Pontryagin's maximum principle

Theory summary and applications

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31-10-2019

1 Preliminary definitions

1.1 Control System

A **control system** is a triple $\Sigma = (\chi, f, U)$, where

1. χ, representing the state of the system, is an **open** subset of $\mathbb{R}^n, \chi \subset \mathbb{R}^n$.
2. U, representing the space of possible (*istantaneous*)controls, is an open subset of $\mathbb{R}^m, U \subset \mathbb{R}^m$
3. $f : \chi \times cl(U) \to f(x, u)$ is a function which dictates the law with which the system evolves. f
   (a) is continuous
   (b) the map $x \to f(x, u)$ is of class C^1 for each $u \in cl(U)$

Since f is function of the current state of the system and of the current control, the system's evolution does not explicitly depends on time. Of course, the law dictating the evolution of the system's status is

$$\dot{\xi}(t) = f(\xi(t), \mu(t)) \tag{1.1}$$

where obviously, for for each $t, \xi(t) \in \chi$ is the "current" (at time t)state of the system, and $\mu(t) \in U$ is the current (at time t) control, dictated by the control law $\mu(t)$.

1.2 Control and Trajectories

The idea here is that we want some limitations on the function $t \to \mu(t)$, because, starting from a certain state, we want the control to originate, through 1.1, trajectories that, at least, "makes sense". So, given a control system $\Sigma = (\chi, f, U)$we define

- An **admissible control** is a *measurable* map $\mu : I \to U$ where I is a (time) intervale $I \subset \mathbb{R}$, and such that $t \to f(x, \mu(t))$ is locally integrable for ceach $x \in \chi$
- we denote the set of admissible control defined on time interval I by $\mathfrak{U}(I)$
- A **controlled trajectory** is a pair (ξ, μ) where, for some time intervale $I \in \mathbb{R}$
  - $\mu \in \mathfrak{U}(I)$ is then a function expressing the control through which the system is driven in the time interval, and is an admissible control. This map will be simply be referred to as the **control**.
  - $\xi : I \to \chi$ is the map linking the times in the interval to their corresponding state, which follows the law 1.1. This map will be referred to as the **trajectory**.
- a **controlled arc** is a controlled trajectory defined on a compact time interval.

The set of the controlled trajectories for a given control system $\Sigma = (\chi, f, U)$is named Ctraj(Σ), the set of the controlled arcs for the control system is named Carc(Σ)

1.3 Lagrangian, costs and optimal control problem(s)

Since we will want to optimize a cost, we first have to define this objective function which has to be minimized. This will be the integral of another function, the lagrangian, along the path of the system in its evolution from the *beginning* to the *end* of the syof another function (Note: the integral is actually calculated by integrating on the time interval associated with the a controlled trajectory, so it's integrated on an interval of $\mathbb{R}$, not along a path in $\mathbb{R}^n$). So, given a control system $\Sigma = (\chi, f, U)$

- A **Lagrangian**for $\Sigma$is a function $L : \chi \times cl(U) \to \mathbb{R}$ such that
  - L is continuous
  - the function $x \to L(x, u)$ is of class C^1 for each $u \in cl(U)$
- given a Lagrangian L, it is said that a controlled trajectory $(\xi, \mu)$with relative time interval $I$is **L-acceptable** if the function $t \to L(\xi(t), \mu(t))$ is integrable.
- given a Lagrangian L, the corresponding **objective function** is the map $J_{\Sigma,L} : Ctraj(\Sigma) \to \overline{\mathbb{R}}$ given by

$$J_{\Sigma,L}(\xi, \mu) = \int_I L(\xi(t), \mu(t))dt \tag{1.2}$$

  where we pose $J_{\Sigma,L} = \infty$ if $(\xi, \mu)$is not L-acceptable (if it's not integrable). The set of L-acceptable controlled trajectories (arcs) for the control system is denoted as Ctraj(Σ, L) (or Carc(Σ, L))

The idea here is that one should seek to minimize the objective function, with the "variable" to be tuned being the controlled trajectory. Usually though the problem faced is such that the system will start his evolution in a certain initial state, which resides in a set of possible initial conditions S_0, and some end conditions will be given, which means that in the end, the state of the system should reside in another set, S_1. Of course $S_0, S_1 \subset \chi$. We thus call $\mathrm{Carc}(\Sigma, L, S_0, S_1)$ the set of controlled arcs for the control system $\Sigma = (\chi, f, U)$with Lagrangian L, which have also the following properties

- every (ξ, μ)in $\mathrm{Carc}(\Sigma, L, S_0, S_1)$is defined on a time interval of the form $[t_0, t_1]$ with $t_0 < t_1; t_0, t_1 \in \mathbb{R}$
- every (ξ, μ)in $\mathrm{Carc}(\Sigma, L, S_0, S_1)$then the said controlled arc is also in $\mathrm{Carc}(\Sigma, L)$, which means it is an L-acceptable controlled arc
- every (ξ, μ)in $\mathrm{Carc}(\Sigma, L, S_0, S_1)$defined on the time interval $[t_0, t_1]$ then $\chi(t_0) \in S_0$ and $\chi(t_1) \in S_1$.

Now we can precisely define the optimization problem. There are actually two of these problem, depending on the fact that t_0, t_1 may or may not be fixed. We are going to consider the demonstration for the fixed interval one. So,

Free interval optimal control problem Let's have

- a control system $\Sigma = (\chi, f, U)$,
- a Lagrangian L,
- $S_0, S_1 \in \chi$ sets,

then a controlled trajectory $(\xi_*, \mu_*) \in$Carc$(\Sigma, L, S_0, S_1)$is a **solution to the free interval optimal control problem** if
$\forall(\xi, \mu) \in$Carc(Σ, L, S_0, S_1), $J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted as $\mathfrak{P}(\Sigma, L, S_0, S_1)$.

Fixed interval optimal control problem Let's have

- a control system $\Sigma = (\chi, f, U)$,
- a Lagrangian L,
- $S_0, S_1 \in \chi$ sets,
- a time interval $[t_0, t_1]$ with $t_0 < t_1; t_0, t_1 \in \mathbb{R}$

then a controlled trajectory $(\xi_*, \mu_*) \in$Carc$(\Sigma, L, S_0, S_1, [t_0, t_1])$is a **solution to the fixed interval optimal control problem** if
$\forall(\xi, \mu) \in$Carc$(\Sigma, L, S_0, S_1, [t_0, t_1])$, $J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted as $\mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$.

A simple example The problem in which the cost is the time with which the system is driven from $S_0$to $S_1$is simply a free interval optimal control problem, in which there is a control system with Lagrangian $L(x,u) = 1$

2 Statement of the maximum Principle

3 Hint of demonstration

4 Moon landing problem