proof that every cubic is projectively equivalent to one of the type Y^2Z = cubic in X and Z.

(d) If the curve F has degree n, and i flexes (all ordinary), and δ simple nodes, and k cusps, and no other singularities, then $i + 6\delta + 8k = 3n(n-2)$. This is one of "Plücker's formulas" (See Walker's "Algebraic Curves" for the others).

CHAPTER 8

RIEMANN-ROCH THEOREM

Throughout this chapter, C will be an irreducible projective curve, $f: X \longrightarrow C$ the birational morphism from the non-singular model X onto C, $K = k(C) = K(X)$ the function field, as in Chapter 7, §5. The points $P \in X$ will be identified with the places of K; ord_P denotes the corresponding order function on K.

1. Divisors

A divisor on X is a formal sum $D = \sum_{P \in X} n_P P$, $n_P \in \mathbf{Z}$, and $n_P = 0$ for all but a finite number of P. The divisors on X form an abelian group - it is just the free abelian group on the set X (Chapter 2, §11).

The degree of a divisor is the sum of its coefficients: $\deg(\Sigma n_P P) = \Sigma n_P$. Clearly $\deg(D+D') = \deg(D) + \deg(D')$. $D = \Sigma n_P P$ is said to be effective (or positive) if each $n_P \geq 0$, and we write $\Sigma n_P P \succ \Sigma m_P P$ if each $n_P \geq m_P$.

Suppose C is a plane curve of degree n, and G

is a plane curve not containing C as a component. Define the <u>divisor of</u> G, $\operatorname{div}(G)$, to be $\sum_{P\in X} \operatorname{ord}_P(G)P$, where $\operatorname{ord}_P(G)$ is defined as in Chapter 7, §5. By Chapter 7, Prop. 2, $\sum_{P\in X} \operatorname{ord}_P(G) = \sum_{Q\in C} I(Q, C \cap G)$. By Bezout's Theorem, $\operatorname{div}(G)$ is a divisor of degree mn, where m is the degree of G. Note that $\operatorname{div}(G)$ contains more information than the intersection cycle $G.C$.

For any non-zero $z \in K$, define the <u>divisor of</u> z, $\operatorname{div}(z)$, to be $\sum_{P\in X} \operatorname{ord}_P(z)P$. Since z has only a finite number of poles and zeros (Problem 4-17), $\operatorname{div}(z)$ is a well-defined divisor. If we let $(z)_0 = \sum_{\operatorname{ord}_P(z)>0} \operatorname{ord}_P(z)P$, the <u>divisor of zeros of</u> z, and $(z)_\infty = \sum_{\operatorname{ord}_P(z)<0} -\operatorname{ord}_P(z)P$, the divisor of poles of z, then $\operatorname{div}(z) = (z)_0 - (z)_\infty$. Note that $\operatorname{div}(zz') = \operatorname{div}(z) + \operatorname{div}(z')$, and $\operatorname{div}(z^{-1}) = -\operatorname{div}(z)$.

PROPOSITION 1. <u>For any non-zero</u> $z \in K$, $\operatorname{div}(z)$ <u>is a divisor of degree zero</u>. <u>A rational function has the same number of zeros as poles, if they are counted properly</u>.

<u>Proof</u>: Take C to be a plane curve of degree n. Let $z = g/h$, g,h forms of the same degree in $\Gamma_h(C)$; say g,h are residues of forms G,H of degree m in $k[X,Y,Z]$. Then $\operatorname{div}(z) = \operatorname{div}(G) - \operatorname{div}(H)$, and we have seen that $\operatorname{div}(G)$ and $\operatorname{div}(H)$ have degree mn.

COROLLARY 1. <u>Let</u> $0 \neq z \in K$. <u>Then the following are equivalent</u>: (i) $\operatorname{div}(z) \succ 0$ (ii) $z \in k$ (iii) $\operatorname{div}(z) = 0$.

<u>Proof</u>: If $\operatorname{div}(z) \succ 0$, $z \in \mathcal{O}_P(X)$ for all $P \in X$. If $z(P_0) = \lambda_0$ for some P_0, then $\operatorname{div}(z-\lambda_0) \succ 0$ and $\deg(\operatorname{div}(z-\lambda_0)) > 0$, a contradiction, unless $z-\lambda_0 = 0$, i.e. $z \in k$.

COROLLARY 2. <u>Let</u> $z, z' \in K$, <u>both non-zero. Then</u> $\operatorname{div}(z) = \operatorname{div}(z')$ if and only if $z' = \lambda z$ for some $\lambda \in k$.

Two divisors D, D' are said to be <u>linearly equivalent</u> if $D' = D + \operatorname{div}(z)$ for some $z \in K$, in which case we write $D' \equiv D$.

PROPOSITION 2. (1) <u>The relation</u> $\equiv$ <u>is an equivalence relation</u>.

(2) $D \equiv 0$ <u>if and only if</u> $D = \operatorname{div}(z)$, $z \in K$.

(3) <u>If</u> $D \equiv D'$, <u>then</u> $\deg(D) = \deg(D')$.

(4) <u>If</u> $D \equiv D'$ and $D_1 \equiv D_1'$, <u>then</u> $D + D_1 \equiv D' + D_1'$.

(5) <u>Let</u> C <u>be a plane curve. Then</u> $D \equiv D'$ <u>if and only if there are two curves</u> G, G' <u>of the same degree with</u> $D + \operatorname{div}(G) = D' + \operatorname{div}(G')$.

<u>Proof</u>: (1)-(4) are left to the reader. For (5) it suffices to write $z = G/G'$, since $\operatorname{div}(z) = \operatorname{div}(G) - \operatorname{div}(G')$ in this case.

The criterion proved in Chapter 7, §5 for Noether's conditions to hold translates nicely into the language of divisors:

Assume C is a plane curve with only ordinary multiple points. For each $Q \in X$, let $r_Q = m_{f(Q)}(C)$. Define the divisor $E = \sum_{Q \in X} (r_Q - 1)Q$. E is an effective divisor of degree $\sum r_Q(r_Q - 1)$. Any plane curve G such that $\operatorname{div}(G) \succ E$ is called an adjoint of C. From Problem 7-19 it follows that a curve G is an adjoint to C if and only if $m_P(G) \geq m_P(C) - 1$ for every (multiple) point $P \in C$. If C is non-singular, every curve is an adjoint.

RESIDUE THEOREM. Let C, E be as above. Suppose D, D' are effective divisors on X, and $D' \equiv D$. Suppose G is an adjoint of degree m such that $\operatorname{div}(G) = D + E + A$, for some effective divisor A. Then there is an adjoint G' of degree m such that $\operatorname{div}(G') = D' + E + A$.

Proof: Let H, H' be curves of the same degree such that $D + \operatorname{div}(H) = D' + \operatorname{div}(H')$. Then $\operatorname{div}(GH) = \operatorname{div}(H') + D' + E + A \succ \operatorname{div}(H') + E$. Let F be the form defining C. Applyint the criterion of Chapter 7, Prop. 3 to F, H', and GH, we see that Noether's conditions are satisfied at all $P \in C$. By Noether's theorem, $GH = F'F + G'H'$ for some F', G', where $\deg(G') = m$. Then $\operatorname{div}(G') = \operatorname{div}(GH) - \operatorname{div}(H') = D' + E + A$, as desired.

Problems. 8-1. Let $X = C = \mathbb{P}^1$, $k(X) = k(t)$, where $t = X_1/X_2$, X_1, X_2 homogeneous coordinates on $\mathbb{P}^1$.

(a) Calculate $\operatorname{div}(t)$.

(b) Calculate $\operatorname{div}(f/g)$, f, g relatively prime in $k[t]$.

(c) Prove Prop. 1 directly in this case.

8-2. Let $X = C = V(Y^2Z - X(X-Z)(X-\lambda Z)) \subset \mathbb{P}^2$, $\lambda \in k$, $\lambda \neq 0,1$. Let $x = X/Z$, $y = Y/Z \in K$; $K = k(x,y)$. Calculate $\operatorname{div}(x)$, $\operatorname{div}(y)$.

8-3. Let $C = X$ be a non-singular cubic.

(a) Let $P, Q \in C$. Show that $P \equiv Q$ if and only if $P = Q$. (Hint: lines are adjoints of degree 1.)

(b) Let $P, Q, R, S \in C$. Show that $P + Q \equiv R + S$ if and only if the line through P and Q intersects the line through R and S in a point on C (If $P = Q$ use the tangent line).

(c) Let P_0 be a fixed point on C, thus defining an addition $\oplus$ on C (Chapter 5, §6). Show that $P \oplus Q = R$ if and only if $P + Q \equiv R + P_0$. Use this to give another proof of Prop. 4 of Chapter 5.

8-4. Let C be a cubic with a node. Show that for any two simple points P, Q on C, $P \equiv Q$.

8-5. Let C be a non-singular quartic, $P_1, P_2, P_3 \in C$. Let $D = P_1 + P_2 + P_3$. Let L, L' lines such that $L.C = P_1 + P_2 + P_4 + P_5$, $L'.C = P_1 + P_3 + P_6 + P_7$. Suppose these seven points are distinct. Show that D is not linearly equivalent to any other effective divisor. (Hint: Apply the residue theorem to the conic LL'.) Investigate in a similar way other divisors of small degree on quartics with various types of multiple points.

8-6. Let $D(X)$ be the group of divisors on X, $D_0(X)$ the subgroup consisting of divisors of degree zero, and $P(X)$ the subgroup of $D_0(X)$ consisting of divisors of rational functions. Let $C_0(X) = D_0(X)/P(X)$ be the quotient group. It is the <u>divisor class group</u> on X.

(a) If $X = \mathbb{P}^1$, then $C_0(X) = 0$.

(b) Let $X = C$ be a non-singular cubic. Pick $P_0 \in C$, defining $\oplus$ on C. Show that the map from C to $C_0(X)$ which sends P to the residue class of the divisor $P-P_0$ is an isomorphism from $(C, \oplus)$ onto $C_0(X)$.

8-7. When do two curves G, H have the same divisor (C and X are fixed)?

2. The Vector Spaces $L(D)$

Let $D = \Sigma n_P P$ be a divisor on X. D picks out a finite number of points, and assigns integers to them. We want to determine when there is a rational function with poles only at the chosen points, and with poles no "worse" than order n_P at P; if so, how many such functions are there?

Define $L(D)$ to be $\{f \in K \mid \text{ord}_P(f) \geq -n_P$ for all $P \in X\}$, where $D = \Sigma n_P P$. Thus a rational function f belongs to $L(D)$ if $\text{div}(f) + D \succ 0$, or if $f = 0$. $L(D)$ forms a vector space over k. Denote the dimension of $L(D)$ by $\ell(D)$; the next proposition shows that $\ell(D)$ is finite.

PROPOSITION 3. (1) <u>If</u> $D \prec D'$, <u>then</u> $L(D) \subset L(D')$, <u>and</u> $\dim_k(L(D')/L(D)) \leq \deg(D'-D)$.

(2) $L(0) = k$; $L(D) = 0$ <u>if</u> $\deg(D) < 0$.

(3) $L(D)$ <u>is finite dimensional for all</u> D. <u>If</u> $\deg(D) \geq 0$, <u>then</u> $\ell(D) \leq \deg(D) + 1$.

(4) <u>If</u> $D \equiv D'$, <u>then</u> $\ell(D) = \ell(D')$.

<u>Proof</u>: (1): $D' = D + P_1 + \ldots + P_s$, and $L(D) \subset L(D + P_1) \subset \ldots \subset L(D + P_1 + \ldots + P_s)$, so it suffices to show that $\dim(L(D + P)/L(D)) \leq 1$ (Problem 2-49). To prove this, let t be a uniformizing parameter in $\mathcal{O}_P(X)$, and let $r = n_P$ be the coefficient of P in D. Define $\varphi: L(D + P) \longrightarrow k$ by letting $\varphi(f) = (t^{r+1}f)(P)$; since $\text{ord}_P(f) \geq -r-1$, this is well-defined. φ is a linear map, and $\text{Ker}(\varphi) = L(D)$, so φ induces a one-to-one linear map $\overline{\varphi}: L(D + P)/L(P) \longrightarrow k$ which gives the result.

(2): This follows from Prop. 1, Cor. 1, and from Prop. 2 (3).

(3): If $\deg(D) = n \geq 0$, choose $P \in X$, and let $D' = D-(n+1)P$. Then $L(D') = 0$, and by (1), $\dim(L(D)/L(D')) \leq n + 1$, so $\ell(D) \leq n + 1$.

(4): Suppose $D' = D + \text{div}(g)$. Define $\psi: L(D) \longrightarrow L(D')$ by letting $\psi(f) = fg$. ψ is an isomorphism of vector spaces, so $\ell(D) = \ell(D')$.

More generally, for any subset S of X, and any divisor $D = \Sigma n_P P$ on X, define $\deg^S(D) = \sum_{P \in S} n_P$, and $L^S(D) = \{f \in K \mid \text{ord}_P(f) \geq -n_P$ for all $P \in S\}$.

LEMMA 1. <u>If</u> $D \prec D'$, <u>then</u> $L^S(D) \subset L^S(D')$. <u>If</u> S <u>is finite, then</u> $\dim(L^S(D')/L^S(D)) = \deg^S(D'-D)$.

<u>Proof</u>: Proceeding as in Prop. 3, we assume $D' = D + P$, and define $\varphi: L^S(D + P) \longrightarrow k$ the same way. We must show that φ maps $L^S(D + P)$ onto k, i.e. $\varphi \neq 0$, for then $\overline{\varphi}$ is an isomorphism. Thus we need to find an $f \in K$ with $\text{ord}_P(f) = -r-1$, and with

$\text{ord}_Q(f) \geq -n_Q$ for all $Q \in S$. But this is easy, since S is finite (Problem 7-21(b)).

The next proposition is an important first step to calculating the dimensions $\ell(D)$. The proof (See Chevalley's "Algebraic Functions of One Variable", Chap. I.) involves only the field of rational functions.

PROPOSITION 4. Let $x \in K$, $x \notin k$. Let $(x)_0$ be the divisor of zeros of x, and let $n = [K: k(x)]$. Then

(1) $(x)_0$ is an effective divisor of degree n.

(2) There is a constant τ such that $\ell(r(x)_0) \geq rn-\tau$ for all r.

Proof: Let $Z = (x)_0 = \Sigma n_P P$, and let $m = \deg(Z)$. We show first that $m \leq n$.

Let $S = \{P \in X \mid n_P > 0\}$. Choose $v_1,\ldots,v_m \in L^S(0)$ so that the residues $\bar{v}_1,\ldots,\bar{v}_m \in L^S(0)/L^S(-Z)$ form a basis for this vector space (Lemma 1). We will show that $v_1,\ldots,v_m$ are linearly independent over $k(x)$. If not (by clearing denominators and multiplying by a power of x), there would be polynomials $g_i = \lambda_i + xh_i \in k[x]$ with $\Sigma g_i v_i = 0$, not all $\lambda_i = 0$. But then

$\Sigma\lambda_i v_i = -x\Sigma h_i v_i \in L^S(-Z)$, so $\Sigma\lambda_i\bar{v}_i = 0$, a contradiction.

So $m \leq n$. Next we prove (2).

Let $w_1,\ldots,w_n$ be a basis of K over $k(x)$ (Chapter 6, Prop. 9). We may assume that each w_i satisfies an equation $w_i^{n_i} + a_{i1}w_i^{n_i-1} + \ldots = 0$, $a_{ij} \in k[x^{-1}]$ (Problem 1-54). Then $\text{ord}_P(a_{ij}) \geq 0$ if

$P \notin S$. If $\text{ord}_P(w_i) < 0$, $P \notin S$, then $\text{ord}_P(w_i^{n_i}) < \text{ord}_P(a_{ij}w_i^{n_i-j})$, which is impossible (Problem 2-29). It follows that for some $t > 0$, $\text{div}(w_i) + tZ \succ 0$, $i = 1,\ldots,n$. Then $w_i x^{-j} \in L((r+t)Z)$ for $i = 1,\ldots,n$, $j = 0, 1,\ldots,r$. Since the w_i are independent over $k(x)$, and $1, x^{-1},\ldots,x^{-r}$ are independent over k, $\{w_i x^{-j} \mid i = 1,\ldots,n,\ j = 0,\ldots,r\}$ are independent over k. So $\ell((r + t)Z) \geq n(r + 1)$. But $\ell((r + t)Z) = \ell(rZ) + \dim(L((r + t)Z)/L(rZ)) \leq \ell(rZ) + tm$ by Prop. 3(1). Therefore $\ell(rZ) \geq n(r + 1)-tm = rn-\tau$, as desired.

Lastly, since $rn-\tau \leq \ell(rZ) \leq rm + 1$ (Prop. 3(3)), if we let r get arbitrarily large, we see that $n \leq m$.

COROLLARY. The following are equivalent:

(1) C is rational.

(2) X is isomorphic to P^1.

(3) There is an $x \in K$ with $\deg(x)_0 = 1$.

(4) For some $P \in X$, $\ell(P) > 1$.

Proof: (4) says that there is non-constant $x \in L(P)$, so $(x)_\infty = P$. Then $\deg((x)_0) = \deg((x)_\infty) = 1$, so $[K: k(x)] = 1$, i.e. $K = k(x)$ is rational. The rest is easy (See Problem 8-1).

Problems. 8-8*. If $D \prec D'$, then $\ell(D') \leq \ell(D) + \deg(D'-D)$, i.e. $\deg(D) -\ell(D) \leq \deg(D') -\ell(D')$.

8-9. Let $X = P^1$, t as in Problem 8-1. Calculate $L(r(t)_0)$ explicitly, and show that $\ell(r(t)_0) = r + 1$.

8-10. Let $X = C$ be a cubic, x,y as in Problem 8-2.

Let $z = x^{-1}$. Show that $L(r(z)_0) \subset k[x,y]$, and show that $\ell(r(z)_0) = 2r$ if $r > 0$.

8-11*. Let D be a divisor. Show that $\ell(D) > 0$ if and only if D is linearly equivalent to an effective divisor.

8-12. Show that $\deg(D) = 0$ and $\ell(D) > 0$ if and only if $D \equiv 0$.

8-13*. Suppose $\ell(D) > 0$, and let $f \neq 0$, $f \in L(D)$. Show that $f \notin L(D-P)$ for all but a finite number of P. So $\ell(D-P) = \ell(D)-1$ for all but a finite number of P.

3. Riemann's Theorem

If D is a large divisor, $L(D)$ should also be large. Proposition 4 shows this for divisors of a special form.

RIEMANN'S THEOREM. *There is a constant* g *such that* $\ell(D) \geq \deg(D) + 1 - g$ *for all divisors* D. *The smallest such* g *is called the genus of* X (*or of* K, *or* C). g *is a non-negative integer*.

Proof: For each D, let $S(D) = \deg(D) + 1 - \ell(D)$ We want to find g so that $S(D) \leq g$ for all D.

(1) $S(0) = 0$, so $g \geq 0$ if it exists.

(2) If $D \equiv D'$, then $S(D) = S(D')$ (Prop. 2 and Prop. 3).

(3) If $D \prec D'$, then $S(D) \leq S(D')$ (Problem 8-8).

Let $x \in K$, $x \notin k$, let $Z = (x)_0$, and let τ be the smallest integer that works for Prop. 4(2). Since $S(rZ) \leq \tau + 1$ for all r, and since $rZ \prec (r+1)Z$, we deduce from (3) that

(4) $S(rZ) = \tau+1$ for all large $r > 0$.

Let $g = \tau+1$. To finish the proof, it suffices (by (2) and (3)) to show:

(5) For any divisor D, there is a divisor $D' \equiv D$, and an integer $r \geq 0$ such that $D' \prec rZ$.

To prove this, let $Z = \Sigma n_P P$, $D = \Sigma m_P P$. We want $D' = D - \operatorname{div}(f)$, so we need $m_P - \operatorname{ord}_P(f) \leq rn_P$ for all P. Let $y = x^{-1}$, and let $T = \{P \in X \mid m_P > 0$ and $\operatorname{ord}_P(y) \geq 0\}$. Let $f = \prod_{P \in T} (y-y(P))^{m_P}$. Then $m_P - \operatorname{ord}_P(f) \leq 0$ whenever $\operatorname{ord}_P(y) \geq 0$. If $\operatorname{ord}_P(y) < 0$, then $n_P > 0$, so a large r will take care of this.

COROLLARY 1. *If* $\ell(D_0) = \deg(D_0) + 1 - g$, *and* $D \equiv D' \succ D_0$, *then* $\ell(D) = \deg(D) + 1 - g$.

COROLLARY 2. *If* $x \in K$, $x \notin k$, *then* $g = \deg(r(x)_0) - \ell(r(x)_0) + 1$ *for all sufficiently large* r.

COROLLARY 3. *There is an integer* N *such that for all divisors* D *of degree* $> N$, $\ell(D) = \deg(D) + 1 - g$.

Proofs: The first two Corollaries were proved on the way to proving Riemann's Theorem. For the third, choose D_0 such that $\ell(D_0) = \deg(D_0) + 1 - g$, and let $N = \deg(D_0) + g$. Then if $\deg(D) \geq N$, $\deg(D-D_0) + 1 - g > 0$, so by Riemann's Theorem, $\ell(D-D_0) > 0$. Then $D-D_0 + \operatorname{div}(f) \succ 0$ for some f, i.e. $D \equiv D + \operatorname{div}(f) \succ D_0$, and the result follows from Cor. 1.

Examples. (1) $g = 0$ if and only if C is rational.

If C is rational, $g = 0$ by Cor. 2 and Problem 8-9 (or Prop. 5 below). Conversely, if $g = 0$, $\ell(P) > 1$ for any $P \in X$, and the result follows from the corollary to Prop. 4.

(2) $g = 1$ if and only if C is birationally equivalent to a non-singular cubic $(\mathrm{char}(k) \neq 2)$.

For if X is a non-singular cubic, the result follows from Cor. 2, Problems 8-10 and 5-24 (or Prop. 5 below). Conversely, if $g = 1$, then $\ell(P) \geq 1$ for all P. By Prop. 3, $\ell(P) = 1$, and by the above Cor. 1, $\ell(rP) = r$ for all $r > 0$.

Let $1, x$ be a basis for $L(2P)$. Then $(x)_\infty = 2P$, since if $(x)_\infty = P$, C would be rational. So $[K: k(x)] = 2$. Let $1, x, y$ be a basis for $L(3P)$. Then $(y)_\infty = 3P$, so $y \notin k(x)$, so $K = k(x,y)$. Since $1, x, y, x^2, xy, x^3, y^2 \in L(6P)$, there is a relation of the form $ay^2 + (bx+c)y = Q(x)$, Q a polynomial of degree ≤ 3. By calculating ord_P of both sides, we see that $a \neq 0$ and $\deg Q = 3$, so we may assume $a = 1$. Replacing y by $y + \frac{1}{2}(bx+c)$, we may assume $y^2 = \prod_{i=1}^{3}(x-\alpha_i)$. If $\alpha_1 = \alpha_2$, then $(y/(x-\alpha_1))^2 = x-\alpha_3$, so $x, y \in k(y/(x-\alpha_1))$; but then X would be rational, which contradicts the first example. So the α_i are distinct.

It follows that $K = k(C)$, where

$C = V(Y^2Z - \prod_{i=1}^{3}(X-\alpha_i Z))$ is a non-singular cubic.

The usefulness of Riemann's Theorem depends on being able to calculate the genus of a curve. By its definition the genus depends only on the non-singular model, or the function field, so two birationally equivalent curves have the same genus. Since we have a method for finding a plane curve with only ordinary multiple points which is birationally equivalent to a given curve, the following proposition is all that we need:

PROPOSITION 5. <u>Let C be a plane curve with only ordinary multiple points. Let n be the degree of C, $r_P = m_P(C)$. Then the genus g of C is given by the formula</u>

$$g = \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{r_P(r_P-1)}{2} .$$

Proof: By the above Cor. 4, we need to find some "large" divisors D for which we can calculate $\ell(D)$. The Residue Theorem allows us to find all effective divisors linearly equivalent to certain divisors D. These two observations lead to the calculation of g.

We may assume that the line $Z = 0$ intersects C in n distinct points $P_1, \ldots, P_n$. Let F be the form defining C.

Let $E = \sum_{Q \in X} (r_Q - 1)Q$, $r_Q = r_{f(Q)} = m_{f(Q)}(C)$ as in §1. Let $E_m = m\sum_{i=1}^{n} P_i - E$. E_m is a divisor of degree $mn - \sum_{P \in C} r_P(r_P-1)$.

Let $V_m = \{$forms G of degree m such that G is adjoint to $C\}$. Since G is adjoint if and only if $m_P(G) \geq r_P - 1$ for all $P \in C$, may apply Chapter 5, Thm. 1 to calculate the dimension of V_m. We find that

$\dim V_m \geq \frac{(m+1)(m+2)}{2} - \Sigma \frac{r_P(r_P-1)}{2}$, with equality if m is large. (Note that V_m is the vector space of forms, not the projective space of curves.)

Let $\varphi: V_m \longrightarrow L(E_m)$ be defined by $\varphi(G) = G/Z^m \in K$. φ is a linear map, and $\varphi(G) = 0$ if and only if G is divisible by F.

We claim that φ is onto. For if $f \in L(E_m)$, write $f = R/S$, R,S forms of the same degree. Then $\mathrm{div}(RZ^m) \succ \mathrm{div}(S) + E$. By Chapter 7, Prop. 3, there is an equation $RZ^m = AS + BF$. So $R/S = A/Z^m$ in $k(F)$, and so $\varphi(A) = f$. (Note that $\mathrm{div}(A) = \mathrm{div}(RZ^m) - \mathrm{div}(S) \succ E$, so $A \in V_m$.)

It follows that the following sequence of vector spaces is exact:

$$0 \longrightarrow W_{m-n} \xrightarrow{\psi} V_m \xrightarrow{\varphi} L(E_m) \longrightarrow 0$$

where W_{m-n} is the space of all forms of degree m-n, and $\psi(H) = FH$ for $H \in W_{m-n}$.

By Chapter 2, Prop. 7, we may calculate $\dim L(E_m)$, at least for m large. It follows that

$\ell(E_m) = \deg(E_m) + 1 - \left(\frac{(n-1)(n-2)}{2} - \Sigma\frac{r_P(r_P-1)}{2}\right)$ for large m. But since $\deg(E_m)$ increases as m increases, Cor. 3 of Riemann's Theorem applies to finish the proof.

COROLLARY 1. <u>Let</u> C <u>be a plane curve of degree</u> n, $r_P = m_P(C)$, $P \in C$. <u>Then</u> $\frac{(n-1)(n-2)}{2} - \Sigma\frac{r_P(r_P-1)}{2} \geq g$.

<u>Proof</u>: The number on the left is what we called $g^*(C)$ in Chapter 7, §4. We saw there that g^* decreases under quadratic transformations, so Chap. 7, Thm. 2 concludes the proof.

COROLLARY 2. <u>As in Cor. 1, if</u> $\Sigma\frac{r_P(r_P-1)}{2} = \frac{(n-1)(n-2)}{2}$, <u>then</u> C <u>is rational</u>.

COROLLARY 3: (a) <u>With</u> E_m <u>as in the proof of the proposition, any</u> $h \in L(E_m)$ <u>may be written</u> $h = H/Z^m$, <u>where</u> H <u>is an adjoint of degree</u> m.

(b) $\deg(E_{n-3}) = 2g-2$. $\ell(E_{n-3}) \geq g$.

<u>Proof</u>: This follows from the exact sequence constructed in proving the proposition. Note that if $m < n$, then $V_m = L(E_m)$.

<u>Examples</u>. Lines and conics are rational. Non-singular cubics have genus one. Singular cubics are rational. Since a non-singular curve of degree n has genus $\frac{(n-1)(n-2)}{2}$, not every curve is birationally equivalent to a non-singular plane curve. For example, $Y^2XZ = X^4 + Z^4$ has one node, so is of genus 2, and no non-singular plane curve has genus 2.

<u>Problems</u>. 8-14. Calculate the genus of each of the following curves:

(a) $X^2Y^2 - Z^2(X^2 + Y^2)$.

(b) $(X^3 + Y^3)Z^2 + X^3Y^2 - X^2Y^3$

(c) The two curves of Problem 7-12.

(d) $(X^2-Z^2)^2 - 2Y^3Z - 3Y^2Z^2$.

8-15. Let $D = \Sigma n_P P$ be an effective divisor, $S = \{P \in X \mid n_P > 0\}$, $U = X-S$. Show that $L(rD) \subset \Gamma(U,\mathcal{O}_X)$ for all $r \geq 0$.

8-16. Let U be any open set on X, $\emptyset \neq U \neq X$. Then $\Gamma(U,\mathcal{O}_X)$ is infinite dimensional over k.

8-17. Let X,Y be non-singular projective curves, $f: X \longrightarrow Y$ a dominating morphism. Prove that $f(X) = Y$. (Hint: If $P \in Y - f(X)$, then $\tilde{f}(\Gamma(Y-\{P\})) \subset \Gamma(X) = k$; apply Problem 8-16.)

8-18. Show that a morphism from a projective curve X to a curve Y is either constant or surjective; if it is surjective, Y must be projective.

8-19. If $f: C \longrightarrow V$ is a morphism from a projective curve to a variety V, then $f(C)$ is a closed subvariety of V. (Hint: Consider C' = closure of $f(C)$ in V.)

8-20. Let C be the curve of Problem 8-14 (b), and let P be a simple point on C. Show that there is a $z \in \Gamma(C-\{P\})$ with $\mathrm{ord}_P(z) \geq -12$, $z \notin k$.

8-21. Let $C_0(X)$ be the divisor class group of X. Show that $C_0(X) = 0$ if and only if X is rational.

4. Derivations and Differentials

This section contains the algebraic background needed to study differentials on a curve.

Let R be a ring containing k, and let M be an R-module. A derivation of R into M over k is a k-linear map $D: R \longrightarrow M$ such that $D(xy) = xD(y) + yD(x)$ for all $x,y \in R$. It follows that for any $F \in k[X_1,\ldots,X_n]$ and $x_1,\ldots,x_n \in R$,

$D(F(x_1,\ldots,x_n)) = \sum_{i=1}^{n} F_{X_i}(x_1,\ldots,x_n)D(x_i)$. Since all rings will contain k, we will omit the phrase "over k".

LEMMA 2. If R is a domain with quotient field K, and M is a vector space over K, then any derivation $D: R \longrightarrow M$ extends uniquely to a derivation $\tilde{D}: K \longrightarrow M$.

Proof: If $z \in K$, and $z = x/y$, $x,y \in R$, then, since $x = yz$, we must have $Dx = y\tilde{D}Z + zDy$. So $\tilde{D}(z) = y^{-1}(Dx-zDy)$, which shows the uniqueness. If we define $\tilde{D}$ by this formula, it is not difficult to verify that $\tilde{D}$ is a well-defined derivation from K to M.

We want to define differentials of R to be elements of the form $\Sigma x_i dy_i$, $x_i,y_i \in R$; they should behave like the differentials of calculus. This is most easily done as follows:

For each $x \in R$ let $[x]$ be a symbol. Let F be the free R-module on the set $\{[x] \mid x \in R\}$. Let N

be the submodule of F generated by the following sets of elements:

(i) $\{[x+y]-[x]-[y] \mid x,y \in R\}$

(ii) $\{[\lambda x]-\lambda[x] \mid x \in R, \lambda \in k\}$

(iii) $\{[xy]-x[y]-y[x] \mid x,y \in R\}$

Let $\Omega_k(R) = F/N$ be the quotient module. Let dx be the residue of $[x]$ in F/N, and let $d: R \longrightarrow \Omega_k(R)$ be the function which takes x to dx. $\Omega_k(R)$ is an R-module, called the <u>module of differentials</u> of R over k, and $d: R \longrightarrow \Omega_k(R)$ is a derivation.

LEMMA 3. <u>For any</u> R-<u>module</u> M, <u>and any derivation</u> $D: R \longrightarrow M$, <u>there is a unique homomorphism of</u> R-<u>modules</u> $\varphi: \Omega_k(R) \longrightarrow M$ <u>such that</u> $D(x) = \varphi(dx)$ <u>for all</u> $x \in R$.

<u>Proof</u>: If we define $\varphi': F \longrightarrow M$ by $\varphi'(\Sigma x_i[y_i]) = \Sigma x_i D(y_i)$, then $\varphi'(N) = 0$, so φ' induces $\varphi: \Omega_k(R) \longrightarrow M$.

If $x_1,\ldots,x_n \in R$, and $G \in k[X_1,\ldots,X_n]$, then $d(G(x_1,\ldots,x_n)) = \sum_{i=1}^{n} G_{X_i}(x_1,\ldots,x_n)dx_i$. It follows that if $R = k[x_1,\ldots,x_n]$, then $\Omega_k(R)$ is generated (as an R-module) by $dx_1,\ldots,dx_n$.

Likewise, if R is a domain with quotient field K, and $z = x/y \in K$, $x,y \in R$, then $dz = y^{-1}dx - y^{-1}zdy$. In particular, if $K = k(x_1,\ldots,x_n)$, then $\Omega_k(K)$ is a finite-dimensional vector space over K, generated by $dx_1,\ldots,dx_n$.

PROPOSITION 6. (1) <u>Let</u> K <u>be an algebraic function field in one variable over</u> k. <u>Then</u> $\Omega_k(K)$ <u>is a one-dimensional vector space over</u> K.

(2) (Char(k) = 0). <u>If</u> $x \in K$, $x \notin k$, <u>then</u> dx <u>is a basis for</u> $\Omega_k(K)$ <u>over</u> K.

<u>Proof</u>: Let $F \in k[X,Y]$ be an affine plane curve with function field K (Chapter 6, Prop. 12, Cor.) Let $R = k[X,Y]/(F) = k[x,y]$; $K = k(x,y)$. We may assume $F_Y \neq 0$, so F doesn't divide F_Y (since F is irreducible), i.e. $F_Y(x,y) \neq 0$. The above discussion shows that dx and dy generate $\Omega_k(K)$ over K. But $0 = d(F(x,y)) = F_X(x,y)dx + F_Y(x,y)dy$, so $dy = udx$, where $u = -F_X(x,y)F_Y(x,y)^{-1}$. Therefore dx generates $\Omega_k(K)$, so $\dim_K(\Omega_k(K)) \leq 1$.

So we must show that $\Omega_k(K) \neq 0$. By Lemmas 2 and 3, it suffices to find a non-zero derivation $D: R \longrightarrow M$ for some vector space M over K. Let $M = K$, and, for $G \in k[X,Y]$, $\overline{G}$ its image in R, let $D(\overline{G}) = G_X(x,y) - uG_Y(x,y)$. It is left to the reader to verify that D is a well-defined derivation, and that $D(x) = 1$, so $D \neq 0$.

It follows (char(k) = 0) that for any $f,t \in K$, $t \notin k$, there is a unique element $v \in k$ such that $df = vdt$. It is natural to write $v = \frac{df}{dt}$, and call v the <u>derivative</u> of f with respect to t.

PROPOSITION 7. <u>With</u> K <u>as in Proposition 6, let</u> $\mathcal{O}$ <u>be a discrete valuation ring of</u> K, <u>and let</u> t <u>be a uniformizing parameter in</u> $\mathcal{O}$. <u>If</u> $f \in \mathcal{O}$, <u>then</u>

$\frac{df}{dt} \in \mathcal{O}$.

Proof: Using the notation of the proof of Prop. 6, we may assume $\mathcal{O} = \mathcal{O}_P(F)$, $P = (0,0)$ a simple point on F. For $z \in K$, write z' instead of $\frac{dz}{dt}$, t being fixed throughout.

Choose N large enough that $\mathrm{ord}_P(x') \geq -N$, $\mathrm{ord}_P(y') \geq -N$. Then if $f \in R = k[x,y]$, $\mathrm{ord}_P(f') \geq -N$, since $f' = f_X(x,y)x' + f_Y(x,y)y'$.

If $f \in \mathcal{O}$, write $f = g/h$, $g,h \in R$, $h(P) \neq 0$. Then $f' = h^{-2}(hg'-gh')$, so $\mathrm{ord}_P(f') \geq -N$.

We can now complete the proof. Let $f \in \mathcal{O}$. Write $f = \sum_{i<N} \lambda_i t^i + t^N g$, $\lambda_i \in k$, $g \in \mathcal{O}$ (Problem 2-30). Then $f' = \sum i\lambda_i t^{i-1} + gNt^{N-1} + t^N g'$. Since $\mathrm{ord}_P(g') \geq -N$, each term belongs to $\mathcal{O}$, so $f' \in \mathcal{O}$, as required.

Problems. 8-22. Generalize Prop. 6 to function fields in n variables.

8-23. With $\mathcal{O}, t$ as in Prop. 7, let $\varphi: \mathcal{O} \longrightarrow k[[T]]$ be the corresponding homomorphism (Problem 2-32). Show that, for $f \in \mathcal{O}$, φ takes the derivative of f to the "formal derivative" of $\varphi(f)$. Use this to give another proof of Prop. 7, and of the fact that $\Omega_k(K) \neq 0$ in Prop. 6.

5. Canonical Divisors

Let C be a projective curve, X its non-singular model, K their function field as before. We let $\Omega = \Omega_k(K)$ be the space of differentials of K over k; elements $\omega \in \Omega$ may also be called *differentials on* X, or on C.

Let $\omega \in \Omega$, $\omega \neq 0$, and let $P \in X$ be a place. We define the *order* of ω at P, $\mathrm{ord}_P(\omega)$, as follows: Choose a uniformizing parameter t in $\mathcal{O}_P(X)$, write $\omega = f\,dt$, $f \in K$, and let $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$. To see that this is well-defined, suppose u were another uniformizing parameter, and $f\,dt = g\,du$, then $f/g = \frac{du}{dt} \in \mathcal{O}_P(X)$ by Prop. 7, and likewise $g/f \in \mathcal{O}_P(X)$, so $\mathrm{ord}_P(f) = \mathrm{ord}_P(g)$.

If $0 \neq \omega \in \Omega$, the *divisor* of ω, $\mathrm{div}(\omega)$, is defined to be $\sum_{P\in X} \mathrm{ord}_P(\omega)P$. In Prop. 8 we shall show that only finitely many $\mathrm{ord}_P(\omega) \neq 0$ for a given ω, so that $\mathrm{div}(\omega)$ is a well-defined divisor.

Let $W = \mathrm{div}(\omega)$. W is called a *canonical divisor*. If ω' is another non-zero differential in Ω, then $\omega' = f\omega$, $f \in K$, so $\mathrm{div}(\omega') = \mathrm{div}(f) + \mathrm{div}(\omega)$, and $\mathrm{div}(\omega') \equiv \mathrm{div}(\omega)$. Conversely if $W' \equiv W$, say $W' = \mathrm{div}(f) + W$, then $W' = \mathrm{div}(f\omega)$. So the canonical divisors form an equivalence class under linear equivalence. In particular, all canonical divisors have the same degree.

PROPOSITION 8. *Assume C is a plane curve of degree $n \geq 3$ with only ordinary multiple points. Let $E = \sum_{Q\in X} (r_Q-1)Q$, as in §1. Let G be any plane curve of degree $n-3$. Then $\mathrm{div}(G)-E$ is a canonical divisor. (If $n = 3$, $\mathrm{div}(G) = 0$.)*

Proof: Choose coordinates X,Y,Z for P^2 in such a way that: $Z.C = \sum_{i=1}^{n} P_i$, P_i distinct; $(1,0,0) \notin C$; and no tangent to C at a multiple point passes through $(1,0,0)$. Let $x = X/Z$, $y = Y/Z \in K$. Let F be the form defining C, and let $f_x = F_X(x,y,1)$, $f_y = F_Y(x,y,1)$.

Let $E_m = m\sum_{i=1}^{n} P_i - E$.

Let $\omega = dx$. Since divisors of the form $\operatorname{div}(G) - E$, $\deg(G) = n-3$, are linearly equivalent, it suffices to show that $\operatorname{div}(\omega) = E_{n-3} + \operatorname{div}(f_y)$. Since $f_y = F_Y/Z^{n-1}$, this is the same as showing

$$(*) \quad \operatorname{div}(dx) - \operatorname{div}(F_Y) = -2\sum_{i=1}^{n} P_i - E.$$

Note first that $dx = -(f_y/f_x)dy = -(F_Y/F_X)dy$, so $\operatorname{ord}_Q(dx) - \operatorname{ord}_Q(F_Y) = \operatorname{ord}_Q(dy) - \operatorname{ord}_Q(F_X)$ for all $Q \in X$.

Suppose Q is a place centered at $P_i \in Z \cap C$. Then $y^{-1} = Z/Y$ is a uniformizing parameter in $\mathcal{O}_{P_i}(X)$, and $dy = -y^2 d(y^{-1})$, so $\operatorname{ord}_Q(dy) = -2$. Since $F_X(P_i) \neq 0$ (Problem 5-16), both sides of (8) have order -2 at Q.

Suppose Q is a place centered at $P = (a,b,1) \in C$. We may assume $P = (0,0,1)$, since $dx = d(x-a)$, and derivatives aren't changed by translation.

Consider the case when Y is tangent to C at P. Then P is not a multiple point (by hypothesis), so x is a uniformizing parameter, and $F_Y(P) \neq 0$. Therefore

$\operatorname{ord}_Q(dx) = \operatorname{ord}_Q(F_Y) = 0$, as desired.

If Y is not tangent, then y is a uniformizing parameter at Q (Chapter 7, §2 , Step (2)), so $\operatorname{ord}_Q(dy) = 0$, and $\operatorname{ord}_Q(f_x) = r_Q - 1$ (Problem 7-4), as desired.

COROLLARY. Let W be a canonical divisor. Then $\deg(W) = 2g-2$ and $\ell(W) \geq g$.

Proof: We may assume $W = E_{n-3}$. Then this is Cor. 3 (b) to Prop. 5.

Problems. 8-24. Show that if $g > 0$, then $n \geq 3$ (notation as in Prop. 8).

8-25. Let $X = P^1$, $K = k(t)$ as in Problem 8-1. Calculate $\operatorname{div}(dt)$, and show directly that the above corollary holds when $g = 0$.

8-26. Show that for any X there is a curve C birationally equivalent to X satisfying the conditions of Prop. 8 (See Problem 7-21).

8-27. Let $X = C$, x,y as in Problem 8-2. Let $\omega = y^{-1}dx$. Show that $\operatorname{div}(\omega) = 0$.

8-28. Show that if $g > 0$, there are effective canonical divisors.

6. Riemann-Roch Theorem

This celebrated theorem finds the missing term in Riemann's Theorem. Our proof follows the classical proof of Brill and Noether.

RIEMANN-ROCH THEOREM. Let W be a canonical divisor on X. Then for any divisor D,

$$\ell(D) = \deg(D) + 1-g + \ell(W-D).$$

Before proving the theorem, notice that we know the theorem for divisors of large enough degree. We can prove the general case if we can compare both sides of the above equation for D and $D + P$, $P \in X$; note that $\deg(D+P) = \deg(D) + 1$, while the other two non-constant terms change either by 0 or 1. The heart of the proof is therefore

NOETHER'S REDUCTION LEMMA. If $\ell(D) > 0$, and $\ell(W-D-P) \neq \ell(W-D)$, then $\ell(D+P) = \ell(D)$.

Proof: Choose C as before with ordinary multiple points, and such that P is a simple point on C (Problem 7-21 (a)), and so $Z.C = \sum_{i=1}^{n} P_i$, P_i distinct. Let $E_m = m\,\Sigma P_i - E$. The terms in the statement of the lemma depend only on the linear equivalence classes of the divisors involved, so we may assume $W = E_{n-3}$, and $D \succ 0$ (Prop. 8 and Problem 8-11). So $L(W-D) \subset L(E_{n-3})$.

Let $h \in L(W-D)$, $h \notin L(W-D-P)$. Write $h = G/Z^{n-3}$, G an adjoint of degree $n-3$. (Cor. 3 to Prop. 5). $\operatorname{div}(G) = D + E + A$, $A \succ 0$, but $A \not\succ P$.

Take a line L such that $L.C = P + B$, where B consists of $n-1$ simple points of C, all distinct from P. $\operatorname{div}(LG) = (D+P) + E + (A+B)$.

Now suppose $f \in L(D+P)$; let $\operatorname{div}(f) + D = D'$. We must show that $f \in L(D)$, i.e. $D' \succ 0$.

Since $D + P \equiv D' + P$, and both these divisors are effective, the Residue Theorem applies: There is a curve H of degree $n-2$ with $\operatorname{div}(H) = (D'+P) + E + (A+B)$.

But B contains $n-1$ distinct collinear points, and H is a curve of degree $n-2$. By Bezout's Theorem, H must contain L as a component. In particular, $H(P) = 0$. Since P does not appear in $E + A + B$, it follows that $D' + P \succ P$, or $D' \succ 0$, as desired.

We turn to the proof of the theorem. For each divisor D, consider the equation

$$(*)_D \qquad \ell(D) = \deg(D) + 1-g + \ell(W-D).$$

Case 1: $\ell(W-D) = 0$. Since $g \leq \ell(W)$ (Cor. to Prop. 8) and $\ell(W) \leq \ell(W-D) + \deg(D)$ (Problem 8-8), we have $\deg(D) \geq g$ in this case. By Riemann's Theorem, $\ell(D) \geq \deg(D) + 1-g \geq 1$, and if $(*)_D$ were false $\ell(D) > 1$.

We prove this case by induction on $\ell(D)$. Choose any P so that $\ell(D-P) = \ell(D) -1$ (Problem 8-13). If $(*)_D$ were false, $\ell(D-P) > 0$, so the Reduction Lemma implies that $\ell(W-(D-P)) = 0$. Applying the induction hypotheses to $D-P$, $\ell(D-P) = \deg(D-P) + 1-g$, so $\ell(D) = \deg(D) + 1-g$, which is $(*)_D$.

Case 2: $\ell(W-D) > 0$. This case can only happen if $\deg(D) \leq \deg(W) = 2g-2$ (Prop. 3 (2)). So we can pick a maximal D for which $(*)_D$ is false; i.e. $(*)_{D+P}$ is true for all $P \in X$. Choose P so that $\ell(W-D-P) = \ell(W-D)-1$ (Problem 8-13). By the Reduction Lemma, $\ell(D+P) = \ell(D)$. Since $(*)_{D+P}$ is true, we get

$\ell(D) = \ell(D+P) = \deg(D+P) + 1-g + \ell(W-D-P) = \deg(D) + 1-g + \ell(W-D)$, as desired.

COROLLARY 1. $\ell(W) = g$ <u>if</u> W <u>is a canonical divisor</u>.

COROLLARY 2. <u>If</u> $\deg(D) \geq 2g-1$, <u>then</u> $\ell(D) = \deg(D) + 1-g$.

COROLLARY 3. If $\deg(D) \geq 2g$, then $\ell(D-P) = \ell(D)-1$ <u>for all</u> $P \in X$.

COROLLARY 4. (<u>Clifford's Theorem</u>) <u>If</u> $\ell(D) > 0$, <u>and</u> $\ell(W-D) > 0$, then $\ell(D) \leq \frac{1}{2}\deg(D) + 1$.

<u>Proofs</u>: The first three are straight-forward applications of the theorem, using Prop. 3. For Cor. 4, we may assume $D \succ 0$, $D' \succ 0$, $D + D' = W$. And we may assume $\ell(D-P) \neq \ell(D)$ for all P, since otherwise we work with $D-P$ and get a better inequality.

Choose $g \in L(D)$ such that $g \notin L(D-P)$ for each $P < D'$. Then it is easy to see that the linear map $\varphi\colon L(D')/L(0) \longrightarrow L(W)/L(D)$ defined by $\varphi(\overline{f}) = \overline{fg}$ (the bar denotes residues) is one-to-one. Therefore $\ell(D') - 1 \leq g - \ell(D)$. Applying Riemann-Roch to D' concludes the proof.

The term $\ell(W-D)$ may also be interpreted in terms of differentials. Let D be a divisor. Define $\Omega(D)$ to be $\{\omega \in \Omega \mid \operatorname{div}(\omega) \succ D\}$. It is a subspace of Ω (over k). Let $\delta(D) = \dim_k \Omega(D)$, the <u>index</u> of D. Differentials in $\Omega(0)$ are called <u>differentials of the first kind</u> (or holomorphic differentials, if $k = \mathbb{C}$).

PROPOSITION 9. (1) $\delta(D) = \ell(W-D)$.

(2) <u>There are</u> g <u>linearly independent differentials of the first kind on</u> X.

(3) $\ell(D) = \deg(D) + 1-g + \delta(D)$.

<u>Proof</u>: Let $W = \operatorname{div}(\omega)$. Define a linear map $\varphi\colon L(W-D) \longrightarrow \Omega(D)$ by $\varphi(f) = f\omega$. φ is an isomorphism, which proves (1). (2) and (3) follow immediately.

<u>Problems</u>. 8-29. Let D be any divisor, $P \in X$. Then $\ell(W-D-P) \neq \ell(W-D)$ if and only if $\ell(D+P) = \ell(D)$.

8-30. (Reciprocity Theorem of Brill-Noether). Suppose D and D' are divisors, and $D + D' = W$ is a canonical divisor. Then $\ell(D)-\ell(D') = \frac{1}{2}(\deg(D)-\deg(D'))$.

8-31. Let D be a divisor with $\deg(D) = 2g-2$ and $\ell(D) = g$. Show that D is a canonical divisor. So these properties characterize canonical divisors.

8-32. Let $P_1,\ldots,P_m \in \mathbb{P}^2$, $r_1,\ldots,r_m$ non-negative integers. Let $V(d;r_1P_1,\ldots,r_mP_m)$ be the space of curves F of degree d with $m_{P_i}(F) \geq r_i$. Suppose there is a curve C of degree n with ordinary multiple points $P_1,\ldots,P_m$, and $m_{P_i}(C) = r_i + 1$; and suppose $d \geq n-3$. Show that (as a projective space)

$$\dim V(d;r_1P_1,\ldots,r_mP_m) = \frac{d(d+3)}{2} - \Sigma\frac{(r_i+1)r_i}{2}.$$

Compare with Chapter 5, Thm. 1.

8-33. (Linear Series). Let D be a divisor, and let V be a subspace of $L(D)$ (as a vector space). The set of effective divisors $\{\mathrm{div}(f) + D \mid f \in V\}$ is called a <u>linear series</u>. If $f_1, \ldots, f_{r+1}$ is a basis for V, then the correspondence $\mathrm{div}(\Sigma\lambda_i f_i) + D \longrightarrow (\lambda_1, \ldots, \lambda_{r+1})$ sets up a one-to-one correspondence between the linear series and P^r. If $\deg(D) = n$, the series is often called a g^r_n. The series is called <u>complete</u> if $V = L(D)$, i.e. every effective divisor linearly equivalent to D appears.

(a) Show that, with C, E as in §1, the series $\{\mathrm{div}(G) - E \mid G$ is an adjoint of degree n not containing $C\}$ is complete.

8-34. Show that there are curves of every positive genus. (Hint: Consider affine plane curves $y^2 a(x) + b(x) = 0$, where $\deg(a) = g$, $\deg(b) = g+2$).

8-35. Show that every curve of genus 2 is birationally equivalent to a plane curve of order 4 with one double point.

8-36. Let $f\colon X \longrightarrow Y$ be a non-constant (therefore surjective) morphism of projective non-singular curves, corresponding to a homomorphism $\tilde{f}$ of $k(Y)$ into $k(X)$. The integer $n = [k(X):k(Y)]$ is called the <u>degree</u> of f. If $P \in X$, $f(P) = Q$, let $t \in \mathcal{O}_Q(Y)$ be a uniformizing parameter. The integer $e(P) = \mathrm{ord}_P(t)$ is called the <u>ramification index</u> of f at P.

(a) For each $Q \in Y$, show that $\sum_{f(P)=Q} e(P)P$ is an effective divisor of degree n. (See Prop. 4).

(b) $(\mathrm{Char}(k) = 0)$. With t as above, show that

$\mathrm{ord}_P(dt) = e(P) - 1$.

(c) $(\mathrm{Char}(k) = 0)$. If g_X (resp. g_Y) is the genus of X (resp. Y), show that

$$2g_X - 2 = (2g_Y - 2)n + \sum_{P \in X} (e(P)-1). \quad \text{(\underline{Hurwitz Formula})}.$$

(d) For all but a finite number of $P \in X$, $e(P) = 1$. The points $P \in X$ (and $f(P) \in Y$) where $e(P) > 1$ are called <u>ramification points</u>. If $Y = P^1$, $n > 1$, show that there are always some ramification points.

If $k = \mathbb{C}$, a non-singular projective curve has a natural structure of a one-dimensional compact complex analytic manifold, and hence a two-dimensional real analytic manifold. From the Hurwitz Formula (c) with $Y = P^1$ it is easy to prove that the genus defined here is the same as the topological genus $(= \frac{1}{2} \dim_{\mathbb{R}} H_1(X,\mathbb{R}))$ of this manifold. (See Lang's "Algebraic Functions".)

8-37. (<u>Weierstrass Points</u>; assume $\mathrm{char}(k) = 0$). Let P be a point on a non-singular curve X of genus g. Let $N_r = N_r(P) = \ell(rP)$.

(a) $1 = N_0 \leq N_1 \leq \cdots \leq N_{2g-1} = g$. So there are exactly g numbers $0 < n_1 < n_2 < \ldots < n_g < 2g$ such that there is no $z \in k(X)$ with pole only at P, and $\mathrm{ord}_P(z) = -n_i$. These n_i are called the <u>Weierstrass gaps</u>, $(n_1, \ldots, n_g)$ the <u>gap sequence</u> at P. The point P is called a Weierstrass point if the gap sequence at P is anything but $(1,2,\ldots,g)$ i.e. if $\sum_{i=1}^{g} (n_i - i) > 0$.

(b) The following are equivalent : (i) P is a Weierstrass point. (ii) $\ell(gP) > 1$. (ii) $\ell(W-gP) > 0$. (iii) There is a differential ω on X with $\operatorname{div}(\omega) \succ gP$.

(c) If r,s are not gaps at P, then $r+s$ is not a gap.

(d) If 2 is not a gap at P, the sequence is $(1,3,\ldots,2g-1)$. Such a Weierstrass point (if $g > 1$) is called <u>hyperelliptic</u>. X has a hyperelliptic Weierstrass point if and only if there is a morphism $f:X \longrightarrow P^1$ of degree 2. Such an X is called a hyperelliptic curve.

(e) n is a gap at P if and only if there is a differential of the first kind ω with $\operatorname{ord}_P(\omega) = n-1$.

8-38. Fix $z \in K$, $z \notin k$. For $f \in K$, denote the derivative of f with respect to z by f'; let $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = (f')'$, etc. For $f_1,\ldots,f_r \in K$, let $W(f_1,\ldots,f_r) = \det(f_j^{(i)})$, $i=0,\ldots,r-1$, $j=1,\ldots,r$. (the "Wronskian"). Let $\omega_1,\ldots,\omega_g$ be a basis of $\Omega(0)$. Write $\omega_i = f_i dz$, and let $h = W(f_1,\ldots,f_g)$.

(a) h is independent of choice of basis, up to multiplication by a constant.

(b) If $t \in K$ and $\omega_i = e_i dt$, then $h = W(e_1,\ldots,e_g)(t')^{1+\cdots+g}$.

(c) There is a basis $\omega_1,\ldots,\omega_g$ for $\Omega(0)$ such that $\operatorname{ord}_P(\omega_i) = n_i-1$, where $(n_1,\ldots,n_g)$ is the gap sequence at P.

(d) $\operatorname{ord}_P(h) = \Sigma(n_i-i) - \frac{1}{2}\, g(g+1)\operatorname{ord}_P(dz)$ (Hint: Let t be a uniformizing parameter at P and look at lowest degree terms in the determinant.)

(e) $\sum_{P,i} (n_i(P)-i) = (g-1)g(g+1)$, so there are a finite number of Weierstrass points. Every curve of genus > 1 has Weierstrass points.