# Perfect state transfer in graphs with an involution

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## 1 Preliminaries

**Lemma 1.** *[4] Let $G$ be a (weighted) graph with an involution $\sigma$, which respects loops and edge weights. Then the characteristic polynomial of the (weighted) adjacency matrix $A$ of $G$ factors into two factors $\Pi_+$ and $\Pi_-$ which are, repsectively, the characteristic polynomials of $A_+ := \begin{bmatrix} A' + A_\sigma & A_\delta \\ 2A_\delta^T & A_S \end{bmatrix}$ and $A_- := A' - A_\sigma$.*

*Furthermore, there is an eigenbasis for $A$ consisting of vectors that take the form $\begin{bmatrix} a & a & b \end{bmatrix}^T$ and $\begin{bmatrix} c & -c & 0 \end{bmatrix}^T$, where $\begin{bmatrix} a & b \end{bmatrix}^T$ is an eigenvector for $A_+$, and $c$ an eigenvector for $A_-$.*

**Theorem 2** (Weyl's Inequality)**.** *If $A$ and $B$ are Hermitian, then $\lambda_{i+j-1}(A+B) \leq \lambda_i(A) + \lambda_j(B)$, and equivalently, $\lambda_{i+j-n}(A+B) \geq \lambda_i(A) + \lambda_j(B)$.*

**Lemma 3.** *[3] Let $u, v$ be vertices of $G$, and $H$ the Hamiltonian. Then perfect state transfer from $u$ to $v$ occurs at some time if and only if the following two conditions are satisfied:*

1. *Every eigenvector $x$ of $H$ satisfies either $x(u) = x(v)$ or $x(u) = -x(v)$.*
2. *If $\{\lambda_i\}$ are the eigenvalues with $x(u) = x(v)$, and $\{\mu_j\}$ are the eigenvalues for the eigenvectors with $x(u) = -x(v)$, and $x(u)$ and $x(v)$ are non-zero, then there exists some time $T$ such that*

   $$e^{iT\lambda_1} = e^{iT\lambda_i} = -e^{iT\mu_j}$$

   *for all $i, j$.*

**Corollary 4.** *[3] Using the notation of Lemma 3, if perfect state transfer occurs from $u$ to $v$, then*

$$\frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_\ell} \in \mathbb{Q}$$

*and*

$$\frac{\lambda_i - \mu_j}{\lambda_k - \lambda_\ell} = \frac{odd}{even}$$

*for all $i, j, k, \ell$.*

**Theorem 5.** *[2] Suppose $X$ is a graph with at least two vertices. Then $X$ is periodic at $a$ if and only if either:*

1. *The eigenvalues in the eigenvalue support of $a$ are integers.*
2. *There is a square-free integer $\Delta$, the eigenvalues in the eigenvalue support of $a$ are quadratic integers in $\mathbb{Q}(\sqrt{\Delta})$, and the difference of any two eigenvalues in the eigenvalue support of $a$ is an integer multiple of $\sqrt{\Delta}$.*

**Theorem 6.** *[1] There is no perfect state transfer on the double star graph $S_{k,\ell}$.*

## 2 Result

**Theorem 7.** *Let $X$ be a graph, $v$ a vertex of $X$, and let $Y$ be the graph formed from two copies of $X$ by adding an edge between the copies of $v$. If $|V(X)| \geq 2$, then $Y$ does not admit perfect state transfer between the copies of $v$.*

*Proof.* Suppose $Y$ admits perfect state transfer between the copies of $v$. By construction, $Y$ has an involution $\sigma$ mapping each vertex to the corresponding vertex in the other copy of $X$. Hence, by Lemma 1, the eigenvalues of $A(Y)$ are the eigenvalues of $A(X) + e_v e_v^T$ and $A(X) - e_v e_v^T$.

Let $\{x_i\}$ be an eigenbasis for $A(X) + e_v e_v^T$ such that the number of eigenvectors for which $x_i(v) = 0$ is maximized. Then every such eigenvector is an eigenvector of $A(X) - e_v e_v^T$. We can extend this set of eigenvectors to an eigenbasis $\{y_j\}$ of $A(X) - e_v e_v^T$ such that the number of eigenvectors for which $y_j(v) = 0$ is maximized. If there is an eigenvector $y_j \notin \{x_i\}$ such that $y_j(v) = 0$, then $y_j$ is an eigenvector of $A(X) + e_v e_v^T$, contradicting maximality. Hence, $A(X) + e_v e_v^T$ and $A(X) - e_v e_v^T$ have the same number of eigenvalues in the eigenvalue support of $v$, and the remaining eigenvalues (with multiplicity) are both eigenvalues of $A(X) + e_v e_v^T$ and $A(X) - e_v e_v^T$.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A(X) + e_v e_v^T$ in the support of $v$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $A(X) - e_v e_v^T$ in the support of $v$. It follows from Weyl's Inequality (Theorem 2) that $\lambda_i \geq \mu_i$ for $1 \leq i \leq n$. Moreover, it follows from Corollary 4 that $\lambda_i \neq \mu_i$. Finally, it follows from Theorem 5 that $\lambda_i \geq \mu_i + 1$. Therefore, if $\{\theta_k\}$ is the multiset of remaining eigenvalues of $A(X) \pm e_v e_v^T$ not in the support of $v$, we have

$$1 = \operatorname{tr}(A(X) + e_v e_v^T) = \sum_{i=1}^{n} \lambda_i + \sum_k \theta_k \geq \sum_{i=1}^{n} (\mu_j + 1) + \sum_k \theta_k = tr(A(X) - e_v e_v^T) + n = n - 1,$$

whence $n \leq 2$.

Suppose $n = 2$, then the inequality holds with equality, and we have $\lambda_i = \mu_i + 1$. As the covering radius of $v$ is at most 1, $v$ is a universal vertex. Now, there exists an eigenbasis of $A(X)$ such that $|V(X)| - 2$ of the vectors $x$ are such that $x(v) = 0$. Then each such eigenvector $x$ is an eigenvector of $X \setminus v$ and $x \cdot \mathbf{1} = 0$. Thus, the remaining eigenvector of $X \setminus v$ is $\mathbf{1}$, so $X \setminus v$ is regular; we assume of degree $k$.

It follows that if $m = |V(X \setminus v)|$, then $\lambda_1, \lambda_2$ are eigenvalues of the quotient matrix

$$\begin{bmatrix} k & \sqrt{m} \\ \sqrt{m} & 1 \end{bmatrix}$$

and $\mu_1, \mu_2$ are eigenvalues of the quotient matrix

$$\begin{bmatrix} k & \sqrt{m} \\ \sqrt{m} & -1 \end{bmatrix}.$$

Hence, we have

$$\begin{aligned} \mu_1 + \mu_2 &= k - 1 \\ \mu_1 \mu_2 &= -k - m \\ \lambda_1 \lambda_2 &= k - m \\ (\mu_1 + 1)(\mu_2 + 1) &= k - m \\ \mu_1 \mu_2 + \mu_1 + \mu_2 + 1 &= k - m \\ -k - m + k - 1 + 1 &= k - m \\ 0 &= k, \end{aligned}$$

so $X \setminus v = \overline{K_m}$ and therefore $Y$ is a double star, which contradicts Theorem 6.

Therefore, $n = 1$, so $X = K_1$, which completes the proof. □

## References

[1] X. Fan, C. Godsil. Pretty good state transfer on double stars. *Linear Algebra and its Applications*, 438: 2346–2358, 2013.

[2] C. Godsil. Periodic Graphs. *Electronic Journal of Combinatorics*, 18(1): P23, 2011.

[3] M. Kempton, G. Lippner, S.-T. Yau. Perfect state transfer on graphs with a potential. *Quantum Information and Computation*, 17 (3 & 4):0303–0327, 2017.

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