Notes on the Average Mixing Matrix

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Abstract

We determine a quadratic formula for the eigenvalues of graphs Γ with exactly two strongly cospectral classes. These imply that for the numbers of vertices v and average degree d we have $v \leq cd^4$ for some constant c.

1 Two Strongly Cospectral Classes

The main focus is to show that there are only finitely many graphs with an average mixing matrix of rank 2. The notation of Chris Godsil's notes is used if not mentioned otherwise.

Remark 1. *The rank of the average mixing matrix of a graph X is bounded by the number of strongly cospectral classes of X.*

Question 2. *When do we have equality in*

$$col(\widehat{M}) \subseteq \langle col(E_i^{\circ 2}) \rangle?$$

My guess is always, but even always for $\dim(\mathrm{col}(\widehat{M})) = 2$ would be helpful (see remark below).

In the following X is always a connected graph with two strongly cospectral classes. We denote the corresponding partition of the vertices by $\{X_1, X_2\}$. We write the adjacency matrix A of X partitioned by $\{X_1, X_2\}$ as

$$A = \begin{bmatrix} A_1 & B \\ B^T & A_2 \end{bmatrix}.$$

Recall that the walk modules of all vertices X_i are the same for $i \in \{1, 2\}$ fixed, so that the following is well-defined. Define $k_1 = k_{11}, k_2 = k_{22}, k_{12}, k_{21}$ as follows. Here $x_i \in X_i$.

$$k_{ij} = |\{x \in X_j : x \sim x_i\}|.$$

Notice that X connected implies that $k_{12}, k_{21} > 0$. Denote $|X_i|$ by v_i (and $v = |X|$). Let $\mathbf{j}_1 \in \mathbb{C}^v$ be the vector with 1 in the first v_1 positions and zeroes otherwise. Set $\mathbf{j}_2 = \mathbf{j} - \mathbf{j}_1$.

The following is a consequence of the partition $\{X_1, X_2\}$ being equitable.

Lemma 3. *Let α_1 and α_2 be the two solutions of*

$$k_{12}\alpha^2 + (k_1 - k_2)\alpha - k_{21} = 0.$$

Set $\lambda_i = k_1 + \alpha_i k_{12}$. Then the following holds:

(a) $\alpha_1, \alpha_2 \in \mathbb{R}$.

(b) $\alpha_1 \neq \alpha_2$.

(c) $\mathbf{j}_1 + \alpha_i \mathbf{j}_2$ *is an eigenvector of X for $i \in \{1, 2\}$.*

(d) λ_i *is an eigenvalue of X for $i \in \{1, 2\}$.*

Proof. The discriminant of the equation is $(k_1 - k_2)^2 + 4k_{12}k_{21} > 0$. This shows (a) and (b).

Suppose that $\mathbf{j}_1 + \alpha \mathbf{j}_2$ is an eigenvector of X. Multiplying with A yields the conditions

$$\begin{aligned} k_1 + k_{12}\alpha &= \lambda, \\ k_{21} + k_2\alpha &= \alpha\lambda. \end{aligned}$$

Solving this equation yields (c) and (d). □

In the following we call $\mathbf{j}_1 + \alpha_i \mathbf{j}_2$ trivial eigenvectors.

Question 4. *In all the examples that I checked it seems to be that there are eigenvectors with eigenvalues corresponding to the solution of*

$$k_{12}\alpha^2 + (k_2 - k_1)\alpha - k_{21} = 0.$$

Can we guarantee the existence of these?

Remark 5. *Notice that if*

$$col(\widehat{M}) = \langle col(E_i^{\circ 2}) \rangle,$$

then in particular

$$col(E_i^{\circ 2}) \subseteq \langle \mathbf{j}_1 + \alpha_1 \mathbf{j}_2, \mathbf{j}_1 + \alpha_2 \mathbf{j}_2 \rangle.$$

This implies that our E_i's have only three different absolute values (depending on if we are looking at A_1, A_2 or B). That we use in the following.

Corollary 6. *Let χ be a non-trivial eigenvector of X. Then the following holds:*

(a) There are non-negative constants α and β such that $|\chi_i| = \alpha$ if $i \in X_1$ and $|\chi_i| = \beta$ if $i \in X_2$.

(b) We have $\mathbf{j}^T \chi_{\restriction X_i} = 0$ for all $i \in \{1, 2\}$ (i.e. α and $-\alpha$ occur the same time).

(c) $|X_i| = 1$ or $|X_i|$ is even for $i \in \{1, 2\}$.

Proof. Part (a) clear due to Remark 5.

Assume WLOG that $\alpha_2 \neq 0$ (we can do this by Lemma 3 (b)). For Part (b) notice that, by Lemma 3, χ is orthogonal to the vectors

$$(\mathbf{j}_1 + \alpha_1 \mathbf{j}_2) - \frac{\alpha_1}{\alpha_2}(\mathbf{j}_1 + \alpha_2 \mathbf{j}_2) = c_1 \mathbf{j}_1 \text{ and}$$
$$(\mathbf{j}_1 + \alpha_1 \mathbf{j}_2) - (\mathbf{j}_1 + \alpha_2 \mathbf{j}_2) = c_2 \mathbf{j}_2.$$

Here c_1, c_2 are some constants with $c_1, c_2 \neq 0$ by Lemma 3 (b).

Part (c) follows from (b). □

Lemma 7. *Let λ be an eigenvalue of X. Let χ be the corresponding eigenvector. Then one of the following occurs:*

(a) $\chi_{\restriction X_i} = 0$ *and* $\lambda \in \{-k_{22}, -k_{22}+2, \ldots, k_{22}-2, k_{22}\}$.

(b) We can scale χ such that $|\chi_1| = 1$ and λ is a solution of a equation of the form

$$a_{11} + \alpha a_{12} = \lambda,$$
$$a_{21} + \alpha a_{22} = \alpha\lambda.$$

for some $a_{ij} \in \{-k_{ij}, -k_{ij}+2, \ldots, k_{ij}-2, k_{ij}\}$.

Proof. As all vertices in X_1, resp., X_2 have the same walk modules, there are constants a_{ij} such that $(A\chi)_h = a_{i1} + \alpha a_{i2}$ for all $h \in X_i$. This yields the given equation. By Corollary 6 (b), the constants a_{ij} are in $\{-k_{ij}, -k_{ij}+2, \ldots, k_{ij}-2, k_{ij}\}$. □

Maybe it is more convenient to see eigenvalues as the roots of a quadratic polynomial. This polynomial is fairly nice looking:

$$\lambda^2 + \left(\frac{a_2 - a_1}{a_{12}}\right)\lambda + \frac{a_1 a_2}{a_{12}} - a_{21}.$$

Corollary 8. *Let X be a connected cubic graph with simple eigenvalues and two strongly cospectral classes. Then $|X| \leq 12$.*

Proof. We apply Lemma 7. Clearly,

$$k_{11} + k_{12} = 3 \text{ and } k_{21} + k_{22} = 3.$$

Furthermore, $k_{12}, k_{21} \geq 1$. If $(k_{11}, k_{12}, k_{21}, k_{22}) = (1, 2, 1, 2)$, then the choices for Lemma 7 (b) are as follows:

a_{11}	a_{12}	a_{21}	a_{22}	Real Solutions
1	2	1	2	$0, 3$
1	2	1	0	$-1, 2$
1	2	1	-2	$-\frac{2\sqrt{17}+10}{\sqrt{17}+3}, -\frac{2\sqrt{17}-10}{\sqrt{17}-3}$
1	2	-1	2	
1	2	-1	0	
1	2	-1	-2	$-1, 0$
1	0	1	2	1
1	0	1	0	1
1	0	1	-2	1
1	0	-1	2	1
1	0	-1	0	1
1	0	-1	-2	1
And so on ...				

- For $(2, 1, 1, 2)$ we obtain 13 different eigenvalues.
- For $(1, 2, 2, 1)$ we obtain 5 different eigenvalues.
- For $(1, 2, 1, 2)$ we obtain 10 different eigenvalues.
- For $(0, 3, 3, 0)$ we obtain 6 different eigenvalues.
- For $(0, 3, 2, 1)$ we obtain 7 different eigenvalues.
- For $(0, 3, 1, 2)$ we obtain 12 different eigenvalues.

So together with the first count in the worst case we obtain 13 simple eigenvalues. As $k_{12} = k_{21}$ implies $|X_1| = |X_2|$, we have $|X|$ even. Hence, at most 12 vertices. $\square$

Together with our computer search this implies that there is no cubic graph with simple eigenvalues and only two strongly cospectral classes (I think). This might be obvious for other reasons. I did not think about that. The same argument shows a bound of 26 for 4-regular graphs.