Pontryagin's maximum principle
Theory summary and applications

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1 Preliminary definitions

1.1 Control System

A **control system** is a triple $\Sigma = (\chi, f, U)$, where

1. χ, representing the states of the system, is an open subset of $\mathbb{R}^n$,
2. U, representing the space of possible (*instantaneous*) controls, is an open subset of $\mathbb{R}^m, U \subset \mathbb{R}^m$,
3. $f : \chi \times cl(U) \to f(x, u)$ is a function which dictates the law with which the system evolves. Moreover, f
   (a) is continuous and
   (b) the map $x \to f(x, u)$ is of class C^1 for each $u \in cl(U)$.

Since f is function of the current state of the system and of the current control, the evolution of the system does not explicitly depend on time. Of course, the law dictating the evolution of the status of the system is

$$\dot{\xi}(t) = f(\xi(t), \mu(t)) \tag{1.1}$$

where obviously, for each $t, \xi(t) \in \chi$ is the "current" (at time t) state of the system, and $\mu(t) \in U$ is the current (at time t) control, dictated by the control law $\mu(t)$.

1.2 Control and trajectories

We want some limitations on the function $t \to \mu(t)$ because, starting from a certain state we want the control to originate through (1.1) trajectories that, at least, "make sense". So, given a control system $\Sigma = (\chi, f, U)$ we define

- An **admissible control** as a *measurable map* $\mu : I \to U$ where I is a (time) interval $I \subset \mathbb{R}$, and such that $t \to f(x, \mu(t))$ is locally integrable for each $x \in \chi$.
- the set of admissible controls defined on the time interval I as $\mathfrak{U}(I)$.
- A **controlled trajectory** as a pair (ξ, μ) where, for some time interval I
  - $\mu \in \mathfrak{U}(I)$ is a function expressing the control through which the system is driven in the time interval, and is an admissible control. This map will be simply called the **control**.
  - $\xi : I \to \chi$ is the map linking the times to their corresponding state, which follows law (1.1). This map will be called the **trajectory**.
- a **controlled arc** as a controlled trajectory defined on a compact time interval.

The set of the controlled trajectories for a given control system $\Sigma = (\chi, f, U)$ is denoted by $\text{Ctraj}(\Sigma)$, the set of the controlled arcs for the control system is denoted by $\text{Carc}(\Sigma)$.

1.3 Lagrangian, costs and optimal control problem(s)

Since we want to optimize a cost, we first have to define an objective function which has to be minimized. This will be the integral of another function called Lagrangian. So, given a control system $\Sigma = (\chi, f, U)$,

- A **Lagrangian** for Σ is a function $L : \chi \times cl(U) \to \mathbb{R}$ such that
  - L is continuous and
  - the function $x \to L(x, u)$ is of class C^1 for each $u \in cl(U)$.
- given a Lagrangian L, we say that a controlled trajectory (ξ, μ) with relative time interval I is **L-acceptable** if the function $t \to L(\xi(t), \mu(t))$ is integrable.
- given a Lagrangian L, the corresponding **objective function** is the map $J_{\Sigma,L} : Ctraj(\Sigma) \to \overline{\mathbb{R}}$ given by
$$J_{\Sigma,L}(\xi, \mu) = \int_I L(\xi(t), \mu(t))dt \tag{1.2}$$
  where we set $J_{\Sigma,L} = \infty$ if (ξ, μ) is not L-acceptable.
  The set of L-acceptable controlled trajectories (resp .arcs) for the control system is denoted by Ctraj(Σ, L) (resp. Carc(Σ, L)).

We should seek to minimize the objective function, with the "parameter" to be tuned being the controlled trajectory (ξ, μ). Usually the problem faced is such that the system will start its evolution in a certain initial state, which lie in a set of possible initial conditions S_0, and some end conditions will be given, which means that in the end, the state of the system should lie in another set, S_1. Of course $S_0, S_1 \subset \chi$. We thus call Carc(Σ, L, S_0, S_1) the set of controlled arcs for the control system $\Sigma = (\chi, f, U)$ with Lagrangian L, which have also the following properties:

- every (ξ, μ) in Carc(Σ, L, S_0, S_1) is defined on a time interval of the form $[t_0, t_1] \subset \mathbb{R}$.
- if $(\xi, \mu)\in$Carc(Σ, L, S_0, S_1) then the controlled arc is also in Carc(Σ, L), which means that it is an L-acceptable controlled arc.
- if $(\xi, \mu) \in$Carc(Σ, L, S_0, S_1) is defined on the time interval $[t_0, t_1]$ then $\chi(t_0) \in S_0$ and $\chi(t_1) \in S_1$.

Now we can precisely define the optimization problem. There are actually two of these problems, depending on the fact that $[t_0, t_1]$ may or may not be fixed. We are only going to consider the proof for the fixed interval case.

Free interval optimal control problem Let us consider

- a control system $\Sigma = (\chi, f, U)$,
- a Lagrangian L,
- $S_0, S_1 \subset \chi$ sets,

then a controlled trajectory $(\xi_*, \mu_*) \in$Carc(Σ, L, S_0, S_1) is a **solution to the free interval optimal control problem** if $\forall(\xi, \mu) \in$Carc(Σ, L, S_0, S_1), $J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted by $\mathfrak{P}(\Sigma, L, S_0, S_1)$.

Fixed interval optimal control problem Let us consider

- a control system $\Sigma = (\chi, f, U)$,
- a Lagrangian L,
- $S_0, S_1 \subset \chi$ sets,

- a time interval $[t_0, t_1]$

then a controlled trajectory $(\xi_*, \mu_*) \in$Carc$(\Sigma, L, S_0, S_1, [t_0, t_1])$is a **solution to the fixed interval optimal control problem** if $\forall(\xi, \mu) \in$Carc$(\Sigma, L, S_0, S_1, [t_0, t_1])$, $J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted by $\mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$.

A simple example The problem in which the cost is the time with which the system is driven from S_0 to $S_1$is simply a free interval optimal control problem, in which there is a control system with Lagrangian $L(x, u) = 1$.

1.4 Hamiltonians

The maximum principle is related with the maximization of a Hamiltonian associated with a control system with a certain Lagrangian, so we have the following definitions.
Let $\Sigma = (\chi, f, U)$ be a control system and L a Lagrangian, then

- the **Hamiltonian** is the function $H_\Sigma : \chi \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H_\Sigma(x, p, u) = \langle p, f(x, u)\rangle$$

- the **extendend Hamiltonian** is the function $H_{\Sigma,L} : \chi \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H_{\Sigma,L}(x, p, u) = \langle p, f(x, u)\rangle + L(x, u) = H_\Sigma(x, p, u) + L(x, u)$$

- the **maximum Hamiltonian** is the function $H^{max}_\Sigma : \chi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$H^{max}_\Sigma = sup\{H_\Sigma(x, p, u)|u \in U\}$$

- the **maximum extended Hamiltonian** is the function $H^{max}_{\Sigma,L} : \chi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$H^{max}_{\Sigma,L} = sup\{H_{\Sigma,L}(x, p, u)|u \in U\}$$

- the variable p is sometimes called **costate**.

1.5 Adjoint response

There are other two quantities, namely the adjoint response and control variation (with the associated adjoint and variational equation), which are very important for the principle. We will now state the part relative to the adjoint response, because it appears in the enunciate of the principle, while keeping for later the part relative to the control variation.
So now let $\Sigma = (\chi, f, U)$ be a control system, and

- $(\xi, \mu) \in Ctraj(\Sigma)$ be a controlled trajectory with time interval I, then we define the **adjoint response** for Σ along (ξ, μ) as a locally absolutely continuous map $\lambda : I \to \mathbb{R}^n$ which satisfies the following differential equations:

$$\begin{aligned} \dot{\xi}(t) &= D_2 H_\Sigma(\xi(t), \lambda(t), \mu(t)) \ (= f(x, u)) \\ \lambda(t) &= -D_1 H_\Sigma(\xi(t), \lambda(t), \mu(t)) \end{aligned} \tag{1.3}$$

  Where, given a vector function a of vector variables x_1, x_2, x_3, we denote by $D_i a$ the partial derivative with respect to the variable x_i.
  The above equation can also be expressed in an equivalent form.

- if L is a Lagrangian and $(\xi, \mu) \in Ctraj(\Sigma, L)$ is an L-acceptable controlled trajectory with time interval I, we then define the **adjoint response** for (Σ, L) along (ξ, μ) as a locally absolutely continous map $\lambda : I \to \mathbb{R}^n$ which also satisfies the following differential equation(s):

$$\begin{aligned} \dot{\xi}(t) &= D_2 H_{\Sigma,L}(\xi(t), \lambda(t), \mu(t)) \ (= f(x, u)) \\ \lambda(t) &= -D_1 H_{\Sigma,L}(\xi(t), \lambda(t), \mu(t)) \end{aligned} \tag{1.4}$$

1.6 Smooth constraint sets

Part of the maximum principle deals with the case in which S_0 and S_1 are "smooth", so we might define a **smooth constraint set S** as a subset of the set of the states space$S \subset \chi$ such that there exists a C^1 function $\Phi : \chi \to \mathbb{R}^k$, such that $S = \Phi^{-1}(0)$ and also $D\Phi(x)$ is surjective for each $x \in S$.

1.7 Reachable sets

Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. (1.1), $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then define

- the **reachable set** from x_0 at t_0 in time $t_1 - t_0$ as

$$\mathfrak{R}(x_0, t_0, t_1) = \{\xi(\mu, x_0, t_0, t_1) | \mu \in \mathfrak{U}([t_0, t_1])\}$$

- the **reachable set** from x_0 at t_0 as

$$\mathfrak{R}(x_0, t_0) = \cup_{t_1 \in [t_0, \infty]} \mathfrak{R}(x_0, t_0, t_1)$$

Remark: since f does not depend explicitly on time, then $\mathfrak{R}(x_0, t_0, t_1)$= $\mathfrak{R}(x_0, 0, t_1 - t_0)$.

2 Statement of the maximum principle

2.1 Maximum principle for free interval problems

Let $\Sigma = (\chi, f, U)$ be a control system,, L a Lagrangian, S_0 and S_1 subsets of χ. A necessary condition for a controlled trajectory (ξ_*, μ_*) defined on $[t_0, t_1]$ to be optimal, that is, a necessary condition so that $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1)$, is the existence of an absolutely continuous map $\lambda_* : [t_0, t_1] \to \mathbb{R}^n$ and of $\lambda_*^0 \in \{-1, 0\}$ that have also the following properties

1. either $\lambda_*^0 = -1$ or $\lambda_*(t_0) \neq 0$,
2. λ_* is an adjoint repsonse for $(\Sigma, \lambda_*^0 L)$ along (ξ_*, μ_*),
3. $H_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$,
   If μ_* is bounded, then
4. $\forall t \in [t_0, t_1]$ $H^{max}_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t)) = 0$.

   Also, if S_1 and S_0 are smooth constraint sets, then λ_* can be chosen such that
5. $\lambda_*(t_0)$ is orthogonal to $\ker(D\Phi_0(\xi(t_0)))$ and $\lambda_*(t_1)$ is orthogonal to $\ker(D\Phi_1(\xi(t_1)))$.

For the fixed interval problem only condition 4 is lost.

2.2 Maximum principle for fixed interval problems

Let $\Sigma = (\chi, f, U)$ be a control system, L a Lagrangian, S_0 and S_1 subsets of $\chi; [t_0, t_1] \subset \mathbb{R}$ an interval. A necessary condition for a controlled trajectory (ξ_*, μ_*) defined on $[t_0, t_1]$ to be optimal, that is, a necessary condition so that $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$, is the existence of an absolutely continuous map $\lambda_* : [t_0, t_1] \to \mathbb{R}^n$ and of $\lambda_*^0 \in \{-1, 0\}$ that have also the following properties

1. either $\lambda_*^0 = -1$ or $\lambda_*(t_0) \neq 0$,
2. λ_* is an adjoint repsonse for (Σ, λ_*^0, L) along (ξ_*, μ_*),
3. $H_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$.
   If μ_* is bounded, then

4. $\forall t \in [t_0, t_1]$ $H^{max}_{\Sigma,\lambda^0_* L}(\xi_*(t), \lambda_*(t))$ is constant.

   Also,if S_1 and S_0 are smooth constraint sets, then λ_* can be chosen such that

5. $\lambda_*(t_0)$ is orthogonal to $\ker(D\Phi_0(\xi(t_0)))$ and $\lambda_*(t_1)$ is orthogonal to $\ker(D\Phi_1(\xi(t_1)))$.

3 Sketch of the proof

The first step we need to take is analyzing the effect of varying a trajectory first. In general, one can expect to vary the trajectory followed by the system in two ways: given a control, varying the initial conditions or, given the initial conditions, varying the control. Nevertheless it is still necessary to develop some tools to *describe* the variation of a trajectory.

3.1 Variational and adjoint equations

Given a control system $\Sigma = (\chi, f, U)$ and an admissible u $\mu : I \to U$ we define the following:

- the **variational equation** equation for Σ with control μ is the differential equation

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t), \mu(t)); \\ \dot{v}(t) &= D_1 f(\xi(t), \mu(t)) \cdot v(t) \\ (\xi(t), \mu(t)) &\in (\chi \times \mathbb{R}^n) \end{aligned} \tag{3.1}$$

- the **adjoint equation** equation for Σ with control μ is the differential equation

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t), \mu(t)); \\ \dot{\lambda}(t) &= -D_1 f^T(\xi(t), \mu(t)) \cdot v(t) \\ (\xi(t), \lambda(t)) &\in (\chi \times \mathbb{R}^n) \end{aligned} \tag{3.2}$$

Interpretation It is straightforward to see that the variational equation describes, through a linearization, the evolution in time of a small (infinitesimal) variation from the original trajectory $\xi(t)$, solution to (1.1).

3.2 Variations and infinitesimal variations

Definitions
Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. (1.1), $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. A **variation** of the trajectory $\xi(\mu, x_0, t_0, \cdot)$ is a map $\sigma : J \times [t_0, t_1] \to \chi$ with the following properties:

- $J \subset \mathbb{R}; 0 \in int(J)$,
- $\sigma(0, t) = \xi(\mu, x_0, t_0, t)$ for each $t \in [t_0, t_1]$,
- $s \to \sigma(s, t)$ is of class C^1 for each $t \in [t_0, t_1]$,
- $t \to \sigma(s, t)$ is a solution of eq. (1.1).

Given a variation and its relative trajectory, there is another important quantity, which is the corresponding **infinitesimal variation**. This is yet another map defined with the following limit

$$\delta\sigma(t) = \left.\frac{d}{ds}\right|_{s=0} \sigma(s, t). \tag{3.3}$$

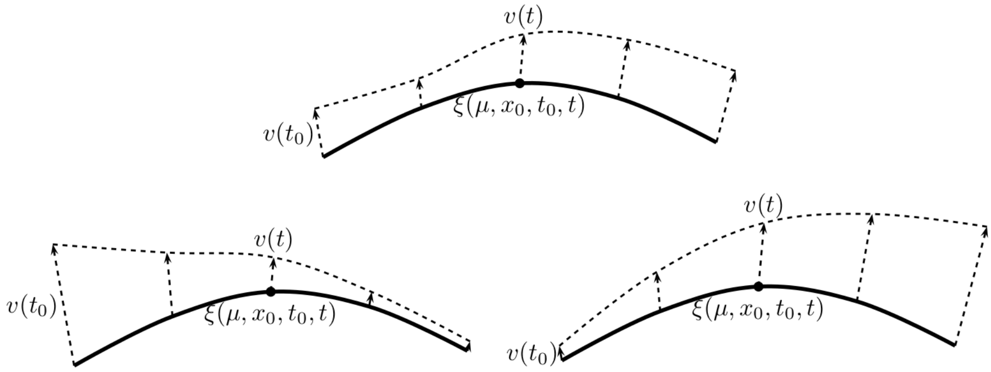


Figure 1: The dotted arrow represent the difference between the variation (at various times and at a given s), and the "original" trajectory, at the same time. The infinitesimal variation is then represented by a vector such that, at a given time $\bar{t}, \sigma(s, \bar{t}) \approx \xi(\mu, x_0, t_0, \bar{t}) + s\delta\sigma(\bar{t})$ Actually, there is another interpretation to the image, in which the dotted arrow are the vectors represent the **infinitesimal** variations, and the dotted line is just the envelope of the dotted arrows calculated at different times. In this case, the dotted part and the continuous-line of the figure have scales that have nothing to do one with each other. Nevertheless, in both interpretations, it is possible to understand the concept of a stable, unstable and "indifferent" trajectory, in which a disturbance in the trajectory is amplified rather than muted.

A theorem about infinitesimal variations Here it will be stated that if a map is solution of equation (3.1) then it is an infinitesimal variation, and vice versa.

Theorem 3.2.1. *Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq.* (1.1)*, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Given a map $v : [t_0, t_1] \to \mathbb{R}^n$ the statements*

- *there exists a map σ which is a variation of $\xi(\mu, x_0, t_0, \cdot)$ such that $v = \delta\sigma$ and*
- *$t \to (\xi(\mu, x_0, t_0, t), v(t))$ satisfies the variational equation 3.1*

are equivalent.

The foundamental matrix Φ We temporarily define an $n \times n$ linear map $\Phi(t) : \mathbb{R}^n \to \mathbb{R}^n$ as a map such that $\Phi(t) \cdot w = D_2\xi(\mu, x_0, t_0, t) \cdot w$.
By differentiation one can see that $\Phi(t)$ satisfies this matrix differential initial value problem

$$\dot{\Phi}(t) = D_1 f(\xi(\mu, x_0, t_0, t), \mu(t)) \circ \Phi(t);\ \Phi(t_0) = Id_{n\times n}.$$

We can re-define, for wider generality, $\Phi(\mu, x_0, t_0, \tau, t)$ by the solution to the matrix initial value problem

$$\dot{\Phi}(t) = D_1 f(\xi(\mu, x_0, t_0, t), \mu(t)) \circ \Phi(t);\ \Phi(\tau) = Id_{n\times n},$$

where of course $t, \tau \in [t_0, t_1]$.

The importance of this matrix lies in the fact that, being the variational equation (3.1) a linear one and being Φ the fundamental matrix of the equation, it is true that, for a variation $v(t)$ (which is also a solution to the variational equation), $v(t) = \Phi(\mu, x_0, t_0, t_0, t) \cdot v(t_0)$.

3.3 Needle variations

We introduced a tool to describe variations of trajectories. Now we are going to analyze a way of causing trajectory variations. This will be of made with control variations, not through the variation of initial conditions.

Needle variation (fixed interval) Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. (1.1), $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then define

- the **fixed interval needle variation data** as a triple $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ for which
  - $\tau_\theta \in (t_0, t_1]$,
  - $l_\theta \in \mathbb{R}_{\geq 0}$,
  - $\omega_\theta \in U$.
- the **control variation** of the control μ associated to the relative fixed interval needle variation data θ is the map $\mu_\theta : J \times [t_0, t_1] \to U$ such that

$$\mu_\theta = \begin{cases} \omega_\theta & \text{if } t \in [\tau_\theta - s l_\theta, \tau_\theta] \\ \mu(t) & \text{otherwise.} \end{cases}$$

  Where $J = [0, s_0]$ is an interval sufficiently small so that $\mu_\theta(s, t)$ is an admissible control for each $s \in J$. Just to have an idea, this is how the function μ_θ can look like for a certain $s > 0$.

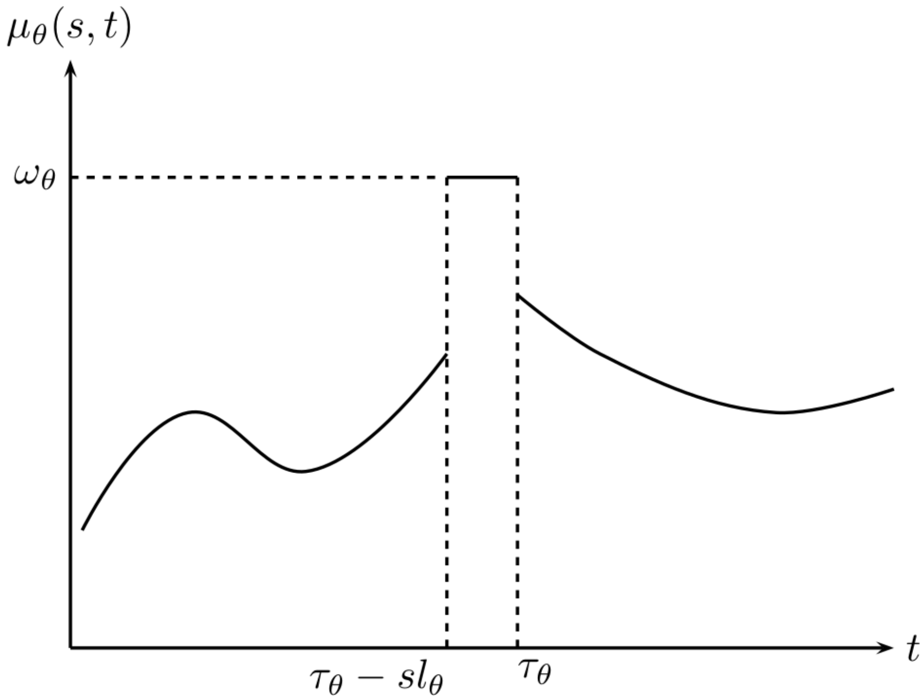


Figure 2

- the **fixed interval needle (infinitesimal) variation** associated with the control μ, the trajectory $\xi(\mu, x_0, t_0, \cdot)$ and the variation data θ as the vector of $\mathbb{R}^n$ defined as

$$v_\theta = \left.\frac{d}{ds}\right|_{s=0} \xi(\mu_\theta(s, \cdot), x_0, t_0, \tau_\theta),$$

  when such derivative exists.

This limit exists at almost any instant, as the next theorem says. Before though we need the definition of $Leb(\mu, x_0, t_0, t)$: it's the set of Lebesgue points of $\tau \to f(\xi(\mu, x_0, t_0, \tau), \mu(\tau))$ in the interval (t_0, t).

Existence and form of fixed interval needle variations

Theorem 3.3.1. *Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. (1.1), $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Let then $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ be a fixed interval needle variation data, with $\tau_\theta \in Leb(\mu, x_0, t_0, t_1)$. Then the fixed interval variation associated with those data exists and it's given by*

$$v_\theta = l_\theta * \bigg(f(\xi(\mu, x_0, t_0, \tau_\theta), \omega_\theta) - f(\xi(\mu, x_0, t_0, \tau_\theta), \mu(\tau_\theta)) \bigg).$$

Variations and cones The real importance of this theorem is not only in the fact that it is (almost) always possible to individuate the infinitesimal variation, but also in the fact that those variations form a cone, which is, if one vector represents a variation, then all of the half-line (originating from $0 \in \mathbb{R}^n$) given by that vector is made up of fixed interval variations. Formally said:

Theorem 3.3.2. *Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. (1.1), $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Let then $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ be a fixed interval needle variation data, with $\tau_\theta \in Leb(\mu, x_0, t_0, t_1)$.*
Then, the set of fixed interval needle variation associated with the data θ form a cone.

Proof. It's just enough to say that, if v_θ is the variation associated with the data $\theta = (\tau_\theta, l_\theta, \omega_\theta)$, then, taken a $k \in \mathbb{R}_{\geq 0}, kv_\theta$ is the variation associated with the data $k\theta = (\tau_\theta, kl_\theta, \omega_\theta)$. Using obvious notation, one could then say that $v_{k\theta} = kv_\theta$. □

It is interesting to note that, in the triple representing the data, only the "lenght of the disturbance" gets multiplied by the scalar. This means, that, referring to Figure 2, only the length of the horizontal step in the figure changes. But in the limit process the disturbance is reduced to a 0-measure point, regardless of its underlying l_θ.
The only effect one may obtain is that, given a certain trajectory, the trajectories associated with varied control depart from the undisturbed one at a higher rate with growing s (at least, as long as s is small enough so that one can linearize the effect of control variation), but in the same direction (this is exactly the same concept as the cone).

3.4 The tangent cone

The proof of maximum principle will make use of cones of variations and relate them to the reachable set.

As usual, let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. (1.1), $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then take $t \in [t_0, t_1]$. Then we denote with $K(\mu, x_0, t_0, t)$ the coned convex hull of the following set:

$$\cup\{\Phi(\mu, x_0, t_0, \tau, t) \cdot v | \tau \in Leb(\mu, x_0, t_0, t) \text{where } v \text{ is a single needle variation at time } \tau.\}$$

This cone will be called **tangent cone**.
In simple words, K is built by taking every single needle variation at every possible time preceding the current one(almost: they still have to be Lebesgue points), making the disturbance evolve from its origin to the current time trough matrix Φ and then by taking the coned convex hull of the set union of all those vectors.

3.5 Significative theorems

Points in the tangent cones are "interior" to the reachable set

Theorem 3.5.1. *Given the usual control system, initial condition and $[t_0, t_1]$, and given a tangent cone $K(\mu, x_0, t_0, t)$, for a $t \in [t_0, t_1]$, if a v_0 exists such that it is in the internal part of K,then there exists a cone $A \subset int(K)$ and an $r > 0$ such that*

- $v_0 \in int(A)$,
- *the intersection B between A and the ball of center $\xi(\mu, x_0, t_0, t)$ and radius r is contained in the reachable set $B \subset \mathfrak{R}(x_0, t_0, t)$. In other words, $\mathfrak{R}(x_0, t_0, t) \supset B = \{\xi(\mu, x_0, t_0, t) + v | \ v \in A, ||v|| < r\}$*

What this theorem says is that basically the tangent cone points "towards", or better to say, "in" the reachable set.

Maximum Hamiltonians and tangent cones In the statement of maximum principle tangent cones do not appear, but maximum Hamiltonian do. Actually, one could just be satisfied with the Hamiltonian part, in the sense that, once the adjoint response is found (not an easy task, but the orthogonality with the smooth set S_1 and S_0 can help), the only remaining part is to find controls that maximizes the Hamiltonian. Only those controls are candidates for optimality.

Nevertheless, it is interesting to see the connections between tangent cones and Hamiltonians, and to have a look at the (geometrical) significance of the following theorems.
The first one is going to be about the binding between maximization of Hamiltonian and the existance of a separating hyperplane between the costate and, in a sense, possible "force fields". This, point to point in time.

Theorem 3.5.2. *Given a control system $\Sigma = (\chi, f, U)$, a costate p and a control $\overline{u}$, then $H_\Sigma(x, p, \overline{u}) = H_\Sigma^{max}(x, p)$ if and only if*

$$\text{for every } v \in \{f(x, u) - f(x, \overline{u}) | u \in U\} \text{ it is valid that } \langle p, v\rangle \leq 0$$

Proof.

$$\begin{gathered}
H_\Sigma(x, p, \overline{u}) = H_\Sigma^{max}(x, p) \iff \\
H_\Sigma(x, p, \overline{u}) \geq H_\Sigma(x, p, u) \iff \\
H_\Sigma(x, p, \overline{u}) - H_\Sigma(x, p, u) \geq 0 \iff \\
\langle p, f(x, u) - f(x, \overline{u}) \leq 0\rangle \iff \\
\langle p, v\rangle \leq 0
\end{gathered}$$

for v defined as before. □

Hamiltonians and adjoint response There is now another theorem, regarding the connection between adjoint response and Hamiltonian. What this theorem says is that one can find a suitable costate p, which maximizes the Hamiltonian up to the present moment.

Theorem 3.5.3. *Let's have the usual control system, initial condition, time interval, and suppose that for a $\tau \in [t_0, t_1]$ there exists a vector $\overline{\lambda} \in \mathbb{R}^n$ ad a cone A, such that $K(\mu, x_0, t_0, \tau) \subset A$ and for each $v \in A \implies \langle \overline{\lambda}, v\rangle \leq 0$. If we then take the adjoint response with initial condition for equation* (3.2) *that $\lambda(\tau) = \overline{\lambda}$ it holds true that for every $t \in Leb(\mu, x_0, t_0, \tau)$*

$$H_\Sigma(\xi(\mu, x_0, t_0, t), \lambda(t), \mu(t)) = H_\Sigma^{max}(\xi(\mu, x_0, t_0, t), \lambda(t)).$$

The idea behind the first one of these last two theorems 3.5.2 and 3.5.3 is that a control is optimal and thus maximizes the Hamiltonian, at any given time, if and only if there exists a supporting hyperplane for the cone C, which is the coned convex hull of $\{f(x, u) - f(x, \overline{u}) | u \in U\}$.
The second theorem tells us that if at any given time τ there exists a supporting hyperplane for the cone, then we can find a function $p : [t_0, \tau] \to \mathbb{R}^n$ such that it maximizes the Hamiltonian at almost every preceding instant.
The bound between the two theorems (the first one doesn't mention the tangent cone!) is the acknowledgment that every variation v defined as in the first theorem is actually a needle variation (with data $(t, 1, u)$ and associated with the control $\overline{u}$). As a last remark: the second theorem tells us only something about the past instants up to the present one. It does not give us ways to predict what the optimal control for the instants after τ.

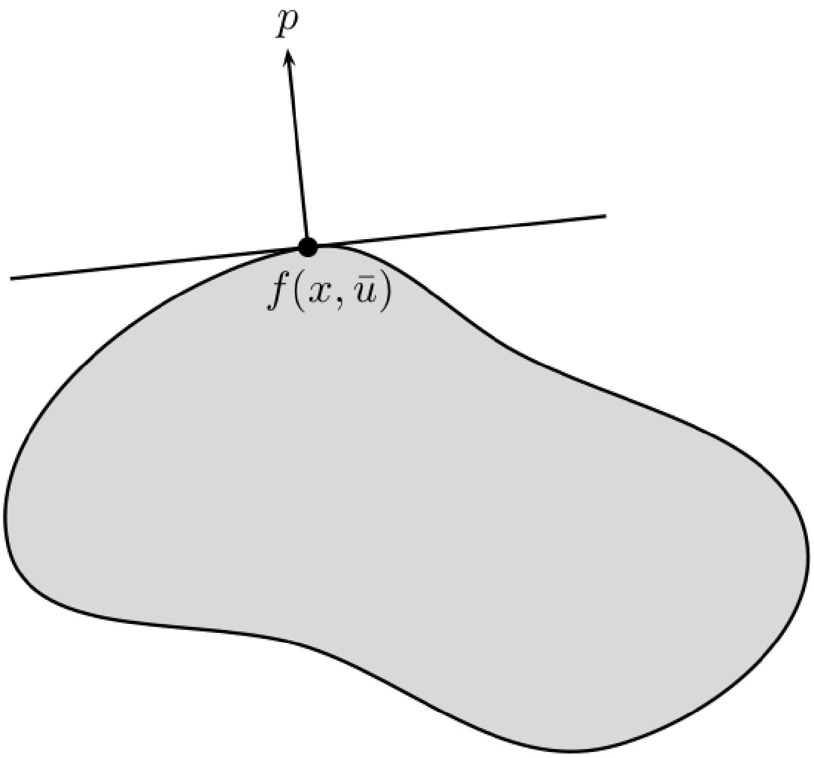


Figure 3: In order to maximize the Hamiltonian, the control should be such that f has the maximum projection in the direction of p.

3.6 Controlled trajectories and boundary of the reachable set

The following very important theorem tell us some important things. The first one is that if in the end (at t_1) the trajectory lies on the boundary of the reachable set, then there exists an adjoint response that maximizes the Hamiltonian for any instant in $[t_0, t_1]$ with that control (or, better to say, in the Lebesgue points for f in that interval). Thus, the control is a candidate for optimality since it maximizes the Hamiltonian.

Theorem 3.6.1. *Let us have the usual control system, initial condition and time interval, a control* $\mu_* \in \mathfrak{U}([t_0, t_1])$*, and* $\xi(\mu_*, x_0, t_0, t_1)$ *(abbreviate* $\xi(\mu_*, x_0, t_0, \cdot)$ *simply with* $\xi_*(\cdot)$*). If* $\xi_*(t_1) \in bd(\mathfrak{R}(x_0, t_0, t_1))$ *then*

- *there exists an adjoint response* $\lambda_* : [t_0, t_1] \to \mathbb{R}^n$ *for* Σ *for the controlled trajectory* (ξ_*, μ_*) *such that*
- $H_\Sigma(\xi(\mu_*, x_0, t_0, t), \lambda_*(t), \mu_*(t)) = H_\Sigma^{max}(\xi(\mu_*, x_0, t_0, t), \lambda_*(t))$ *for almost every* $t \in [t_0, t_1]$

Proof. Since $\xi_*(t1)$ is on the boundary of the reachable set, then there exists a sequence $\{x_j\}$ in $\chi \backslash closure(\mathfrak{R}(x_0, t_0, t_1))$ that converges to that point. Now, let's take another sequence $v_j = \frac{x_j - \xi_*(t_1)}{||x_j - \xi_*(t_1)||}$. Since this sequence lies on the unitary sphere in R^n and this set is compact, there exists a converging subsequence, that converges to v_0.
We now claim that $v_0 \notin int(K(\mu_*, x_0, t_0, t_1))$:

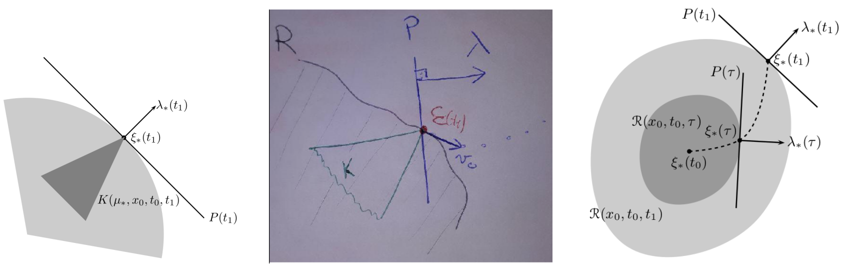
Supposing that v_0 lies in the internal part of the cone, then there exists a sufficiently large N such that, for $j > N$, every v_j lies in the internal of the cone. Theorem 3.5.1 applies then to this vector.
Obviously, if v_j lies in the internal of the cone A, also $\alpha v_j, \alpha > 0$ does.
Now, since $\{x_j\}$ converges to $\xi_*(t1)$, we can also take that N so that it holds true $||x_j - \xi_*(t_1)|| < \frac{1}{2}r$ for every j.
Since $\alpha v_j \in int(A), \alpha v_j \in A$. We can rescale each v_j such that its module is no longer one, but $||x_j - \xi_*(t_1)||$. Thus, $\xi_*(t1) + \alpha v_j = x_j$, and because of theorem 3.5.1, $x_j \in \mathfrak{R}(x_0, t_0, t_1)$, thus violating the hypothesis $\{x_j\} \subset \chi \backslash closure(\mathfrak{R}(x_0, t_0, t_1))$.

Given this, a topological result states that there exists a separating hyperplane $P(t_1)$ such that vector v_0 is contained in a half-space and the cone is contained in the other one.
Now take a the vector called $\lambda_*(t_1)$ orthogonal to $P(t_1)$ lying in the half space not containing the cone, and thus

$$\langle \lambda_*(t_1), v \rangle \leq 0; \quad v \in K(\mu_*, x_0, t_0, t_1).$$

To conclude, we just need to take the unique adjoint response that has the value $\lambda_*(t_1)$ at time t_1, and use the preceding theorem 3.5.3: the supporting hyperplane for the cone exists a certain instant, so there exists an adjoint response that maximizes the Hamiltonian with the given control. □

Interpretation In the following figures we can have a deeper insight of the passages and geometrical constructions behind this theorem. In the left one from Figure4 one can see that the adjoint response points somehow "outside" of the reachable set. Indeed, if the objective is staying on the boundary of the expanding reachable set, it sounds logical to choose a control, and thus an $f(\xi, \mu)$ that "points outside the reachable set, as much as it can". This effort is represented in the maximization of the Hamiltonian, which is by the maximization of the projection of the possible f and λ, as depicted in Figure 3.
Important remark:the hyperplane actually separates the vector v_0 from the cone K, so the it does not have to be tangent to the reachable set, as seen in the central part of the figure.
Finally, the last of the three figures shows us the evolution of the adjoint response, of the separating plane P (which are guaranteed to exists in the past instants) and of the dotted trajectory.



(a) "Zoom" on the cone, part of the reachable set and final point of the trajectory at the final instant.

(b) The separating hyperplane must not be necessarily tangent to the reachable set.

(c) Depiction of the situation at two different instants, the final one and one intermediate, τ.

Figure 4:

There is a last theorem for this section, which states that once a system is fallen from the border of the reachable set, it remains in its internal for all the successive instants. Basically, in a crowd of runners where everybody runs at the (same and uniform) maximum speed, once a runner diminishes the speed for an amount of time that is not Lebesgue-neglettable, it will fall inside the crowd, and won't be able to return on the border anymore.

Trajectories in the interior of the reachable set remain in its interior

Theorem 3.6.2. *Let's have the usual control system, initial condition, time interval and admissible control. If, for a certain instant* $\tau \in [t_0, t_1)$ *it happens that* $\xi(\mu, x_0, t_0, \tau) \in int(\mathfrak{R}(x_0, t_0, \tau))$ *then*$\xi(\mu, x_0, t_0, t) \in int(\mathfrak{R}(x_0, t_0, t))$ *for all* $t \in (\tau, t_1]$.

3.7 Extended system, extended reachable set and optimal trajectories

Before stating the penultimate theorem, we briefly define one last object, the extended system. It is just the extension from n to n+1 dimensions for the state of the system, where the new, 0-th dimension is the value of the cost function so far.

Definition: Extended System Let L be a Lagrangian for control system $\Sigma = (\chi, f, U)$. We define the **extended system** as $\hat{\Sigma} = (\hat{\chi}, \hat{f}, U)$ just by asking that

- $\hat{\chi} = \mathbb{R} \times \chi$,
- $\hat{f}((x^0, x), u) = (L(x, u), f(x, u))$.

Note that now the equations governing the extended system are

$$\dot{\xi}^0(t) = L(\xi(t), \mu(t))$$
$$\dot{\xi}(t) = f(\xi(t), \mu(t))$$
$$\Longrightarrow$$
$$\xi^0(\tau) = \int_{t_0}^{\tau} L(\xi(t), \mu(t))dt.$$

This is precisely the cost function.
There now will be an important theorem, that lies an important layer in the proof of the principle. In fact, so far, the optimality of a trajectory has always been related to the maximization of an Hamiltonian, as is stated in the proof of the principle. But, looking deeper into the previous theorems, the maximization of the Hamiltonian just ensures that the trajectory lies on the boundary of the reachable set (on the "frontier of runners", so to say). The fact that a trajectory runs through the state space as fast as possible possible has nothing to do, a-priori, with the paid cost associated to that controlled trajectory.

Optimal trajectories lie on the boundary of the extended reachable set

Theorem 3.7.1. *Let L be a Lagrangian for control system* $\Sigma = (\chi, f, U)$*,* $S_0, S_1 \subset \chi$ *be sets, and suppose that* (ξ_*, μ_*) *is a solution to the fixed interval problem (which is,* $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$*), then* $\hat{\xi}_*(t_1) \in boundary(\hat{\mathfrak{R}}(\hat{\xi}_*(t_0), t_0, t_1))$.

Proof. Since $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$, then the corresponding extended trajectory has the obvious property that the cost $\xi^0(t_1)$ is the lowest possible among all the possible controlled arcs (ξ, μ) that steer the system from $\xi_*(t_0)$ to $\xi_*(t_1)$. So given the set of possible costs , the first element of $\hat{\xi}_*(t_1)$ is obviously on its boundary (at least, as long as the trajectories are L-acceptable, which is, the cost function is finite). This must then imply that also $\hat{\xi}_*(t_1)$ is on the boundary of the extended reachable set: let's take a neighbourhood of $\hat{\xi}_*(t_1)$ in the extended state space $\hat{\chi}$. Any neighbourhood will then contain points of the form $(c, \xi_*(t_1))$ with $c < \xi^0_*(t_1)$. But those points, with a cost which is littler than the minimum one, are not in the extended reachable set by hypothesis, since the controlled trajectory (ξ_*, μ_*) is optimal. So, since there is no neighbourhood of $\hat{\xi}_*(t_1)$ contained in $\hat{\mathfrak{R}}(\hat{\xi}_*(t_0), t_0, t_1)$, this point must lie on the boundary, as wanted. $\square$

3.8 Adjoint response, Hamiltonian and maximum principle

Finally, we will now prove the points 1 to 3 in the statement of the principle.

Theorem 3.8.1. *Let L be a Lagrangian for control system $\Sigma = (\chi, f, U)$, $S_0, S_1 \subset \chi$ be sets, and suppose that (ξ_*, μ_*) is a solution to the fixed interval problem (which is, $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$), then there exists an absolutely continuous map $\lambda_* : [t_0, t_1] \to \mathbb{R}^n$ and a number $\lambda_*^0 \in \{-1, 0\}$ with the following properties:*

1. *either $\lambda_*^0 = -1$ or $\lambda_*(t_0) \neq 0$,*
2. *λ_* is an adjoint repsonse for $(\Sigma, \lambda_*^0 L)$ along (ξ_*, μ_*),*
3. *$H_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$.*

Proof. First, we observe that the vector $(-1, 0) \in \mathbb{R} \times \mathbb{R}^n$ cannot lie in the interior of the extended tangent cone $\hat{K}(\mu, \hat{x}_0, t_0, t_1)$. If this were not the case, by means of theorem 3.5.1, there would be points $(a^0, a) \in \hat{\mathfrak{R}}(\hat{\xi}_*(t_0), t_0, t_1)$ at a lower cost, whose state is the same, at the same time, as the optimal trajectory $(a = \xi_*(t_1))$. This would violate the optimality of (ξ_*, μ_*).
Since this vector does not lie in the interior of $\hat{K}(\mu, \hat{x}_0, t_0, t_1)$, there exists a separating hyperplane $\hat{P}(t_1)$ between this cone and the vector.
Take a vector $\hat{\lambda}_*(t_1)$ orthogonal to $\hat{P}(t_1)$, in the half space not containing the cone. This implies

$$\langle \hat{\lambda}_*(t_1), (-1, 0) \rangle \geq 0,$$
$$\langle \hat{\lambda}_*(t_1), \hat{v} \rangle \leq 0; \quad \hat{v} \in \hat{K}(\mu, \hat{x}_0, t_0, t_1)$$

This implies then that $\hat{\lambda}_*^0(t_1) \leq 0$.
Define then $\hat{\lambda}_*$ as the adjoint response whose value at t_1 is equal to $\hat{\lambda}_*(t_1)$.
From the equations for the adjoint response (3.2) we obtain that $\dot{\lambda}_*^0(t) = 0$. This is because since $\hat{f}$ does not depend on x_0, then the Jacobian matrix $D_1 \hat{f}$ has the first column 0, and thus the transposed matrix has the first row of zeros. This is of course by imagining the (extended) adjoint response as a coloumn vector.
So $\lambda_*^0(t) = 0$ is constant and nonpositive. So, if $\lambda_*^0(t) \neq 0$, we can rescale the whole extended adjoint response so that $\hat{\lambda}_*$ becomes $\frac{\hat{\lambda}_*}{-(\lambda_*^0)}$, so that now λ_*^0 is either 0 or -1.
The Hamiltonian for the extended system is very similar to the extended Hamiltonian for the system (plus the Lagrangian):

$$\hat{H}_{\hat{\Sigma}}\left((x^0, x), (p^0, p), u\right) = \langle p, f(x, u) \rangle + p^0 L(x, u) = H_{\Sigma, p^0 L}(x, p, u)$$

Now, since the trajectory is optimal, by virtue of theorem 3.7.1, it lies on the boundary of the extended reachable se at time t_1. Then it hold s true that $H_{\Sigma, p^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma, p^0 L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$.
The fact that at least one between , $\lambda_*(t)$ and λ_*^0 is not zero derives from the linearity of the adjoint equation. □