mathfrak{Z}$ is rational over the fields $k_0((f))$, where k_0 is the prime field.*

Proof. Without any loss of generality we can assume that the form $f(V^0, \cdots, V^r)$ is irreducible over the field $K_0 = k_0((f))$, for otherwise we can apply the same argument to each irreducible factor of $f(V^0, \cdots, V^r)$ over K_0; then our assumption about the multiplicities of the components in $\mathfrak{Z}$ implies that the form $f(V^0, \cdots, V^r)$ has no multiple factors over $\bar{K}_0$. It follows that $f(V^0, \cdots, V^r)$ is the product of a number s of absolutely irreducible forms, all distinct and conjugate to each other over K_0, and each one of these forms is rational over a finite extension of degree s over K_0. In order to prove our assertion, it is therefore sufficient to restrict ourselves to the case where the form $f(V^0, \cdots, V^r)$ is absolutely irreducible and the positive cycle $\mathfrak{Z}$ is a variety defined over a field K containing K_0, and to show in this case the variety $\mathfrak{Z}$ is also defined over K_0. Consider the equation $f(V^0, \cdots, V^r) = 0$; since the form $f(V^0, \cdots, V^r)$ is absolutely irreducible, this equation defines a variety $\mathfrak{T}$ of dimension $(r+1)m-1$ in the $(r+1)$-fold product space $S_m \times \cdots, \times S_m$, defined over the field K_0. Let (x) be a generic point of $\mathfrak{Z}$ over K, and let $(v^0), \cdots, (v^r)$ be $r+1$ independent generic solutions of the linear equation $\sum_{j=0}^{m} x_j V_j = 0$ over the field $K((x))$; if r' $(\geq r)$ is the dimension of (x) over K_0, then the point $(v^0, \cdots, v^r)$ in $\mathfrak{T}$ has the dimension

$$(r+1)(m-1)+r' = (r+1)m-1+(r'-r) \geq (r+1)m-1$$

over K_0, from which it follows that $r = r'$ and that $(v^0, \cdots, v^r)$ is a generic point of $\mathfrak{T}$ over K_0. Since the point $(v^0, \cdots, v^r)$ has the same dimension $(r+1)m-1$ over both K and K_0, the fields $K_0((v^0), \cdots, (v^r))$ and K must be independent with respect to each other over K_0, and since $K_0((v^0), \cdots, (v^r))$ is a regular extension of K_0, this means that $K_0((v^0), \cdots, (v^r))$ and K are linearly disjoint with respect to each other over K_0. Without any loss of generality, we can assume that $x_0 \neq 0$ and that $x_m = 0$ in case any one of the coordinates x_i vanishes; then we can take all $v^i{}_j$ $(j \neq 0)$ to be independent variables over $K((x))$ and we have $v^i{}_m \neq 0$ for all $i = 0, 1, \cdots, r$. We have then the relations $\sum_{j=1}^{m} (v^i{}_j/v^i{}_m)(x_j/x_0) = -v^i{}_0/v^i{}_m$, $i = 0, 1, \cdots, r$; the first r equations show ([8], Chapter V, Theorem 1) that (x) is separably algebraic over $K_0((v^0), \cdots, (v^{r-1}))$, and then the last equation shows that (x) is rational over $K_0((v^0), \cdots, (v^r))$. Since $K_0((x))$ is subfield in $K_0((v^0), \cdots, (v^r))$, it is also linearly disjoint with respect to K over K_0; this shows that the variety $\mathfrak{Z}$ is also the locus of (x) over K_0 and hence is defined over K_0.

2. The problem. Let $\mathfrak{C}$ be a curve in the projective space $\mathfrak{S}_m$ of dimension m, defined over a field k, and let h and g be the degree and genus of $\mathfrak{C}$ respectively. A positive divisor $\mathfrak{p}$ of degree n in $\mathfrak{C}$ can be considered as a positive 0-cycle in $\mathfrak{S}_m$ and as such it can be represented by its associated form $p(V)$. Let $\Omega_0(V), \Omega_1(V), \cdots, \Omega_t(V)$ be the set of all power products of degree n in (V), arranged in an order which is arbitrary but fixed once for all; if we set $p(V) = \sum_{j=0}^{t} p_j\Omega_j(V)$, then the set of coefficients $(p) = (p_0, p_1, \cdots, p_t)$, arranged in this fixed order, determines a point in the projective space $\mathfrak{S}_t$ of dimension t, so that every positive divisor of degree n in $\mathfrak{C}$ is represented in a one-to-one manner by a point in $\mathfrak{S}_t$. We shall assume that the curve $\mathfrak{C}$ is free of singularities, so that every 0-cycle in $\mathfrak{S}_m$

which is contained in $\mathfrak{C}$ can also be considered as a divisor in $\mathfrak{C}$; then the set of all points in $\mathfrak{S}_t$ which represent positive divisors of degree n in $\mathfrak{C}$ is a variety $\mathfrak{C}^n$ of dimension n, defined over k. In fact, let $\mathfrak{y}$ be the positive divisor of degree n consisting of n independent generic points $(x^1), \cdots, (x^n)$ of $\mathfrak{C}$ over k, and let (y) be the point in $\mathfrak{S}_t$ representing $\mathfrak{y}$; then, for any positive divisor $\mathfrak{p}$ of degree n in $\mathfrak{C}$, the point (p) is a specialization of the point (y) over k, and conversely, by the main theorem of associated form, every such specialization (p) determines a positive 0-cycle of degree n in $\mathfrak{S}_m$ which is contained in $\mathfrak{C}$ and hence is a positive divisor of degree n in $\mathfrak{C}$. Since $k((y))$ is contained in the regular extension $k((x^1), \cdots, (x^n))$ of k, it is also a regular extension of k, and since $k((x^1), \cdots, (x^n))$ has the degree of transcendency n over k and is an algebraic extension of $k((y))$, the field $k((y))$ must have also the degree of transcendency n over k. It follows that the point (y) has a locus over k which is a variety of dimension n and which evidently coincides with $\mathfrak{C}^n$. As we shall see later, this variety $\mathfrak{C}^n$ is also free of singularities, but we shall not need this fact for the present. It is clear that if a positive divisor $\mathfrak{p}$ is rational over a field K, then the point (p) representing it is also rational over K; the converse of this statement is also true, as we have shown in [3], so that for any point (p) in $\mathfrak{C}^n$ the positive divisor $\mathfrak{p}$ determined by it is rational over $k((p))$.

Two divisors $\mathfrak{p}$, $\mathfrak{q}$ in $\mathfrak{C}$ are said to be linearly equivalent and we shall write $\mathfrak{p} \sim \mathfrak{q}$, if $\mathfrak{p} - \mathfrak{q}$ is the divisor of a rational function on $\mathfrak{C}$, or in other words, if $\mathfrak{p} - \mathfrak{q}$ is a principal divisor in $\mathfrak{C}$. It is well known that the set of all principal divisors in $\mathfrak{C}$ forms a subgroup $\mathfrak{G}_l(\mathfrak{C})$ in the group $\mathfrak{G}(\mathfrak{C})$ of all divisors in $\mathfrak{C}$, and that the quotient group of $\mathfrak{G}(\mathfrak{C})$ over $\mathfrak{G}_l(\mathfrak{C})$ is the group $\mathfrak{D}$ of divisor classes in $\mathfrak{C}$, each class consisting of all divisors which are linearly equivalent to each other. In this group $\mathfrak{D}$ the classes of degree zero, sometimes called the zero classes, again form a subgroup $\mathfrak{D}_0$, while the classes of any one fixed degree n form a coset $\mathfrak{D}_n$ of $\mathfrak{D}_0$ in $\mathfrak{D}$; if we denote by $\mathfrak{G}_a(\mathfrak{C})$ the group of all divisors of degree zero in $\mathfrak{C}$, then $\mathfrak{D}_0$ is the quotient group of $\mathfrak{G}_a(\mathfrak{C})$ over $\mathfrak{G}_l(\mathfrak{C})$. If we let any one fixed divisor class $\mathfrak{O}$ of $\mathfrak{D}_n$ correspond to the principal class of $\mathfrak{D}_0$, then the relation $\mathfrak{D}_n = \mathfrak{O} + \mathfrak{D}_0$ establishes a one-to-one correspondence between the classes of $\mathfrak{D}_n$ and those of $\mathfrak{D}_0$. Through this correspondence the group addition of $\mathfrak{D}_0$ will induce in $\mathfrak{D}_n$ the following operation: Given any two classes $\mathfrak{P}, \mathfrak{Q}$ in $\mathfrak{D}_n$, the class $\mathfrak{P} + \mathfrak{Q} - \mathfrak{O}$, also belonging to $\mathfrak{D}_n$, is called the sum of $\mathfrak{P}$ and $\mathfrak{Q}$. We shall call this operation in $\mathfrak{D}_n$ also addition (with respect to the class $\mathfrak{O}$).

A divisor class $\mathfrak{P}$ is said to be rational over a field K, if it contains a divisor $\mathfrak{p}$ which is rational over K. It is well known that the set of positive divisors in the class $\mathfrak{P}$ constitutes a linear system; that is, there exists a linear system of hypersurfaces in $\mathfrak{S}_m$ which cut out on $\mathfrak{C}$, besides a fixed component, exactly the positive divisors of the class $\mathfrak{P}$. If the class $\mathfrak{P}$ is rational over K, then ([8], Chapter VIII, Theorem 10) this linear system of hypersurfaces has a basis consisting of hypersurfaces which are defined over K. We recall that according to the Theorem of Riemann-Roch, the linear system of positive divisors of a class of degree $n > 2g - 2$ has exactly the dimension $n - g$.

A variety $\mathfrak{J}$, defined over a field K containing k, is called a Jacobian variety of the curve $\mathfrak{C}$, if it has the following properties: (1) $\mathfrak{J}$ is an Abelian variety defined over K; (2) there is a homomorphic mapping Φ of $\mathcal{G}_a(\mathfrak{C})$ onto $\mathfrak{J}$, called the canonical homomorphism, whose kernel is precisely $\mathcal{G}_l(\mathfrak{C})$; (3) the homomorphism Φ is rational and is defined over K in the following sense: if $\mathfrak{p}_u$ is a rational divisor over $K(u)$, where (u) is any set of elements (in the universal domain), then the point $\Phi(\mathfrak{p}_u)$ is also rational over $K(u)$, and if $\mathfrak{p}_\mu$ is any specialization of $\mathfrak{p}_u$ over a specialization $(u) \rightarrow (\mu)$ over K, then $\Phi(\mathfrak{p}_\mu)$ is also a specialization of $\Phi(\mathfrak{p}_u)$ over the specialization $(u) \rightarrow (\mu)$ over K; (4) the rational homomorphism Φ has the "universal maping" property: if Ψ is any rational homomorphism of $\mathcal{G}_a(\mathfrak{C})$ into any Abelian variety $\mathfrak{A}$, then Ψ is the product of Φ and a rational homomorphism of $\mathfrak{J}$ into $\mathfrak{A}$. It can be easily seen from the "universal mapping" property (4) that the Jacobian variety $\mathfrak{J}$, it it exists, is uniquely determined up to a birational isomorphism, so that we can sometimes speak of *the* Jacobian variety. However, the question whether such a Jacobian variety exists at all is not a simple one, and this existence problem of the Jacobian variety is one of principal importance not only for the theory of algebraic curves, but also for the theory of Abelian varieties. In this paper we shall offer a general solution of this problem.

In the classical case, where the ground field is the field of complex numbers, a proof of the existence of the Jacobian variety can be obtained from the known results in the literature in the following way. (The terminology and notations in this one paragraph do not follow strictly the conventions set forth in section 1.) Let $I_1, \cdots, I_g$ be a system of g independent Abelian differentials of the first kind on $\mathfrak{C}$, and let $\gamma_1, \cdots, \gamma_{2g}$ be a base of the first Betti group of $\mathfrak{C}$ and we set $p_{ij} = \int_{\gamma_i} I_j$; let $\mathfrak{E}_g$ be the complex linear space of dimension g, considered as an analytic (= complex-analytic) group variety, and let $[P]$ be the discrete subgroup generated by the $2g$ rows of the matrix $P=(p_{ij})$, each row being considered as a point in E_g, and let $\mathfrak{E}_g/[P]$ be the quotient group variety of $\mathfrak{E}_g$ over $[P]$. Let (x_0) be any fixed point in $\mathfrak{C}$ and let ϕ be the analytic mapping of $\mathfrak{C}$ into $\mathfrak{E}_g/[P]$ defined by the formula $\phi(x)=(\int_{x_0}^{x} I_1,\cdots,\int_{x_0}^{x} I_g)$ (mod. $[P]$); then, if $\mathfrak{p}=\sum n_i(x_i)$ is any divisor in $\mathfrak{G}_a(\mathfrak{C})$, the correspondence $\mathfrak{p}\rightarrow\sum n_i\phi(x_i)$ defines a homomorphism Φ of $\mathfrak{G}_a(\mathfrak{C})$ onto $\mathfrak{E}_g/[P]$, whose kernel is, by the Abel's Theorem, precisely the group $\mathfrak{G}_l(\mathfrak{C})$. If Ψ is any homomorphism of $\mathfrak{G}_a(\mathfrak{C})$ into an analytic commutative group variety $\mathfrak{A}$, then $\psi(x)=\Psi((x)-(x_0))$ is a mapping of $\mathfrak{C}$ into $\mathfrak{A}$ and we have $\Psi(\mathfrak{p})=\sum n_i\psi(x_i)$; we shall say that the homomorphism Ψ is analytic if the mapping ψ is analytic, and we observe that, in case $\mathfrak{A}$ is algebraic, the homomorphism Ψ is rational if and only if ψ is rational (over the complex ground field). It is clear that the homomorphism Φ defined above is analytic; we shall show that if Ψ is analytic and if $\mathfrak{A}$ is analytically isomorphic to a complex torus, then Ψ is the product of Φ and an analytic homomorphism of $\mathfrak{E}_g/[P]$ into $\mathfrak{A}$. We set $\mathfrak{A}=\mathfrak{E}_r/[Q]$, where $Q=(q_{ij})$ is a $(2r,r)$-matrix such that the matrix $(Q,\bar{Q})$ is non-singular, and let $z_1,\cdots,z_r$ be the coordinates of a variable point in $\mathfrak{E}_r$, so that $dz_1,\cdots,dz_r$ can be considered as Abelian differentials of the first kind on $\mathfrak{E}_r/[Q]$; it is well known that a base $\delta_1,\cdots,\delta_{2r}$ for the first Betti group of $\mathfrak{E}_r/[Q]$ can be so chosen that we have $q_{ij}=\int_{\delta_i} dz_j$. Let $d\psi_1,\cdots,d\psi_r$ be the Abelian differentials of the first kind on $\mathfrak{C}$ induced by $dz_1,\cdots,dz_r$ respectively through the mapping ψ, and let $\psi(\gamma_1),\cdots,\psi(\gamma_{2g})$ be the image cycles of $\gamma_1,\cdots,\gamma_{2g}$ respectively under ψ; then there exist a (g,r)-matrix $M=(m_{ij})$ and an integral $(2g,2r)$-matrix $N=(n_{ij})$ such that we have the relations $d\psi_i=\sum_{j=1}^{g} m_{ji}I_j$, $\psi(\gamma_i)=\sum_{j=1}^{2r} n_{ij}\delta_j$, and

$$PM=\sum_{k=1}^{g} p_{ik}m_{kj}=\sum_{k=1}^{g} m_{kj}\int_{\gamma_i} I_k=\int_{\gamma_i} d\psi_j=\int_{\psi(\gamma_i)} dz_j=\sum_{k=1}^{2r} n_{ik}\int_{\delta_k} dz_j$$
$$=\sum_{k=1}^{2r} n_{ik}q_{kj}=NQ.$$

This shows that the linear mapping of $\mathfrak{E}_g$ into $\mathfrak{E}_r$ determined by the matrix M induces an analytic homomorphism Θ of $\mathfrak{E}_g/[P]$ into $\mathfrak{E}_r/[Q]$, and we have the relation

$$\psi(x)=(\int_{x_0}^{x} d\psi_1,\cdots,\int_{x_0}^{x} d\psi_r)=(\int_{x_0}^{x} I_1,\cdots,\int_{x_0}^{x} I_g)M \quad (\text{mod. } [Q]),$$

or $\psi(x) = \Theta\phi(x)$, from which it follows that $\Psi = \Theta\Phi$. Now, according to a result of Lefschetz ([5], p. 368), there exists an analytic isomorphism σ of $\mathfrak{C}_g/[P]$ onto an Abelian variety $\mathfrak{J}$; if we assume that there is an analytic isomorphism τ of $\mathfrak{C}_r/[Q]$ onto an Abelian variety $\mathfrak{A}$, and if we now write ϕ, Φ, ψ, Ψ, Θ in place of $\sigma\phi$, $\sigma\Phi$, $\tau\psi$, $\tau\Psi$, $\tau\Theta\sigma^{-1}$, respectively, then we have again the relations $\psi = \Theta\phi$ and $\Psi = \Theta\Phi$. Since ϕ, ψ, and Θ are now analytic mappings of an algebraic variety into another algebraic variety, it follows from a theorem of ours proved elsewhere ([2], Theorem VII) that they are all rational transformations, and hence the homomorphisms Φ and Ψ are also rational. This shows that $\mathfrak{J}$ is the Jacobian variety of $\mathfrak{C}$ with Φ as the canonical homomorphism.

In the general case of an arbitrary ground field, it is clear that the above highly transcendental proof cannot be carried over and that new methods of a purely algebraic nature have to be introduced. A. Weil, who first proposed and studied this problem, obtained a solution by a generalization of the concept of an algebraic variety ([9], § V); in fact, the attempt to solve this problem is one of the main reasons which led Weil to introduce his notation of an "abstract" variety, an algebraic variety in abstracto in contrast to the usual ones which are embedded in a projective space. As Weil has been able to extend most of the fundamental results in algebraic geometry to the "abstract" varieties, his "abstract" Jacobian variety proves to be almost just as useful as the projective Jacobian variety in the classical case and thus constitutes a satisfactory solution of the problem. Nevertheless, it remains an important and interesting question whether a Jacobian variety exists as a projective algebraic variety in the usual sense; furthermore, what is probably even more important from the algebraic viewpoint, there remains still unsolved the question of the field of definition for the Jacobian variety and the canonical homomorphism, as has been pointed out by Weil himself ([9], p. 68). Our solution of the problem will provide satisfactory answers to both these questions; we shall show that the Jacobian variety exists as a variety in a projective space, and that both the Jacobian variety and the canonical homomorphism are defined over the defining field of the curve $\mathfrak{C}$. This latter fact, which seems to be new even in the classical case, has important applications in the theory of Picard varieties over arbitrary ground fields, as will be shown in a forthcoming paper of ours.

We shall describe briefly the main idea underlying our construction of the Jacobian variety. Since the positive divisors in $\mathfrak{C}$ can be represented by the points of an algebraic variety, it is natural to try to represent the divisor classes by associating with them certain positive divisors. According to the Riemann-Roch Theorem, each class of $\mathfrak{D}_g$ contains *in general* only one positive divisor; thus each point of $\mathfrak{C}^g$ will represent a class in $\mathfrak{D}_g$, and the representation is one-to-one except for the points of a proper subvariety in $\mathfrak{C}^g$. It is therefore natural that this variety $\mathfrak{C}^g$ has been usually taken as the starting point in the construction of the Jacobian variety; thus Weil's "abstract" Jacobian variety is essentially a collection of birational transforms of $\mathfrak{C}^g$ patched together in a suitable manner. Also van der Waerden, in a paper [7] which deals in reality with certain aspects of the Jacobian variety, has taken this variety $\mathfrak{C}^g$ as the starting point and has constructed by means of the associated forms a variety in a projective space, the points of which are in one-to-one correspondence with the classes in $\mathfrak{D}_g$; however, his results do not show that the so constructed variety is the Jacobian variety, as, for one thing, there is no proof that the variety is non-singular. We observe that in both cases the main difficulty lies in the existence of special divisor classes of degree g which are represented not by points but by subvarieties in $\mathfrak{C}^g$; one has then to cope with the fact that the divisor classes are represented by different types of geometrical entities. If now, instead of $\mathfrak{C}^g$, we consider the variety $\mathfrak{C}^n$ for sufficiently high n (say $n > 2g - 2$), then there will be no special divisor classes and every divisor class of degree n will be represented by a subvariety of dimension $n - g$ in $\mathfrak{C}^n$; the method of associated forms then will enable us to represent the divisor classes by ponts of a certain variety, and from the homogeneous nature of the construction one would expect the so constructed variety to be non-singular. This is the underlying idea of the method by which we shall construct the Jacobian variety in the next section; however, the proof that the so constructed variety is non-singular is not so simple and will be given later in section 5 as an application of a general theorem proved by us elsewhere. In the final section we then introduce the canonical homomorphism and complete the proof of the existence of the Jacobian variety.

3. Construction of the variety $\mathfrak{B}$. We start with the variety $\mathfrak{C}^n$ of n dimensions, defined over the field k, for a fixed $n > 2g - 2$. For any point (p) of $\mathfrak{C}^n$, such that the corresponding positive divisor $\mathfrak{p}$ is rational over a field K containing k, the Theorem of Riemann-Roch, applied to the curve $\mathfrak{C}$ over the field K, shows that the complete linear system of positive divisors on $\mathfrak{C}$ determined by the divisor $\mathfrak{p}$ has the dimension $n - g$ and is rational over the field K. In other words, the set of all positive divisors in the class $\mathfrak{P}$ determined by $\mathfrak{p}$ consists exactly of those divisors cut out on the cuve $\mathfrak{C}$, apart from a fixed component, by a linear system of $n - g + 1$ hypersurfaces $\sum_{j=0}^{n-g} U_j\psi_j(X) = 0$, where the $\psi_j(X)$ are forms of the same degree in $K[X]$ and the $(U) = (U_0, \cdots, U_{n-g})$ is a system of indeterminates; this implies also that no hypersurface of this system contains the curve $\mathfrak{C}$. Let $(u) = (u_0, \cdots, u_{n-g})$ be a set of independent variables over K, and let $\mathfrak{p}_u$ be the divisor cut out on $\mathfrak{C}$, apart from the fixed component, by the generic hypersurface $\sum_{j=0}^{n-g} u_j\psi_j(X) = 0$ of the system; then the divisor $\mathfrak{p}_u$ is rational over $K(u)$ and hence is represented by a rational point (p_u) over $K(u)$ in $\mathfrak{C}^n$. It is well known that over any specialization $(u) \rightarrow (\mu)$ over K, the divisor $\mathfrak{p}_u$ specializes into the divisor $\mathfrak{p}_\mu$ cut out on $\mathfrak{C}$, apart from the fixed component, by the hypersurface $\sum_{j=0}^{n-g} \mu_j\psi_j(X) = 0$. It follows that over any specialization $(u) \rightarrow (\mu)$ over K, there is a uniquely determined specialization (p_μ) of (p_u), which represents a divisor in $\mathfrak{P}$, and that in this way all positive divisors in $\mathfrak{P}$ can be obtained. Since $K((p_u))$, being a subfield of $K(u)$, is a regular extension of K, this shows that if we denote by $\mathfrak{G}_{\mathfrak{P}}$ the set of all points in $\mathfrak{C}^n$ which represent positive divisors in $\mathfrak{P}$, then $\mathfrak{G}_{\mathfrak{P}}$ is a variety of dimension $n - g$, defined over K, and the point (p_u) is a generic point of $\mathfrak{G}_{\mathfrak{P}}$ over K. Furthermore, the correspondence $u \rightarrow (p_u)$ defines a birational transformation of the projective space $\mathfrak{S}_{n-g}$ onto $\mathfrak{G}_{\mathfrak{P}}$, defined over K; in fact, the divisor $\mathfrak{p}_u$ consists of n distinct points $(\xi^1), \cdots, (\xi^n)$, which form a complete set of separable conjugates over $K((p_u))$, and the point (u) is determined rationally by the linear conditions $\sum_{j=0}^{n-g} u_j\psi_j(\xi^i) = 0$, $i = 1, \cdots, n$. We shall denote by $G_{\mathfrak{P}}(W) = G_{\mathfrak{P}}(W^0, \cdots, W^{n-g})$ the associated form of the variety $\mathfrak{G}_{\mathfrak{P}}$, where for each $i = 0, 1, \cdots, n - g$, the $(W^i) = (W^i{}_0, W^i{}_1, \cdots, W^i{}_t)$ is a system of indeterminates in $\mathfrak{S}_t$; since the variety $\mathfrak{G}_{\mathfrak{P}}$ is defined over K, the form $G_{\mathfrak{P}}(W)$ is rational over K.

Let (y) be a generic point of $\mathfrak{C}^n$ over k, and let $\mathfrak{y}$ be the corresponding positive divisor on $\mathfrak{C}$. Since $\mathfrak{y}$ is rational over $k((y))$ and consequently the class $\mathfrak{Y}$ determined by $\mathfrak{y}$ is also rational over $k((y))$, it follows that the associated form $G_{\mathfrak{Y}}(W)$ of the variety $\mathfrak{G}_{\mathfrak{Y}}$ is rational over $k((y))$. Let d be the degree of the variety $\mathfrak{G}_{\mathfrak{Y}}$ (as a subvariety in $\mathfrak{S}_t$), and let $\Delta_0(W)$, $\Delta_1(W), \cdots, \Delta_l(W)$ be the set of all power products of degree d in (W), arranged in an order which is arbitrary but fixed once for all; if we set $G_{\mathfrak{Y}}(W) = \sum_{j=0}^{l} G_j\Delta_j(W)$, then the set of coefficients $(G_{\mathfrak{Y}}) = (G_0, G_1, \cdots, G_l)$, arranged in this fixed order, determines a point in the projective space $\mathfrak{S}_l$

of dimension l, rational over $K((y))$. Since the variety $\mathfrak{C}^n$ is defined over k, it follows that the point $(G_{\mathfrak{Y}})$ defines a variety in $\mathfrak{S}_l$ over k, which we shall denote by $\mathfrak{B}$; and the correspondence $(y) \rightarrow (G_{\mathfrak{Y}})$ determines a rational transformation ϕ of $\mathfrak{C}^n$ onto $\mathfrak{B}$, defined over k. We maintain that the rational transformation has a unique value $\phi(p)$ at every point (p) in $\mathfrak{C}^n$, and that the positive cycle $\mathfrak{G}'_{\mathfrak{P}}$ of dimension $n-g$ and degree d in $\mathfrak{S}_t$ determined by the point $\phi(p)$ consists of exactly the one variety $\mathfrak{G}_{\mathfrak{P}}$, possibly with a certain multiplicity. To show this, we observe ([7], § 14) that the property that any two points (p) and (q) in $\mathfrak{C}^n$ represent linearly equivalent divisors is an algebraic condition over k; it follows then from this and Lemma 1 (section 1) that the property that a positive cycle $\mathfrak{G}$ of dimension $n-g$ and degree d in $\mathfrak{S}_t$ is contained in the variety $\mathfrak{G}_{\mathfrak{P}}$ can be expressed as an algebraic conidtion over k in terms of the point (p) in $\mathfrak{S}_t$ and the point (G) in $\mathfrak{S}_l$ representing $\mathfrak{G}$. In other words, there exists a set of forms $A_j(Y,Z)$ in $k[Y,Z]$, where the (Y) and (Z) are systems of indeterminates in $\mathfrak{S}_t$ and $\mathfrak{S}_l$ respectively, such that the cycle $\mathfrak{G}$ is contained in $\mathfrak{G}_{\mathfrak{P}}$ if and only if the equations $A_j(p,G)=0$ hold. It is evident that the equations $A_j(y,G_{\mathfrak{Y}})=A_j(y,\phi(y))=0$ hold; if $\phi(p)$ is any specialization of $\phi(y)$ over the specialization $(y)\rightarrow(p)$ over k, then we have the relations $A_j(p,\phi(p))=0$. This shows that the positive cycle $\mathfrak{G}'_{\mathfrak{P}}$ determined by $\phi(p)$ can only have component varieties which are contained in $\mathfrak{G}_{\mathfrak{P}}$; and since every component in $\mathfrak{G}'_{\mathfrak{P}}$ has the same dimension $n-g$ as that of $\mathfrak{G}_{\mathfrak{P}}$, it must coincide with $\mathfrak{G}_{\mathfrak{P}}$, so that the positive cycle $\mathfrak{G}'_{\mathfrak{P}}$ is a multiple of the variety $\mathfrak{G}_{\mathfrak{P}}$ (we shall show in section 5 that $\mathfrak{G}'_{\mathfrak{P}}$ actually coincides with $\mathfrak{G}_{\mathfrak{P}}$). This evidently also shows that the specialization $\phi(p)$ is uniquely determined.

Thus we have shown that the points of the variety $\mathfrak{B}$ represent in a one-to-one manner the divisor classes of $\mathfrak{C}$, and that for any two divisors $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathfrak{C}$, we have $\phi(p)=\phi(q)$ if and only if $\mathfrak{p}\sim\mathfrak{q}$. The next step is to define a composition function in $\mathfrak{B}$ which corresponds to the group addition in $\mathfrak{D}_0$, or rather the addition in $\mathfrak{D}_n$ with respect to some one fixed divisor class $\mathfrak{O}$. For this assume the integer n so chosen that there exists a divisor class $\mathfrak{O}$ in $\mathfrak{D}_n$ which is rational over k; it is easily seen that such an integer n exists. The corresponding point (G^0) on $\mathfrak{B}$ is then also rational over k. Let (G^1) and (G^2) be any two points in $\mathfrak{B}$, and let $\mathfrak{P}^1$ and $\mathfrak{P}^2$ be the divisor classes determined by (G^1) and (G^2) respectively; if (G^3) is the point in $\mathfrak{B}$ which represents the divisor class $\mathfrak{P}^3=\mathfrak{P}^1+\mathfrak{P}^2-\mathfrak{O}$, then we define (G^3) as the sum of (G^1) and (G^2), and write $(G^3)=(G^1)+(G^2)$. We shall show that if (G^1) and (G^2) are two independent generic points of $\mathfrak{B}$ over k, then the point (G^3) is rational over the field $K=k((G^1),(G^2))$. We observe first that since both (G^1) and (G^2) are generic points of $\mathfrak{V}$ over k, the positive cycles $\mathfrak{G}^1$ and $\mathfrak{G}^2$ determined by them are subvarieties in $\mathfrak{C}^n$, which by Lemma 2 (section 1), are defined over the fields $k((G^1))$ and $k((G^2))$ respectively, so that both are defined over K. Let (p^1) and (p^2) be independent generic points of the varieties $\mathfrak{G}^1$ and $\mathfrak{G}^2$ respectively over K; since the divisor class $\mathfrak{P}^1$ is rational over $K((p^1))$ and the divisor class $\mathfrak{P}^2$ is rational over $K((p^2))$, the class $\mathfrak{P}^3 = \mathfrak{P}^1 + \mathfrak{P}^2 - \mathfrak{O}$ is rational over the field $K((p^1), (p^2))$ and consequently we have $k((G^3)) \subset K((p^1), (p^2))$. Now, let (q^1) and (q^2) be another pair of such independent generic points of the varieties $\mathfrak{G}^1$ and $\mathfrak{G}^2$ respectively, both independent with respect to the field $K((p^1), (p^2))$ over K; then we must also have $k((G^3)) \subset K((q^1), (q^2))$. Since both $K((p^1), (p^2))$ and $K((q^1), (q^2))$ are regular extensions of K and since they are independent with respect to each other over K, it follows that $K((p^1), (p^2)) \cap K((q^1), (q^2)) = K$. Therefore we conclude that $k((G^3)) \subset K = k((G^1), (G^2))$. In order to prove that the addition on $\mathfrak{V}$ is a rational function, we have only to show that $(G^a) + (G^b)$ is the only specialization of $(G^1) + (G^2)$ over any specialization $(G^1) \to (G^a)$, $(G^2) \to (G^b)$ over k. Here again our assertion will follow immediately from the fact that the relation $(G^c) = (G^a) + (G^b)$ is an algebraic condition over k between the three points. To show this, let $\mathfrak{P}^a$, $\mathfrak{P}^b$, $\mathfrak{P}^c$ be the divisor classes determined by the points (G^a), (G^b), (G^c) respectively, and let $\mathfrak{p}^a$, $\mathfrak{p}^b$, $\mathfrak{p}^c$ be any positive divisors in the classes $\mathfrak{P}^a$, $\mathfrak{P}^b$, $\mathfrak{P}^c$ respectively; let $\mathfrak{o}$ be a rational positive divisor over k in the divisor class $\mathfrak{O}$. We recall again ([7], § 14) that linear equivalence between positive divisors is an algebraic condition over k; since the positive divisor $\mathfrak{o}$ is rational over k and hence is represented by a rational point over k in $\mathfrak{C}^n$, the relation $\mathfrak{p}^a + \mathfrak{p}^b \sim \mathfrak{p}^c + \mathfrak{o}$ can be expressed as an algebraic condition over k in terms of the points (p^a), (p^b), (p^c). It follows then by Lemma 1 (the case where the points are not involved at all) that the relation $(G^c) = (G^a) + (G^b)$ is an algebraic condition over k.

Thus we have shown that the addition in $\mathfrak{D}_n$ (or the addition in $\mathfrak{D}_0$) induces on the variety $\mathfrak{V}$ an addition of points, which is a rational function, defined over the ground field k, on the product variety $\mathfrak{V} \times \mathfrak{V}$ with values in $\mathfrak{V}$. Moreover, this function is not only defined for a generic point of $\mathfrak{V} \times \mathfrak{V}$, but also has a uniquely determined value for any point on $\mathfrak{V} \times \mathfrak{V}$. From this it follows that the points $\mathfrak{V}$ form an abelian group under this addition operation, with the point (G^0) as the unit element. The inverse of any point (G) corresponding to the class $\mathfrak{P}$ is the point $(G)^{-1}$ corresponding to the class $2\mathfrak{O} - \mathfrak{P}$; and in case (G) is a generic point of V over k, the point $(G)^{-1}$ is rational over $k((G))$. From this it follows, by a similar argument as before, that the inverse of a point is a rational function on $\mathfrak{V}$, defined over k. We shall write $(G^1) - (G^2)$ for $(G^1) + (G^2)^{-1}$. Finally, it is also easily seen that, for any fixed point (G^a) on $\mathfrak{V}$, the mapping $(G) \rightarrow (G^a) + (G)$ defines a birational transformation of $\mathfrak{V}$ onto itself, which is one-to-one for every point of $\mathfrak{V}$. In order to prove that $\mathfrak{V}$ is an Abelian variety, it remains to shows that the function $(G^1) + (G^2)$ is defined at every point of the product variety $\mathfrak{V} \times \mathfrak{V}$; since a rational function is defined at every simple point for which it has a uniquely determined value, it is sufficient to show that the variety $\mathfrak{V}$ is non-singular. This will be done in section 5, where this result is obtained as an application of a general theorem proved by us elsewhere; in the final section we shall then introduce the canonical homomorphism and show that it has the desired properties, thus completing the proof that $\mathfrak{V}$ is a Jacobian variety of the curve $\mathfrak{C}$. Before we proceed with this, we shall insert a digression in the next section, where we shall show that a derived (absolutely) normal model $\mathfrak{W}$ of the variety $\mathfrak{V}$ is an Abelian variety and that the birational correspondence between $\mathfrak{V}$ and $\mathfrak{W}$ is one-to-one without exception; this result, together with the results of the final section on the canonical homomorphism, gives us an alternate proof of the existence of a Jacobian variety, though not necessarily defined over the original ground field k, except in case k is perfect. Although this alternate proof is now superseded by the more complete result in section 5, it still has the advantage of simplicity; furthermore, the method of proof used here may have possible applications to other similar problems.*

4. The derived normal model $\mathfrak{W}$. Consider a derived (absolutely) normal variety $\mathfrak{W}$ of $\mathfrak{V}$, which is defined over a purely inseparable extension K of the field k. The birational correspondence $\Gamma(\mathfrak{W}) = \mathfrak{V}$ between $\mathfrak{W}$ and $\mathfrak{V}$, also defined over K, has the property that to each point (H) of $\mathfrak{W}$ corresponds exactly one point $\Gamma(H)$ of $\mathfrak{V}$, while conversely to each point (G) of $\mathfrak{V}$ correspond at most a finite number of points $\Gamma^{-1}(G)$ of $\mathfrak{W}$. Let (H^1) and (H^2) be two independent generic points on $\mathfrak{W}$ over K, and let $(G^1) = \Gamma(H^1)$

* [*Added in proof, March* 10, 1954] T. Matsusaka, in a recent paper "On the algebraic construction of Picard variety (II)," *Japanese Journal of Mathematics*, vol. 22, pp. 51-62, has successfully applied this method to prove the existence of the Picard variety over an arbitrary field; the application of this method was made possible by a crucial result, proved by Matsusaka in another paper, concerning the existence of a "regular maximal algebraic family" of divisors. Matsusaka has also shown that it is sufficient to take a derived normal model with reference to k, which can then be shown to be also absolutely normal; this removes the restriction on the field of definition stated above.

and $(G^2) = \Gamma(H^2)$ be the corresponding points on $\mathfrak{B}$ under the correspondence Γ; then (G^1) and (G^2) are also generic and independent with respect to each other over K. The sum $(G^3) = (G^1) + (G^2)$ of the two independent generic points (G^1) and (G^2) on $\mathfrak{B}$, being a rational function on $\mathfrak{B} \times \mathfrak{B}$ defined over K, is carried by the birational cooorespondence Γ into the rational function $(H^3) = \Gamma^{-1}(G^3) = \Gamma^{-1}(\Gamma(H^1) + \Gamma(H^2))$ on $\mathfrak{W} \times \mathfrak{W}$, also defined over K, which we shall define as the sum $(H^1) + (H^2)$ of the two independent generic points (H^1) and (H^2) on $\mathfrak{W}$. Now, what is the behavior of this function $(H^1) + (H^2)$ for a pair of special points (H^a) and (H^b) on $\mathfrak{W}$? By definition, any value of the function $(H^1) + (H^2)$ is a specialization (H^c) of the point (H^3) over the specialization $(H^1) \rightarrow (H^a)$, $(H^2) \rightarrow (H^b)$ over K. It is well known that any such specialization $(H^1, H^2, H^3) \rightarrow (H^a, H^b, H^c)$ can always be extended to a specialization of (G^1, G^2, G^3), while on the other hand it is clear from the above that any such extension of specialization is already uniquely determined by the specialization $(H^1, H^2) \rightarrow (H^a, H^b)$ alone. In fact, since the transformation $\Gamma(H)$ has always uniquely determined specialization at every point of $\mathfrak{W}$, we have $(G^1) \rightarrow \Gamma(H^a)$, $(G^2) \rightarrow \Gamma(H^b)$; from this it follows that since the sum $(G^1) + (G^2)$ has always a uniquely determined specialization for every pair of points on $\mathfrak{B}$, we must have $(G^3) \rightarrow \Gamma(H^a) + \Gamma(H^b)$. Since, in particular, the relation $\Gamma(H^3) = (G^3)$, which defines the correspondence Γ, must be preserved under the extended specialization, we have the relation $\Gamma(H^c) = \Gamma(H^a) + \Gamma(H^b)$. It follows then that any specialization (H^c) of (H^3), over the specialization $(H^1, H^2) \rightarrow (H^a, H^b)$ over K, must be one of the finite number of points of $\mathfrak{W}$ which are carried over by Γ into the point $\Gamma(H^a) + \Gamma(H^b)$ on $\mathfrak{B}$. In similar manner, we can consider a specialization (H^b) of the point $(H^2) = \Gamma^{-1}(\Gamma(H^3) - \Gamma(H^1))$ over any given specialization $(H^1, H^3) \rightarrow (H^a, H^c)$, where the (H^1) and (H^3) are now a pair of independent generic points of $\mathfrak{W}$. Since the specialization can also be extended in a unique way to the specialization $(G^1, G^3, G^2) \rightarrow (\Gamma(H^a), \Gamma(H^c), \Gamma(H^c) - \Gamma(H^a))$, we conclude that the relation $\Gamma(H^b) = \Gamma(H^c) - \Gamma(H^a)$ must hold for any such specialization, and hence (H^b) must be one of the finite number of points of $\mathfrak{W}$ which are carried over by Γ into the point $\Gamma(H^c) - \Gamma(H^a)$ of $\mathfrak{B}$.

Thus we have shown that for any specialization $(H^1, H^2, H^3) \rightarrow (H^a, H^b, H^c)$ the relation $\Gamma(H^c) = \Gamma(H^a) + \Gamma(H^b)$ always holds, and that in this specialization any two of the three points can be chosen arbitrarily on $\mathfrak{W}$, while the third one is then one of the finite number of possible points determined by this relation. Let now (H^a) be any given point of $\mathfrak{W}$, and let (H) be a generic point of $\mathfrak{W}$ over $K((H^a))$. Since the point $\Gamma(H^a) + \Gamma(H)$ is a generic point of $\mathfrak{V}$ over $K((H^a))$, the relation $\Gamma(H') = \Gamma(H^a) + \Gamma(H)$ determines uniquely a specialization $(H^3) \to (H')$ over the specialization $(H^1, H^2) \to (H^a, H)$ over K, and this point (H') is also a generic point of $\mathfrak{W}$ over $K((H^a))$. Conversely, given a generic point (H') of $\mathfrak{W}$ over $K((H^a))$, the same relation determines uniquely a specialization $(H^2) \to (H)$ over the specialization $(H^1, H^3) \to (H^a, H')$ over K, and the point (H) is also generic over $K((H^a))$. In other words, for any given point (H^a) of $\mathfrak{W}$, the relation $(H^a) + (H) = (H')$ determines a birational correspondence $\Theta_a(H) = (H')$ of the variety $\mathfrak{W}$ onto itself; in fact we have $(H') = \Gamma^{-1}(\Gamma(H^a) + \Gamma(H))$ for any generic (H) over $K((H^a))$ and $(H) = \Gamma^{-1}(\Gamma(H') - \Gamma(H^a))$ for generic (H') over $K((H^a))$. This birational correspondence Θ_a between the normal variety $\mathfrak{W}$ and itself has the property that to each specialization of (H) correspond at most a finite number of specializations of (H'), and vice versa. For, any specialization $(H, H') \to (H^b, H^c)$ of this correspondence Θ_a over K is contained in a specialization $(H^1, H^2, H^3) \to (H^a, H^b, H^c)$ over K, and we have seen above that there is only a finite number of the latter when two of the three points (H^a, H^b, H^c) are given. This means that the birational correspondence Θ_a has no fundamental points on $\mathfrak{W}$, neither for the direct transformation Θ_a nor for the inverse transformation Θ_a^{-1}. Since for a birational transformation, or more generally for a rational transformation of a normal variety, a point is either fundamental or regular ([10], Th. 9 and Th. 10, note the difference in terminology), we conclude that the birational correspondonce Θ_a must be biregular everywhere on $\mathfrak{W}$.

From the biregularity of the correspondence Θ_a for every point (H^a) we can draw some important conclusions. Let (H^c) be a simple point on $\mathfrak{W}$ with the property that it is the only point on $\mathfrak{W}$ which is carried by Γ into the point $\Gamma(H^c)$ on $\mathfrak{V}$; we can take, for example, a generic point of $\mathfrak{W}$ over K. Given any point (H^b), let (H^a) be any point such that $\Gamma(H^c) = \Gamma(H^a) + \Gamma(H^b)$. The birational correspondence Θ_a, determined by the so chosen point (H^a), will then carry the point (H^b) into a point with the image $\Gamma(H^a) + \Gamma(H^b)$ on $\mathfrak{V}$ under the correspondence Γ, which can only be the point (H^c). Since the correspondence Θ_a is biregular and the point (H^c) is simple, it follows that the point (H^b) is also simple. As (H^b) is any point on $\mathfrak{W}$, this shows that the variety $\mathfrak{W}$ is free of singularities. Furthermore, let (H^β) be another point such that $\Gamma(H^\beta) = \Gamma(H^b)$, then the same birational correspondence Θ_a must also carry (H^β), into (H^c), from which it follows, on account of the biregularity of Θ_a, that $(H^\beta) = (H^b)$. This shows that the correspondence Γ between $\mathfrak{W}$ and $\mathfrak{B}$ is one-to-one for all points without exception.

Thus we have shown the existence of an Abelian variety $\mathfrak{W}$ which is in one-to-one birational correspondence without exception with the variety $\mathfrak{B}$; as we have mentioned in our paper [1], this is our original proof of the existence of a Jacobian variety for the curve $\mathfrak{C}$. The underlying idea of the proof is the simple observation that if a variety is homogeneous in the sense that there is a transitive group of rational transformations which are everywhere determined, and if the variety is normal, then it is non-singular. That a variety which is homogeneous in this sense, but not normal, need not be non-singular, is shown by the simple example of a plane cubic curve with a cusp, which is in one-to-one birational correspondence without exception with the projective line. The question naturally arises as to whether the variety $\mathfrak{B}$ is itself also non-singular and is already a Jacobian variety. This question, besides being of some interest in itself, is significant not only on account of the fact that $\mathfrak{B}$ is defined over k, but also on account of the fact, proved in the next section, that for any point (G) in $\mathfrak{B}$ the corresponding variety $\mathfrak{G}$ in $\mathfrak{C}^n$ is defined over the field $k((G))$.

5. Proof that the variety $\mathfrak{B}$ is non-singular. Consider the rational transformation ϕ of $\mathfrak{C}^n$ onto $\mathfrak{B}$, defined over k; if (z) is a generic point of $\mathfrak{B}$ over k, then the inverse image $\phi^{-1}(z)$ is a subvariety $\mathfrak{G}(z)$ of dimension $n - g$ in $\mathfrak{C}^n$, defined over $k((z))$, and for any specialization $(z) \rightarrow (\zeta)$ over k, the variety $\mathfrak{G}(z)$, considered as a positive cycle of dimension $n - g$, has a uniquely determined specialization $\mathfrak{G}(\zeta)$, which is a multiple of the variety $\phi^{-1}(\zeta)$. In the terminology introduced in our paper [1], the rational transformation ϕ induces an involutional system of positive cycles $|\mathfrak{G}(z)|$ on $\mathfrak{C}^n$, and the variety $\mathfrak{B}$ is the associated variety of this involutional system; according to the main theorem in that paper ([1], p. 258), the variety $\mathfrak{B}$ is simple at every point (ζ) for which the positive cycle $\mathfrak{G}(\zeta)$ coincides with the variety $\phi^{-1}(\zeta)$ and contains a simple point in $\mathfrak{C}^n$. In order to prove that the variety $\mathfrak{B}$ is non-singular, it is therefore sufficient to show that (1) the variety $\phi^{-1}(\zeta)$ has the same degree (as a subvariety in $\mathfrak{S}_t$) for every point (ζ) in $\mathfrak{B}$, i. e. the degree of the variety $\mathfrak{G}_{\mathfrak{P}}$ is the same for every divisor class $\mathfrak{P}$ of $\mathfrak{C}$, and (2) the variety $\mathfrak{C}^n$ is non-singular.

We shall show that the degree of the variety $\mathfrak{G}_{\mathfrak{P}}$, for any divisor class $\mathfrak{P}$ of degree n, is equal to h^{n-g}. By definition, the degree of $\mathfrak{G}_{\mathfrak{P}}$ is the number of intersections of $\mathfrak{G}_{\mathfrak{P}}$ with $n-g$ independent generic hyperplanes $\sum_{j=0}^{t} w^i{}_j Y_j = 0$ $(i=1,\cdots,n-g)$ in $\mathfrak{S}_t$ over K, where K is a field over which the divisor class $\mathfrak{P}$ is rational. Consider the $n-g$ independent generic hyperplanes $\sum_{j=0}^{m} v^i{}_j X_j = 0$ $(i=1,\cdots,n-g)$ in $\mathfrak{S}_m$ over K, and for each i $i=1,\cdots,n-g$, let $(\xi^{i,1}),\cdots,(\xi^{i,h})$ be the h intersections of the i-th hyperplane with the curve $\mathfrak{C}$; let $\mathfrak{p}^\lambda$, where $\lambda=(m_1,\cdots,m_{n-g})$ runs through all the h^{n-g} possible ordered sets of $n-g$ positive integers $m_i \leq h$, be the positive divisor of the class $\mathfrak{P}$ which contains the $n-g$ points (ξ^{i,m_i}), $i=1,\cdots,n-g$. It is then easily seen that the h^{n-g} points (p^λ) in $\mathfrak{G}_{\mathfrak{P}}$ corresponding to the h^{n-g} positive divisors $\mathfrak{p}^\lambda$ constitute all the intersections of $\mathfrak{G}_{\mathfrak{P}}$ with the $n-g$ special hyperplanes $\sum_{j=0}^{t} \Omega_j(v^i) Y_j = 0$ $(i=1,\cdots,n-g)$ in $\mathfrak{S}_t$. In fact, let (q) be the point in $\mathfrak{C}^n$ corresponding to a positive divisor $\mathfrak{q}$ of degree n consisting of the n points $(\eta^1),\cdots,(\eta^n)$, then we have $\sum_{j=0}^{t} \Omega_j(v^i) q_j = \prod_{k=1}^{n} \left(\sum_{j=0}^{m} v^i{}_j \eta^k{}_j \right)$, $i=1,\cdots,n-g$. Hence, all the h^{n-g} points (p^λ) of $\mathfrak{G}_{\mathfrak{P}}$ lie on the $n-g$ special hyperplanes; and conversely, if a point of $\mathfrak{G}_{\mathfrak{P}}$ lies on these $n-g$ special hyperplanes, then the corresponding positive divisors must contain at least one point from each of the $n-g$ sets $(\xi^{i,1}),\cdots,(\xi^{i,h})$, $i=1,\cdots,n-g$, and hence must be one of the h^{n-g} positive divisors $\mathfrak{p}^\lambda$. Therefore, in order to prove our assertion, we need only show that each point (p^λ) is a simple intersection of $\mathfrak{G}_{\mathfrak{P}}$ with these $n-g$ special hyperplanes. Since the point (p^λ) is a generic point of $\mathfrak{G}_{\mathfrak{P}}$ over K (it has the dimension $n-g$ over K), it is a simple point of $\mathfrak{G}_{\mathfrak{P}}$; hence, we need only show that the linear variety of dimension $t-n+g$ in $\mathfrak{S}_t$ defined by the $n-g$ special hyperplanes is transversal to the tangent linear variety of $\mathfrak{G}_{\mathfrak{P}}$ at the point (p^λ). As we shall from now on only deal with the one divisor $\mathfrak{p}^\lambda$ or point (p^λ), for an arbitrary but fixed λ, we can drop the superscript λ and simply write $\mathfrak{p}$ or (p).

Let $(x^1),\cdots,(x^n)$ be the n points contained in the divisor $\mathfrak{p}$, where we can assume that the first $n-g$ points are the points (ξ^{i,m_i}), $i=1,\cdots,n-g$. Since $\mathfrak{p}$ is a generic positive divisor of the class $\mathfrak{P}$ over K, all the n points $(x^1),\cdots,(x^n)$ are generic points of the curve $\mathfrak{C}$ over K and distinct from each other; moreover, any $n-g$ of these points are independent with respect to each other over K and determine the remaining g points uniquely. Hence, there is an automorphism of the field $K((x^1),\cdots,(x^n))$ over the field $K((p))$ which carries any set of $n-g$ of the n points $(x^1),\cdots,(x^n)$ into any other such set. If $\sum_{j=0}^{m} v_j X_j = 0$ is a hyperplane in $\mathfrak{S}_m$ containing one of the n points $(x^1), \cdots, (x^n)$, then the hyperplane $\sum_{j=0}^{t} \Omega_j(v) Y_j = 0$ in $\mathfrak{S}_t$ evidently contains the point (p); we shall call such a hyperplane in $\mathfrak{S}_t$ a derived hyperplane of the hyperplane in $\mathfrak{S}_m$. The $n-g$ special hyperplanes $\sum_{j=0}^{t} \Omega_j(v^i) Y_j = 0$ $(i = 1, \cdots, n-g)$ considered above are all derived hyperplanes, since each hyperplane $\sum_{j=0}^{m} v^i{}_j X_j = 0$ in $\mathfrak{S}_m$ contains the point (x^i), $i = 1, \cdots, n-g$. Moreover, for each $i = 1, \cdots, n-g$, the hyperplane $\sum_{j=0}^{m} v^i{}_j X_j = 0$ is a generic hyperplane over $K((x^1), \cdots, (x^n))$ of the system of all hyperplanes in $\mathfrak{S}_m$ which contain the point (x^i); and the $n-g$ hyperplanes $\sum_{j=0}^{m} v^i{}_j X_j = 0$ $(i = 1, \cdots, n-g)$, considered as generic hyperplanes of the $n-g$ corresponding systems over $K((x^1), \cdots, (x^n))$, are independent with respect to each other over $K((x^1), \cdots, (x^n))$.

Consider now the set of all derived hyperplanes in $\mathfrak{S}_t$; it can be easily seen that the point (p) is the only point in $\mathfrak{S}_t$ common to all these hyperplanes. For, each point in $\mathfrak{S}_t$ corresponds to a form of degree n in (V), and the form $\sum_{j=0}^{t} p_j \Omega_j(V) = \prod_{i=1}^{n} (\sum_{j=0}^{m} x^i{}_j V_j)$ is obviously the only form of degree n in (V) which vanishe for all the solutions of any one of the n linear equations $\sum_{j=0}^{m} x^i{}_j V_j = 0$, $i = 1, \cdots, n$. Thus there exist t linearly independent derived hyperplanes in $\mathfrak{S}_t$, and consequently there is a set of $n-g$ hyperplanes $\sum_{j=0}^{m} v^i{}_j X_j = 0$ $(i = 1, \cdots, n-g)$ in $\mathfrak{S}_m$, each containing a point (x^{l_i}) of the divisor $\mathfrak{p}$, such that the $n-g$ derived hyperplanes $\sum_{j=0}^{t} \Omega_j(v^i) Y_j = 0$ $(i = 1, \cdots, n-g)$ define a linear variety of dimension $t-n+g$ transversal to the tangent linear variety of $\mathfrak{G}_{\mathfrak{P}}$ at (p). The $n-g$ points (x^{l_i}), $i = 1, \cdots, n-g$, are distinct; for, otherwise there would be infinitely many positive divisors in the class $\mathfrak{P}$ which contain all the (less than $n-g$ distinct) points (x^{l_i}), $i = 1, \cdots, n-g$, and the corresponding points in $\mathfrak{G}_{\mathfrak{P}}$ would then be contained in the $n-g$ derived hyperplanes $\sum_{j=0}^{t} \Omega_j(v^i) Y_j = 0$ $(i = 1, \cdots, n-g)$, and consequently these derived hyperplanes could not define a linear variety transversal to the tangent linear variety of $\mathfrak{G}_{\mathfrak{P}}$ at (p). Thus, for each $i = 1, \cdots, n-g$, the set

(ν^i) is a solution of the linear equation $\sum_{j=0}^{m} x^{l_i}{}_j V_j = 0$. Let $(\bar{v}^i)$, $i = 1, \cdots, n - g$, be respectively generic solutions of the linear equations $\sum_{j=0}^{m} x^{l_i}{}_j V_j = 0$, $i = 1, \cdots, n - g$, over $K((x^1), \cdots, (x^n))$, independent with respect to each other over $K((x^1), \cdots, (x^n))$; then the $n - g$ derived hyperplanes $\sum_{j=0}^{t} \Omega_j(\bar{v}^l) Y_j = 0$ $(i = 1, \cdots, n - g)$ define a linear variety which is also transversal to the tangent linear variety of $\mathfrak{G}_{\mathfrak{P}}$ at (p). For, the set of $n - g$ hyperplanes $\sum_{j=0}^{t} \Omega_j(\nu^i) Y_j = 0$ $(i = 1, \cdots, n - g)$ is a specialization over $K((x^1), \cdots, (x))$ of the set of $n - g$ hyperplanes $\sum_{j=0}^{t} \Omega_j(\bar{v}^i) Y_j = 0$ $(i = 1, \cdots, n - g)$, and the former set defines a linear variety transversal to the tangent linear variety of $\mathfrak{G}_{\mathfrak{P}}$ at (p). Now, there is an automorphism of the field $K((x^1), \cdots, (x^n))$ over $K((p))$, which carries the $n - g$ points $(x^{l_1}), \cdots, (x^{l_{n-g}})$ into the $n - g$ points $(x^1), \cdots, (x^{n-g})$ respectively, and this automorphism can be extended into an isomorphism between the fields $K((x^1), \cdots, (x^n), (\bar{v}^1), \cdots, (\bar{v}^{n-g}))$ and $K((x^1), \cdots, (x^n), (v^1), \cdots, (v^{n-g}))$ by the correspondence $(\bar{v}^i) \leftrightarrow (v^i)$, $i = 1, \cdots, n - g$. In this isomorphism the derived hyperplanes $\sum_{j=0}^{t} \Omega_j(\bar{v}^i) Y_j = 0$ $(i = 1, \cdots, n - g)$ are carried over into the derived hyperplanes $\sum_{j=0}^{t} \Omega_j(v^i) Y_j = 0$ $(i = 1, \cdots, n - g)$; since the former set defines a linear variety of $\mathfrak{G}_{\mathfrak{P}}$ at (p), the same must be true of the latter. This completes the proof of the assertion that the degree of $\mathfrak{G}_{\mathfrak{P}}$ is h^{n-g}.

Next, we shall show that the variety $\mathfrak{C}^n$ is non-singular. We observe first that in a similar manner as above, by taking (p) to be a generic point of $\mathfrak{C}^n$ over k and considering n instead of $n - g$ derived hyperplanes in $\mathfrak{S}_t$, we can show that the degree of the variety $\mathfrak{C}^n$ is equal to h^n. In fact, the proof in this case is somewhat simpler than the above, for there it is not necessary to consider any automorphisms of the field $k((x^1), \cdots, (x^n))$. Now, let (q) be any point in $\mathfrak{C}^n$ and let the corresponding positive divisor $\mathfrak{q}$ be composed of the n (not necessarily distinct) points $(\eta^1), \cdots, (\eta^n)$ in $\mathfrak{C}$. Consider the n linear equations $\sum_{j=0}^{m} \eta^i{}_j V_j = 0$ $(i = 1, \cdots, n)$; let (ρ^i) $(i = 1, \cdots, n)$ be n generic solutions of these n equations, respectively, over $k((\eta^1), \cdots, (\eta^n))$, independent with respect to each other over $k((\eta^1), \cdots, (\eta^n))$; for each $i = 1, \cdots, n$, let $(\eta^{i,1}) = (\eta^i), (\eta^{i,2}), \cdots, (\eta^{i,h})$

be the h intersections of the hyperplane $\sum_{j=0}^{m} \rho^{i}{}_{j} X_j = 0$ with the curve $\mathfrak{C}$. Since $\mathfrak{C}$ is non-singular (and the hyperplanes have been chosen as above), the points $(\eta^{i,j})$ $(i = 1, \cdots, n; j = 2, \cdots, h)$ are all distinct from each other and different from the points $(\eta^{i,1})$ $(i = 1, \cdots, n)$. It follows then that all the h^n positive divisors $\mathfrak{q}^{\lambda} = ((\eta^{1,m_1}), \cdots, (\eta^{n,m_n}))$, where the $\lambda = (m_1, \cdots, m_n)$ runs through all the h^n possible ordered sets of n positive integers $m_i \leq h$ $(i = 1, \cdots, n)$, are distinct from each other. It is easily seen that the h^n corresponding distinct points (q^{λ}) are precisely the intersections of $\mathfrak{C}^n$ with the linear variety of dimension $t - n$ in $\mathfrak{S}_t$ defined by the hyperplanes $\sum_{j=0}^{t} \Omega_j(\rho^i) Y_j = 0$ $(i = 1, \cdots, n)$. Since the degree of $\mathfrak{C}^n$ is h^n, it follows that each point (q^{λ}) and hence in particular the point (q) is a simple intersection of $\mathfrak{C}^n$ with this linear variety. This shows that (q) is a simple point in $\mathfrak{C}^n$, and as (q) is an arbitrary point in $\mathfrak{C}^n$, this means that the variety $\mathfrak{C}^n$ is non-singular.

Finally, we observe that in exactly the same manner as above, we can show that there exists a linear variety of dimension $t - n + g$ in $\mathfrak{S}_t$ which intersects simply any given $\mathfrak{G}_{\mathfrak{P}}$ at any given point (q); it follows then that the variety $\mathfrak{G}_{\mathfrak{P}}$ is also non-singular for every $\mathfrak{P}$.

6. The canonical homomorphism. Consider again the rational transformation ϕ of $\mathfrak{C}^n$ onto $\mathfrak{B}$; we shall show that it generates in a natural way (by linear extension, in the terminology of Weil) a rational homomorphism Φ of $\mathfrak{G}_a(\mathfrak{C})$ onto the Abelian variety $\mathfrak{B}$ with the subgroup $\mathfrak{G}_l(\mathfrak{C})$ as the kernel, and that the so defined homomorphism Φ is defined over k and has the "universal mapping" property. We observe first that if $\mathfrak{p}^1, \cdots, \mathfrak{p}^s$ and $\mathfrak{q}^1, \cdots, \mathfrak{q}^s$ are two sets of positive divisors of degree n in $\mathfrak{C}$, then we have the relation $\sum_{i=1}^{s} \phi(p^i) = \sum_{i=1}^{s} \phi(q^i)$ if and only if $\sum_{i=1}^{s} \mathfrak{p}^i \sim \sum_{i=1}^{s} \mathfrak{q}^i$; in fact, let $\mathfrak{o}$ be any divisor in the class $\mathfrak{D}$ and let $\mathfrak{p}^0$ and $\mathfrak{q}^0$ be respectively positive divisors such that the relations $\sum_{i=1}^{s} \mathfrak{p}^i \sim (s-1)\mathfrak{o} + \mathfrak{p}^0$ and $\sum_{i=1}^{s} \mathfrak{q}^i \sim (s-1)\mathfrak{o} + \mathfrak{q}^0$ hold respectively, then we have, by definition, the relations $\phi(p^0) = \sum_{i=1}^{s} \phi(p^i)$ and $\phi(q^0) = \sum_{i=1}^{s} \phi(q^i)$, and it is clear that $\phi(p^0) = \phi(q^0)$ if and only if $\sum_{i=1}^{s} \mathfrak{p}^i \sim \sum_{i=1}^{s} \mathfrak{q}^i$. We shall call this relation the compatibility condition for the transformation ϕ. Now, let $(\xi^1), \cdots, (\xi^{n-1})$ be $n-1$ independent generic points over k in $\mathfrak{C}$, and let (x) be a generic point of $\mathfrak{C}$ over $k((\xi^1), \cdots, (\xi^{n-1}))$; let (y) be the point in $\mathfrak{C}^n$ representing the divisor consisting of the points $(\xi^1), \cdots, (\xi^{n-1})$ and (x), and let ϕ' be the rational transformation of $\mathfrak{C}$ into $\mathfrak{B}$ determined by the correspondence $(x) \to \phi(y)$, which is defined over $k((\xi^1), \cdots, (\xi^{n-1}))$. For any divisor $\mathfrak{p} = \sum_{i=1}^{s} n_i(x^i)$ in $\mathfrak{G}_a(\mathfrak{C})$, we define $\Phi(\mathfrak{p}) = \sum_{i=1}^{s} n_i\phi'(x^i)$; it is clear that the so defined mapping Φ is a rational homomorphism of $\mathfrak{G}_a(\mathfrak{C})$ into $\mathfrak{B}$, defined over $k((\xi^1), \cdots, (\xi^{n-1}))$. It can be easily seen from the compatibility condition that the kernel of the homomorphism Φ is precisely the subgroup $\mathfrak{G}_l(\mathfrak{C})$, and that this definition of Φ is actually independent of the choice of the points $(\xi^1), \cdots, (\xi^{n-1})$. If $(\eta^1), \cdots, (\eta^{n-1})$ are $n-1$ independent generic points of $\mathfrak{C}$ over k, independent with respect to the points $(\xi^1), \cdots, (\xi^{n-1})$ over k, then the rational homomorphism Φ is also defined over $k((\eta^1), \cdots, (\eta^{n-1}))$; it follows then that Φ must be defined over

$$k((\xi^1), \cdots, (\xi^{n-1})) \cap k((\eta^1), \cdots, (\eta^{n-1})) = k.$$

In fact, if $\mathfrak{p}$ is any divisor in $\mathfrak{C}$, rational over any extension K of k, then the point $\Phi(\mathfrak{p})$ is rational over both $K((\xi^1), \cdots, (\xi^{n-1})$ and $K((\eta^1), \cdots, (\eta^{n-1}))$ and hence must be rational over

$$K((\xi^1), \cdots, (\xi^{n-1})) \cap K((\eta^1), \cdots, (\eta^{n-1})) = K.$$

Consider now a rational homomorphism Ψ of $\mathfrak{G}_a(\mathfrak{C})$ into an Abelian variety $\mathfrak{A}$, defined over an extension K of k; let (z) be a generic point of $\mathfrak{B}$ over K, and let (q) be a generic point of the variety $\phi^{-1}(z)$ over $K((z))$ and $\mathfrak{q}$ be the positive divisor represented by (q). Since the divisor class $\mathfrak{D}$ is rational over K, it contains a divisor $\mathfrak{o}$ which is rational over K; the point $\Psi(\mathfrak{q}-\mathfrak{o})$ is then rational over $K((q), (z))$. The correspondence $(q) \to \Psi(\mathfrak{q}-\mathfrak{o})$ then determines over $K((z))$ a rational transformation of $\phi^{-1}(z)$ into $\mathfrak{A}$; since $\phi^{-1}(z)$ is birationally equivalent to a projective space, it follows from a result of Weil ([9], Th. 8, Cor.) that this rational transformation must be a constant, so that the point $\Psi(\mathfrak{q}-\mathfrak{o})$ is rational over $K((z))$. The correspondence $(z) \to \Psi(\mathfrak{q}-\mathfrak{o})$ then determines a rational transformation Ψ_1 of $\mathfrak{B}$ into $\mathfrak{A}$, defined over K, which carries the unit element in $\mathfrak{B}$ into the unit element in $\mathfrak{A}$; according to another result of Weil ([9], Th. 9), this rational transformation Ψ_1 must be a homomorphism of $\mathfrak{B}$ into $\mathfrak{A}$. Let $\mathfrak{p} = \sum_{i=1}^{s} n_i(x^i)$ be again any divisor in $\mathfrak{G}_a(\mathfrak{C})$, and for each

$i=1,\cdots,n$, let $\mathfrak{y}^i$ be the positive divisor of degree n consisting of the points $(\xi^1),\cdots,(\xi^{n-1})$ and (x^i); since $\sum_{i=1}^{s} n_i = 0$, we have evidently the equation $\mathfrak{p} = \sum_{i=1}^{s} n_i(\mathfrak{y}^i - \mathfrak{o})$. We have then

$$\Psi(\mathfrak{p}) = \sum_{i=1}^{s} n_i \Psi(\mathfrak{y}^i - \mathfrak{o}) = \sum_{i=1}^{s} n_i \Psi_1(\phi'(x^i)) = \Psi_1(\sum_{i=1}^{s} n_i \phi'(x^i)) = \Psi_1 \Phi(\mathfrak{p}).$$

Thus we have shown that Φ is the canonical homomorphism, and this completes the proof that $\mathfrak{V}$ is the Jacobian variety of the curve $\mathfrak{C}$.

Once we have obtained the canonical homomorphism Φ, we can derive the rational transformation ϕ from it by the formula $\phi(q) = \Phi(\mathfrak{q} - \mathfrak{o})$, and this formula holds for every positive integer $n > 2g - 2$, provided there exists a rational divisor $\mathfrak{o}$ over k of degree n. In fact, in order that the rational transformation ϕ be defined over k, it is sufficient that $\mathfrak{o}$ is contained in a class $\mathfrak{O}$ such that the variety $\mathfrak{G}_{\mathfrak{O}}$ is defined over k. In general, we shall say that a divisor class $\mathfrak{P}$ in $\mathfrak{D}_n$ $(n > 2g - 2)$ is rationally defined over a field K if the variety $\mathfrak{G}_{\mathfrak{P}}$ is defined over K; and an arbitrary divisor class is said to be rationally defined over K if it can be expressed as the difference of two divisor classes which are rationally defined over K. It is clear that a rational divisor class over K is also rationally defined over K, but the converse of this statement is not generally true, as can be shown by examples. It can be easily shown that, for any positive integer n, if there exists a rationally defined divisor class $\mathfrak{O}$ over k of degree n, then the formula $\phi(q) = \Phi(\mathfrak{q} - \mathfrak{o})$ determines a rational transformation ϕ of $\mathfrak{C}^n$ into $\mathfrak{V}$, defined over k. We maintain now that, conversely, if there is a rational transformation of $\mathfrak{C}^n$ into $\mathfrak{V}$, defined over k, which generates by linear extension the canonical homomorphism Φ, then there exists a rationally defined divisor class over k of degree n in $\mathfrak{C}$. In fact, for every positive integer s, the rational transformation ϕ induces a rational transformation ϕ_s of $\mathfrak{C}^{ns}$ into $\mathfrak{V}$, determined by the formula $\phi_s(p) = \sum_{i=1}^{s} \phi(p^i)$, where (p) is any point in $\mathfrak{C}^{ns}$ and $(p^1),\cdots,(p^s)$ are s points in $\mathfrak{C}^n$ such that we have $\mathfrak{p} = \sum_{i=1}^{s} \mathfrak{p}^i$ for the corresponding positive divisors, the particular choice of the points $(p^1),\cdots,(p^s)$ being immaterial on account of the compatibility condition. It is clear that the rational transformation ϕ_s is defined over k, and that, again by virtue of the compatibility condition, we have the equation $\phi_s^{-1}(\phi^s(p)) = \mathfrak{G}_{\mathfrak{P}}$, where $\mathfrak{P}$ is the divisor class determined by $\mathfrak{p}$. It is also clear that for sufficiently large s, say $s \geq g$, the rational transformation ϕ_s is onto; if $\mathfrak{P}_s$ is the divisor class of degree ns such that $\phi_s^{-1}(0) = \mathfrak{G}_{\mathfrak{P}_s}$, then $\mathfrak{P}_s$ is evidently rationally defined over k. The divisor class $\mathfrak{P}_{s+1} - \mathfrak{P}_s$, for any sufficiently large s, is then a rationally defined divisor class over k of degree n. In case $n = 1$, we have the "canonical function" of Weil, and our result offers an answer to the question raised by Weil ([9], p. 68) as to when a field of definition is "complete" for a curve.

Finally, we shall add a remark concerning the special case when the curve $\mathfrak{C}$ contains a rational divisor over k of degree 1; such is always the case, for example, when the field k is finite, as has been shown by F. K. Schmidt ([6], p. 27). We shall show that in this case every rationally defined divisor class over any field K (containing k) is also rational over K, so that a divisor class $\mathfrak{P}$ is rational over the field $k((G_{\mathfrak{P}}))$. To prove this, let $\mathfrak{o}$ be a divisor of degree 1, rational over k, and let $\mathfrak{O}$ be the class determined by $\mathfrak{o}$. Let $\mathfrak{P}$ be a rationally defined class of degree n over K, and let (p) be a generic point of $\mathfrak{G}_{\mathfrak{P}}$ over K. Then the corresponding positive divisor $\mathfrak{p}$ is rational over $K((p))$, and consequently the class $\mathfrak{P}$ is rational over $K((p))$. There exists an integer r, such that the class $\mathfrak{Q} = \mathfrak{P} - r\mathfrak{O}$ of degree $n - r$ has exactly the dimension 0 and hence contains exactly one positive divisor $\mathfrak{q}$. Since the class $\mathfrak{Q}$ is rational over $K((p))$, the divisor $\mathfrak{q}$ is rational over $K((p))$. Now, let (p') be another generic point of $\mathfrak{G}_{\mathfrak{P}}$ over K, independent with respect to (p) over K; then, by a similar argument, we can conclude that $\mathfrak{q}$ is rational over $K((p'))$. It follows then that the positive divisor $\mathfrak{q}$ is rational over $K((p)) \cap K((p')) = K$, and consequently the class $\mathfrak{Q}$ is rational over K. This implies that the class $\mathfrak{P} = \mathfrak{Q} + r\mathfrak{O}$ is also rational over K.

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