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On the Laplacian Eigenvalues of Signed Graphs

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A signed graph is a graph with a sign attached to each edge. This article extends some fundamental concepts of the Laplacian matrices from graphs to signed graphs. In particular, the largest Laplacian eigenvalue of a signed graph is investigated, which generalizes the corresponding results on the largest Laplacian eigenvalue of a graph.

Keywords: Signed graph; Laplacian matrix; Largest eigenvalue; Balanced signed graph

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1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(d_{v_1}, d_{v_2}, \ldots, d_{v_n})$ and $A(G) = (a_{ij})$ are the diagonal matrix of degrees and the adjacency matrix of G, respectively. There is a long history of results which relate the Laplacian matrix of a graph. The first of these is the celebrated 1847 result of Kirchoff referred to as the Matrix-Tree Theorem. More recent investigations were stimulated by the results of Fiedler [6], and there has been a lot of activity in this area since then. See for example, [12] or a book [4] for a survey of some the recent work.

A signed graph $\Gamma = (G, \sigma)$ consists of an unsigned graph $G = (V, E)$ and a mapping $\sigma : E \to \{+, -\}$, the edge labelling. We may write $V(\Gamma)$ for the vertex set and $E(\Gamma)$ for the edge set if necessary. The *signed degree* $\mathrm{sdeg}(v)$ of a vertex v of Γ is the number of positive edges incident with v minus the number of negative edges incident with v. Thus if v is incident with d_v^+ positive edges and d_v^- negative edges, then $\mathrm{sdeg}(v) = d^+ - d^-$. However, in the signed graph Γ, the degree of v is defined as $d_v = d_v^+ + d_v^-$. Consequently, a signed graph $\Gamma = (G, \sigma)$ and its underlying graph G

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have the same degree sequence. Signed graphs were introduced by Harary [8] in connection with the study of theory of social balance in social psychology proposed by Heider [9], and the matroids of graphs were extended to matroids of signed graphs by Zaslavsky [16]. The Matrix-Tree Theorem for signed graph were obtained by Chaiken [5] and by Zaslavsky [16], respectively. More recent results on signed graphs can be found [3].

Let $\Gamma = (G, \sigma)$ be a signed graph. The diagonal matrix of degrees and the *signed adjacency matrix* of the signed graph Γ are denoted by $D(\Gamma)$, and $A(\Gamma) = (a^{\sigma}_{ij})$, respectively, where $a^{\sigma}_{ij} = \sigma(v_i v_j)a_{ij}$ and $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Then the *Laplacian matrix* of Γ, denoted by $L(\Gamma)$ or $L(G, \sigma)$, is defined as $D(\Gamma) - A(\Gamma)$. It is easy to see that $L(\Gamma)$ is a symmetric matrix and its row sum vector is $2(d^-_{v_1}, d^-_{v_2}, \ldots, d^-_{v_n})^t$.

From the above definitions, it follows that $L(G) = L(G, +)$, and $D(G) + A(G) = L(G, -)$, where $\sigma = +$ and $\sigma = -$ are the all-positive and all-negative edge labelling, respectively. Thus $L(G, \sigma)$ may be viewed as a common generalization of the Laplacian matrix $L(G)$ and $D(G) + A(G)$ of a graph G. The aim of this article is to extend the concept of Laplacian matrix of a graph to a signed graph. In Section 2 we recall some results on signed graphs and obtain some elementary properties on the Laplacian matrices of signed graphs. In Section 3 we turn our attention to the eigenvalues of Laplacian matrix of a signed graph, in particular, we give some bounds on the largest Laplacian eigenvalue of a signed graph, which generalize the corresponding results on a unsigned graph. In Section 4 we give some examples and present some problems for further research.

2. PRELIMINARY

Let $\Gamma = (G, \sigma)$ be a signed graph and C a cycle of Γ, the sign of C is denoted by sgn(C)= $\prod_{e \in C} \sigma(e)$. A cycle whose sign is $+$ (respectively, $-$) is called *positive* (respectively, *negative*). A signed graph is called *balanced* if all its cycles are positive.

Suppose $\Gamma = (G, \sigma)$ is a signed graph and $\theta : V \to \{+, -\}$ is any sign function. *Switching* Γ by θ means forming a new signed graph $\Gamma^{\theta} = (G, \sigma^{\theta})$ whose underlying graph is the same as G, but whose sign function is defined on an edge $e = v_i v_j$ by $\sigma^{\theta}(e) = \theta(v_i)\sigma(e)\theta(v_j)$. Observe that $\sigma^{\theta}(e) \neq \sigma(e)$ only if $e = v_i v_j$ extends between $X = \theta^{-1}(-)$ and $Y = \theta^{-1}(+)$. Thus we may also speak of *switching* Γ by X. Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graph with the same underlying graph. We call Γ_1 and Γ_2 *switching equivalent*, write $\Gamma_1 \sim \Gamma_2$, if there exists a *switching function* θ such that $\Gamma_2 = \Gamma_1^{\theta}$. Switching leaves the many signed-graphic invariant, such as the set of positive cycles. Switching was first introduced by Seidel (see [3]) and plays an important role in the discussions of signed graphs.

We call two matrices M_1 and M_2 of order n *signature similar* if there exists a signature matrix, that is, a diagonal matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ with diagonal entries $s_i = \pm 1$ such that $M_2 = SM_1S$. From the definitions of switching equivalent and signature similar, we obtain

LEMMA 2.1 *Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be signed graphs on the same underlying graph G. Then $\Gamma_1 \sim \Gamma_2$ if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are signature similar.*

The following lemma appeared in [16], and it is very useful for us in discussing signed graphs.

LEMMA 2.2 ([16]) *Let G be a graph and T a maximal forest of G. Then each switching equivalent class of signed graphs on the graph G has a unique representative which is $+$ on T. Indeed, given any prescribed sign function $\sigma_T : T \to \{+,-\}$, each switching class has a single representative which agrees with σ_T on T.*

By Lemmas 2.1 and 2.2, we have

COROLLARY 2.3 *Let G be a tree. Then all signed graphs on G are switching equivalent. Therefore $L(G,\sigma)$ and $L(G)$ are signature similar for any sign function σ on the tree G.*

COROLLARY 2.4 *Let G be a unicyclic graph. Then there are two different switching equivalent classes of all signed graphs on G, one contains the positive cycle and the other contains the negative cycle.*

THEOREM 2.5 *Let $\Gamma = (G,\sigma)$ be a signed graph. Then the following conditions are equivalent*:

(1) Γ *is balanced.*
(2) $\Gamma = (G,\sigma) \sim (G,+)$.
(3) *There exists a signature matrix S such that $SL(\Gamma)S$ has all off-diagonal entries of $SL(\Gamma)S$ 0 or -1.*
(4) *There exists a partition $V(\Gamma) = V_1 \cup V_2$ such that every edge between V_1 and V_2 is negative and every edge within V_1 or V_2 is positive.*

Proof The equivalence of (1), (2) and (3) follows immediately from Lemmas 2.1 and 2.2 (see [16]). In what follows we prove the equivalence of (3) and (4). We assume that Γ is connected, since the general result can be obtain by treating the connected components separately.

Let S be a signature matrix such that $SL(\Gamma)S$ has all off-diagonal entries 0 or -1. Let $V_1 = \{v_i: s_i > 0\}$ and $V_2 = \{v_i: s_i < 0\}$. Since the off-diagonal (i,j)-entry of $SL(\Gamma)S$ is $-s_i\sigma(v_iv_j)s_j$, it can be seen that any edge connecting a vertex in V_1 and a vertex in V_2 must be negative, but the remaining edges must be all positive. Conversely, the assertion is easily proved by defining a signature matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, where $s_i = 1$ when $v_i \in V_1$ and $s_i = -1$ else. ■

Similar to Theorem 2.5 we can prove

THEOREM 2.6 *Let $\Gamma = (G,\sigma)$ be a signed graph. Then the following conditions are equivalent:*

(1) $\Gamma = (G,\sigma)$ *is a signed graph such that all odd cycles are negative and all even cycles are positive.*
(2) $\Gamma = (G,\sigma) \sim (G,-)$.
(3) *There exists a signature matrix S such that $SL(\Gamma)S$ has all off-diagonal entries of $SL(\Gamma)S$ 0 or 1.*
(4) *There exists a partition $V(\Gamma) = V_1 \cup V_2$ such that every edge between V_1 and V_2 is positive and every edge within V_1 or V_2 is negative.*

Similar to unsigned graphs, we may define the Laplacian matrix of a signed graph $\Gamma = (G,\sigma)$ by means of the *incidence matrix* of Γ. For each edge $e_k = (v_i, v_j)$ of G, we choose one of v_i or v_j to be the head of e_k and the other to be the tail. We call this an orientation of Γ. The vertex-edge incidence matrix $C = C(\Gamma)$ afforded by a fixed orientation of Γ is the n-by-m matrix $C = (c_{ij})$ given by

$$c_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the head of } e_j; \\ -1, & \text{if } v_i \text{ is the tail of } e_j, \text{ and } \sigma(e_j) = +; \\ +1, & \text{if } v_i \text{ is the tail end of } e_j, \text{ and } \sigma(e_j) = -; \\ 0 & \text{otherwise.} \end{cases}$$

The general rule behind this is: for each edge $e = uv$ of Γ, $c_{ue} = -\sigma(e)c_{ve}$. While C depends on the orientation of Γ, CC^t does not, and it is easy to verify that CC^t is always the Laplacian matrix $L(\Gamma) = D(\Gamma) - A(\Gamma)$.

We call a subgraph H of a connected signed graph $\Gamma = (G, \sigma)$ an *essential spanning subgraph* of Γ if either Γ is balanced and H is a spanning tree of G, or else Γ is not balanced, $V(H) = V(\Gamma)$, and every component of H is a unicyclic graph in which the unique cycle is negative. The following is the Matrix-Tree Theorem for signed graphs:

LEMMA 2.7 (Matrix-Tree theorem for signed graphs [5,16]) *Let Γ be a connected signed graph with n vertices and let b_l be the number of essential spanning subgraphs which contain l (negative) cycles. Then*

$$\det L(\Gamma) = \sum_{l=0}^{n} 4^l b_l.$$

From the Matrix-Tree Theorem for signed graphs, it follows that for a unicyclic signed graph Γ whose cycle is negative, then $\det L(\Gamma) = 4$.

We may also describe $L(\Gamma)$ by means of its quadratic form:

$$x^t L(\Gamma)x = x^t CC^t x = \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j)x_j)^2,$$

where $x = (x_1, x_2, \ldots, x_n)^t \in \mathbf{R}^n$. Hence $L(\Gamma)$ is a symmetric, positive semidefinite matrix. By Matrix-Tree theorem for signed graphs we obtain:

COROLLARY 2.8 *Let $\Gamma = (G, \sigma)$ be a connected signed graph and $L(\Gamma)$ be its Laplacian matrix. Then $\det L(\Gamma) = 0$ if and only if Γ is balanced.*

3. THE LARGEST EIGENVALUE OF THE LAPLACIAN MATRIX OF A SIGNED GRAPH

Let $\Gamma = (G, \sigma)$ be a signed graph and $L(\Gamma)$ its Laplacian matrix. Since $L(\Gamma)$ is positive semidefinite, all eigenvalues of $L(\Gamma)$ are nonnegative, we call the eigenvalues of $L(\Gamma)$ Laplacian eigenvalues of Γ. From Lemma 2.1, it follows that the eigenvalues of $L(\Gamma)$ and $L(G)$ are the same if Γ is a tree. Let $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$ denote the Laplacian eigenvalues of the signed graph $\Gamma = (G, \sigma)$ with n vertices, and $\lambda_1(\Gamma)$ or $\lambda_1(\sigma)$ denote the largest Laplacian eigenvalue of the signed graph $\Gamma = (G, \sigma)$. By means of *Rayleigh quotient* we have

$$\lambda_1(\sigma) = \max\left\{ \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j)x_j)^2 \colon \sum_{i=1}^{n} x_i^2 = 1, \ x \in R^n \right\}.$$

LEMMA 3.1 *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then*

$$\lambda_1(\sigma) \leq \lambda_1(-),$$

with equality if and only if $(G, \sigma) \sim (G, -)$.

Proof Let $x = (x_1, x_2, \ldots, x_n)^t$ be a unit eigenvector of $L(\Gamma)$ corresponding to the eigenvalue $\lambda_1(\sigma)$. Taking absolute values of x's, one gets a unit vector $y = (|x_1|, |x_2|, \ldots, |x_n|)^t$, and

$$\begin{aligned}\lambda_1(\sigma) = x^t L(\Gamma)x &= \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j)x_j)^2 \\ &\leq \sum_{v_i v_j \in E(\Gamma)} (|x_i| + |x_j|)^2 \leq \lambda_1(-).\end{aligned}$$

If $\lambda_1(\sigma) = \lambda_1(-)$, then $-\sigma(v_i v_j)x_i x_j = |x_i||x_j|$ and y is an eigenvector corresponding to $\lambda_1(-)$ of the signed graph $(G, -)$. Since $L(G, -)$ is a nonnegative matrix and G is connected, x has no zero entries. Let $V_1 = \{v_i: x_i > 0\}$ and $V_2 = \{v_i: x_i < 0\}$. From $-\sigma(v_i v_j)x_i x_j = |x_i||x_j|$ it follows that if $\sigma(v_i v_j) < 0$ then $x_i x_j > 0$, and if $\sigma(v_i v_j) > 0$ then $x_i x_j < 0$. Hence all positive edges of Γ extend between V_1 and V_2, and all negative edges of Γ are contained in V_1 or V_2. Thus all odd cycles of Γ are negative and all even cycles of Γ are positive. Hence $(G, \sigma) \sim (G, -)$ by Theorem 2.6.

Conversely, if $(G, \sigma) \sim (G, -)$, then $L(\Gamma)$ and $L(G, -)$ are signature similar from Lemma 2.1, and they have the same Laplacian eigenvalues. ■

From Lemma 3.1, in order to obtain the upper bounds for $\lambda_1(\sigma)$ one may make use of the upper bounds for the largest eigenvalue of $L(G, -) = D(G) + A(G)$. Since $D(G) + A(G)$ is a nonnegative symmetric matrix and has the same nonzero eigenvalues as $2I + B$, where B is the adjacency matrix of the line graph L_G of G and I denotes the unit matrix (see [12]). Thus the results on nonnegative matrices may be used for finding the upper bound of $\lambda_1(\sigma)$. The following lemma is a well-known fact.

LEMMA 3.2 ([2]) *Let M be an irreducible nonnegative matrix of order $n, r = (r_1, r_2, \ldots, r_n)^t$ be its row sum vector, and λ_1 be the largest eigenvalue of M. Then*

$$\min\{r_1, r_2, \ldots, r_n\} \leq \lambda_1 \leq \max\{r_1, r_2, \ldots, r_n\}.$$

Equality in one side implies the other, and this occurs if and only if $r_1 = r_2 = \cdots = r_n$.

Recall that a graph G is (r, s)-semiregular if it is bipartite with a bipartition $V = V_1 \cup V_2$ such that all vertices in V_1 have degree r and all vertices in V_2 have degree s. In order to obtain the upper bounds for $\lambda_1(\Gamma)$, we need

LEMMA 3.3 *Let $G = (V, E)$ be a connected graph. Then the line graph L_G of G is regular if and only if G is either regular or semiregular bipartite.*

Proof If G is either regular or semiregular bipartite, then L_G is regular from the fact that edge $e = uv$ has degree $d_u + d_v - 2$ in L_G. Now suppose that L_G is regular. For any two adjacent edges $e = uv$ and $e_1 = uw$, then $d_u + d_v - 2 = d_u + d_w - 2$ by the regularity of L_G. Thus $d_v = d_w$. Therefore if G has a odd cycle, then all vertices have the same degree, and if G has no odd cycles, then G is bipartite and all vertices belong to the same partition set have the same degree. Thus Lemma 3.3 holds. ∎

The next results generalize the corresponding upper bound for the Laplacian matrix of a unsigned graph (see [1,11,13]).

THEOREM 3.4 *Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices. Then*

$$\lambda_1(\Gamma) \leq 2(n-1),$$

with equality if and only if G is the complete graph K_n of order n and $\Gamma \sim (K_n, -)$.

Proof Since $L(\Gamma) = D(\Gamma) - A(\Gamma)$, $\lambda_1(\Gamma) \leq \lambda_1(D(\Gamma)) + \lambda_1(-A(\Gamma)) \leq (n-1)+(n-1) = 2(n-1)$. If $\Gamma \sim (K_n, -)$ then clearly $\lambda_1(\Gamma) = 2(n-1)$. Conversely, if $\lambda_1(\Gamma) = 2(n-1)$, then $\lambda_1(D(\Gamma)) = \lambda_1(-A(\Gamma)) = n-1$. Thus $A(\Gamma) = -(J-I)$, where J is the all one matrix. Hence $\Gamma \sim (K_n, -)$. ∎

Let $\Gamma = (G, \sigma)$ be a signed graph and $v \in V(\Gamma)$. $m_v = (1/d_v)\sum_{uv \in E} d_u$ is called the 2-degree of the vertex v.

THEOREM 3.5 *Let $\Gamma = (G, \sigma)$ be a connected signed graph. Then*

(1) $\lambda_1(\sigma) \leq \max\{d_u + d_v \colon uv \in E\}$.
(2) $\lambda_1(\sigma) \leq \max\{d_u + m_u \colon u \in V\}$.
(3) $\lambda_1(\sigma) \leq \max\{(d_u(d_u + m_u) + d_v(d_v + m_v))/(d_u + d_v) \colon uv \in E\}$.

And any of the above equalities holds if and only if $(G, \sigma) \sim (G, -)$ and G is regular bipartite or semiregular bipartite.

Proof

(1) The result follows by applying Lemmas 3.2 and 3.3 to the matrix $2I + B$, where B is the adjacency matrix of the line graph L_G of G and I denotes the unit matrix.
(2) The result follows by applying Lemma 3.2 to the matrix $D^{-1}(\Gamma)(D(\Gamma) + A(G))D(\Gamma)$ and notice that the row sum of row v is $d_v + \sum_{uv \in E} d_u/d_v = d_v + m_v$.
(3) The result follows by applying Lemma 3.2 to the matrix $(2I + D(L_G))^{-1} \times (2I + B)(2I + D(L_G))$, where $D(L_G)$ is the diagonal matrix of degrees of the line graph L_G, and notice that the row sum of the matrix $(2I + D(L_G))^{-1} \times (2I + B)(2I + D(L_G))$ corresponding to edge $e = uv$ is

$$2 + \sum_{\substack{ux \in E \\ x \neq v}} \frac{d_u + d_x}{d_u + d_v} + \sum_{\substack{yv \in E \\ y \neq u}} \frac{d_v + d_y}{d_u + d_v} = \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}.$$

∎

The last statement follows from Lemma 3.1 and the results of [14].

The upper bounds given in Theorem 3.5 are not very good because they are the common bounds for the largest eigenvalue of $L(G, \sigma)$, perhaps it is better to obtain the bounds which depend on the sign function σ, such as in [15]. We next turn our attention to the lower bound for $\lambda_1(\sigma)$. Unlike $\lambda_1(\sigma) \leq \lambda_1(-)$, $\lambda_1(\sigma) \geq \lambda_1(+)$ does not hold in general, for example, for bipartite graphs. One may conjecture that $\lambda_1(\sigma) \geq \lambda_1(+)$ holds for all nonbipartite graphs, but this is not true in general.

(See the examples in Section 4.) In next we prove this is true when Γ has a vertex of degree $n-1$, we need the following lemmas.

LEMMA 3.6 (Cauchy-Poincare separation theorem) *Let A be an n-by-n Hermitian matrix with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. For a given integer r, $1 \leq r \leq n$, Let A_r denote any r-by-r principal submatrix of A (obtained by deleting $n-r$ rows and the corresponding columns from A). For each integer k such that $1 \leq k \leq r$, we have*

$$\lambda_{k+n-r}(A) \leq \lambda_k(A_r) \leq \lambda_k(A).$$

LEMMA 3.7 *Let $\Gamma = (G, \sigma)$ be a signed graph on n vertices, and E' a subset of E consisting of k edges. The signed subgraph of Γ obtained by deleting all the edges in E' is denoted by $\Gamma' = \Gamma - E'$. Then for each $i = 1, 2, \ldots, n-k$, we have*

$$\lambda_i(\Gamma) \geq \lambda_i(\Gamma') \geq \lambda_{i+k}(\Gamma).$$

Proof Clearly, the nonzero eigenvalues of $L(\Gamma) = C(\Gamma)C(\Gamma)^t$ and $K(\Gamma) = C(\Gamma)^t C(\Gamma)$ are the same. The lemma follows from Lemma 3.6, because $K(\Gamma - E')$ is a principal submatrix of $K(\Gamma)$. ■

Lemma 3.7 tells us that if Γ is obtained from Γ' by adding a signed edge between two nonadjacent vertices, then $L(\Gamma) \geq L(\Gamma')$ and so every eigenvalue of $L(\Gamma)$ is greater than or equal to the corresponding eigenvalue of $L(\Gamma')$.

COROLLARY 3.8 *Let Γ be a signed graph, Γ^+ and Γ^- denote the induced signed subgraphs by all positive edges and all negative edges, respectively. Then*

$$\max\{\lambda_1(\Gamma^+), \lambda_1(\Gamma^-)\} \leq \lambda_1(\Gamma) \leq \lambda_1(\Gamma^+) + \lambda_1(\Gamma^-).$$

THEOREM 3.9 *Let $\Gamma = (G, \sigma)$ be an n-vertex signed graph with the largest degree $n-1$. If Γ has a negative cycle, then $\lambda_1(\Gamma) > n = \lambda_1(G, +)$.*

Proof Suppose that vertex u has degree $n-1$. Then G has an n-vertex star T with the center u as its spanning tree. From Lemmas 2.1 and 2.2, we may assume that the signs on T are all-positive. Since Γ has a negative cycle, there exist vertices $v, w \neq u$ such that $vw \in E(\Gamma)$ and $\sigma(vw) = -$. Let $G_1 = T + vw$ and Γ_1 be the induced signed subgraph by edges of G_1. It is not too difficult to obtain the characteristic polynomial of $L(\Gamma_1)$ is $f(\lambda) = (\lambda - 1)^{n-3}(\lambda^3 - (n+3)\lambda^2 + 3n\lambda - 4)$, and $n^3 - (n+3)n^2 + 3n^2 - 4 = -4 < 0$. Thus $\lambda_1(\Gamma_1) > n$. Hence the theorem follows from that $L(G, +)$ has the largest eigenvalue n. ■

THEOREM 3.10 *Let $\Gamma = (G, \sigma)$ be a signed graph. Then*

$$\lambda_1(\Gamma) \geq \max\{d_v + 1, v \in V(\Gamma)\}.$$

Proof Suppose that $d_u + 1 = \max\{d_v + 1, v \in V(\Gamma)\}$. Considering the induced signed subgraph Γ_1 of Γ by the edges which incident to vertex u, by Lemma 3.7 then $\lambda_1(\Gamma) \geq \lambda_1(\Gamma_1)$. Hence $\lambda_1(\Gamma) \geq d_u + 1$, since Γ_1 is a tree and $\lambda_1(\Gamma_1) = d_u + 1$. ■

For a connected unsigned graph G, let the degree sequence of G be $d_1 \geq d_2 \geq \cdots \geq d_n$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the Laplacian eigenvalues of G, respectively. Grone in [7] proved the following inequalities, which is conjectured by Merris [12], and improved the result on majorization which is the strongest general statement that can be made about the relationship between the eigenvalues and diagonal entries,

$$\sum_{i=1}^{k} \lambda_i \geq 1 + \sum_{i=1}^{k} d_i, \quad \text{for all } k = 1, 2, \ldots, n-1.$$

But for connected signed graphs, in general, the above inequalities do not hold for all $k = 1, 2, \ldots, n-1$. For an example, let G be a connected graph with a cycle and with a pendant vertex, and $\Gamma = (G, \sigma)$ be a signed graph containing a negative cycle. Then $\lambda_n(\Gamma) > 0$. Thus

$$\sum_{i=1}^{n-1} \lambda_i < \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i = 1 + \sum_{i=1}^{n-1} d_i.$$

We conjecture that the following inequalities hold:

$$\sum_{i=1}^{k} \lambda_i > \sum_{i=1}^{k} d_i, \quad \text{for all } k = 1, 2, \ldots, n-1.$$

In [10], Li and Pan showed that $\lambda_2(G, +) \geq d_2$, where d_2 is the second largest degree, but this is not true for signed graphs in general. For example, let $\Gamma = (K_n, -)$. Then $\lambda_2(\Gamma) = n - 2 < n - 1 = d_2$. Nevertheless, similar to [10] we can prove that $\lambda_2(\Gamma) \geq d_2 - 1$ holds.

4. EXAMPLES AND PROBLEMS

In this section we will give some examples and then obtain some results from those examples.

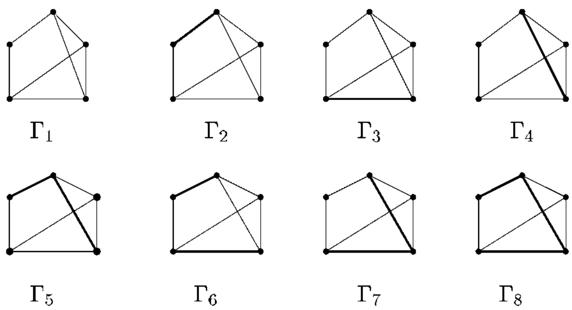

FIGURE 1 Graph $G = \Gamma_1$ and all nonequivalent signed graphs on it. Thinline: +, thickline: –.

The following table gives some data on the above example.

Signed graphs	$\det L(\Gamma)$	*Eigenvalues of* $L(\Gamma)$
Γ_1	0	5, 4, 3, 2, 0
Γ_2	32	4.56, 4, 4, 1, 0.44
Γ_3–Γ_6	44	5.42, 4.28, 2.63, 1.39, 0.54
Γ_7	48	5.78, 3, 2.71, 2, 0.51
Γ_8	64	5.56, 4, 2, 1.44, 1

From the above examples we know that neither $\lambda_1(+) \geq \lambda_1(\sigma)$ nor $\lambda_n(\sigma) \leq \lambda_n(-)$ hold in general (In Fig. 1, $\Gamma_7 \sim (G, -)$). On the other hand, the above example illustrates that in this case, $L(\sigma_1)$ and $L(\sigma_2)$ are cospectral if and only if $\det L(\sigma_1) = \det L(\sigma_2)$.

Of the several problems that arose in our investigations, the following two seem to be challenging and significant importance.

Let G be a graph and $\Gamma = (G, \sigma)$ be a signed graphs on G. Note that if Γ is balanced then Γ and $(G, +)$ are Laplacian cospectral. A natural problem is the following

PROBLEM 1 *Let G be a graph, $\Gamma_1 = (G, \sigma_1)$, $\Gamma_2 = (G, \sigma_2)$ be two signed graphs on G, and* $\det L(\Gamma_1) = \det L(\Gamma_2)$. *Are $L(\Gamma_1)$ and $L(\Gamma_2)$ cospectral?*

If a signed graph $\Gamma = (G, \sigma)$ is not balanced, then $\det L(\Gamma) > 0$. It is interesting to find the switching equivalent class among all signed graphs on G, such that each of this class has the largest Laplacian determinant.

PROBLEM 2 *Do there exist pairs $\Gamma_1 = (G_1, \sigma_1), \Gamma_2 = (G_2, \sigma_2)$ of signed graphs that have either of the following properties* (i) *and* (ii)?

(i) *Γ_1 and Γ_2 are not balanced but Laplacian cospectral such that G_1 and G_2 are non-isomorphic.*
(ii) *Γ_1 and Γ_2 are not balanced but Laplacian cospectral such that G_1 and G_2 are not cospectral.*

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