Exampls for Graphs with Average Mixing Matrices of Rank 2

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Abstract

Descriptions of graphs with average mixing matrices of rank 2 and at least 11 vertices (so those that are not easily found by doing a computer search).

1 16 Vertices

1.1 Explicit List

Format is graph6.

Matrices on 16 vertices which were found by taking a group of order 16 and a subgroup of order 8 (like Cayley graphs, just with two degrees):

```
Os_raXPI_wPCTEGvGLCSH
O}amrwUQQqEDqCaz?PWcp
O}ambWqYOqAdrCAz?VGc\
O}AmpwqQQQAfqKaZ?rHcT
O}QmdWUROUEDyCEZ?Pwdp
O}amrwuYQqEdrCaz?VWc|
O}a~bXrYoyRdvEI~G^Ks\
O}A~pxrQqYRfuMi^GzLsT
O}a~rxvYqyVdvEi~G^[s|
OvOekY{F@UkcWCERpBoFd
O~qkdY}^PueLycFJPVydp
OvoeeY{N@uckXcFrPFqDl
O~qmfYuZOuedzCEz@Vwd|
O~QmlY}VPUmdyCEZpRwft
O~qmfY}^PuelzcFzPVyd|
```

The complements have rank 2 as well, but they might not be in this list.

1.2 Group Description

All of the examples above where found using groups of order 16, so it is straightforward to give a description in these terms.

Let G be the group with presentation

$$\langle a, b, c : a^2 = (ab)^2 = b^4 = c^2 = (ac)^2 = (bc)^2 \rangle.$$

Then G is isomorphic to $C_2 \times D_8$ (where D_8 has order 8). Let H be the subgroup generated by $\{a, b^2, c\}$ (isomorphic to C_2^3).

Let

$$\Gamma_Y = \{a, ba, b^2, cb, cb^3\}, \qquad \Gamma_Z = \{b^2 c\}.$$

Notice that $b^2 c \in H$. Let Y be the 16-vertices Cayley graph generated by G and Γ_Y, let Z be the 8-vertices Cayley graph generated by H and Γ_Z. As H is a subgroup of G, we can see the vertices of Z as vertices of Y. Then $X = Y \cup Z$ is a graph on 16 vertices with degrees 5 and 6. The average mixing matrix of X has rank 2.

1.3 A Randomized Description

Here a short description on how to find one example (and example on 4 and 8) vertices with a randomized algorithm. Let A be the (8×8)-adjacency matrix of the union of two disjoint Hamming graphs on 4 vertices. Let B be the (8×8)-adjacency matrix of a Hamming graph on 8 vertices. Let

$$\begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix}$$

be the (16×16)-adjacency matrix of the union of two disjoint Hamming graphs of order 8. Here C is a (8×8)-matrix.

Let P be a random (8×8)-permutation matrix. Then

$$\begin{bmatrix} A & PC \\ (PC)^T & B \end{bmatrix}$$

sometimes has an average mixing matrix of rank 2. Experimentally, the probability is about 10^{-3}. Question: Is there such a matrix for a graph on 32 vertices, i.e. what would be the probability there if such a graph exists? (just double all the numbers in the construction)

2 32 Vertices

2.1 Explicit List

A computer search with the help of Anton Betten and his Orbiter software, we obtained the following graphs with an average mixing matrix of rank 2.

```
 _JP?hKgEJ_OxaM_WOQKWB@K??sGU?WEC[IA?AO_GHG@_a_FK?DgOAPIG@DAg?RKSO?YPeo?HQ\??ajC_?dRC
_G\rrg_OK}Oe_k_\PES_NCc?GgAG?xA@GQCJ@?cY?Cq_Q_bCD@fAAPP`CHM?@QPPP?ahQ`?LXSA@CJt?@KDs
_IOxocKWFBq]b[m?WoKBbH_?E?ao??oA?c?nG_Jq_A]H?Hw?WDc?QDD_DBDOBORB?D``Y?S@i[?cCks?KQRK
_J\_xPPi]b{of``wOK{KrE?CD_J?@H?BX?@l@PAj?AC_c_Z?X@Y?cbQ_PPsWCoAw?EalEW?HfsO?kps_?c]s
_BXrBOAn]zPsbgcxQ[V?NCK?e?IO?gKGCaCL?QGo_aYaCGL?KR|Q?_lC_`hgOCiqO?egM@GItKA@Ojs?PKS{
_@CiyyslEFw]zTzh^JFtOO?o?wH?@wOECOgFCEBaOARH?PiCK?]@`A|?g`N?EGI^@oBby_?U^L?@L\c?@rdk
_@GZz~?ovPyftLyMvkD|@?CgAgI?GgIAWGAbCIBwcODAS@DW@@|W?EVC?bhgOcWrACHza@@InSI?NbsGAS\w
_@GZ}FA}^jzlz\nW^wN|AOAG?dGE_H?PCPOJAHBWPAZAW@Z@?f^A@OuAQ@cz?O]jW?JyY?AZ|CG?kn{GCLVw
_?Kr}FA|]z~Lz\~gVkN}@Vz|^}r~\~^vFs~v}vu~uzn]u~Zj~zTj}~Y]m~c~vy}vvzn^I~za~nxn~Mf}ywv[
_@GYyywmEFw]|TzU]sf\ON~k~|r~x~J~jf^y|V}~Y~j^^wV~h|e~i}f^u^{}v}zju~zZb^~BY\~xy]N~FzE[
_?Krz~?ouhyfyLyMvkD|@V^|^vrznl^}un|uv\}^mvr}r}{|m|p}~in|~jivv^Xm}znqj}Zww~~\Mx\~y|Ug
_B\bJ?@nm|Q[bWbXQ[V?N^y|nzz^^t|zfy|pm~\Mn|m~yzK~Vt~zy|j^mzE]~|]_|~jI^V~ilTt~qh^ynmhc
_J\PZ?Xw[ryoj``wOK{KrF~|^~P~}n~Rn}ly~i~N|r}\~tB|~iX}\~L~i~hfuneTx~le]^~XuVq~dmNN~N@s
_JDjZo_OK}OU_[`LQESONN}|nzx^~d^vf}}bv~g~fzb]znE}v\L~\tp~{zrE||wU~ZvYJr^qW~zNuME|~WWk
_IOxr?IDCVqmw[m?WoKBb^{|}vz|vv|fn~lh~vYV}Zl{~uIh~}Il~kdt~uduv}ek}^vA^\~[eMr~eBn}^LiC
_JCPIKgQJ_PF`qb?OIKDBV~k~}w~~f^}F~vBf~k~^mw\}|`~M{L|N{q~\maflvqP|]|HNz\yG~}ZuEFu^heC
```


Notice that the search was less exhaustive than our search for 16 vertcies, so the fact that we do not know more examples does not tell us anything.

2.2 Group Description

Consider the affine group Γ that is generated by the following elements:

$$a = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right], \qquad b = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right],$$

$$c = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right], \qquad d = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right],$$

$$e = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right].$$

Let

$$\Gamma_Y = \{a, c, d, ad, bcd, e, ea, eb\}, \qquad \Gamma_Z = \{acd\}.$$

Let H be the subgroup of G generated by $\{a, b, c, d\}$.

Let Y be the 32-vertices Cayley graph generated by G and Γ_Y, let Z be the 16-vertices Cayley graph generated by H and Γ_Z. As H is a subgroup of G, we can see the vertices of Z as vertices of Y. Then $X = Y \cup Z$ is a graph on 32 vertices with degrees 8 and 9. The average mixing matrix of X has rank 2.

All the other graphs in the previous section can be described in a similar way. For 16 vertices this is a possible choice:

$$a = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right], \qquad b = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right],$$
$$c = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right], \qquad d = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right].$$

with

$$\Gamma_Y = \{a, c, d, da, db\} \qquad \Gamma_Z = \{bc\}.$$

3 64 Vertices

The explicit list is in a seperate file. The structure is similar to 32 vertices, see the general section.

4 General Case

4.1 Group Point of View

The working conjecture is that the general group of order 2^n has (in GAP notation) the form $C_2^{n-1} : C_2$ and that we can write this as an affine group G in the following way: Define the $(n-1) \times (n-1)$ matrix M over $\mathbb{F}_2$ by

$$(M)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let k be the smallest natural number such that $M^k = I$. Then k is even. Set $M_n = M^{k/2}$. Now the generators of M are the canonical basis $\{g_1, \dots, g_{n-1}\}$ of C_2^{n-1} and $g_n = (M_n, 0)$.

We found examples with 64 vertices using this group.

4.2 Spectral Point of View

Chris Godsil is not a fan of relying too much on group while investigating graphs, so here are some notes on the spectrum. Also see `notes_rank2_averagemixing_matrix...` which will be refered to by [1].

There is no specific working conjecture on how to define the Cayley graph now. One should take slightly more than $1/4$ of the elements, all of them involutions. It seems to be good for them to include $\{g_1, \dots, g_{n-2}\}$ and g_n.

Let A_1, B and A_2 be as in [1]. For the smallest degrees cases the graph

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

seems to be a disjoint union of hypercubes of dimension 2, 3, 3, resp., 4 for 8, 16, 32, resp., 64 vertices. I would guess that the series looks like this: 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, ... The graphs A_1 and A_2 are always cube-like and either are cubes or appear to be double covers of cubes.

For the investigated example the spectrum is always a (for larger examples: proper) subset of $\{-k_1, -k_1+2, \ldots, k_1-2, k_1\}$ and $\{-k_2, -k_2+2, \ldots, k_2-2, k_2\}$, where WLOG k_1 is odd and k_2 is even. The spectrum of the graph belongng to B is also in $\{-k_{12}, -k_{12}+2, \ldots, k_{12}-2, k_{12}\}$. This is similar to coefficients of the linear equation used in [1] to calculate the possible eigenvalues of the entire graph.

A common feature for all examples is that the automorphism group of the graphs acts sharply transitive on each of the two strongly cospectral classes. Due to Chris Godsil's notes this already implies that all eigenvalues are simple and that there are only two strongly cospectral classes. So one can rephrase:

Problem 1. *Find a graph with an automorphism group with two orbits of vertices that acts sharply transitive on both of these orbits.*