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CHIRAL SYMMETRY ON THE LATTICE WITH WILSON FERMIONS(*)

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- **Abstract**

The chiral properties of the continuum limit of lattice QCD with Wilson fermions are studied. We show that a partially conserved axial current can be defined, satisfying the usual current algebra requirements.

A proper definition of the chiral symmetry order parameter, $\langle 0|\bar{\psi}\psi|0\rangle$, is given, and the chiral properties of composite operators are investigated. The implications of our analysis to the lattice determination of non leptonic weak amplitudes are also discussed.

1. - Introduction and Presentation of the Results

Approximate chiral symmetry plays an important role in particle physics. The notion of approximate, spontaneously broken, chiral symmetry allows a general understanding of the meson and baryon spectrum, and it allows to express the amplitudes for emission and absorption of soft pions in terms of few phenomenological parameters. Chiral currents couple the hadrons to the weak interactions, so that useful, soft pion theorems can be obtained also for the weak semi-leptonic and non-leptonic transitions. Finally, chiral symmetry is crucial to include the hadrons in the unified, electro-weak theory.

Although at the classical level the conservation of vector and axial vector currents are conceptually similar, a deep difference arises between the two cases, when quantum corrections are considered. This comes about because the conservation of the axial charges is potentially broken by the same regularization procedure needed to define the theory beyond the formal level. Chiral symmetry is broken by any non vanishing fermion mass term, such as those needed to regularize the theory, e.g. in the Pauli-Villars scheme, and it is not at all guaranteed that after removing the cut-off, the chiral current conservation properties will remain as they are at the classical level. The occurrence of the Adler-Bell-Jackiw anomaly is the best known example of the imperfect decoupling of the unphysical degrees of freedom needed to regularize the theory.

Lattice QCD with Wilson fermions [1] provides another example where the regularization procedure conflicts with the naive conservation of the axial currents. Although it complicates the matter, such a conflict is necessary if lattice QCD has to reproduce the correct chiral anomalies needed for the $\pi^0 \rightarrow \gamma\gamma$ decay, in the continuum limit. But it is by no means obvious that the continuum limit of lattice QCD does give anomalies only where they are needed; and that otherwise it goes to the correct chiral limit, as it has been observed in the real world.

In the present paper we investigate the Ward identities of the approximate chiral symmetry in lattice QCD, in the presence of both the physical breaking due to quark masses and of the explicit breaking introduced by the Wilson term. We discuss how a suitable definition of the physical quantities allows one to get a correct chiral continuum limit, unaffected by the chiral breaking introduced by the regularization. We substantiate our discussion with the results of one-loop calculations, in perturbation theory.

We start from the result, essentially derived earlier in ref. [2], that the anomalous term in the axial current divergence can be reabsorbed into a rescaling factor, Z_A, of the current itself, and into the definition of the "current" quark masses. Subsequently, we show that the rescaled axial currents satisfy the appropriate Ward identities, implied by the chiral current commutators.

Based on this discussion, we propose a non perturbative method, to determine Z_A and the current quark-masses. The proposed procedure is accessible to practical Montecarlo calculations.

The existence of the correct chiral limit depends crucially on the absence of mass singularities in the rescaling factor Z_A. This property holds true in one-loop; further investigation is called for, in view of the relevance of the issue.

Assuming that the chiral limit can be defined, the next relevant problem is the chiral classification of composite operators. We start from the simplest case of scalar and pseudoscalar quark densities. On the lattice, a relative rescaling is required, to construct the correct $(3,\bar{3})$ representation.

In the absence of elementary scalar fields, the spontaneous breaking of the chiral symmetry is a non perturbative phenomenon, which has not been possible to study, until now, in the continuum QCD theory. The lattice theory provides a scheme in which such a question can be meaningfully investigated. We give the appropriate lattice definition of $\langle 0|\bar{\psi}\psi|0\rangle$, and show that it vanishes to all orders in perturbation theory.

Another important problem within reach of present Montecarlo simulations, is the determination of the matrix elements of the weak, non-leptonic hamiltonian [3,4]. As a preliminary contribution to this program, we study the chiral properties of four-fermion operators. We show that it is possible to construct operators with definite chiral properties, which correspond to the various components of the weak hamiltonian of the continuum theory. This construction requires, in general, the subtraction of operators bilinear in quark fields, with coefficients which must be determined in a non perturbative way, due to their severe divergence for vanishing lattice spacing.

Other formulations have been proposed which implement chiral symmetry in a different way, notably the Kogut-Susskind fermions [5]. In what follows, these possibilities are not considered and we shall deal exclusively with Wilson fermions.

The paper is organized as follows. After reviewing the basic formulae and notations in Sect.2, we discuss the partial conservation of the axial current in Sect. 3. Sect. 4 is devoted to the derivation of current algebra, and to the discussion of the rescaling factor Z_A. Chiral properties of composite operators are discussed in Sect. 5 and Sect. 7. The definition of the chiral order parameter is given in Sect. 6. The results of the one-loop perturbative calculations are all reported in Appendix.

2. - Basic Notations and Formulae

We start by recalling the basic formulae and notations for Wilson fermions and chiral transformations.

The QCD action on the lattice is defined by:

$$S = S_U + S_F$$

with S_U the usual, pure Yang-Mills action on the lattice[1], and:

$$S_F = \sum_x \left\{ -\frac{1}{2a} \sum_\mu \left[\bar{\psi}(x)(r-\gamma_\mu) U_\mu(x) \psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu}) U_\mu^+(x)(r+\gamma_\mu)\psi(x) \right] + \bar{\psi}(x)\left(M_0 + \frac{4r}{a}\right)\psi(x) \right\} \tag{2.1}$$

The partition function is expressed as:

$$Z = \int d[U]d[\psi]d[\bar{\psi}]\, e^{-S} \tag{2.2}$$

We have suppressed flavour and colour indices, M_0 (the bare mass) is diagonal in flavour and is a colour singlet, a is the lattice spacing and r the Wilson parameter. The euclidean Dirac matrices are choosen to be hermitian and such that:

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu}$$

Under vector and axial vector flavour transformations, the fermion fields behave according to:

$$\delta\psi = i\left[\alpha_V^a \frac{\lambda^a}{2} + \alpha_A^a \frac{\lambda^a}{2}\gamma_5\right]\psi \tag{2.3}$$

$$\delta\bar{\psi} = -i\bar{\psi}\left[\alpha_V^a \frac{\lambda^a}{2} - \alpha_A^a \frac{\lambda^a}{2}\gamma_5\right] \tag{2.4}$$

where λ^a are the flavour matrices, normalized so that

$$\mathrm{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$$

In the limit of equal masses, the action (2.1) is invariant under flavour vector transformations, while the axial transformations are not a symmetry of (2.1) even in the limit of vanishing bare masses, due to the Wilson term. In the absence of the latter (r=0) the vector and axial vector currents are defined by:

$$V_\mu^a(x) = \frac{1}{2}\left\{ \bar{\psi}(x+\hat{\mu})\frac{\lambda^a}{2}\gamma_\mu U_\mu(x)\psi(x) + h.c.\right\} \tag{2.5}$$

$$A_\mu^a(x) = \frac{1}{2}\left\{ \bar{\psi}(x+\hat{\mu})\frac{\lambda^a}{2}\gamma_\mu\gamma_5 U_\mu(x)\psi(x) + h.c.\right\} \tag{2.6}$$

The Wilson term induces a non conservation of all currents. For the vector case, one could still define a new current:

$$\tilde{V}^a_\mu = V^a_\mu - \frac{r}{2}\left\{\bar{\psi}(x)U_\mu(x)\psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu})U^\dagger_\mu(x)\psi(x)\right\} \quad (2.7)$$

which is exactly conserved for degenerate bare masses. This is however not possible for A^a_μ, and we prefer to stay with the definition (2.5), in order to preserve a simple current algebra.

Along with the flavour currents, it will be necessary to consider the scalar and pseudoscalar density operators, defined as:

$$S^a(x) = \bar{\psi}(x)\frac{\lambda^a}{2}\psi(x) \quad (2.8)$$

$$P^a(x) = \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_5\psi(x) \quad (2.9)$$

The Ward identities involving flavour currents and local (or multilocal) operators, can be derived as usual by a change of variables in the functional integral which defines the (time-ordered) product of local operators:

$$\langle O(x_1,\ldots,x_m)\rangle = Z^{-1}\int d[U]d[\psi]d[\bar{\psi}]\,O(x_1,\ldots,x_m)e^{-S} \quad (2.10)$$

Restricting to the axial case for simplicity, we perform the substitutions (2.3), (2.4) in (2.10) obtaining:

$$\left\langle \frac{\delta O(x_1,\dots,x_m)}{\delta \alpha_A^a(x)} \right\rangle - \left\langle O(x_1,\dots,x_m) \frac{\delta S}{\delta \alpha_A^a(x)} \right\rangle = 0 \qquad (2.11)$$

where

$$i \left\langle O(x_1,\dots,x_m) \frac{\delta S}{\delta \alpha_A^a(x)} \right\rangle = \nabla_x^\mu \langle O(x_1,\dots,x_m) A_\mu^a(x) \rangle -$$

$$\left\langle O(x_1,\dots,x_m) \left[\bar\psi(x) \left\{ \frac{\lambda^a}{2}, M_0 \right\} \gamma_5 \psi(x) + X^a(x) \right] \right\rangle \qquad (2.12)$$

The functional derivative of O is defined so that:

$$\delta O(x_1,\dots,x_m) = \int dx \left[\frac{\delta O(x_1,\dots,x_m)}{\delta \alpha_A^a(x)} \alpha_A^a(x) \right]$$

and we have used the abbreviation:

$$\nabla^\mu f(x) = \frac{f(x) - f(x-\hat\mu)}{a}$$

The last term in the r.h.s. of eq.(2.12) is just the chiral variation of the Wilson term, i.e.:

$$\chi^a(x) = -\frac{r}{2a}\sum_\mu \Big[\bar\psi(x)\frac{\lambda^a}{2}\gamma_5 U_\mu(x)\psi(x+\hat\mu) + \bar\psi(x+\hat\mu)\frac{\lambda^a}{2}\gamma_5 U^+_\mu(x)\psi(x) + (x \to x-\hat\mu) - 4\bar\psi(x)\frac{\lambda^a}{2}\gamma_5\psi(x)\Big] \tag{2.13}$$

In the following, we shall study eq.(2.11) in a number of interesting cases.

i) On-shell matrix elements of A^a_μ and of its divergence. In this case $\mathcal{O}(x_1,\dots,x_n)$ is a product of the operators producing the required initial and final states from the vacuum. In perturbation theory, the external states are quark and gluons, and $\mathcal{O}$ is the appropriate product of the corresponding fields, at different points. Hadronic states, on the other hand, are created by composite operators (i.e. products of quark and gluon fields at the same point). Multiple insertions of such objects are affected by additional divergences, which are not removed by the process of renormalization which makes the S-matrix finite. However, in configuration space, the additional counter-terms contribute only in the form of local distributions, and are irrelevant when the large distance limit of eq.(2.11) is taken, which corresponds to matrix elements on-the-mass-shell.

ii) Check of the current algebra sum rules between on-shell quark states. In this case:

$$O = \bar{\Psi}_\alpha(x_1)\,\Psi_\beta(x_2)\,A^a_\mu(x_3)$$

iii) The order parameter of the spontaneous chiral breaking. In this case:

$$O = \bar{\psi}(x)\frac{\lambda^a}{2}\gamma_5\psi(x) \equiv P^a$$

iv) Matrix elements of four quark operators between pseudoscalar meson states.

3. - Partial conservation of the axial current

The partial conservation equation for the on-shell matrix elments of A^a_μ as obtained from eq.(2.11), reads:

$$\langle\alpha|\nabla^\mu A^a_\mu|\beta\rangle = \langle\alpha|\left[\bar{\psi}\left\{\frac{\lambda^a}{2}, M_0\right\}\gamma_5\psi + X^a\right]|\beta\rangle \qquad (3.1)$$

In the tree approximation, eq.(3.1) has a smooth continuum limit, of the form expected in QCD. In fact eq.(2.13) shows that X^a vanishes for vanishing lattice spacing. For example, the matrix element of X^a between two free quark states is:

$$\langle p_2|X^a|p_1\rangle_{tree} = r a\left(m_1^2 + m_2^2\right)\bar{u}(p_2)\frac{\lambda^a}{2}\gamma_5 u(p_1) + O(a^2)$$

In higher orders, the vanishing with a of the bare vertex can be compensated by the ultraviolet divergence of the loop integrations. This is the potential source of anomalies to the continuum limit of eq.(3.1).

To analyze this situation, one defines an operator $\bar{X}^a$, such that it is multiplicatively renormalizable, and its on-the-mass-shell matrix elements vanish in the continuum limit. In general the latter condition is not sufficient to specify completely the operator $\bar{X}^a$, and another condition must be given, to be discussed shortly.

The operator $\bar{X}^a$ is obtained by subtracting from X^a operators of lower dimensionality, with which X^a mixes because of the interaction. In the continuum limit, only operators with dimension three or four may contribute.(f1) Here, and in the following, we restrict to the non-singlet operator(f2) X^a, where a simple analysis of the allowed quantum numbers, including charge-conjugation invariance, leads to the general result:

$$\bar{X}^a = X^a + \bar{\psi}\left\{\frac{\lambda^a}{2},\bar{M}\right\}\gamma_5\psi + (Z_A-1)\,\partial^\mu A^a_\mu \qquad (3.2)$$

Using eq.(3.2), we may rearrange eq.(3.1) as follows:

$$Z_A\langle\alpha|\nabla^\mu A^a_\mu|\beta\rangle = \langle\alpha|\bar{\psi}\left\{\frac{\lambda^a}{2},M_0-\bar{M}\right\}\gamma_5\psi|\beta\rangle + \langle\alpha|\bar{X}^a|\beta\rangle \qquad (3.3)$$

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f1) Recall that X equals the lattice spacing, a, times a dimension five operator, to leading order in a.

f2) For the singlet operator, the additional term $G_{\mu\nu}\tilde{G}_{\mu\nu}$ contributes[2,6,7], giving rise to the anomaly needed to solve the U(1) problem. Contrary to the non-singlet case, the U(1) axial current suffers an infinite renormalization.

Keeping into account the decoupling condition:

$$\langle\alpha|\,\bar{X}^a\,|\beta\rangle \xrightarrow[(a\to 0)]{} 0 \tag{3.4}$$

we obtain the continuum limit equation:

$$Z_A\langle\alpha|\partial_\mu A^a_\mu|\beta\rangle = \langle\alpha|\,\bar{\psi}\left\{\frac{\lambda^a}{2}, M_0-\bar{M}\right\}\gamma_5\psi\,|\beta\rangle \tag{3.5}$$

We discuss now the determination of $\bar{M}$ and Z_A, needed to specify completely the operator $\bar{X}^a$, in eq. (3.2).

It is already apparent from (3.5) that the vanishing of the matrix elements of $\bar{X}^a$ on-the-mass-shell determines only the combination:

$$\rho = Z_A^{-1}\,(M_0 - \bar{M}) \tag{3.6}$$

so that we must supplement the definition of $\bar{X}^a$ with other conditions.

For the purpose of orientation we discuss first the case where QCD is treated in perturbation theory. In this case, we may consider the insertion of X^a with products of quark and gluon field operators at different points. By a simple inspection of the superficial degree of divergence, we conclude that X^a has a non vanishing insertion only in irreducible diagrams with one external ψ and one external $\bar{\psi}$ lines. Moreover, only constant and linear terms in the momentum associated with X^a are non vanishing so that, comparing with (3.2), we see that we can use $\bar{M}$ and Z_A to enforce the stronger condition that the insertion of $\bar{X}^a$ does vanish, in the continuum limit, also for the off-the-mass-shell Green functions:

$$\langle \bar{X}^a(x)\,\psi(x_1)\,\bar{\psi}(x_2)\rangle \xrightarrow[(a\to 0)]{} 0 \qquad (3.7)$$

Eq.(3.7) specifies both Z_A and $\bar{M}$, which can be computed explicitely in perturbation theory (the results of the one-loop approximation are reported in Appendix).

It should be stressed that $\bar{M}$ as determined from eq.(3.7) is itself a function of M_o and of the other parameters of the theory:

$$\bar{M} = \bar{M}(M_o, r, g_o)$$

Eq.(3.3) shows that the currents A^a_μ are conserved for a critical value, M_{cr}, of M_o, such that

$$M_{cr} - \bar{M}(M_{cr}, r, g_o) = 0 \qquad (3.8)$$

The matrix elements of A^a_μ are, however, r-dependent, as an explicit one-loop calculation shows [2,8,9]. To prove that eq.(3.3) leads to the partial conservation equation of the continuum QCD, we still have to show that the matrix elements of the rescaled currents:

$$\hat{A}^a_\mu = Z_A A^a_\mu \tag{3.9}$$

do satisfy the correct normalization condition, in the chiral limit, namely:

$$\langle p|\hat{A}^a_\mu|p\rangle = Z_A \langle p|A^a_\mu|p\rangle = \bar{u}\,\frac{\lambda^a}{2}\gamma_\mu\gamma_5 u \tag{3.10}$$

for the on-shell, zero momentum transfer, quark matrix element. This is shown by using the off-shell version of eq.(3.1), which reads:

$$\nabla^\mu \langle A^a_\mu(x)\,\bar{\psi}(y)\,\psi(z)\rangle = \langle \left[\bar{\psi}(x)\left\{\frac{\lambda^a}{2},M_0\right\}\gamma_5\,\psi(x) + X^a(x)\right]\bar{\psi}(y)\psi(z)\rangle +$$
$$\delta(x-y)\,\langle \left(\bar{\psi}\,\frac{\lambda^a}{2}\gamma_5\right)(y)\,\psi(z)\rangle \;+$$
$$\delta(x-z)\,\langle\, \bar{\psi}(y)\left(\frac{\lambda^a}{2}\gamma_5\,\psi\right)(z)\rangle \tag{3.11}$$

Using eq.(3.2), we may rewrite (3.11) as follows:

$$\nabla^{\mu}\langle \hat{A}^{a}_{\mu}(x)\bar{\psi}(y)\psi(z)\rangle =$$
$$\langle \left[\bar{\psi}(x)\left\{\frac{\lambda^{a}}{2}, M_{0}-\bar{M}\right\}\gamma_{5}\psi(x) + \bar{X}^{a}(x)\right]\bar{\psi}(y)\psi(z)\rangle + \qquad (3.12)$$
$$\delta(x-y)\langle\left(\bar{\psi}\frac{\lambda^{a}}{2}\gamma_{5}\right)(y)\psi(z)\rangle + \delta(x-z)\langle\bar{\psi}(y)\left(\frac{\lambda^{a}}{2}\gamma_{5}\psi\right)(z)\rangle$$

Going to the chiral limit (3.8) and using the condition (3.7) we see that the equal time commutators of the quark fields with the rescaled currents $\hat{A}^{a}_{\mu}$, represented by the last two terms in eq.(3.12), have indeed the canonical form. From this point, using standard arguments [10], it is immediate to recover the result (3.10).

Finally, going back to eq.(3.3), we conclude that the r.h.s. must be finite (for $a \to 0$) and r-independent,(f) with $M_o-\bar{M}$ proportional to the physical quark mass, as defined by the pole in the propagator, with the same proportionality constant which makes finite the insertion of $\bar{\psi}\frac{\lambda^{a}}{2}\gamma_{5}\psi$ (see also the Appendix). This completes the discussion of the perturbative case.

In the confined situation, we must consider Green's functions of $\bar{X}^{a}$ with composite, color singlet, operators, needed to create the physical hadrons. The decoupling condition (3.4) is still necessary for the partial conservation of the axial currents, and it guarantees that $\bar{X}^{a}$ does not propagate physical states. This means that, when inserted in off-shell Green's functions, $\bar{X}^{a}$ gives rise at most to completely localized contact terms. Turning the argument around, we can use the condition (3.4), or the localization condition, to determine the combination of Z_A and $\bar{M}$ given in eq.(3.6).

(f) It is non trivial that the r-independence of the matrix elements of $\hat{A}^{a}_{\mu}$ extends to non-vanishing momentum transfer. We have checked (Appendix) that everything is correct in the one loop approximation.

In the perturbative case, a separate determination of Z_A and $\bar{M}$ was possible because we required eq.(3.7) to hold, namely the vanishing of the coefficients of the contact terms. It was precisely this condition which led to the correct normalization of the physical currents $\hat{A}^a_\mu$, eq.(3.9). In the confined situation, it is therefore natural to determine Z_A from the requirement that $Z_A A^a_\mu$ obey the correct current algebra. We shall discuss how this is possible in the next Section.

Entirely analogous condiderations can be made concerning the partial conservation of the vector currents V^a_μ, eq.(2.5). In analogy with eq.(3.9), one needs to define a rescaled vector current:

$$\hat{V}^a_\mu = Z_V V^a_\mu \tag{3.13}$$

with an r-dependent Z_V which is different from Z_A as an explicit, one loop calculation shows [8,9]. The on-shell matrix elements of $\hat{V}^a_\mu$, in the continuum limit, coincide with those of the "good" vector current $\tilde{V}^a_\mu$, defined in eq.(2.7). The normalization factor Z_V can be computed explicitly, by comparing the matrix elements of the two currents, eqs.(2.5) and (2.7). Alternatively, enforcing the algebra of currents gives the value of Z_V at the same time as Z_A, as we shall see in the next Section.

Before concluding this Section we want to add a few remarks, concerning the determination of ρ, as defined in eq.(3.6). For definiteness, we assume that chiral symmetry is spontaneously broken, so that the confined hadron spectrum contains low mass pseudoscalars which become the Goldstone bosons for $M_o = M_{cr}$.

Montecarlo simulations can be used to compute the matrix elements:

$$\langle 0 | \bar{\psi} \frac{\lambda^a}{2} \gamma_5 \psi | \pi^b \rangle \tag{3.14}$$

$$\langle 0 | A_\mu^a | \pi^b \rangle \tag{3.15}$$

and then eq.(3.5) can be used to determine ρ.

A comment is in order about the number of equations versus the number of arbitrary parameters. In flavour SU(3), M_o contains only three independent parameters, which are to be fixed from the five independent experimental pseudoscalar masses(f) ($\pi^+, \pi^o, \eta, K^+, K^o$). This implies two constraints on the physical masses. In the limit of first-order SU(3) $\otimes$ SU(3) breaking, these relations are the Gell-Mann-Okubo sum rule and one additional relation, which reduces to:

$$m_{\pi^+} = m_{\pi^o}$$

in the limit:

$$m_{u,d} \ll m_s$$

As for ρ, again only three parameters are available, which means that the matrix elements of the currents and of the pseudoscalar densities will satisfy appropriate relations, so as to make the linear system eqs.(3.5) compatible.

(f) After subtraction of $\Delta I = 2$, electromagnetic mass-shifts[11].

4. - Current Algebra

We discuss now the way in which current algebra is recovered on the lattice, for the well normalized currents $\hat{A}^a_\mu$ and $\hat{V}^a_\mu$. In principle, problems could arise because the chiral variations of the fields in eqs.(2.3) and (2.4) transform A^a_μ in V^a_μ and viceversa, neither of which is correctly normalized.

We start the discussion within perturbation theory. To be definite, we set ourselves at the chiral limit, $M_o=M_{cr}$, and consider the Ward identity eq.(2.11) with $\mathcal{O} = \overline{\Psi}_\alpha(z)\,\Psi_\beta(w)\,A^b_\nu(y)$. Putting the quarks on the mass-shell and at equal momenta, and multiplying by a factor Z_A, we obtain:

$$\nabla^\mu \langle p|\, T\left(A^a_\mu(x)\, \hat{A}^b_\nu(y)\right)|p\rangle =$$

$$Z_A \langle p|\, T\left[\left(X^a(x) + \overline{\psi}(x)\left\{\frac{\lambda^a}{2}, M_o\right\}\gamma_5\,\psi(x)\right)A^b_\nu(y)\,|p\rangle + \right.$$

$$i f^{abc} Z_A\ \delta(x-y)\ \langle p|\, V^c_\nu(y)\,|p\rangle \qquad (4.1)$$

Our aim is to see if eq.(4.1) reproduces, for $a \to 0$, the current algebra relation:

$$\partial^\mu \langle p|\, T\left(\hat{A}^a_\mu(x)\,\hat{A}^b_\nu(y)\right)|p\rangle = i f^{abc}\,\delta(x-y)\langle p|\,\hat{V}^c_\nu(y)|p\rangle \qquad (4.2)$$

For vanishing current momentum, eq.(4.2) is the Adler-Weisberger relation for quark states.

We replace in eq.(4.1) X^a by $\bar{X}^a$ as defined in eq.(3.2). In the chiral limit, rearranging the terms and using eqs.(3.9) and (3.13), we obtain:

$$\nabla^\mu \langle p| T\left(\hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\right)|p\rangle =$$

$$Z_A \langle p| T\left(\bar{X}^a(x) A^b_\nu(y)\right)|p\rangle + i f^{abc} \frac{Z_A}{Z_V} \delta(x-y)\langle p| \hat{V}^c_\nu(y)|p\rangle =$$

$$i f^{abc} \delta(x-y)\langle p| \hat{V}^c_\nu(y)|p\rangle + \tag{4.3}$$

$$Z_A \langle p| T\left(\bar{X}^a(x) A^b_\nu(y)\right)|p\rangle + i f^{abc}\left(\frac{Z_A}{Z_V}-1\right)\delta(x-y)\langle p| \hat{V}^c_\nu(y)|p\rangle$$

Eq.(4.3) agrees with eq.(4.2) if the last two terms in the r.h.s. cancel each other.

As discussed in Section 3, $\bar{X}^a$ can contribute contact terms, when inserted with other local operators. Thus, in the continuum limit, the T-product of $\bar{X}^a$ with A^b_ν can be replaced by a combination of local operators times δ-functions.

By dimensional counting, only operators with dimension three or less can appear, with a-independent coefficients. In fact, the only operators with the correct quantum numbers are[f1] $V^c_\nu(x)$. Therefore:

$$\langle\alpha| T\left(\bar{X}^a(x) A^b_\nu(y)\right)|\beta\rangle = i C^{abc} \delta(x-y)\langle\alpha| V^c_\nu(y)|\beta\rangle \tag{4.4}$$

(f1) The identity operator is not considered here, because we are interested in connected Green's functions.

Consistency with eq.(4.2), requires:

$$C^{abc} = \left(\frac{Z_V - Z_A}{Z_V Z_A} \right) f^{abc} \tag{4.5}$$

That eq.(4.5) is valid is not a miracle: it follows from the fact that eq.(4.2) for zero momentum of the currents is indeed trivially satisfied. If we let the momentum k_μ of the currents go to zero, only pole terms survive in the l.h.s. of eq.(4.2), corresponding to the propagation of a massless quark. However, we have already seen in Sect. 3 that the mass-shell zero momentum matrix elements of $\hat{A}^a_\mu$ are well-normalized, so that their commutator on the pole equals trivially $if^{abc}\frac{\lambda^c}{2}\gamma_\mu$, i.e. the matrix elements of $if^{abc}\hat{V}^c_\mu$. Thus eq.(4.5) is satisfie, because eq.(4.2) is obeyed at $k_\mu = 0$. Besides, we learn from eq.(4.4) that no other counterterms are required, which means that eq.(4.2) is valid at all k's, and it remains true even in the non-forward direction, $p' \neq p$.

For completeness, we have computed the coefficients C^{abc} in the one loop approximation, confirming the validity of eq.(4.5), see Appendix.

In the non perturbative case Z_A cannot be determined by a normalization condition for some matrix element of $\hat{A}^a_\mu$, because chiral symmetry is spontaneously broken. We can, however, use the current algebra constraints. The simplest non-trivial and non anomalous vertex, is the A-A-V vacuum expectation value. In the continuum limit, we should recover the following Ward identity:

$$\partial^\mu \langle \hat{A}^a_\mu(x) \hat{A}^b_\nu(y) \hat{V}^c_\rho(0) \rangle =$$
$$\langle \bar{\psi}(x) \left\{ \frac{\lambda^a}{2}, M_0 - \bar{M} \right\} \gamma_5 \psi(x) \hat{A}^b_\nu(y) \hat{V}^c_\rho(0) \rangle + \tag{4.6}$$
$$i f^{abd} \delta(x-y) \langle \hat{V}^d_\nu(y) \hat{V}^c_\rho(0) \rangle +$$
$$i f^{acd} \quad \delta(x) \quad \langle \hat{A}^b_\nu(y) \hat{A}^d_\rho(0) \rangle$$

On the other hand, the Ward identity one derives from (2.11) is:

$$\nabla^\mu \langle \hat{A}^a_\mu(x) \hat{A}^b_\nu(y) \hat{V}^c_\rho(0) \rangle =$$
$$\langle \bar{\psi}(x) \left\{ \frac{\lambda^a}{2}, M_0 - \bar{M} \right\} \gamma_5 \psi(x) \hat{A}^b_\nu(y) \hat{V}^c_\rho(0) \rangle +$$
$$Z_A Z_V \langle \bar{X}^a(x) A^b_\nu(y) V^c_\rho(0) \rangle \qquad +$$
$$i f^{abd} \frac{Z_A}{Z_V} \delta(x-y) \langle \hat{V}^d_\nu(y) \hat{V}^c_\rho(0) \rangle \qquad + \tag{4.7}$$
$$i f^{acd} \frac{Z_V}{Z_A} \delta(x) \quad \langle \hat{A}^b_\nu(y) \hat{A}^d_\rho(0) \rangle$$

Agreement of (4.7) with (4.6) requires:

$$\langle \bar{X}^a(x) A^b_\nu(y) V^c_\rho(0) \rangle =$$
$$i f^{abd} \left(\frac{Z_V}{Z_A} - 1 \right) \delta(x-y) \langle V^d_\nu(y) V^c_\rho(0) \rangle + \tag{4.8}$$
$$i f^{acd} \left(\frac{Z_A}{Z_V} - 1 \right) \delta(x) \quad \langle A^b_\nu(y) A^d_\rho(0) \rangle$$

It is simpler to rewrite $\bar{X}^a$ using eq.(3.2) once again:

$$\bar{X}^a = -\left(\partial_\mu A_\mu^a - \bar{\Psi}\{\tfrac{\lambda^a}{2}, M^0\}\gamma_5\Psi - X^a \right) + Z_A\left(\partial_\mu A_\mu^a - \bar{\Psi}\{\tfrac{\lambda^a}{2}, \rho\}\gamma_5\Psi \right) \equiv -\eta^a + Z_A \xi^a \qquad (4.9)$$

with ρ given by eq.(3.6) and already determined, as indicated in Sect. 3.

Insertion of η^a by itself gives a contact term which, by eq.(2.11) is just the chiral variation of the product $A_\nu^b(y)\, V_\rho^c(0)$. The insertion of ξ^a should also give a contact term, and in fact it does, since ρ was precisely choosen so as to decouple ξ^a from the physical states. We define two coefficients B_1, B_2 according to:

$$\langle \xi^a(x) A_\nu^b(y) V_\rho^c(0)\rangle = i f^{abd} B_1\, \delta(x-y) \langle V_\nu^d(y) V_\rho^c(0)\rangle + i f^{acd} B_2\, \delta(x) \langle A_\nu^b(y) A_\rho^d(0)\rangle + \dots \qquad (4.10)$$

where dots represent possible terms, completely localized at x=y=0. Putting all together, eq.(4.6) is reproduced for:

$$B_1 = \frac{Z_V}{Z_A^2} \qquad\qquad B_2 = \frac{1}{Z_V} \qquad (4.11)$$

The operator ξ^a is known and a measure of B_1 and B_2 on the lattice is possible (keeping $y \neq 0$, contributions from the dots are eliminated). This allows, at the same time, a determination of Z_A^2 and Z_V:

$$Z_V = \frac{1}{B_2} \qquad ; \qquad Z_A^2 = \frac{1}{B_1 B_2}$$

The sign of Z_A is undetermined, since the chiral algebra is invariant under $V \longrightarrow V$; $A \longrightarrow -A$. With this non perturbative determination of Z_A, lattice QCD can give a unique prediction for all chiral parameters, such as f_π and g_A/g_V.

The existence of a chiral limit restricts the dependence of Z_A from the basic parameters of the theory: when expressed in terms of the bare coupling constant, Z_A must be cut-off independent:

$$Z_A = f(g_o, r) \tag{4.12}$$

The reason is that a cutoff dependence in Z_A may appear only in logarithmic terms, of the form $\ln(am)$, m being the physical quark mass. Such terms, if present, would prevent the definition of the axial charge in the chiral limit, $m \to 0$.

The above property has been checked at one-loop level[2,8]. We have no systematic proof that it holds true in general; a two loop calculation of Z_A would shed some light on this point.

Eq.(4.12) allows a partial resolution of the paradoxal situation, caused by the need of a non trivial rescaling of the axial current. In continuum field theories, the renormalization of a softly broken symmetry current is <u>forbidden</u>, and the physical current operator coincides with the bare one. For an asymptotically free theory, and barring possible non perturbative singularities in eq.(4.12), when the cut-off is removed g_o goes to zero, i.e. to the critical value, and one recovers the non-renormalization theorem.

The approach to unity of Z_A is only logarithmic which means that in actual calculations Z_A might deviate significantly from one, and it must be directly measured, as discussed above, to obtain a correct measurement e.g. of f_π.

5. - Normalization of scalar and pseudoscalar densities

In the continuum theory, chiral symmetry implies that the insertion of scalar and pseudoscalar densities is made finite by the same renormalization constant. The presence of the Wilson term makes the relation more complicated, as we shall see presently.

In the chiral limit, the relevant Ward identity, for the relative normalization of scalar and pseudoscalar densities is:

$$\nabla^\mu \langle\alpha| T\left(\hat{A}^a_\mu(x) P^b(0)\right)|\beta\rangle = \langle\alpha| T\left(\bar{X}^a(x) P^b(0)\right)|\beta\rangle + i\, d^{abc}\,\delta(x)\,\langle\alpha| S^c(0)|\beta\rangle \tag{5.1}$$

As usual, the first term in the r.h.s. is a local operator for $a \to 0$. It is easily seen that:

$$\langle\alpha| T\left(\bar{X}^a(x) P^b(0)\right)|\beta\rangle = i\, C\, d^{abc}\,\delta(x)\langle\alpha| S^c(0)|\beta\rangle \tag{5.2}$$

The above equations can be interpreted as saying that the true $(3,\bar{3})$ multiplet is formed by:

$$(1+C)S^a,\ P^a$$

These are the operators which suffer a common, multiplicative renormalization. So that, if $Z_S S^a$ and $Z_p P^a$ are the finite operators, chiral symmetry implies:

$$Z_S = Z_P (1 + C) \tag{5.3}$$

It may be convenient to adopt a conventional normalization of S^a and P^a, which makes contact with the usual definition of the "current" quark masses in the continuum theory. Following ref. [12], we define:

$$Z_S \langle N | \frac{1}{3} (\bar{\psi}\psi) | N \rangle = \bar{u}(p) u(p) \tag{5.4}$$

for the matrix elements of S^o over one baryon states at zero momentum transfer. With this definition, the "current" quark masses are related to the chiral breaking matrix $M_o - \bar{M}$ defined previously according to:

$$(m)_{\text{current quark}} = \frac{1}{Z_P} (M_o - \bar{M})$$

with Z_p given by (5.3) and (5.4). In practice, a non perturbative evaluation of the coefficient C in eq.(5.2) can be obtained by considering the three-point function A_μ - P - S and applying the same considerations discussed in Sect.4, for the determination of Z_A and Z_V.

6. - The Order Parameter of Chiral Breaking

It is usually assumed in QCD that the spontaneous breaking of chiral symmetry is indicated by a non-vanishing vacuum expectation value of the scalar density, $\bar{\psi}\psi$. In the continuum theory, this is a purely formal statement. We want to discuss in this Section how the statement can be made operative in lattice-regularized QCD.

The relevant Ward identity is obtained from eq.(2.11), with $O = P^a \equiv \bar{\psi}\frac{\lambda^a}{2}\gamma_5\psi$. Rearranging the various terms, in a by now familiar way, one obtains:

$$\nabla^\mu \langle \hat{A}^a_\mu(x) P^b(0)\rangle = \langle\left[\bar{X}^a(x) + \bar{\psi}(x)\left\{\frac{\lambda^a}{2}, M_0 - \bar{M}\right\}\gamma_5\psi(x)\right]P^b(0)\rangle + i\, d^{abc}\,\delta(x)\,\langle S^c(0)\rangle \tag{6.1}$$

where brackets denote ensemble averages (i.e. vacuum expectation values of a suitably defined, time-ordered, product).

To analyze eq.(6.1), we first go to the chiral limit, $M_0 = M_{cr}$, and multiply both sides by the renormalization factor, Z_P introduced in Sect. 5. We obtain:

$$\nabla^\mu \langle \hat{A}^a_\mu(x) Z_P P^b(0)\rangle = \langle \bar{X}^a(x) Z_P P^b(0)\rangle + \frac{1}{3}\delta^{ab} Z_P\,\delta(x)\,\langle\bar{\psi}\psi\rangle \tag{6.2}$$

where we have specialized to flavour SU(3), and assumed that there is no spontaneous breaking of the vector symmetry.

As it happened before, the term containing $\bar{X}^a$ gives rise to contact terms, which, for $a \rightarrow 0$, can be easily shown to be of the form:

$$\langle \bar{X}^a(x) P^b(0) \rangle = \left[\frac{C_0}{3} \delta(x) + C_1 \Box\delta(x) \right] \delta^{ab} \tag{6.3}$$

C_1 is needed to compensate a term proportional to $\partial^\mu \delta(x)$ which arises in the T-product in the l.h.s. of eq.(6.1). Bringing the C_1 term to the l.h.s., thus redefining a subtracted, finite A_μ - P correlation, we find:

$$\nabla^\mu \langle \hat{A}^a_\mu(x) Z_P P^b(0) \rangle_{sub} = \frac{1}{3} \delta^{ab} \delta(x) \langle \bar{\psi}\psi \rangle_{sub} \tag{6.4}$$

with:

$$\langle \bar{\psi}\psi \rangle_{sub} = Z_P \left(\langle \bar{\psi}(0)\psi(0) \rangle + C_0 \right) \tag{6.5}$$

and:

$$\langle \hat{A}^a_\mu(x) Z_P P^b(0) \rangle_{sub} = \langle \hat{A}^a_\mu(x) Z_P P^b(0) \rangle - \Box\delta(x) C_1 \delta^{ab} \tag{6.6}$$

If the chiral symmetry is not spontaneously broken there are no Goldstone particles. The l.h.s. of eq.(6.4) vanishes upon integration over x, and so does $\langle \bar{\psi}\psi \rangle_{sub}$. This is the situation which applies to every order in perturbation theory.

In the spontaneously broken phase, the contribution of a massless pion to the l.h.s. gives, after integration:

$$f_\pi \langle 0 | Z_P P^b(0) | \pi \rangle = \frac{1}{3} \delta^{ab} \langle \bar{\psi} \psi \rangle_{sub} \tag{6.7}$$

Thus the definition eq.(6.5) gives the correct order parameter of chiral symmetry, whose value in the broken phase is finite and non-vanishing.

We stress that the subtraction term, C_0, in eq.(6.5) makes the order parameter different from the one used previously in Montecarlo simulations with Wilson fermions. The latter is obtained by considering:

$$\langle \bar{\psi}(0) \psi(0) \rangle = \langle \mathrm{Tr} [G(0,0)] \rangle \tag{6.8}$$

where G(x,y) is the quark propagator in the background color field, and subtracting from eq.(6.8) its perturbative expansion, in the chiral limit. This is in principle incorrect, because C_0 may well have non perturbative contributions which have to be removed exactly.

The above discussion suggests two, physically equivalent, ways to measure $\langle \bar{\psi} \psi \rangle_{sub}$ on the lattice, so as to assess whether chiral symmetry is spontaneously broken in QCD.

i) start with a non vanishing, explicit chiral breaking, and use eq.(6.1) integrated over space-time, to yield:

$$\delta^{ab} \frac{1}{3} \langle \bar{\psi}(0) \psi(0) \rangle_{sub} = - \lim_{M_0 \to M_c} \sum_x a^4 \langle \bar{\psi}(x) \left\{ \frac{\lambda^a}{2}, M_0 - \bar{M} \right\} \gamma_5 \psi(x) Z_P P^b(0) \rangle$$

ii) use eq.(6.5) where all terms in the r.h.s. are computed in the chiral limit.

7. - Four Fermion Operators

We discuss in this Section the renormalization and chiral properties of four fermion operators.

Operators of this kind are directly measured in non leptonic weak interactions. In practical calculations, soft pion theorems are necessary to relate physical amplitudes (e.g. $K \to \pi\pi$) to more tractable matrix elements, such as the $K \to \pi$ or $K \to$ vacuum ones. This makes the understanding of the chiral properties of the four fermion operators quite a crucial step for computing weak amplitudes from the basic $SU(3) \otimes SU(2) \otimes U(1)$ theory. Besides that, the problems we shall meet are quite illustrative of the situation for the general case of composite operators.

We start by considering a basis of four fermion operators:

$$O^{(n)}_{\alpha} \qquad (7.1)$$

where n denotes the chiral representation according to which the operators transform under the variations given in eqs.(2.3), (2.4). α labels the components of the representation n. These transformation properties are however purely nominal. Due to the Wilson term, they are not respected by higher order QCD radiative corrections. As shown by explicit perturbative calculations, the operators (7.1) mix without respecting the (nominal) chiral selection rules [3,13]. Of course, due to the fact that we can define a conserved vector current, flavour is conserved by the mixing matrix.

The arguments given in the previous Sections support the existence of exactly conserved axial currents, in the chiral limit. If this is the case, one should be able to find suitable linear combinations of the operators (7.1) with all the other operators with the same flavour quantum numbers, which truly transform as the representation n under the algebra generated by $\hat{A}^a_\mu$ and $\hat{V}^a_\mu$.

Therefore, one should be able to define a new basis:

$$\hat{O}^{(m)}_\alpha = C^{(m,m')}_{\alpha,\alpha'} O^{(m')}_{\alpha'} \tag{7.2}$$

such that the mixing among the $\hat{O}$ respects all the chirality selection rules.

The coefficients in eq.(7.2) can be determined in perturbation theory. Independently from that, however, we can adjust their values, up to a common constant, precisely from the requirement that $\hat{O}^{(n)}$ satisfy the Ward identity appropriate for the representation n (for simplicity, we are assuming that in the continuum theory there is only one multiplet of operators belonging to the representation n; more general cases will be dealt with later).

For arbitrary external states, one has:

$$\begin{aligned}
\nabla^\mu \langle\alpha| T\left(\hat{A}^a_\mu(x)\,\hat{O}^{(m)}_\alpha(0)\right)|\beta\rangle = {} \\
\langle\alpha| T\left(\bar{\psi}(x)\left\{\frac{\lambda^a}{2}, M_0-\bar{M}\right\}\gamma_5\psi(x)\,\hat{O}^{(m)}_\alpha(0)\right)|\beta\rangle + {} \\
\langle\alpha| T\left(\bar{X}^a(x)\,\hat{O}^{(m)}_\alpha(0)\right)|\beta\rangle + {} \\
i\,\langle\alpha| \frac{\delta\hat{O}^{(m)}_\alpha(0)}{\delta\alpha^a_A(x)}|\beta\rangle
\end{aligned} \tag{7.3}$$

For the $\hat{O}^{(m)}_{\alpha}$ to be bona-fide chiral operators, the T-product of $\bar{X}^a$ with $\hat{O}$ (which we recall gives a contact term in the continuum limit) must combine with the last term in the r.h.s. of eq.(7.3) so as to give the correct value of the equal time commutator of the axial charge with $\hat{O}$:

$$\langle\alpha|\,T\left(\bar{X}^a(x)\,\hat{O}^{(m)}_{\alpha}(0)\right)|\beta\rangle + i\,\langle\alpha|\,\frac{\delta\hat{O}^{(m)}_{\alpha}}{\delta\alpha^a_A(x)}\,|\beta\rangle = \tag{7.4}$$

$$\left[A^a_{\alpha\alpha'}\right]^{(m)}\langle\alpha|\,\hat{O}^{(m)}_{\alpha'}\,|\beta\rangle$$

where $\left[A^a_{\alpha\alpha'}\right]^{(n)}$ are the axial charges in the representation n. By varying the external states, eq.(7.4) generates a system of linear equations for the coefficients $c^{(n,n')}$. When the coefficients $c^{(n,n')}$ mix together operators of the same dimensionality d, eq.(7.4) itself implies the $c^{(n,n')}$ to be mass independent. This is because the insertion of $\bar{X}^a$ with $\hat{O}$, when expanded in the operator basis, gives δ-functions times operators of dimension d, therefore implying adimensional coefficients. A possible mass dependence through logarithmic factors should not be present, <u>if</u> a chiral limit exists, similarly to the case of Z_A, discussed in Sect. 4.

The overall normalization of $\hat{O}_{\alpha}^{n}$ can be provisionally fixed according to the (conventional) prescription

$$\hat{O}^{(m)}_{\alpha} = 1 \times O^{(m)}_{\alpha} + \cdots \tag{7.5}$$

The operator defined above has still divergent matrix elements, for $a \to 0$, and we must multiply it by a normalization factor, $Z^{(m)}_{LATT}$, so that:

$$\langle\alpha|\, Z_{LATT}^{(m)}\, \hat{O}_{\alpha}^{(m)}\, |\beta\rangle = 1 \qquad (7.6)$$

for some, well specified, external states, at the subtraction point. In a similar way, the bare operator of the continuum theory requires a normalization constant, $Z_{CONT}^{(m)}$, to achieve the same normalization given in (7.6), so that the operator:

$$\left[\hat{O}_{\alpha}^{(m)}\right]_{BARE} = \frac{Z_{LATT}^{(m)}}{Z_{CONT}^{(m)}}\, \hat{O}_{\alpha}^{(m)} \qquad (7.7)$$

corresponds precisely to the bare operator of the continuum limit.

In the case where the continuum theory has more than one operator transforming according to the same chiral representation, eq.(7.5) can be replaced by:

$$\begin{aligned} \hat{O}_{1}^{(m)} &= O_{1}^{(m)} + \cdots \\ \hat{O}_{2}^{(m)} &= O_{2}^{(m)} + \cdots \\ &\text{etc.} \end{aligned} \qquad (7.8)$$

where dots indicate operators with "wrong" naive chirality. More explicitly, the dots in the first equation do not include terms proportional to $O_{2}^{(n)}$, etc. To obtain the bare operator, in sense of eq.(7.7), we have to multiply the operators in (7.8) by the reduced matrix $(Z_{CONT})^{-1}\, Z_{LATT}$, with Z_{LATT} having entries which only mix equivalent chiral representations.

The absolute normalization is relevant in the case of the weak interactions, where the underlying theory gives us the weak hamiltonian as a sum of operators in the continuum at a mass-scale $\mu = a^{-1}$, with well defined coefficients.

The procedure outlined above, to determine the relative coefficients $C^{(n,n')}$, is independent from the use of perturbation theory. This is not true for the absolute normalization, however. Truly enough, we could fix the latter by using one weak amplitude as experimental input, but this means we would give up the idea of explaining the $\Delta I = 1/2$ rule. On the other hand, since the whole program of computing non-leptonic amplitudes relies on the idea that perturbation theory controls the dynamics from $\mu \simeq M_W$ down to $\mu \simeq a^{-1}$, it is consistent to compute the ratio $(Z_{CONT})^{-1} Z_{LATT}$ in perturbation theory.

Finally, let us comment about the case in which the definition of $\hat{O}$ requires the subtraction of lower dimension operators. In flavour SU(3), this is the case for the (8,1) operators, whose flavour content can be found also in quark-antiquark bilinears. If we consider, in eq.(7.4), the insertion of $\bar{X}^a$ with the dimension six operators, we shall obtain dimension three operators, besides the already discussed, dimension six ones. By dimensional counting, the coefficients of the additional operators contain mass-independent terms (diverging as a^{-3}) as well as less-divergent or finite, but mass-dependent terms[f]. Similarly to the previous case, enforcing the Ward identity, determines all coefficients needed, up to the ambiguity of the overall normalization. Perturbation theory does not help, in this subtraction; we are confronted here with a case similar to the determination of M_{cr}, eq.(3.8) where it is numerically known that perturbation theory is unreliable.

(f) This subtraction is different from the one discussed in ref. [14]. In fact, the terms to be subtracted in our case are non vanishing also in the chiral limit. For instance, the most divergent (a^{-3}) subtractions arise from the mass-independent Wilson term in the numerator of quark propagators.

8. - Conclusions

In this paper we have presented a method to explore questions related to chirality in theories whose regularization explicitely breaks chiral symmetry.

In particular we have shown how a partially conserved axial current can be recovered in the continuum limit. The crucial point of this construction is the finiteness of the normalization constant Z_A. If this condition is not met the continuum theory will not have a recognizable chiral limit. Although not a sufficient condition, it would be important to explore the finiteness of Z_A in higher orders of perturbation theory.

In any case one can determine Z_A non-perturbatively using the method outlined in Sect. 4 and compare with the available (one-loop) perturbative calculations.

Under the hypothesis that one can define a partially conserved axial current, we have discussed the correct way of defining the chiral order parameter $\langle 0|\bar{\psi}\psi|0\rangle$ on the lattice.

We have also discussed the construction of four quark composite operators with well defined chiral properties. This is a crucial, preliminary, step in the program of computing non-leptonic weak amplitudes on the lattice. Our analysis confirms the method of calculation proposed in ref. [3], for what concerns operators with which, even on the lattice, operators bilinear in quark fields cannot mix. This includes the important cases of $\Delta I = 3/2$ weak amplitudes and of the K_o-$\bar{K}_o$, $\Delta S=2$, matrix element. Octet weak amplitudes, on the other hand, require a more sophisticated non-perturbative subtraction of dimension three operators. We estimate that this subtraction is not technically more involved than the lattice determination of the unsubtracted matric elements.

APPENDIX

In this Appendix we present the results of the one loop computations mentioned in the text.

i) We start with the $0(\alpha_s)$ corrections to the partial conservation equation, eq.(3.1) of the text. To order α_s, corrections to the axial current A^a_μ have been presented in refs.[2,8,9]. Here we explicitely report the complete computation of the first order corrections to the r.h.s. of eq.(3.1).

Let us consider the matrix element of X^a between two quark states with momenta p and p'and mass m_1 and m_2. The relevant diagrams to order α_s are given in fig. (1). The operator X^a will mix with lower dimension operators because of diagrams (a, b, c, d). On general grounds we can write:

$$\langle p', m_2 | X^a | p, m_1 \rangle_1 = -\langle p', m_2 | \bar{\psi} \gamma_5 \left\{ \frac{\lambda^a}{2}, \bar{M} \right\} \psi | p, m_1 \rangle_0$$

$$+ (1 - Z_A) \langle p', m_2 | \partial_\mu A^a_\mu | p, m_1 \rangle_0 + \cdots \qquad (A1)$$

Dots denote terms which are of order a (the lattice spacing) with respect to those explicitely given in eq.(A1). $\langle \cdot | O | \cdot \rangle_i$, i=0,1 indicate the matrix elements of the operator O to order $(\alpha_s)^i$. At order α_s we find for $\bar{M}$:

$$\bar{M} = \frac{g^2}{16\pi^2} C_F \Big[r\Sigma_0 + \frac{M_0 r^2}{(2\pi)^2} \int_{-\pi}^{\pi} d^4q \Big[-\frac{2}{\Delta_2} +$$

$$\frac{4\sum_\mu \sin^2 q_\mu + \left(\sum_\mu \sin^2 q_\mu/_2\right)\left(32r^2 - 8(1+r^2)\sum_\mu \cos^2 q_\mu/_2\right)}{\Delta_2^2} \Big] \tag{A2}$$

where $C_F = 4/3$, M_o is the bare quark mass matrix. Σ_o was computed in refs. [15, 16] and it is given by:

$$\Sigma_0 = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d^4q \Big[\frac{-\sum_\mu \sin^2 q_\mu}{\Delta_1 \Delta_2} + \frac{8}{\Delta_2} - \frac{2(1+r^2)\sum_\mu \sin^2 q_\mu/_2}{\Delta_2} + \tag{A3}$$

$$+ \frac{2}{\Delta_1} \Big]$$

with:

$$\Delta_1 = \sum_\mu \sin^2 q_\mu/_2 \qquad\qquad \Delta_2 = \sum_\mu \sin^2 q_\mu + 4r^2\left(\sum_\mu \sin^2 q_\mu/_2\right)^2$$

$1-Z_A$ is given by the following expression:

$$1 - Z_A = \frac{g^2}{16\pi^2} C_F \frac{r^2}{(2\pi)^2} \int_{-\pi}^{\pi} d^4q \left(-\frac{4\Delta^*}{\Delta_2^2} - \frac{2}{\Delta_2} \right) \tag{A4}$$

where:

$$\Delta^{*}= -\frac{1}{4}\sum_{\mu} \sin^2 q_\mu \cos q_\mu + \frac{1}{4}\Big(\sum_{\mu} \sin^2 q_\mu\Big)\Big(\sum_{\mu} \cos q_\mu\Big) +$$

$$\Big(\sum_{\mu} \sin^2 q_\mu/2\Big)\Big[\sum_{\mu} \cos^2 q_\mu/2 \cos q_\mu - \frac{1}{2}\Big(\sum_{\mu} \cos^2 q_\mu/2\Big)\Big(\sum_{\mu} \cos q_\mu\Big)$$

$$-\frac{r^2}{4}\sum_{\mu} \sin^2 q_\mu\Big] - \frac{1}{2}\sum_{\mu} \cos^2 q_\mu/2 \sin^2 q_\mu +$$

$$\frac{1}{4}\Big(\sum_{\mu} \cos^2 \frac{q_\mu}{2}\Big)\Big(\sum_{\mu} \sin^2 q_\mu\Big) + \frac{r^2}{2}\Big(\sum_{\mu} \sin^2 q_\mu/2\Big)^2\Big(\sum_{\mu} \cos q_\mu\Big) \tag{A5}$$

To complete the computation we need the one loop corrections to the matrix elements of the operator $\bar{\Psi}_2\, \gamma_5\, \Psi_1$ appearing on the r.h.s. of eq.(3.1). These are given by diagrams a,e,f in fig. (1), ref.[17].

To first order, we find:

$$\langle p', m_2 | \bar{\psi} \{ \tfrac{\lambda^a}{2}, M^0 \} \gamma_5 \psi | p, m_1 \rangle_1 =$$

$$(1 - Z_P) \langle p', m_2 | \bar{\psi} \{ \tfrac{\lambda^a}{2}, M_0 \} \gamma_5 \psi | p, m_1 \rangle_0 \tag{A6}$$

where $1-Z_p$ is given by:

$$1-Z_P = \frac{g^2}{16\pi^2} C_F \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d^4q \left(\frac{-4}{\Delta_1 \Delta_2(M_0)} + \frac{(1-r^2)}{\Delta_2} \right) + \dots \quad \text{(A7)}$$

where dots indicate wave function renormalization contributions which we need not to report explicitely.

In eq.(A7):

$$\Delta_2(M_0) = \sum_\mu \sin^2 q_\mu + \left(2r \sum_\mu \sin^2 q_\mu/2 + M_0 \right)^2$$

Since the first term of the integrand in eq.(A7) is logarithmically divergent, we cannot put to zero the quark mass in Δ_2. Collecting together all the terms we find:

$$\begin{aligned} M_R \equiv Z_P^{-1}(M_0 - \bar{M}) = M_0 - \frac{g^2}{16\pi^2} C_F \Big\{ r \Sigma_0 + \\ \frac{M_0}{(2\pi)^2} \int_{-\pi}^{\pi} d^4q \Big[\frac{4}{\Delta_1 \Delta_2(M_0)} - \frac{(1+r^2)}{\Delta_2} + \\ \frac{r^2}{\Delta_2^2} \Big(4 \sum_\mu \sin^2 q_\mu + \Big(\sum_\mu \sin^2 q_\mu/2 \Big) \Big(32 r^2 - 8(1+r^2) \sum_\mu \cos^2 q_\mu/2 \Big) \Big) \Big] \\ + \Sigma_1 \Big\} \end{aligned} \quad \text{(A8)}$$

Σ_1 is the contribution coming from the quark wave function renormalization, as computed in ref. [16].

M_R given in eq.(A8) is precisely the one loop expression for the renormalized quark mass on the lattice found in ref. [16,18] (f) and defined as the pole in the quark propagator. By numerical integration of the results of ref. [8] we checked that $1 - Z_A$ given in eq.(A4) coincides with the one loop correction to A^a_μ given in ref. [8]. This computation also proves that the r.h.s. of eq.(A4) is finite and r independent, in the limit $a \to 0$.

ii) We have stated in Sect. 4 the condition under which the Adler-Weisberger relation is satisfied. Defining C^{abc} as:

$$\langle\alpha| T\left(\bar{X}^a(x) A^b_\nu(y)\right)|\beta\rangle = i\, C^{abc}\, \delta(x-y) \langle\alpha| V^c_\nu(x)|\beta\rangle \tag{A9}$$

the following equality must hold:

$$C^{abc} = \left(\frac{Z_V - Z_A}{Z_V Z_A}\right) f^{abc} \tag{A10}$$

We have computed the coefficient C^{abc} with the diagrams given in fig.(2). One particle reducible diagrams do not contribute, due to eq.(3.7) of the text. Note that to the order we are considering, $\bar{X}^a = X^a$ in the diagrams of fig.2.

The result is:

(f) In ref. [16] the authors overlooked the term proportional to $r^2/(\Delta_2)^2$ in eq.(A8). This error was subsequently corrected in ref. [18].

$$C^{abc} = -f^{abc}\frac{g^2}{16\pi^2}C_F\frac{r^2}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^4q}{\Delta_2^2}\Big[16\sum_\mu \sin^2 q_\mu/2 - 12\Big(\sum_\mu \sin^2 q_\mu/2\Big)^2 - 8\Big(\sum_\mu \sin^2 q_\mu/2\Big)\Big(\sum_\mu \sin^4 q_\mu/2\Big) + 4\Big(\sum_\mu \sin^2 q_\mu/2\Big)^3 - \sum_\mu \sin^2 q_\mu - 2\sum_\mu \sin^2 q_\mu/2 \sin^2 q_\mu + \Big(\sum_\mu \sin^2 q_\mu\Big)\Big(\sum_\mu \sin^2 q_\mu/2\Big)\Big] \tag{A11}$$

Numerical values of $Z_{V,A}$ have been given in ref. [8]. Using their analytical expressions (unpublished) we have checked that (A10) is satisfied.

iii) The renormalization of scalar and pseudoscalar densities has been discussed in Sect. 5. We have computed to one loop the coefficient C appearing in eq.(5.2) of the text. The result is

$$C = -\frac{g^2}{16\pi^2}C_F\frac{r^2}{(2\pi)^2}\int_{-\pi}^{\pi}d^4q\left[\frac{8\Delta_1(\Delta_1-4)+2\sum_\mu \sin^2 q_\mu}{\Delta_2^2}\right] \tag{A12}$$

The coefficient C is related by eq.(5.3) of the text to the renormalization constants Z_S and Z_P. Again one may compare the expression (A12) with results of ref. [17] to see that (5.3) is satisfied.

Figure Captions

Fig. 1 - One loop diagrams for the renormalization of any extended operator, such as X^a, bilinear in quark fields. For local bilinear operators, such as $\bar{\psi}_2 \gamma_5 \psi_1$; diagrams b, c and d are absent.

Fig. 2 - One loop diagrams for the double insertion of $\bar{X}^a$ and A^b_ν. To the order considered, $\bar{X}^a = X^a$.

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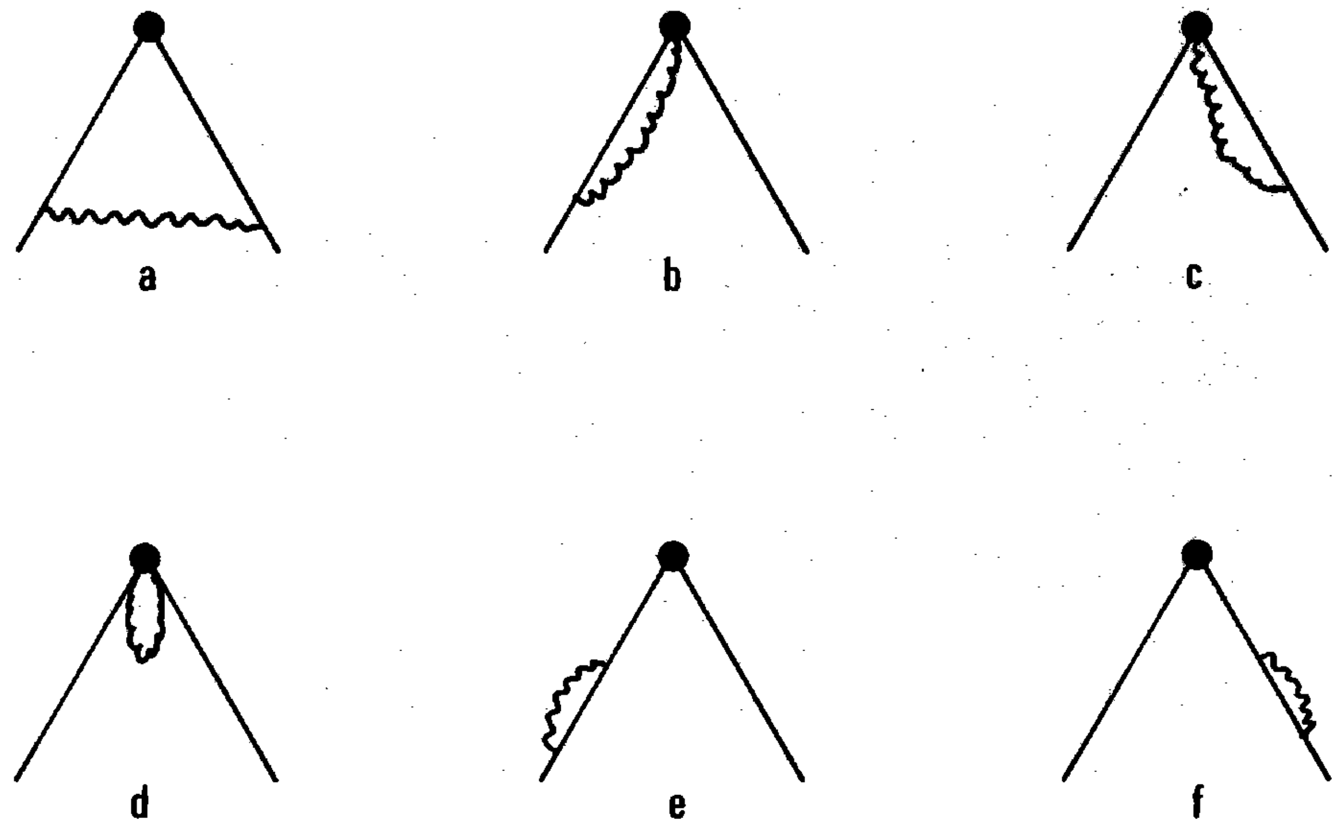

Fig. 1

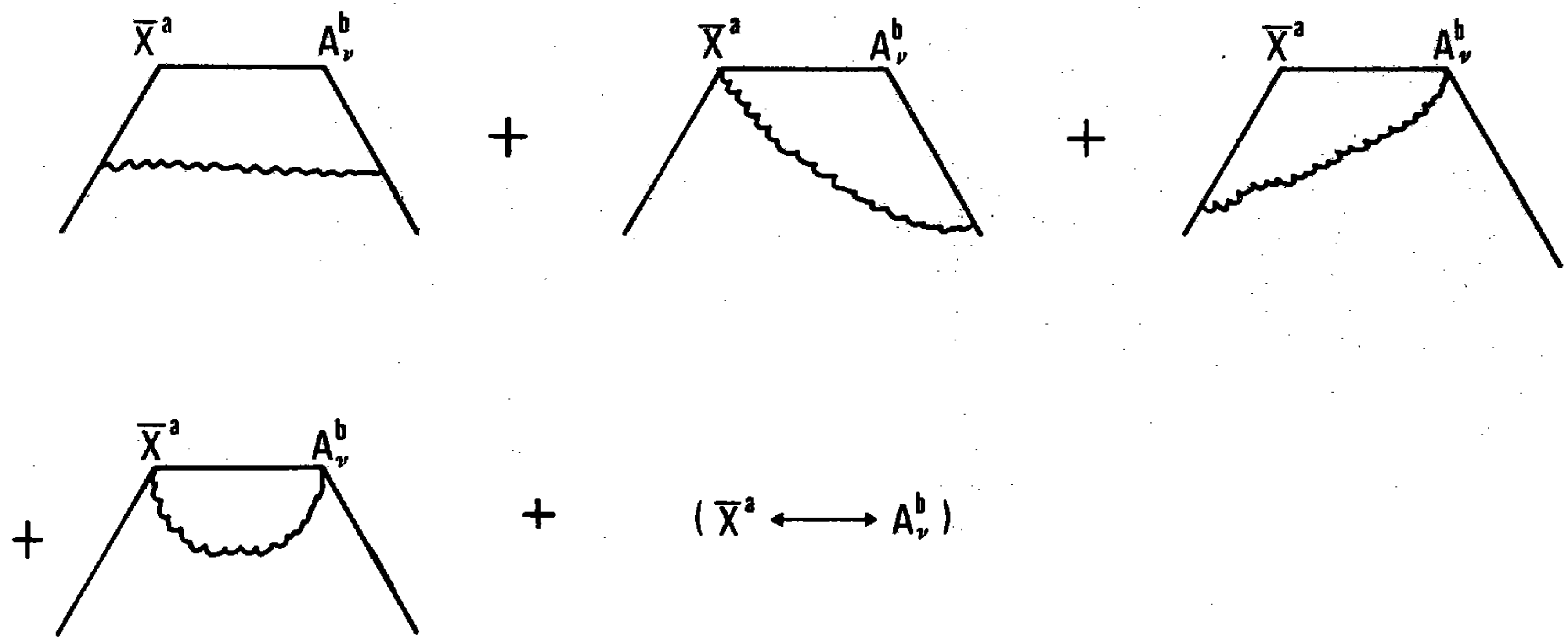

Fig. 2