Game Transformations That Preserve Nash Equilibria or Best Response Sets

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Abstract

In the literature on simultaneous non-cooperative games, it is a widely used fact that a positive affine (linear) transformation of the utility payoffs neither changes the best response sets nor the Nash equilibrium set. We investigate which other game transformations also possess one of these two properties when being applied to an arbitrary N-player game ($N \geq 2$):

(i) The Nash equilibrium set stays the same.

(ii) The best response sets stay the same.

For game transformations that operate player-wise and strategy-wise, we prove that (i) implies (ii) and that transformations with property (ii) must be positive affine. The resulting equivalence chain gives an explicit description of all those game transformations that always preserve the Nash equilibrium set (or, respectively, the best response sets). Simultaneously, we obtain two new characterizations of the class of positive affine transformations.

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1. Introduction

A classic tool in game theory is to transform some given game into another strategically equivalent game that is easier to handle [1]. Positive affine (linear) transformations (PATs) for normal-form games have been particularly useful in that regards [2, 3]. For a PAT example, take any two player game in which the players utility payoffs are measured in amounts of dollars. Then the best response strategies of player 1 do not change if her utility payoffs are multiplied by 5. Moreover, they don't change if 10 dollars are added to all outcomes that involve player 2 playing his, say, third strategy. More generally, PATs have the power to rescale the utility payoffs of each player and to add constant terms to the utility payoffs for each player i and strategy choice k_{-i} of her opponents. PATs have been leveraged in the literature to significantly extend the applicability of efficient Nash equilibrium solvers beyond the classes of zero-sum and rank-1 games[1] (see [4, 5, 6, 7, 8] for the NE solvers and [9, 10, 11] for their extensions). The key to success of these extensions was the well-known property of PATs that they do not change the Nash equilibrium set and best response sets when being applied to an arbitrary input game [11, 12, 13, 14].

This paper addresses the question whether there are game transformations H other than PATs with that same property. Such transformations are of key interest for two major reasons:

1. They open up methods of generating (infinitely many) games that share key game-theoretic characteristics among themselves, and therefore,

2. They also allow us to straight-forwardly extend any results we have about Nash equilibria and best response sets of normal-form games to any of their transformations.

Throughout this paper, we restrict our attention to game transformations that transform utilities player-wise and strategy-wise.

1.1. Our Contributions

Our mathematical analysis starts with considering all those game transformations that preserve the Nash equilibrium set when being applied to an arbitrary N-player game. Proposition 3.1 shows that such game transformations must also preserve the best response sets when being applied to an arbitrary N-player game. We proceed to prove over a series of results

[1] A 2-player game, represented by its payoff matrices $A, B \in \mathbb{R}^{m \times n}$, is said to have rank-1 if $\text{rank}(A + B) = 1$.

that any best response preserving game transformation must (in particular) be a positive affine transformation (PAT). To close the proof cycle, we restate in Lemma 2.6 the popular textbook result that PATs preserve the Nash equilibrium set. Therefore, we have found two new equivalent definitions of PATs that point out their special status among game transformations: PATs are the only types of game transformations that always preserve the Nash equilibrium set (or, respectively, the best response sets). This main result of our paper is summarized in Theorem 1. Any proofs can be found in the appendix.

Setup. Let $N \in \mathbb{N}, N \geq 2$, be the number of players and $S^i = \{1, \ldots, m_i\} =: [m_i]$ be the strategy sets for all player i, where $m_i \in \mathbb{N}, m_i \geq 2$. These parameters shall be fixed for the rest of this paper. Denote all strategy profiles as $S := S^1 \times \ldots \times S^N$. A game G is then determined by the utility functions $u_i : S \longrightarrow \mathbb{R}$ for each player i; even after we allow the players to randomize over their pure strategies.

A *positive affine transformation* H_{PAT} specifies a scaling parameter $\alpha^i \in \mathbb{R}$, $\alpha^i > 0$, for each player i, as well as a translation constant $c^i_{\mathbf{k}_{-i}}$ for each player i and each opponent's strategy choice $\mathbf{k}_{-i} = (k_j \in S^j)_{j \neq i}$. The transformation H_{PAT} then takes a game $G = \{u_i\}_{i \in [N]}$ as an input and returns the transformed game $H_{\text{PAT}}(G) = \{u'_i\}_{i \in [N]}$ where

$$\begin{aligned} u'_i : S &\longrightarrow \mathbb{R} \\ \mathbf{k} &\longmapsto \alpha_i \cdot u_i(\mathbf{k}) + c^i_{\mathbf{k}_{-i}} \,. \end{aligned}$$

A *game transformation* $H = \{H^i\}_{i \in [N]}$, on the other hand, specifies for each player i a collection of functions $H^i := \left\{h^i_{\mathbf{k}} : \mathbb{R} \longrightarrow \mathbb{R}\right\}_{\mathbf{k} \in S}$ that are indexed by the different pure strategies $\mathbf{k} = (k_1, \ldots, k_N) \in S$. The game transformation H then takes a game $G = \{u_i\}_{i \in [N]}$ as an input and returns the transformed game $H(G) = \{H^i(u_i)\}_{i \in [N]}$ where

$$\begin{aligned} H^i(u_i) : S &\longrightarrow \mathbb{R} \\ \mathbf{k} &\longmapsto h^i_{\mathbf{k}}\big(u_i(\mathbf{k})\big) \,. \end{aligned}$$

Intuitively speaking, H operates player-wise and strategy-wise through its maps $h^i_{\mathbf{k}}$.

Main Result. Abstractly speaking, we say that a transformation H *universally preserves* some special set $\mathcal{Z}$ if applying transformation H to an arbitrary input G does not change G's special set, that is, $\mathcal{Z}_{H(G)} = \mathcal{Z}_G$. Definitions 2.9 and 2.10 make this concept precise for $\mathcal{Z} = \{\text{Nash equilibria}\}$ and $\mathcal{Z} = \{\text{Best responses}\}$.

Theorem (Main Result, Restatement of Theorem 1)**.**
Let $H = \{H^i\}_{i\in[N]}$ be a game transformation. Then:

$$\begin{aligned} &H \textit{ universally preserves the Nash equilibrium set} \\ \iff &\textit{ for each player } i\textit{, map } H^i \textit{ universally preserves best responses} \\ \iff &H \textit{ is a positive affine transformation.} \end{aligned}$$

Therefore, if we intend to utilize game transformations to generate games with the same Nash equilibrium set or the same best responses as some input game G, we must do at least one of the following:

1. rely on positive affine transformations only,
2. take advantage of properties that are specific to the input game G, or
3. consider other notions of transforming a game altogether.

1.2. Illustration of Game Transformations on Bimatrix Games

In the case of two-player games ($N = 2$), a normal-form game can be represented by its pair of payoff matrices $(A, B) = \Big((a_{ij})_{ij}, (b_{ij})_{ij}\Big) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$. A PAT of bimatrix games is then determined by positive scalars $\alpha := \alpha_1 > 0$ and $\beta := \alpha_2 > 0$, and translation vectors $c := (c^1_{k_2})_{k_2\in[n]} \in \mathbb{R}^n$ and $d := (c^2_{k_1})_{k_1\in[m]} \in \mathbb{R}^m$. It transforms the game (A, B) into the game

$$H_{\text{PAT}}(A, B) = \Big((\alpha \cdot a_{ij} + c^1_j)_{ij}, (\beta \cdot b_{ij} + c^2_i)_{ij}\Big) = (\alpha A + \mathbf{1}_m c^T, \beta B + d\mathbf{1}_n^T)\,.$$

A general game transformation H, on the other hand, specifies $H^1, H^2 : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$ as matrices of single-variable functions $(h^1_{ij})_{ij}$ and $(h^2_{ij})_{ij}$. Transforming a game (A, B) through H yields the bimatrix game

$$H(A, B) = \Big(\big(h^1_{ij}(a_{ij})\big)_{ij}, \big(h^2_{ij}(b_{ij})\big)_{ij}\Big)\,. \tag{1}$$

Example 1.1. Consider the transformation H_{Ex} that takes a 2×2 bimatrix game (A, B) as an input and returns the transformed game

$$A' = \begin{pmatrix} -3 \cdot a_{11} + 10 & a_{12}^5 \\ e^{a_{21}} & 0 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} |b_{11}| & \text{sign}(b_{12}) \\ \sqrt{|b_{22}|} & \arctan(b_{21}) \end{pmatrix}\,.$$

Then H_{Ex} is a game transformation[2] that is not positive affine. Therefore, by our main result, there exists a 2×2 bimatrix game (A, B) that has a different Nash equilibrium set and different best response sets than its transformed game $H_{\text{Ex}}(A, B)$.

1.3. Literature Review

Strategic Similarity. Much work has gone into identifying when two games are strategically similar/identical.

Strategic similarity, for example, is an important aspect of *Potential Games* [15]. Morris and Ui [16] noted that a game G is a weighted potential game if and only if it is the PAT transformation of an identical interest game[3]. This characterization has been used to analyze the Nash equilibria and solvers of potential games. The main contribution of Morris and Ui [16], however, was to characterize when two given input games are *best-response equivalent*, *better-response equivalent* or *von Neumann-Morgenstern equivalent.* Best-response equivalency essentially means that the two input games have the same best response sets and von Neumann-Morgenstern equivalency requires that the input games only differ by a PAT. An attempt from our side to prove (ii) $\implies$ (iii) in Theorem 1 based on their characterization for best-response equivalency [16, Prop. 4-5, Cor. 6] failed because their characterization only holds for input games that satisfy specific properties. One of them (property G3) requires knowledge of the player's preferences over non-best-response strategies; knowledge that we do not have in our analysis.

Hammond [17] described that the strategic decision-making in a game with mixed strategies does not depend on the player's numerical utility values, but solely on the preferences that the utility functions induce. This is based on the observation that the utility functions $\{u_i\}_{i \in [N]}$ of a game reveal the preferences of each player over her strategy set, given the strategy choices of her opponents. Appendix D gives some further background in utility theory that may help put Hammond's work into our context. Using the Expected Utility Theorem [18, Prop 6.B.2-3], he deduced that utility functions that induce the same preferences can only differ up to a positive affine transformation. From our perspective, Hammond therefore showed

[2]Note that H_{Ex} is not even continuous because of the sign function in B'. Nonetheless, our definition still considers it to be a game transformation.

[3]Definition of an identical interest game: Given an action profile s, each player receives the same utility from s.

that PATs are the only game transformations that preserve the underlying preferences of each player. Note, however, that the property of preserving the player's preferences is - a priori - much harder to satisfy than preserving best responses (and, hence, Nash equilibria). Thus, our Theorem 1 generalizes his result to broader questions of interest.

Du [19] proved that it is **NP**-hard to decide whether two input games share a common Nash equilibrium, and that it is **co**-**NP**-hard to decide whether two input games have the same Nash equilibrium set.

On the more broader related work, Gabarró et al. [20, 21] gave several complexity-theoretic results to the problem of deciding whether two pure strategy games are *isomorphic* w.r.t. a notion of game transformation that can help us understand the symmetries within a game [13, Chapter 3]. McKinsey [22] and Chang and Tijs [23] studied two notions of game equivalency specific to the research area of cooperative games.

Game Transformations. Another related line of research consists of work that utilizes different notions of transforming a game while preserving its "strategic structure". These papers work with similar notions of game transformations as we do, but their purpose and conclusions are non-comparable to ours.

Abbott et al. [24] and Brandt et al. [25] obtained complexity-theoretic results for computing Nash equilibria and other solution concepts. Their methods are based on so-called *Nash homomorphisms*. A Nash homomorphism resembles a reduction in the complexity-theoretic sense[4]. PATs would be a simple example of a Nash homomorphism. Note, however, that their complexity analysis requires their game transformations to always be efficiently computable.

Pottier and Nessah [26] worked with a notion of a general game transformation that is close to ours, but their study of interest are game transformations that convert the Berge-Vaisman equilibria of an input game to the Nash equilibria of the transformed game.

Other transformations that preserve strategic features were also studied by Wu et al. [27] for Bayesian games and in [28, 29, 30, 31] for extensive form games.

[4] Namely, "a mapping h from a set of games $\mathcal{A}$ into a set of games $\mathcal{B}$, such that there exists a polynomial-time computable function f that, when given a game Γ and an Nash equilibrium of $h(\Gamma)$, returns an Nash equilibrium of Γ."

2. Preliminaries

For $n \in \mathbb{N}$, we denote $[n] := \{1, \ldots, n\}$ and the vector with all ones as $\mathbf{1}_n \in \mathbb{R}^n$.

2.1. Multiplayer Games

A multiplayer game G specifies (a) the number of players $N \in \mathbb{N}, N \geq 2$, (b) a set of pure strategies $S^i = [m_i]$ for each player i where $m_i \in \mathbb{N}, m_i \geq 2$, and (c) the utility payoffs for each player i given as a function $u_i : S = S^1 \times \ldots \times S^N \longrightarrow \mathbb{R}$. Throughout this paper, all multiplayer games considered shall have the same number of players N and the same strategy sets $S^1, \ldots, S^N$. Hence, any game G will be determined by its utility functions $\{u_i\}_{i \in [N]}$. The players choose their strategies simultaneously and they cannot communicate with each other. A utility function u_i can be summarized by its pure strategy outcomes for player i, captured as an N-dimensional array $\big\{u_i(\mathbf{k})\big\}_{\mathbf{k} \in S}$.

Example 2.1. 2-player games are better known as bimatrix games because their 2-dimensional payoff arrays in become matrices $A, B \in \mathbb{R}^{m \times n}$.

As usual, we allow the players to randomize over their pure strategies. Then, player i's strategy space extends to the set of probability distributions over S^i. We identify this set with $\Delta(S^i) := \Big\{ s^i = (s^i_k)_k \in \mathbb{R}^{m_i} \;\Big|\; s^i_k \geq 0 \;\forall k \in [m_i] \text{ and } \sum_{k \in [m_i]} s^i_k = 1 \Big\}$ and refer to the probability distributions as mixed strategies. A tuple $\mathbf{s} = (s^1, \ldots, s^N) \in \Delta(S^1) \times \ldots \times \Delta(S^N) =: \Delta(S)$ of mixed strategies is called a strategy profile in G[5]. The utility payoff of player i for the strategy profile $\mathbf{s}$ is defined as the player's utility payoff in expectation

$$u_i(\mathbf{s}) := \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k}) \,.$$

The goal of each player is to maximize her utility.

We will abbreviate with S^{-i} the set that consists of all possible pure strategy choices $\mathbf{k}_{-i} = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_N)$ of the opponent players (resp. $\Delta(S^{-i})$ for the set of mixed strategy choices $\mathbf{s}^{-i} = (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^N)$).

[5] Note that in our notation, $\Delta(S)$ is not a simplex of higher dimensions itself but only the product of N simplices.

We will also often use $u_i(k_i, \mathbf{k}_{-i})$ instead of $u_i(\mathbf{k})$ to stress how player i can only influence her own strategy when it comes to her payoff (resp. $u_i(s^i, \mathbf{s}^{-i})$ instead of $u_i(\mathbf{s})$).

Definition 2.2. The best response set of player i to the opponents' strategy choices $\mathbf{s}^{-i}$ is defined as $\mathrm{BR}_{u_i}(\mathbf{s}^{-i}) := \operatorname{argmax}_{t^i \in \Delta(S^i)} \left\{ u_i(t^i, \mathbf{s}^{-i}) \right\}$.

Best response strategies capture the idea of optimal play against the other player's strategy choices. The most popular equilibrium concept in non-cooperative games is based on best responses.

Definition 2.3. A strategy profile $\mathbf{s} \in \Delta(S)$ to a game $G = \{u_i\}_{i \in [N]}$ is called a Nash equilibrium if for all player $i \in [N]$ we have $s^i \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i})$.

By [32], any multiplayer game G admits at least one Nash equilibrium.

2.2. Positive Affine Transformations

The following lemma is a well-known result for 2-player games[6]:

Lemma 2.4. *Let (A, B) be a $m \times n$ bimatrix game and take arbitrary $\alpha, \beta > 0$ and $c \in \mathbb{R}^n, d \in \mathbb{R}^m$. Define $A' = \alpha A + \mathbf{1}_m c^T$ and $B' = \beta B + d\mathbf{1}_n^T$.*

Then the game (A', B') has the same best response sets as the game (A, B). Consequently, both games have the same Nash equilibrium set.

The intuition behind this lemma is as follows: player 1 wants to maximize her utility given the strategy that player 2 chose. A positive rescaling of u_1 will change the utility payoffs but will not change the sets of best response strategies. The same holds true if we add utility payoffs to u_1 that are only dependent on the strategy choice s^2 of her opponent. In the notation of bimatrix games, this intuition yields that the transformation $A \mapsto \alpha A + \mathbf{1}_m c^T$ does not affect the best response sets of player 1. The analogous result holds for player 2 and the transformation $B \mapsto \beta B + d\mathbf{1}_n^T$.

Let us generalize PATs to multiplayer games.

Definition 2.5. A positive affine transformation (PAT) specifies for each player i a scaling parameter $\alpha^i \in \mathbb{R}, \alpha^i > 0$, and translation constants $C^i := (c_{\mathbf{k}_{-i}})_{\mathbf{k}_{-i} \in S^{-i}}$ for each choice of pure strategies from the opponents. The PAT $H_{\mathrm{PAT}} = \left\{\alpha^i, C^i\right\}_{i \in [N]}$ applied to an input game $G = \{u_i\}_{i \in [N]}$ returns the

[6] See [11, Lemma 2.1], [12, Theorem 5.35], [13, Chapter 3] or [14, Proposition 3.1].

transformed game $H_{\mathrm{PAT}}(G) = \{u'_i\}_{i\in[N]}$ in which (only) the utility functions changed to

$$\begin{aligned} u'_i : S &\longrightarrow \mathbb{R} \\ \mathbf{k} &\longmapsto \alpha_i \cdot u_i(\mathbf{k}) + c^i_{\mathbf{k}_{-i}} \,. \end{aligned} \tag{2}$$

We could not find multiplayer PATs defined in the literature, so we came up with the natural generalization above. As shown in Section 1.2, they indeed generalize the 2-player PATs from Lemma 2.4 to multiplayer settings. Moreover, multiplayer PATs also preserve the best response sets and Nash equilibrium set.

Lemma 2.6. *Take a PAT $H_{\mathrm{PAT}} = \left\{\alpha^i, C^i\right\}_{i\in[N]}$ and any game $G = \{u_i\}_{i\in[N]}$. Then, the transformed game $H_{\mathrm{PAT}}(G) = \{u'_i\}_{i\in[N]}$ has the same best response sets as the input game G. Consequently, $H_{\mathrm{PAT}}(G)$ also has the same Nash equilibrium set as G.*

Proof. See Appendix A. □

PATs have found much success as a tool for simplifying an input game precisely because of this property. We want to investigate which other game transformations also preserve the best response sets or the Nash equilibrium set. If we found more of these transformations, we could use them to, e.g., further increase the class of efficiently solvable games.

2.3. Game Transformations

There are various ways in which we could define the concept of a game transformation. Section 1.3 gives an overview of some definitions from the literature that are useful for different purposes. A key component of PATs are that they operate player-wise and strategy-wise, that is, they do not change the player set nor the players' strategy sets. This allows for a direct comparison of the strategic structure between a game and its PAT-transform. We argue that this is a natural desideratum for a definition of more general game transformation.

Definition 2.7. A game transformation $H = \{H^i\}_{i\in[N]}$ specifies for each player i a collection of functions $H^i := \left\{h^i_{\mathbf{k}} : \mathbb{R} \longrightarrow \mathbb{R}\right\}_{\mathbf{k}\in S}$, indexed by the different pure strategy profiles $\mathbf{k}$.

The transformation H can then be applied to any N-player game $G = \{u_i\}_{i\in[N]}$ to construct the transformed game $H(G) = \{H^i(u_i)\}_{i\in[N]}$ where

$$H^i(u_i) : S \to \mathbb{R}, \quad \mathbf{k} \mapsto h^i_{\mathbf{k}}\big(u_i(\mathbf{k})\big) . \tag{3}$$

Observe that the utility payoff of player i in the transformed game $H(G)$ from the pure strategy outcome $\mathbf{k}$ is only a function of the utility payoff from *that same* player in *that same* pure strategy outcome of the input game G.

We extend the utility functions $H^i(u_i)$ to mixed strategy profiles $\mathbf{s} \in \Delta(S)$ as usual through $H^i(u_i)(\mathbf{s}) := \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^N_{k_N} \cdot h^i_{\mathbf{k}}\big(u_i(\mathbf{k})\big)$. To simplify future notation, we will often use $h^i_{k_i,\mathbf{k}_{-i}}$ to refer to $h^i_{\mathbf{k}}$.

Remark 2.8. A multiplayer positive affine transformation $H_{\mathrm{PAT}} = \big\{\alpha^i, C^i\big\}_{i\in[N]}$ makes a game transformation $H = \{H^i\}_{i\in[N]}$ in the above sense by setting $h^i_{\mathbf{k}} :\ \mathbb{R} \to \mathbb{R}$, $z \mapsto \alpha^i \cdot z + c^i_{\mathbf{k}_{-i}}$.

Definition 2.9. Let $H = \{H^i\}_{i\in[N]}$ be a game transformation. Then we say that H universally preserves Nash equilibrium sets if for all input games $G = \{u_i\}_{i\in[N]}$, the transformed game $H(G) = \{H^i(u_i)\}_{i\in[N]}$ has the same Nash equilibrium set as the input game G.

Definition 2.10. Let map H^i come from a game transformation H. Then we say that H^i universally preserves best responses if for all utility functions $u_i : S \longrightarrow \mathbb{R}$ and for all opponents' strategy choices $\mathbf{s}^{-i} \in \Delta(S^{-i})$:

$$\mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i}) = \operatorname*{argmax}_{t^i\in\Delta(S^i)} \big\{H^i(u_i)(t^i, \mathbf{s}^{-i})\big\} = \operatorname*{argmax}_{t^i\in\Delta(S^i)} \big\{u_i(t^i, \mathbf{s}^{-i})\big\} = \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) .$$

Definition 2.11. Let map H^i come from a game transformation H. Then we say that H^i only depends on the strategy choice of the opponents if for all pure strategy choices $\mathbf{k}_{-i} \in S^{-i}$ of the opponents, we have the map identities

$$h^i_{1,\mathbf{k}_{-i}} = \ldots = h^i_{m_i,\mathbf{k}_{-i}} : \mathbb{R} \to \mathbb{R} .$$

3. Transformations that preserve Nash Equilibrium Sets or Best Response Sets

To our knowledge, the results of this section are all novel unless explicitly stated otherwise. The proofs can be found in Appendix C.

We showed in Lemma 2.6 that the maps H^i of a PAT universally preserve best responses. Note moreover that by the definition of a Nash equilibrium, if for all player i the map H^i universally preserves best responses, then the game transformation $H = \{H^i\}_{i\in[N]}$ universally preserves Nash equilibrium sets. The main result of our paper states the reverse:

Theorem. *Let $H = \{H^i\}_{i\in[N]}$ be a game transformation. Then:*

$$
\begin{aligned}
&H \text{ universally preserves Nash equilibrium sets} && \text{(i)}\\
&\iff \text{for each player } i, \text{ map } H^i \text{ universally preserves best responses} && \text{(ii)}\\
&\iff H \text{ is a positive affine transformation.} && \text{(iii)}
\end{aligned}
$$

In this section, we develop the directions (i) $\implies$ (ii) $\implies$ (iii). The key property that enables us to derive this theorem is that the game transformation $H = \{H^i\}_{i\in[N]}$ of consideration needs to be *universally* applicable, no matter the input game $G = \{u_i\}_{i\in[N]}$ we have at hand.

Proposition 3.1. *Let $H = \{H^i\}_{i\in[N]}$ be a game transformation that universally preserves Nash equilibrium sets and consider the map H^i of a player i. Then H^i only depends on the strategy choice of the opponents. Moreover, H^i universally preserves best responses.*

We prove the second conclusion of Proposition 3.1 by analyzing H on games where all other players $j \neq i$ receive constant utilities. In such games, the Nash equilibrium profiles are exactly those in which player i plays a best response. The first conclusion of Proposition 3.1 relies on the following intuition: If the maps $h^i_{\mathbf{k}}$ from H^i would depend on the strategy choice of player i, then in the transformed game $H(G)$, player i must adjust her strategy choice to those $h^i_{\mathbf{k}}$ that yield a high transformed payoff. This would affect the strategic decision making and therefore the Nash equilibrium set. Similar reasoning provides us with a related result:

Lemma 3.2. *Suppose a map H^i universally preserves best responses. Then H^i only depends on the strategy choice of the opponents.*

Due to Proposition 3.1, we can transition to the analysis of transformation maps H^i that universally preserve best responses. Thus from now on, our results also become relevant for the game theory literature that focuses on best response sets, such as best response dynamics or fictitious play.

Proposition 3.1 also allows us to restrict our analysis to the map H^1 for player 1 w.l.o.g. because any results for H^1 will, by symmetry, also hold for maps $H^2, \dots, H^N$. By Lemma 3.2, we can also drop the dependence of H^1 on k_1 and write

$$H^1 := \left\{ h^1_{\mathbf{k}_{-1}} : \mathbb{R} \longrightarrow \mathbb{R} \right\}_{\mathbf{k}_{-1} \in S^{-1}} .$$

For each (pure strategy)-specific map $h^1_{\mathbf{k}_{-1}}$, we introduce its *distance distortion* function which takes two utility values and measures their distance after a $h^1_{\mathbf{k}_{-1}}$-transformation:

$$\begin{aligned} \Delta h^1_{\mathbf{k}_{-1}} : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto h^1_{\mathbf{k}_{-1}}(z) \, - \, h^1_{\mathbf{k}_{-1}}(w) \, . \end{aligned} \tag{4}$$

The next lemma relates the (pure strategy)-specific maps $h^1_{\mathbf{k}_{-1}}$ to each other through their distance distortion functions. The subsequent lemma then shines some light on how those maps $h^1_{\mathbf{k}_{-1}}$ behave. Both lemmas are based on another technical result that we present and prove in Appendix C.1 due to its more complicated formulation and interpretation. In all those results it is also crucial that H^1 is assumed to preserve best responses *universally*. Only due to that we are able to deduce those global properties of the $h^1_{\mathbf{k}_{-1}}$ in H^1.

Lemma 3.3. *Suppose transformation H^1 universally preserves best responses. Then the (pure strategies)-specific maps within H^1 equally distort distances:*

$$\forall \mathbf{k}_{-1} \in S^{-1} \, : \, \Delta h^1_{\mathbf{k}_{-1}} = \Delta h^1_{\mathbf{1}_{-1}}$$

where $\mathbf{1}_{-1} := (1, \dots, 1) \in S^{-1}$.

Lemma 3.4. *Suppose transformation H^1 universally preserves best responses. Then we obtain for all $\mathbf{k}_{-1} \in S^{-1}$ that*

1. *map $h^1_{\mathbf{k}_{-1}}$ is strictly increasing, and that*

2. *map $h^1_{\mathbf{k}_{-1}}$ distorts distances independently of their reference points:*

$$\forall z, z', \lambda \in \mathbb{R} \, : \, \Delta h^1_{\mathbf{k}_{-1}}(z + \lambda, z) = \Delta h^1_{\mathbf{k}_{-1}}(z' + \lambda, z') \, .$$

With Lemmas 3.3 and 3.4, we can finally show that positive affine transformations are the only game transformations that universally preserve best responses. Intuitively speaking, the second conclusion of Lemma 3.4 states that a step of length λ in the domain space consistently maps to a step of some (other) length in the range space independently of the base point z from which we take that step. This brings us to a known result from the analysis literature. Recall that a function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is called linear if there exists some $a \in \mathbb{R}$ such that $\forall z \in \mathbb{R} : h(z) = az$. A function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be additive if it satisfies $\forall x, y \in \mathbb{R} : h(x + y) = h(x) + h(y)$.

Lemma 3.5 (Known result from [33, 34])**.** *If a map $h : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone and additive, then it is also linear.*

Proof. Reproven in Appendix B. □

Corollary 3.6. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be monotone and satisfy for all $z, z', \lambda \in \mathbb{R}$:*

$$h(z + \lambda) - h(z) = h(z' + \lambda) - h(z')\,.$$

Then h is affine linear, i.e., there exist some $a, c \in \mathbb{R}$ such that for all $z \in \mathbb{R} : h(z) = az + c$.

We can conclude with the main result:

Theorem 1. *Let $H = \{H^i\}_{i\in[N]}$ be a game transformation. Then:*

H universally preserves Nash equilibrium sets (i)

$\iff$ *for each player i, map H^i universally preserves best responses* (ii)

$\iff$ *H is a positive affine transformation.* (iii)

Let us additionally state the result for 2-player games as a special case of interest:

Corollary 3.7. *PATs are all the game transformations in the sense of (1) that can be applied to any bimatrix game without changing the Nash equilibrium set (resp., the best response sets).*

4. Conclusion

First, we introduced game transformations and positive affine transformations (PATs) for multiplayer games. Next, we defined the properties (i) *universally preserving Nash equilibrium sets* and (ii) *universally preserving best responses*. It is well-known that PATs universally preserve Nash equilibrium sets. We showed that game transformations which universally preserve

Nash equilibrium sets also universally preserve best responses. In the subsequent results, we derived further that if a game transformation universally preserves best responses then it is a positive affine transformation. Therefore, we gave two equivalent characterisations for a game transformation to be a PAT. They highlight how special PATs are with regard to Nash equilibrium sets and best response sets.

Implications for future work. The current literature is lacking in methods for generating strategically equivalent games to a given input game. Finding such methods would greatly benefit the field of game theory.

Among the game transformations of our definition, and aside from PATs, such methods can only exist if we restrict our focus to smaller classes of N-player games (i.e., games of some specific characteristic). Preferably, such a utilized method would be computationally efficient, and the corresponding class of games would contain "most" games.

Alternatively, we could relax the notion of a game transformation. Future work may, for example, examine game transformations that can freely change the strategy sets of the players or the number of players. Note however, that in that case, it is not straightforward anymore what it means for two games to be strategically equivalent.

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Appendix A. Helping Lemmas

For player i with pure strategy set $S^i = [m_i]$ in a game G, we will also refer to pure strategy $k \in [m_i]$ as the k-th standard basis vector $e_k \in \mathbb{R}^{m_i}$ (with a 1 in its k-th entry and 0's anywhere else) because that is its corresponding mixed strategy vector in $\Delta(S^i)$.

Denote the restriction of a best response set to its pure strategies as $\mathrm{PBR}_{u_i}(\mathbf{s}^{-i}) := \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) \cap \{e_1, \ldots, e_{m_i}\}$. Then, we have that best responses are always convex combinations of pure best responses:

Lemma. *Take a game $G = \{u_i\}_{i\in[N]}$, fix a player $i \in [N]$ and a strategy profile $\mathbf{s}^{-i} \in \Delta(S^{-i})$ of the opponents. Then, we have for $t^i \in \Delta(S^i)$:*

$$t^i \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) \iff \forall k \in [m_i] : \; t^i_k = 0 \;\text{ or }\; e_k \in \mathrm{PBR}_{u_i}(\mathbf{s}^{-i})\,. \tag{A.1}$$

Proof. We can observe

$$\begin{aligned} u_i(t^i, \mathbf{s}^{-i}) &= \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k}) \\ &= \sum_{k_i=1}^{m_i} t^i_{k_i} \cdot \sum_{\mathbf{k}_{-i}\in S^{-i}} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k}) \\ &= \sum_{k_i=1}^{m_i} t^i_{k_i} \cdot u_i(e_{k_i}, \mathbf{s}^{-i}) = \sum_{k=1}^{m_i} t^i_k \cdot u_i(e_k, \mathbf{s}^{-i})\,. \end{aligned} \tag{A.2}$$

Thus, the mixed strategy t^i of player i only determines the convex combination of the attainable utility values $\Big(u_i(e_k, \mathbf{s}^{-i})\Big)_k$. Therefore, any best response strategy t^i must only randomize over maximal values within $\Big(u_i(e_k, \mathbf{s}^{-i})\Big)_k$, that is, over pure best response strategies. $\square$

Corollary Appendix A.1. *Two best response sets (of possibly different games) are equal if and only if they contain the same pure best responses.*

Lemma. *Take a PAT $H_{\mathrm{PAT}} = \left\{\alpha^i, C^i\right\}_{i\in[N]}$ and any game $G = \{u_i\}_{i\in[N]}$. Then, the transformed game $H_{\mathrm{PAT}}(G) = \{u'_i\}_{i\in[N]}$ has the same best response sets as the input game G. Consequently, $H_{\mathrm{PAT}}(G)$ also has the same Nash equilibrium set as G.*

Proof. The proof is an appropriate generalization of the known proof for Lemma 2.4.

Take a game $\{u_i\}_{i\in[N]}$, fix a player i and the opponents' strategy choices $\mathbf{s}^{-i}$. Then, we have

$$\begin{aligned}
\mathrm{BR}_{u_i'}(\mathbf{s}^{-i}) &= \operatorname*{argmax}_{t^i\in\Delta(S^i)}\Big\{\, u_i'(t^i,\mathbf{s}^{-i}) \,\Big\} \\
&= \operatorname*{argmax}_{t^i\in\Delta(S^i)}\Big\{ \sum_{\mathbf{k}\in S} s^1_{k_1}\cdot\ldots\cdot s^{i-1}_{k_{i-1}}\cdot t^i_{k_i}\cdot s^{i+1}_{k_{i+1}}\cdot\ldots\cdot s^N_{k_N}\cdot u_i'(\mathbf{k}) \Big\} \\
&\overset{(2)}{=} \operatorname*{argmax}_{t^i\in\Delta(S^i)}\Big\{ \sum_{\mathbf{k}\in S} s^1_{k_1}\cdot\ldots\cdot s^{i-1}_{k_{i-1}}\cdot t^i_{k_i}\cdot s^{i+1}_{k_{i+1}}\cdot\ldots\cdot s^N_{k_N}\cdot \big(\alpha^i\cdot u_i(\mathbf{k}) + c^i_{\mathbf{k}_{-i}}\big)\Big\} \\
&\overset{(*)}{=} \operatorname*{argmax}_{t^i\in\Delta(S^i)}\Big\{ \alpha^i\cdot\sum_{\mathbf{k}\in S} s^1_{k_1}\cdot\ldots\cdot s^{i-1}_{k_{i-1}}\cdot t^i_{k_i}\cdot s^{i+1}_{k_{i+1}}\cdot\ldots\cdot s^N_{k_N}\cdot u_i(\mathbf{k}) \\
&\qquad\qquad + \sum_{\mathbf{k}_{-i}\in S^{-i}} s^1_{k_1}\cdot\ldots\cdot s^{i-1}_{k_{i-1}}\cdot s^{i+1}_{k_{i+1}}\cdot\ldots\cdot s^N_{k_N}\cdot c^i_{\mathbf{k}_{-i}}\cdot 1\Big\} \\
&\overset{(\dagger)}{=} \operatorname*{argmax}_{t^i\in\Delta(S^i)}\Big\{ \sum_{\mathbf{k}\in S} s^1_{k_1}\cdot\ldots\cdot s^{i-1}_{k_{i-1}}\cdot t^i_{k_i}\cdot s^{i+1}_{k_{i+1}}\cdot\ldots\cdot s^N_{k_N}\cdot u_i(\mathbf{k})\Big\} \\
&= \operatorname*{argmax}_{t^i\in\Delta(S^i)}\Big\{\, u_i(t^i,\mathbf{s}^{-i}) \,\Big\} = \mathrm{BR}_{u_i}(\mathbf{s}^{-i})
\end{aligned}$$

We obtain the second summand in $(*)$ by changing the order of summation and multiplication such that $\sum_{k_i=1}^{m_i} t_i$ remains as the most inner sum. Since $\sum_{k_i=1}^{m_i} t_i = 1$, this factor can be dropped. We get line $(\dagger)$ because the argmax operator is neither affected by a constant in t_i (such as the secoond summand) nor by rescaling with a positive factor (such as α_i).

Finally, the definition of a Nash equilibrium immediately implies that strategy profile s is a Nash equilibrium for the PAT transformed game $\{u_i'\}_{i\in[N]}$ if and only if it was one for the original game $\{u_i\}_{i\in[N]}$. □

Appendix B. Monotone and additive implies linear

The proof of the following lemma is taken from [35, 36]:

Lemma. *Take a map $h:\mathbb{R}\longrightarrow\mathbb{R}$ which is monotone and additive. Then:*

1. $h(0) = 0$.
2. $\forall x\in\mathbb{R}:\quad -h(-x) = h(x)$.

3. $\forall n \in \mathbb{N}, x \in \mathbb{R}: \quad h(n \cdot x) = n \cdot h(x)$.

4. $\forall p \in \mathbb{Z}, x \in \mathbb{R}: \quad h(p \cdot x) = p \cdot h(x)$.

5. $\forall r \in \mathbb{Q}, x \in \mathbb{R}: \quad h(r \cdot x) = r \cdot h(x)$.

6. $\forall x \in \mathbb{R}: \quad h(x) = x \cdot h(1)$.

In particular, the last conclusion yields that h is linear.

Proof. The first three conclusions follow from h being additive.

1.

$$h(0) = h(0) + h(x) - h(x) = h(0 + x) - h(x) = 0\,.$$

2.

$$\forall x \in \mathbb{R}: \quad -h(-x) = -\Big(h(-x)+h(x)\Big)+h(x) = -h(-x+x)+h(x) = -h(0)+h(x) = h(x)\,.$$

3. Proof by induction. The induction start $n = 1$ is clear, so assume it to be true for $n \in \mathbb{N}$.
Then, for all $x \in \mathbb{R}$:

$$h\Big((n+1)\cdot x\Big) = h(n\cdot x+x) = h(n\cdot x)+h(x) = n\cdot h(x)+h(x) = (n+1)\cdot h(x)\,.$$

4. The statement for the case $p \in \mathbb{Z} \cap \{z \geq 0\}$ follows from the first and third conclusion. If $p \in \mathbb{Z} \cap \{z < 0\}$, we can use the second and third conclusion to obtain for all $x \in \mathbb{R}$:

$$h(p \cdot x) = h\Big((-p) \cdot (-x)\Big) = (-p) \cdot h(-x) = (-p) \cdot \Big(-h(x)\Big) = p \cdot h(x)\,.$$

5. Write $r = \frac{p}{q}$ where $p \in \mathbb{Z}, q \in \mathbb{N}$. Then, by the fourth conclusion:

$$h(r \cdot x) = \frac{1}{q} \cdot q \cdot h\Big(\frac{p}{q} \cdot x\Big) = \frac{1}{q}h\Big(q \cdot \frac{p}{q} \cdot x\Big) = \frac{1}{q}h(p \cdot x) = \frac{1}{q} \cdot p \cdot h(x) = r \cdot h(x)\,.$$

6. Suppose $x \in \mathbb{Q}$. Then, the fifth conclusion yields

$$h(x) = h(x \cdot 1) = x \cdot h(1)\,.$$

Therefore, suppose $x \in \mathbb{R} \setminus \mathbb{Q}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can take an increasing sequence $(r_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ that converges to x (from below) and a decreasing sequence $(s_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ that converges to x (from above). In the case where h is an increasing function, we have for all $n \in \mathbb{N}$:

$$r_n \leq x \leq s_n \implies h(r_n) \leq h(x) \leq h(s_n) \implies r_n \cdot h(1) \leq h(x) \leq s_n \cdot h(1)\,.$$

Taking the limit $n \to \infty$ in the last inequality chain yields

$$x \cdot h(1) \leq h(x) \leq x \cdot h(1)\,.$$

If h is a decreasing function instead of an increasing one, we get the same implications but with reverse inequalities in the second and last inequality chains. The end result, however, will be the same. Putting everything together yields the sixth conclusion.

□

Corollary. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be monotone and satisfy for all $z, z', \lambda \in \mathbb{R}$:*

$$h(z+\lambda) - h(z) = h(z'+\lambda) - h(z')\,. \tag{B.1}$$

Then h is affine linear, i.e., there exist some $a, c \in \mathbb{R}$ such that for all $z \in \mathbb{R} : h(z) = az + c$.

Proof. Define $h'(z) := h(z) - h(0)$, which is still a monotone function. By our assumption on h, we have for all $x, y \in \mathbb{R}$:

$$\begin{aligned} h'(x+y) &= h(x+y) - h(0) = h(x+y) - h(y) + h(y) - h(0) \\ &= h(x) - h(0) + h(y) - h(0) = h'(x) + h'(y)\,. \end{aligned}$$

Therefore, we can apply Lemma 3.5 to h' to get $a \in \mathbb{R}$ such that for all $z \in \mathbb{R}$

$$h(z) = h(z) - h(0) + h(0) = h'(z) + h(0) = az + h(0) =: az + c\,.$$

□

Appendix C. Proofs of Novel Results

To our knowledge, the results of this section (and therefore their proofs) are all novel. All proofs can be found in the appendix.

Throughout this section, we write $u \equiv \lambda$ if we want to define a function $u : \mathcal{D} \longrightarrow \mathbb{R}$ as a constant function on its domain $\mathcal{D}$ set to the value $\lambda \in \mathbb{R}$.

Proposition. *Let $H = \{H^i\}_{i\in[N]}$ be a game transformation that universally preserves Nash equilibrium sets and consider the map H^i of a player i. Then H^i only depends on the strategy choice of the opponents. Moreover, H^i universally preserves best responses.*

Proof. Take a game transformation $H = \{H^i\}_{i\in[N]}$ that universally preserves Nash equilibrium sets and fix some player i.

1. Fix a pure strategy choice $\mathbf{k}_{-i} \in S^{-i}$ of the opponent players and take some arbitrary value $z \in \mathbb{R}$. Consider the game $G = \{u_j\}_{j\in[N]}$ with constant utility functions $u_j \equiv z$ for all $j \in [N]$. Then, the Nash equilibrium set will be the whole strategy space $\Delta(S)$. By assumption on H, the transformed game $H(G)$ also has the full strategy space as its set of Nash equilibria. In particular, each of the strategy profiles $(1, \mathbf{k}_{-i}), \dots, (m_i, \mathbf{k}_{-i})$ will be a Nash equilibrium of the transformed game $H(G)$. Hence, for all $2 \leq l \leq m_i$:

$$
\begin{aligned}
h^i_{1,k_{-i}}(z) &\overset{u_i \equiv z}{=} h^i_{1,\mathbf{k}_{-i}}\Big(u_i(1,\mathbf{k}_{-i})\Big) \overset{(3)}{=} H^i(u_i)(1,\mathbf{k}_{-i}) \\
&\overset{\text{Nash-Eq}}{=} \max_{t^i \in \Delta(S^i)} \Big\{ H^i(u_i)(t^i,\mathbf{k}_{-i}) \Big\} \overset{\text{Nash-Eq}}{=} H^i(u_i)(l,\mathbf{k}_{-i}) \\
&= h^i_{l,\mathbf{k}_{-i}}\Big(u_i(l,\mathbf{k}_{-i})\Big) = h^i_{l,\mathbf{k}_{-i}}(z)\,.
\end{aligned}
$$

Since z and l were chosen arbitrarily, we get $h^i_{1,l,\mathbf{k}_{-i}} = \ldots = h^i_{m_i,l,\mathbf{k}_{-i}}$.

2. Fix player i's utility function u_i and the opponents' strategy choices $\mathbf{s}^{-i} \in \Delta(S^{-i})$. Then by Appendix A.1, it suffices to identify the pure strategies in the best response sets $\mathrm{BR}_{u_i}(\mathbf{s}^{-i})$ and $\mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i})$.

Complete the prefixed u_i to a full game $G = \{u_j\}_{j\in[N]}$ by setting $u_j \equiv 0$ for the other players $j \neq i$. Then, the best response set of a player $j \neq i$ is her whole strategy space $\Delta(S^j)$. By assumption on the game transformation H, we get for a pure strategy $e_l = l \in S^i$:

$$
\begin{aligned}
& e_l \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) \\
& \iff (e_l, \mathbf{s}^{-i}) \text{ is a Nash equilibrium for the game } G \\
& \iff (e_l, \mathbf{s}^{-i}) \text{ is a Nash equilibrium for the game } H(G) \\
& \overset{\text{def}}{\iff} e_l \in \mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i}) \\
& \qquad \text{and } \forall j \neq i \,:\, s^j \in \mathrm{BR}_{H^j(u_j)}(s^1, \ldots, s^{j-1}, s^{j+1}, \ldots, s^{i-1}, e_l, s^{i+1}, \ldots, s^N) \\
& \overset{(*)}{\iff} e_l \in \mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i})
\end{aligned}
$$

Let us give some further explanation for step $(*)$. Recall the definition for a strategy s^j, $j \neq i$, to be a best response to the opponents' strategy choices $(s^1, \ldots, s^{j-1}, s^{j+1}, \ldots, s^{i-1}, s^i := e_l, s^{i+1}, \ldots s^N)$:

$$
s^j \in \operatorname*{argmax}_{t^j \in \Delta(S^j)} \Big\{ \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot h^j_{\mathbf{k}}\big(u_j(\mathbf{k})\big) \Big\} .
$$

We can show that the term in the argmax is constant in t^j. First, note that the maps $h^j_{\mathbf{k}}$ are independent of player j's action, which, in particular, implies $h^j_{\mathbf{k}} = h^j_{1,\mathbf{k}_{-j}}$. Then, rearranging yields

$$
\begin{aligned}
& \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot h^j_{\mathbf{k}}\big(u_j(\mathbf{k})\big) \\
& \overset{u_j \equiv 0}{=} \sum_{\mathbf{k}_{-j}} \left(s^1_{k_1} \cdot \ldots \cdot s^{j-1}_{k_{j-1}} \cdot s^{j+1}_{k_{j+1}} \cdot \ldots \cdot s^N_{k_N} \cdot h^j_{1,\mathbf{k}_{-j}}(0) \cdot \sum_{k_j=1}^{m_j} t^j_{k_j} \right) \\
& \overset{(\dagger)}{=} \sum_{\mathbf{k}_{-j}} s^1_{k_1} \cdot \ldots \cdot s^{j-1}_{k_{j-1}} \cdot s^{j+1}_{k_{j+1}} \cdot \ldots \cdot s^N_{k_N} \cdot h^j_{1,\mathbf{k}_{-j}}(0) \,.
\end{aligned}
$$

Since the term in the argmax is constant in t^j, any strategy of player j is a best response to $(s^1, \ldots, s^{j-1}, s^{j+1}, \ldots, s^{i-1}, e_l, s^{i+1}, \ldots s^N)$. Therefore, we obtain the equivalence $(*)$ by removing/adding the redundant condition on each s^j, $j \neq i$, to be a best response.

All in all, we proved that the sets $\mathrm{BR}_{u_i}(\mathbf{s}^{-i})$ and $\mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i})$ contain the same pure strategies. Appendix A.1 therefore yields set equality. $\square$

Lemma. *Suppose a map H^i universally preserves best responses. Then H^i only depends on the strategy choice of the opponents.*

Proof. Let the pure strategy choice of the opponents be $\mathbf{k}_{-i} \in S^{-i}$. Pick some $z \in \mathbb{R}$ and set $u_i \equiv z$. Then we can reformulate

$$\begin{aligned}
&h^i_{1,\mathbf{k}_{-i}}(z) = \ldots = h^i_{m_i,\mathbf{k}_{-i}}(z) \\
&\iff \forall l \in [m_i] : \; h^i_{l,\mathbf{k}_{-i}}(z) = \max_{p\in[m_i]} h^i_{p,\mathbf{k}_{-i}}(z) \\
&\overset{u_i\equiv z}{\iff} \forall l \in [m_i] : \; h^i_{l,\mathbf{k}_{-i}}(u_i(l,\mathbf{k}_{-i})) = \max_{p\in[m_i]} h^i_{p,\mathbf{k}_{-i}}(u_i(p,\mathbf{k}_{-i})) \\
&\iff \forall l \in [m_i] : \; H^i(u_i)(l,\mathbf{k}_{-i}) = \max_{p\in[m_i]} H^i(u_i)(p,\mathbf{k}_{-i}) \\
&\iff \forall l \in [m_i] : \; e_l = l \in \mathrm{BR}_{H^i(u_i)}(s^{-i} = \mathbf{k}_{-i}) \\
&\overset{(*)}{\iff} \forall l \in [m_i] : \; e_l = l \in \mathrm{BR}_{u_i}(s^{-i} = \mathbf{k}_{-i}) \\
&\iff \forall l \in [m_i] : \; u_i(l,\mathbf{k}_{-i}) = \max_{p\in[m_i]} u_i(p\mathbf{k}_{-i}) \\
&\overset{u_i\equiv z}{\iff} \forall l \in [m_i] : \; z = \max_{p\in[m_i]} z\,.
\end{aligned}$$

In $(*)$, we use that H^i is universally best response preserving.

With the last line of the equivalence chain above being a universally true statement, we obtain that the first line also holds true. Since z was chosen arbitrarily, we can conclude $h^i_{1,k_{-i}} = \ldots = h^i_{m_i,k_{-i}}$. $\square$

Remark. A distance distortion function $\Delta h^1_{\mathbf{k}_{-1}}$, as defined in (4), is skew-symmetric:

$$\forall\, z, w \in \mathbb{R} : \quad \Delta h^1_{\mathbf{k}_{-1}}(z,w) = -\Delta h^1_{\mathbf{k}_{-1}}(w,z)\,. \tag{C.1}$$

The upcoming lemma reveals an important preliminary observation on how the distance distortion functions $\Delta h^1_{\mathbf{k}_{-1}}$ relate to each other. It highlights how the distorted utility distances are affected by a strategy change of a player $j \neq 1$ from, e.g., some pure strategy $k_j \in [m_j] \setminus \{1\}$ to their pure strategy $1 \in [m_j]$.

We formulate the lemma with index variables $\mathbf{p}_{-1} = (p_2, \ldots, p_N)$ instead of $\mathbf{k}_{-1} = (k_2, \ldots, k_N)$ in order to avoid confusion in the proof of the subsequent Lemma 3.3.

Lemma Appendix C.1. *Suppose transformation map H^1 universally preserves best responses. Take a player $r \in [N] \setminus \{1\}$ and let $\mathbf{p}_{-1} \in S^{-1}$ be such that $p_r \neq 1$. Define $\mathbf{p}'_{-1} \in S^{-1}$ to have r-th entry $p'_r = 1$ and, otherwise, to have the same entries as $\mathbf{p}_{-1}$. Then, for all $z, z', w, w' \in \mathbb{R}$:*

$$z - w \geq z' - w' \quad \iff \quad \Delta h^1_{\mathbf{p}_{-1}}(z,w) \geq \Delta h^1_{\mathbf{p}'_{-1}}(z',w')\,.$$

Proof. Take a transformation map H^1 that universally preserves the best response sets. Then by Lemma 3.2, its maps $h^1_{\mathbf{k}}$ only depend on the strategy choices $\mathbf{k}_{-1}$ of the opponents. Fix $r, \mathbf{p}_{-1}, \mathbf{p}'_{-1}$ and z, z', w, w' as described in the lemma statement. We will construct a utility function u_1 for these parameters such that a universally best response preserving H^1 reveals to satisfies the property of this lemma.

Set $u_1(1, \mathbf{p}_{-1}) := z$ and $u_1(1, \mathbf{p}'_{-1}) := w'$. Additionally, for all pure strategies $l \in [m_1] \setminus \{1\}$, set $u_1(l, \mathbf{p}_{-1}) := w$ and $u_1(l, \mathbf{p}'_{-1}) := z'$. All these utility value assignments are possible because of $p_r \neq 1 = p'_r$. The utility payoffs of player 1 (i.e., the values of u_1) from other pure strategy outcomes $\mathbf{k} \in S$ can be set arbitrarily. Finally, consider the opponents' mixed strategy profile $\mathbf{s}^{-1} := \frac{1}{2}\mathbf{p}_{-1} + \frac{1}{2}\mathbf{p}'_{-1} \in \Delta(S^{-1})$. Then we derive:

$$z - w \geq z' - w'$$

$$\iff \forall l \in [m_1] \setminus \{1\} \, : \, u_1(1, \mathbf{p}_{-1}) - u_1(l, \mathbf{p}_{-1}) \geq u_1(l, \mathbf{p}'_{-1}) - u_1(1, \mathbf{p}'_{-1})$$

Reorder and divide by 2

$$\iff \forall l \in [m_1] \setminus \{1\} \, : \, \frac{1}{2}u_1(1, \mathbf{p}_{-1}) + \frac{1}{2}u_1(1, \mathbf{p}'_{-1}) \geq \frac{1}{2}u_1(l, \mathbf{p}_{-1}) + \frac{1}{2}u_1(l, \mathbf{p}'_{-1})$$
$$\iff \forall l \in [m_1] \setminus \{1\} \, : \, u_1(e_1, \mathbf{s}^{-1}) \geq u_1(e_l, \mathbf{s}^{-1})$$
$$\iff e_1 \in \mathrm{BR}_{u_i}(\mathbf{s}^{-1})$$

H^1 *is universally preserves best responses*

$$\iff e_1 \in \mathrm{BR}_{H^1(u_1)}(\mathbf{s}^{-1})$$
$$\iff \forall l \in [m_1] \setminus \{1\} \, : \, H^1(u_1)(e_1, \mathbf{s}^{-1}) \geq H^1(u_1)(e_l, \mathbf{s}^{-1})$$
$$\begin{aligned}\iff \forall l \in [m_1] \setminus \{1\} \, : \, & \frac{1}{2}h^1_{1,\mathbf{p}_{-1}}(u_1(1, \mathbf{p}_{-1})) + \frac{1}{2}h^1_{1,\mathbf{p}'_{-1}}(u_1(1, \mathbf{p}'_{-1})) \\ & \geq \frac{1}{2}h^1_{l,\mathbf{p}_{-1}}(u_1(l, \mathbf{p}_{-1})) + \frac{1}{2}h^1_{l,\mathbf{p}'_{-1}}(u_1(l, \mathbf{p}'_{-1}))\end{aligned}$$
$$\iff \forall l \in [m_1] \setminus \{1\} \, : \, h^1_{1,\mathbf{p}_{-1}}(z) + h^1_{1,\mathbf{p}'_{-1}}(w') \geq h^1_{l,\mathbf{p}_{-1}}(w) + h^1_{l,\mathbf{p}'_{-1}}(z')$$

H^1 *does not depend on the pure strategy choice of player 1*

$$\iff h^1_{\mathbf{p}_{-1}}(z) + h^1_{\mathbf{p}'_{-1}}(w') \geq h^1_{\mathbf{p}_{-1}}(w) + h^1_{\mathbf{p}'_{-1}}(z')$$
$$\iff h^1_{\mathbf{p}_{-1}}(z) - h^1_{\mathbf{p}_{-1}}(w) \geq h^1_{\mathbf{p}'_{-1}}(z') - h^1_{\mathbf{p}'_{-1}}(w')$$
$$\iff \Delta h^1_{\mathbf{p}_{-1}}(z, w) \geq \Delta h^1_{\mathbf{p}'_{-1}}(z', w')$$

□

Lemma. *Suppose transformation H^1 universally preserves best responses. Then the (pure strategies)-specific maps within H^1 equally distort distances:*

$$\forall \mathbf{k}_{-1} \in S^{-1} \,:\, \Delta h^1_{\mathbf{k}_{-1}} = \Delta h^1_{\mathbf{1}_{-1}}$$

where $\mathbf{1}_{-1} := (1, \ldots, 1) \in S^{-1}$.

Proof. Take a transformation map H^1 that universally preserves the best response sets. Then by Lemma 3.2, its maps $h^1_{\mathbf{k}}$ only depend on the strategy choices $\mathbf{k}_{-1}$ of the opponents. Fix $\mathbf{k}_{-1} \in S^{-1}$. Recall that the elements $r \geq 2$ and $\mathbf{p} \in S^{-1}$ in Appendix C.1 can be chosen arbitrarily[7]. So we can apply Appendix C.1 on a trivially true statement to get for all $z, w \in \mathbb{R}$:

$$\begin{aligned}
&z - w \geq z - w \\
&\quad \Longrightarrow \forall r \in [N] \setminus \{1\} \,:\, \Delta h^1_{k_2,\ldots,k_{r-1},k_r,1,\ldots,1}(z,w) \geq \Delta h^1_{k_2,\ldots,k_{r-1},1,1,\ldots,1}(z,w) \\
&\quad \Longrightarrow \Delta h^1_{k_2,\ldots,k_{N-1},k_N}(z,w) \geq \Delta h^1_{k_2,\ldots,k_{N-1},1}(z,w) \geq \ldots \geq \Delta h^1_{1,\ldots,1}(z,w)\,.
\end{aligned}$$

With skew-symmetry, we similarly obtain

$$\begin{aligned}
&w - z \geq w - z \\
&\quad \Longrightarrow \forall r \in [N] \setminus \{1\} \,:\, \Delta h^1_{k_2,\ldots,k_{r-1},k_r,1,\ldots,1}(w,z) \geq \Delta h^1_{k_2,\ldots,k_{r-1},1,1,\ldots,1}(w,z) \\
&\quad \Longrightarrow \Delta h^1_{k_2,\ldots,k_{N-1},k_N}(w,z) \geq \Delta h^1_{k_2,\ldots,k_{N-1},1}(w,z) \geq \ldots \geq \Delta h^1_{1,\ldots,1}(w,z) \\
&\quad \overset{\cdot(-1)}{\Longrightarrow} \Delta h^1_{k_2,\ldots,k_{N-1},k_N}(z,w) \leq \Delta h^1_{1,\ldots,1}(z,w)\,.
\end{aligned}$$

Putting both together, we have for all $z, w \in \mathbb{R}$:

$$\Delta h^1_{\mathbf{k}_{-1}}(z,w) = \Delta h^1_{k_2,\ldots,k_{N-1},k_N}(z,w) = \Delta h^1_{1,\ldots,1}(z,w) = \Delta h^1_{\mathbf{1}_{-1}}(z,w)\,.$$

□

Lemma. *Suppose transformation H^1 universally preserves best responses. Then we obtain for all $\mathbf{k}_{-1} \in S^{-1}$ that*

1. *map $h^1_{\mathbf{k}_{-1}}$ is strictly increasing, and that*
2. *map $h^1_{\mathbf{k}_{-1}}$ distorts distances independently of their reference points:*

$$\forall z, z', \lambda \in \mathbb{R} \,:\, \Delta h^1_{\mathbf{k}_{-1}}(z + \lambda, z) = \Delta h^1_{\mathbf{k}_{-1}}(z' + \lambda, z')\,. \tag{C.2}$$

Proof. Take a transformation map H^1 that universally preserves the best response sets. Then by Lemma 3.2, its maps $h^1_{\mathbf{k}}$ only depend on the strategy choices $\mathbf{k}_{-1}$ of the opponents.

[7] We required $p_r \neq 1$, but this is irrelevant for the argument we are making here.

1. Let us first consider $h^1_{2,1,\ldots,1}$ that is associated to the pure strategy profile $(2,1,\ldots,1) \in S^{-1}$. Apply Appendix C.1 in the upcoming line $(*)$ with parameters $r=2$, $\mathbf{p}_{-1}=(2,1,\ldots,1)$, and $z'=w'\in\mathbb{R}$ to get for arbitrary $z,w\in\mathbb{R}$:

$$\begin{aligned} z \geq w &\iff z-w \geq 0 = z'-w' \\ &\overset{(*)}{\iff} \Delta h^1_{2,1,\ldots,1}(z,w) \geq \Delta h^1_{\mathbf{1}_{-1}}(z',w') \overset{z'=w'}{\equiv} 0 \\ &\iff h^1_{2,1,\ldots,1}(z) \geq h^1_{2,1,\ldots,1}(w)\,. \end{aligned}$$

Consequently, we have for arbitrary $\bar{z},\bar{w}\in\mathbb{R}$:

$$\begin{aligned} \bar{z} > \bar{w} &\iff \bar{z} \geq \bar{w} \text{ and } \bar{w} \not\geq \bar{z} \\ &\overset{\text{by above}}{\iff} h^1_{2,1,\ldots,1}(\bar{z}) \geq h^1_{2,1,\ldots,1}(\bar{w}) \text{ and } h^1_{2,1,\ldots,1}(\bar{w}) \not\geq h^1_{2,1,\ldots,1}(\bar{z}) \\ &\iff h^1_{2,1,\ldots,1}(\bar{z}) > h^1_{2,1,\ldots,1}(\bar{w})\,. \end{aligned}$$

This shows that $h^1_{2,1,\ldots,1}$ is strictly increasing.
For arbitrary $\mathbf{k}_{-1}\in S^{-1}$, we can then use Lemma 3.3 to obtain

$$\begin{aligned} \bar{z} > \bar{w} &\iff h^1_{2,1,\ldots,1}(\bar{z}) > h^1_{2,1,\ldots,1}(\bar{w}) \\ &\iff \Delta h^1_{2,1,\ldots,1}(\bar{z},\bar{w}) > 0 \\ &\iff \Delta h^1_{\mathbf{k}_{-1}}(\bar{z},\bar{w}) = \Delta h^1_{\mathbf{1}_{-1}}(\bar{z},\bar{w}) = \Delta h^1_{2,1,\ldots,1}(\bar{z},\bar{w}) > 0 \\ &\iff h^1_{\mathbf{k}_{-1}}(\bar{z}) > h^1_{\mathbf{k}_{-1}}(\bar{w})\,. \end{aligned}$$

Thus, $h^1_{\mathbf{k}_{-1}}$ is strictly increasing as well.

2. Because of Lemma 3.3, we only need to show that the map $\Delta h^1_{\mathbf{1}_{-1}}$ satisfies property (C.2), which would consequently imply the property for all maps $\Delta h^1_{\mathbf{k}_{-1}}$.

Fix $z,z',\lambda\in\mathbb{R}$. Then the upcoming equivalence chain uses skew-symmetry (C.1) in $(*)$, Lemma 3.3 in $(\dagger)$, and Appendix C.1 in $(\star)$ for parameters $r=2$

and $\mathbf{p}_{-1} = (2, 1, \ldots, 1)$:

$$\begin{aligned}
&\Delta h^1_{\mathbf{1}_{-1}}(z+\lambda, z) = \Delta h^1_{\mathbf{1}_{-1}}(z'+\lambda, z') \\
&\quad \overset{(*)}{\iff} \Delta h^1_{\mathbf{1}_{-1}}(z+\lambda, z) \geq \Delta h^1_{\mathbf{1}_{-1}}(z'+\lambda, z') \\
&\qquad\quad \text{and } \Delta h^1_{\mathbf{1}_{-1}}(z, z+\lambda) \geq \Delta h^1_{\mathbf{1}_{-1}}(z', z'+\lambda) \\
&\quad \overset{(\dagger)}{\iff} \Delta h^1_{2,\ldots,1}(z+\lambda, z) \geq \Delta h^1_{\mathbf{1}_{-1}}(z'+\lambda, z') \\
&\qquad\quad \text{and } \Delta h^1_{2,\ldots,1}(z, z+\lambda) \geq \Delta h^1_{\mathbf{1}_{-1}}(z', z'+\lambda) \\
&\quad \overset{(\star)}{\iff} z+\lambda-z \geq z'+\lambda-z' \quad \text{and} \quad z-(z+\lambda) \geq z'-(z'+\lambda)\,.
\end{aligned}$$

The last line is a true statement and thus, the first line as well. Because $z, z', \lambda \in \mathbb{R}$ were taken arbitrarily, map $h^1_{\mathbf{1}_{-1}}$ satisfies property (C.2). □

Theorem. *Let $H = \{H^i\}_{i\in[N]}$ be a game transformation. Then:*

$$\begin{aligned}
&H \text{ universally preserves Nash equilibrium sets} && \text{(i)}\\
&\iff \text{for each player } i, \text{ map } H^i \text{ universally preserves best responses} && \text{(ii)}\\
&\iff H \text{ is a positive affine transformation.} && \text{(iii)}
\end{aligned}$$

Proof. (iii) $\implies$ (i):
By Lemma 2.6.

(i) $\implies$ (ii):
By Proposition 3.1.

(ii) $\implies$ (iii):
Consider map H^1 first, that is, the perspective of player 1. By Lemma 3.4, maps $h^1_{\mathbf{k}_{-1}}$ are monotone functions and they satisfy the distance distortion property (B.1). Hence, by Corollary 3.6, there exist parameters $a^1_{\mathbf{k}_{-1}}, c^1_{\mathbf{k}_{-1}} \in \mathbb{R}$ for each $\mathbf{k}_{-1} \in S^{-1}$ such that for all $z \in \mathbb{R}$:

$$h^1_{\mathbf{k}_{-1}}(z) = a^1_{\mathbf{k}_{-1}} \cdot z + c^1_{\mathbf{k}_{-1}}\,.$$

Lemma 3.3 implies $a^1_{\mathbf{k}_{-1}} = a^1_{\mathbf{1}_{-1}}$ for all $\mathbf{k}_{-1} \in S^{-1}$. Therefore, we only have to keep track of one scaling parameter for all the maps within H^1; let us denote it with α^1. With the first conclusion of Lemma 3.4, we obtain $\alpha^1 > 0$. Therefore, we have shown that H^1 is a positive affine transformation.

After Lemma 3.2, we mentioned that the same results as above can be analogously derived for the other player's transformation maps $H^2, \ldots, H^N$. There is only one difference for their analysis, in that for H^i, we don't index over $\mathbf{k}_{-1} \in S^{-1}$ but over $\mathbf{k}_{-i} \in S^{-i}$ instead. $\square$

Appendix D. Utility Theory in Game Theory

Let us revise some utility theory, as to be found in e.g. Mas-Colell et al. [18].

Preferences and Utility Functions. Suppose a decision maker can choose one outcome from a space C of N-many outcomes (where N finite). Moreover, the decision maker prefers some outcomes over others which is captured by her preference relation $\succeq$ on C.

We typically describe the preferences of the decision maker through utility functions:

Definition Appendix D.1. A utility function $u : C \longrightarrow \mathbb{R}$ is said to represent a preference relation $\succeq$ if for all $c, d \in C$, we have $c \succeq d \iff u(c) \geq u(d)$.

Multiple utility functions can represent the same preference relation. Their practical use is that they translate the preference relation $\succeq$ into comparisons of numerical values.

On the other hand, starting with a utility function u yields an induced preference relation $\succeq$ through

$$\forall\, c, d \in C \quad : \quad c \succeq d :\iff u(c) \geq u(d)\,.$$

Lotteries and the Expected Utility. Now suppose we want to allow the decision maker to choose each outcome in C with some probability. Call such a probability distribution $L = (p_1, \ldots, p_N)$ over C a lottery. The i-th outcome in C can then be represented by the lottery $e_i \in \mathbb{R}^n$. Thus, we extended the choice space of the decision maker from C to the space $\mathcal{L}$ of lotteries. We can also extend Definition Appendix D.1 to preference relations $\succeq$ over $\mathcal{L}$ by requiring $u : \mathcal{L} \longrightarrow \mathbb{R}$ and $\forall\, L, M \in \mathcal{L} \,:\, L \succeq M \iff u(L) \geq u(M)$.

We will be especially interested in those utility functions that simply compute the expected utility of randomly choosing an outcome according to input L.

Definition. A *von Neumann-Morgenstern (NM)* expected utility function is a map $U : \mathcal{L} \longrightarrow \mathbb{R}$ that is determined by its values $U(e_i)$ on the outcomes $e_i \in C, i \in [N]$, and by

$$\forall L = (p_1, \ldots, p_N) : \quad U(L) = \sum_{i=1}^{N} p_i \cdot U(e_i) \,.$$

The following theorem describes the preference relations that can be represented by a NM expected utility function. The theorem relies on four properties - called *axioms* - that a preference relation $\succeq$ can satisfy: Completeness[8], Transitivity[9], Continuity[10] and Independence[11].

Theorem 2 (Expected Utility Theorem)**.** *Let preference relation $\succeq$ satisfy the four axioms mentioned above. Then $\succeq$ can be represented by a NM expected utility function U. Moreover, the representing U is unique up to a positive affine transformation. That is, if U and U' are NM expected utility functions representing $\succeq$, then there exist $\alpha, c \in \mathbb{R}$ such that for all $L \in \mathcal{L}$, we have $U'(L) = \alpha \cdot U(L) + c$.*

Proof. See Proposition 6.B.2 and 6.B.3 from Mas-Colell et al. [18]. □

In contrast to Theorem 2, suppose we start with an arbitrary NM expected utility function U. Then U induces a preference relation $\succeq$ on $\mathcal{L}$ by

$$\forall L, L' \in \mathcal{L} \quad : \quad L \succeq L' :\Longleftrightarrow U(L) \geq U(L') \,.$$

By construction, U represents $\succeq$. One can also show that this induced preference relation $\succeq$ satisfies the four axioms. Therefore, by Theorem 2, U uniquely represents the induced $\succeq$ up to a PAT.

Connections to Game Theory. Take a multiplayer game $G = \big(N, \{S^i\}_{i\in[N]}, \{u_i\}_{i\in[N]}\big)$. Then, the utility functions u_i induce each player's preferences through the following:

[8] For all $L, M \in \mathcal{L}$, we have $L \succeq M$ or $L \preceq M$ (or both, in which case we write $L \sim M$).

[9] For all $L, M, N \in \mathcal{L}$, if $L \succeq M$ and $M \succeq N$, then $L \succeq N$.

[10] For all $L, M, N \in \mathcal{L}$ with $L \succeq M \succeq N$, there exists probability $p \in [0, 1]$ such that $p \cdot L + (1 - p) \cdot N \sim M$.

[11] For all $L, M, N \in \mathcal{L}$ and $p \in [0, 1]$, we have $L \succeq M$ if and only if $p \cdot L + (1 - p) \cdot N \succeq p \cdot M + (1 - p) \cdot N$.

Consider a game that only allows for pure strategy play. Then, given some player i and the pure strategy profile s^{-i} of the opponents, the "sliced" utility function $u_i(\cdot, s^{-i})$ induces a preference relation $\succeq$ for player i over her strategy set S^i.

Now suppose that the input game allows for mixed strategy play. In that case, each element in $\Delta(S^i)$ can be viewed as a lottery over the choice set $C := S^i$. Moreover, player i's utility payoff from a mixed strategy profile $s \in \bigtimes_{i=1}^{N} \Delta(S^i)$ is

$$u_i(s^i, s^{-i}) = \sum_{k_i=1}^{m_i} s^i_{k_i} \cdot u_i(e_{k_i}, s^{-i}) \, .$$

Therefore, $u_i(\cdot, s^{-i})$ has the form of a NM expected utility function. This induces a preference relation $\succeq_{i,s^{-i}}$ on the space of lotteries $\Delta(S^i)$ with $\succeq_{i,s^{-i}}$ satisfying the four axioms. Hence, $u_i(\cdot, s^{-i})$ represents the induced preference relation $\succeq_{i,s^{-i}}$ uniquely up to a PAT.

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