Pontryagin's maximum principle

Theory summary and applications

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1 Preliminary definitions

1.1 Control System

A **control system** is a triple $\Sigma = (\chi, f, U)$, where

1. χ, representing the state of the system, is an **pen** subset of $\mathbb{R}^n, \chi \subset \mathbb{R}^n$.
2. U, representing the space of possible (*istantaneous*)controls, is an open subset of $\mathbb{R}^m, U \subset \mathbb{R}^m$
3. $f : \chi \times cl(U) \to f(x, u)$ is a function which dictates the law with which the system evolves. f
   (a) is continuous
   (b) the map $x \to f(x, u)$ is of class C^1 for each $u \in cl(U)$

Since f is function of the current state of the system and of the current control, the system's evolution does not explicitly depends on time. Of course, the law dictating the evolution of the system's status is

$$\dot{\xi}(t) = f(\xi(t), \mu(t)) \tag{1.1}$$

where obviously, for for each $t, \xi(t) \in \chi$ is the "current" (at time t)state of the system, and $\mu(t) \in U$ is the current (at time t) control, dictated by the control law $\mu(t)$.

1.2 Control and Trajectories

The idea here is that we want some limitations on the function $t \to \mu(t)$, because, starting from a certain state, we want the control to originate, through 1.1, trajectories that, at least, "makes sense". So, given a control system $\Sigma = (\chi, f, U)$we define

- An **admissible control** is a *measurable* map $\mu : I \to U$ where I is a (time) intervale $I \subset \mathbb{R}$, and such that $t \to f(x, \mu(t))$ is locally integrable for ceach $x \in \chi$
- we denote the set of admissible control defined on time interval I by $\mathfrak{U}(I)$
- A **controlled trajectory** is a pair (ξ, μ) where, for some time intervale $I \in \mathbb{R}$
  - $\mu \in \mathfrak{U}(I)$ is then a function expressing the control through which the system is driven in the time interval, and is an admissible control. This map will be simply be referred to as the **control**.
  - $\xi : I \to \chi$ is the map linking the times in the interval to their corresponding state, which follows the law 1.1. This map will be referred to as the **trajectory**.
- a **controlled arc** is a controlled trajectory defined on a compact time interval.

The set of the controlled trajectories for a given control system $\Sigma = (\chi, f, U)$is named Ctraj(Σ), the set of the controlled arcs for the control system is named Carc(Σ)

1.3 Lagrangian, costs and optimal control problem(s)

Since we will want to optimize a cost, we first have to define this objective function which has to be minimized. This will be the integral of another function, the lagrangian, along the path of the system in its evolution from the *beginning* to the *end* of the syof another function (Note: the integral is actually calculated by integrating on the time interval associated with the a controlled trajectory, so it's integrated on an interval of $\mathbb{R}$, not along a path in $\mathbb{R}^n$). So, given a control system $\Sigma = (\chi, f, U)$

- A **Lagrangian**for $\Sigma$is a function $L : \chi \times cl(U) \to \mathbb{R}$ such that
  - L is continuous
  - the function $x \to L(x, u)$ is of class C^1 for each $u \in cl(U)$
- given a Lagrangian L, it is said that a controlled trajectory $(\xi, \mu)$with relative time interval $I$is **L-acceptable** if the function $t \to L(\xi(t), \mu(t))$ is integrable.
- given a Lagrangian L, the corresponding **objective function** is the map $J_{\Sigma,L} : Ctraj(\Sigma) \to \overline{\mathbb{R}}$ given by

  $$J_{\Sigma,L}(\xi, \mu) = \int_I L(\xi(t), \mu(t))dt \tag{1.2}$$

  where we pose $J_{\Sigma,L} = \infty$ if $(\xi, \mu)$is not L-acceptable (if it's not integrable). The set of L-acceptable controlled trajectories (arcs) for the control system is denoted as Ctraj(Σ, L) (or Carc(Σ, L))

The idea here is that one should seek to minimize the objective function, with the "variable" to be tuned being the controlled trajectory. Usually though the problem faced is such that the system will start his evolution in a certain initial state, which resides in a set of possible initial conditions S_0, and some end conditions will be given, which means that in the end, the state of the system should reside in another set, S_1. Of course $S_0, S_1 \subset \chi$. We thus call $\mathrm{Carc}(\Sigma, L, S_0, S_1)$ the set of controlled arcs for the control system $\Sigma = (\chi, f, U)$with Lagrangian L, which have also the following properties

- every (ξ, μ)in $\mathrm{Carc}(\Sigma, L, S_0, S_1)$is defined on a time interval of the form $[t_0, t_1]$ with $t_0 < t_1; t_0, t_1 \in \mathbb{R}$
- every (ξ, μ)in $\mathrm{Carc}(\Sigma, L, S_0, S_1)$then the said controlled arc is also in $\mathrm{Carc}(\Sigma, L)$, which means it is an L-acceptable controlled arc
- every (ξ, μ)in $\mathrm{Carc}(\Sigma, L, S_0, S_1)$defined on the time interval $[t_0, t_1]$ then $\chi(t_0) \in S_0$ and $\chi(t_1) \in S_1$.

Now we can precisely define the optimization problem. There are actually two of these problem, depending on the fact that t_0, t_1 may or may not be fixed. We are going to consider the demonstration for the fixed interval one. So,

Free interval optimal control problem Let's have

- a control system $\Sigma = (\chi, f, U)$,
- a Lagrangian L,
- $S_0, S_1 \in \chi$ sets,

then a controlled trajectory $(\xi_*, \mu_*) \in \mathrm{Carc}(\Sigma, L, S_0, S_1)$is a **solution to the free interval optimal control problem** if
$\forall (\xi, \mu) \in \mathrm{Carc}(\Sigma, L, S_0, S_1),\ J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted as $\mathfrak{P}(\Sigma, L, S_0, S_1)$.

Fixed interval optimal control problem Let's have

- a control system $\Sigma = (\chi, f, U)$,
- a Lagrangian L,
- $S_0, S_1 \in \chi$ sets,
- a time interval $[t_0, t_1]$ with $t_0 < t_1; t_0, t_1 \in \mathbb{R}$

then a controlled trajectory $(\xi_*, \mu_*) \in \mathrm{Carc}(\Sigma, L, S_0, S_1, [t_0, t_1])$is a **solution to the fixed interval optimal control problem** if
$\forall (\xi, \mu) \in \mathrm{Carc}(\Sigma, L, S_0, S_1, [t_0, t_1]),\ J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted as $\mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$.

A simple example The problem in which the cost is the time with which the system is driven from $S_0$to $S_1$is simply a free interval optimal control problem, in which there is a control system with Lagrangian $L(x, u) = 1$.

1.4 Hamiltonians

The maximum principle is related with the maximization of a Hamiltonian associated with a control system with a certain Lagrangian, so we have the following definitions.
Let $\Sigma = (\chi, f, U)$be a control system, L a Lagrangian, then

- the **Hamiltonian** is the function $H_\Sigma : \chi \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H_\Sigma(x, p, u) = \langle p, f(x, u)\rangle$$

- the **extendend Hamiltonian** is the function $H_{\Sigma,L} : \chi \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H_\Sigma(x, p, u) = \langle p, f(x, u)\rangle + L(x, u) = H_\Sigma(x, p, u) + L(x, u)$$

- the **maximum Hamiltonian** is the function $H_\Sigma^{max} : \chi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$H_\Sigma^{max} = sup\{H_\Sigma(x, p, u)|u \in U\}$$

- the **maximum extended Hamiltonian** is the function $H_{\Sigma,L}^{max} : \chi\times\mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$H_{\Sigma,L}^{max} = sup\{H_{\Sigma,L}(x, p, u)|u \in U\}$$

- the variable p is sometimes called **costate**

1.5 Adjoint response

There are another two quantities, namely the adjoint response and the control variations (with the associated variational equation), which are very important in the understanding of the maximum principle. The part relative to adjoint response will now be stated, because it appears in the statement of the principle, while the part concerning brings along concepts that are useful to the demonstration and to the understanding of the geometrical significance of the principle.
So now let $\Sigma = (\chi, f, U)$be a control system, and

- $(\xi, \mu) \in Ctraj(\Sigma)$ be a controlled trajectory with time interval I, then we define the **adjoint response** for Σ along (ξ, μ)as a locally absolutely continous map $\lambda : I \to \mathbb{R}^n$ which also satisfies the following differential equation(s):

$$\begin{aligned} \dot{\xi}(t) &= \boldsymbol{D}_2 H_\Sigma(\xi(t), \lambda(t), \mu(t)) \ (= f(x,u)) \\ \lambda(t) &= -\boldsymbol{D}_1 H_\Sigma(\xi(t), \lambda(t), \mu(t)) \end{aligned} \tag{1.3}$$

we will see later that this last equation can reach another equivalent form.

- if L is a Lagrangian and $(\xi, \mu) \in Ctraj(\Sigma, L)$is an L-acceptable controlled trajectory with time interval I, we then define the **adjoint response** for (Σ, L) along (ξ, μ)as a locally absolutely continous map $\lambda : I \to \mathbb{R}^n$ which also satisfies the following differential equation(s):

$$\begin{aligned} \dot{\xi}(t) &= \boldsymbol{D}_2 H_{\Sigma,L}(\xi(t), \lambda(t), \mu(t)) \ (= f(x,u)) \\ \lambda(t) &= -\boldsymbol{(}D_1 H_{\Sigma,L}(\xi(t), \lambda(t), \mu(t)) \end{aligned} \tag{1.4}$$

1.6 Smooth constraint sets

Part of the maximum principle deals with the case in which $S_0$and $S_1$are "smooth", so we might define a **smooth constraint set S** as a subset of the set of possible (control)system states $S \subset \chi$ such that there exists a C^1 function $\Phi : \chi \to \mathbb{R}^k$, such that $S = \Phi^{-1}(0)$ and also $\boldsymbol{D}\Phi(x)$ is surjective for each $x \in S$

1.7 Reachable set

Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then define

- the **reachable set** from x_0 at t_0 in time $t_1 - t_0$ as

$$\mathfrak{R}(x_0, t_0, t_1) = \{\xi(\mu, x_0, t_0, t_1) | \mu \in \mathfrak{U}([t_0, t_1])\}$$

- the **reachable set** from x_0 at t_0 as

$$\mathfrak{R}(x_0, t_0) = \cup_{t_1 \in [t_0, \infty]} \mathfrak{R}(x_0, t_0, t_1)$$

Little note: since f does not depend explicitly on time, then $\mathfrak{R}(x_0, t_0, t_1)$= $\mathfrak{R}(x_0, 0, t_0)$.

2 Statement of the maximum Principle

2.1 Maximum principle for free interval problems

Let $\Sigma = (\chi, f, U)$be a control system,, L a Lagrangian, $S_0$and S_1subsets of χ. A necessary condition for a controlled trajectory (ξ_*, μ_*) defined on $[t_0, t_1]$ to be optimal, that is, a necessary condition so that $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1)$ is the existence of an absolutely continuous map $[t_0, t_1] : [t_0, t_1] \to \mathbb{R}^n$ and of $\lambda_*^0 \in \{-1, 0\}$ that have also the following properties

1. either $\lambda_*^0 = .1$ or $\lambda_*(t_0) \neq 0$
2. λ_* is an adjoint repsonse for (Σ, λ_*^0, L) along (ξ_*, μ_*)
3. $H_{\Sigma,\lambda_*^0,L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma,\lambda_*^0,L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$

   If μ_* is bounded, then

4. $\forall t \in [t_0, t_1]$ $H^{max}_{\Sigma,\lambda_*^0,L}(\xi_*(t), \lambda_*(t)) = 0$

   If, $S_1$and $S_0$are smooth constraint sets, then $[t_0, t_1]$ can be chosen such that

5. $\lambda_*(t_0)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_0(\xi(t_0)))$ and $\lambda_*(t_1)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_1(\xi(t_1)))$

For the fixed interval problem only condition 4 is lost.

2.2 Maximum principle for fixed interval problems

Let $\Sigma = (\chi, f, U)$be a control system,, L a Lagrangian, $S_0$and S_1subsets of $\chi; [t_0, t_1] \subset \mathbb{R}$ an interval. A necessary condition for a controlled trajectory (ξ_*, μ_*) defined on $[t_0, t_1]$ to be optimal, that is, a necessary condition so that $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$ is the existence of an absolutely continuous map $[t_0, t_1] : [t_0, t_1] \to \mathbb{R}^n$ and of $\lambda_*^0 \in \{-1, 0\}$ that have also the following properties

1. either $\lambda_*^0 = .1$ or $\lambda_*(t_0) \neq 0$
2. λ_* is an adjoint repsonse for (Σ, λ_*^0, L) along (ξ_*, μ_*)
3. $H_{\Sigma,\lambda_*^0,L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma,\lambda_*^0,L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$

   If, $S_1$and $S_0$are smooth constraint sets, then $[t_0, t_1]$ can be chosen such that

4. $\lambda_*(t_0)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_0(\xi(t_0)))$ and $\lambda_*(t_1)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_1(\xi(t_1)))$

3 Hint of demonstration

the first step we need to take is analyzing the effect of varying a trajectory first. In general, one can expect to vary trajectory followed by the state of a control system in two ways: given a control, varying the initial conditions or, given the initial conditions, varying the control. Nevertheless it is still necessary to develop some tools to *describe* the variation of the trajectory in some ways.

3.1 Variations and adjoint response

Variational and adjoint equations
Given a control system $\Sigma = (\chi, f, U)$and am admissible control $\mu : I \to U$ we have

- the **variational equation** equation for Σ with control μ is the differential equation

$$\dot{\xi}(t) = f(\xi(t), \mu(t));$$
$$\dot{v}(t) = \boldsymbol{D}_1 f(\xi(t), \mu(t)) \cdot v(t) \tag{3.1}$$
$$(\xi(t), \mu(t)) \in (\chi \times \mathbb{R}^n)$$

- the **variational equation** equation for Σ with control μ is the differential equation

$$\dot{\xi}(t) = f(\xi(t), \mu(t));$$
$$\dot{\lambda}(t) = -\boldsymbol{D}_1 f^T(\xi(t), \mu(t)) \cdot v(t) \tag{3.2}$$
$$(\xi(t), \lambda(t)) \in (\chi \times \mathbb{R}^n)$$

interpretation It is straightforward to see that the variational equation describes, through a linearization, the evolution in time of a small (infinitesimal) variation from the original trajectory $\xi(t)$, which is one of the solutions to the equation 1.1.
Obviously, it is hoped that, given a certain control and initial condition, the solution to 1.1 is unique, but a-priori it cannot be said.
The geometrical interpretation of the adjoint equation is more subtle, but, in a naive way, one could say that, given an optimal trajectory, the adjoint response is a vector orthogonal to the hyerplane given by (directions of the)the possibile (infinitesimal) variations to that trajectory.

3.2 Variations and infinitesimal variations

Definitions
Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. a **variation** of the trajectory $\xi(\mu, x_0, t_0, \cdot)$ is map $\sigma : J \times [t_0, t_1] \to \chi$ with the following properties

- $J \subset \mathbb{R}; 0 \in int(J)$ is an interval for which zero reside in its internal part
- $\sigma(0, t) =\xi(\mu, x_0, t_0, t)$ for each $t \in [t_0, t_1]$
- $s \to \sigma(s, t)$is of class $C^1$for each $t \in [t_0, t_1]$
- $t \to \sigma(s, t$ is a solution of eq. 1.1.

Given a variation and it's relative "original" trajectory, there is another important quantity, which is he corresponding **infinitesimal variation**. This is yet another map defined with the following limit

$$\delta\sigma(t) = \frac{d}{ds}\bigg|_{s=0} \sigma(s,t). \tag{3.3}$$

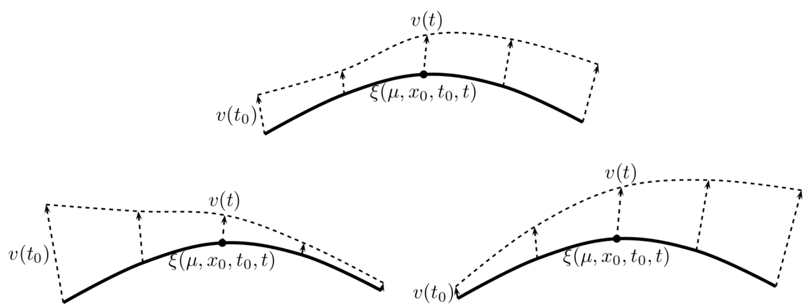

Figure 1:

The dotted arrow represent the difference between the variation (at various times and at a given s), and the "original" trajectory, at the same time. The infinitesimal variation is then represented by a vector such that, at a given time $\bar{t}, \sigma(s,\bar{t}) \approx \xi(\mu, x_0, t_0, \bar{t}) + \delta\sigma(\bar{t}) * s$ Actually, there is another interpretation to the image, in which the dotted arrow are the vectors represent the **infinitesimal** variations, and the dotted line is just the envelope of the dotted arrows calculated at different times. In this case, the dotted part and the continuous-line of the figure have scales that have nothing to do one with each other. Nevertheless, in both interpretations, it is possible to understand the concept of a stable, unstable and "indifferent" trajectory, in which a disturbance in the trajectory is amplified rather than muted.

A theorem about infinitesimal variations Here it will be stated that if a map is solution of equation 3.1 then it is an infinitesimal variation , and vice versa.
Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Given a map $v : [t_0, t_1] \to \mathbb{R}^n$ the statements

- there exists a map σ which is a variation of $\xi(\mu, x_0, t_0, \cdot)$ such that $v = \delta\sigma$;
- $t \to (\xi(\mu, x_0, t_0, t), v(t))$ satisfies the variational equation 3.1.

are equivalent.

The foundamental matrix Φ During the demonstration of the preceding theorem, an $n \times n$ linear map $\Phi(t) : \mathbb{R}^n \to \mathbb{R}^n$ has been defined, as a map such that $\Phi(t)\cdot = \boldsymbol{D}_2\xi(\mu, x_0, t_0, t)\cdot w$.
By differentiation one can see that $\Phi(t)$ satisfies this matrix differential initial value problem:

$$\dot{\Phi}(t) = \boldsymbol{D}_1 f(\xi(\mu, x_0, t_0, t), \mu(t))\circ;\ \Phi(t_0) = Id_{n\times n}$$

We thus define $\boldsymbol{\Phi(\mu, x_0, t_0, \tau, t)}$ as the solution to the matrix initial value problem

$$\dot{\Phi}(t) = \boldsymbol{D}_1 f(\xi(\mu, x_0, t_0, t), \mu(t))\circ;\ \Phi(\tau) = Id_{n\times n}...$$

Where of course $t, \tau \in [t_0, t_1]$.

The importance of this matrix resides in the fact that, being the variational equation 3.1 a linear one and being Φ the foundamental matrix of the equation, it is true that, for a variation $v(t)$ (which is also a solution to the variational equation, because of the theorem), $v(t) = \Phi(\mu, x_0, t_0, t_0, t) \cdot v(t_0)$
Φ also has some properties, about the composition of the

3.3 Properties of adjoint repsonse

What this theorem says is that seeing the adjoint repsonse as a solution of the adjoint equation 3.2 and of the adjoint equation defined with the hamiltonians 1.3 is the same exact thing.

Hamilton's and adjoint equation
Theorem: given a control system $\Sigma = (\chi, f, U)$, an admissible control $\mu : I \to U$,and two maps $\xi : I \to \chi; \lambda : I \to \mathbb{R}^n$, those two statements are equivalent:

- the curve $t \to (\xi(t), \lambda(t))$ satisfies the adjoint equation 3.2
- the curve $t \to (\xi(t), \lambda(t))$ satisfies the differential equation 1.3 recalled here:

$$\dot{\xi}(t) = \boldsymbol{D}_2 H_\Sigma(\xi(t), \lambda(t), \mu(t))$$
$$\lambda(t) = -\boldsymbol{D}_1 H_\Sigma(\xi(t), \lambda(t), \mu(t))$$

This theorem is demonstrated simpy by differentiating the given quantities, and by using the properties of dot product.

Adjoint response and variations
Theorem Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. If v and λ are two maps $v, \lambda : [t_0, t_1] \to \mathbb{R}^n$ that satisfy the variational and adjoint equation respectively (toghether with the trajectory $t \to \xi(\mu, x_0, t_0, t)$), then the standard scalar product between those two vectors is constant with time, which is

$$\langle \lambda(t), v(t) \rangle = \langle \lambda(t_0), v(t_0) \rangle$$

Proof:

$$\frac{d}{dt}\langle\lambda(t), v(t)\rangle =$$
$$= \langle\dot{\lambda}(t), v(t)\rangle + \langle\lambda(t), \dot{v}(t)\rangle =$$
$$= -\langle \boldsymbol{D}_1 f^T((\xi,\mu)) \cdot \lambda(t), v(t)\rangle + \langle\lambda(t), \boldsymbol{D}_1 f^T((\xi,\mu)) \cdot v(t)\rangle =$$
$$= 0$$

A corollary An important consequence of this fact is that if the two vectors v, λ are orthogonal at the initial moment, they remain orthogonal throughout the whole evolution of the system.

Foundamental matrix for the adjoint response Theorem Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$, and let $\tau \in [t_0, t_1]$. Then the solution of the initial value problem

$$\dot{\lambda}(t) = -\boldsymbol{D}_1 f^T(\xi(t), \mu(t)) \cdot v(t),\ \ \lambda(\tau) = \lambda_\tau$$

is the following one:

$$t \to \lambda(t) = \Phi(\mu, x_0, t_0, t, \tau)^T \cdot \lambda_\tau$$

3.4 Needle variations

In the beginning of section 3 we talked about the variations of trajectories, and analyzed those variations *a-posteriori*, trying to describe them. Now we are going to see, so to say, *the elementary (infinitesimal) way of producing said variations.* This will be of course through control variations, not through the variation of initial condition.

Needle variation (fixed interval) et $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then define

- **fixed interval needle variation** <u>**data**</u> as a triple $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ for which
  - $\tau_\theta \in (t_0, t_1]$
  - $l_\theta \in \mathbb{R}_{\geq 0}$
  - $\omega_\theta \in U$
- the **control variation** of the control μ associated to the relative fixed interval needle variation data θ is the map μ_θ is the map $mu_\theta : J \times [t_0, t_1] \to U$ such that

$$\mu_\theta = \begin{cases} \omega_\theta & \text{if } t \in [\tau_\theta - s * l_\theta, \tau_\theta] \\ \mu(t) & \text{otherwise.} \end{cases}$$

Where $J = [0, s_0]$ is an interval sufficiently small so that $\mu_\theta(s,t)$ is an admissible control for each $s \in J$ Just to have an idea, this is how the function μ_θ can look like for a certain $s > 0$

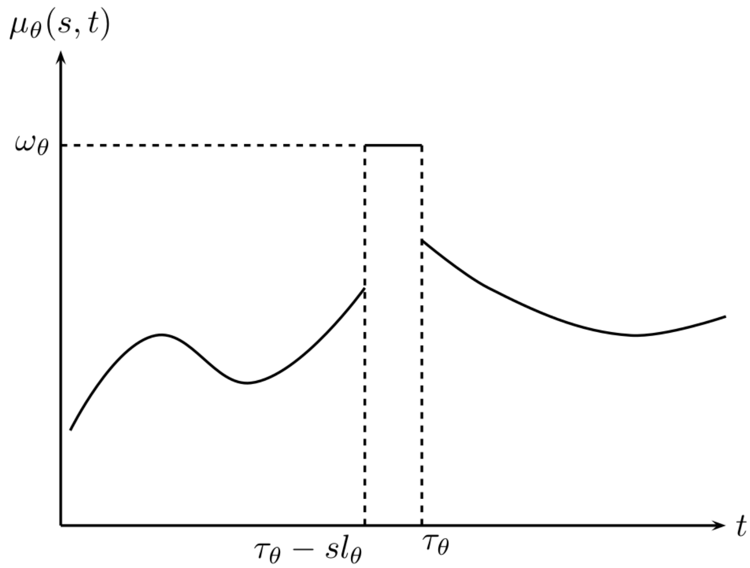

Figure 2:

- **fixed interval needle (*infinitesimal*)variation** associated with the control μ, the trajectory $\xi(\mu, x_0, t_0, \cdot)$ and the variation data θ as a vector of $\mathbb{R}^n$ defined as

$$v_\theta = \frac{d}{ds}\bigg|_{s=0} \xi(\mu_\theta(s,\cdot), x_0, t_0, \tau_\theta),$$

  when such derivative exists

This limit exists at almost any instant, since it exists for every instant that is a Lebesgue point for $t \to f(\xi(\mu, x_0, t_0, t), \mu(t))$. Before stating this formally, we need the definition of $Leb(\mu, x_0, t_0, t)$: it's the set of Lebesgue points if $\tau \to f(\xi(\mu, x_0, t_0, \tau), \mu(\tau)), \tau \in (t_0, t)$. Then,

Theorem:existence and form of fixed interval needle variations
Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Let then $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ be a fixed interval needle variation data, with $\tau_\theta \in Leb(\mu, x_0, t_0, t_1)$. Then the fixed interval variation associated with those data exists and it's given by

$$v_\theta = l_\theta * \Big(f(\xi(\mu, x_0, t_0, \tau_\theta), \omega_\theta) - f(\xi(\mu, x_0, t_0, \tau_\theta), \mu(\tau_\theta)) \Big)$$

Variations and cones The real importance of this theorem is not only in the fact that it is (almost) always possible to individuate the infinitesimal variation, but also in the fact that those variations form a cone, which is, if one vector represents a variation, then all of the half-line (originating from $0 \in \mathbb{R}^n$) given by that vector is made up of fixed interval variations. Formally said,

Proposition: Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Let then $\theta = (\tau_\theta, l_\theta, \omega_\theta)$be a fixed interval needle variation data, with $\tau_\theta \in Leb(\mu, x_0, t_0, t_1)$. Then, the set of fixed interval needle variation associated with the data θ form a cone.

Proof: It's just enough to say that, if v_θ is the variation associated with the data $\theta = (\tau_\theta, l_\theta, \omega_\theta)$, then, taken a $k \in \mathbb{R}_{\geq 0}, kv_\theta$ is the variation associated with the data $k\theta = (\tau_\theta, k * l_\theta, \omega_\theta)$. Using obvious notation, one could then say that $v_{k\theta} = kv_\theta$.

It is interesting to note that, in the triple representing the data, only the *"lenght of the disturbance"* gets multiplied by the scalar. This means, that, referring to 2, the final control is practically the same. In fact, the actual length of the horizontal step in the figure doesn't really matter, since s tends to zero, and the actual disturbance gets reduced to a (Lebesgue-neglettable) single point in time.
The only effect one may obtain is that, given a certain trajectory, the trajectories associated with varied control depart from the undisturbed one at a higher rate with growing s (at least, as long as s is small enough so that one can linearize the effect of control variation), but in the same direction. This is the same exact meaning of the length of the dotted arrows in 1.

3.5 Multi-needle variations

The concept behind multi needle variations is that they are made of single needle variations which sum up(linearly, as can be see in the relative theorem). We need this object basically because although single needle variations do form a cone, this is not generally convex. Convexity is needed in the proof, to find a hyperplane that separates the half space containing cone from another one. This is because a variational cone originating from the boundary of the reachable set cone will point toward the "less optimal", this half space is the non optimal one.

So now let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We want to define

- the **fixed interval multi-needle variation data** Θ as the collection $\Theta = \{\theta_1, .., \theta_k\}$ of fixed interval needle variation data $\theta_j = (\tau_j, l_j, \omega_j), j = 1, .., k$, with the times τ_j all distinct.

- the **control variation** of the control μ associated with the relative data Θ as the map $\mu_\Theta : J \times [t_0, t_1] \to U$ such that

$$\mu_\Theta = \begin{cases} \omega_j & \text{if } t \in [\tau_\Theta - s * l_\Theta, \tau_\Theta], j = 1, .., k \\ \mu(t) & \text{otherwise.} \end{cases}$$

  Where $J = [0, s_0]$ is an interval sufficiently small so that $\mu_\Theta(s,t)$ is an admissible control for each $s \in J$.

- the **fixed interval multi-needle variation** associated with the control μ, the trajectory $\xi(\mu, x_0, t_0, \cdot)$, the time $t \in [t_0, t_1]$ taken such that $t > \max_j(\tau_j)$ and the variation data Θ as a vector of $\mathbb{R}^n$ (a vectorial function of time actually) defined like this:

$$v_\theta(t) = \left.\frac{d}{ds}\right|_{s=0} \xi(\mu_\Theta(s, \cdot), x_0, t_0, t),$$

  when such derivative exists

Theorem: existence of multi-needle fixed interval variations
Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Then, let $\Theta = \{\theta_1, .., \theta_k\}$ be a multi needle variation data ("fixed interval" will from now on be omitted, since this case is the only one treated in this essay),Θ such that the times $\tau_j \in Leb(\mu, x_0, t_0, t)$, are all distinct, and also $t > \max_j(\tau_j)$.
Then, the multi needle variation is given by

$$v_\Theta(t) = \sum_{j=1}^{k} \Phi(\mu, x_0, t_0, \tau_j, t) \cdot v_{\theta_j}$$

As one can see, the variations caused by the various single needle variations first are let evolve in time from the instant in which they are applied to the "current" one, t, and then their effects sum up linearly.

Taking this into account, and also considering the fact that $v_{k\theta_j} = kv_{\theta_j}$ for whatever $k \geq 0$ real, and remembering that the matrix-vector product is linear with respect to scalar multiplication, one can easily get the following

Corollary: coned convex combination of multi needle variations
With slight abuse of notation, given the usual control system, admissible control, initial condition, interval, given a multi needle variation data with its distinct times being Lebesgue points for f, and with a $t > \max_j(\tau_j)$, given a set $\lambda = \{\lambda_1, .., \lambda_k\} \subset \mathbb{R}_{\geq 0}$ then

$$v_{\lambda\Theta(t)} = \sum_{j=1}^{k} \lambda_j \Phi(\mu, x_0, t_0, \tau_j, t) \cdot v_{\theta_j}$$

3.6 Cones and reachable set

The proof of maximum principle will deal with the *"approximation of reachable set using cones"*, which is to say that at its boundary the reachable set will be confused, at least in a neighborhood of a frontier point, with a (variational) cone. The variations used will be both single and multi needle. We're going to see that the two are, in some sense, equal.
We just need two more definitions.
The first one is going to be the

Fixed interval tangent cone As usual, let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then take $t \in [t_0, t_1]$. Then we denote by $K(\mu, x_0, t_0, t)$ the coned convex hull of the following set:

$\cup\{\Phi(\mu, x_0, t_0, \tau, t) \cdot v | \tau \in Leb(\mu, x_0, t_0, t)$where v is a single needle variation at time $\tau.\}$

4 Moon landing problem