Applied Practice

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Week 5

2. Comparing Estimators

Say that $X_1, ..., X_n$ is an i.i.d. sample from an exponential distribution, with common density,

$$f_X(x) = \lambda e^{-\lambda x}$$

$$E[X] = \int x \cdot f_X(x)dx = \frac{1}{\lambda}$$

$$V[X] = \frac{1}{\lambda^2}$$

You are considering the following estimators.

1. $\hat{\theta}_1 = \frac{X_1+X_2+...+X_n}{n}$
2. $\hat{\theta}_2 = \frac{X_1+X_2}{2}$
3. $\hat{\theta}_3 = \frac{X_1+X_2+...+X_n}{n+1}$

a. Compute the bias of each estimator, $E[\hat{\theta}_i] - E[X]$.
b. Compute the sampling variance of each estimator.
c. Compute the MSE of each estimator.

Solution $\hat{\theta}_1$

Bias $\hat{\theta}_1$

$$\begin{aligned}
E[\hat{\theta}_1] &= E\left[\frac{X_1 + X_2 + ... + X_n}{n}\right] \\
&= \frac{1}{n}E[X_1 + X_2 + ... + X_n] \\
&= \frac{1}{n}\left(E[X_1] + E[X_2] + ... + E[X_n]\right) \\
&= \frac{1}{n}\left(E[X] + E[X] + ... + E[X]\right) \\
&= \frac{1}{n}nE[X] \\
&= E[X] \\
&= \frac{1}{\lambda}
\end{aligned}$$

Therefore, the bias for $\hat{\theta}_1$ will be:

$$\begin{aligned} E[\hat{\theta}_1] - E[X] &= E[X] - E[X] \\ &= \frac{1}{\lambda} - \frac{1}{\lambda} \\ &= 0 \end{aligned}$$

Sampling Variance $\hat{\theta}_1$

$$\begin{aligned} V[\hat{\theta}_1] &= V\left[\frac{X_1 + X_2 + ... + X_n}{n}\right] \\ &= \frac{1}{n^2} V[X_1 + X_2 + ... + X_n] \\ &= \frac{1}{n^2}\left(V[X_1] + V[X_2] + ... + V[X_n]\right) && \text{Since } X_i's \text{ are independent} \\ &= \frac{1}{n^2}\left(V[X] + V[X] + ... + V[X]\right) \\ &= \frac{1}{n^2} nV[X] \\ &= \frac{V[X]}{n} \\ &= \frac{\left(\frac{1}{\lambda^2}\right)}{n} \\ &= \frac{1}{n\lambda^2} \end{aligned}$$

MSE $\hat{\theta}_1$

$$\begin{aligned} MSE[\hat{\theta}_1] &= V[\hat{\theta}_1] + \left(E[\hat{\theta}_1] - E[X]\right)^2 \\ &= \frac{1}{\lambda^2 n} + (0)^2 \\ &= \frac{1}{\lambda^2 n} \end{aligned}$$

Solution $\hat{\theta}_2$

Bias $\hat{\theta}_2$

$$\begin{aligned}
E[\hat{\theta}_2] &= E\left[\frac{X_1 + X_2}{2}\right] \\
&= \frac{1}{2}E[X_1 + X_2] \\
&= \frac{1}{2}\left(E[X_1] + E[X_2]]\right) \\
&= \frac{1}{2}\left(E[X] + E[X]\right) \\
&= \frac{1}{2}2E[X] \\
&= E[X] \\
&= \frac{1}{\lambda}
\end{aligned}$$

Therefore, the bias for $\hat{\theta}_2$ will be:

$$\begin{aligned}
E[\hat{\theta}_2] - E[X] &= E[X] - E[X] \\
&= \frac{1}{\lambda} - \frac{1}{\lambda} \\
&= 0
\end{aligned}$$

Sampling Variance $\hat{\theta}_2$

$$\begin{aligned}
V[\hat{\theta}_2] &= V\left[\frac{X_1 + X_2}{2}\right] \\
&= \frac{1}{2^2}V[X_1 + X_2] \\
&= \frac{1}{4}\left(V[X_1] + V[X_2]\right) && \text{Since } X_1 \perp\!\!\!\perp X_2 \\
&= \frac{1}{4}\left(V[X] + V[X]\right) \\
&= \frac{1}{4}2V[X] \\
&= \frac{V[X]}{2} \\
&= \frac{\left(\frac{1}{\lambda^2}\right)}{2} \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

MSE $\hat{\theta}_2$

$$
\begin{aligned}
MSE[\hat{\theta}_1] &= V[\hat{\theta}_2] + \Big(E[\hat{\theta}_2] - E[X]\Big)^2 \\
&= \frac{1}{2\lambda^2} + (0)^2 \\
&= \frac{1}{2\lambda^2}
\end{aligned}
$$

Solution $\hat{\theta}_3$

Bias $\hat{\theta}_3$

$$
\begin{aligned}
E[\hat{\theta}_3] &= E\left[\frac{X_1 + X_2 + ... + X_n}{n+1}\right] \\
&= \frac{1}{n+1}E[X_1 + X_2 + ... + X_n] \\
&= \frac{1}{n+1}\left(E[X_1] + E[X_2] + ... + E[X_n]\right) \\
&= \frac{1}{n+1}\left(E[X] + E[X] + ... + E[X]\right) \\
&= \frac{1}{n+1}nE[X] \\
&= \frac{n}{n+1}E[X] \\
&= \frac{n}{(n+1)\lambda}
\end{aligned}
$$

Therefore, the bias for $\hat{\theta}_3$ will be:

$$
\begin{aligned}
E[\hat{\theta}_3] - E[X] &= \frac{n}{n+1}E[X] - E[X] \\
&= \frac{n}{(n+1)\lambda} - \frac{1}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{n}{(n+1)} - 1\right) \\
&= \frac{1}{\lambda}\left(\frac{n}{(n+1)} - \frac{n+1}{(n+1)}\right) \\
&= \frac{1}{\lambda}\left(-\frac{1}{(n+1)}\right) \\
&= -\frac{1}{(n+1)\lambda}
\end{aligned}
$$

Sampling Variance $\hat{\theta}_3$

$$\begin{aligned}
V[\hat{\theta}_3] &= V\left[\frac{X_1 + X_2 + ... + X_n}{n+1}\right] \\
&= \frac{1}{(n+1)^2} V[X_1 + X_2 + ... + X_n] \\
&= \frac{1}{(n+1)^2}\left(V[X_1] + V[X_2] + ... + V[X_n]\right) && \text{Since } X_i's \text{ are independent} \\
&= \frac{1}{(n+1)^2}\left(V[X] + V[X] + ... + V[X]\right) \\
&= \frac{1}{(n+1)^2} nV[X] \\
&= \frac{nV[X]}{(n+1)^2} \\
&= \frac{n\left(\frac{1}{\lambda^2}\right)}{(n+1)^2} \\
&= \frac{n}{(n+1)^2\lambda^2}
\end{aligned}$$

MSE $\hat{\theta}_3$

$$\begin{aligned}
MSE[\hat{\theta}_3] &= V[\hat{\theta}_3] + \left(E[\hat{\theta}_3] - E[X]\right)^2 \\
&= \frac{n}{(n+1)^2\lambda^2} + \left(-\frac{1}{(n+1)\lambda}\right)^2 \\
&= \frac{n}{(n+1)^2\lambda^2} + \frac{1}{(n+1)^2\lambda^2} \\
&= \frac{n+1}{(n+1)^2\lambda^2} \\
&= \frac{1}{(n+1)\lambda^2}
\end{aligned}$$

d. Explain in your own words, why estimator 3 has the highest bias, but the lowest MSE

Even if $\hat{\theta}_3$ is biased because of the +1 in the denominator, this same denominator makes the sampling variance smaller, meaning that $V[\hat{\theta}_3] = \frac{n}{(n+1)^2\lambda^2} < \frac{1}{n\lambda^2} = V[\hat{\theta}_1]$ as long as $n > 0$ (which is the case here).

Even if the terms don't look alike we can prove it as follows:

$$
\begin{aligned}
V[\hat{\theta}_3] &= \frac{n}{(n+1)^2\lambda^2} \\
&= \frac{n}{n+1}\left(\frac{1}{(n+1)\lambda^2}\right) \\
&< \frac{1}{(n+1)\lambda^2} && \text{Since } \frac{n}{n+1} < 1 \\
&< \frac{1}{n\lambda^2} = V[\hat{\theta}_1] && \text{Since } \frac{1}{n+1} < \frac{1}{n}
\end{aligned}
$$

And as long as $n > 1$, it will also be smaller than $V[\hat{\theta}_2]$ (which, by the way, is fixed).

On the other hand, the bias of the estimator $E[\hat{\theta}_3] - E[X]$ even if it's not zero as the other two, it tends to zero as n grows larger: $\lim_{n\to\infty} -\frac{1}{(n+1)\lambda} = 0$.

By summing these two factors we end up with a lower $MSE[\hat{\theta}_3] = V[\hat{\theta}_3] + \left(E[\hat{\theta}_3] - E[X]\right)^2$, (since the variance is lower and the bias tends to the same value) meaning a more efficient estimator (it converges faster as n grows larger).