Graphs, Groups and Pseudo-Similar Vertices

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GRAPHS, GROUPS AND PSEUDO-SIMILAR VERTICES

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Abstract

Vertices $u_0, u_1, \ldots, u_{k-1}$ of a graph X are mutually *pseudo-similar* if $X - u_0 \cong X - u_1 \cong \cdots \cong X - u_{k-1}$, but no two of the vertices are related by an automorphism of X. We describe a method for constructing graphs with a set of $k \geq 2$ mutually pseudo-similar vertices, using a group with a special subgroup. We show that in all graphs with pseudo-similar vertices, the vertices are pseudo-similar due to the action of a group on the cosets of some subgroup.

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1. Introduction

In Godsil and Kocay [4], a method was given for constructing graphs with a pair of pseudo-similar vertices. It was also shown that all graphs with a pair of pseudo-similar vertices could be constructed in this way. In Godsil and Kocay [5], this method was generalized somewhat, and a method was given for constructing graphs with three mutually pseudo-similar vertices. The method relied on finding a group and a subgroup with special properties.

In this paper we give a general method for constructing graphs with a set of $k \geq 2$ mutually pseudo-similar vertices, using an extension of the ideas of [5], based on a group with a certain subgroup. We also show that all graphs with k mutually pseudo-similar vertices can be constructed in this way. In a certain sense then, we can say that graphs with pseudo-similar vertices exist because of the existence of groups with subgroups with certain properties. This more or less answers the question of why do pseudo-similar vertices exist, a question which

was raised in Harary and Palmer [6], when investigations of the reconstruction conjecture resulted in pseudo-similar vertices being discovered. It leaves open the question of when do such groups with special subgroups exist, and how does one find them.

The main construction is based on Schreier coset diagrams, defined in Section 2 following. We refer the reader to Chapter 8 of Bollobas [1] for a concise statement of their properties.

We shall use the graph-theoretic terminology of Bondy and Murty [2].

2. Schreier coset diagrams and partial permutations

Let G be a group generated by elements $p_1, p_2, \ldots, p_k \in G$. Let $p = \{p_1, p_2, \ldots, p_k\}$, and let $K \leq G$ be a subgroup of G. Let $A = \{Ka_1, Ka_2, \ldots, Ka_n\}$ be a complete set of right cosets of K in G, where we assume that $a_1 = e$, the identity in G. The *Schreier coset diagram* $G \bmod K$ is an edge-coloured directed graph with vertex set A. The edge colours are the symbols $p_1, p_2, \ldots, p_k$. Vertex Ka_i is joined to $Ka_i p_j$ by a directed edge of colour p_j, for all $i = 1, 2, \ldots, n$, and all $j = 1, 2, \ldots, k$.

It is well known that walks in $G \bmod K$ correspond to words in the generators P of G. Moreover, if T is any spanning tree of $G \bmod K$, then the unique path in T joining $K = Ka_1$ to any vertex Ka_i defines a representative for the coset Ka_i. The edges of $G \bmod K$ which are not in the spanning tree T define a set of generators for K, as follows. If an edge $(Ka_i, Ka_i p_j)$ not in T is added to T, this creates a unique closed walk of base K in $G \bmod K$ using the edge $(Ka_i, Ka_i p_j)$ and edges from T. If Ka_i and $Ka_i p_j$ have representatives r_i and r_j, respectively, then the word corresponding to this closed walk is $r_i p_j r_j^{-1}$. As the closed walks so defined by a spanning tree T generate all the closed walks of base K of the graph and since every word representing an element in K corresponds to a closed walk of base K in $G \bmod K$ it follows that the edges of $G \bmod K$ not in T define words which generate the subgroup K. This is the essence of the Nielson-Schreier Theorem (see Bollobas [1], Imrich [7], or Johnson [8]).

2.1. DEFINITION. Let X be a graph. Let $u, v \in V(X)$ be two vertices such that $X - u \cong X - v$, but no automorphism of X maps u to v. Then u and v are *pseudo-similar* vertices of X. If u and v are related by an automorphism, then they are *similar*.

Now let X be a graph with a set $U = \{u_0, u_1, \ldots, u_{k-1}\} \subseteq V(X)$ of $k \geq 2$ mutually pseudo-similar vertices. Then $X - u_0 \cong X - u_1 \cong \cdots \cong X - u_{k-1}$, but

no two of the vertices of U are related by an automorphism of X. Many examples of such graphs for $k = 2$ are given in Kocay [10]. Kimble, Stockmeyer, and Schwenk [9] given an infinite family of such graphs, one for every $k \geqslant 2$. Krishnamoorthy and Parthasarathy [14] also given an infinite family, for k a power of two. Godsil and Kocay [5] and Kocay [12] give several other examples for $k = 3$. Graphs with large sets of k mutually pseudo-similar vertices are hard to find. We explain why this is so in Section 4.

Let $p_i: X - u_i \to X - u_0$ be an isomorphism, for each $i = 1, 2, \ldots, k - 1$. Each p_i maps $V(X) - u_i$ to $V(X) - u_0$, so that each p_i is "almost" a permutation of $V(X)$, in the sense that only u_i has no image under p_i, and only u_0 is not an image. We call p_i a *partial permutation* of $V(X)$. For the sake of formality, we define this.

2.2. DEFINITION. Let V be any finite set. Let p be a one-to-one map from a proper subset of V into V, such that Im $p \neq$ Dom p. Then p is called a *partial permutation* of V.

Partial permuations p for which $|\text{Im } p| = |V| - 1$ turn out to be important in the pseudo-similar vertex problem.

We shall extend each partial permutation p_i of $V(X)$ above to a permutation p_i^* of a larger set $V(Y)$. The action of the group generated by the p_i^*, $i = 1, 2, \ldots, k - 1$, will then make it clear why the orginal vertices U are pseudo-similar in X.

3. Orbits of partial permutations

Given the set $P = \{p_1, p_2, \ldots, p_{k-1}\}$ of partial permutations, we define a graph Γ analogously to a Schreier coset diagram. Γ has coloured, directed edges. The edge colours are the symbols $p_1, p_2, \ldots, p_{k-1}$. $V(\Gamma) = V(X)$. Vertex $u \in V(\Gamma)$ is joined by an edge of colour p_i to $(u)p_i$, if u has an image under p_i, for all $i = 1, 2, \ldots, k - 1$.

Now if the p_i were permutations, then the connected components of Γ would be the orbits of $\langle P \rangle$, the group generated by P. It is easy to see that each connected component (or orbit) of Γ would then be a Schreier coset diagram $\langle P \rangle$ mod K, where K is any point-stabiliser subgroup for that orbit. In fact, this is often used as the basis for computer algorithms which determine the order of a permutation group defined by a set of generating permutations, in terms of a nested sequence of stabiliser subgroups.

We accordingly call the connected components of Γ the *orbits* of the set of partial permutations P. An orbit A is *closed* if every vertex u in A has an in-edge

and an out-edge of colour p_i, for every $i = 1, 2, \ldots, k - 1$. (A loop is counted as both an in-edge and an out-edge, in case u is a fixed point of p_i.) Otherwise an orbit is *open*.

3.1. THEOREM. *P has exactly one open orbit.*

PROOF. Consider $p_i: V(X) - u_i \to V(X) - u_0$. p_i maps u_0 to $(u_0)p$. If $(u_0)p_i \neq u_i$, then we can find $(u_0)p_i^2$, and so on. Eventually we must have $(u_0)p_i^l = u_i$, for some positive integer l, since p_i is one-to-one, and only u_i has no image under p. Thus there is a directed path of colour p_i from u_0 to u_i in Γ, for every $i = 1, 2, \ldots, k - 1$. Thus all the $u_i \in U$ lie in one connected component of Γ. This is the only open orbit of P.

Let $\overline{U}$ denote the orbit of P containing U, and let $V' = V(X) - \overline{U}$.

If, $V' \neq \varnothing$, then $p_i | V'$, the restriction of p_i to V', is an automorphism of the induced graph $X[V']$. The set $P | V' = \{ p_i | V' \}$ generates a subgroup of Aut $X[V']$. Furthermore each connected component of $\Gamma[V']$ is a Schreier coset diagram for a point-stabilizer subgroup of the group of automorphisms generated by $P | V'$

We can make the open orbit $\overline{U}$ into a closed orbit, thereby completing the partial permutations, as follows.

Let G denote the free group (see Johnson [8]) generated by the symbols of $P = \{ p_1, p_2, \ldots, p_{k-1} \}$. Let T denote a spanning tree of $\Gamma[\overline{U}]$. We shall define a subgroup K of G as follows. Label the vertex u_0 in $\Gamma[\overline{U}]$ by the coset Ke, where e is the identity in G. For each vertex $u \in \overline{U}$, label u with $Kw(u)$, where $w(u)$ is the unique word in the p_i corresponding to the unique path in T from u_0 to u. For each directed edge uv of colour p_i in $\Gamma[\overline{U}]$ define the *label* of uv as $w(u)p_iw(v)^{-1}$. Let K be the subgroup of G generated by the labels of the edges of $\Gamma[\overline{U}]$. Using Schreier's method (see Bollobas [1]), we have:

3.2. THEOREM. *The Schreier coset diagram G mod K contains $\Gamma[\overline{U}]$ as an induced subgraph.*

This gives the following theorem.

3.3. THEOREM. *Let X be a graph with a set $U = \{u_0, u_1, \ldots, u_{k-1}\}$ of k mutually pseudo-similar vertices. Let $p_i: X - u_i \to X - u_0$ be isomorphisms, for $i = 1, 2, \ldots, k - 1$. Then X can be extended to a graph Y, and each p_i can be extended to a permutation p_i^* of $V(Y)$ such that:*
  (i) *X is an induced subgraph of Y;*
  (ii) *each p_i^* is an automorphism of Y;*
  (iii) *the vertices of U are all similar in Y;*
  (iv) *the vertices of $V(Y) - V(X)$ are all in the same orbit of Aut Y as U.*

PROOF. Let G and K be as above. Let $A = \{Ka_i | i \in I\}$ be a complete set of right cosets of K in G. Let $V(Y) = V' \cup A$. Notice that $V(X) \subseteq V(Y)$ since $V' \subseteq V(X)$, and $\bar{U} \subseteq A$. We define the edge set of Y as follows.

If $uv \in E(X)$, where $u, v \in V'$, then $uv \in E(Y)$.

Let u_0 be joined to vertices M of V' in X. Then join each coset $Ka \in V(Y)$, where a is a word in the p_i, to $(M)a$. This is possible, since $p_i | V'$ is an automorphism of $X[V']$. Thus arbitrary words in the p_i define permutations of V'. If the coset Ka is in $V(X)$, this agrees with edges already present in X, since each p_i is an isomorphism from $X - u_i$ to $X - u_0$, that is, it adds no new edges to X. Moreover, this includes all edges from V' to $\bar{U}$ in X.

We have left only edges in $X[\bar{U}]$. Let uv be an edge of X in $X[\bar{U}]$. Suppose that u and v correspond in $V(Y)$ to cosets Ka_1 and Ka_2 of K. For each word $a \in G$, define an edge (Ka_1a, Ka_2a) in Y. Clearly this includes all edges of $X[\bar{U}]$. It adds no new edges to X, since each p_i is an isomorphism, and since the only circuits in the Schreier coset diagram $G \bmod K$ are contained in $\Gamma[\bar{U}]$, by the definition of K.

It follows that X is an induced subgraph of Y. Moreover each p_i extends to a permutation p_i^* in the permutation representation of G in terms of the right cosets of K. Thus each p_i^* is an automorphism of Y. Since $(u_0)p_i^{l_i} = u_i$, for each $i = 1, 2, \ldots, k - 1$, where l_i is a positive integer, it follows that all the vertices of U are similar in Y, and clearly all the new vertices added to X to get Y lie in one orbit, the orbit of $\bar{U}$.

Theorem 3.3 is the k-analog of the main theorem of Godsil and Kocay [4] for pairs of pseudo-similar vertices. The main difference is that in [4] the group G was made finite, so that the resultant graph Y was finite. In Theorem 3.3 above, the graph Y is always infinite.

If we adjoin relators to G, consistent with Aut $X[V']$, to make G into a finite group, then K will become finite, too. So long as the cosets of K corresponding to the vertices of $\Gamma[\bar{U}]$ are all distinct, and $G \bmod K$ does not add any new edges to $\Gamma[\bar{U}]$, then we can use this finite G and K to create a finite graph Y. If $X[\bar{U}]$ contains no edges, then this will work fine. But if $X[\bar{U}]$ contains edges, then we cannot be sure that a finite G will not add new edges to X.

3.4. PROBLEM. Do there exist graphs X with $k \geqslant 3$ mutually pseudo-similar vertices for which the graph Y cannot be made finite?

3.5. PROBLEM. Do there exist graphs X with $k \geqslant 3$ mutually pseudo-similar vertices for which $V' = \varnothing$?

In [4] (see also [11]), it was proved that when $k = 2$, we must always have $|V'| \geqslant 6$. All known examples of graphs with $k \geqslant 3$ have $V \neq \varnothing$. It would seem reasonable to expect that when $k \geqslant 3$, we can find graphs X with $V' = \varnothing$.

In the same way as we defined the orbits of the set P of partial permuations, we can define the orbits of P on the pairs of vertices of X, that is, the *pair-orbits* of P. Unlike Theorem 3.1, P will in general have several open pair orbits. A solution to Problems 3.4 and 3.5 would apparently require a study of the pair-orbits of P.

4. Constructing graphs with k mutually pseudo-similar vertices

In [5] a method was given for constructing graphs with three mutually pseudo-similar vertices, based on a group G with a special subgroup K. Section 3 above shows that corresponding to any graph with k mutually pseudo-similar vertices, we can construct a group G with a subgroup K, such that the vertices are pseudo-similar due to the action of G on the cosets of K.

In this section we show how to use these ideas to construct graphs with pseudo-similar vertices.

4.1. EXAMPLE. Let $k = 2$. Let $U = \{u_0, u_1\}$ and let $p: X - u_1 \to X - u_0$. Then $\Gamma[\overline{U}]$ is easily determined. It must be a directed path of length $l \geqslant 1$ from u_0 to u_1. When $k = 2$, we must have $|V'| \geqslant 6$ (see [4] or [11]).

Since $\Gamma[\overline{U}]$ is a tree, we must have $T = \Gamma[\overline{U}]$. It follows that G is the free group generated by p, $G = \langle p \rangle$, and K is the identity. If we make G finite it becomes a cyclic group, and we then easily construct pseudo-similar vertices.

Only in the case of $k = 2$ can we completely classify all groups G giving rise to pseudo-similar vertices, that is, all those which contain an element of period three or more.

When $k \geqslant 3$, it is impossible to classify, in general, all the possible equivalence classes for the coset diagram $\Gamma[\overline{U}]$, where we define equivalence of two coset diagrams $\Gamma_1[\overline{U}]$ and $\Gamma_2[\overline{U}]$ as follows.

If $U = \{u_0, u_1, \ldots, u_{k-1}\}$ and $P = \{p_1, p_2, \ldots, p_{k-1}\}$, defined $p_{ij} = p_i p_j^{-1}$, for $1 \leqslant i < j \leqslant k - 1$, where the product of isomorphisms is read from left to right. Then $p_{ij}: X - u_i \to X - u_j$. Let $P' = P \cup \{p_{ij} | 1 \leqslant i < j \leqslant k - 1\}$.

For each $p_{ij} \in P'$, we add edges of colour p_{ij} to $\Gamma[\overline{U}]$, in the same way as edges corresponding to p_i, $i = 1, 2, \ldots, k - 1$, were introduced. This results in the *completed coset diagram* $\Gamma'[\overline{U}]$.

4.2. DEFINITION. Two coset diagrams $\Gamma_1[\overline{U}]$ and $\Gamma_2[\overline{U}]$ with sets of partial permutations P_1 and P_2, respectively, are *equivalent* if there exist two bijections,

both denoted θ, between the vertex-sets, and edge-sets of $\Gamma_1'[\bar{U}]$ and $\Gamma_2'[\bar{U}]$ such that:

(i) for each colour $p \in P_1'$, θ maps the edges of colour p in $\Gamma_1'[\bar{U}]$ to the edges of some colour $p' \in P_2'$ in $\Gamma_2'[\bar{U}]$;

(ii) for each colour $p \in P_1'$, θ either preserves the direction of all the edges coloured p, or reverses the direction of all of them.

4.3. EXAMPLE. Let $k = 3$. Let $U = \{u_0, u_1, u_2\}$ and let $p_1 \colon X - u_1 \to X - u_0$ and $p_2 \colon X - u_2 \to X - u_0$. Then $p_{12} = p_1 p_2^{-1} \colon X - u_1 \to X - u_2$, and $P' = \{p_1, p_2, p_{12}\}$. If $|\bar{U}| = 3$, so that $\bar{U} = U$, there are exactly two possible equivalence classes for $\Gamma[\bar{U}]$, shown in Figures 4.1 and 4.2.
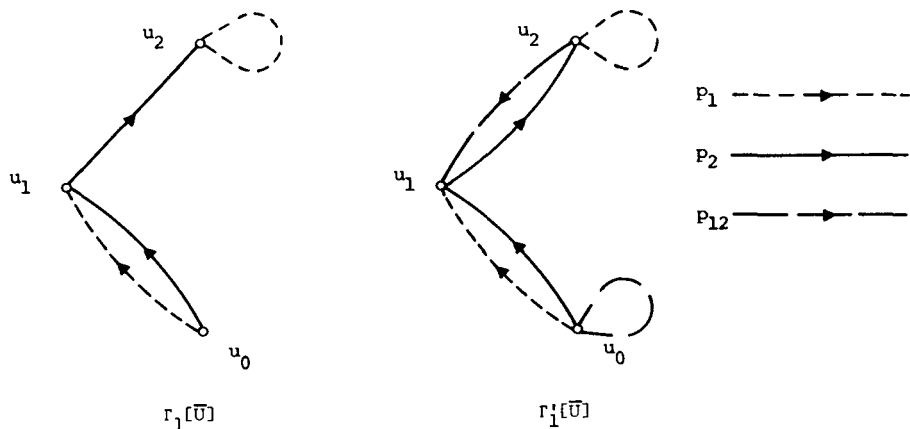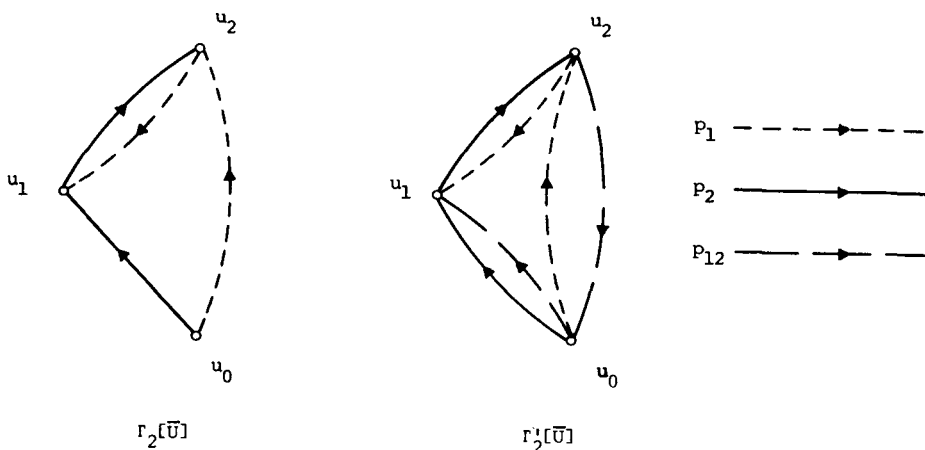

Figure 4.1.   Type I

Figure 4.2.   Type II

Both of these configurations have been used in [5] to construct graphs with three mutually pseudo-similar vertices. If we choose the solid edges in each coset diagram as the spanning tree T, then we get the following subgroups K.

I. $K = \langle p_1 p_2^{-1}, p_2^2 p_1 p_2^{-2} \rangle$.

II. $K = \langle p_1 p_2^{-2}, p_2^2 p_1 p_2^{-1} \rangle$.

If $k = 3$ and $|\overline{U}| = 4$, there are exactly 8 equivalence classes for $\Gamma[\overline{U}]$. If $k = 4$ and $|\overline{U}| = 4$, there are exactly 19 equivalence classes for $\Gamma[\overline{U}]$. These are enumerated in Kocay [13].

A basic recipe for constructing a graph X with k mutually pseudo-similar vertices can then be given as follows. We first need the definition of a Cayley graph.

4.4. DEFINITION. Let G be a group with a generating set H such that $e \notin H$. The *Cayley graph* $C(G, H)$ has vertex set G and edge set $\{(g, hg)|g \in G, h \in H\}$. Thus G acts as a group of automorphisms of $C(G, H)$, by right multiplication. In case $C(G, H)$ has no other automorphisms, $C(G, H)$ is called a *graphical regular representation*, or GRR, of G (see Godsil [3]). Except for 10 exceptional graphs of order at most 32, all finite, non-abelian groups which are not generalized dicyclic groups have GRR's (see [3]).

4.5. CONSTRUCTION.

1. Given k, let $U = \{u_0, u_1, \ldots, u_{k-1}\}$.

2. Choose a set $\overline{U}$ of vertices so that $U \subseteq \overline{U}$.

3. Given colours $p_1, p_2, \ldots, p_{k-1}$, draw a directed path of colour p_i from u_0 to u_i in $\overline{U}$, for $i = 1, 2, \ldots, k - 1$.

4. If there are any vertices of $\overline{U}$ not on the directed path from u_0 to u_i of colour p_i, then add edges of colour p_i so that a disjoint union of cycles is induced on these vertices by the edges of colour p_i.

5. Choose a spanning tree T of the resultant $\Gamma[\overline{U}]$, and label the vertices of $\Gamma[\overline{U}]$ with the coset representatives of K, given by the unique path in T from u_0 to each vertex.

6. Read off a set of generators for K.

7. Find a finite group G generated by $P = \{p_1, p_2, \ldots, p_{k-1}\}$ and identify K as a subgroup.

8. Draw a Cayley graph for G, and call it $X[V']$.

9. Find the pair-orbits or P on $\overline{U}$. Make some of the pair-orbits into edges in X, if desired.   .

The resultant graph X satisfies $X - u_0 \cong X - u_1 \cong \cdots \cong X - u_{k-1}$, since G acts on the right cosets of K in $X[V']$ by right multiplication. If $X[V']$ is a GRR, then we can be sure there will be no automorphisms of X connecting any of the

$u_i \in U$. For then the only automorphisms of $X[V']$ are those arising from G. It follows that the vertices u_i, $i = 0, 1, \ldots, k - 1$, are mutually pseudo-similar in X.

The reason graphs with $k \geq 4$ mutually pseudo-similar vertices are hard to find is because of the difficulty with step 7, namely finding a suitable group G and identifying the subgroup K. The author knows of no general algorithm for finding such a G and K.

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