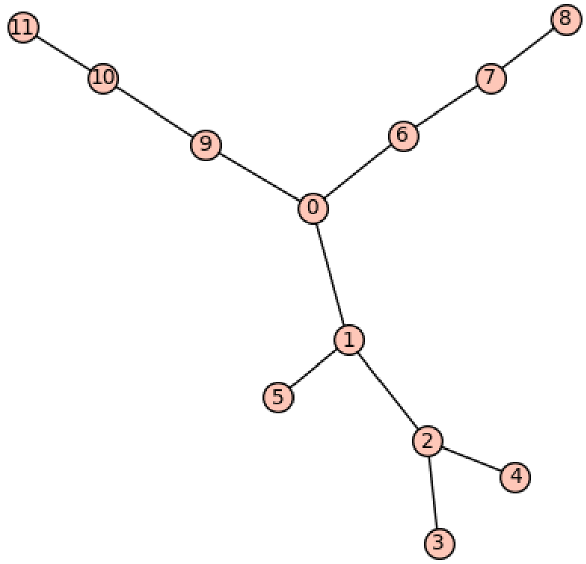
Strongly Cospectral Vertices on Trees



At the above tree, vertices 0 and 1 are strongly cospectral.

Goals are:

1. Proof that vertices 0 and 1 continue strongly cospectral if we add a path P_n, $\forall n > 0$ between then.
2. Proof that the equidistant vertices of that path are also strongly cospectral: $\{(P_n)_k, (P_n)_{n-1-k}\}$ are strongly cospectral.
3. Try to characterize, using conditions about trees T_1 and T_2, when vertices on the path $T_1 - - - - T_2$ are strongly cospectral.

Answer 1. For simplicity, we will name vertices 0 as b and 1 as a.

So, at our original graph, after deleting edge ab from the tree, we get two disjoint trees T_1 that contains a and T_2 that contains b. Then, the graph T can be decomposed in $T = T_1 \cup P_{n+2} \cup T_2$, where a and b are end vertices of P_{n+2}

We claim that: a and b are cospectral if and only if

$$\phi(T_1 \backslash a, t)\phi(T_2, t) = \phi(T_2 \backslash b, t)\phi(T_1, t)$$

To proof the claim, we use the lemma 4.7.1: *If Z is the 1-sum of X and Y at a, then*

$$\phi(Z, t) = \phi(X \backslash a, t)\phi(Y, t) + \phi(X, t)\phi(Y \backslash a, t) - t\phi(X \backslash a, t)\phi(Y \backslash a, t)$$

First, we can rewrite $\phi(T \backslash a, t)$ as $\phi(T_1 \backslash a, t)\phi(P_{n+1} \cup T_2, t)$, since they form different connected components.

And now, from the above lemma, set

$$Z = P_{n+1} \cup T_2 \qquad X = P_n \qquad Y = T_2$$

Then

$$\phi(T\backslash a,t)=\phi(T_1\backslash a,t)(\phi(P_n,t)\phi(T_2,t)+\phi(P_{n+1},t)\phi(T_2\backslash b,t)-t\phi(P_n,t)\phi(T_2\backslash b,t))$$

And if we set

$$Z'=P_{n+1}\cup T_1 \qquad X'=P_n \qquad Y'=T_1$$

$$\phi(T\backslash b,t)=\phi(T_2\backslash b,t)(\phi(P_n,t)\phi(T_1,t)+\phi(P_{n+1},t)\phi(T_1\backslash a,t)-t\phi(P_n,t)\phi(T_1\backslash a,t))$$

After distributing the multiplication, we can see that some terms of the two identities are equal, so

$$\begin{aligned}\phi(T\backslash a,t)=\phi(T\backslash b,t) &\iff \phi(T_1\backslash a,t)\phi(P_{n+1}\cup T_2,t)=\phi(T_2\backslash b,t)\phi(P_{n+1}\cup T_1,t)\\ &\iff \phi(T_1\backslash a,t)\phi(P_n,t)\phi(T_2,t)=\phi(T_2\backslash b,t)\phi(P_n,t)\phi(T_1,t)\\ &\iff \phi(T_1\backslash a,t)\phi(T_2,t)=\phi(T_2\backslash b,t)\phi(T_1,t)\end{aligned}$$

Since this terms don't depend on the paths, we only need to look at the characteristic polynomial of T_1, T_2, $T_1\backslash a$ and $T_2\backslash b$:

$$\begin{aligned}&\phi(T_1,t)=t(t^4-4t^2+2)\\ &\phi(T_1\backslash a,t)=t(t(t^2-2))\\ &\phi(T_2,t)=t(t^2-2)(t^4-4t^2+2)\\ &\phi(T_2\backslash b,t)=(t(t^2-2))^2\end{aligned}$$

And so,

$$\begin{aligned}\phi(T_1\backslash a,t)\phi(T_2,t)&=t(t(t^2-2))t(t^2-2)(t^4-4t^2+2)\\ &=(t(t^2-2))^2t(t^4-4t^2+2)\\ &=\phi(T_2\backslash b,t)\phi(T_1,t)\end{aligned}$$

Now, since we know that a and b are cospectral, we can use Theorem 6.7.1:

Let X be the graph obtained from vertex-disjoint graphs Y and Z by joining a vertex a in Y to a vertex b in Z by a path P of length at least one. If a and b are cospectral in X, they are strongly cospectral.

Answer 2. Using a similar idea, we will proof that all equidistant vertices of P_n are cospectral.

Again, we divide T in $T_1\cup P_m\cup P_k\cup P'_m\cup T_2$ where $T_1\cap P_m=\{a\}$, $P_m\cap P_k=\{x\}$, $P_k\cap P'_m=\{y\}$ and $P'_m\cap T_2=\{b\}$ and $|P_m|=|P'_m|$. We want to show that x and y are strongly cospectral.

Using the claim of Answer 1, this happens if and only if

$$\phi(T_1\cup P_m\backslash x,t)\phi(P'_m\cup T_2,t)=\phi(P'_m\backslash y\cup T_2,t)\phi(T_1\cup P_m,t)$$

So, using lemma 4.7.1, and taking $Z=P'_m\cup T_2$:

$$\phi(T_1\cup P_m\backslash x,t)\phi(P'_m\cup T_2,t)$$

$$= \phi(T_1 \cup P_m \backslash x, t)(\phi(P'_m \backslash b, t)\phi(T_2, t) + \phi(P'_m, t)\phi(T_2 \backslash b, t) - t\phi(P_m \backslash b, t)\phi(T_2 \backslash b, t))$$

And taking $Z = T_1 \cup P_m \backslash x$

$$\phi(T_1 \cup P_m \backslash x, t)\phi(P'_m \cup T_2, t)$$

$$= (\phi(P_m \backslash ax, t)\phi(T_1, t) + \phi(P_m \backslash x, t)\phi(T_1 \backslash a, t) - t\phi(P_m \backslash ax, t)\phi(T_1 \backslash a, t))$$
$$(\phi(P'_m \backslash b, t)\phi(T_2, t) + \phi(P'_m, t)\phi(T_2 \backslash b, t) - t\phi(P'_m \backslash b, t)\phi(T_2 \backslash b, t))$$

If we apply similar steps at $\phi(P'_m \backslash y \cup T_2, t)\phi(T_1 \cup P_m, t)$, first taking $Z = T_1 \cup P_m$ and then taking $Z = P'_m \backslash y \cup T_2$ we reach a similar equality:

$$\phi(P'_m \backslash y \cup T_2, t)\phi(T_1 \cup P_m, t)$$

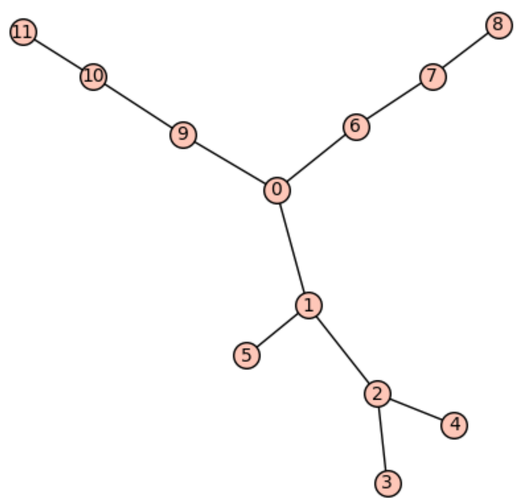
$$= (\phi(P'_m \backslash by, t)\phi(T_2, t) + \phi(P'_m \backslash y, t)\phi(T_2 \backslash b, t) - t\phi(P'_m \backslash by, t)\phi(T_2 \backslash b, t))$$
$$(\phi(P_m \backslash a, t)\phi(T_1, t) + \phi(P_m, t)\phi(T_1 \backslash a, t) - t\phi(P_m \backslash a, t)\phi(T_1 \backslash a, t))$$

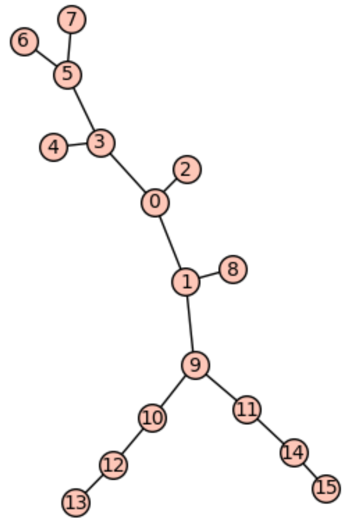
And after comparing each term, we have that if $\phi(T_1 \backslash a, t)\phi(T_2, t) = \phi(T_2 \backslash b, t)\phi(T_1, t)$ then the two terms are equal.

So, they are cospectral and we can also use Theorem 6.7.1 to say that they are strongly cospectral.

Answer 3. The condition $\phi(T_1 \backslash a, t)\phi(T_2, t) = \phi(T_2 \backslash b, t)\phi(T_1, t)$ can be a starting point to try to characterize the vertices of the path $T_1 - - - - - T_2$, since, as we proved, if the condition holds then the equidistant vertices are strongly cospectral, for example, if T_1 and T_2 are isomorphic.

Here are some examples of trees that can be divided in T_1 and T_2 such that they are non-isomorphic and the condition holds (0 and 1 are the strongly cospectral vertices):

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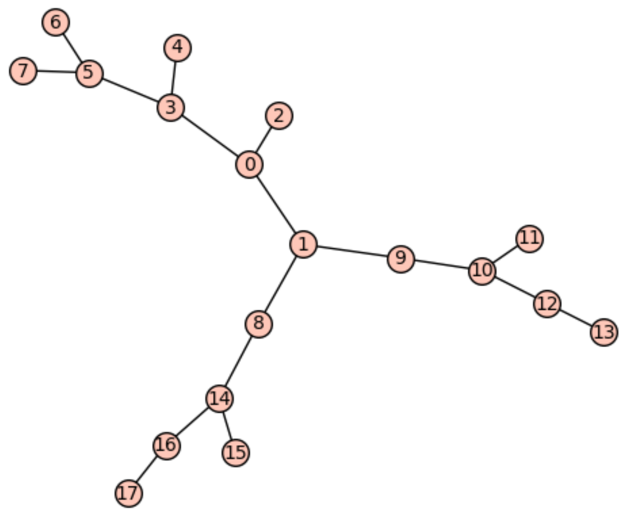
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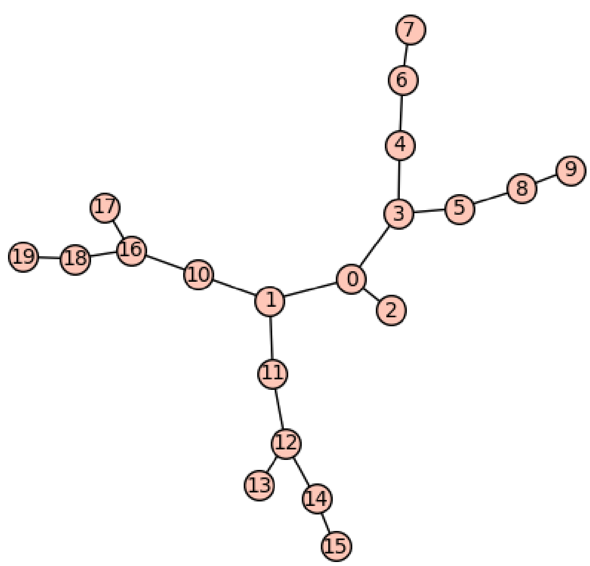
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Now, to find three strongly cospectral vertices, the goals are:

4. Change the path joining T_1 and T_2 by a star or a general tree, keeping the cospectrality.

5. Extend the results adding T_3 and verifying what is necessary for a and b to be parallel

6. Approach Lemma 5.6.1 without assuming that the matrices are similar, but using the fact that $\phi(T_1\backslash a,t)/\phi(T_1,t) = \phi(T_2\backslash b,t)/\phi(T_2,t)$

Answer 4. We claim: assuming that $\phi(T_1\backslash a,t)/\phi(T_1,t) = \phi(T_2\backslash b,t)/\phi(T_2,t)$, if T_v is a tree join T_1 and T_2 such that $\{a,b\} \subseteq V(T_v)$ and $\phi(T_v\backslash a,t) = \phi(T_v\backslash b,t)$, then a and b are cospectral.

The proof is straightforward using the 1-sums lemma:

$$\phi(T\backslash a,t) = \phi(T_1\backslash a,t)(\phi(T_2\backslash b,t)\phi(T_v\backslash a,t) + \phi(T_2,t)\phi(T_v\backslash ab,t) - t\phi(T_2\backslash b,t)\phi(T_v\backslash ab,t))$$

$$\phi(T\backslash b,t) = \phi(T_2\backslash b,t)(\phi(T_1\backslash a,t)\phi(T_v\backslash b,t) + \phi(T_1,t)\phi(T_v\backslash ab,t) - t\phi(T_1\backslash a,t)\phi(T_v\backslash ab,t))$$

If $\phi(T_v\backslash a,t) = \phi(T_v\backslash b,t)$ then the two terms are equal.

- Comment: Here I don't carry the iff condition, but I guess it will be nice if we want to proof that the triple of cospectral vertices don't exists.

Answer 5. The above results can be easily extended if we have T_3 such that $\frac{\phi(T_1\backslash a,t)}{\phi(T_1,t)} = \frac{\phi(T_2\backslash b,t)}{\phi(T_2,t)} = \frac{\phi(T_3\backslash c,t)}{\phi(T_3,t)}$, $\{a,b,c\} \subseteq V(T_v)$ and $\phi(T_v\backslash a,t) = \phi(T_v\backslash b,t) = \phi(T_v\backslash c,t)$ then a, b and c are cospectral.

And we can construct a tree with 3 cospectral vertices by letting T_v be a star with tree leaves: (a, b and c) and a central vertex v.

But we still need to discovery when a, b and c will be parallel, for this, let's try a similar approach used at theorem 6.7.1, but with our construction:

a and b are parallel if and only if $\phi(T\backslash ab,t)/\phi(T,t)$ have simple poles:

$$\begin{aligned}
\frac{\phi(T\backslash ab,t)}{\phi(T,t)} &= \phi(T_3 \cup v,t)\frac{\phi(T_1\backslash a,t)\phi(T_2\backslash b,t)}{\phi(T,t)} \\
&= \frac{\phi(T_3 \cup v,t)}{\phi(T_3,t)}\frac{\phi_{ab}(T,t)}{\phi(T,t)} \\
&= (\phi(v,t) - \frac{\phi(T_3\backslash c,t)}{\phi(T_3,t)})\frac{\phi_{ab}(T,t)}{\phi(T,t)}
\end{aligned}$$

After run some tests at all trees up to 15 vertices, I found 400 triples of trees T_1, T_2 and T_3, but using this construction none of the resulting trees had 3 strongly cospectral vertices.

- Comment: I suspect that this equality prevents a and b to be parallel if $\frac{\phi(T_3\backslash c,t)}{\phi(T_3,t)} = \frac{\phi(T_1\backslash a,t)}{\phi(T_1,t)}$, or something a little more general.

Answer 6. At this part I became a little confused, because we cannot state much about all the eigenvalues of $A(T_1)$ and $A(T_2)$, so I have to think more.

Instead, we can say something about the eigenvalue support of a at T_1 and of b at T_2:

From equation 4.3.4,

$$(E_r)_{a,a} = \frac{\phi(X \backslash a, t)(t - \theta_r)}{\phi(X, t)}\Big|_{t=\theta_r}$$

And $(E_r)_{a,a} \neq 0 \iff \theta_r$ is a pole of $\frac{\phi(X\backslash a,t)}{\phi(X,t)}$. Since $\frac{\phi(T_1\backslash a,t)}{\phi(T_1,t)} = \frac{\phi(T_2\backslash b,t)}{\phi(T_2,t)}$, a have the same eigenvalue support at T_1 as b at T_2.