Lyon PhD Course Actuarial Science

Chapter 5 - Multivariate distributions

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1 Introduction

Since the mid 1990s, we observe in actuarial science and in quantitative risk management an increasing interest in modeling the dependence between the risks.

It becomes essential for the actuary to be familiar with multivariate models and dependence modeling.

The dependence between risks has an impact on the risk pooling.

The actuary also needs to develop aggregation methods for dependent risks.

2 Univariate and multivariate Laplace-Stieltjes Transforms

Let X be positive rv with cdf F_X. Its Laplace-Stieltjes (LS) transform is defined by

$$\mathcal{L}_X(t) = E\left[\mathrm{e}^{-tX}\right] = \int_0^\infty \mathrm{e}^{-tx}\mathrm{d}F_X(x),$$

for $t > t^*$ (for some $t^* \leq 0$).

The LS of a rv exists for any distribution of X for $t \geq 0.$, which is not the case for the mgf.

If X a positive discrete rv defined on $\mathbb{N}$, we have

$$\mathcal{L}_X(t) = E\left[\mathrm{e}^{-tX}\right] = \sum_{k=0}^{\infty} \mathrm{e}^{-tk} f_X(k),$$

where $f_X(k) = \Pr(X = k)$ is the pmf of the rv X.

If X is a positive continuous rv with a pdf f_X, we have

$$\mathcal{L}_X(t) = E\left[\mathrm{e}^{-tX}\right] = \int_0^{\infty} \mathrm{e}^{-tk} f_X(k)\, \mathrm{d}x.$$

For a vector of positive rvs $\underline{X}$, we define

$$\mathcal{L}_{\underline{X}}(t_1, ..., t_n) = E\left[\mathrm{e}^{-t_1 X_1} ... \mathrm{e}^{-t_n X_n}\right],$$

which exists for any multivariate distribution when $t_1, ..., t_n \geq 0$

3 Multivariate distributions and aggregation

3.1 General approach for continuous rvs

Let (X_1, X_2) be a pair of positive continuous rvs with bivariate pdf f_{X_1,X_2}.

We define $S = X_1 + X_2$

Then, the pdf of the rv S is given by

$$f_S(s) = \int_0^s f_{X_1,X_2}(x_1, s - x_1)\, \mathrm{d}x_1. \tag{1}$$

We consider now the case for $n > 2$ risks.

Let $\underline{X} = (X_1, ..., X_n)^t$ be a vector of positive continuous rvs with multivariate pdf $f_{\underline{X}}$.

We define $S = \sum_{i=1}^{n} X_i$.

Then we obtain

$$
\begin{aligned}
& f_S(s) \\
= \; & \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-\sum\limits_{i=1}^{n-2} x_i} f_{\underline{X}} \left(x_1, x_2, \ldots, s - \sum_{i=1}^{n-1} x_i \right) \mathrm{d}x_{n-1} ... \mathrm{d}x_2 \mathrm{d}x_1.
\end{aligned}
$$

If the multivariate mgf $\underline{X}$ exists, then we have

$$
\begin{aligned}
M_S(t) &= E\left[\mathrm{e}^{tS}\right] = E\left[\mathrm{e}^{t(X_1+...+X_n)}\right] \\
&= E\left[\mathrm{e}^{tX_1}...\mathrm{e}^{tX_n}\right] = M_{X_1,...,X_n}(t, ..., t), \qquad (2)
\end{aligned}
$$

from which we may identify the distribution of the rv S.

Similarly, we have

$$\begin{aligned}\mathcal{L}_S(t) &= E\left[\mathrm{e}^{-tS}\right] = E\left[\mathrm{e}^{-t(X_1+...+X_n)}\right] \\ &= E\left[\mathrm{e}^{-tX_1}...\mathrm{e}^{-tX_n}\right] = \mathcal{L}_{\underline{X}}(t,...,t).\end{aligned}$$

3.2 General approach for discrete rvs

Let (X_1, X_2) be a pair of positive discrete rvs defined on the arithmetical support, i.e. $X_i \in \{0, 1h, 2h, ...\}$ with $h > 0$

The multivariate pmf is given by $f_{X_1,X_2}(m_1h, m_2h)$.

We define $S = X_1 + X_2$.

The pmf of the rv S is given by

$$f_S(kh) = \sum_{m_1=0}^{k} f_{X_1,X_2}(m_1 h, kh - m_1 h). \tag{3}$$

Let $\underline{X} = (X_1, ..., X_n)^t$ be a vector of positive discrete rvs with multivariate pmf $f_{\underline{X}}$.

The multivariate pgf of $\underline{X}$ is defined by

$$\begin{aligned} P_{\underline{X}}(t_1, ..., t_n) &= E\left[t_1^{X_1} ... t_n^{X_n}\right] \\ &= \sum_{k_1=0}^{\infty} ... \sum_{k_n=0}^{\infty} t_1^{k_1 h} ... t_n^{k_n h} f_{\underline{X}}(k_1 h, ..., k_n h). \end{aligned}$$

We define $S = \sum_{i=1}^{n} X_i$.

Then, we have

$$f_{S_n}(kh) = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} \dots \sum_{k_{n-1}=0}^{k-k_1-\dots-k_{n-2}} f_{\underline{X}}\left(k_1 h, k_2 h, \dots, k_{n-1} h, \left(k - \sum_{j=1}^{n-1} k_j\right) h\right).$$

If the mgf

$$M_{X_1,\dots,X_n}(t_1,\dots,t_n) = E\left[\mathrm{e}^{t_1 X_1} \dots \mathrm{e}^{t_n X_n}\right]$$

of $\underline{X} = (X_1, \dots, X_n)$ exists, then we obtain

$$M_S(t) = E\left[\mathrm{e}^{t(X_1+\dots+X_n)}\right] = E\left[\mathrm{e}^{tX_1} \dots \mathrm{e}^{tX_n}\right] = M_{X_1,\dots,X_n}(t,\dots,t). \tag{4}$$

Similarly, we have

$$\begin{aligned}\mathcal{L}_S(t) &= E\left[\mathrm{e}^{-tS}\right] = E\left[\mathrm{e}^{-t(X_1+...+X_n)}\right] \\ &= E\left[\mathrm{e}^{-tX_1}...\mathrm{e}^{-tX_n}\right] = \mathcal{L}_{\underline{X}}(t,...,t).\end{aligned}$$

We also have

$$P_S(t) = P_{X_1,...,X_n}(t,...,t),$$

where

$$P_S(t) = \sum_{k=0}^{\infty} f_S(kh)\, t^{kh}$$

for $t \geq 0$.

3.3 Simulation methods

To simulate sampled values of a vector of continuous rvs $\underline{X}$, we can select two approaches :

- conditional approach
- approach based on the stochastic representation of $\underline{X}$ or the specific definition of $F_{\underline{X}}$.

The general approch requires the conditional cdf given by

$$F_{X_2|X_1=x_1}(x_2) = \frac{\frac{\partial}{\partial x_1} F_{X_1,X_1}(x_1, x_2)}{f_{X_1}(x_1)}.$$

Algorithm 1 *General approach to simulate a sampled value of the pair of continuous rvs* (X_1, X_2)

1. *Simulate* $\left(U_1^{(j)}, U_2^{(j)}\right)$ *of the pair of iid rvs* (U_1, U_2) *where* $U_1 \sim U_2 \sim Unif\,(0,1)$.
2. *Simulate* $\left(X_1^{(j)}, X_2^{(j)}\right)$ *of* (X_1, X_2) *as follows :*
   - $X_1^{(j)} = F_{X_1}^{-1}\left(U_1^{(j)}\right)$ *;*
   - $X_2^{(j)}$ *is a solution of the equation in* x_2 *given by*

$$F_{X_2|X_1=U_1^{(j)}}(x_2) = \left.\frac{\frac{\partial}{\partial x_1} F_{X_1,X_1}(x_1,x_2)}{f_{X_1}(x_1)}\right|_{x_1=X_1^{(j)}} = U_2^{(j)},$$

i.e.

$$X_2^{(j)} = F^{-1}_{X_2|X_1=U_1^{(j)}}\left(U_2^{(j)}\right).$$

3. *Repeat for* $j = 1, 2, ..., m$.

An illustration of this general approach is provided with the EFGM's bivariate exponential distribution.

4 Continuous bivariate distributions

There is impressive number of multivariate continuous distributions.

In this chapter, we present only a few of them.

5 Bivariate exponential distributions

5.1 Preliminaries

In this section, we assume that the couple of rvs (X_1, X_2) has a bivariate distribution with exponential marginals with mean $1/\beta_i$, for $i = 1, 2$.

We define the Fréchet class $\Gamma\left(F_{X_1}, F_{X_2}\right)$ as the set of all joint cdf's F_{X_1,X_2} with exponential marginals $F_{X_i}(x_i) = 1 - e^{-\beta_i x_i}$, $i = 1, 2$.

The elements of $\Gamma\left(F_{X_1}, F_{X_2}\right)$ are bounded above and below by the Fréchet upper and lower bounds, meaning

$$F^{-}_{X_1,X_2}(x_1, x_2) \leq F_{X_1,X_2}(x_1, x_2) \leq F^{+}_{X_1,X_2}(x_1, x_2),$$

where

$$F^{-}_{X_1,X_2}(x_1,x_2) = \max\left(F_{X_1}(x_1) + F_{X_2}(x_2) - 1; 0\right)$$

and

$$F^{+}_{X_1,X_2}(x_1,x_2) = \min\left(F_{X_1}(x_1); F_{X_2}(x_2)\right).$$

The Pearson correlation coefficient is a measure of association for two rvs that captures their level of linear correlation.

It is defined as $\rho_P(X_1,X_2) = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)Var(X_2)}}$.

Note that the bounds on $\rho_P(X_1,X_2)$ are

$$\rho_{\min} = 1 - \frac{\pi^2}{6} \leq \rho_P(X_1,X_2) \leq 1 = \rho_{\max},$$

where

- the upper bound ρ_{max} is attained when the components of (X_1, X_2) are comonotonic
- the lower bound ρ_{min} is attained when the components of (X_1, X_2) are countermonotonic
- see e.g. Denuit et al. (2005), McNeil et al. (2005), Bladt and Nielsen (2010b)), and the next chapter.

We will provide the expression of Pearson's correlation coefficient for the different families of bivariate distributions considered to capture the degree of linear relationship within (X_1, X_2).

5.2 Special case – independence

One important member of the Fréchet class $\Gamma\left(F_{X_1}, F_{X_2}\right)$ corresponds to the case where X_1 and X_2 are independent.

In the following proposition, we recall in this context expressions for the cdf of $S = X_1 + X_2$ and the expectation terms in the TVaR.

These are well known results but we restate them to establish the notations.

Proposition 2 *Let X_1 and X_2 be independent exponentially distributed rvs with mean $1/\beta_i$, $(i = 1, 2)$, with*

$$f_{X_1,X_2}(x_1, x_2) = \beta_1 e^{-\beta_1 x_1} \beta_2 e^{-\beta_2 x_2}, \tag{5}$$

and let $S = X_1 + X_2$. Then, we have

$$F_S(x) = H(x; \beta_1, \beta_2) = \begin{cases} 1 - \mathrm{e}^{-\beta x} \sum_{j=0}^{2-1} \frac{(\beta x)^j}{j!}, & \beta_1 = \beta_2 = \beta \\ \sum_{i=1}^{2} \left(\prod_{j=1, j \neq i}^{2} \frac{\beta_j}{\beta_j - \beta_i} \right) \left(1 - \mathrm{e}^{-\beta_i x}\right), & \beta_1 \neq \beta_2 \end{cases}, \tag{6}$$

$$E\left[S \times 1_{\{S>b\}}\right] = \zeta(b; \beta_1, \beta_2) = \begin{cases} \frac{2}{\beta} \overline{H}(b; \beta, \beta) = \frac{2}{\beta} \left(\mathrm{e}^{-\beta b} \sum_{j=0}^{2} \frac{(\beta b)^j}{j!} \right), & \beta_1 \\ \sum_{i=1}^{2} \left(\prod_{j=1, j \neq i}^{2} \frac{\beta_j}{\beta_j - \beta_i} \right) \left(b\mathrm{e}^{-\beta_i b} + \frac{\mathrm{e}^{-\beta_i b}}{\beta_i} \right), & \beta_1 \end{cases} \tag{7}$$

Preuve. The expressions are obtained straightforwardly from their definition.

■

Remark 3 *In (6), the rv S follows an Erlang-2 distribution if $\beta_1 = \beta_2 = \beta$ and a generalized Erlang distribution if $\beta_1 \neq \beta_2$. Both expressions in (6) are provided in e.g. Gerber and Shiu (2005). The computation of the TVaR and the contribution based on the TVaR allocation rule for non-negative independent rvs is treated e.g. in section 2 of Furman and Landsman (2005). They also discuss, in sections 3 and 4, the particular case of the sum of independent gamma rvs.*

5.3 Eyraud - Farlie - Gumbel - Morgenstern (EFGM) bivariate exponential distribution

5.3.1 Definition and properties

The cdf is given by

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= \left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right) \\ &\quad + \theta\left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right)\mathrm{e}^{-\beta_1 x_1}\mathrm{e}^{-\beta_2 x_2}, \qquad (8) \end{aligned}$$

with a dependence parameter $-1 \leq \theta \leq 1$ and with $\beta_i > 0$.

We also have

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= (1+\theta)\left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right) \\ &\quad -\theta\left(1-\mathrm{e}^{-2\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right) \\ &\quad -\theta\left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-2\beta_2 x_2}\right) \\ &\quad +\theta\left(1-\mathrm{e}^{-2\beta_1 x_1}\right)\left(1-\mathrm{e}^{-2\beta_2 x_2}\right). \end{aligned}$$

We find that the marginals of X_1 and X_2 are exponential i.e. $X_i \sim Exp(\beta_i)$ $(i = 1, 2)$.

Special case : $\theta = 0$ corresponds to the independence.

The bivariate pdf is

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= (1+\theta)\,\beta_1\mathrm{e}^{-\beta_1 x_1}\beta_2\mathrm{e}^{-\beta_2 x_2} + \theta 2\beta_1\mathrm{e}^{-2\beta_1 x_1}2\beta_2\mathrm{e}^{-2\beta_2 x_2} \\ &\quad -\theta 2\beta_1\mathrm{e}^{-2\beta_1 x_1}\beta_2\mathrm{e}^{-\beta_2 x_2} - \theta\beta_1\mathrm{e}^{-\beta_1 x_1}2\beta_2\mathrm{e}^{-2\beta_2 x_2}. \end{aligned}$$

Comment: Notice that $f_{X_1,X_2}(x_1,x_2)$ is the linear combination of four 4 terms and each term is the product of the pdfs of two exponential distributions.

This bivariate allows for moderate dependence relation between the rvs X_1 and X_2, positive ou negative.

This distribution can be seen as a perturbation of the bivariate exponential distribution with independence.

5.3.2 Covariance of (X_1, X_2)

First, we need to find the expression for

$$E[X_1X_2] = \int_0^\infty \int_0^\infty x_1x_2f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2$$

which becomes

$$\begin{aligned}\int_0^\infty \int_0^\infty x_1 x_2 f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 &= (1+\theta)\int_0^\infty \int_0^\infty x_1 x_2 \beta_1 \mathrm{e}^{-\beta_1 x_1} \beta_2 \mathrm{e}^{-\beta_2 x_2} \\ &\quad +\theta \int_0^\infty \int_0^\infty x_1 x_2 2\beta_1 \mathrm{e}^{-2\beta_1 x_1} 2\beta_2 \mathrm{e}^{-2\beta_2 x_2} \\ &\quad -\theta \int_0^\infty \int_0^\infty x_1 x_2 2\beta_1 \mathrm{e}^{-2\beta_1 x_1} \beta_2 \mathrm{e}^{-\beta_2 x_2} \mathrm{d}x \\ &\quad -\theta \int_0^\infty \int_0^\infty x_1 x_2 \beta_1 \mathrm{e}^{-\beta_1 x_1} 2\beta_2 \mathrm{e}^{-2\beta_2 x_2} \mathrm{d}x\end{aligned}$$

Then, we have

$$\begin{aligned}
E\left[X_1X_2\right] &= (1+\theta)\left(\int_0^\infty x_1\beta_1\mathrm{e}^{-\beta_1x_1}\mathrm{d}x_1\right)\left(\int_0^\infty x_2\beta_2\mathrm{e}^{-\beta_2x_2}\mathrm{d}x_2\right)\\
&\quad +\theta\left(\int_0^\infty x_1 2\beta_1\mathrm{e}^{-2\beta_1x_1}\mathrm{d}x_1\right)\left(\int_0^\infty x_2 2\beta_2\mathrm{e}^{-2\beta_2x_2}\mathrm{d}x_2\right)\\
&\quad -\theta\left(\int_0^\infty x_1 2\beta_1\mathrm{e}^{-2\beta_1x_1}\mathrm{d}x_1\right)\left(\int_0^\infty x_2\beta_2\mathrm{e}^{-\beta_2x_2}\mathrm{d}x_2\right)\\
&\quad -\theta\left(\int_0^\infty x_1\beta_1\mathrm{e}^{-\beta_1x_1}\mathrm{d}x_1\right)\left(\int_0^\infty x_2 2\beta_2\mathrm{e}^{-2\beta_2x_2}\mathrm{d}x_2\right)\\
&= (1+\theta)\frac{1}{\beta_1}\frac{1}{\beta_2}+\theta\frac{1}{2\beta_1}\frac{1}{2\beta_2}-\theta\frac{1}{2\beta_1}\frac{1}{\beta_2}-\theta\frac{1}{\beta_1}\frac{1}{2\beta_2}\\
&= \frac{1}{\beta_1}\frac{1}{\beta_2}\left(1+\theta\left(1+\frac{1}{4}-\frac{1}{2}-\frac{1}{2}\right)\right)\\
&= \frac{1}{\beta_1}\frac{1}{\beta_2}\left(1+\frac{\theta}{4}\right).
\end{aligned}$$

Finally, we find

$$\begin{aligned} Cov\left(X_1, X_2\right) &= E\left[X_1 X_2\right] - E\left[X_1\right] E\left[X_2\right] \\ &= \frac{1}{\beta_1}\frac{1}{\beta_2}\frac{\theta}{4}. \end{aligned}$$

5.3.3 Pearson's cofficient of (X_1, X_2)

The Pearson correlation coefficient is

$$\rho_P\left(X_1, X_2\right) = \frac{Cov\left(X_1, X_2\right)}{\sqrt{Var\left(X_1\right) Var\left(X_2\right)}} = \frac{\theta}{4}.$$

We conclude

$$-\frac{1}{4} \leq \rho_P\left(X_1, X_2\right) \leq \frac{1}{4}.$$

Also, the bivariate mgf of (X_1, X_2) is

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= (1+\theta)\left(\frac{\beta_1}{\beta_1-t_1}\right)\left(\frac{\beta_2}{\beta_2-t_2}\right) \\ &\quad -\theta\left(\frac{2\beta_1}{2\beta_1-t_1}\right)\left(\frac{\beta_2}{\beta_2-t_2}\right) \\ &\quad -\theta\left(\frac{\beta_1}{\beta_1-t_1}\right)\left(\frac{2\beta_2}{2\beta_2-t_2}\right) \\ &\quad +\theta\left(\frac{2\beta_1}{2\beta_1-t_1}\right)\left(\frac{2\beta_2}{2\beta_2-t_2}\right). \end{aligned}$$

5.3.4 Simulation

We recall the expression of $F_{X_1,X_2}(x_1,x_2)$:

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= (1+\theta)\left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right) \\ &\quad -\theta\left(1-\mathrm{e}^{-2\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right) \\ &\quad -\theta\left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-2\beta_2 x_2}\right) \\ &\quad +\theta\left(1-\mathrm{e}^{-2\beta_1 x_1}\right)\left(1-\mathrm{e}^{-2\beta_2 x_2}\right). \end{aligned}$$

The simulation method is based on the conditional cdf

$$
\begin{aligned}
F_{X_2|X_1=x_1}(x_2) &= \frac{\frac{\partial}{\partial x_1} F_{X_1,X_1}(x_1, x_2)}{f_{X_1}(x_1)} \\
&= (1+\theta) \times \left(1 - \mathrm{e}^{-\beta_2 x_2}\right) \\
&\quad -\theta \times 2\mathrm{e}^{-\beta_1 x_1}\left(1 - \mathrm{e}^{-\beta_2 x_2}\right) \\
&\quad -\theta \times \left(1 - \mathrm{e}^{-2\beta_2 x_2}\right) \\
&\quad +\theta \times 2\mathrm{e}^{-\beta_1 x_1}\left(1 - \mathrm{e}^{-2\beta_2 x_2}\right).
\end{aligned}
$$

The value x_1 is assumed to be known.

We isolate x_2.

With $v_1 = \mathrm{e}^{-\beta_1 x_1}$, $v_2 = \mathrm{e}^{-\beta_2 x_2}$, $u_2 = F_{X_2|X_1=x_1}(x_2)$, we have

$$u_2 = (1+\theta)\times(1-v_2) - \theta\times 2v_1(1-v_2) - \theta\times(1-v_2) + \theta\times 2v_1\left(1-v_2^2\right)$$

We observe

$$c_2 v_2^2 + c_1 v_2 + c_0 = 0$$

with

$$\begin{aligned}
c_0 &= u_2 - (1+\theta) + \theta\times 2v_1 + \theta - \theta 2u_1 \\
&= u_2 - 1 + \theta\times 2v_1 - \theta\times 2v_1 \\
&= u_2 - 1 \\
c_1 &= (1+\theta) - \theta 2v_1 - \theta \\
&= 1 - \theta 2v_1 \\
c_2 &= \theta\times 2v_1
\end{aligned}$$

We conclude

$$\begin{aligned} v_2 &= \frac{-2c_1 + \sqrt{c_1^2 - 4c_2c_0}}{2c_2} \\ &= \frac{-2\left(1-\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}\right) + \sqrt{\left(1-\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}\right)^2 - 4\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}\left(u_2-1\right)}}{2\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} x_2 &= -\frac{1}{\beta_2}\ln\left(v_2\right) \\ &= -\frac{1}{\beta_2}\ln\left(\frac{-2\left(1-\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}\right) + \sqrt{\left(1-\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}\right)^2 - 4\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}\left(u_2-1\right)}}{2\theta\times 2\times \mathrm{e}^{-\beta_1 x_1}}\right) \quad (9) \\ &= F^{-1}_{X_2|X_1=x_1}\left(u_2\right). \quad (10) \end{aligned}$$

The simulation has two steps.

First, we simulate a sample of the rv X_1.

Second, we use (9) to simulate of sample of the rv X_2.

Algorithm 4 Simulation for the EFGM bivariate exponential distribution.

1. *Simulate the samples* $U_1^{(j)}$ *and* $U_2^{(j)}$ *of the independent rvs* $U_1 \sim U_2 \sim Unif\,(0,1)$.
2. *Calculate the sample* $\left(X_1^{(j)}, X_2^{(j)}\right)$ *of* (X_1, X_2) *with*

$$X_1^{(j)} = -\frac{1}{\beta_1} \ln \left(1 - U_1^{(j)}\right)$$

et

$$
\begin{aligned}
X_2^{(j)} &= F^{-1}_{X_2|X_1=X_1^{(j)}}\left(U_2^{(j)}\right) \\
&= -\frac{1}{\beta_2}\ln\left(\frac{-2\left(1-\theta\times 2\times \mathrm{e}^{-\beta_1 X_1^{(j)}}\right)+\sqrt{\left(1-\theta\times 2\times \mathrm{e}^{-\beta_1 X_1^{(j)}}\right)^2 - 4\theta\times 2\times \mathrm{e}^{-\beta_1 X_1^{(j)}}\left(U_2^{(j}\right.}}{2\theta\times 2\times \mathrm{e}^{-\beta_1 X_1^{(j)}}}\right. \\
&= -\frac{1}{\beta_2}\ln\left(\frac{-2\left(1-\theta\times 2\times\left(1-U_1^{(j)}\right)\right)+\sqrt{\left(1-\theta\times 2\times\left(1-U_1^{(j)}\right)\right)^2 - 4\theta\times 2\times\left(1 -\right.}}{2\theta\times 2\times\left(1-U_1^{(j)}\right)}\right.
\end{aligned}
$$

5.3.5 Risk aggregation

We define $S = X_1 + X_2$.

Then, the mgf of the rv S is

$$\begin{aligned} M_S(t) &= (1+\theta)\left(\frac{\beta_1}{\beta_1-t}\right)\left(\frac{\beta_2}{\beta_2-t}\right) \\ &- \theta\left(\frac{2\beta_1}{2\beta_1-t}\right)\left(\frac{\beta_2}{\beta_2-t}\right) \\ &- \theta\left(\frac{\beta_1}{\beta_1-t}\right)\left(\frac{2\beta_2}{2\beta_2-t}\right) \\ &+ \theta\left(\frac{2\beta_1}{2\beta_1-t}\right)\left(\frac{2\beta_2}{2\beta_2-t}\right). \end{aligned}$$

From $M_S(t)$, we can find the following expression for $F_S(s)$:

$$\begin{aligned} F_S(x) &= (1+\theta)\,G(x;\beta_1;\beta_2)+\theta G(x;2\beta_1;2\beta_2) \\ &- \theta G(x;2\beta_1;\beta_2)-\theta G(x;\beta_1;2\beta_2), \end{aligned}$$

where

$$G(x;\gamma_1,\gamma_2)=\begin{cases} 1-\mathrm{e}^{-\gamma x}\sum_{j=0}^{1}\frac{(\gamma x)^j}{j!}, \ \gamma_1=\gamma_2=\gamma \\ \sum_{i=1}^{2}\left(\prod_{j=1,j\neq i}^{2}\frac{\gamma_j}{\gamma_j-\gamma_i}\right)\left(1-\mathrm{e}^{-\gamma_i x}\right), \ \gamma_1\neq\gamma_2 \end{cases}.$$

The distribution of the rv S is linear combination of Erlang distribution ($\gamma_1 = \gamma_2 = \gamma$) or/and Generalized Erlang distributions ($\gamma_1 \neq \gamma_2$).

The expression for $TVaR_\kappa(S)$ is

$$\begin{aligned} TVaR_\kappa(S) &= \frac{1}{1-\kappa}(1+\theta)\zeta\left(VaR_\kappa(S);\beta_1,\beta_2\right) \\ &\quad +\frac{1}{1-\kappa}\theta\zeta\left(VaR_\kappa(S);2\beta_1,2\beta_2\right) \\ &\quad -\frac{1}{1-\kappa}\theta\zeta\left(VaR_\kappa(S);2\beta_1,\beta_2\right) \\ &\quad -\frac{1}{1-\kappa}\theta\zeta\left(VaR_\kappa(S);\beta_1,2\beta_2\right), \end{aligned}$$

where

$$\zeta(b;\gamma_1,\gamma_2) = \begin{cases} \frac{2}{\gamma}\mathrm{e}^{-\gamma b}\sum_{j=0}^{2}\frac{(\gamma b)^j}{j!}, & \gamma_1=\gamma_2=\gamma \\ \sum_{i=1}^{2}\left(\prod_{j=1,j\neq i}^{2}\frac{\gamma_j}{\gamma_j-\gamma_i}\right)\mathrm{e}^{-\gamma_i b}\left(b+\frac{1}{\gamma_i}\right), & \gamma_1\neq\gamma_2 \end{cases}.$$

5.4 Marshall-Olkin Bivariate Exponential distribution

5.4.1 Definition and properties

Let Y_0, Y_1, Y_2 be three independent rvs with $Y_i \sim Exp(\lambda_i)$ for $i = 0, 1, 2$.

We define les v.a. X_1 et X_2 par $X_i = \min(Y_i; Y_0)$ pour $i = 1, 2$.

For $i = 1, 2$, we observe that

$$
\begin{aligned}
\overline{F}_{X_i}(x_i) &= \Pr(X_i > x_i) \\
&= \Pr(\min(Y_i; Y_0) > x_i) = \Pr(Y_i > x_i, Y_0 > x_i) \\
&= \Pr(Y_i > x_i)\Pr(Y_0 > x_i) \\
&= \overline{F}_{Y_i}(x_i)\,\overline{F}_{Y_0}(x_i) = \exp(-(\lambda_i + \lambda_0)\,x_i)\,,
\end{aligned}
$$

It implies that

$$X_i \sim Exp\left(\lambda_i + \lambda_0\right), \; i = 1, 2.$$

The bivariate survival function of the pair of rvs (X_1, X_2) is

$$\begin{aligned}
\overline{F}_{X_1,X_2}(x_1, x_2) &= \Pr\left(X_1 > x_1, X_2 > x_2\right) \\
&= \Pr\left(\min\left(Y_1; Y_0\right) > x_1, \min\left(Y_2; Y_0\right) > x_2\right) \\
\blacksquare \quad &= \Pr\left(Y_1 > x_1, Y_0 > x_1, Y_2 > x_2, Y_0 > x_2\right) \\
&= \Pr\left(Y_1 > x_1, Y_2 > x_2, Y_0 > \max\left(x_1; x_2\right)\right) \\
&= \Pr\left(Y_1 > x_1\right) \Pr\left(Y_2 > x_2\right) \Pr\left(Y_0 > \max\left(x_1; x_2\right)\right) \\
&= \mathrm{e}^{-\lambda_1 x_1} \mathrm{e}^{-\lambda_2 x_2} \mathrm{e}^{-\lambda_0 \max(x_1; x_2)}
\end{aligned}$$

which becomes

$$\begin{aligned}
\overline{F}_{X_1,X_2}(x_1, x_2) &= \mathrm{e}^{-\lambda_1 x_1} \mathrm{e}^{-\lambda_2 x_2} \mathrm{e}^{-\lambda_0 (x_1 + x_2 - \min(x_1; x_2))} \\
&= \mathrm{e}^{-(\lambda_1 + \lambda_0) x_1} \mathrm{e}^{-(\lambda_2 + \lambda_0) x_2} \mathrm{e}^{\lambda_0 \min(x_1; x_2)} \\
&= \mathrm{e}^{-\beta_1 x_1} \mathrm{e}^{-\beta_2 x_2} \mathrm{e}^{\lambda_0 \min(x_1; x_2)}
\end{aligned}$$

We fix

$$\beta_i = \lambda_i + \lambda_0 \ (i = 1, 2)$$

and

$$0 \leq \lambda_0 \leq \min(\beta_1; \beta_2).$$

Then, $\overline{F}_{X_1,X_2}(x_1, x_2)$ becomes

$$\begin{aligned}\overline{F}_{X_1,X_2}(x_1, x_2) &= \mathrm{e}^{-\beta_1 x_1} \mathrm{e}^{-\beta_2 x_2} \mathrm{e}^{\lambda_0 \min(x_1; x_2)} \\ &= \overline{F}_{X_1}(x_1) \overline{F}_{X_2}(x_2) \mathrm{e}^{\lambda_0 \min(x_1; x_2)}.\end{aligned}$$

Interpretation : the Marshall-Olkin bivariate exponential distribution can be seen as a form of perturbation of the bivariate exponential distribution with independence.

This distribution induces a positive dependence relation only.

The bivariate pdf of the pair of rvs (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \beta_1 \mathrm{e}^{-\beta_1 x_1} (\beta_2 - \lambda_0) \mathrm{e}^{-(\beta_2-\lambda_0)x_2}, & x_1 > x_2, \\ (\beta_1 - \lambda_0) \mathrm{e}^{-(\beta_1-\lambda_0)x_1} \beta_2 \mathrm{e}^{-\beta_2 x_2}, & x_1 < x_2, \\ \lambda_0 \mathrm{e}^{-\beta_1 x} \mathrm{e}^{-\beta_2 x} \mathrm{e}^{\lambda_0 x}, & x_1 = x_2 = x, \end{cases}$$

with a singularity on the diagonal $x_1 = x_2 = x$.

To obtain f_{X_1,X_2}, we proceed as follows.

For the continuous part, we have

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{\partial^2}{\partial x_2 \partial x_1} \overline{F}_{X_1,X_2}(x_1,x_2) \\ &= \frac{\partial^2}{\partial x_2 \partial x_1} \mathrm{e}^{-\beta_1 x_1} \mathrm{e}^{-\beta_2 x_2} \mathrm{e}^{\lambda_0 \min(x_1;x_2)} \end{aligned}$$

If $x_1 > x_2$, we have

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{\partial^2}{\partial x_2 \partial x_1} \mathrm{e}^{-\beta_1 x_1} \mathrm{e}^{-\beta_2 x_2} \mathrm{e}^{\lambda_0 x_2} \\ &= \frac{\partial^2}{\partial x_2 \partial x_1} \mathrm{e}^{-\beta_1 x_1} \mathrm{e}^{-(\beta_2 - \lambda_0) x_2} \\ &= \beta_1 \mathrm{e}^{-\beta_1 x_1} (\beta_2 - \lambda_0) \mathrm{e}^{-(\beta_2 - \lambda_0) x_2}. \end{aligned}$$

If $x_1 < x_2$, we have

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{\partial^2}{\partial x_2 \partial x_1} \mathrm{e}^{-\beta_1 x_1} \mathrm{e}^{-\beta_2 x_2} \mathrm{e}^{\lambda_0 x_1} \\ &= \frac{\partial^2}{\partial x_2 \partial x_1} \mathrm{e}^{-(\beta_1 - \lambda_0) x_1} \mathrm{e}^{-\beta_2 x_2} \\ &= (\beta_1 - \lambda_0) \mathrm{e}^{-(\beta_1 - \lambda_0) x_1} \beta_2 \mathrm{e}^{-\beta_2 x_2}. \end{aligned}$$

For the singularity part (i.e. when $x_1 = x_2 = x$), we have

$$\begin{aligned} f_{X_1,X_2}(x,x)\,\mathrm{d}x &\simeq \Pr(x < X_1 \leq x + \mathrm{d}x, x < X_2 \leq x + \mathrm{d}x) \\ &= \Pr(x < Y_0 \leq x + \mathrm{d}x, Y_1 > x, Y_2 > x) \\ &= \left(f_{Y_0}(x)\,\mathrm{d}x\right) \Pr(Y_1 > x) \Pr(Y_2 > x) \\ &= \lambda_0 \mathrm{e}^{-\lambda_0 x} \mathrm{e}^{-(\beta_1 - \lambda_0)x} \mathrm{e}^{-(\beta_2 - \lambda_0)x} \\ &= \lambda_0 \mathrm{e}^{-(\beta_1 + \beta_2 - \lambda_0)x}. \end{aligned}$$

The Pearson correlation coefficient is $\rho_P(X_1, X_2) = \frac{\lambda_0}{\beta_1 + \beta_2 - \lambda_0}$.

Consequently, we observe

$$0 \leq \rho_P(X_1, X_2) \leq \frac{\min(\beta_1; \beta_2)}{\beta_1 + \beta_2 - \min(\beta_1; \beta_2)}.$$

If $\beta_1 \leq \beta_2$, we have

$$0 \leq \rho_P(X_1, X_2) \leq \frac{\beta_1}{\beta_2}.$$

We conclude

$$0 \leq \rho_P(X_1, X_2) \leq \min\left(\frac{\beta_1}{\beta_2}; \frac{\beta_2}{\beta_1}\right).$$

The methode used to build the Marshall-Olkin bivariate exponential distribution is called the *common shock* method.

This method can be easily adapted to construct multivariate exponential distributions.

5.4.2 Simulation

Algorithm 5 Simulation for lhe Marshall-Olkin bivariate exponential distribution.

1. *Simulate sample values* $Y_0^{(j)}$, $Y_1^{(j)}$ *and* $Y_2^{(j)}$ *of the rvs* Y_0, Y_1 *and* Y_2 *where* $Y_0 \sim Exp\left(\lambda_0\right)$, $Y_1 \sim Exp\left(\beta_1 - \lambda_0\right)$ *and* $Y_2 \sim Exp\left(\beta_2 - \lambda_0\right)$.
2. *Calculate the pair of sample values* $\left(X_1^{(j)}, X_2^{(j)}\right)$ *of the pair of rvs* (X_1, X_2) *with*

$$X_i^{(j)} = \min\left(Y_0^{(j)}; Y_i^{(j)}\right),$$

for $i = 1, 2$.

6 Bivariate gamma distributions

6.1 Introduction

There are several multivariate gamma distributions.

In this section, we only consider the CRMM gamma distribution.

6.2 Cheriyan - Ramabhadran - Mathai - Moschopoulos (CRMM) bivariate gamma distribution

6.2.1 Definition and properties

Let Y_0, Y_1 and Y_2 are independent rvs where $Y_0 \sim Ga(\gamma_0, \beta_0)$, $Y_1 \sim Ga(\alpha_1 - \gamma_0, \beta_1)$ and $Y_2 \sim Ga(\alpha_2 - \gamma_0, \beta_2)$, with $0 \leq \gamma_0 \leq \min(\alpha_1; \alpha_2)$.

We define the rvs X_1 and X_2 by

$$X_i = \frac{\beta_0}{\beta_i} Y_0 + Y_i$$

for $i = 1, 2$.

First, we observe that $X_i \sim Ga(\alpha_i, \beta_i)$, $i = 1, 2$.

For $i = 1$, we have

$$
\begin{aligned}
M_{X_1}(t) &= E[\exp(X_1 t)] \\
&= E\left[e^{t\frac{\beta_0}{\beta_1}Y_0} e^{tY_1}\right] \\
&= E\left[e^{t\frac{\beta_0}{\beta_1}Y_0}\right] E\left[e^{tY_1}\right] \\
&= \left(\frac{1}{1-\frac{1}{\beta_0}\frac{\beta_0}{\beta_1}t}\right)^{\gamma_0} \left(\frac{1}{1-\frac{1}{\beta_1}t}\right)^{\alpha_1-\gamma_0} \\
&= \left(\frac{1}{1-\frac{1}{\beta_1}t}\right)^{\alpha_1}.
\end{aligned}
$$

The covariance is

$$\begin{aligned}
Cov\left(X_1, X_2\right) &= Cov\left(\frac{\beta_0}{\beta_1}Y_0 + Y_1, \frac{\beta_0}{\beta_2}Y_0 + Y_2\right) \\
&= \frac{\beta_0}{\beta_1}\frac{\beta_0}{\beta_2}Cov\left(Y_0, Y_0\right) \\
&= \frac{\beta_0}{\beta_1}\frac{\beta_0}{\beta_2}Var\left(Y_0\right) \\
&= \frac{\beta_0}{\beta_1}\frac{\beta_0}{\beta_2} \times \frac{\gamma_0}{\beta_0^2} \\
&= \frac{\gamma_0}{\beta_1\beta_2}.
\end{aligned}$$

It followns that the pair of rvs (X_1, X_2) follows a gamma bivariate distribution with $X_i \sim Ga\left(\alpha_i, \beta_i\right)$, $i = 1, 2$, and $\rho_P\left(X_1, X_2\right) = \frac{\gamma_0}{\sqrt{\alpha_1\alpha_2}}$.

We observe

$$0 \leq \rho_P(X_1, X_2) \leq \frac{\min(\alpha_1; \alpha_2)}{\sqrt{\alpha_1 \alpha_2}} = \min\left(\sqrt{\frac{\alpha_1}{\alpha_2}}; \sqrt{\frac{\alpha_2}{\alpha_1}}\right).$$

The parameter γ_0 corresponds to the dependence paramater,

The expression of the mgf of (X_1, X_2) is given by

$$\begin{aligned} M_{X_1,X_2}(t_1, t_2) &= E\left[e^{t_1 X_1} e^{t_2 X_2}\right] \\ &= E\left[e^{t_1 Y_1}\right] E\left[e^{t_2 Y_2}\right] E\left[e^{\left(\frac{\beta_0}{\beta_1} t_1 + \frac{\beta_0}{\beta_2} t_2\right) Y_0}\right] \\ &= \left(1 - \frac{t_1}{\beta_1}\right)^{-(\alpha_1 - \gamma_0)} \left(1 - \frac{t_2}{\beta_2}\right)^{-(\alpha_2 - \gamma_0)} \\ &\times \left(1 - \frac{t_1}{\beta_1} - \frac{t_2}{\beta_2}\right)^{-\gamma_0}. \end{aligned}$$

6.2.2 Simulation

Algorithm 6 Sampling from Cheriyan - Ramabhadran - Mathai - Moschopoulos (CRMM) bivariate gamma distribution

1. *Simulate sample values* $Y_0^{(j)}$, $Y_1^{(j)}$ *and* $Y_2^{(j)}$ *of the independent rvs* Y_0, Y_1 *and* Y_2 *where* $Y_0 \sim Ga\left(\gamma_0, \beta_0\right)$, $Y_1 \sim Ga\left(\alpha_1 - \gamma_0, \beta_1\right)$ *and* $Y_2 \sim Ga\left(\alpha_2 - \gamma_0, \beta_2\right)$.
2. *Calculate the sampling* $\left(X_1^{(j)}, X_2^{(j)}\right)$ *of* (X_1, X_2) *with* $X_i^{(j)} = \frac{\beta_0}{\beta_i} Y_0^{(j)} + Y_i^{(j)}$, *for* $i = 1, 2$.

6.2.3 Aggregation

We define $S = X_1 + X_2$.

Since $M_S(t) = M_{X_1,X_2}(t,t)$, we find that

$$M_S(t) = \left(1 - \frac{t}{\beta_1}\right)^{-(\alpha_1-\gamma_0)} \left(1 - \frac{t}{\beta_2}\right)^{-(\alpha_2-\gamma_0)} \left(1 - \frac{t}{\beta_1} - \frac{t}{\beta_2}\right)^{-\gamma_0},$$

which corresponds to the mgf for the sum of the three following independent rvs :

$$W_1 \sim Gamma(\alpha_1 - \gamma_0, \beta_1),$$

$$W_2 \sim Gamma(\alpha_2 - \gamma_0, \beta_2),$$

and

$$W_0 \sim Gamma\left(\gamma_0, \frac{1}{\frac{1}{\beta_1} + \frac{1}{\beta_2}}\right).$$

We have

$$S = W_1 + W_2 + W_0$$

where the independent rvs W_1, W_2 and W_0 follow gamma distribution with different scale parameters.

Then, it implies that the rv S does not follow a gamma distribution.

We apply the result provided in Proposition **??** to conclude that the rv S follows a mixture of gamma distributions with

$$f_S(x) = \sum_{k=0}^{\infty} p_k h(x; \alpha + k, \beta),$$

where

$$\beta = \max\left(\beta_1; \beta_2; \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)^{-1}\right)$$

and $\alpha = \alpha_1 + \alpha_2 - \gamma_0$.

The probabilities p_k, $k \in \mathbb{N}$, are defined by $p_k = \sigma \times \xi_k$ where

$$\sigma = \beta^{-\gamma_0} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right)^{-\gamma_0} \prod_{i=1}^{2} \left(\frac{\beta_i}{\beta} \right)^{\alpha_i - \gamma_0},$$

and

$$\xi_0 = 1,\ \xi_k = \frac{1}{k} \sum_{i=1}^{k} i \zeta_i \xi_{k-i},\ k \in \mathbb{N}^+,$$

with

$$\zeta_k = \frac{\gamma_0}{k} \left(1 - \left(\beta \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right)^{-1} \right)^k + \sum_{i=1}^{2} \frac{(\alpha_i - \gamma_0)}{k} \left(1 - \frac{\beta_i}{\beta} \right)^k,$$

for $k \in \mathbb{N}^+$.

We obtain

$$F_S(x) = \sum_{k=0}^{\infty} p_k H(x; \alpha + k, \beta)$$

and

$$E\left[S \times 1_{\{S>b\}}\right] = \sum_{k=0}^{\infty} p_k \frac{\alpha + k}{\beta} \overline{H}(x; \alpha + k + 1, \beta).$$

Example 7 *Let* (X_1, X_2) *be a pair of rvs which follows a CRMM gamma bivariate distribution with* $\beta_1 = 0.1$, $\beta_2 = 0.2$, $\alpha_1 = 2$, $\alpha_2 = 4$.

For $S = X_1 + X_2$, we find the following values :

γ_0	κ	$VaR_\kappa(S)$	$TVaR_\kappa(S)$
0	0.95	72.2301	84.5060
0	0.995	100.2088	111.6268
0.5	0.95	75.0652	89.2894
0.5	0.995	107.6104	121.4649
1	0.95	77.7790	93.4458
1	0.995	113.6537	128.8391

Note that the values of Var (S) *are 300, 350 and 400 for* $\gamma_0 =$ *0, 0.5 and 1.*
$\square$

7 Multivariate normal distribution and its extensions

7.1 Introduction

The multivariate normal distribution is very well known.

We also treat some of its extensions.

7.2 Multivariate normal distribution

7.2.1 Definition and properties

We consider the multivariate normal distribution for the vector of rvs $\underline{X} = (X_1, ..., X_n)^t$ with the vector of means $\underline{\mu} = (\mu_1, ..., \mu_n)^t$ and the variance-covariance matrix

$$\underline{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix},$$

where

- $\underline{\Sigma}$ is a positive semi-definite (halfo-definite) matrix
- and $()^t$ corresponds to the transpose of a matrix or a vector.

Also, we have

- $E[X_i] = \mu_i$,
- $\mathrm{Var}(X_i) = \sigma_i^2$ $(i = 1, 2, ..., n)$
- and $\mathrm{Cov}(X_i, X_{i'}) = \sigma_{ii'} = \rho_{ii'}\sigma_i\sigma_{i'}$, $(i,\ i' = 1, 2, ..., n)$,
- where $\rho_{ii'}$ is the Pearson's coefficient of correlation for the pair $(X_i, X_{i'})$.

The expression for the multivariate pdf of $\underline{X}$ is

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\underline{\boldsymbol{\Sigma}}|^{\frac{1}{2}}} \mathrm{e}^{-\frac{1}{2}(\underline{x}-\underline{\mu})^t \underline{\boldsymbol{\Sigma}}^{-1}(\underline{x}-\underline{\mu})}, \quad x \in \mathbb{R}^n,$$

where $|\underline{\boldsymbol{\Sigma}}|$ is the determinant of $\underline{\boldsymbol{\Sigma}}$.

The multivariate mgf of $\underline{X}$ is

$$\begin{aligned} M_{\underline{X}}(\underline{s}) &= \mathrm{e}^{\underline{s}^t\underline{\mu}+\frac{1}{2}\underline{s}^t\underline{\boldsymbol{\Sigma}}\underline{s}} \\ &= \mathrm{e}^{\sum_{i=1}^n \mu_i s_i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \sigma_{i,j} s_i s_j} \end{aligned}$$

The covariance-variance matrix can also be written as follows :

$$\underline{\boldsymbol{\Sigma}} = \underline{\sigma}^t \underline{\sigma}$$

where

$$\underline{\sigma}^t = (\sigma_1, ..., \sigma_n)$$

and

$$\underline{\rho} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}.$$

In the univariate case, if $X \sim N\left(\mu, \sigma^2\right)$, then $X = \mu + \sigma Z$, where $Z \sim N(0, 1)$.

In the multivariate case, that relation becomes

$$\underline{X} = \underline{\mu} + \underline{\sigma}^t \underline{Z}, \tag{11}$$

where $\underline{Z}$ follows a multivariate normal standard distribution with

- a vector of means $(0, ..., 0)^t$
- and the variance-covariance matrix $\underline{\rho}$.

The multivariate cdf of $\underline{Z}$ is denoted by the symbol $\overline{\Phi}_{\underline{\rho}}$ such that

$$\overline{\Phi}_{\underline{\rho}}(x_1, ..., x_n) = F_{Z_1,...,Z_n}(x_1, ..., x_n).$$

Also, for $\underline{X} = \underline{\mu} + \underline{\sigma}^t \underline{Z}$, we have

$$F_{X_1,...,X_n}(x_1, ..., x_n) = \overline{\Phi}_{\underline{\rho}}\left(\frac{x_1 - \mu_1}{\sigma_1}, ..., \frac{x_n - \mu_n}{\sigma_n}\right).$$

7.2.2 Choleski decomposition

Let

$$\underline{Z} = (Z_1, ..., Z_n)^t$$

be a vector of rvs which follow a multivariate standard normal distribution with $Z_i \sim N(0, 1)$ $(i = 1, 2, ..., n)$ and a correlation matrix (assumed to be positive definite)

$$\underline{\rho} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}.$$

Then, we can write $\underline{\rho} = \underline{B}\ \underline{B}^t$ where $\underline{B}^t$ is the transpose of $\underline{B}$.

The matrix $\underline{B}$ can be obtained the Choleski decomposition

$$\underline{B} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

witj

$$bij = \frac{\rho_{ij} - \sum_{l=1}^{j-1} b_{il} b_{jl}}{\sqrt{1 - \sum_{l=1}^{j-1} b_{jl}^2}},$$

where $1 \leq j \leq i \leq n$ and $\sum_{l=1}^{0} () = 0$.

Then, we have

$$\underline{Z} = \underline{B}\ \underline{Y}.$$

7.2.3 Simulation

The Choleski decomposition is useful for sampling from the multivariate standard normal distribution.

Algorithm 8 Simulation for the multivariate standard normal distribution.

1. *We simulate the samplings* $Y_1^{(j)}, ..., Y_n^{(j)}$ *of the independent standard normal rvs* $Y_1, ..., Y_n$.
2. *We calculate*

$$\underline{Z}^{(j)} = \underline{B}\ \underline{Y}^{(j)}$$

where

$$\underline{Z}^{(j)} = \left(Z_1^{(j)}, ..., Z_n^{(j)}\right)^t$$

and

$$\underline{Y}^{(j)} = \left(Y_1^{(j)}, ..., Y_n^{(j)}\right)^t.$$

The algorithm for the simulation of sampling of $\underline{X}$ which follows a multivariate normal distribution with parameters $\left(\underline{\mu}, \underline{\mathbf{\Sigma}}\right)$ is based on

$$\underline{X} = \underline{\mu} + \underline{\sigma}^t \underline{Z}.$$

Algorithm 9 Simulation pour la loi normale multivariée.

1. *We simulate a sampling* $\underline{Z}^{(j)} = \left(Z_1^{(j)}, ..., Z_n^{(j)}\right)^t$ *of* $\underline{Z} = (Z_1, ..., Z_n)^t$.

2. *We calculate a sampling of $\underline{X}^{(j)}$ of $\underline{X}$ with*

$$\underline{X}^{(j)} = \underline{\mu} + \underline{\sigma}^t \underline{Z}^{(j)}.$$

7.3 Risk aggregation

We define $S = \sum_{i=1}^{n} X_i$.

Using the mgf of $\underline{X}$, we find the mgf of S

$$\begin{aligned} M_S(s) &= M_{X_1,...,X_n}(s,...,s) \\ &= \mathrm{e}^{s\left(\sum_{i=1}^{n}\mu_i\right)+\frac{1}{2}s^2\left(\sum_{i=1}^{n}\sigma_i^2+\sum_{i=1}^{n}\sum_{i'=1,i'\neq i}^{n}\sigma_{ii'}.\right)}. \end{aligned}$$

Then, from the mgf of the rv S, we conclude that

$$S \sim N\left(\mu_S, \sigma_S^2\right)$$

where

- $\mu_S = \sum_{i=1}^n \mu_i$
- $\sigma_S^2 = \sum_{i=1}^n \sigma_i^2 + \sum_{i=1}^n \sum_{i'=1, i' \neq i}^n \sigma_{ii'}$.

8 Bivariate and multivariate discrete distributions

Many bivariate and multivariate discrete distributions whose marginals are Poisson, binomial or negative binomial.

In the present version of the document, we only present some of them.

8.1 Teicher bivariate Poisson distribution

8.1.1 Definition and properties

The Teicher bivariate Poisson distribution (voir [**?**]) is the simplest bivariate distribution of the pair of rvs (M_1, M_2) with

$$\begin{aligned} M_1 &\sim Poisson\,(\lambda_1) \\ M_2 &\sim Poisson\,(\lambda_2)\,. \end{aligned}$$

The dependence parameter is $0 \leq \alpha_0 \leq \min(\lambda_1; \lambda_2)$.

Let K_0, K_1, K_2 be independent rvs with $K_i \sim Pois\,(\alpha_i)$, $i = 0, 1, 2$, where $0 \leq \alpha_0 \leq \min(\lambda_1; \lambda_2)\,,\ \alpha_1 = \lambda_1 - \alpha_0$ and $\alpha_2 = \lambda_2 - \alpha_0$.

We define

$$M_1 = K_1 + K_0 \text{ et } M_2 = K_2 + K_0.$$

Clearly,

$$M_i \sim Pois(\lambda_i), i = 1, 2.$$

The covariance is

$$\begin{aligned} \mathsf{Cov}(M_1, M_2) &= Cov(K_1 + K_0, K_2 + K_0) \\ &= Cov(K_0, K_0) \\ &= \mathsf{Var}(K_0) = \alpha_0. \end{aligned}$$

Pearson's correlation coefficient is

$$\begin{aligned}\rho_P(M_1, M_2) &= \frac{Cov(M_1, M_2)}{\sqrt{Var(M_1)\,Var(M_2)}} \\ &= \frac{\alpha_0}{\sqrt{\lambda_1\lambda_2}}.\end{aligned}$$

It implies that the dependence relatition induces by this construction is always positive.

We observe that

$$0 \leq \rho_P(M_1, M_2) \leq \frac{\min(\lambda_1; \lambda_2)}{\sqrt{\lambda_1\lambda_2}} = \min\left(\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}}; \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}\right).$$

If $\lambda_1 = \lambda_2 = \lambda$, then

$$0 \leq \rho_P(M_1, M_2) \leq 1.$$

If $\lambda_1 = m\lambda_2$ $(m > 1)$, then

$$0 \leq \rho_P(M_1, M_2) \leq \frac{\min(m\lambda_2; \lambda_2)}{\sqrt{m\lambda_2\lambda_2}} = \frac{\lambda_2}{\sqrt{m\lambda_2\lambda_2}} = \frac{1}{\sqrt{m}}.$$

The mgf of (M_1, M_2) is

$$\begin{aligned}
M_{M_1,M_2}(t_1, t_2) &= E\left[\mathrm{e}^{t_1 M_1}\mathrm{e}^{t_2 M_2}\right] \\
&= E\left[\mathrm{e}^{t_1(K_1+K_0)}\mathrm{e}^{t_2(K_2+K_0)}\right] \\
&= E\left[\mathrm{e}^{t_1 K_1}\right] E\left[\mathrm{e}^{t_2 K_2}\right] E\left[\mathrm{e}^{(t_1+t_2)K_0}\right]. \\
&= \mathrm{e}^{(\lambda_1-\alpha_0)\left(\mathrm{e}^{t_1}-1\right)}\mathrm{e}^{(\lambda_2-\alpha_0)\left(\mathrm{e}^{t_2}-1\right)}\mathrm{e}^{\alpha_0\left(\mathrm{e}^{t_1+t_2}-1\right)}.
\end{aligned}$$

Similarly, the bivariate pgf is

$$\begin{aligned}
P_{M_1,M_2}(t_1, t_2) &= E\left[t_1^{M_1} t_2^{M_2}\right] \\
&= \mathrm{e}^{(\lambda_1-\alpha_0)(t_1-1)}\mathrm{e}^{(\lambda_2-\alpha_0)(t_2-1)}\mathrm{e}^{\alpha_0(t_1 t_2-1)}.
\end{aligned}$$

The bivariate pmf of (M_1, M_2) is

$$f_{M_1,M_2}(m_1,m_2) = \mathrm{e}^{-\lambda_1-\lambda_2+\alpha_0} \sum_{j=0}^{\min(m_1;m_2)} \frac{\alpha_0^j}{j!} \frac{(\lambda_1-\alpha_0)^{m_1-j}}{(m_1-j)!} \frac{(\lambda_2-\alpha_0)^{m_2-j}}{(m_2-j)!}, \tag{12}$$

Indeed, we observe

- let $m_1 = 0$, $m_2 = 0$:

$$\begin{aligned} f_{M_1,M_2}(0,0) &= \Pr(M_1 = 0, M_2 = 0) \\ &= \Pr(K_1 + K_0 = 0, K_2 + K_0 = 0) \\ &= \Pr(K_1 + K_0 = 0, K_2 + K_0 = 0 | K_0 = 0) \Pr(K_0 = 0) \\ &= \Pr(K_1 = 0, K_2 = 0) \Pr(K_0 = 0) \\ &= \Pr(K_1 = 0) \Pr(K_2 = 0) \Pr(K_0 = 0) \end{aligned}$$

- let $m_1, m_2 \in \mathbb{N}$:

$$
\begin{aligned}
f_{M_1,M_2}(m_1,m_2) &= \Pr(K_1 + K_0 = m_1, K_2 + K_0 = m_2) \\
&= \sum_{j=0}^{\min(m_1;m_2)} \Pr(K_0 = j) \Pr(K_1 + K_0 = m_1, K_2 + K_0 = \\
&= \sum_{j=0}^{\min(m_1;m_2)} \Pr(K_0 = j) \Pr(K_1 = m_1 - j) \Pr(K_2 = m_2 \\
&= \sum_{j=0}^{\min(m_1;m_2)} e^{-\alpha_0} \frac{\alpha_0^j}{j!} e^{-\lambda_1 - \alpha_0} \frac{(\lambda_1 - \alpha_0)^{m_1 - j}}{(m_1 - j)!} \frac{(\lambda_2 - \alpha_0)^{m_2}}{(m_2 - j)!}
\end{aligned}
$$

This method of construction is simple.

However, it only allows a positive dependence relation between the components of (M_1, M_2).

We can also show that

$$E[M_1|M_2 = m_2] = (\lambda_1 - \alpha_0) + \frac{\alpha_0}{\lambda_2} m_2.$$

Notation : $(M_1, M_2) \sim PBiv(\lambda_1, \lambda_2, \alpha_0)$ with $\lambda_1,\ \lambda_2 > 0$, $0 \leq \alpha_0 \leq \min(\lambda_1; \lambda_2)$.

8.1.2 Simulation

Algorithm 10 Simulation for the Teicher bivariate Poisson distribution.

1. *Simulate the sample values $K_0^{(j)}$, $K_1^{(j)}$ and $K_2^{(j)}$ of the independent rvs K_0, K_1 and K_2 where $K_0 \sim Pois(\alpha_0)$, $K_1 \sim Pois(\lambda_1 - \alpha_0)$ and $K_2 \sim Pois(\lambda_2 - \alpha_0)$.*

2. *Compute the sample value* $\left(M_1^{(j)}, M_2^{(j)}\right)$ *of the pair of rvs* (M_1, M_2) *with* $M_i^{(j)} = K_0^{(j)} + K_i^{(j)}$*, for* $i = 1, 2$*.*

8.1.3 Risk aggregation

Let $(M_1, M_2) \sim PBiv\,(\lambda_1, \lambda_2, \alpha_0 = 1)$.

We define $N = M_1 + M_2$.

Notice that

$$\begin{aligned} N &= M_1 + M_2 \\ &= K_1 + K_2 + 2K_0. \end{aligned}$$

Then, we can show that

$$N \sim PComp\left(\lambda_N, F_B\right)$$

where $\lambda_N = \lambda_1 + \lambda_2 - \alpha_0$ and

$$f_B\left(k\right) = \begin{cases} \frac{\lambda_1+\lambda_2-2\alpha_0}{\lambda_N} & , k = 1 \\ \frac{\alpha_0}{\lambda_N} & , k = 2 \\ 0 & , k \in N \backslash \left\{1, 2\right\} \end{cases}.$$

Indeed, we have

$$P_N\left(t\right) = P_{M_1,M_2}\left(t, t\right) = \mathrm{e}^{(\lambda_1-\alpha_0)(t-1)}\mathrm{e}^{(\lambda_2-\alpha_0)(t-1)}\mathrm{e}^{\alpha_0\left(t^2-1\right)}, \qquad (13)$$

which becomes

$$\begin{aligned}
P_N(t) &= \mathrm{e}^{(\lambda_1-\alpha_0)t+(\lambda_2-\alpha_0)t+\alpha_0 t^2-((\lambda_1-\alpha_0)+(\lambda_2-\alpha_0)+\alpha_0)} \\
&= \mathrm{e}^{((\lambda_1-\alpha_0)+(\lambda_2-\alpha_0))t+\alpha_0 t^2-\lambda_N} \\
&= \mathrm{e}^{\lambda_N\left(\frac{((\lambda_1-\alpha_0)+(\lambda_2-\alpha_0))}{\lambda_N}t+\frac{\alpha_0}{\lambda_N}t^2-1\right)} \\
&= \mathrm{e}^{\lambda_N\left(f_B(1)t+f_B(2)t^2-1\right)},
\end{aligned}$$

leading to the desired result.

To evaluate f_N, we can use the Panjer recursive algorithm (since

$$N \sim PComp\left(\lambda_N, F_B\right).$$

Example 11 *Let* $(M_1, M_2) \sim PBiv\left(\lambda_1 = 2, \lambda_2 = 3, \alpha_0 = 1\right)$.

We define $N = M_1 + M_2$ *where* $E[N] = 5$ *and* $\mathrm{Var}(N) = 7$.

Also, $N \sim PComp(\lambda = 4, F_B)$ *where* $f_B(1) = 0.75$, $f_B(2) = 0.25$ *and* $f_B(k) = 0$ *for* $k \neq 1$ *or 2.*

Using Panjer's algorithm, we found the following values of the pmf of the rv N *:* $f_N(0) = 0.01831564$, $f_N(5) = 0.14698300$ *and* $f_N(10) = 0.02644007$. $\square$

Let $a_1, a_2 \in N^+$, with $a_1 < a_2$ (to simplify the presentation).

Then, for $S = a_1 M_1 + a_2 M_2$, we conclude $S \sim PComp(\lambda_S, F_B)$ where $\lambda_S = \lambda_1 + \lambda_2 - \alpha_0$ and

$$f_B(k) = \begin{cases} \frac{\lambda_1 - \alpha_0}{\lambda_N} & , k = a_1 \\ \frac{\lambda_2 - \alpha_0}{\lambda_N} & , k = a_2 \\ \frac{\alpha_0}{\lambda_N} & , k = a_1 + a_2 \\ 0 & , k \in N \backslash \{a_1, a_2, a_1 + a_2\} \end{cases}.$$

8.1.4 Extension

(Omitted in the present version).

8.2 Bivariate mixed Poisson distribution

Let $\underline{\Theta} = (\Theta_1, \Theta_2)$ be a pair of mixing strictly positive rvs with a bivariate mgf

$$M_{\Theta_1,\Theta_2}(t_1, t_2).$$

Let (M_1, M_2) be pair of rvs with

$$(M_1|\Theta_1 = \theta_1, \Theta_2 = \theta_2) = (M_1|\Theta_1 = \theta_1) \sim Pois(\lambda_1\theta_1)$$

and

$$(M_2|\Theta_1 = \theta_1, \Theta_2 = \theta_2) = (M_2|\Theta_2 = \theta_2) \sim Pois\,(\lambda_2\theta_2)$$

Also, given $(\underline{\Theta} = \underline{\theta})$, the rvs M_1 and M_2 are contidionally independent i.e.

$$(M_1|\Theta_1 = \theta_1)$$

and

$$(M_2|\Theta_2 = \theta_2)$$

are independent.

Additional assumption: the parameters of the distribution of $\underline{\Theta}$ are fixed such that

$$E\,[\Theta_1] = E\,[\Theta_2] = 1.$$

Firstm the covariance of (M_1, M_2) is

$$\begin{aligned}
Cov(M_1, M_2) &= Cov(E[M_1|\underline{\Theta}], E[M_2|\underline{\Theta}]) + E[Cov(M_1, M_2|\underline{\Theta})] \\
&= Cov(E[M_1|\Theta_1], E[M_2|\Theta_2]) + 0 \\
&= Cov(\lambda_1\Theta_1, \lambda_2\Theta_2) \\
&= \lambda_1\lambda_2 Cov(\Theta_1, \Theta_2).
\end{aligned}$$

The pgf of (M_1, M_2) is

$$\begin{aligned}
P_{M_1,M_2}(t_1, t_2) &= E\left[t_1^{M_1} t_2^{M_2}\right] \\
&= E\left[E\left[t_1^{M_1} t_2^{M_2}|\underline{\Theta}\right]\right] \\
&= E\left[E\left[t_1^{M_1}|\underline{\Theta}\right] E\left[t_2^{M_2}|\underline{\Theta}\right]\right] \\
&= E\left[E\left[t_1^{M_1}|\Theta_1\right] E\left[t_2^{M_2}|\Theta_2\right]\right] \\
&= E\left[e^{\lambda_1\Theta_1(t_1-1)} e^{\lambda_2\Theta_2(t_2-1)}\right] \\
&= M_{\underline{\Theta}}(\lambda_1(t_1-1), \lambda_2(t_2-1)).
\end{aligned}$$

The expression for the pmf of $\underline{M}$ depends on the distribution chosen for $\underline{\Theta}$.

Recall that

$$\Pr(M_1 = 0, M_2 = 0) = P_{M_1,M_2}(0,0).$$

Also,

$$\Pr(M_1 = k_1, M_2 = k_2) = \frac{\partial^{k_1+k_2}}{\partial^{k_1}\partial^{k_2}} P_{M_1,M_2}(t_1,t_2) \frac{1}{k_1!k_2!}\bigg|_{t_1=t_2=0}.$$

8.2.1 Simulation

Algorithm 12 Exercise

8.2.2 Bivariate Poisson-Gamma CRMM distribution

Let (Θ_1, Θ_2) follow the CRMM bivariate gamma distribution with parameters

$$(\alpha_1 = r_1, \alpha_2 = r_2, \beta_1 = r_1, \beta_2 = r_2, \gamma_0).$$

From Chapter 2, it implies that

$$\begin{aligned} M_1 &\sim NBin(r_1, q_1) \\ M_2 &\sim NBin(r_2, q_2) \end{aligned}$$

where

$$q_1 = \frac{1}{1 + \frac{\lambda_1}{r_1}} \text{ and } q_2 = \frac{1}{1 + \frac{\lambda_2}{r_2}}.$$

The expression of the mgf of (Θ_1, Θ_2) be given by

$$
\begin{aligned}
M_{(\Theta_1,\Theta_2)}(t_1,t_2) &= E\left[\mathrm{e}^{t_1\Theta_1}\mathrm{e}^{t_2\Theta_2}\right] \\
&= \left(1-\frac{t_1}{\alpha_1}\right)^{-(\alpha_1-\gamma_0)}\left(1-\frac{t_2}{\alpha_2}\right)^{-(\alpha_2-\gamma_0)} \times \left(1-\frac{t_1}{\alpha_1}-\frac{t_2}{\alpha_2}\right)^{-\gamma} \\
&= \left(1-\frac{t_1}{r_1}\right)^{-(r_1-\gamma_0)}\left(1-\frac{t_2}{r_2}\right)^{-(r_2-\gamma_0)} \times \left(1-\frac{t_1}{r_1}-\frac{t_2}{r_2}\right)^{-\gamma_0}.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
P_{M_1,M_2}(t_1,t_2) &= E\left[t_1^{M_1}t_2^{M_2}\right] \\
&= M_{\underline{\Theta}}\left(\lambda_1\left(t_1-1\right),\lambda_2\left(t_2-1\right)\right).
\end{aligned}
$$

which becomes

$$\begin{aligned} P_{M_1,M_2}(t_1,t_2) &= \left(1-\frac{\lambda_1(t_1-1)}{r_1}\right)^{-(r_1-\gamma_0)}\left(1-\frac{\lambda_2(t_2-1)}{r_2}\right)^{-(r_2-\gamma_0)} \\ &\quad\times\left(1-\frac{\lambda_1(t_1-1)}{r_1}-\frac{\lambda_2(t_2-1)}{r_2}\right)^{-\gamma_0}. \end{aligned}$$

The covariance between M_1 and M_2 is given

$$Cov(M_1,M_2)=\lambda_1\lambda_2\frac{\gamma_0}{r_1r_2}=\frac{\gamma_0}{(1-q_1)(1-q_2)}.$$

Let $N = M_1 + M_2$. The expression for P_N is

$$\begin{aligned} P_N(t) &= P_{M_1,M_2}(t,t) \\ &= \left(1 - \frac{\lambda_1(t-1)}{r_1}\right)^{-(r_1-\gamma_0)} \left(1 - \frac{\lambda_2(t-1)}{r_2}\right)^{-(r_2-\gamma_0)} \\ &\quad \times \left(1 - \frac{\lambda_1(t-1)}{r_1} - \frac{\lambda_2(t-1)}{r_2}\right)^{-\gamma_0} \end{aligned}$$

which becomes

$$P_N(t) = P_{J_1}(t) \times P_{J_2}(t) \times P_{J_0}(t)$$

where

$$P_{J_1}(t) = \left(1 - \frac{\lambda_1(t-1)}{r_1}\right)^{-(r_1-\gamma_0)} \quad \text{(pgf of NBin dist)}$$

$$P_{J_2}(t) = \left(1 - \frac{\lambda_2 (t-1)}{r_2}\right)^{-(r_2-\gamma_0)} \text{ (pgf of NBin dist)}$$

$$P_{J_0}(t) = \left(1 - \left(\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2}\right)(t-1)\right)^{-\gamma_0} \text{ (pgf of NBin dist)}$$

It means that the rv N can be represented as the sum of the independent rvs J_1, J_2, and J_0. The pmf of N can be obtained by convolution product or with FFT.

9 Multivariate compound distributions

9.0.3 Definition

Let $\underline{X} = (X_1, ..., X_n)$ be a vector of rvs where the rv X_i is defined by

$$X_i = \begin{cases} \sum_{k=1}^{M_i} B_{i,k}, & M_i > 0 \\ 0, & M_i = 0 \end{cases} \tag{14}$$

for $i = 1, 2, ..., n$.

The assumptions are :

- for each i, the rvs $B_{i,1}$, $B_{i,2}$, ... form a sequence of iid rvs ($B_{i,k} \sim B_i$);

- the sequences $\left\{B_{1,k}, k \in \mathbb{N}^{+}\right\}$, ..., $\left\{B_{n,k}, k \in \mathbb{N}^{+}\right\}$ are independent betweene each other and they are independent to $(M_1, ..., M_n)$.

The dependence relation between the components of $\underline{X}$ is introduced via the vector of rvs $\underline{M}$.

The covariance between X_i and X_j is given

$$Cov\left(X_i, X_j\right) = E\left[B_i\right] E\left[B_j\right] Cov\left(M_i, M_j\right)$$

for $i \neq j$.

Exercise: Prove that the multivariate mgf of $\underline{X} = (X_1, ..., X_n)$ is

$$\begin{aligned}
M_{\underline{X}}(t_1, ..., t_n) &= E_{\underline{M}}\left[E\left[\mathrm{e}^{t_1 X_1}...\mathrm{e}^{t_n X_n}|\underline{M}\right]\right] \\
&= E_{\underline{M}}\left[E\left[\mathrm{e}^{t_1 X_1}|\underline{M}\right] \times ... \times E\left[\mathrm{e}^{t_n X_n}|\underline{M}\right]\right] \\
&= E_{\underline{M}}\left[E\left[\mathrm{e}^{t_1 X_1}|M_1\right] \times ... \times E\left[\mathrm{e}^{t_n X_n}|M_n\right]\right] \\
&= E_{\underline{M}}\left[E\left[\mathrm{e}^{t_1 B_1}\right]^{M_1} \times ... \times E\left[\mathrm{e}^{t_n B_n}\right]^{M_n}\right] \\
&= P_{\underline{M}}\left(M_{B_1}(t_1), ..., M_{B_n}(t_n)\right).
\end{aligned}$$

9.0.4 Simulation

Let $\underline{M} = (M_1, ..., M_n)$

Algorithm 13 *Simulate a sample $\underline{M}^{(j)}$ of $\underline{M}$.*

2. *Let* $X_1^{(j)} = 0$*, ...,* $X_n^{(j)} = 0$*.*
3. *For each* $i = 1, 2, ..., n$*, if* $M_i^{(j)} > 0$*,*
   (a) *simulate the sampled value of*

$$\left(B_{i,1}, ..., B_{i,M_i^{(j)}}\right)$$

   (b) *simulate the sample value of* $\left(X_i | M_i = M_i^{(j)}\right)$ *with*

$$X_i^{(j)} = \left(B_{i,1} + ... + B_{i,M_i^{(j)}}\right)$$

9.0.5 Special case #1

Fix $n = 2$. Let (M_1, M_2) follow the Teicher bivariate distribution with λ_1, λ_2 et γ_0.

Assume that B_1 and B_2 are discrete rvs with

$$f_{B_i}(k_i h) = \Pr(B_i = k_i h)$$

for $k_i \in N$ and $i = 1, 2$.

Let $S = X_1 + X_2$.

We want to compute the values of $f_S(kh)$, for $k = 0, 1, 2, \ldots$ using Panjer's algorithm.

Clearly, we have

$$\begin{aligned} P_S(t) &= P_{X_1,X_2}(t,t) \\ &= P_{M_1,M_2}\left(P_{B_1}(t), P_{B_2}(t)\right) \\ &= \mathrm{e}^{(\lambda_1-\alpha_0)\left(P_{B_1}(t)-1\right)}\mathrm{e}^{(\lambda_2-\alpha_0)\left(P_{B_2}(t)-1\right)}\mathrm{e}^{\alpha_0\left(P_{B_1}(t)\times P_{B_2}(t)-1\right)} \end{aligned}$$

We define

$$\lambda_S = \lambda_1 + \lambda_2 - \alpha_0$$

and

$$\begin{aligned} P_C(t) &= \frac{(\lambda_1 - \alpha_0)}{\lambda_S}\Big(P_{B_1}(t) - 1\Big) + \frac{(\lambda_2 - \alpha_0)}{\lambda_S}\Big(P_{B_2}(t) - 1\Big) \\ &\quad + \frac{\alpha_0}{\lambda_S}\Big(P_{B_1}(t)\, P_{B_2}(t) - 1\Big). \end{aligned}$$

Then, we obtain the following expression for $P_S(t)$:

$$P_S(t) = \mathrm{e}^{\lambda_S (P_C(t) - 1)}$$

It means that $S \sim CompPois(\lambda_s; F_C)$ with

$$f_C(k) = \frac{(\lambda_1 - \alpha_0)}{\lambda_S} f_{B_1}(k) + \frac{(\lambda_2 - \alpha_0)}{\lambda_S} f_{B_2}(k) + \frac{\alpha_0}{\lambda_S} f_{B_1 + B_2}(k)$$

for $k = 0, 1, 2, \ldots$.

10 References

(...)