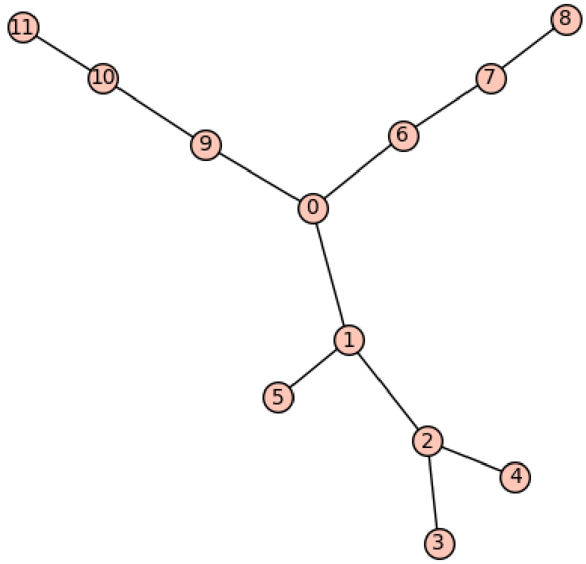
Strongly Cospectral Vertices on Trees



At the above tree, vertices 0 and 1 are strongly cospectral.
Goals are:

1. Proof that vertices 0 and 1 continue strongly cospectral if we add a path P_n, $\forall n > 0$ between then.
2. Proof that the equidistant vertices of that path are also strongly cospectral: $\{(P_n)_k, (P_n)_{n-1-k}\}$ are strongly cospectral.
3. Change the path joining T_1 and T_2 by a star or a general tree, keeping the cospectrality.
4. Extend the results adding T_3 and verifying what is necessary for a and b to be parallel
5. Proof that three mutually pseudo-similar vertices cannot be strongly cospectral
6. Approach Lemma 5.6.1 without assuming that the matrices are similar, but using the fact that $\phi(T_1 \backslash a, t)/\phi(T_1, t) = \phi(T_2 \backslash b, t)/\phi(T_2, t)$
7. If a and b are strongly cospectral, there is an orthogonal symmetry Q that swaps them. Does it also swap the path between them? Even more, does it also swaps T_1 and T_2?
8. * We are trying to find 3 strongly cospectral vertices in trees or show that they don't exist. Try to show this, for instance for the special case with simple eigenvalues only in the graph.
9. * Extra from another meeting I had with Chris: show that controllable graphs (chapter 5) don't have cospectral vertices (the result is true for irreducible characteristic polynomial, which are a subset of controllable graphs)... this would be a major result

1 Asymmetric strongly cospectral vertices

For simplicity, we name vertices 0 as b and 1 as a.

At our original graph, after deleting edge ab from the tree, two disjoint trees are formed, T_1 that contains a and T_2 that contains b. Let T be the graph we get by adding a path between a and b, thus, T can be decomposed in $T = T_1 \cup P_{n+2} \cup T_2$. Thus,

Theorem 1.1. *a and b are strongly cospectral at T if and only if*

$$\frac{\phi(T_1 \backslash a, t)}{\phi(T_1, t)} = \frac{\phi(T_2 \backslash b, t)}{\phi(T_2, t)} \tag{1}$$

For this proof we use **Lemma 4.7.1** and **Theorem 6.7.1** from the course notes:

Lemma 1.2 (Lemma 4.7.1)**.** *If Z is the 1-sum of X and Y at a, then*

$$\phi(Z, t) = \phi(X \backslash a, t)\phi(Y, t) + \phi(X, t)\phi(Y \backslash a, t) - t\phi(X \backslash a, t)\phi(Y \backslash a, t)$$

Theorem 1.3 (Theorem 6.7.1)**.** *Let X be the graph obtained from vertex-disjoint graphs Y and Z by joining a vertex a in Y to a vertex b in Z by a path P of length at least one. If a and b are cospectral in X, they are strongly cospectral.*

Proof of Theorem 1.1. It follows that:

$$\phi(T \backslash a, t) = \phi(T_1 \backslash a, t)(\phi(P_n, t)\phi(T_2, t) + \phi(P_{n+1}, t)\phi(T_2 \backslash b, t) - t\phi(P_n, t)\phi(T_2 \backslash b, t))$$

$$\phi(T \backslash b, t) = \phi(T_2 \backslash b, t)(\phi(P_n, t)\phi(T_1, t) + \phi(P_{n+1}, t)\phi(T_1 \backslash a, t) - t\phi(P_n, t)\phi(T_1 \backslash a, t))$$

After distributing the multiplication, some terms of the two identities are equal, so

$$\begin{aligned} \phi(T \backslash a, t) = \phi(T \backslash b, t) &\iff \phi(T_1 \backslash a, t)\phi(P_{n+1} \cup T_2, t) = \phi(T_2 \backslash b, t)\phi(P_{n+1} \cup T_1, t) \\ &\iff \phi(T_1 \backslash a, t)\phi(T_2, t) = \phi(T_2 \backslash b, t)\phi(T_1, t) \end{aligned}$$

Since these terms don't depend on the paths, the characteristic polynomials of T_1, T_2, $T_1 \backslash a$ and $T_2 \backslash b$ determine if a and b are cospectral.

Thus, for this particular graph:

$$\begin{aligned} \phi(T_1 \backslash a, t)\phi(T_2, t) &= t(t(t^2 - 2))t(t^2 - 2)(t^4 - 4t^2 + 2) \\ &= (t(t^2 - 2))^2 t(t^4 - 4t^2 + 2) \\ &= \phi(T_2 \backslash b, t)\phi(T_1, t) \end{aligned}$$

Since a and b are cospectral and joined by a path, by Theorem 1.3 they are strongly cospectral. $\square$

2 Equidistant strongly cospectral vertices on paths

Using a similar idea, we prove that

Theorem 2.1. *Let a and b be two vertices of a tree connected by a path P and let T_1 and T_2 be the limbs of a and b at T. If T_1 and T_2 satisfy equation* (1) *for vertices a and b then all equidistant vertices of P are strongly cospectral.*

Proof. Again, we divide T in $T_1 \cup P_m \cup P_k \cup P'_m \cup T_2$ where $T_1 \cap P_m = \{a\}$, $P_m \cap P_k = \{x\}$, $P_k \cap P'_m = \{y\}$ and $P'_m \cap T_2 = \{b\}$ and $|P_m| = |P'_m|$. We show that x and y are strongly cospectral.

By Theorem 1.1, this happens if and only if

$$\phi(T_1 \cup P_m \backslash x, t)\phi(P'_m \cup T_2, t) = \phi(P'_m \backslash y \cup T_2, t)\phi(T_1 \cup P_m, t)$$

So, using lemma 1.2, and taking $Z = P'_m \cup T_2$:

$$\phi(T_1 \cup P_m \backslash x, t)\phi(P'_m \cup T_2, t)$$

$$= \phi(T_1 \cup P_m \backslash x, t)(\phi(P'_m \backslash b, t)\phi(T_2, t) + \phi(P'_m, t)\phi(T_2 \backslash b, t) - t\phi(P_m \backslash b, t)\phi(T_2 \backslash b, t))$$

And taking $Z = T_1 \cup P_m \backslash x$

$$\phi(T_1 \cup P_m \backslash x, t)\phi(P'_m \cup T_2, t)$$

$$= (\phi(P_m \backslash ax, t)\phi(T_1, t) + \phi(P_m \backslash x, t)\phi(T_1 \backslash a, t) - t\phi(P_m \backslash ax, t)\phi(T_1 \backslash a, t))$$
$$(\phi(P'_m \backslash b, t)\phi(T_2, t) + \phi(P'_m, t)\phi(T_2 \backslash b, t) - t\phi(P'_m \backslash b, t)\phi(T_2 \backslash b, t))$$

If we apply similar steps at $\phi(P'_m \backslash y \cup T_2, t)\phi(T_1 \cup P_m, t)$, first taking $Z = T_1 \cup P_m$ and then taking $Z = P'_m \backslash y \cup T_2$ we reach a similar equality:

$$\phi(P'_m \backslash y \cup T_2, t)\phi(T_1 \cup P_m, t)$$

$$= (\phi(P'_m \backslash by, t)\phi(T_2, t) + \phi(P'_m \backslash y, t)\phi(T_2 \backslash b, t) - t\phi(P'_m \backslash by, t)\phi(T_2 \backslash b, t))$$
$$(\phi(P_m \backslash a, t)\phi(T_1, t) + \phi(P_m, t)\phi(T_1 \backslash a, t) - t\phi(P_m \backslash a, t)\phi(T_1 \backslash a, t))$$

After comparing each term, if $\phi(T_1 \backslash a, t)\phi(T_2, t) = \phi(T_2 \backslash b, t)\phi(T_1, t)$ then the two terms are equal.

So, they are cospectral and by Theorem 1.3 they are strongly cospectral. □

3 General case

Theorem 3.1. *Let T_1 and T_2 be two trees such that $a \in V(T_1)$ and $b \in V(T_2)$ and let T_v be a tree joining T_1 and T_2 such that $\{a, b\} \subseteq V(T_v)$. Assume that $\phi(T_1 \backslash a, t)/\phi(T_1, t) = \phi(T_2 \backslash b, t)/\phi(T_2, t)$. Then, a and b are cospectral at T_v if and only if a and b are cospectral at $T = T_1 \cup T_v \cup T_2$.*

The proof is straightforward using lemma 1.2:

Proof.

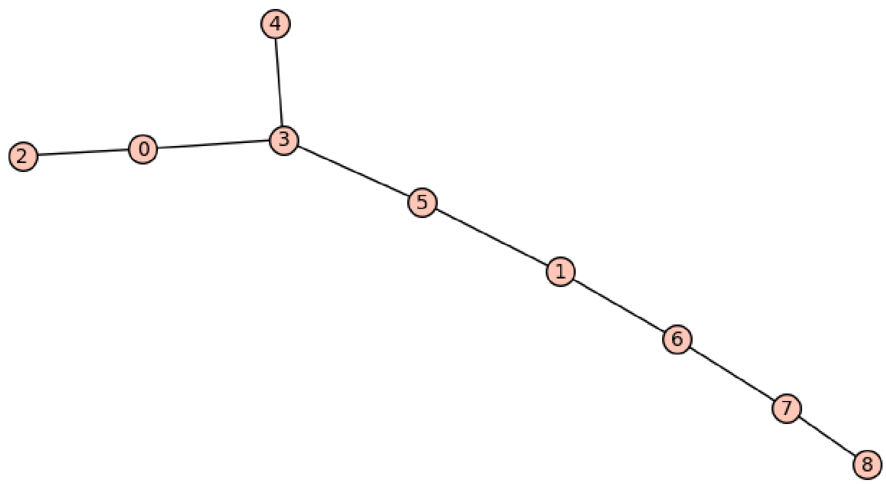
$$\phi(T\backslash a,t) = \phi(T_1\backslash a,t)(\phi(T_2\backslash b,t)\phi(T_v\backslash a,t) + \phi(T_2,t)\phi(T_v\backslash ab,t) - t\phi(T_2\backslash b,t)\phi(T_v\backslash ab,t))$$

$$\phi(T\backslash b,t) = \phi(T_2\backslash b,t)(\phi(T_1\backslash a,t)\phi(T_v\backslash b,t) + \phi(T_1,t)\phi(T_v\backslash ab,t) - t\phi(T_1\backslash a,t)\phi(T_v\backslash ab,t))$$

If $\phi(T_v\backslash a,t) = \phi(T_v\backslash b,t)$ then the two terms are equal □

Corollary 3.1.1. *Assume that a and b are cospectral at T. Then a and b are cospectral at T_v if and only if they satisfy equation* (1). □

- However there are trees where none of these two conditions are satisfied, for example at this graph showed by Chris, where 0 and 1 are cospectral:



4 Three cospectral vertices joined by a tree

The above results can be easily extended if we have T_3 such that $\frac{\phi(T_1\backslash a,t)}{\phi(T_1,t)} = \frac{\phi(T_2\backslash b,t)}{\phi(T_2,t)} = \frac{\phi(T_3\backslash c,t)}{\phi(T_3,t)}$, $\{a,b,c\} \subseteq V(T_v)$ and $\phi(T_3 \cup T_v\backslash a,t) = \phi(T_3 \cup T_v\backslash b,t)$, $\phi(T_2 \cup T_v\backslash a,t) = \phi(T_2 \cup T_v\backslash c,t)$, $\phi(T_1 \cup T_v\backslash b,t) = \phi(T_1 \cup T_v\backslash c,t)$ then a, b and c are cospectral.

And we can construct a tree with 3 cospectral vertices by letting T_v be a star with tree leaves: (a, b and c) and a central vertex v.

But we still need to discovery if a, b and c are parallel, for this, we use **Lemma 1** from *L. Lovász & J. Pelikán (1973). On the Eigenvalue of Trees*:

Lemma 4.1 (Lemma 1)**.** *if T is a forest and $e \in E(T)$, then*

$$\phi(T,t) = \phi(T\backslash e,t) - \phi(T\backslash[e],t)$$

Theorem 4.2. *If T_1, T_2 and T_3 are defined as above and $T = T_1 \cup T_2 \cup T_3 \cup S_3$, then a, b and c are mutually cospectral at T but not parallel.*

Proof. $\phi(T,t)$ can be written as:

$$\begin{aligned}\phi(T,t) &= \phi(T_1,t)\phi(T_3 \cup v \cup T_2,t) - \phi(T_1\backslash a,t)\phi(T_3,t)\phi(T_2,t)\\ &= \phi(T_1,t)(\phi(T_2,t)\phi(T_3 \cup v,t) - \phi(T_2\backslash b,t)\phi(T_3,t)) - \phi(T_1\backslash a,t)\phi(T_3,t)\phi(T_2,t)\end{aligned}$$

As a and b are parallel if and only if all roots of $\phi(T,t)/\phi(T\backslash ab,t)$ have multiplicity one:

$$\begin{aligned}\frac{\phi(T,t)}{\phi(T\backslash ab,t)} &= \frac{\phi(T,t)}{\phi(T_1\backslash a,t)\phi(T_2\backslash b,t)\phi(T_3 \cup v,t)}\\ &= \frac{\phi(T_1,t)(\phi(T_2,t)\phi(T_3 \cup v,t) - \phi(T_2\backslash b,t)\phi(T_3,t)) - \phi(T_1\backslash a,t)\phi(T_3,t)\phi(T_2,t)}{\phi(T_1\backslash a,t)\phi(T_2\backslash b,t)\phi(T_3 \cup v,t)}\\ &= (\frac{\phi(T_1,t)}{\phi(T_1\backslash a,t)})^2 - 2\frac{\phi(T_1,t)\phi(T_3,t)}{\phi(T_1\backslash a,t)\phi(T_3 \cup v,t)}\\ &= (\frac{\phi(T_1,t)}{\phi(T_1\backslash a,t)} - 2\frac{\phi(T_1,t)}{\phi(T_1 \cup v,t)})\frac{\phi(T_1,t)}{\phi(T_1\backslash a,t)}\end{aligned}$$

Let T' be constructed as T but with $T'_1 \simeq T'_2 \simeq T'_3 \simeq T_1$ it is easy to verify that $\phi(T,t)/\phi(T\backslash ab,t) = \phi(T',t)/\phi(T'\backslash ab,t)$. Since a and b are not parallel at T' (since there is an automorphism mapping a to c and fixing b) they cannot be parallel at T. $\square$

We can extend this proof and show that:

Theorem 4.3. *Let T' be constructed as T, but replacing T_2 with a copy of T_1, then a, b and c are strongly cospectral at T if and only if they are strongly cospectral at T'*

Proof. Let a and b be connected at vertices x and y at T_v respectively. Applying Lemma 4.1 at edges $\{a,x\}$ and $\{b,y\}$ we decompose $\phi(T,t)$ in:

$$\begin{aligned}\phi(T,t) =& \phi(T_1,t)(\phi(T_2,t)\phi(T_v\backslash ab,t) - t\phi(T_2\backslash b,t)\phi(T_v\backslash aby,t))\\ &- t\phi(T_1\backslash a,t)(\phi(T_2,t)\phi(T_v\backslash abx,t) - t\phi(T_2\backslash b,t)\phi(T_v\backslash abxy,t))\end{aligned}$$

Thus,

$$\frac{\phi(T,t)}{\phi(T\backslash ab,t)} = (\frac{\phi(T_1,t)}{\phi(T_1\backslash a,t)})^2 - t\frac{\phi(T_1,t)}{\phi(T_1\backslash a,t)}(\frac{\phi(T_v\backslash aby,t)}{\phi(T_v\backslash ab,t)} + \frac{\phi(T_v\backslash abx,t)}{\phi(T_v\backslash ab,t)}) + t^2\frac{\phi(T_v\backslash abxy,t)}{\phi(T_v\backslash ab,t)}$$

$\square$

5 Pseudo-Similar vertices

At *On Pseudo-Similar Vertices in Trees*, David G. Kirkpatrick, Maria M. Klawe, and D. G. Corneil show that it is not possible to have three or more mutually strictly pseudo-similar vertices in trees, so we restrict our study to similar vertices.

Theorem 5.1. *If a, b and c are three mutually similar vertices at T they cannot be mutually strongly cospectral.*

Proof. Let $\Gamma(T)$ be the group of automorphism of T. We show by induction that exists an automorphism $g \in \Gamma(T)$ that fixes one vertex of $\{a, b, c\}$ without fixing the others. If $|V(T)| = 4$ then T is S_3 with leaves $\{a, b, c\}$ and there is an automorphism that swaps a and b, but fixes c.

Assume $|V(T)| > 4$, we split in two cases:

- T has a vertex x fixed by every automorphism with branches be $B_1, \ldots, B_{\text{degree}(x)}$.

  Assume that a, b and c are at the same branch B_1, let x_1 be the vertex adjacent to x at B_1, then, every automorphism of T restricted to B_1 fixes x_1. Thus, we apply induction at B_1.

  Otherwise, assume that a and b are at different branches, assume that $a \in B_1$, $b \in B_2$, thus $B_1 \cong B_2$. If $c \in B_1$ then there is an automorphism $g \in \Gamma(T)$ such that $cg = a$ with g acting as identity at every branch $B_i, i \neq 1$. By symmetry, if $c \in B_2$ then there is g such that $cg = b$ with g acting as identity on other branches. If $c \notin B_1$ and $c \notin B_2$ then there is an automorphism g that transposes B_1 and B_2, but fixes every other branch. In either case g fixes one vertex of $\{a, b, c\}$, but not the others.

- T has bicentral vertices $\{x, y\}$ and we can partition the vertices of T in two disjoint connected subgraphs $T_x \ni x$ and $T_y \ni y$, thus $T_x \cong T_y$.

  Assume that a, b and c are at the same branch T_x, since every automorphism restricted to T_x must fix x we apply induction on T_x.

  Otherwise, assume that $a \in T_x$ and $b \in T_y$. If $c \in T_x$ there is an automorphism g restricted to T_x with $cg = a$ that fixes x and T_y. Similarly, if $c \in T_y$ there is an automorphism g with $cg = b$ that fixes y and T_x.

□

Theorem 5.2. *Let T_1, T_2 and T_3 be disjoint trees with $a \in T_1$, $b \in T_2$ and $c \in T_3$. Let T_v be a tree with vertices a, b and c mutually similar. Thus, a, b and c cannot be mutually strongly cospectral at $T = T_1 \cup T_2 \cup T_3 \cup T_v$.*

Proof. Let g be an automorphism of T_v that transposes a and b but fixes c. Thus, g is also an automorphism when extended to $T_v \cup T_3$. Thus a and b are cospectral at $T_v \cup T_3$ and by Corollary 3.1.1, this cospectrality implies that if a and b are cospectral at T, then T_1 and T_2 satisfy equation (1) for vertices a and b.

Let T' be constructed as T but with a copy of T_1 in place of T_2. By Theorem 4.3, a, b and c are strongly cospectral at T if and only if they are strongly cospectral at T'. But at T' a and b are similar and there is an automorphism that transposes both and fixes c. □

6 Eigenvalue support

Lemma 5.6.1 from the course notes states that:

Lemma 6.1 (Lemma 5.6.1)**.** *Assume A and B are similar matrices, with spectral decompositions*

$$A = \sum_{r=0}^{d} \sigma_r E_r, \qquad B = \sum_{r=0}^{d} \sigma_r F_r$$

Consider vectors y and z. The following are equivalent.

(a) $y^T E_r y = z^T F_r z$, for all r.

(b) $y^T (I - tA)^{-1} y = z^T (I - tB)^{-1} z$.

(c) $W_{Y,z}^T W_{Y,z} = W_{X,y}^T W_{X,y}$.

The proof of this lemma uses an useful equation, **equation 4.3.4** from the course notes:

$$(E_r)_{a,a} = \frac{\phi(X \backslash a, t)(t - \theta_r)}{\phi(X, t)}\Big|_{t=\theta_r} \tag{2}$$

Inspired by that lemma, we prove that:

Theorem 6.2. *Let T_1 and T_2 be two graphs with spectral decompositions:*

$$A(T_1) = \sum_{r=0}^{d} \sigma_r E_r, \qquad A(T_2) = \sum_{r=0}^{d} \sigma_r F_r$$

If $\frac{\phi(T_1 \backslash a, t)}{\phi(T_1, t)} = \frac{\phi(T_2 \backslash b, t)}{\phi(T_2, t)}$ then the following holds:

(a) $e_a^T E_r e_a = e_b^T F_r e_b$, for all r

(b) The eigenvalue support of a at T_1 is the same as the eigenvalue support of b at T_2

(c) $e_a^T (I - tA(T_1))^{-1} e_a = e_b^T (I - tA(T_2))^{-1} e_b$

Proof. From equation 4.3.4, $(E_r)_{a,a} \neq 0 \iff \theta_r$ is a pole of $\frac{\phi(X \backslash a, t)}{\phi(X, t)}$.

Since $\frac{\phi(T_1 \backslash a, t)}{\phi(T_1, t)} = \frac{\phi(T_2 \backslash b, t)}{\phi(T_2, t)}$, the two polynomials have the same poles. Thus,

(a) $e_a E_r e_a = e_b^T (I - tA(T_2))^{-1} e_b$

(b) a have the same eigenvalue support at T_1 as b at T_2

(c) Since $\sum_{r=0}^{d} e_a^T E_r e_a = e_a^T (I - tA)^{-1} e_a$, it follows that

$$e_a^T (I - tA)^{-1} e_a = e_b^T (I - tB)^{-1} e_b$$

□

7 Matrix Q

Theorem 7.1. *Let T_1 and T_2 be two trees satisfying equation (1) for vertex a and b. Let $T = T_1 \cup T_2 \cup P$, where a and b are end vertices of P. Then, there is an orthogonal matrix Q that swaps a and b along with P.*

Proof. Let $\sigma_r \in \{-1, 1\}$. $E_r e_a = \sigma_r E_r e_b$ if and only if there is a polynomial defined by $p(\theta_r) = \sigma$ such that $p(A)e_a = e_b$.

Let l be the length of the path that joins a and b, let x and y be strongly cospectral vertices of the path such that $d(x, a) = d(y, b) \leq l/2$ and let $\sigma'_r \in \{-1, 1\}$ such that $E_r e_x = \sigma'_r E_r e_y$. Then if we define $Q := p(A)$ a matrix that swaps a and b, we want to show that $Qe_x = e_y$. Therefore, we only need to show that $\sigma_r = \sigma'_r \; \forall r$.

Observing the (a, x) and the (b, y) entries of the idempotents:

$$(E_r)_{a,x} = (E_r e_a)^T (E_r e_x) = (\sigma_r E_r e_b)^T (\sigma'_r E_r e_y) = \sigma_r \sigma'_r (E_r)_{b,y}$$

We verify that $(E_r)_{a,x} = (E_r)_{b,y} \iff \sigma_r = \sigma'_r$.

From Theorem 4.4.3:

$$(E_r)_{a,x} = \frac{\sqrt{\phi(X \backslash a, t)\phi(X \backslash x, t) - \phi(X \backslash ax, t)\phi(X, t)}(t - \theta_r))}{\phi(X, t)}\Bigg|_{t=\theta_r}$$

For $(E_r)_{a,x}$ to be equal to $(E_r)_{b,y}$ it is sufficient for the terms inside the square root remain the same after changing a for b and x for y, but since a is cospectral to b and x is cospectral to y:

$$\phi(X \backslash ax, t) = \phi(X \backslash by, t) \implies (E_r)_{a,x} = (E_r)_{b,y}$$

We show by induction that $\phi(X \backslash ax, t) = \phi(X \backslash by, t)$:

For $l = 2$ a and b are adjacent, therefore $\phi(X \backslash ax, t) = \phi(X \backslash ab, t) = \phi(X \backslash by, t)$.

For $l > 2$, use induction, let $d = d(a, x)$.

$$\begin{aligned}
\phi(X \backslash ax, t) &= \phi(T_1 \backslash a, t)\phi(P_{l-d-1}, t)\phi(T_2 \cup P_d, t) \\
&= \phi((T_1 \cup P_d \cup T_2) \backslash ax, t)\phi(P_{l-d-1}, t) \\
&= \phi((T_1 \cup P_d \cup T_2) \backslash by, t)\phi(P_{l-d-1}, t) \\
&= \phi(X \backslash by, t)
\end{aligned}$$

□

8 Simple Eigenvalues

9 Controllable graphs