JOURNAL OF COMBINATORIAL THEORY (B) 22, 10–25 (1977)

On the Automorphism Group of a Tree

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Communicated by W. T. Tutte

Received October 17, 1974

It is shown that $H = \Gamma(T)_v$ is normal in $G = \Gamma(T_v)$ for any tree T and any vertex v, if and only if, for all vertices u in the neighborhood N of v, the set of images of u under G is either contained in N or has precisely the vertex u in common with N and every vertex in the set of images is fixed by H. Further, if S is the smallest normal subgroup of G containing H then G/S is the direct product of the wreath products of various symmetric groups around groups of order 1 or 2. The degrees of the symmetric groups involved depend on the numbers of isomorphic components of T_v and the structure of such components.

1. INTRODUCTION

If $\mathbf{G}$ is any graph, let $V(\mathbf{G})$ denote the vertex set of $\mathbf{G}$ and let $\Gamma(\mathbf{G})$ denote its automorphism group considered as a group of permutations on $V(\mathbf{G})$. If $v \in V(\mathbf{G})$, we denote by $\mathbf{G}_v$ the subgraph of $\mathbf{G}$ obtained by deleting v and all edges incident with v. We denote the stabilizer of v by $\Gamma(\mathbf{G})_v = \{g \in \Gamma(\mathbf{G}) \mid vg = v\}$. Note that $\Gamma(\mathbf{G})_v$ can be embedded in $\Gamma(\mathbf{G}_v)$. G is said to be semistable at $v \in V(\mathbf{G})$ if and only if $\Gamma(\mathbf{G})_v = \Gamma(\mathbf{G}_v)$. This concept was introduced by Holton [2] and, with the related concept of stability, has been extensively studied [4].

In this paper we examine other relationships between the groups $\Gamma(T_v)$ and $\Gamma(T)_v$ for any tree T at any vertex $v \in V(T)$. In particular, we derive necessary and sufficient conditions that $\Gamma(T)_v$ be a normal subgroup of $\Gamma(T_v)$ and, denoting by $S(T)_v$ the smallest normal subgroup of $\Gamma(T_v)$ containing $\Gamma(T)_v$, we find the quotient group $\Gamma(T_v)/S(T)_v$.

As the symbols $\Gamma(T_v)$, $\Gamma(T)_v$, and $S(T)_v$ are cumbersome, we simplify these to G, H, and S whenever this is unambiguous. Basic graph–theoretic properties and terminology can be found in [1] and group–theoretic terminology can be found in [5]. Throughout T denotes a tree and $N_{\mathbf{G}}(v)$ denotes the set of vertices of $\mathbf{G}$ adjacent to $v \in V(\mathbf{G})$.

2. Preliminary Results

The most useful characterization for the semistability of a vertex of any graph was proved in [3] and is given in Lemma 2.1. This criterion will be used extensively throughout this paper.

Lemma 2.1. *$N_T(v)$ is fixed by $g \in G$ if and only if $g \in H$. $H = G$ if and only if $N_T(v)$ is fixed by every $g \in G$.*

Let $z \in V(T)$ and let $A_1, \ldots, A_n$ be the branches of T at z. Further, let F be any forest and let $B_1, \ldots, B_m$ be the components of F.

Definition 2.2. A *z-branch transposition* of T is an automorphism $g \in \Gamma(T)_z$ which is either

(i) the identity mapping on all branches other than A_i for some $i \in \{1, \ldots, n\}$, or

(ii) an involution which interchanges A_i and A_j for some $1 \leqslant i < j \leqslant n$ and which acts as the identity on A_k for each $k \neq i, j$.

A *branch transposition* of F is an automorphism $g \in \Gamma(F)$ which is either

(i) the identity mapping on all components except B_i for some $i \in \{1, \ldots, n\}$, or

(ii) an involution which interchanges B_i and B_j for some $1 \leqslant i < j \leqslant n$ and which acts as the identity on B_k for each $k \neq i, j$.

We omit the straightforward proof of the following lemma.

Lemma 2.3. *If $g \in \Gamma(T)_z$, there exists a z-branch transposition which acts like g on A_i for any given $i \in \{1, \ldots, n\}$. If $f \in \Gamma(F)$ there is a branch transposition which acts like f on B_i for any given $i \in \{1, \ldots, n\}$.*

Lemma 2.4. *Every $g \in \Gamma(T)_z$ is a product of z-branch transpositions of T. Every $f \in \Gamma(F)$ is a product of branch transpositions of F.*

Proof. We will give a proof by induction for $g \in \Gamma(T)_z$. The proof for $f \in \Gamma(F)$ follows in a similar way. Let T have n branches, $A_1, \ldots, A_n$ at z. If g is the identity map on at least $n - 1$ branches, then g is a z-branch transposition and we are done. Now assume that g is the identity map on all but two branches A_i and A_j. Firstly, if $A_i g = A_i$ and $A_j g = A_j$ then, as in Lemma 2.3, we may define z-branch transpositions g_i and g_j by $g_i = g$ on A_i and $g_i = e$ elsewhere whilst $g_j = g$ on A_j and $g_j = e$ elsewhere. Then $g = g_i \cdot g_j$. Secondly, if $A_i g = A_j$ then $A_j g = A_i$ and by Lemma 2.3 there exist z-branch transpositions g_{ij} acting like g on A_i and g_i acting like g^2 on A_i. Then if $x \in A_i$, $xg_{ij}g_i = xgg_i = xg$ whilst if $x \in A_j$, $xg_{ij}g_i = xg^{-1}g_i = xg$. Consequently $g = g_{ij}g_i$.

Now let us assume that the lemma is true for all automorphisms in $\Gamma(T)_z$ which are not the identity map on at most $m - 1$ branches and assume that $g \in \Gamma(T)_z$ is not the identity map on m branches. If $A_i g = A_i$ for some i, let g_i be the z-branch transposition which acts as g on A_i. Then $gg_i^{-1} = e$ on A_i so, by the induction hypothesis, gg_i^{-1} and hence g is a product of z-branch transpositions. If g fixes no A_i for $i \in \{1,..., n\}$ then, relabeling if necessary, $A_1 g = A_2$, and there exists a z-branch transposition $g_{12} \in \Gamma(T)_z$ which acts like g on A_1. Then $xg_{12}g = xg^{-1}g = x$ for $x \in A_2$ so that $g_{12}g$ is not the identity map on at most $m - 1$ branches. By the induction hypothesis, $g_{12}g$ and hence g is a product of z-branch transpositions.

COROLLARY 2.5. *Every $g \in G$ is a product of branch transpositions of T_v. Every $h \in H$ is a product of branch transpositions of T_v belonging to H.*

LEMMA 2.6. *S is generated by automorphisms of the form ghg^{-1} where $g \in G$ and $h \in H$.*

Proof. By the definition of S.

LEMMA 2.7. *Every $s \in S$ is a product of branch transpositions of T_v which belong to S.*

Proof. By Lemma 2.6, it suffices to show that for all $g \in G$, $h \in H$,

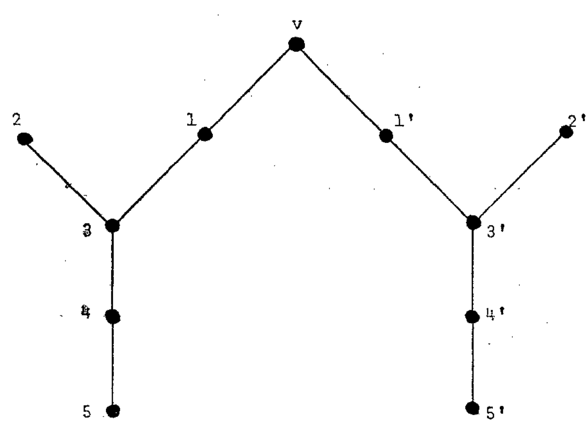


FIGURE 1

ghg^{-1} is a product of branch transpositions in S. By Corollary 2.5, h is a product of branch transpositions $h_1, h_2, \ldots, h_m$ in H so that $ghg^{-1} = (gh_1 g^{-1})(gh_2 g^{-1}) \cdots (gh_m g^{-1})$. Each $gh_i g^{-1} \in S$ and it is easily checked that $gh_i g^{-1}$ is a branch transposition.

In passing, we note that the analog of Lemma 2.3 is not true for S. An example is provided by the tree in Fig. 1 for which

$$S = \{e, (12)(1'2'), (11')(22')(33')(44')(55'), (12')(21')(33')(44')(55')\}.$$

3. The Normality of H in G

In this section necessary and sufficient conditions under which H is a normal subgroup of G are found. This may be considered as a generalization for trees of the criterion that $H = G$ given in Lemma 2.1. We denote $N_T(v)$ by N in this section.

Lemma 3.1. *If $x \in V(T_v)$ then the orbit of x under G, denoted by xG, is the union of disjoint H-orbits. If H is normal in G, every $g \in G$ permutes these H-orbits (as sets) and the H-orbits have the same cardinality.*

Proof. Since $H \leqslant G$, $xG = \bigcup_{i=1}^{m} x_i H$ for some vertices $x_i \in xG$. If $H \trianglelefteq G$ then for each $i \in \{1,\ldots, m\}$ and $g \in G$, $(x_i H)g = x_i(Hg) = x_i gH = x_j H$ for some $j \in \{1,\ldots, m\}$. For every pair i, j such that $1 \leqslant i \leqslant j \leqslant m$ there exists $g_{ij} \in G$ such that $x_i g_{ij} = x_j$ so that $x_i H g_{ij} = x_j H$. Since g_{ij} is a permutation of the vertices of T_v, $x_i H$ and $x_j H$ have the same cardinality.

Proposition 3.2. *If $H \ntrianglelefteq G$ then for some $u \in N$, there exists $x \in uG \backslash N$ which is not fixed by H.*

Proof. If $H \ntrianglelefteq G$, there exist $g \in G$, $h \in H$ such that $ghg^{-1} \notin H$. Hence, by Lemma 2.1, there exists $u \in N$ such that $ughg^{-1} = y \notin N$. Since $u \neq y$, $ug \neq yg = ugh$ so the proposition is proved by putting $x = ug$ if $ug \notin N$.

If, on the other hand, $ug \in N$ then $ugh \in N$ and so ug and ugh belong to different components B_1 and B_2 of T_v. Consequently h moves both B_1 and B_2 so if $y \in B_1$ or $y \in B_2$, $y \neq yh$ and putting $x = y$ satisfies the proposition. If, however, y belongs to a third component B_3 and $\{z\} = N \cap B_3$ then $zg \in B_3 g = B_2$ but $zg \notin N$ as $zg \neq yg = ugh \in N \cap B_2$. The proposition is now satisfied by replacing u by z and x by zg because $x = zg \in zG \backslash N$ and $zg \neq zgh^{-1}$ since $zg \in B_2$ and $zgh^{-1} \in B_1$.

Proposition 3.3. *If there exists $u \in N$ and $x \in uG \backslash N$ such that x is not fixed by H, then $H \ntrianglelefteq G$.*

Proof. Let $x = ug$ be moved by $h \in H$. If $x = ug$ and $xh = ugh$ both belong to the same component of T_v, then $u = (ug)g^{-1}$ and $ughg^{-1} = (ugh)g^{-1}$ must belong to the same component. But $u = ugg^{-1} \neq ughg^{-1}$ so that $ughg^{-1} \notin N$, and so by Lemma 2.1, $ghg^{-1} \notin H$ and $H \ntrianglelefteq G$.

If x and xh belong to different components B_1 and B_2 of T_v, we may assume without loss of generality that $u \notin B_2$. (Otherwise replace x by xh and h by h^{-1}.) Then, by Lemma 2.3, there exists $g_0 \in G$ such that $x = ug_0$ and $xhg_0 = xh$, and so $ug_0hg_0^{-1} = xhg_0^{-1} = xh \notin N$. Thus $g_0hg_0^{-1} \notin H$ and $H \ntrianglelefteq G$.

On combining Propositions 3.2 and 3.3 and using Lemma 3.1, we obtain the following characterization.

THEOREM 3.4. *$H \trianglelefteq G$ if and only if, for all $u \in N$, either*

(i) *$uG \subseteq N$, or*

(ii) *$uG \cap N = \{u\}$ and every vertex of uG is fixed by H.*

Proof. By Propositions 3.2 and 3.3, $H \trianglelefteq G$ if and only if, for all $u \in N$ and for all $x \in uG \backslash N$, $xH = \{x\}$. If $H \trianglelefteq G$ and $uG \nsubseteq N$, then as $xH = \{x\}$ we conclude by Lemma 3.1 that each H-orbit in uG has size one and consequently every vertex in uG is fixed by H.

If $uG \cap N \supseteq \{u, y\}$ and $uG \nsubseteq N$ then there exists $g \in G$ such that $yg = u$, so by Lemma 2.3 there is a branch transposition $g_0 \in G$ such that $yg_0 = u$ and $ug_0 = y$. But g_0 then fixes N so, by Lemma 1.1, $g_0 \in H$. Thus, as we have shown above that u is fixed by H, it follows that $u = y$. As $uG \cap N$ is nonempty, it has exactly one element, u.

4. THE GROUP G/S WHEN T_v IS CONNECTED

We now turn to determine the group $\Gamma(T_v)/S(T)_v$ for all trees T and vertices v. In this section, we consider the special case when v is an end-vertex of T so that T_v is connected. $N_T(v) \cap V(T_v)$ then contains only one vertex, which we denote by w.

LEMMA 4.1. $|G/S| = |wG|/|wS|$.

Proof. In both S and G, the subgroups of automorphisms which fix w is H. Hence, by the orbit-stabilizer relation [5, Theorem 3.2],

$$|G| = |wG| \cdot |H| \qquad \text{and} \qquad |S| = |wS| \cdot |H|.$$

LEMMA 4.2. *If $x, y \in V(T_v)$, $xH = \{x\}$, $yH = \{y\}$, and $xG = \{x, y\}$, then $xS = \{x\}$ and $yS = \{y\}$.*

Proof. We show that every generator of S (as in Lemma 2.6) fixes x. Let $g \in G$ and $h \in H$. Then $xg = x$ implies that $xghg^{-1} = xhg^{-1} = xg^{-1} = x$ whilst $xg = y$ implies that $xghg^{-1} = yhg^{-1} = yg^{-1} = x$.

LEMMA 4.3. *Let Z be the set of vertices in $V(T_v)$ which are fixed by G. If Z is nonempty and $w \notin Z$ then the vertex $z \in Z$ such that $d(z, w) = \min\{d(u, w) \mid u \in Z\}$ is unique.*

Proof. Assume that $d(z_1, w) = d(z_2, w) = \min\{d(u, w) \mid u \in Z\}$ and that $z_1, z_2 \in Z$. Let $P(u_1, u_2)$ be the set of vertices of T_v along the shortest path between any vertices u_1 and u_2 of $V(T_v)$. As z_1 and z_2 are fixed by G, every vertex of $P(z_1, z_2)$ is fixed by G, so that if $u \in P(z_1, z_2)$, $d(u, w) \geqslant d(z_1, w)$. Every vertex y of $P(w, z_1)$ has $d(y, w) \leqslant d(z_1, w)$ so that $P(z_1, z_2) \cap P(w, z_1) = \{z_1\}$ and hence $P(w, z_2) = P(w, z_1) \cup P(z_1, z_2)$ and $d(w, z_2) > d(w, z_1)$, unless $z_1 = z_2$.

PROPOSITION 4.4. *Let $z \neq w$ be a vertex of T_v fixed by G and let $A_1, \ldots, A_n$ be the branches of T_v at z, with $w \in A_1$.*

(i) *If there exists $\theta \in G$ such that $A_1\theta = A_i$ for some $i \in \{2, \ldots, n\}$ then $wG \cap A_1 = wS \cap A_1$ and $wG \cap A_i = w\theta S \cap A_i$.*

(ii) *If $A_1 G = A_1 \cup \cdots \cup A_r$ (relabeling if necessary) and $r \geqslant 3$, then $wG = wS$.*

Proof. (i) Let $w_i \in wG \cap A_i$ for some $i \in \{2, \ldots, n\}$. As $w\theta \in wG$ there exists $g \in G$ such that $w_i g = w\theta$ and, by Lemma 2.3, there exists a z-branch transposition h which acts like g on A_i and as the identity elsewhere. Since $w \in A_1$, $wh = w$ and $h \in H$. As $w_i h = w\theta$ it follows that $wG \cap A_i \subseteq w\theta H \cap A_i$. But as $H \leqslant S$ and $S \leqslant G$, $w\theta S \cap A_i \subseteq wG \cap A_i \subseteq w\theta H \cap A_i \subseteq w\theta S \cap A_i$ so that $w\theta S \cap A_i = wG \cap A_i$. If $w_1 \in wG \cap A_1$ then $w_1\theta \in w\theta G \cap A_i = wG \cap A_i = w\theta S \cap A_i$ so that $w_1 \in w\theta S \cap A_i$. Hence $w_1 \in (w\theta S \cap A_i)\,\theta^{-1} = w\theta S\theta^{-1} \cap A_1 = wS \cap A_1$ so that $wG \cap A_1 \subseteq wS \cap A_1$. The equality of $wG \cap A_1$ and $wS \cap A_1$ follows by noting again that $S \leqslant G$.

(ii) Let us now assume that $A_1, \ldots, A_r$ $(r \geqslant 3)$ are isomorphic branches at z. Choose any $w_i \in wG \cap A_i$ for some $i \in \{1, \ldots, r\}$. By part (i) if $i = 1$ then $w_i \in wS$. If, on the other hand, $i > 1$, there exists a z-branch transposition f_i such that $w_i = wf_i$. Choose $j \in \{1, \ldots, r\} \backslash \{1, i\}$ and a z-branch transposition g_j which interchanges A_1 and A_j. Now $A_i f_i^{-1} g_j = A_j$ so there is a z-branch transposition h_{ij} with the same action on A_i as $f_i^{-1} g_j$. Further, $h_{ij} \in H$ since h_{ij} is the identity map on A_1 and thus $g_j h_{ij} g_j^{-1} \in S$. But $w_i(g_j h_{ij} g_j^{-1}) = w_i h_{ij} g_j^{-1} = w_i f_i^{-1} g_j g_j^{-1} = w$ so that

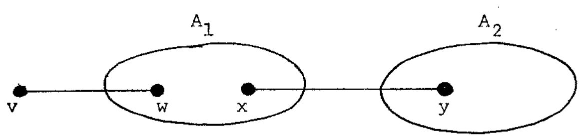
$wg_j h_{ij}^{-1} g_j^{-1} = w_i$, $w_i \in wS$ and so $wG \subseteq wS$. As $S \leqslant G$, $wS \subseteq wG$ and so $wS = wG$.

The next theorem uses the preceding results to determine the group G/S for all connected T_v.

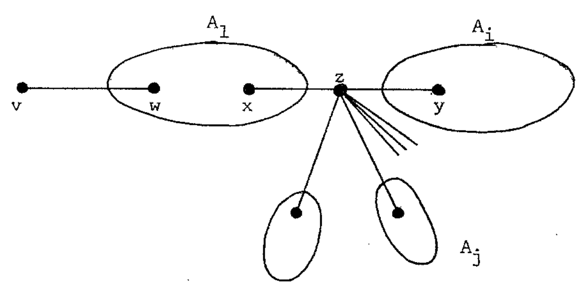
THEOREM 4.5. *$G = S$ unless either*

(a) *T_v is a bicentral tree with center $\{x, y\}$ which possesses a map $\theta \in G$ which interchanges x and y, or*

(b) *T_v is a tree with a vertex $z \neq w$ fixed by G and with branches $A_1, \ldots, A_n$ at z such that $w \in A_1$ and $A_1 G = A_1 \cup A_i$ for exactly one $i \in \{2, \ldots, n\}$.*



(a)



(b)

FIGURE 2

In Cases (a) and (b) $G/S \simeq C_2$, the cyclic group with two elements. The vertices x and y are in both cases fixed by S but interchanged by some $\theta \in G$.

Proof. (i) Let us first consider trees T_v which have no vertices fixed by G. Such a tree is bicentral with center $\{x, y\}$ and there exists a map $\theta \in G$ such that $x\theta = y$. This is Case (a) of the theorem and we now use the notation introduced there. Although this is not exactly the situation described in Proposition 4.4, the result that $wS \cap A_1 = wG \cap A_1$

is still valid. The proof of this requires only a slight modification of the proof of Proposition 4.4, so it is omitted. An application of Lemma 4.2 shows that $xS = \{x\}$ and since every $s \in S$ preserves adjacencies, this means that $wS \subseteq A_1$. Consequently $wS = wS \cap A_1 = wG \cap A_1$ so $|wS| = |wG \cap A_1| = |wG \cap A_2| = \frac{1}{2}|wG|$ and Lemma 4.1 applies to show that $|G/S| = 2$. Hence $G/S \simeq C_2$.

(ii) Let us now assume that some vertex of T_v is fixed by G. If $wG = \{w\}$ then $G = H = S$ by Lemma 2.1. If w is not fixed by G, let z be the vertex fixed by G at minimum distance from w. By Lemma 4.3, z is uniquely defined. Let us now adopt the notation of Case (b) of the theorem. Our choice of z now ensures that the vertex x in $N_T(z) \cap A_1$ is not fixed by G so that, reordering if necessary,

$$A_1G = A_1 \cup A_2 \cup \cdots \cup A_r \qquad (r \geqslant 2).$$

If $r > 2$, Proposition 4.4(ii) applies and $wG = wS$ so that, by Lemma 4.1, $G = S$.

If $r = 2$, let $y = N_T(z) \cap A_2$ and let $\theta \in G$ interchange A_1 and A_2. Proposition 4.4(i) can now be applied and we have $wG \cap A_1 = wS \cap A_1$ and $wG \cap A_2 = w\theta S \cap A_2$. Since $|wG \cap A_1| = |wG \cap A_2| = \frac{1}{2}|wG|$ we will show that $|G/S| = 2$ by showing that $wS = wG \cap A_1$ so that $|wS| = \frac{1}{2}|wG|$ and then applying Lemma 4.1.

If $w_1 \in wS \cap A_1$ then $w\theta, w_1\theta \in wG \cap A_2$. Now there exists $g \in G$ such that $w\theta = w_1\theta g$ and since g preserves adjacencies, $yg = y$. Hence, by Lemma 2.3, there is a y-branch transposition h such that $w\theta = w_1\theta h$ and clearly $h \in H$. Thus $w_1 = w\theta h^{-1}\theta^{-1} \in wS \cap A_1$ and so $wG \cap A_1 \subseteq wS \cap A_1$. But as every $h \in H$ fixes w, it must fix both x and y, so Lemma 4.2 applies and shows that $wS \subseteq A_1$. Now, as $S \leqslant G$, $wS \cap A_1 \subseteq wG \cap A_1$ so that $wS = wS \cap A_1 \subseteq wG \cap A_1 \subseteq wS \cap A_1 = wS$ and $wS = wG \cap A_1$ as required.

Corollary 4.6. *If T_v is connected and $G \neq S$, then the orbits of G are not the same as the orbits of S.*

Proof. Consider the vertices x and y in both cases of Theorem 4.5.

5. The Group G/S for any Tree T

We will now use and extend Theorem 4.5 to obtain the group G/S for all trees. Before deriving the general result, we must investigate trees T such that T_v consists of n isomorphic components $B_1, \ldots, B_n$. We denote by w_i the vertex of T_v in $N_T(v) \cap B_i$ for $i \in \{1, \ldots, n\}$. Since all components are isomorphic under G, by Lemma 2.3 we may choose for every pair $i, j \in \{1,..., n\}$ a branch transposition ϕ_{ij} of T_v such that ϕ_{ij} interchanges B_i and B_j. We also define the subgroup G_i of G to be the set of all $g \in G$ which are identity maps on all components other than B_i. H_i is the subgroup of G_i which fixes w_i, and S_i is the smallest normal subgroup of G_i containing H_i. F_i is the subgroup of G_i generated by elements of the form ghg^{-1} where $g \in G$ and, if $B_k = B_i g$ then $h \in H_k$. Theorem 4.5 gives the relationship between S_i and G_i for all i. We note the following easy lemmas.

LEMMA 5.1. *Either $F_i = S_i$ or $F_i = G_i$.*

Proof. By Theorem 4.5, $|G_i/S_i| \leqslant 2$ and $S_i \leqslant F_i \leqslant G_i$. Hence $|G_i/F_i| \leqslant 2$.

LEMMA 5.2. *$G_i \simeq G_j$ and $F_i \simeq F_j$ for all $i, j \in \{1,..., n\}$.*

Proof. $G_j = \phi_{ij} G_i \phi_{ij}^{-1}$ and $F_j = \phi_{ij} F_i \phi_{ij}^{-1}$.

LEMMA 5.3. *If $G_i = S_i$ for one $i \in \{1,..., n\}$ then $G_j = F_j$ for all $j \in \{1,..., n\}$.*

Proof. If $G_i = S_i$ then $F_i = S_i$ by Lemma 5.1, and the result follows via Lemma 5.2.

LEMMA 5.4. *F_i is the group generated by $\phi_{ij} S_j \phi_{ij}^{-1}$ for all $j \in \{1,..., n\}$.*

Proof. Obviously $S_i \leqslant F_i$. If $i \neq j$, S_j is generated by automorphisms of the form ghg^{-1} where $g \in G_j$, $h \in H_j$. Let $f = \phi_{ij} ghg^{-1} \phi_{ij}^{-1} = (\phi_{ij} g) h (\phi_{ij} g)^{-1}$. Now $B_i \phi_{ij} g = B_j g = B_j$ so it follows that $f \in F_i$. We deduce that $\phi_{ij} S_j \phi_{ij}^{-1} \subseteq F_i$.

Conversely, every generator of F_i is of the form ghg^{-1} where, if $B_j = B_i g$ then $h \in H_j$. Now if $g = g_0$ on B_i, $ghg^{-1} = g_0 h g_0^{-1}$ since for $x \in B_k$ $(k \neq i)$, $xghg^{-1} = x = xg_0 h g_0^{-1}$ and for $x \in B_i$, $xghg^{-1} = xg_0 h g_0^{-1} = (xg_0 h) g_0^{-1}$. Now $\phi_{ij} g^{-1}$ maps B_i to B_i so there exists $\sigma \in G_i$ such that $\phi_{ij} g^{-1} = \sigma$ on B_i. Now $ghg^{-1} = \phi_{ij} \sigma^{-1} h \sigma \phi_{ij}^{-1}$ and since $\sigma \in G_i$ and $h \in H_j$, $\sigma^{-1} h \sigma = h$ so $ghg^{-1} = \phi_{ij} h \phi_{ij}^{-1}$. Hence, $F_i \leqslant \phi_{ij} S_j \phi_{ij}^{-1}$ and we have proved the lemma.

We note the following generalization of Lemma 4.2.

LEMMA 5.5. *Let $x_i, y_i \in B_i$ and let $\{x_i, y_i\} \phi_{ij} = \{x_j, y_j\}$ for all i, j. If, for all i, $x_i H_i = \{x_i\}$, $y_i H_i = \{y_i\}$ and $x_i G_i = \{x_i, y_i\}$, then $x_k F_k = \{x_k\}$ and $y_k F_k = \{y_k\}$ for all $k \in \{1,..., n\}$.*

Proof. By Lemma 4.2, $x_k S_k = \{x_k\}$ for every $k \in \{1, \ldots, n\}$. Now $\{x_i, y_i\} \phi_{ij} = \{x_j, y_j\}$ so if $x_i \phi_{ij} = x_j$ then $x_i \phi_{ij} S_j \phi_{ij}^{-1} = \{x_i\}$ whilst if $x_i \phi_{ij} = y_i$ then $x_i \phi_{ij} S_j \phi_{ij}^{-1} = \{x_i\}$.

PROPOSITION 5.6. *If B_1 is a bicentral tree with bicenter $\{x_1, y_1\}$ and $\theta_1 \in G_1$ interchanges x_1 and y_1 then $| G_i/F_i | = 2$ for all i.*

Proof. Let $\{x_r, y_r\}$ be the bicenter of B_r. Lemma 5.5 applies to the set of vertices $\{x_1, y_1, x_2, \ldots, y_n\}$ so that each x_i is fixed by F_i. Now by Lemma 5.1, $F_i = S_i$ or $F_i = G_i$. But by Theorem 4.5, x_i is not fixed by G_i. Hence $F_i \neq G_i$ and so $| G_i/F_i | = 2$.

PROPOSITION 5.7. *Let B_1 have a vertex fixed by G_1. Then $| G_i/F_i | = 2$ for all i, if and only if the following two conditions are satisfied*:

(i) *Each component B_i is of the form described in Theorem* 4.5(b) *so that each $| G_i/S_i | = 2$.*

(ii) *Denote by z_1 the fixed vertex of B_1 at minimum distance from w_1 and by $A_1{}^1$ the branch of B_1 at z_1 containing w_1. Then, for every r, the vertex w_r in $N_T(v) \cap B_r$ must belong to $(A_1{}^1 \backslash \{z_1\}) g$ for some $g \in G$.*

Proof. By Lemma 4.3, z_1 is uniquely determined. Let $z_r = z_1 \phi_{1r}$ for each r. If $g: B_1 \to B_r$, then $z_1 g = z_r$, otherwise z_1 is not fixed by $\phi_{1r} g^{-1}$ and hence is not fixed by the branch transposition $g_1 \in G_1$ which acts like $\phi_{1r} g^{-1}$ on B_1 and is the identity map elsewhere. Hence z_r is defined independently of the ϕ_{ij}.

If each $| G_i/S_i | = 2$ and condition (ii) holds, then on labeling the two branches at z_r which are images of $A_1{}^1$ by $A_1{}^r$ and $A_2{}^r$ we see that

$$A_1{}^1 G = A_1{}^1 \cup A_2{}^1 \cup \cdots \cup A_1{}^n \cup A_2{}^n.$$

Since $G_i \neq S_i$ and $w_i \in A_1{}^i \cup A_2{}^i$, every $s_i \in S_i$ fixes both $A_1{}^i$ and $A_2{}^i$ as sets (by Theorem 4.5). Consequently $\phi_{1i} s_i \phi_{1i}^{-1}$ fixes both $A_1{}^i$ and $A_2{}^i$ so every $f \in F_i$ fixes both $A_1{}^i$ and $A_2{}^i$ (by Lemma 5.4). But, by Theorem 4.5, there exists $\theta_i \in G_i$ which interchanges $A_1{}^i$ and $A_2{}^i$ so that $F_i \neq G_i$.

If $| G_i/F_i | = 2$ then by Lemma 5.1, $F_i = S_i$ for all i and $| G_i/S_i | = 2$ so that condition (i) must hold. If condition (ii) does not hold, then for some j, $w_j \notin (A_1{}^1 \cup A_2{}^1) G$. Hence $w_j \notin A_1{}^1 \phi_{ij} \cup A_2{}^1 \phi_{ij}$ and, using Lemma 2.3, there exists a z_j-branch transposition h_j such that $A_1{}^1 \phi_{ij} h_j = A_2{}^1 \phi_{ij}$ and h_j is the identity map on other branches and components including the branch of B_j containing w_j.

Thus $h_j \in H_j$ and $f = \phi_{ij} h_j \phi_{ij}^{-1} \in F_1$. But f interchanges $A_1{}^1$ and $A_2{}^1$ so that $F_i \neq S_i$ and consequently $F_i = G_i$. This demonstrates that condition (ii) is implied by $| G_i/F_i | = 2$.

Propositions 5.6 and 5.7 together completely characterize those situations in which $G_i \neq F_i$. This is because if each B_i has a vertex fixed by G_i then Proposition 5.7 applies whilst if no vertex of B_i is fixed by G_i, B_i is a bicentral tree and G_i contains an automorphism θ_i which interchanges the vertices in the center. We summarize this in the following theorem.

THEOREM 5.8. *For all i, $G_i = F_i$ unless each B_i is a bicentral tree as described in Proposition 5.6 or each B_i has a vertex fixed by G_i and is as described in Proposition 5.7. In these cases $G_i/F_i \simeq C_2$, the cyclic group of order 2.*

Corollary 5.9 shows that $G_i = F_i$ if and only if G_i and F_i have the same orbits.

COROLLARY 5.9. *If $G_i \neq F_i$, there exist vertices x_i and y_i in B_i such that $x_iF_i = \{x_i\}$, $y_iF_i = \{y_i\}$, $x_iG_i = \{x_i, y_i\}$ and $x_iG = \{x_1, y_1, x_2, y_2, \ldots, y_n\}$.*

Proof. If $G_i \neq F_i$ then $F_i = S_i$ and we choose x_i, y_i to be the vertices of the center of a bicentral B_i as in Proposition 5.6, or the vertices in $N_T(z_i) \cap (A_1{}^i \cap A_2{}^i)$, when B_i is as described in Proposition 5.7.

We can now use these results to find the group G/S for a tree T when all the components of T_v are isomorphic under G.

THEOREM 5.10. *If no pair of the n components of T_v are isomorphic under H, then $G/S \simeq \mathrm{S}_n[G_i/F_i]$, i.e., the wreath product of the symmetric group on n symbols around G_i/F_i, for any $i \in \{1, \ldots, n\}$.*

Proof. If there are no automorphisms in H which interchange components of T_v, then S is generated by $F_1, F_2, \ldots, F_n$.

(i) If $G_i = F_i$ for every $i \in \{1, \ldots, n\}$, define the mapping $\sigma\colon G/S \to \mathrm{S}_n$ by $(gS)\sigma = \pi$ where π is the permutation of $\{1, \ldots, n\}$ which maps i to j when $B_i g = B_j$. The mapping σ is well defined since if $g_1S = g_2S$ then $g_1 g_2^{-1}$ fixes every B_i (setwise) and so $B_i g_1 = B_i g_2$. Since all components B_i are isomorphic under G, σ is a bijection. If $B_i g_1 = B_j$ and $B_j g_2 = B_k$ then $i(g_1S)\sigma(g_2S)\sigma = j \cdot (g_2S)\sigma = k = i(g_1 g_2S)\sigma$. Consequently, $(g_1Sg_2S)\sigma = (g_1S)\,\sigma(g_2S)\sigma$ and so σ is an isomorphism.

(ii) Now let us assume $G_i \neq F_i$ and choose $\theta_i \in G_i$ such that G_i is generated by F_i and θ_i. By Corollary 5.9, there are vertices $x_i, y_i \in B_i$ such that $x_iF_j = \{x_i\}$, $y_iF_j = \{y_i\}$ for all j and $x_i\theta_i = y_i$, $y_i\theta_i = x_i$. We show that G/S is the wreath product $\mathrm{S}_n[C_2]$ by exhibiting an isomorphism σ from G/S to the group of all permutations of the sets $\{x_1, y_1\}, \{x_2, y_2\}, \ldots,$

$\{x_n, y_n\}$. $(gS)\sigma$ is defined as that permutation of $U = \{x_1, y_1, ..., y_n\}$ which acts in the same way as g acts on U. Since S is generated by $F_1, ..., F_n$, if $g_1S = g_2S$ then $g_1 g_2^{-1}$ fixes every x_i and y_i so that g_1 and g_2 have the same action on U and σ is well defined. As before, σ is a bijection and an isomorphism since

$$x_i(g_1S)\,\sigma(g_1S)\sigma = (x_i g_2)(g_1S)\sigma = x_i g_2 g_1 = x_i(g_2 g_1 S)\sigma.$$

The next lemma provides a useful method of showing that an automorphism does not belong to S.

LEMMA 5.11. *Let the vertices x_i, y_i be chosen to satisfy the conditions of Corollary* 5.9. *Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$*. Then if $s \in S$, $| X \cap Xs | \equiv n \pmod 2$ *and* $| Y \cap Ys | \equiv n \pmod 2$.

Proof. By Lemma 2.7, every $s \in S$ is a product of branch transpositions of the form ghg^{-1} where $g \in G$ and $h \in H$. Thus the lemma will be proved if we show that each branch transposition of the form ghg^{-1} either leaves X and Y unchanged or interchanges two members of X with two members of Y.

First consider a branch transposition which acts as the identity map on all branches other than B_k. For all $z \in V(B_i)$ where $i \in \{1, ..., n\} \setminus \{k\}$, $zghg^{-1} = z$ so that $zgh = zg$ and h fixes all vertices of T_v not in $B_k g = B_m$. Thus $h \in H_m < F_m$. By Corollary 5.9, $x_k g \in \{x_m, y_m\}$. As both x_m and y_m are fixed by F_m (and therefore by h) $x_k ghg^{-1} = x_k$. Similarly $y_k ghg^{-1} = y_k$. Consequently, every member of X and Y is fixed by ghg^{-1}.

Secondly consider a branch transposition ghg^{-1} which interchanges B_i and B_j. As ghg^{-1} fixes all vertices of T_v other than those of B_i and B_j, it fixes all vertices in the sets $X \setminus \{x_i, x_j\}$ and $Y \setminus \{y_i, y_j\}$. Since h is a branch transposition $h^2 = e$ and $ghg^{-1} = (ghg^{-1})^{-1}$. Hence if $x_i ghg^{-1} = x_j$ then $x_j ghg^{-1} = x_i$ whilst if $x_i ghg^{-1} = y_i$ then $y_i ghg^{-1} = x_i$ and consequently $x_j ghg^{-1} = y_j$. This argument applied to the several possibilities shows that ghg^{-1} either fixes X and Y or interchanges two elements from each set.

The next theorem describes the group G/S when Theorem 5.10 does not apply.

THEOREM 5.12. *If the components B_1 and B_2 are isomorphic under H, then $G/S = G_1/S_1$. In other words, if H does not fix every component of T_v, then $G/S = G_i/S_i$ for all $i \in \{1, 2, ..., n\}$.*

Proof. Let $h_{12} \in H$ be a branch transposition interchanging B_1 and B_2. For any B_i $(i \neq 1, 2)$ there exists $g \in G$ which interchanges B_i and B_2.

Since $gh_{12}g^{-1}$ then interchanges B_1 and B_i, there is a map $s_i \in S$ which interchanges B_1 and B_i for every $i \in \{1,..., n\}$. Given any $g \in G$, define the permutation π by $i\pi = j$ when $B_i g = B_j$. As $\mathbf{S}_n$ is generated by the transpositions $(12),..., (1n)$, we can write

$$\pi = (1i_1) \cdots (1i_t).$$

Hence $g_0 = gs_{i_t}^{-1} \cdots s_{i_1}^{-1}$ fixes each B_i (as a set) and so $g_0 = \alpha_1 \cdots \alpha_n$ for suitable $\alpha_i \in G_i$. Hence $g \in \alpha_1 \cdots \alpha_n S$. If $G_i = F_i$ then each $\alpha_i \in F_i \subseteq S$ so that $g \in S$ and consequently $G = S$. If $G_i \neq F_i$, then $F_i = S_i$ and there exists $\theta \in G_1 \backslash F_1$ which interchanges $A_1{}^1$ and $A_2{}^1$ (in the notation of Propositions 5.6 and 5.7). As $s_i \theta s_i^{-1}$ interchanges $A_1{}^i$ and $A_2{}^i$, G_i is generated by F_i and $s_i \theta s_i^{-1}$ and so by all s_i and $s_i \theta s_i^{-1}$. As $\theta^2 \in S$ and $g = \alpha_1 \cdots \alpha_n S$ where $\alpha_i \in G_i$ either $g \in S$ or $g \in S\theta$. We complete the proof by showing $\theta \notin S$, so that $| G/S | = 2$. This follows by using Lemma 5.11 on the set of vertices provided by Corollary 5.9 when we note that $\{x_1, ..., x_n\}\theta = \{y_1, x_2, ..., x_n\}$ so that

$$|\{x_1, ..., x_n\}\theta \cap \{x_1, ..., x_n\}| \equiv n - 1 \pmod 2.$$

The results above now readily combine to give the group $\Gamma(T_v)/S(T)_v$ for any tree T and any vertex v. Let the components B_{ij} of T_v belong to m isomorphism classes $\{B_{11}, ..., B_{1n_1}\}, \{B_{21}, ..., B_{2n_2}\}, ..., \{B_{m_1}, ..., B_{mn_m}\}$ and define the graph T^i to be the subgraph of T containing v and all components $B_{i1}, ..., B_{in_i}$. It is well known that, as no automorphism of T_v interchanges B_{ij} and B_{kl} (for $i \neq k$),

$$\Gamma(T_v) \simeq \Gamma(T_v{}^1) \times \cdots \times \Gamma(T_v{}^m).$$

Similarly $S(T)_v$ is constructed from permutation groups on disjoint sets of vertices so that

$$S(T)_v \simeq S(T^1)_v \times \cdots \times S(T^m)_v.$$

Now $S(T^i)_v \trianglelefteq \Gamma(T_v{}^i)$ so that

$$\Gamma(T_v)/S(T)_v \simeq \Gamma(T_v{}^1)/S(T^1)_v \times \cdots \times \Gamma(T_v{}^m)/S(T^m)_v.$$

Each $\Gamma(T_v{}^i)/S(T^i)_v$ is known from the earlier results so that we have the following theorem.

THEOREM 5.13. *Let the components of T_v belong to m different isomorphism classes each containing n_i $(1 \leqslant i \leqslant m)$ components. Then*

$$\Gamma(T_v)/S(T)_v \simeq \mathbf{S}_{p_1}[Z_1] \times \cdots \times \mathbf{S}_{p_m}[Z_n]$$

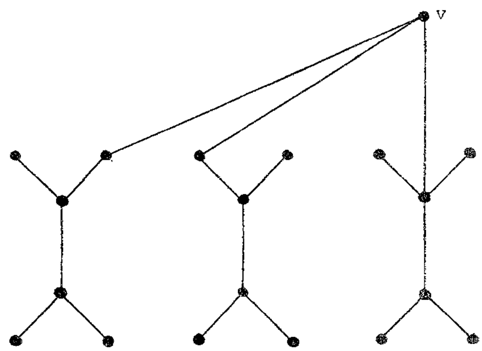


FIGURE 3

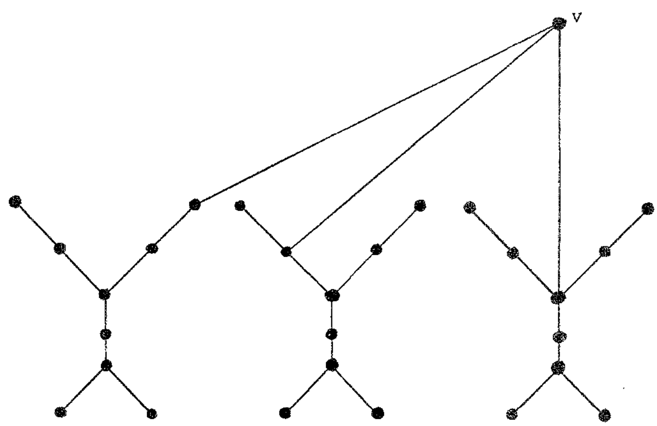


FIGURE 4

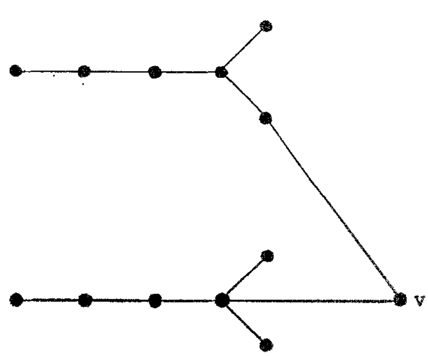


FIGURE 5

where the groups Z_i have order either 1 *or* 2 *and* $p_i = n_i$ *or* $p_i = 1$. *The values of* p_i *and* Z_i *are found from Theorems* 5.10 *and* 5.12.

We conclude with an illustration of Theorem 5.13. For the tree in Fig. 3, $G/S \cong G_i/F_i$ because Theorem 5.12 applies. As each component is a bicentral tree which satisfies Proposition 5.6, $| G_i/F_i | = 2$ so that $G/S \cong C_2$. For the tree in Fig. 4, $G/S \cong S_3[G_i/F_i]$ because Theorem 5.10

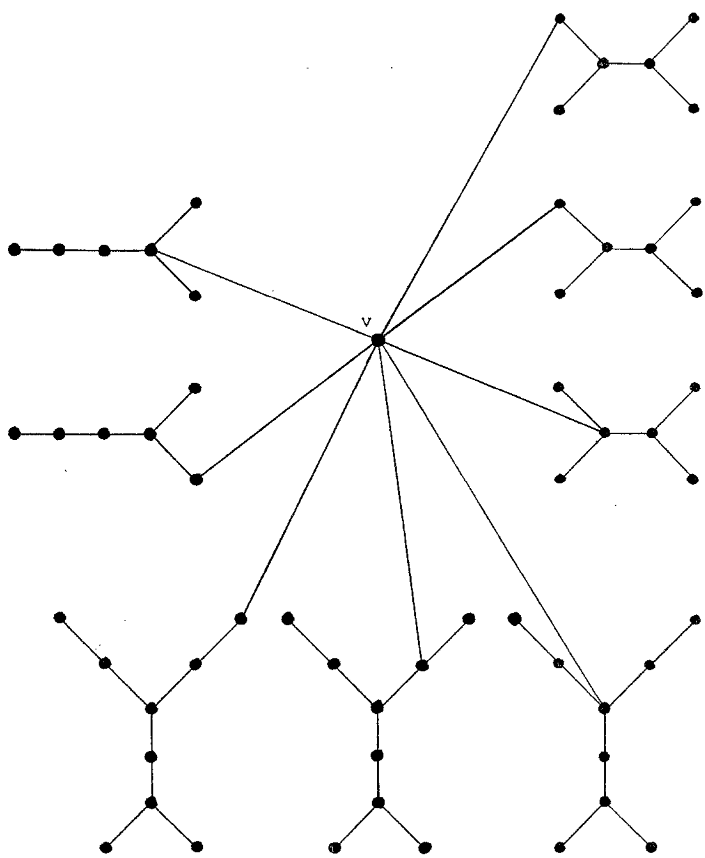


FIGURE 6

applies. As each component satisfies the requirements for Proposition 5.7, $| G_i/F_i | = 2$ and so $G/S \cong S_3[C_2]$. The tree in Fig. 5 has $G/S \cong S_2[G_i/F_i]$. This tree does not satisfy condition (ii) of Proposition 5.7, so $G_i = F_i$ and thus $G/S \simeq S_2$. Using these results we see that, for the tree in Fig. 6,

$$G/S \cong C_2 \times S_3[C_2] \times S_2 .$$

REFERENCES

1. M. BEHZAD AND G. CHARTRAND, "Introduction to the Theory of Graphs," Allyn and Bacon, Boston, 1971.
2. D. A. HOLTON, Two applications of semi-stable graphs, *Discrete Math.* 4 (1973), 151–158.
3. D. A. HOLTON AND D. D. GRANT, Regular graphs and stability, *J. Austal. Math. Soc.* **20** (*Series A*) (1975), 377–384.
4. D. A. HOLTON, A survey of stability, *Graph Theory Newsletter*, Western Michigan University, **4** (1974), 25–32.
5. H. WIELANDT, "Finite Permutation Groups," Academic Press, New York, 1964.