Pontryagin's maximum principle
Theory summary and applications

Emanuele Pallaoro

31-10-2019

1 Preliminary definitions

1.1 Control System

A **control system** is a triple $\Sigma = (\chi, f, U)$, where

1. χ, representing the state of the system, is an **pen** subset of $\mathbb{R}^n, \chi \subset \mathbb{R}^n$.

2. U, representing the space of possible (*istantaneous*)controls, is an open subset of $\mathbb{R}^m, U \subset \mathbb{R}^m$

3. $f : \chi \times cl(U) \to f(x, u)$ is a function which dictates the law with which the system evolves. f

   (a) is continuous

   (b) the map $x \to f(x, u)$ is of class C^1 for each $u \in cl(U)$

Since f is function of the current state of the system and of the current control, the system's evolution does not explicitly depends on time. Of course, the law dictating the evolution of the system's status is

$$\dot{\xi}(t) = f(\xi(t), \mu(t)) \tag{1.1}$$

where obviously, for for each $t, \xi(t) \in \chi$ is the "current" (at time t)state of the system, and $\mu(t) \in U$ is the current (at time t) control, dictated by the control law $\mu(t)$.

1.2 Control and Trajectories

The idea here is that we want some limitations on the function $t \to \mu(t)$, because, starting from a certain state, we want the control to originate, through 1.1, trajectories that, at least, "makes sense". So, given a control system $\Sigma = (\chi, f, U)$we define

- An **admissible control** is a *measurable* map $\mu : I \to U$ where I is a (time) intervale $I \subset \mathbb{R}$, and such that $t \to f(x, \mu(t))$ is locally integrable for ceach $x \in \chi$

- we denote the set of admissible control defined on time interval I by $\mathfrak{U}(I)$

- A **controlled trajectory** is a pair (ξ, μ) where, for some time intervale $I \in \mathbb{R}$

    - $\mu \in \mathfrak{U}(I)$ is then a function expressing the control through which the system is driven in the time interval, and is an admissible control. This map will be simply be referred to as the **control**.

    - $\xi : I \to \chi$ is the map linking the times in the interval to their corresponding state, which follows the law 1.1. This map will be referred to as the **trajectory**.

- a **controlled arc** is a controlled trajectory defined on a compact time interval.

The set of the controlled trajectories for a given control system $\Sigma = (\chi, f, U)$is named Ctraj(Σ), the set of the controlled arcs for the control system is named Carc(Σ)

1.3   Lagrangian, costs and optimal control problem(s)

Since we will want to optimize a cost, we first have to define this objective function which has to be minimized. This will be the integral of another function, the lagrangian, along the path of the system in its evolution from the *beginning* to the *end* of the syof another function (Note: the integral is actually calculated by integrating on the time interval associated with the a controlled trajectory, so it's integrated on an interval of $\mathbb{R}$, not along a path in $\mathbb{R}^n$). So, given a control system $\Sigma = (\chi, f, U)$

- A **Lagrangian** for Σ is a function $L : \chi \times cl(U) \to \mathbb{R}$ such that

  – L is continuous

  – the function $x \to L(x, u)$ is of class C^1 for each $u \in cl(U)$

- given a Lagrangian L, it is said that a controlled trajectory (ξ, μ) with relative time interval I is **L-acceptable** if the function $t \to L(\xi(t), \mu(t))$ is integrable.

- given a Lagrangian L, the corresponding **objective function** is the map $J_{\Sigma,L} : Ctraj(\Sigma) \to \overline{\mathbb{R}}$ given by

$$J_{\Sigma,L}(\xi, \mu) = \int_I L(\xi(t), \mu(t))dt \qquad (1.2)$$

  where we pose $J_{\Sigma,L} = \infty$ if (ξ, μ) is not L-acceptable (if it's not integrable).
  The set of L-acceptable controlled trajectories (arcs) for the control system is denoted as $Ctraj(\Sigma, L)$ (or $Carc(\Sigma, L)$)

The idea here is that one should seek to minimize the objective function, with the "variable" to be tuned being the controlled trajectory. Usually though the problem faced is such that the system will start his evolution in a certain initial state, which resides in a set of possible initial conditions S_0, and some end conditions will be given, which means that in the end, the state of the system should reside in another set, S_1. Of course $S_0, S_1 \subset \chi$. We thus call $Carc(\Sigma, L, S_0, S_1)$ the set of controlled arcs for the control system $\Sigma = (\chi, f, U)$ with Lagrangian L, which have also the following properties

- every (ξ, μ) in $Carc(\Sigma, L, S_0, S_1)$ is defined on a time interval of the form $[t_0, t_1]$ with $t_0 < t_1; t_0, t_1 \in \mathbb{R}$

- every (ξ, μ) in $Carc(\Sigma, L, S_0, S_1)$ then the said controlled arc is also in $Carc(\Sigma, L)$, which means it is an L-acceptable controlled arc

- every (ξ, μ) in $Carc(\Sigma, L, S_0, S_1)$ defined on the time interval $[t_0, t_1]$ then $\chi(t_0) \in S_0$ and $\chi(t_1) \in S_1$.

Now we can precisely define the optimization problem. There are actually two of these problem, depending on the fact that t_0, t_1 may or may not be fixed. We are going to consider the demonstration for the fixed interval one. So,

Free interval optimal control problem   Let's have

- a control system $\Sigma = (\chi, f, U)$,

- a Lagrangian L,

- $S_0, S_1 \in \chi$ sets,

then a controlled trajectory $(\xi_*, \mu_*) \in Carc(\Sigma, L, S_0, S_1)$ is a **solution to the free interval optimal control problem** if
$\forall (\xi, \mu) \in Carc(\Sigma, L, S_0, S_1), J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted as $\mathfrak{P}(\Sigma, L, S_0, S_1)$.

Fixed interval optimal control problem   Let's have

- a control system $\Sigma = (\chi, f, U)$,

- a Lagrangian L,

- $S_0, S_1 \in \chi$ sets,

- a time interval $[t_0, t_1]$ with $t_0 < t_1; t_0, t_1 \in \mathbb{R}$

then a controlled trajectory $(\xi_*, \mu_*) \in \mathrm{Carc}(\Sigma, L, S_0, S_1, [t_0, t_1])$ is a **solution to the fixed interval optimal control problem** if
$\forall (\xi, \mu) \in \mathrm{Carc}(\Sigma, L, S_0, S_1, [t_0, t_1]), J_{\Sigma,L}(\xi_*, \mu_*) < J_{\Sigma,L}(\xi, \mu)$.
The set of all the possible solutions is denoted as $\mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$.

A simple example   The problem in which the cost is the time with which the system is driven from S_0 to S_1 is simply a free interval optimal control problem, in which there is a control system with Lagrangian $L(x, u) = 1$.

1.4   Hamiltonians

The maximum principle is related with the maximization of a Hamiltonian associated with a control system with a certain Lagrangian, so we have the following definitions.
Let $\Sigma = (\chi, f, U)$ be a control system, L a Lagrangian, then

- the **Hamiltonian** is the function $H_\Sigma : \chi \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H_\Sigma(x, p, u) = \langle p, f(x, u) \rangle$$

- the **extendend Hamiltonian** is the function $H_{\Sigma,L} : \chi \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H_\Sigma(x, p, u) = \langle p, f(x, u) \rangle + L(x, u) = H_\Sigma(x, p, u) + L(x, u)$$

- the **maximum Hamiltonian** is the function $H_\Sigma^{max} : \chi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$H_\Sigma^{max} = sup\{H_\Sigma(x, p, u) | u \in U\}$$

- the **maximum extended Hamiltonian** is the function $H_{\Sigma,L}^{max} : \chi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$H_{\Sigma,L}^{max} = sup\{H_{\Sigma,L}(x, p, u) | u \in U\}$$

- the variable p is sometimes called **costate**

1.5   Adjoint response

There are another two quantities, namely the adjoint response and the control variations (with the associated variational equation), which are very important in the understanding of the maximum principle. The part relative to adjoint response will now be stated, because it appears in the statement of the principle, while the part concerning brings along concepts that are useful to the demonstration and to the understanding of the geometrical significance of the principle.
So now let $\Sigma = (\chi, f, U)$ be a control system, and

- $(\xi, \mu) \in Ctraj(\Sigma)$ be a controlled trajectory with time interval I, then we define the **adjoint response** for Σ along (ξ, μ) as a locally absolutely continous map $\lambda : I \to \mathbb{R}^n$ which also satisfies the following differential equation(s):

$$\dot{\xi}(t) = \boldsymbol{D}_2 H_\Sigma(\xi(t), \lambda(t), \mu(t)) \ (= f(x, u))$$
$$\lambda(t) = -\boldsymbol{D}_1 H_\Sigma(\xi(t), \lambda(t), \mu(t))$$

(1.3)

we will see later that this last equation can reach another equivalent form.

- if L is a Lagrangian and $(\xi, \mu) \in Ctraj(\Sigma, L)$is an L-acceptable controlled trajectory with time interval I, we then define the **adjoint response** for (Σ, L) along (ξ, μ)as a locally absolutely continous map $\lambda : I \to \mathbb{R}^n$ which also satisfies the following differential equation(s):

$$\dot{\xi}(t) = \boldsymbol{D}_2 H_{\Sigma, L}(\xi(t), \lambda(t), \mu(t)) \ (= f(x, u))$$
$$\lambda(t) = -\big(D_1 H_{\Sigma, L}(\xi(t), \lambda(t), \mu(t)\big)$$

(1.4)

1.6 Smooth constraint sets

Part of the maximum principle deals with the case in which $S_0$and $S_1$are "smooth", so we might define a **smooth constraint set S** as a subset of the set of possible (control)system states $S \subset \chi$ such that there exists a C^1 function $\Phi : \chi \to \mathbb{R}^k$, such that $S = \Phi^{-1}(0)$ and also $\boldsymbol{D}\Phi(x)$ is surjective for each $x \in S$

1.7 Reachable set

Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then define

- the **reachable set** from x_0 at t_0 in time $t_1 - t_0$ as

$$\mathfrak{R}(x_0, t_0, t_1) = \{\xi(\mu, x_0, t_0, t_1) | \mu \in \mathfrak{U}([t_0, t_1])\}$$

- the **reachable set** from x_0 at t_0 as

$$\mathfrak{R}(x_0, t_0) = \cup_{t_1 \in [t_0, \infty]} \mathfrak{R}(x_0, t_0, t_1)$$

Little note: since f does not depend explicitly on time, then $\mathfrak{R}(x_0, t_0, t_1)$= $\mathfrak{R}(x_0, 0, t_0)$.

2 Statement of the maximum Principle

2.1 Maximum principle for free interval problems

Let $\Sigma = (\chi, f, U)$be a control system,, L a Lagrangian, $S_0$and S_1subsets of χ. A necessary condition for a controlled trajectory (ξ_*, μ_*) defined on $[t_0, t_1]$ to be optimal, that is, a necessary condition so that $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1)$ is the existence of an absolutely continuous map $[t_0, t_1] : [t_0, t_1] \to \mathbb{R}^n$ and of $\lambda_*^0 \in \{-1, 0\}$ that have also the following properties

1. either $\lambda_*^0 = .1$ or $\lambda_*(t_0) \neq 0$

2. λ_* is an adjoint repsonse for (Σ, λ_*^0, L) along (ξ_*, μ_*)

3. $H_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H_{\Sigma, \lambda_*^0 L}^{max}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$

   If μ_* is bounded, then

4. $\forall t \in [t_0, t_1] \ H_{\Sigma, \lambda_*^0 L}^{max}(\xi_*(t), \lambda_*(t)) = 0$

   If, $S_1$and $S_0$are smooth constraint sets, then $[t_0, t_1]$ can be chosen such that

5. $\lambda_*(t_0)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_0(\xi(t_0)))$ and $\lambda_*(t_1)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_1(\xi(t_1)))$

For the fixed interval problem only condition 4 is lost.

2.2 Maximum principle for fixed interval problems

Let $\Sigma = (\chi, f, U)$ be a control system,, L a Lagrangian, S_0 and S_1 subsets of χ; $[t_0, t_1] \subset \mathbb{R}$ an interval. A necessary condition for a controlled trajectory (ξ_*, μ_*) defined on $[t_0, t_1]$ to be optimal, that is, a necessary condition so that $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$ is the existence of an absolutely continuous map $[t_0, t_1] : [t_0, t_1] \to \mathbb{R}^n$ and of $\lambda_*^0 \in \{-1, 0\}$ that have also the following properties

1. either $\lambda_*^0 = .1$ or $\lambda_*(t_0) \neq 0$

2. λ_* is an adjoint repsonse for (Σ, λ_*^0, L) along (ξ_*, μ_*)

3. $H_{\Sigma, \lambda_*^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H_{\Sigma, \lambda_*^0 L}^{max}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$

   If, S_1 and S_0 are smooth constraint sets, then $[t_0, t_1]$ can be chosen such that

4. $\lambda_*(t_0)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_0(\xi(t_0)))$ and $\lambda_*(t_1)$ is orthogonal to $\ker(\boldsymbol{D}\Phi_1(\xi(t_1)))$

3 Hint of demonstration

the first step we need to take is analyzing the effect of varying a trajectory first. In general, one can expect to vary trajectory followed by the state of a control system in two ways: given a control, varying the initial conditions or, given the initial conditions, varying the control. Nevertheless it is still necessary to develop some tools to *describe* the variation of the trajectory in some ways.

3.1 Variations and adjoint response

Variational and adjoint equations
Given a control system $\Sigma = (\chi, f, U)$ and am admissible control $\mu : I \to U$ we have

- the **variational equation** equation for Σ with control μ is the differential equation

$$\dot{\xi}(t) = f(\xi(t), \mu(t));$$
$$\dot{v}(t) = \boldsymbol{D}_1 f(\xi(t), \mu(t)) \cdot v(t) \qquad (3.1)$$
$$(\xi(t), \mu(t)) \in (\chi \times \mathbb{R}^n)$$

- the **variational equation** equation for Σ with control μ is the differential equation

$$\dot{\xi}(t) = f(\xi(t), \mu(t));$$
$$\dot{\lambda}(t) = -\boldsymbol{D}_1 f^T(\xi(t), \mu(t)) \cdot v(t) \qquad (3.2)$$
$$(\xi(t), \lambda(t)) \in (\chi \times \mathbb{R}^n)$$

interpretation It is straightforward to see that the variational equation describes, through a linearization, the evolution in time of a small (infinitesimal) variation from the original trajectory $\xi(t)$, which is one of the solutions to the equation 1.1.
Obviously, it is hoped that, given a certain control and initial condition, the solution to 1.1 is unique, but a-priori it cannot be said.
The geometrical interpretation of the adjoint equation is more subtle, but, in a naive way, one could say that, given an optimal trajectory, the adjoint response is a vector orthogonal to the hyerplane given by (directions of the)the possibile (infinitesimal) variations to that trajectory.

3.2 Variations and infinitesimal variations

Definitions
Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. a **variation** of the trajectory $\xi(\mu, x_0, t_0, \cdot)$ is map $\sigma : J \times [t_0, t_1] \to \chi$ with the following properties

- $J \subset \mathbb{R}; 0 \in int(J)$ is an interval for which zero reside in its internal part

- $\sigma(0,t) = \xi(\mu, x_0, t_0, t)$ for each $t \in [t_0, t_1]$

- $s \rightarrow \sigma(s,t)$ is of class C^1 for each $t \in [t_0, t_1]$

- $t \rightarrow \sigma(s,t$ is a solution of eq. 1.1.

Given a variation and it's relative "original" trajectory, there is another important quantity, which is he corresponding **infinitesimal variation**. This is yet another map defined with the following limit

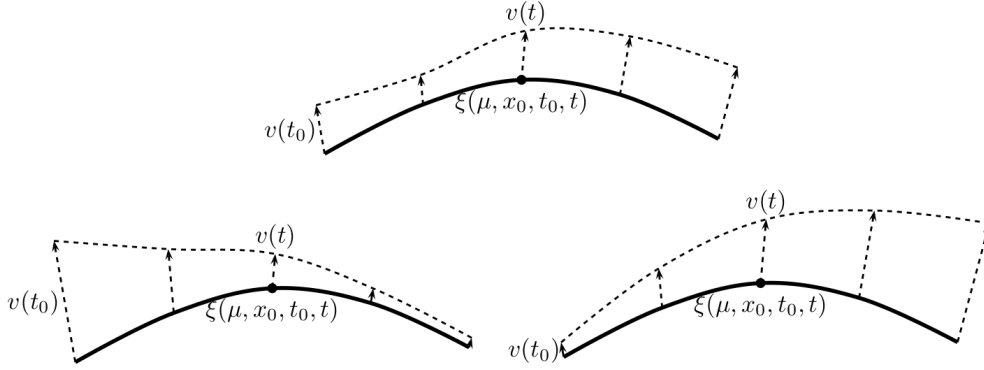$$\delta\sigma(t) = \left.\frac{d}{ds}\right|_{s=0} \sigma(s,t). \tag{3.3}$$

Figure 1:

The dotted arrow represent the difference between the variation (at various times and at a given s), and the "original" trajectory, at the same time. The infinitesimal variation is then represented by a vector such that, at a given time $\bar{t}, \sigma(s,\bar{t}) \approx \xi(\mu, x_0, t_0, \bar{t}) + \delta\sigma(\bar{t}) * s$ Actually, there is another interpretation to the image, in which the dotted arrow are the vectors represent the **infinitesimal** variations, and the dotted line is just the envelope of the dotted arrows calculated at different times. In this case, the dotted part and the continuous-line of the figure have scales that have nothing to do one with each other. Nevertheless, in both interpretations, it is possible to understand the concept of a stable, unstable and "indifferent" trajectory, in which a disturbance in the trajectory is amplified rather than muted.

A theorem about infinitesimal variations Here it will be stated that if a map is solution of equation 3.1 then it is an infinitesimal variation , and vice versa.
Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Given a map $v : [t_0, t_1] \rightarrow \mathbb{R}^n$ the statements

- there exists a map σ which is a variation of $\xi(\mu, x_0, t_0, \cdot)$ such that $v = \delta\sigma$;

- $t \rightarrow (\xi(\mu, x_0, t_0, t), v(t))$ satisfies the variational equation 3.1.

are equivalent.

The foundamental matrix Φ During the demonstration of the preceding theorem, an $n \times n$ linear map $\Phi(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has been defined, as a map such that $\Phi(t) \cdot = \boldsymbol{D}_2\xi(\mu, x_0, t_0, t) \cdot w$. By differentiation one can see that $\Phi(t)$ satisfies this matrix differential initial value problem:

$$\dot{\Phi}(t) = \boldsymbol{D}_1 f(\xi(\mu, x_0, t_0, t), \mu(t)) \circ; \ \Phi(t_0) = Id_{n \times n}$$

We thus define $\boldsymbol{\Phi}(\boldsymbol{\mu}, \boldsymbol{x_0}, \boldsymbol{t_0}, \boldsymbol{\tau}, \boldsymbol{t})$ as the solution to the matrix initial value problem

$$\dot{\Phi}(t) = \boldsymbol{D}_1 f(\xi(\mu, x_0, t_0, t), \mu(t)) \circ; \ \Phi(\tau) = Id_{n \times n} ...$$

Where of course $t, \tau \in [t_0, t_1]$.

The importance of this matrix resides in the fact that, being the variational equation 3.1 a linear one and being Φ the foundamental matrix of the equation, it is true that, for a variation $v(t)$ (which is also a solution to the variational equation, because of the theorem), $v(t) = \Phi(\mu, x_0, t_0, t_0, t) \cdot v(t_0)$ Φ also has some properties, about the composition of the

3.3 Properties of adjoint repsonse

What this theorem says is that seeing the adjoint repsonse as a solution of the adjoint equation 3.2 and of the adjoint equation defined with the hamiltonians 1.3 is the same exact thing.

Hamilton's and adjoint equation
Theorem: given a control system $\Sigma = (\chi, f, U)$, an admissible control $\mu : I \to U$,and two maps $\xi : I \to \chi; \lambda : I \to \mathbb{R}^n$, those two statements are equivalent:

- the curve $t \to (\xi(t), \lambda(t))$ satisfies the adjoint equation 3.2

- the curve $t \to (\xi(t), \lambda(t))$ satisfies the differential equation 1.3 recalled here:

$$\dot{\xi}(t) = \boldsymbol{D}_2 H_\Sigma(\xi(t), \lambda(t), \mu(t))$$
$$\lambda(t) = -\boldsymbol{D}_1 H_\Sigma(\xi(t), \lambda(t), \mu(t))$$

This theorem is demonstrated simpy by differentiating the given quantities, and by using the properties of dot product.

Adjoint response and variations
Theorem Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. If v and λ are two maps $v, \lambda : [t_0, t_1] \to \mathbb{R}^n$ that satisfy the variational and adjoint equation respectively (toghether with the trajectory $t \to \xi(\mu, x_0, t_0, t)$), then the standard scalar product between those two vectors is constant with time, which is

$$\langle \lambda(t), v(t) \rangle = \langle \lambda(t_0), v(t_0) \rangle$$

Proof:

$$\frac{d}{dt} \langle \lambda(t), v(t) \rangle =$$
$$= \langle \dot{\lambda}(t), v(t) \rangle + \langle \lambda(t), \dot{v}(t) \rangle =$$
$$= -\langle \boldsymbol{D}_1 f^T((\xi, \mu)) \cdot \lambda(t), v(t) \rangle + \langle \lambda(t), \boldsymbol{D}_1 f^T((\xi, \mu)) \cdot v(t) \rangle =$$
$$= 0$$

A corollary An important consequence of this fact is that if the two vectors v, λ are orthogonal at the initial moment, they remain orthogonal throughout the whole evolution of the system.

Foundamental matrix for the adjoint response **Theorem** Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$, and let $\tau \in [t_0, t_1]$. Then the solution of the initial value problem

$$\dot{\lambda}(t) = -\boldsymbol{D}_1 f^T(\xi(t), \mu(t)) \cdot v(t), \ \lambda(\tau) = \lambda_\tau$$

is the following one:

$$t \to \lambda(t) = \Phi(\mu, x_0, t_0, t, \tau)^T \cdot \lambda_\tau$$

3.4 Needle variations

In the beginning of section 3 we talked about the variations of trajectories, and analyzed those variations *a-posteriori*, trying to describe them. Now we are going to see, so to say, *the elementary (infinitesimal) way of producing said variations.* This will be of course through control variations, not through the variation of initial condition.

Needle variation (fixed interval) et $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then define

- **fixed interval needle variation <u>data</u>** as a triple $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ for which

  - $\tau_\theta \in (t_0, t_1]$
  - $l_\theta \in \mathbb{R}_{\geq 0}$
  - $\omega_\theta \in U$

- the **control variation** of the control μ associated to the relative fixed interval needle variation data θ is the map μ_θ is the map $mu_\theta : J \times [t_0, t_1] \to U$ such that

$$\mu_\theta = \begin{cases} \omega_\theta & \text{if } t \in [\tau_\theta - s * l_\theta, \tau_\theta] \\ \mu(t) & \text{otherwise.} \end{cases}$$

  Where $J = [0, s_0]$ is an interval sufficiently small so that $\mu_\theta(s, t)$ is an admissible control for each $s \in J$ Just to have an idea, this is how the function μ_θ can look like for a certain $s > 0$
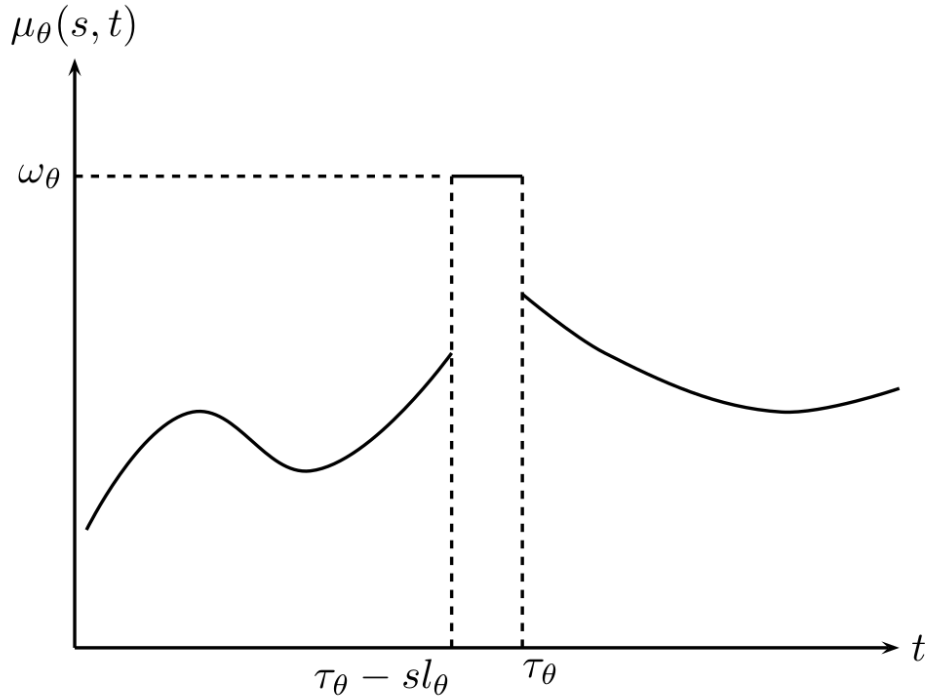

Figure 2:

- **fixed interval needle (*infinitesimal*)variation** associated with the control μ, the trajectory $\xi(\mu, x_0, t_0, \cdot)$ and the variation data θ as a vector of $\mathbb{R}^n$ defined as

$$v_\theta = \frac{d}{ds}\Big|_{s=0} \xi(\mu_\theta(s, \cdot), x_0, t_0, \tau_\theta),$$

  when such derivative exists

This limit exists at almost any instant, since it exists for every instant that is a Lebesgue point for $t \to f(\xi(\mu, x_0, t_0, t), \mu(t))$. Before stating this formally, we need the definition of $Leb(\mu, x_0, t_0, t)$: it's the set of Lebesgue points if $\tau \to f(\xi(\mu, x_0, t_0, \tau), \mu(\tau)), \tau \in (t_0, t)$. Then,

Theorem:existence and form of fixed interval needle variations
Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Let then $\theta = (\tau_\theta, l_\theta, \omega_\theta)$ be a fixed interval needle variation data, with $\tau_\theta \in Leb(\mu, x_0, t_0, t_1)$. Then the fixed interval variation associated with those data exists and it's given by

$$v_\theta = l_\theta * \left(f(\xi(\mu, x_0, t_0, \tau_\theta), \omega_\theta) - f(\xi(\mu, x_0, t_0, \tau_\theta), \mu(\tau_\theta)) \right)$$

Variations and cones The real importance of this theorem is not only in the fact that it is (almost) always possible to individuate the infinitesimal variation, but also in the fact that those variations form a cone, which is, if one vector represents a variation, then all of the half-line (originating from $0 \in \mathbb{R}^n$) given by that vector is made up of fixed interval variations. Formally said,

Proposition: Let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Let then $\theta = (\tau_\theta, l_\theta, \omega_\theta)$be a fixed interval needle variation data, with $\tau_\theta \in Leb(\mu, x_0, t_0, t_1)$.
Then, the set of fixed interval needle variation associated with the data θ form a cone.

Proof: It's just enough to say that, if v_θ is the variation associated with the data $\theta = (\tau_\theta, l_\theta, \omega_\theta)$, then, taken a $k \in \mathbb{R}_{\geq 0}, kv_\theta$ is the variation associated with the data $k\theta = (\tau_\theta, k * l_\theta, \omega_\theta)$. Using obvious notation, one could then say that $v_{k\theta} = kv_\theta$.

It is interesting to note that, in the triple representing the data, only the *"lenght of the disturbance"* gets multiplied by the scalar. This means, that, referring to 2, the final control is practically the same. In fact, the actual length of the horizontal step in the figure doesn't really matter, since s tends to zero, and the actual disturbance gets reduced to a (Lebesgue-neglettable) single point in time.
The only effect one may obtain is that, given a certain trajectory, the trajectories associated with varied control depart from the undisturbed one at a higher rate with growing s (at least, as long as s is small enough so that one can linearize the effect of control variation), but in the same direction. This is the same exact meaning of the length of the dotted arrows in 1.

3.5 Multi-needle variations

The concept behind multi needle variations is that they are made of single needle variations which sum up(linearly, as can be see in the relative theorem). We need this object basically because although single needle variations do form a cone, this is not generally convex. Convexity is needed in the proof, to find a hyperplane that separates the half space containing cone from another one. This is because a variational cone originating from the boundary of the reachable set cone will point toward the "less optimal", this half space is the non optimal one.

So now let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We want to define

- the **fixed interval multi-needle variation data** Θ as the collection $\Theta = \{\theta_1, .., \theta_k\}$ of fixed interval needle variation data $\theta_j = (\tau_j, l_j, \omega_j), j = 1, .., k$, with the times τ_j all distinct.

- the **control variation** of the control μ associated with the relative data Θ as the map $\mu_\Theta : J \times [t_0, t_1] \to U$ such that

$$\mu_\Theta = \begin{cases} \omega_j & \text{if } t \in [\tau_\Theta - s * l_\Theta, \tau_\Theta], j = 1, .., k \\ \mu(t) & \text{otherwise.} \end{cases}$$

Where $J = [0, s_0]$ is an interval sufficiently small so that $\mu_\Theta(s, t)$ is an admissible control for each $s \in J$.

- the **fixed interval multi-needle variation** associated with the control μ, the trajectory $\xi(\mu, x_0, t_0, \cdot)$, the time $t \in [t_0, t_1]$ taken such that $t > \max_j(\tau_j)$ and the variation data Θ as a vector of $\mathbb{R}^n$ (a vectorial function of time actually) defined like this:

$$v_\theta(t) = \left.\frac{d}{ds}\right|_{s=0} \xi(\mu_\Theta(s, \cdot), x_0, t_0, t),$$

when such derivative exists

Theorem: existence of multi-needle fixed interval variations

Let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. Then, let $\Theta = \{\theta_1, .., \theta_k\}$ be a multi needle variation data ("fixed interval" will from now on be omitted, since this case is the only one treated in this essay), Θ such that the times $\tau_j \in Leb(\mu, x_0, t_0, t)$, are all distinct, and also $t > \max_j(\tau_j)$.

Then, the multi needle variation is given by

$$v_\Theta(t) = \sum_{j=1}^{k} \Phi(\mu, x_0, t_0, \tau_j, t) \cdot v_{\theta_j}$$

As one can see, the variations caused by the various single needle variations first are let evolve in time from the instant in which they are applied to the "current" one, t, and then their effects sum up linearly.

Taking this into account, and also considering the fact that $v_{k\theta_j} = kv_{\theta_j}$ for whatever $k \geq 0$ real, and remembering that the matrix-vector product is linear with respect to scalar multiplication, one can easily get the following

Corollary: coned convex combination of multi needle variations With slight abuse of notation, given the usual control system, admissible control, initial condition, interval, given a multi needle variation data with its distinct times being Lebesgue points for f, and with a $t > \max_j(\tau_j)$, given a set $\lambda = \{\lambda_1, .., \lambda_k\} \subset \mathbb{R}_{\geq 0}$ then

$$v_{\lambda\Theta(t)} = \sum_{j=1}^{k} \lambda_j \Phi(\mu, x_0, t_0, \tau_j, t) \cdot v_{\theta_j}$$

3.6 Cones and reachable set

The proof of maximum principle will deal with the *"approximation of reachable set using cones"*, which is to say that at its boundary the reachable set will be confused, at least in a neighborhood of a frontier point, with a (variational) cone. The variations used will be both single and multi needle. We're going to see that the two are, in some sense, equal.
We just need two more definitions.
The first one is going to be the

Tangent cone As usual, let $\Sigma = (\chi, f, U)$ be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then take $t \in [t_0, t_1]$. Then we denote by $K(\mu, x_0, t_0, t)$ the coned convex hull of the following set:

$$\cup\{\Phi(\mu, x_0, t_0, \tau, t) \cdot v | \tau \in Leb(\mu, x_0, t_0, t) \text{where } v \text{ is a single needle variation at time } \tau.\}$$

In simple words, K is built in the following way: by taking every single needle variation at every possible time preceding the current one(almost: they still have to be Lebesgue points), making the disturbance evolve from its origin to the current time trough matrix Φ and then by taking the coned convex hull of the set union of all those vectors.
We now define the analogue to $K(\mu, x_0, t_0, t)$ but for multi needle variations.

Tangent r-simplex cone(and relative data)   As usual, let $\Sigma = (\chi, f, U)$be a control system, $x_0 \in \chi$ be an initial condition for eq. 1.1, $[t_0, t_1] \subset \mathbb{R}$ a time interval, $\mu \in \mathfrak{U}(x_0, t_0, [t_0, t_1])$. We then take $t \in [t_0, t_1]$. We then define

- the **tangent r-simplex cone data** as a collection $\{\Theta_1, ..., \Theta_r\}$ of multi needle variations. By using this notation:$\Theta_i = \{\theta_i j\}; i = 1, .., r; j = 1, .., k_i$, we require that the times $\tau_i j$ are all distincts, are all Lebesgue points-

- given the multi needle variations v_{Θ_i}associated with the relativa r-simplex cone data, the coned convex hull of v_{Θ_i} is actually an r-simplex cone and defined as the **tangent r-simplex cone**.

Of course, when not explicitly said, the previous definitions where all relative to the fixed-interval case.

Theorem: various forms for the tangent cone   The following sets are actually the same set:

- $K(\mu, x_0, t_0, t)$

- given r=$dim(K(\mu, x_0, t_0, t))$, the closure of the set formed by the union of all the possible tangent r-simplex cones

- the closure of the set formed by the union of all the sets $\{\Phi(\mu, x_0, t_0, \tau, t) \cdot K(\mu, x_0, t_0, \tau) | \tau \in Leb(\mu, x_0, t_0, t)\}$

Despite the difficulty of interpretation, this theorem tells us that in the end it does not matter if variations to the control are obtained through single needle or multi needle variations. The equivalence of the first one and the third one is easier to justify and understand, using the property of concatenation of Φ for which $\Phi(\mu, x_0, t_0, a, b) \cdot \Phi(\mu, x_0, t_0, b, c) = \Phi(\mu, x_0, t_0, a, c)$.

3.7   Significative theorems

Theorem: points in the tangent cones are "interior" to the reachable set   Given the usual control system, initial condition and $[t_0, t_1]$, and given a tangent cone $K(\mu, x_0, t_0, t)$, for a $t \in [t_0, t_1]$, if a v_0 exists such that it is in the internal part of K,then
There exists: a cone $A \subset int(K)$ and an $r > 0$ such that

- $v_0 \in int(A)$

- the intersection B between A and the ball of center $\xi(\mu, x_0, t_0, t)$ (which is the "current" state of the system) and radius r is contained in the reachable set $B \in \mathfrak{R}(x_0, t_0, t)$.

Basically what this theorem says is not properly that the tangent cone points towards the interior of the reachable set. Although this is in a certain sense true, his theorem says something a slightly less powerful. The first step is to understand that, since λv_0 is in the internal part of the cone for every $\lambda > 0$, then one can choose a variational vector with the same direction of v_0 such that by moving the trajectory in that direction (which is obtainable by applying the corresponding control variation) one still remains in the reachable set.

Maximum Hamiltonians and tangent cones   In the statement of maximum principle tangent cones do not appear, but maximum Hamiltonian do. Actually, one Could just be satisfied with the Hamiltonian part, in the sense that, once the adjoint response is found (not an easy task, but the orthogonality with the smooth set $S_1$and S_0can help), the only remaining part is to find a control that maximizes the Hamiltonian. Those and only those controls are candidates for optimality.
Nevertheless, it is interesting to see the connections between tangent cones and Hamiltonians, and to have a look at the (geometrical) significance of the following theorems.
The first one is going to be about the binding between maximization of Hamiltonian and the existance of a separating hyperplane between the costate and, in a sense, possible "controls". This, point to point in time.

Theorem: Given a control system $\Sigma = (\chi, f, U)$, a costate p and a control $\overline{u}$, then $H_\Sigma(x, p, \overline{u}) = H_\Sigma^{max}(x, p)$ if and only if

$$\text{for every } v \in \{f(x, u) - f(x, \overline{u}) | u \in U\} \text{it is valid that } \langle p, v \rangle < 0$$

Proof:

$$\begin{aligned}
H_\Sigma(x, p, \overline{u}) = H_\Sigma^{max}(x, p) &\iff \\
H_\Sigma(x, p, \overline{u}) \geq H_\Sigma(x, p, u) &\iff \\
H_\Sigma(x, p, \overline{u}) - H_\Sigma(x, p, u) \geq 0 &\iff \\
\langle p, f(x, u) - f(x, \overline{u}) \leq 0 \rangle &\iff \\
\langle p, v \rangle \leq 0 &
\end{aligned}$$

for v defined as before.

Hamiltonians and adjoint response There is now another theorem, regarding the connection between adjoint response and Hamiltonian. What this theorem says is that one can find a suitable costate p, with which he may start to try to maximize the Hamiltonian, by simply taking the adjoint response coming from eq. 3.2, provided a suitable initial condition for this equation is found. So

Theorem: Given the usual control system, initial condition, time interval, and supposing that for a $\tau \in [t_0, t_1]$ there exists a vector $\overline{\lambda} \in \mathbb{R}^n$ ad a cone A that contains $K(\mu, x_0, t_0, \tau) \subset A$ such that
for each $v \in A \implies \langle \overline{\lambda}, v \rangle$, then
by taking the adjoint response with initial condition for equation 3.2 that $\lambda(\tau) = \overline{\lambda}$ then it is true that, for every $t \in Leb(\mu, x_0, t_0, \tau)$

$$H_\Sigma(\xi(\mu, x_0, t_0, t), \lambda(t), \mu(t)) = H_\Sigma^{max}(\xi(\mu, x_0, t_0, t), \lambda(t))$$

As already anticipated, the first of the two preceding lemmas tells us that, given and certain p and a certain state of the system, at a certain time t, the control that maximizes the Hamiltonian is the one that, if we refer single needle variations to that base control, the variational cone $K(\mu, x_0, t_0, t)$ possesses a support hyperplane and that is individuated as the one being perpendicular to p. This tells us two things: first, that trajectories can only be moved "one way", by varying the control. This is also trivial because if a control maximizes the Hamiltonian, than by varying it, there is no other option than to get a lower valued of that Hamiltonian. The second thing is that, generally, optimal controls will be found on the boundary of the set of possible control (at a given state and time).
The second theorem is more related to the properties of the adjoint equation. First, one has to acknowledge that v is a needle variation (with data $(t, 1, w)$). Secondly, the system lies at a certain state at time τ. The system has been steered from the initial conditions to there by the control $\mu(t)$. If at τ the variational tangent cone *(still)* possesses a supporting hyperplane, then we are able to find a function $p : [t_0, \tau] \to \mathbb{R}^n$ such that it maximizes the Hamiltonian. This function is, because of the properties of the adjoint and variational equations and the fundamental matrix, the adjoint response. But besides having found the costate that maximizes the Hamiltonian at any given moment, this theorem tells us also something about the control being, if not optimal, at least "extremal" in some sense. At least one can conclude, via the first lemma (all the implications in the demonstration are double sided) that the variational tangent cone $K(\mu, x_0, t_0, a)$ possessed a supporting hyperplane $\forall a \in [t_0, \tau]$.
As a last remark: the last of those two theorems tells us only something about the past, which is, from the initial instant to τ. It does not give us ways to predict what the optimal control for the instants after τ.

TODO: DECIDE WHETHER TO LEAVE THIS   The following theorem tells us that, if for a given control and at a given time τ there exists a supporting hyperplane for $K(\mu, x_0, t_0, \tau)$ then the corresponding Hamiltonian is zero. Which is, the supporting hyperplane for the cone "touches" the set of possible $f(x, u)$. This is evdent by referring to figure 3.
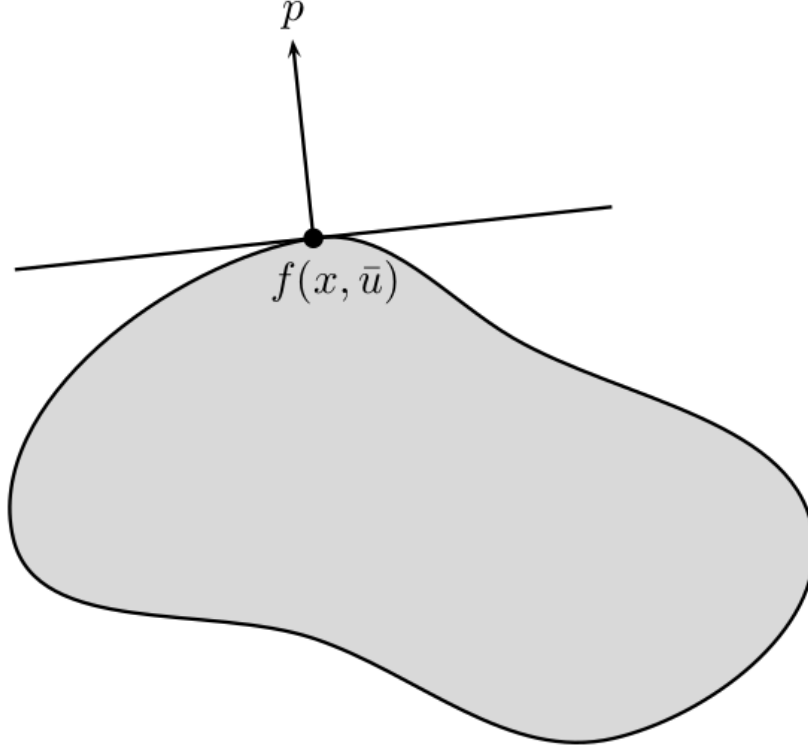

Figure 3: text

So now,

Theorem: Demonstration of theorem 5-14 is not that clear in the book, so for now, it is skipped.

3.8   Controlled trajectories and boundary of reachable set

The following very important theorem tell us two very important things. If, in the end (at t_1) the trajectory lies on the boundary of the reachable set, then first, there exists an adjoint response that maximizes the Hamiltonian for any instant in $[t_0, t_1]$ (or, better to say, in the Lebesgue points for f in that interval) and thus, secondly, the control is optimal, since it maximizes the Hamiltonian. Formally said,

Theorem: Given the usual control system, initial condition and time interval, given a control $\mu_* \in \mathfrak{U}([t_0, t_1])$, and given $\xi(\mu_*, x_0, t_0, t_1)$ (abbreviate $\xi(\mu_*, x_0, t_0, \cdot$ simply with ξ_*), if $\xi_*(t_1) \in bd(\mathfrak{R}(x_0, t_0, t_1))$ then

- there exists an adjoint response $\lambda_* : [t_0, t_1] \to \mathbb{R}^n$ for Σ for the controlled trajectory (ξ_*, μ_*),

- $H_\Sigma(\xi(\mu_*, x_0, t_0, t), \lambda_*(t), \mu_*(t)) = H_\Sigma^{max}(\xi(\mu_*, x_0, t_0, t), \lambda_*(t))$ for (almost) every $t \in [t_0, t_1]$

Proof: Since $\xi_*(t1)$ is on the boundary of the reachable set, then there exists a sequence $\{x_j\}$ in $\chi \backslash closure(\mathfrak{R}(x_0, t_0, t_1))$ that converges to that point. Now, let's take another sequence $v_j =$

$\frac{x_j - \xi_*(t_1)}{||x_j - \xi_*(t_1)||}$. Since this sequence lies on the unitary sphere in R^n and this set is compact, there exists a converging subsequence, that converges to v_0.

We now claim that $v_0 \notin int(K(\mu_*, x_0, t_0, t_1))$:

> Supposing that v_0 lies in the internal part of the cone, then there exists a suffiecntly large N such that, for $j > N$, every v_j lies in the internal of the cone. Using the first theorem from the section "significative theorems" implies that for a certain N also $x_{j>N} \in int(K(\mu_*, x_0, t_0, t_1))$. Indeed, a whole ball with center $\xi_*(t1) + v_j$ and radius r is contained in the cone. In order to prove the last claim, the center of the ball is moved along the direction of v_j until it coincides con x_j. One could (and should, in the case that $||x_j - \xi_*(t_1)|| < 1$) actually also re-scale the radius, and the re-scaled ball still lies in the cone.

Given this, a topological lemma states that there exists a separating hyperplane $P(t_1)$ such that vector v_0 is contained in a half-space and the cone is contained in the other one.

Now take a the vector called $\lambda_*(t_1)$ orthogonal to $P(t_1)$ lying in the half space not containing the cone, and thus

$$\langle \lambda_*(t_1), v \rangle \ v \in K(\mu_*, x_0, t_0, t_1).$$

To conclude, we just need to take the unique adjoint response that has the value $\lambda_*(t_1)$ at time t_1, and use the preceding theorems.

Interpretation   In the following figures we can have a deeper insight of the passages and geometrical constructions behind this theorem. In the first figure, 4 one can see that the adjoint response points somehow "outside" of the reachable set. Indeed, if the objective is staying on the boundary of the expanding reachable set, it sounds logical to choose a control, and thus an $f(\xi, \mu)$ that "points outside the reachable set, as much as it can". This effort is represented in the maximization of the Hamiltonian, which is, by the maximization of the scalar product between the possible f and λ, as depicted in 3.
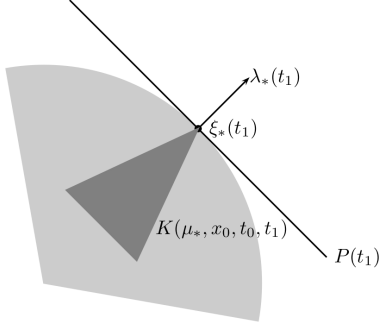

Figure 4: "Zoom" on the cone, part of the reachable set and final point of the trajectory at the final instant

An important observation must be made, which is that the hyperplane actually separates the cone from the vector v_0 from the cone K, so the hyper plane does not have to be tangent to the reachable set, as seen in figure Finally, the last figure, shows us the evolution of the adjoint response, of the separating plane P and of the trajectory (dotted in the figure). As already said, the control should be such that, instant by instant, the function steering the system $\dot{\xi}(t)$ has maximal projection in the direction of λ. There is a last theorem for this section, which states that once a system is fallen from the border of the reachable set, it remains in its internal for all the successive instants. Basically, in a crowd of runners where everybody runs at the (same and uniform) maximum speed, once a runner diminishes the speed for an ammount of time that is not Lebesgue-neglettable, it will fall inside the crowd, and won't be able to return on the border anymore.

Theorem:trajectories in the interior of the reachable set remain in the interior
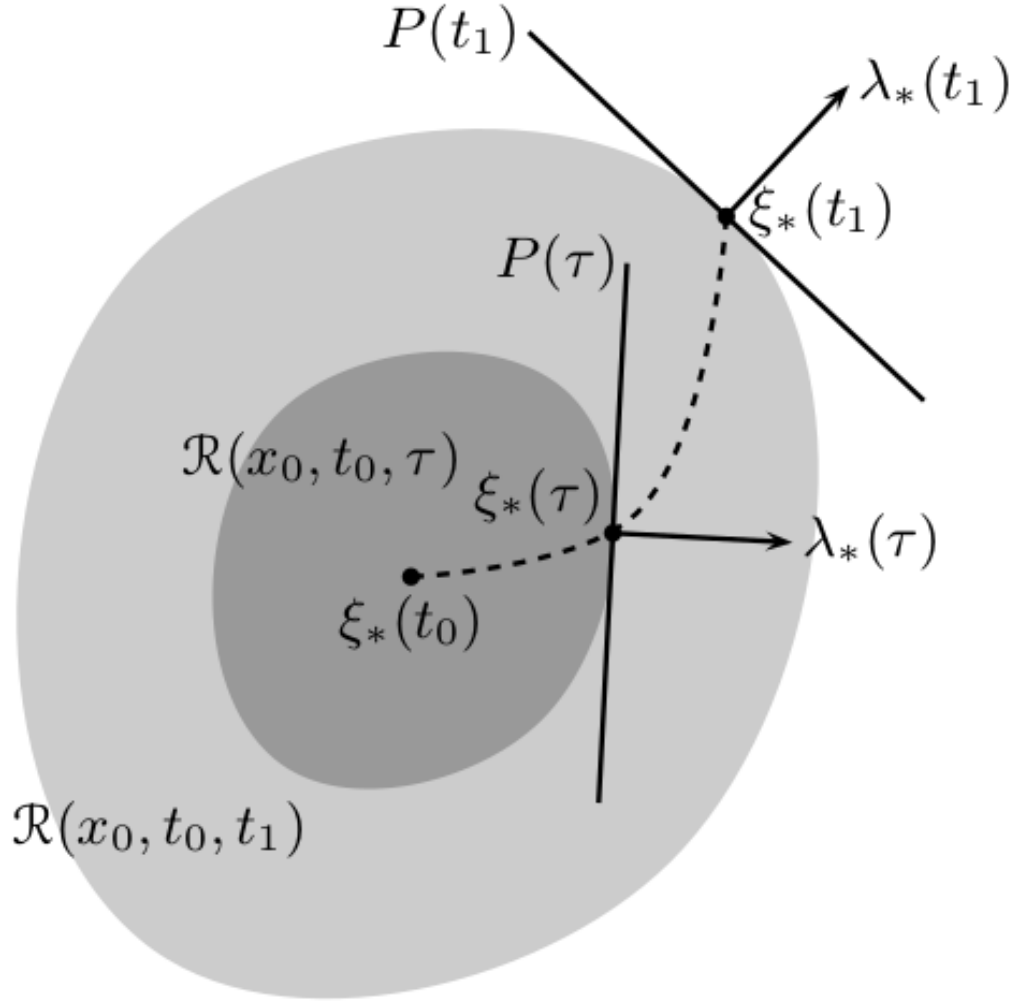Let's have the usual control system, initial condition, time interval and admissible control. If, for

Figure 5: Depiction of the situation at two different instants, the final one and one intermediate, τ

a certain instant $\tau \in [t_0, t_1)$ it happens that $\xi(\mu, x_0, t_0, \tau) \in int(\mathfrak{R}(x_0, t_0, \tau))$ then $\xi(\mu, x_0, t_0, t) \in int(\mathfrak{R}(x_0, t_0, t))$ for all $t \in (\tau, t_1]$ **TODO:PROOF. it requires to give as good that the map F is a diffeomorphism.**

3.9 Extended system, extended reachable set and optimal trajectories

Before stating the penultimate theorem, there is just another small definition to be made, which is the one of extended system, which is just the extension from n to n+1 dimensions for the state of the system, where the new, 0-th dimension is the value of the cost function so far.

Definition: Extended System Let L be a Lagrangian for control system $\Sigma = (\chi, f, U)$. We define the **extended system** as $\hat{\Sigma} = (\hat{\chi}, \hat{f}, U)$ just by asking that

- $\hat{\chi} = \mathbb{R} \times \chi$

- $\hat{f}((x^0, x), u) = (L(x, u), f(x, u))$

Note that now the equations governing the extended system are

$$\dot{\xi}^0(t) = L(\xi(t), \mu(t))$$
$$\dot{\xi}(t) = f(\xi(t), \mu(t))$$
$$\implies$$
$$\xi^0(\tau) = \int_{t_0}^{\tau} L(\xi(t), \mu(t))dt$$

Which is precisely the cost function.

There now will be an important theorem, that lies an important layer in the proof of the principle. In fact, so far, the optimality of a trajectory has always been related to the maximization of an Hamiltonian, as is stated in the proof of the principle. But, looking deeper into the previous theorems, the maximization of the Hamiltonian just ensures that the trajectory lies on the boundary of the reachable set (that it's always on the "frontline of runners, so to say". But the fact that the trajectory runs through the space of admissible states as possibile has nothing to do, a-priori, with the cost associated to that controlled trajectory.

Theorem: Let L be a Lagrangian for control system $\Sigma = (\chi, f, U)$, $S_0, S_1 \subset \chi$ be sets, and suppose that (ξ_*, μ_*) is a solution to the fixed interval problem (which is, $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$), then
$\hat{\xi}_*(t_1) \in boundary(\hat{\mathfrak{R}}(\hat{\xi}_*(t_0), t_0, t_1))$

Proof: Since $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$, then the corresponding extended trajectory has the obvious property that the cost $\xi^0(t_1)$ is the lowest possible among all the possible controlled arcs (ξ, μ) that steer the system from $\xi_*(t_0)$ to $\xi_*(t_1)$. So given the set of "possible costs", the first element of $\hat{\xi}_*(t_1)$ is obviously on the boundary of the said set (at least, as long as the trajectories are L-acceptable, which is, the integral represented by the cost function is finite). This must then imply that also $\hat{\xi}_*(t_1)$ is on the boundary of the extended reachable set: let's take a neighbourhood of $\hat{\xi}_*(t_1)$ in the extended state space $\hat{\chi}$. Any neighbourhood will then contain points of the form $(c, \xi_*(t_1))$ with $c < \xi^0_*(t_1)$. But those points, with a cost which is littler than the minimum one, are not in the extended reachable set by hypothesis, since the controlled trajectory (ξ_*, μ_*) is optimal. So, since there is no neighbourhood of $\hat{\xi}_*(t_1)$ contained in $\hat{\mathfrak{R}}(\hat{\xi}_*(t_0), t_0, t_1)$, th is point must lie on the boundary, as wanted.

The fact that $\hat{\xi}_*(t_1)$ is on the boundary of the extended reachable set implies that the trajectory $\xi(t_1)$ is on the boundary of $\mathfrak{R}(x_0, t_0, t_1)$, thus implying the bond with cost-wise optimality and being on the boundary of the reachable set, which is an Hamiltonian-wise optimality. Indeed, it will be used in the following theorem.

3.10 Adjoint response, Hamiltonian and maximum principle

Finally, we will now prove the points 1 to 3 in the statement of the principle.

Theorem: Let L be a Lagrangian for control system $\Sigma = (\chi, f, U)$, $S_0, S_1 \subset \chi$ be sets, and suppose that (ξ_*, μ_*) is a solution to the fixed interval problem (which is, $(\xi_*, \mu_*) \in \mathfrak{P}(\Sigma, L, S_0, S_1, [t_0, t_1])$), then
There exists an absolutely continous map $\lambda_* : [t_0, t_1] \to \mathbb{R}^n$ and a number $\lambda^0_* \in \{-1, 0\}$ with the following properties:

1. either $\lambda^0_* = .1$ or $\lambda_*(t_0) \neq 0$

2. λ_* is an adjoint repsonse for (Σ, λ^0_*, L) along (ξ_*, μ_*)

3. $H_{\Sigma, \lambda^0_* L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H^{max}_{\Sigma, \lambda^0_* L}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$

Proof: First, we observe that the vector $(-1, 0_v) \in \mathbb{R} \times \mathbb{R}^n$ cannot lie in the interior of the extended tangent cone $K(\mu, \hat{x}_0, t_0, t_1)$. If this were not the case, by means of the preceding theorems, there would be points a^0, a in $\hat{\mathfrak{R}}(\hat{\xi}_*(t_0), t_0, t_1)$ at a lower cost, whose position in the state space is actually the same ($a = \xi_*(t_1)$). This would violate the optimality of (ξ_*, μ_*).

Since this vector does not lie in the internal of $K(\mu, \hat{x}_0, t_0, t_1)$, there exists a separating hyperplane $\hat{P}(t_1)$ between this cone and the vector.

Take a vector $\hat{\lambda}_*(t_1)$ orthogonal to $\hat{P}(t_1)$, in the half space not containing the cone. This implies

$$\langle \hat{\lambda}_*(t_1), (-1, 0) \rangle \geq 0;$$
$$\langle \hat{\lambda}_*(t_1), \hat{v} \rangle \leq 0; \ \hat{v} \in K(\mu, \hat{x}_0, t_0, t_1)$$

This implies then that $\hat{\lambda}_*^0(t_1) \leq 0$.

Define then $\hat{\lambda}_*$ as the adjoint response whose value at t_1 is equal to $\hat{\lambda}_*(t_1)$.

From the equations for the adjoint response 3.2 we obtain that $\dot{\lambda}_*^0(t) = 0$. This is because since $\hat{f}$ does not depend on x_0, then the Jacobian matrix $\boldsymbol{D}_1 \hat{f}$ has the first column 0, and thus the transposed matrix has the first row of zeros. This is of course by imagining the (extended) adjoint response as a coloumn vector.

So $\lambda_*^0(t) = 0$ is constant and nonpositive. So, if $\lambda_*^0(t) \neq 0$, we can rescale the whole extended adjoint response so that $\hat{\lambda}_*$ becomes $\frac{\hat{\lambda}_*}{-(\lambda_*^0)}$, so that now λ_*^0 is either 0 or -1.

Now, the Hamiltonian for the extended system is very similar to the extended Hamiltonian for the system (plus the Lagrangian):

$$\hat{H}_{\hat{\Sigma}}\left((x^0, x), (p^0, p), u\right) = \langle p, f(x, u) \rangle + p^0 L(x, u) = H_{\Sigma, p^0 L}(x, p, u)$$

Now, since the trajectory is optimal, by virtue of the preceeding theorem, it lies on the boundary of the extended reachable se at time t_1. By means of the preceeding theorem it holds that $H_{\Sigma, p^0 L}(\xi_*(t), \lambda_*(t), \mu_*(t)) = H_{\Sigma, p^0 L}^{max}(\xi_*(t), \lambda_*(t))$ for almost every $t \in [t_0, t_1]$

4 Moon landing problem