Homework 4

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1 Task 4:Bézier Techniques

The points

$$B_0(1|1), \quad B_1(2|4), \quad B_2(4|5), \quad B_3(5|3), \quad B_4(7|5)$$

are the control points of a Bézier curve defined on the parameter domain [0, 1].

1.1

Evaluate the curve at parameter $t_0 = \frac{1}{2}$ and sketch the curve in rough strokes.

Solution Let's start by calculating the curve using the Bernstein polynomial

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t) = \binom{1}{1} * 1 * (1-t)^4 + \binom{2}{4} * 4 * t^1(1-t)^3 + \binom{4}{5} * 6 * t^2(1-t)^2 + \binom{5}{3} * 4 * t^3(1-t)^1 + \binom{7}{5} * 1 * t^4$$

Therefore if we evaluate it when $t_0 = \frac{1}{2}$ we will obtain

$$\mathbf{p}(\frac{1}{2}) = \binom{3.75}{4}$$

. In figure 1 we can see a sketch of the curve.

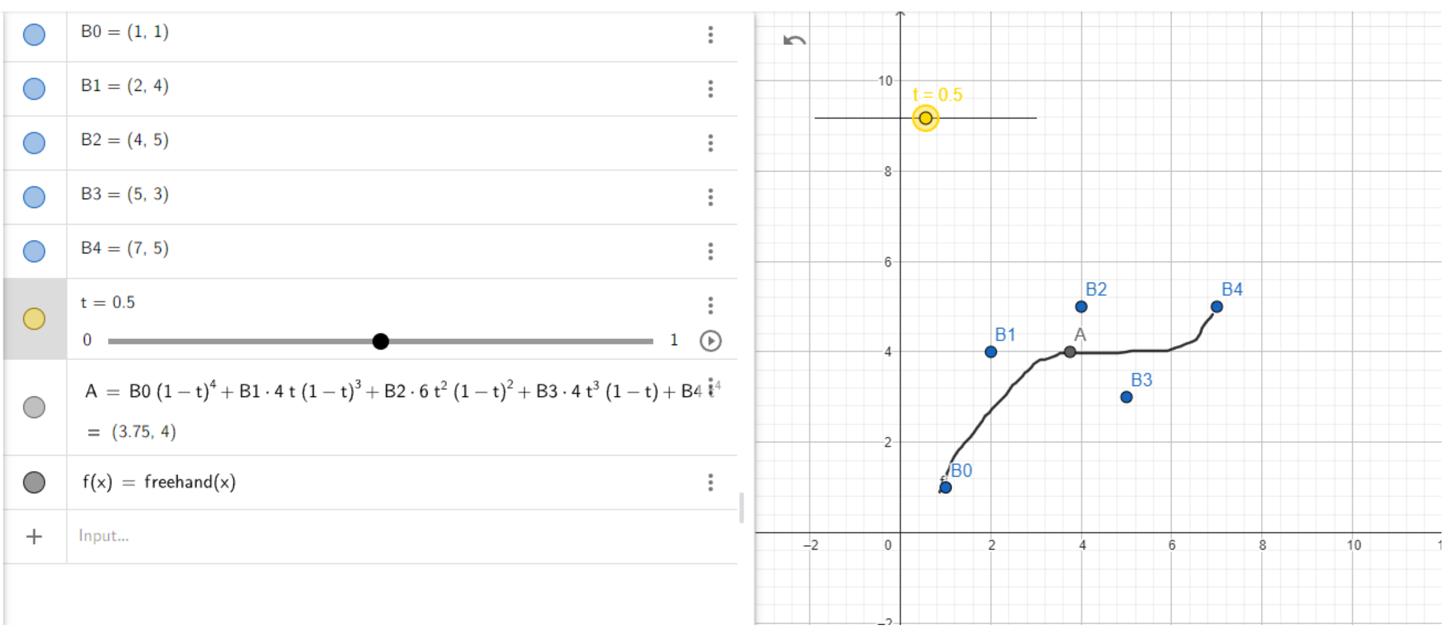

Figure 1: Sketch of the curve

1.2

If one divides the Bézier curve in the curve point of the parameter t0, the result is two partial curves. Specify the Bézier points of both partial curves.

Solution To divide the Bézier curve at the parameter value $t_0 = \frac{1}{2}$, we can use the de Casteljau's algorithm. This algorithm recursively computes the intermediate control points to split the curve.

Given the control points:

$B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

To obtain the partial curves, we perform the following steps:

Step 1: Compute the first set of intermediate control points:

$C_0 = B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C_1 = \frac{1}{2}(B_0 + B_1) = \frac{1}{2}(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}) = \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix}$

$C_2 = \frac{1}{2}(B_1 + B_2) = \frac{1}{2}(\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix}) = \begin{pmatrix} 3 \\ 4.5 \end{pmatrix}$

$C_3 = \frac{1}{2}(B_2 + B_3) = \frac{1}{2}(\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}) = \begin{pmatrix} 4.5 \\ 4 \end{pmatrix}$

$C_4 = \frac{1}{2}(B_3 + B_4) = \frac{1}{2}(\begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix}) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Step 2: Compute the second set of intermediate control points:

$D_0 = C_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$D_1 = \frac{1}{2}(C_0 + C_1) = \frac{1}{2}(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix}) = \begin{pmatrix} 1.25 \\ 1.75 \end{pmatrix}$

$D_2 = \frac{1}{2}(C_1 + C_2) = \frac{1}{2}(\begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} + \begin{pmatrix} 3 \\ 4.5 \end{pmatrix}) = \begin{pmatrix} 2.25 \\ 3.5 \end{pmatrix}$

$D_3 = \frac{1}{2}(C_2 + C_3) = \frac{1}{2}(\begin{pmatrix} 3 \\ 4.5 \end{pmatrix} + \begin{pmatrix} 4.5 \\ 4 \end{pmatrix}) = \begin{pmatrix} 3.75 \\ 4.25 \end{pmatrix}$

$D_4 = \frac{1}{2}(C_3 + C_4) = \frac{1}{2}(\begin{pmatrix} 4.5 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix}) = \begin{pmatrix} 5.25 \\ 4 \end{pmatrix}$

Step 3: Compute the final control points for the partial curves:

Partial Curve 1:

$B_{0_1} = B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B_{1_1} = D_0 = \begin{pmatrix} 1.25 \\ 1.75 \end{pmatrix}$

$B_{2_1} = D_1 = \begin{pmatrix} 2.25 \\ 3.5 \end{pmatrix}$

$B_{3_1} = D_2 = \begin{pmatrix} 3.75 \\ 4.25 \end{pmatrix}$

$B_{4_1} = D_3 = \begin{pmatrix} 5.25 \\ 4 \end{pmatrix}$

Partial Curve 2:

$B_{0_2} = D_4 = \begin{pmatrix} 5.25 \\ 4 \end{pmatrix}$

$B_{1_2} = C_4 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

$B_{2_2} = C_3 = \begin{pmatrix} 4.5 \\ 4 \end{pmatrix}$

$B_{3_2} = C_2 = \begin{pmatrix} 3 \\ 4.5 \end{pmatrix}$

$B_{4_2} = B_4 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

Therefore, the Bézier points for the two partial curves are:

Partial Curve 1: $B_{0_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_{1_1} = \begin{pmatrix} 1.25 \\ 1.75 \end{pmatrix}, \quad B_{2_1} = \begin{pmatrix} 2.25 \\ 3.5 \end{pmatrix}, \quad B_{3_1} = \begin{pmatrix} 3.75 \\ 4.25 \end{pmatrix}, \quad B_{4_1} = \begin{pmatrix} 5.25 \\ 4 \end{pmatrix}$

Partial Curve 2: $B_{0_2} = \begin{pmatrix} 5.25 \\ 4 \end{pmatrix}, \quad B_{1_2} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad B_{2_2} = \begin{pmatrix} 4.5 \\ 4 \end{pmatrix}, \quad B_{3_2} = \begin{pmatrix} 3 \\ 4.5 \end{pmatrix}, \quad B_{4_2} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

1.3

Represent the Bézier curve with the control points B0, ..., B4 by a Bézier curve with degree 5. Determine the necessary Bézier points of the degree elevated curve (geometrically or algebraically).

Solution Suppose we have a Bézier curve of degree n defined by n + 1 control points $B_0, B_1, B_2, ..., B_n$ and we want to increase the degree of this curve to $n + 1$ without changing its shape. Since a degree $n + 1$ Bézier curve is defined by $n + 2$ control points, we need to find such a new set of control points. Obviously, B_0 and B_n must be in the new set because the new curve also passes through them. Therefore, what we need are only n new control points. Let the new set of control points be $Q_0, Q_1, Q_2, ..., Q_{n+1}$. As mentioned above, $Q_0 = P_0$ and $Q_{n+1} = P_n$. The other control points are computed as follows:

$$Q_i = \frac{i}{n+1} P_{i-1} + (1 - \frac{i}{n+1}) P_i$$

. In our case where n=4 we would need to calculate 6 Q points.

$$\begin{aligned} Q_0 = B_0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ Q_1 = \frac{1}{5} B_0 + (1 - \frac{1}{5}) B_1 &= \begin{pmatrix} \frac{9}{5} \\ \frac{17}{5} \end{pmatrix} \\ Q_2 = \frac{2}{5} B_1 + (1 - \frac{2}{5}) B_2 &= \begin{pmatrix} \frac{16}{5} \\ \frac{23}{5} \end{pmatrix} \\ Q_3 = \frac{3}{5} B_2 + (1 - \frac{3}{5}) B_3 &= \begin{pmatrix} \frac{22}{5} \\ \frac{21}{5} \end{pmatrix} \\ Q_4 = \frac{4}{5} B_3 + (1 - \frac{4}{5}) B_4 &= \begin{pmatrix} \frac{27}{5} \\ \frac{17}{5} \end{pmatrix} \\ Q_5 = B_4 &= \begin{pmatrix} 7 \\ 5 \end{pmatrix} \end{aligned}$$

1.4 Continuity

How must the points of a curve (we will call the ne points of this curve S_i connecting in the point B4 be chosen, so that the junction at B4 is

1. G^1 continuous
   In order to be G^1 continuous we have to respect two properties
   - $S_0 = B_4$: in this way, we are assuming that the continuity of the function is respected
   - The slope of the curve in B_4 must be continuous. This means that we need to avoid having points in which the curve look similar to the absolute function like in figure 2

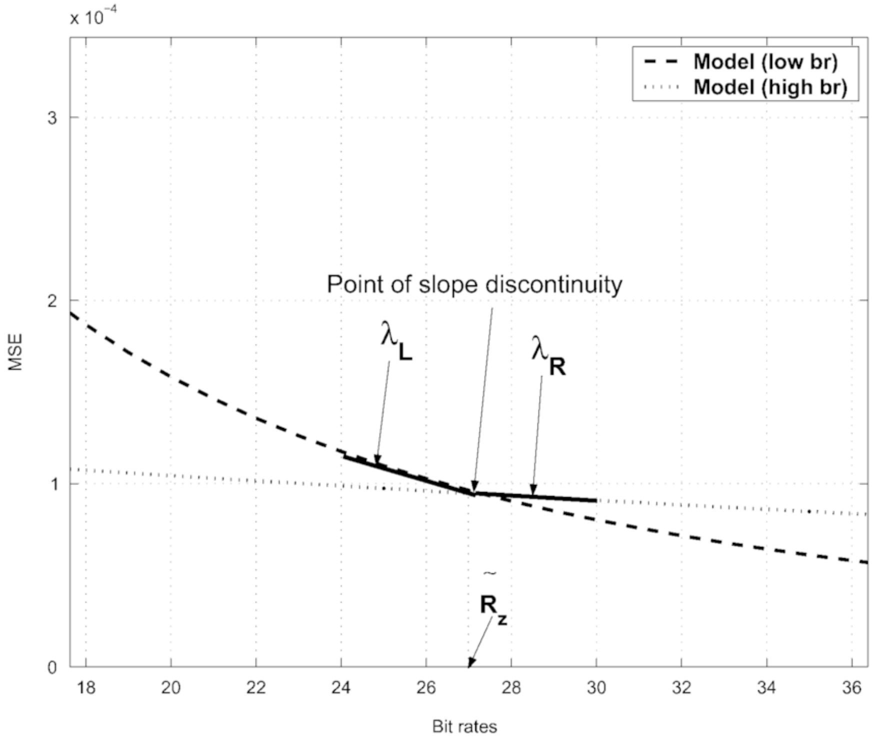

Figure 2: Slope Discontinuity

2. C^1 continuous
   In order to be C^1 continuous we need to impose that the derivative is continuos therefore we would need:
   - $S_0 = B_4$ for the same reason as before
   - The tangent vector in B_4 for our original curve must be colinear with the tangent vector of the new curve.
3. C^2 continuous
   In order to be C^2 continuous we need to impose that the second derivative is continuous therefore we would need:
   - $S_0 = B_4$ for the same reason as before
   - The tangent vector in B_4 for our original curve must be colinear with the tangent vector of the new curve.
   - Second Derivative Equality at the Junction Point: the values of the second derivative calculated at the junction point should be equal.