Applied Practice

Emanuel Mejía

Week 5

1. Safety First through Statistics

Suppose the strength of a particular metal beam is given by,

$$S = 5 + 2T^2 \cdot P$$

Where T is a random variable representing the forging temperature and P is a random variable representing purity. Suppose the following statements are true about these random variables:

- T has a uniform distribution on $[0, 2]$.
- Conditional on a value for T, P is has a normal distribution with mean $T/2$ and standard deviation $T/12$.

For example, if $T = 1$, then,

$$E[P|T = 1] = \frac{1}{2}$$

and,

$$\sigma[P|T = 1] = \sqrt{V[P|T = 1]} = \frac{1}{12}$$

1. Compute the expectation of S.

Solution

Since $S = 5 + 2T^2P$, we know:

$$E[S] = E[5 + 2T^2P] = 5 + 2E[T^2P]$$

Now, to calculate $E[T^2P]$ we'll use the law of iterated expectations as follows:

$$
\begin{aligned}
E[T^2P] &= E_T[E[T^2P|T]] \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} t^2 p f_{P|T}(p|t) dp \right) f_T(t) dt \\
&= \int_{-\infty}^{\infty} t^2 \left(E[P|T=t] \right) f_T(t) dt \\
&= \int_{-\infty}^{\infty} t^2 \left(\frac{t}{2} \right) f_T(t) dt \\
&= \frac{1}{2} \int_{-\infty}^{\infty} t^3 f_T(t) dt \\
&= \frac{1}{2} \int_{0}^{2} t^3 (\frac{1}{2}) dt && \text{Since } T \sim U(0,2) \\
&= \frac{1}{4} \left[\frac{t^4}{4} \right]_0^2 \\
&= \frac{1}{4} [4] \\
&= 1
\end{aligned}
$$

Substituting this back in our result for $E[S]$ we have that:

$$
\begin{aligned}
E[S] &= 5 + 2E[T^2P] \\
&= 5 + 2(1) \\
&= 7
\end{aligned}
$$