Lyon PhD Course Actuarial Science

Chapter 3 - Risk Pooling and aggregation methods

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Contents

1 Context

Risk pooling is fundamental in actuarial science.

Insurance operations are based on the possibility that they can pool risks.

We consider a portfolio of risks represented by the rvs $X_1, ..., X_n$.

The rvs $X_1, ..., X_n$ may be independent or dependent.

The aggregate claim amount for the entire portfolio is represented by the rv S where

$$S = \sum_{i=1}^{n} X_i.$$

It is a traditionnal research topic in actuarial science.

Challenge : Find the distribution of the sum of independent or dependent rvs.

2 Risk aggregation

2.1 Definitions and basic results

The aggregate claim amount for the entire portfolio is represented by the rv S where

$$S = \sum_{i=1}^{n} X_i.$$

The expectation of S is given by

$$E[S] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]. \tag{1}$$

The relation in (1) is true, if the risks $X_1, ..., X_n$ are independent or not.

The variance of the rv S is

$$\begin{aligned}
\mathsf{Var}(S) &= \mathsf{Var}\left(\sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n} \mathsf{Var}(X_i) + \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} \mathsf{Cov}\left(X_i, X_j\right) \\
&= \sum_{i=1}^{n} \mathsf{Var}(X_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathsf{Cov}\left(X_i, X_j\right).
\end{aligned}$$

2.2 Agregation

The main objective is to compute F_S.

For specific contexts, we can find closed-form expressions for F_S.

In other contexts, we can identify the distribution of S.

Generally, we need to use approximation methods in order to compute F_S and the associated risk measures.

In actuarial science and quantitative risk management, a number of situations arise where the moments of a distribution can be rather easily obtained but the analytical form of this distribution can not be derived.

An accurate knowledge of this distribution is however crucial to compute the popular risk measures VaR and TVaR, as well as risk quantities such as stop-loss premiums.

3 Sum of n independent arithmetic rvs

3.1 Algorithm

Let X_1, ..., X_n be n independent discretes rvs defined over an arithmetic support $\{0, 1h, 2h, ...\}$, with

$$f_{X_i}(kh) = \Pr(X_i = kh)$$

for $k \in \mathbb{N}$.

Note: $h =$ discretization span.

Later, we explain the discretization methods.

We define the rv

$$S_n = \sum_{i=1}^{n} X_i.$$

Objective : we want to compute the pmf f_{S_n} of the rv S_n i.e. compute the values of

$$f_{S_n}(kh) = \Pr(S_n = kh)$$

for $k \in \mathbb{N}$.

General relations (recall) :

- no assumption is made about an eventual dependence relation between the rvs X_1 and X_2 :

$$
\begin{aligned}
f_{S_2}(kh) &= \Pr(S_2 = kh) \\
&= \Pr(X_1 + X_2 = kh) \\
&= \sum_{j=0}^{k} f_{X_1,X_2}((k-j)h, jh),
\end{aligned}
$$

- the rvs X_1 and X_2 are independent :

$$
\begin{aligned}
f_{S_2}(kh) &= \sum_{j=0}^{k} f_{X_1,X_2}((k-j)h, jh). \\
&= \sum_{j=0}^{k} f_{X_1}((k-j)h) f_{X_2}(jh).
\end{aligned}
$$

Recurvice procedure for the computation of f_{S_n}

- $n = 2$: We start to compute the values of $f_{S_2}(kh)$:

$$f_{S_2}(kh) = \sum_{j=0}^{k} f_{X_1}((k-j)h) f_{X_2}(jh).$$

- $n = 3$: Then, we compute the values of $f_{S_3}(kh)$:

$$f_{S_3}(kh) = \sum_{j=0}^{k} f_{S_2}((k-j)h) f_{X_3}(jh).$$

- $n = 4$: Then, we compute the values of $f_{S_4}(kh)$:

$$f_{S_4}(kh) = \sum_{j=0}^{k} f_{S_3}((k-j)h) f_{X_4}(jh).$$

- Etc

- General recursive relation for $n = 2, 4, \ldots$:

$$\begin{aligned} f_{S_n}(kh) &= \sum_{j=0}^{k} \Pr(S_{n-1} = (k-j)h) \Pr(X_n = jh) \\ &= \sum_{j=0}^{k} f_{S_{n-1}}((k-j)h) f_{X_n}(jh), \end{aligned} \tag{2}$$

  for $k \in \mathbb{N}$ and $S_1 = X_1$

- Consequence : To compute the values of f_{S_n}, we need to know the values of $f_{S_{n-1}}$:

3.2 Example

Let X_1, ..., X_n be independent rvs with

$$X_i \sim BN\left(2, 1 - 0.01 \times i\right) \ \left(i = 1, 2, ..., 10\right)$$

We define the rv

$$S_{10} = \sum_{i=1}^{10} X_i.$$

We apply (2) with $h = 1$.

- We fix m (e.g. $m = 100$).
- For $k = 0, 1, 2, ..., m$, we compute $f_{S_2}\left(kh\right) = \sum_{j=0}^{k} f_{X_1}\left(\left(k-j\right)h\right) f_{X_2}\left(jh\right)$.

- For $k = 0, 1, 2, ..., m$, we compute $f_{S_3}(kh) = \sum_{j=0}^{k} f_{S_2}((k-j)h) f_{X_3}(jh)$.
- For $k = 0, 1, 2, ..., m$, we compute $f_{S_4}(kh) = \sum_{j=0}^{k} f_{S_3}((k-j)h) f_{X_4}(jh)$.
- ...
- For $k = 0, 1, 2, ..., m$, we compute $f_{S_{10}}(kh) = \sum_{j=0}^{k} f_{S_9}((k-j)h) f_{X_{10}}(jh)$.

We obtain the following values of $f_{S_{10}}(k)$, for $k = 0, 1, ..., 11$:

k	0	1	2	3	4	5
$f_{S_{10}}(k)$	0.319610	0.351571	0.205669	0.085080	0.027928	0.007742
k	6	7	8	9	10	11
$f_{S_{10}}(k)$	0.001884	0.000413	0.000083	0.000016	0.000003	0.000000

With those values, we obtain $E[S_{10}] = 1.183605$ and $\mathrm{Var}(S_{10}) = 1.274424$ which can be verified easily. $\square$

Remark 1 *It is important to verify the obtained values.*

- $\sum_{k=0}^{m} f_{S_n}(kh) = 1$
- $\sum_{k=0}^{m} kh f_{S_n}(kh) = E[S_n] = \sum_{i=1}^{n} E[X_i]$
- $\sum_{k=0}^{m} kh^2 f_{S_n}(kh) = E\left[S_n^2\right] = Var(S_n) + E[S_n]^2$ with $Var(S_n) = \sum_{i=1}^{n} Var(X_i)$

3.3 Function R

```
directconvo<-function(ff1,ff2)
{
    # convolution of two pmfs
    l1<-length(ff1)
    l2<-length(ff2)
    ffs<-ff1[1]*ff2[1]
    smax<-l1+l2-2
    ff1<-c(ff1,rep(0,smax-l1+1))
    ff2<-c(ff2,rep(0,smax-l2+1))
    for (i in 1:smax)
    {
        j<-i+1
        ffs<-c(ffs,sum(ff1[1:j]*ff2[j:1]))
    }
    return(ffs)
}
```

4 Portfolios of homogeneous risks

In this section, we consider an homogeneous portfolios of m risks.

It means that the rvs X_1, ..., X_m share the same stochastic behavior.

First, we examine some cases of portfolios with iid risks.

We derive the distribution of S.

4.1 Portfolio of 1-period term life insurance contracts

We consider a portfolio of m 1-period term life insurance contracts

The aggregate claim amount for the insurance contract i is defined by the rv

$$X_i = bI_i$$

where b is the death benefit, for $i = 1, 2, ..., m$.

Let I_1, ..., I_m be iid rvs with

$$I_i \sim I \sim Bern\,(q)$$

for $i = 1, 2, ..., m$.

The rv I_i corresponds to the occurrence rv where

$$I_i = \begin{cases} 1 & \text{, if the insured dies over the next period} \\ 0 & \text{, otherwise} \end{cases},$$

for $i = 1, 2, ..., m$.

Then, the aggregate claim amount for the entire portfolio can be represented as follows

$$\begin{aligned} S &= \sum_{i=1}^{m} X_i \\ &= b \sum_{i=1}^{m} I_i \sim bN, \end{aligned}$$

where the rv N is binomially distributed i.e.

$$N = \sum_{i=1}^{m} I_i \sim Binom\,(m; q)\,.$$

Interpretation: Since it is homogeneous, we can consider the entire portfolio as a "large" single risk i.e.

$$S = bN$$

where

$$N \sim Binom\left(m; q\right).$$

For computation purpose, it doesn't matter if the death benefit comes from contract i or contract i' (i and $i' \in \{1, 2, ..., m\}$).

4.2 Portfolio of 1-period non-life insurance contracts

We consider an homogeneous portfolio of m non-life insurance contracts (or m risks).

The aggregate claim amount for the contract i is defined by the rv X_i where

$$X_i = \begin{cases} B_i & , I_i = 1 \\ 0 & , I_i = 0 \end{cases} ,$$

for $i = 1, 2, ..., m$.

Let I_1, ..., I_m be iid rvs with

$$I_i \sim I \sim Bern\,(q)$$

for $i = 1, 2, ..., m$.

Let B_1, ..., B_m be iid rvs with

$$B_i \sim B$$

for $i = 1, 2, ..., m$.

To simplify the presentation, we assume that the mgf of B exists.

The vectors of rvs $(I_1, ..., I_m)$ and $(B_1, ..., B_m)$ are independent.

To identify the distribution of the rv $S = \sum_{i=1}^{m} X_i$, we first derive its mgf

$$\begin{aligned} M_S(t) &= \prod_{i=1}^{m} M_{X_i}(t) \\ &= (M_X(t))^m \\ &= (1 - q + qM_B(t))^m . \end{aligned} \tag{3}$$

It is clear from (3) that the rv S follows a Compound Binomial distribution i.e.

$$S = \sum_{i=1}^{m} X_i \sim CBinom\left(m, q; F_B\right) \tag{4}$$

with the usual assumptions for a compound distribution.

From (4), it means that the rv S can be represented as a rv sum i.e.

$$S = \begin{cases} \sum_{k=1}^{N} B'_k & , N > 0 \\ 0 & , N = 0 \end{cases},$$

where

- the usual assumptions for a compound distribution are satisfied,
- the rv N is binomially distributed i.e.

$$N \sim Binom\left(m, q\right);$$

- and $B_k^{'} \sim B$ corresponds to the kth claim from the portfolio; since it is an homogeneous portfolio and for computation purposes, it doesn't matter from which contracts the claims come.
-

4.3 Portfolio of 1-period non-life insurance contracts (again)

Again, we consider an homogeneous portfolio of m non-life insurance contracts (or m risks).

For $i = 1, 2, ..., m$, the aggregate claim amount for the contract i is defined by the rv X_i with

$$X_i = \begin{cases} \sum_{k=1}^{M} B_{i,k} & , M_i > 0 \\ 0 & , M_i = 0 \end{cases} ,$$

where

- rv $M_i =$ number of claims for insurance contract i;
- rv $B_{i,k} =$ amount of the kth claim for insurance contract i;
- $\left\{B_{i,k}, k = 1, 2, ...\right\}$ forms a sequence of iid rvs where $B_{i,k} \sim B_i$ ....

Let M_1, ..., M_m be iid rvs with

$$M_i \sim M,$$

for $i = 1, 2, ..., m$.

Let $\left\{B_{1,k}, k = 1, 2, ...\right\}$, ..., $\left\{B_{m,k}, k = 1, 2, ...\right\}$ be independent sequences of rvs with

$$B_{i,k} \sim B_i \sim B,$$

for $i = 1, 2, ..., m$.

We assume that the mgf of B exists (to simplify the presentation).

The vectors of rvs $(M_1, ..., M_m)$ and the sequences of $\left\{B_{1,k}, k = 1, 2, ...\right\}$, ..., $\left\{B_{m,k}, k = 1, 2, ...\right\}$ are independent.

The distribution of the rv $S = \sum_{i=1}^{m} X_i$ can be found by developing its mgf

$$\begin{aligned} M_S(t) &= \prod_{i=1}^{m} M_{X_i}(t) \\ &= (M_X(t))^m \\ &= P_M(M_B(t))^m . \end{aligned} \tag{5}$$

Let N be a counting rv with pgf defined by

$$P_N(s) = P_M(s)^m . \tag{6}$$

Combining (5) and (6), it follows that the rv S follows a compound distribution and that the rv S can be represented as a rv sum i.e.

$$S = \begin{cases} \sum_{k=1}^{N} B_k' & , N > 0 \\ 0 & , N = 0 \end{cases} ,$$

where

- the usual assumptions for a compound distribution are satisfied,
- the rv N is a counting distribution and it can be represented as

$$N \sim M_1 + ... + M_m;$$

- and $B_k' \sim B$ corresponds to the kth claim from the portfolio; for computation purposes, it doesn't matter from which contracts the claim comes.

Here are some special cases:

Distribution of M	$P_M(t)$	$P_N(t)$	Distribution of N
$M \sim Pois(\lambda)$	$e^{\lambda(t-1)}$	$e^{m\lambda(t-1)}$	$N \sim Pois(m \times \lambda)$
$M \sim Bin(n, q)$	$(1-q+qt)^n$	$(1-q+qt)^{m\times n}$	$N \sim Bin(m \times n, q)$
$M \sim NB(r, q)$	$\left(\frac{q}{1-(1-q)t}\right)^r$	$\left(\frac{q}{1-(1-q)t}\right)^{m\times r}$	$N \sim NB(m \times r, q)$

5 Portfolios of heterogeneous risks

In this section, we consider examples of portfolios of m heterogeneous risks.

5.1 Portfolio of heterogeneous risks and compound Poisson distribution

5.1.1 Context

We consider an heterogeneous portfolio of m non-life insurance contracts (or m risks).

For $i = 1, 2, ..., m$, the aggregate claim amount for the contract i is defined by the rv X_i with

$$X_i = \begin{cases} \sum_{k=1}^{M} B_{i,k} & , M_i > 0 \\ 0 & , M_i = 0 \end{cases} ,$$

where

- rv $M_i =$ number of claims for insurance contract i;
- rv $B_{i,k} =$ amount of the kth claim for insurance contract i;
- $\left\{B_{i,k}, k = 1, 2, ...\right\}$ forms a sequence of iid rvs where $B_{i,k} \sim B_i$ ....

Let M_1, ..., M_m be independent rvs with

$$M_i \sim Pois\left(\lambda_i\right),$$

for $i = 1, 2, ..., m$.

Let $\left\{B_{1,k}, k = 1, 2, ...\right\}$, ..., $\left\{B_{m,k}, k = 1, 2, ...\right\}$ be independent sequences of rvs with

$$B_{i,k} \sim B_i,$$

for $i = 1, 2, ..., m$.

It implies that

$$X_i \sim PComp\left(\lambda_i; F_{B_i}\right),$$

for $i = 1, 2, ..., m$.

The vectors of rvs $(M_1, ..., M_m)$ and the sequences of $\left\{B_{1,k}, k = 1, 2, ...\right\}$, ..., $\left\{B_{m,k}, k = 1, 2, ...\right\}$ are independent.

Then, the rvs X_1, ..., X_n are independent.

5.1.2 Fundamental result for the aggregate claim amount

The distribution of the rv $S = \sum_{i=1}^{m} X_i$ is provided in the following result which is fundamental in actuarial science.

Proposition 2 *Let X_1, ..., X_m be independent rvs where*

$$X_i \sim PComp\left(\lambda_i; F_{B_i}\right), i = 1, 2, ..., m.$$

Then, we have

$$S = \sum_{i=1}^{m} X_i \sim PComp\left(\lambda_S, F_C\right),$$

where $\lambda_S = \sum_{i=1}^{m} \lambda_i$ and

$$F_C(x) = \frac{\lambda_1}{\lambda_S} F_{B_1}(x) + \frac{\lambda_2}{\lambda_S} F_{B_2}(x) + ... + \frac{\lambda_m}{\lambda_S} F_{B_m}(x).$$

The result of Proposition 2 is valid regardless if the mgfs of the rvs B_1, ..., B_m exist or not.

However, we choose to use the mgfs in order to prove the result.

A proof using Laplace transforms is provided in [**?**].

Proof. The mgf of X_i is given by

$$M_{X_i}(t) = P_{M_i}\left(M_{B_i}(t)\right) = \mathrm{e}^{\left(\left\{\lambda_i\left(M_{B_i}(t)-1\right)\right\}\right)}, \qquad (i = 1, 2, ..., m).$$

The expression of the mgf of $S = \sum_{i=1}^{m} X_i$ is

$$\begin{aligned} M_S(t) &= E\left[\mathrm{e}^{tS_m}\right] = E\left[\mathrm{e}^{t\left(\sum_{i=1}^{m} X_i\right)}\right] = \prod_{i=1}^{m} M_{X_i}(t) \\ &= \prod_{i=1}^{m} \mathrm{e}^{\lambda_i\left(M_{B_i}(t)-1\right)} = \mathrm{e}^{(\{\lambda_S(M_C(t)-1)\})}, \end{aligned}$$

where

$$\lambda_S = \lambda_1 + ... + \lambda_m$$

and

$$M_C(t) = \frac{\lambda_1}{\lambda_S} M_{B_1}(t) + \frac{\lambda_2}{\lambda_S} M_{B_2}(t) + ... + \frac{\lambda_m}{\lambda_S} M_{B_m}(t). \tag{7}$$

From (7), it is clear that the cdf F_C is a mixture of $F_{B_1}, ..., F_{B_m}$, e.g.

$$F_C(x) = \frac{\lambda_1}{\lambda_S} F_{B_1}(x) + \frac{\lambda_2}{\lambda_S} F_{B_2}(x) + ... + \frac{\lambda_m}{\lambda_S} F_{B_m}(x).$$

Since

$$M_S(t) = \mathrm{e}^{\lambda_S(M_C(t)-1)} = P_N(M_C(t)) \tag{8}$$

is the mgf of a Compound Poisson distribution, it follows from (8) that $S = \sum_{i=1}^{m} X_i \sim PComp(\lambda_S, F_C)$ such that

$$S = \begin{cases} \sum_{k=1}^{N} C_k, & N > 0 \\ 0, & N = 0 \end{cases},$$

with the usual assumptions, $N = \sum_{i=1}^{m} M_i \sim Pois(\lambda_S)$ and $C_k \sim C$ for $k \in \mathbb{N}^{+}$. ■

This result has important consequences :

- The entire portfolio can be seen as a single risk.
- The aggregate claim amount (i.e. rv S) for the entire portfolio follows a compound Poisson distribution.
- This is interesting for computation purposes but also for estimation.

- When a claim occurs within the portfolio, the claim amount follows a mixed distribution.
- When a claim occurs within the portfolio, it doesn't matter from which contracts (or risks) it comes.

5.1.3 Application in general insurance and Tweedie's compound Poisson distribution

In general insurance, the compound Poisson distribution is frequently used for modelling the aggregate claim amount for an insurance contract, notably in car insurance.

In North America, the Tweedie's compound Poisson distribution is frequently used by general insurance companies.

The estimation of its parameters are easy to implement through the GLM (GLM = Generalized Linear Models) framework, using all the tools and technics provided within this framework.

Under the Tweedie's compound Poisson distribution, the aggregate claim amount for the insurance contract i follows a compound Poisson distribution where the claim amount has a gamma distribution i.e.

$$X_i \sim PComp\left(\lambda_i; F_{B_i}\right),$$

for $i = 1, 2, ..., m$ where

- the usual assumptions are supposed
- $B_i \sim Gamma\left(\alpha_i, \beta_i\right)$ $(i = 1, 2, ..., m)$.

It implies that

$$\begin{aligned} F_{X_i}(x) &= f_{M_i}(0) + \sum_{k=1}^{\infty} f_{M_i}(k) F_{B_{i,1}+\ldots+B_{i,k}}(x) \\ &= e^{-\lambda_i} + \sum_{k=1}^{\infty} e^{-\lambda_i} \frac{\lambda_i^k}{k!} H(x; \alpha_i k, \beta_i). \end{aligned}$$

Then, we obtain

$$S = \sum_{i=1}^{m} X_i \sim PComp(\lambda_S, F_C),$$

where $\lambda_S = \sum_{i=1}^{m} \lambda_i$ and

$$F_C(x) = \frac{\lambda_1}{\lambda_S} H(x; \alpha_1, \beta) + \frac{\lambda_2}{\lambda_S} H(x; \alpha_2, \beta) + \ldots + \frac{\lambda_m}{\lambda_S} H(x; \alpha_m, \beta).$$

Tweedy's Compound Poisson distribution is applied in the context of non-life insurance pricing and reserving.

5.2 Portfolio of heterogeneous risks and other compound distributions

We consider a portfolio of m heteregeneous risks.

We don't have nice results as the one provide in subsection 5.1 for compound Poisson distributions.

We skip that part.

6 Random sum and Panjer's algorithm

6.1 Preliminaries

Let X be a rv defined by

$$X = \begin{cases} \sum_{i=1}^{M} B_i, & M > 0 \\ 0, & M = 0 \end{cases}, \tag{9}$$

where

- M is a discrete counting rv,
- $B_1, B_2, ...$ is a sequence of iid rvs defined on $\{0h, 1h, 2h, ...\}$ with $B_i \sim B$;

- and the sequence of rvs $B_1, B_2, ...$ is independent of the rv M.

Then, the support of the rv X is also $\{0h, 1h, 2h, ...\}$.

The mpf of X is defined by $f_X(kh) = \Pr(X = kh)$, for $k \in \mathbb{N}$.

The general approach to compute $f_X(kh)$ is

$$\begin{aligned} f_X(0) &= f_M(0) + \sum_{j=1}^{\infty} f_M(j) f_{B_1+...+B_j}(0) \\ &= f_M(0) + \sum_{j=1}^{\infty} f_M(j) f_B^{*j}(0) \\ &= \sum_{j=0}^{\infty} f_M(j) (f_B(0))^j = P_M(f_B(0)) \end{aligned} \quad (10)$$

and

$$f_X(kh) = \sum_{j=1}^{\infty} f_M(j)\, f_{B_1+\ldots+B_j}(kh) = \sum_{j=1}^{\infty} f_M(j)\, f_B^{*j}(kh), \tag{11}$$

for $k \in \mathbb{N}^+$.

Even if (10) is easy to apply, (11) requires a lot of computer time.

Indeed ! We need to apply Proposition **??** to compute the values of $f_B^{*j}(kh)$ for each $j \in \mathbb{N}^+$ and each $k \in \mathbb{N}^+$.

Panjer (1981) proposed a recursive algorithm algorithme de Panjer»

It is recursive relation allowing the computation of the values of f_X porovided that the rv M belongs to the family $(a, b, 0)$.

6.2 $(a, b, 0)$ class of counting distributions

6.2.1 Definition

Definition 3 *A counting ditribution of the rv M belongs to $(a, b, 0)$ class of counting distributions if f_M satisfies the following recursive relation :*

$$f_M(k) = \left(a + \frac{b}{k}\right) f_M(k-1),$$

for $k \in \mathbb{N}^+$.

The starting (initial) value is $f_M(0) > 0$.

The members of the $(a, b, 0)$ class of counting distributions are only the following ones :

- Poisson distribution
- Binomial distribution
- Negative binomial distribution.

We provide the values of a and b for the members of $(a, b, 0)$ class of counting distributions :

- Poisson distribution : $a = 0$ et $b = \lambda$;
- Negative binomial distribution (with r, q) : $a = 1-q$ et $b = (1-q)(r-1)$;
- Negative binomial distribution (with r, β) : $a = \frac{\beta}{1+\beta}$ et $b = \frac{\beta}{1+\beta}(r-1)$;
- Binomial distribution : $a = -\frac{q}{1-q}$ et $b = (n+1)\frac{q}{1-q}$.

We have

$$P_M(t) = E\left[t^M\right] = \sum_{k=0}^{\infty} f_M(k)\, t^k. \tag{12}$$

6.2.2 Result

The following recursive relation for P_M is applied in the proof of the Panjer's algorithm.

Proposition 4 Recursive relation. *For a distribution from $(a, b, 0)$ class of counting distributions, we have*

$$P'_M(t) = a \times t \times P'_M(t) + (a + b) P_M(t), \tag{13}$$

where $P'_M(t) = \frac{\mathrm{d}\, P_M(t)}{\mathrm{d}t}$.

6.2.3 Proof of the result

Proof. We take the derivative of (12) with respect to t

$$
\begin{aligned}
P_M'(t) &= \sum_{k=1}^{\infty} f_M(k)\, kt^{k-1} \\
&= \sum_{k=1}^{\infty} \left(a + \frac{b}{k}\right) f_M(k-1)\, kt^{k-1}.
\end{aligned}
$$

We rearrange the terms

$$
\begin{aligned}
P_M'(t) &= a \sum_{k=1}^{\infty} f_M(k-1)\, kt^{k-1} + b \sum_{k=1}^{\infty} f_M(k-1)\, t^{k-1} \\
&= a \sum_{k=1}^{\infty} f_M(k-1) \times k \times t^{k-1} + bP_M(t).
\end{aligned}
$$

We replace k by $k - 1 + 1$

$$\begin{aligned} P_M'(t) &= a \sum_{k=1}^{\infty} f_M(k-1) \times (k-1+1) \times t^{k-1} + bP_M(t) \\ &= a \sum_{k=1}^{\infty} f_M(k-1) \times (k-1) \times t^{k-1} \\ &\quad + a \sum_{k=1}^{\infty} f_M(k-1) \times t^{k-1} + bP_M(t) \end{aligned}$$

We slightly rearrange

$$\begin{aligned} P_M'(t) &= at \sum_{k=2}^{\infty} f_M(k-1) \times (k-1) \times t^{k-2} \\ &\quad + aP_M(t) + bP_M(t) \end{aligned}$$

We observe

$$\sum_{k=2}^{\infty} f_M(k-1) \times (k-1) \times t^{k-2} = \sum_{k=1}^{\infty} f_M(k) \, k t^{k-1} = P'_M(t)$$

Then, we obtain

$$P'_M(t) = a t P'_M(t) + a P_M(t) + b P_M(t)$$

which corresponds to (13). ■

6.3 Panjer's algorithm

6.3.1 Result

The Panjer's algorithm is a recursive relation allowing the computation of the pmf f_X.

Condition: the distribution of the counting rv M has to belong to $(a, b, 0)$ class of counting distributions.

Algorithm 5 Panjer's algorithm.

- *Starting (initial) value of the algorithm :*

$$f_X(0) = M_M\{\ln[f_B(0)]\} = P_M\{f_B(0)\}$$

- *Recursive relation :*

$$f_X(kh) = \frac{\sum_{j=1}^{k} \left(a + b\frac{jh}{kh}\right) f_B(jh) f_X(\left(k-j\right)h)}{1 - af_B(0)},$$

for $k \in \mathbb{N}^+$.

6.3.2 Basic ingredients to the proof

To prove the desired result, it is preferable to proceed as follows.

We define the rvs Y and C on $\mathbb{N}$ such that

$$X = hY$$

and

$$B = hC,$$

with

$$Y = \begin{cases} \sum_{i=1}^{M} C_i, & M > 0 \\ 0, & M = 0 \end{cases},$$

under the usual assumptions for a random sum and where C_1, C_2, ..., form a sequence of iid rvs with $C_i \sim C$, for $i = 1, 2, \ldots$.

We use the following relations:

$$f_B(kh) = f_C(k)$$

and

$$f_X(kh) = f_X(k), \tag{14}$$

for $k \in \mathbb{N}$.

First, we prove the recursive relation for $f_Y(k)$, with $k \in \mathbb{N}$.

Then, we apply (14).

We also use the pgf of the rv Y, defined by

$$P_Y(t) = E\left[t^Y\right] = P_M(P_C(t)).$$

We know that

$$f_Y(0) = P_M(f_C(0)),$$

which provides the starting value of the Panjer's algorihm.

6.3.3 Proof of the result

We take the derivative of the P_Y with respect to t

$$P_Y^{'}(t) = P_M^{'}(P_C(t)) \times P_C^{'}(t)\,. \tag{15}$$

We replace the recursive relation in (13)

$$P_M'(t) = a \times t \times P_M'(t) + (a+b)\,P_M(t)\,,$$

within (15) and we obtain

$$\begin{aligned} P_Y^{'}(t) &= a \times P_C(t) \times P_M'(P_C(t))\,P_C'(t) + (a+b)\,P_M(P_C(t))\,P_C'(t) \\ &= a \times P_C(t) \times P_Y'(t) + (a+b)\,P_Y(t)\,P_C'(t) \end{aligned} \tag{16}$$

We recall

$$P_C(t) = \sum_{j=0}^{\infty} f_C(j)\, t^j, \qquad P'_C(t) = \sum_{j=0}^{\infty} j f_C(j)\, t^{(j-1)}$$

and

$$P_Y(t) = \sum_{k=0}^{\infty} f_Y(k)\, t^k, \qquad P'_Y(t) = \sum_{k=0}^{\infty} k f_Y(k)\, t^{(k-1)}.$$

We use those expressions within (16)

$$\begin{aligned}\sum_{k=0}^{\infty} k f_Y(k)\, t^{(k-1)} &= a \times \sum_{j=0}^{\infty} f_C(j)\, t^j \times \sum_{k=0}^{\infty} k f_Y(k)\, t^{(k-1)} \\ &\quad + (a+b) \sum_{j=0}^{\infty} f_Y(j)\, t^j \sum_{k=0}^{\infty} k f_C(k)\, t^{(k-1)}\end{aligned}$$

We multiply by t

$$\sum_{k=0}^{\infty} k f_Y(k) t^k = a \times \sum_{j=0}^{\infty} f_C(j) t^j \times \sum_{k=0}^{\infty} k f_Y(k) t^k + (a+b) \sum_{j=0}^{\infty} f_Y(j) t^j \sum_{k=0}^{\infty} k f_C(k) t^k. \tag{17}$$

Be careful ! We examine the coefficients of t^k associated to $f_Y(k)$ for $k \in \mathbb{N}$.

From the left-hand side of (17), we identify $k f_Y(k)$.

Within the first term of the right-hand side of (17), we find

$$a \times \sum_{j=0}^{k} f_C(j) \times (k-j) \times f_Y((k-j)) \tag{18}$$

Within the second term of the right-hand side of (17), we observe

$$(a+b) \times \sum_{j=0}^{k} f_C(j) \times j \times f_Y((k-j)). \tag{19}$$

We combine (17), (18) and (19)

$$\begin{aligned} k f_Y(k) &= a \times \sum_{j=0}^{k} f_C(j) \times (k-j) \times f_Y((k-j)) \\ &\quad + (a+b) \times \sum_{j=0}^{k} f_C(j) \times j \times f_Y((k-j)) \end{aligned}$$

We rearrange for a first time

$$\begin{aligned} kf_Y(k) &= akf_C(0)\,f_Y(k) \\ &\quad +a\sum_{j=1}^{k} f_C(j)\times((k-j))\,f_Y((k-j)) \\ &\quad +(a+b)\times\sum_{j=1}^{k} f_C(j)\times j\times f_Y((k-j)) \end{aligned}$$

We rearrange for a second time

$$\begin{aligned} kf_Y(k) &= akf_C(0)\,f_Y(k) \\ &\quad +\sum_{j=1}^{k}(a(k-j)+(a+b)\,j)\,f_C(j)\times f_Y((k-j)). \end{aligned}$$

Then, we obtain

$$f_Y(k) = \frac{1}{1 - af_C(0)} \sum_{j=1}^{k} \left(a + b\frac{j}{k}\right) f_C(j) \times f_Y((k-j)).$$

Finally, we find the desired result

$$f_X(kh) = \frac{1}{1 - af_B(0)} \sum_{j=1}^{k} \left(a + b\frac{jh}{kh}\right) f_B(jh) \times f_X((k-j)h).$$

6.3.4 Special case – compound Poisson

Algorithm 6 Poisson composée. *Let* $M \sim Pois(\lambda)$.

- *Starting (initial) value :* $f_X(0) = \mathrm{e}^{\lambda(f_B(0)-1)}$
- *Recursive relation :*

$$f_X(kh) = \frac{\lambda}{k} \sum_{j=1}^{k} j f_B(jh) f_X((k-j)\,h),$$

for $k \in \mathbb{N}^+$. $\square$

```
panjer.poisson<-function(lam,ff,smax)
{
 # Algorithme de Panjer
 # Loi discrete pour B
 aa<-0
 bb<-lam
 ll<-length(ff)
 ffs<-exp(lam*(ff[1]-1))
 ff<-c(ff,rep(0,smax-ll+1))
 for (i in 1:smax)
 {
  j<-i+1
  ffs<-c(ffs,(1/(1-aa*ff[1]))*sum(ff[2:j]*ffs[i:1]*(bb*(1:i)/i+aa)))
 }
 return(ffs)
}
```

6.3.5 Special case – compound negative binomial

Algorithm 7 Compound negative binomial *(with r and q). Let $M \sim BN(r,q)$.*

- *Starting (initial) value:*

$$f_X(0) = \left(\frac{q}{1-(1-q)\,f_B(0)}\right)^r$$

- *Recursive relation :*

$$f_X(kh) = \frac{\sum_{j=1}^{k}\left(1-q+\frac{(1-q)(r-1)j}{k}\right) f_B(jh) f_X((x-j)\,h)}{1-(1-q)\,f_B(0)},$$

for $k \in \mathbb{N}^+$. $\square$

```
panjer.nbinom1<-function(rr,qq,ff,smax)
{
 # Algorithme de Panjer
 # Loi discrete pour B
 aa<-1-qq
 bb<-aa*(rr-1)
 ll<-length(ff)
 ffs<-(qq/(1-(1-qq)*ff[1]))^rr
 ff<-c(ff,rep(0,smax-ll+1)))
 for (i in 1:smax)
 {
  j<-i+1
  ffs<-c(ffs,(1/(1-aa*ff[1]))*sum(ff[2:j]*ffs[i:1]*(bb*(1:i)/i+aa)))
 }
 return(ffs)
}
```

6.3.6 Special case – compound binomial

Algorithm 8 Binomial. *Let* $M \sim Bin(n, q)$.

- *Starting value:*

$$f_X(0) = (1 - q + q f_B(0))^n$$

- *Recursive relation:*

$$f_X(kh) = \frac{\sum_{j=1}^{k} \left(-q + \frac{(n+1)qj}{k}\right) f_B(j) f_X((k-j)h)}{1 - q + q f_B(0)},$$

for $k \in \mathbb{N}^+$. $\square$

```
panjer.binom<-function(nn,qq,ff,smax)
{
 # Algorithme de Panjer
 # Loi discrete pour B
 aa<--qq/(1-qq)
 bb<--(nn+1)*aa
 ll<-length(ff)
 ffs<-(1-qq+qq*ff[1])^nn
 ff<-c(ff,rep(0,smax-ll+1)))
 for (i in 1:smax)
 {
  j<-i+1
  ffs<-c(ffs,(1/(1-aa*ff[1]))*sum(ff[2:j]*ffs[i:1]*(bb*(1:i)/i+aa)))
 }
 return(ffs)
}
```

6.3.7 Numerical illustration of the Panjer's algorithm

Let the rv X be defined as (9) where

$$B \in \{1000, 2000, ..., 6000\}$$

with the following values $f_B(1000k)$:

k	0	1	2	3	4	5	6
$f_B(1000k)$	0	0.20	0.30	0.20	0.15	0.I0	0.05

We find that

- $E[B] = 2800$
- $\text{Var}(B) = 2\ 060\ 000$.

We consider three assumptions for the distribution of the rv M :

- Assumption 1:
  - $M \sim Pois\left(\lambda = 1.25\right)$
  - $E\left[X\right] = 3500$
  - $\mathrm{Var}\left(X\right) = 12\ 375\ 000.$
- Assumption 2:
  - $M \sim Bin\left(n = 10, q = 0.125\right)$
  - $E\left[X\right] = 3500$
  - $\mathrm{Var}\left(X\right) = 11\ 150\ 000.$
- Assumption 3
  - $M \sim BN\left(r = 0.5, \beta = 2.5\right)$
  - $E\left[X\right] = 3500$
  - $\mathrm{Var}\left(X\right) = \mathrm{Var}\left(X\right) = 36\ 875\ 000.$

For the assumption 1, we have

$$\begin{aligned} f_X\left(0\right) &= P_M\left(f_B\left(0\right)\right) = \exp\left(\lambda\left(f_B\left(0\right) - 1\right)\right) \\ &= \exp\left(-\lambda\right) = \exp\left(-1.25\right) = 0.286505 \end{aligned}$$

$$
\begin{aligned}
f_X(1000) &= \frac{\lambda}{1}\sum_{j=1}^{1} j f_B(1000j) f_X(1000(1-j)) \\
&= 1.25\left(1 \times f_B(1000) \times f_X(0)\right) \\
&= 1.25\left(1 \times 0.2 \times 0.286505\right) = 0.071626
\end{aligned}
$$

and

$$
\begin{aligned}
f_X(2000) &= \frac{\lambda}{2}\sum_{j=1}^{2} j f_B(1000j) f_X(1000(2-j)) \\
&= \frac{1.25}{2}\left(1 \times f_B(1000) \times f_X(1000) + 2 \times f_B(2000) \times f_X(0)\right) \\
&= \frac{1.25}{2}\left(1 \times 0.2 \times 0.071626 + 2 \times 0.3 \times 0.286505\right) \\
&= 0.116393.
\end{aligned}
$$

For the assumption 2, we have

$$
f_X(0) = P_M\left(f_B(0)\right) = \left(1 - q + q f_B(0)\right)^n = 0.263076
$$

$$
\begin{aligned}
f_X(1000) &= \frac{\sum_{j=1}^{1}\left(-q+\frac{(n+1)qj}{1}\right) f_B(1000j) f_X(1000(1-j))}{1-q+qf_B(0)} \\
&= \frac{\left(-0.125+\frac{11\times 0.125\times 1}{1}\right) f_B(1000) f_X(0)}{1-0.125} \\
&= 0.075164
\end{aligned}
$$

and

$$
\begin{aligned}
f_X(2000) &= \frac{\sum_{j=1}^{2}\left(-q+\frac{(n+1)qj}{2}\right) f_B(1000j) f_X(1000(2-j))}{1-q+qf_B(0)} \\
&= \frac{\left(-0.125+\frac{11\times 0.125\times 1}{2}\right) 0.20\times 0.075164}{1-0.125} \\
&\quad +\frac{\left(-0.125+\frac{11\times 0.125\times 2}{2}\right) 0.30\times 0.263076}{1-0.125} \\
&= 0.122411.
\end{aligned}
$$

For the assumption 3, we have

$$
\begin{aligned}
f_X(0) &= P_M(f_B(0)) = \left(\frac{1}{(1-\beta((f_B(0))-1))}\right)^r \\
&= \left(\frac{1}{1+\beta}\right)^r = \left(\frac{1}{1+2.5}\right)^{0.5} = 0.534522
\end{aligned}
$$

$$
\begin{aligned}
f_X(1000) &= \frac{\sum_{j=1}^{1}\left(\beta+\frac{\beta(r-1)j}{1}\right) f_B(1000j) f_X(1000(1-j))}{1+\beta-\beta f_B(0)} \\
&= \frac{\left(2.5+\frac{2.5(0.5-1)1}{1}\right) 0.2\times 0.534522}{1+2.5-2.5\times 0} \\
&= 0.038180
\end{aligned}
$$

and

$$
\begin{aligned}
f_X(2000) &= \frac{\sum_{j=1}^{2}\left(\beta + \frac{\beta(r-1)j}{2}\right) f_B(1000j) f_X(1000(2-j))}{1+\beta-\beta f_B(0)} \\
&= \frac{\left(2.5 + \frac{2.5(0.5-1)1}{2}\right) 0.2 \times 0.038180}{1+2.5-2.5\times 0} \\
&\quad + \frac{\left(2.5 + \frac{2.5(0.5-1)2}{2}\right) 0.3 \times 0.534522}{1+2.5-2.5\times 0} \\
&= 0.061361.
\end{aligned}
$$

In the following table, we provide values of $f_X(1000k)$:

Assumption	k	0	5	10	20	30
1	$f_X(1000k)$	0.286505	0.083659	0.020898	0.000368	0.000002
2	$f_X(1000k)$	0.263076	0.088471	0.020159	0.000177	0.000000
3	$f_X(1000k)$	0.534522	0.042620	0.016593	0.003770	0.000981

$\square$

6.3.8 Additional Remark

The Panjer's algorithm provide exact values of probability.

6.3.9 Summary

Distribution of M	Initial value $(f_X(0))$	Recursive relation $(f_X(kh))$
$Pois\,(\lambda)$	$\mathrm{e}^{\lambda(f_B(0)-1)}$	$\frac{\lambda}{k}\sum_{j=1}^{k} j f_B(jh) f_X((k-j)\,h)$
$Bin\,(n,q)$	$(1-q+qf_B\,(0))^n$	$\frac{\sum_{j=1}^{k}\left(-q+\frac{(n+1)qj}{k}\right)f_B(j)f_X((k-j)h)}{1-q+qf_B(0)}$
$BN\,(r,q)$	$\left(\frac{q}{1-(1-q)f_B(0)}\right)^r$	$\frac{\sum_{j=1}^{k}\left(1-q+\frac{(1-q)(r-1)j}{k}\right)f_B(jh)f_X((x-j)h)}{1-(1-q)f_B(0)}$

7 Discretization methods

7.1 Preliminaries

The recursive algorithms of convolution (including both DePril's and Panjer's algorithms) work for sum (finite or random) of rvs defined over an arithmetic support.

We propose two simple approaches to discretize a continuous rv (distribution).

These two approachs show to define a rv over an arithmetic support which can be used to approximate the initial continuous rv.

7.2 Finite sum of rvs

Let B_1, ..., B_n be independent continuous rvs.

We define the rv

$$S = \sum_{i=1}^{n} B_i.$$

We approximate the continuous rv B_i by a discrete rv $\widetilde{B}_i$ defined on the support

$$A_h = \{0, 1h, 2h, 3h, ...\},$$

where $h > 0$ is called the discretization span.

The pmf of $\widetilde{B}_i$ is

$$f_{\widetilde{B}_i}(kh) = \Pr\left(\widetilde{B}_i = kh\right), k \in \mathbb{N},$$

for $i = 1, 2, ..., n$.

We define the rv

$$\widetilde{S} = \sum_{i=1}^{n} \widetilde{B}_i,$$

which is also defined over A_h.

The pmf of $\widetilde{S}$ is

$$f_{\widetilde{S}}(kh) = \Pr\left(\widetilde{S} = kh\right),$$

for $k \in \mathbb{N}$,.

Then, the values of $f_{\widetilde{S}}$ can be calculated with the appropriated aggregation's recursive algorithms.

We approximate the desired quantities associated to the rv S (e.g. F_S, $VaR_\kappa(S)$, $TVaR_\kappa(S)$, stop-loss premium) by the corresponding quantities associated to the rv $\widetilde{S}$.

7.3 Random sum of rvs

Let the rv X be defined by the random sum

$$X = \begin{cases} \sum_{j=1}^{M} B_j, M > 0 \\ 0, M > 0 \end{cases},$$

where

- the rvs $B_1, B_2, ...$ form a sequence of iid, positive and continuous rvs with cdf F_B
- the rv M belongs to the $(a, b, 0)$ class of counting distributions.

We approximate the rvs B_1, B_2, ... by the discrete rvs $\widetilde{B}_1, \widetilde{B}_2$,... defined over A_h.

By assumption of the model, $\widetilde{B}_1 \sim \widetilde{B}_2 \sim \ldots \sim \widetilde{B}$.

The pmf of $\widetilde{B}$ is

$$f_{\widetilde{B}}(kh) = \Pr\left(\widetilde{B} = kh\right),$$

for $k \in \mathbb{N}$.

We define the rv $\widetilde{X}$ by

$$\widetilde{X} = \begin{cases} \sum_{j=1}^{M} \widetilde{B}_j, M > 0 \\ 0, M > 0 \end{cases},$$

keeping the same assumptions.

Obviously, rv $\widetilde{X}$ is also defined on A_h.

The pmf of $\widetilde{B}$ is

$$f_{\widetilde{B}}(kh) = \Pr\left(\widetilde{B} = kh\right),$$

for $k \in \mathbb{N}$.

The values of f are calculated with Panjer's algorithm or another algorithm.

Then, we approximate the quantities associated to the rv X (e.g. F_X, $VaR_\kappa(X)$, $TVaR_\kappa(X)$, stop-loss premium) by the corresponding quantities associated to the rv $\widetilde{X}$.

7.4 List discretization methods

The following methods can be used to define a rv $\widetilde{B}$ which approximate the rv B:

- Upper method
- Lower method
- Dispersion method with preserved expectation
- Rounding method

We only present the first two methods.

7.4.1 Upper method of discretization

We have

- $f_{\widetilde{B}}(0) = \Pr(B \leq h) = F_B(h)$
- $f_{\widetilde{B}}(kh) = \Pr(kh \leq B < (k+1)h) = F_B((k+1)h) - F_B(kh)$, for $k \in \mathbb{N}^+$.

The cdf $F_{\widetilde{B}}(x)$ is a step function where the steps (jumps) are at $0h$, $1h$, $2h$, ... i.e.

$$F_{\widetilde{B}}(x) = \begin{cases} F_B(h), & 0 \leq x < h \\ F_B(2h), & h \leq x < 2h \\ F_B(3h), & 2h \leq x < 3h \\ & \ldots \end{cases}$$

According to that method, the following relation is satisfied

$$F_B(x) \leq F_{\widetilde{B}}(x), \ x \geq 0.$$

7.4.2 Lower method of discretization

We have

- $f_{\widetilde{B}}(0) = 0$
- $f_{\widetilde{B}}(kh) = \Pr((k-1)h \leq B < kh) = F_B(kh) - F_B((k-1)h)$, for $k \in \mathbb{N}^+$.

The cdf $F_{\widetilde{B}}(x)$ is a step function where the steps (jumps) are at $1h$, $2h$, $3h$, ... i.e.

$$F_{\widetilde{B}}(x) = \begin{cases} 0, & 0 \leq x < h \\ F_B(h), & h \leq x < 2h \\ F_B(2h), & 2h \leq x < 3h \\ & \ldots \end{cases}.$$

We observe the following relation:

$$F_{\widetilde{B}}(x) \leq F_B(x), \; x \geq 0.$$

7.4.3 Illustration

On Figure 1, we illustrate the upper and the lower methods of discretization applied to the cdf of the exponential distribution.

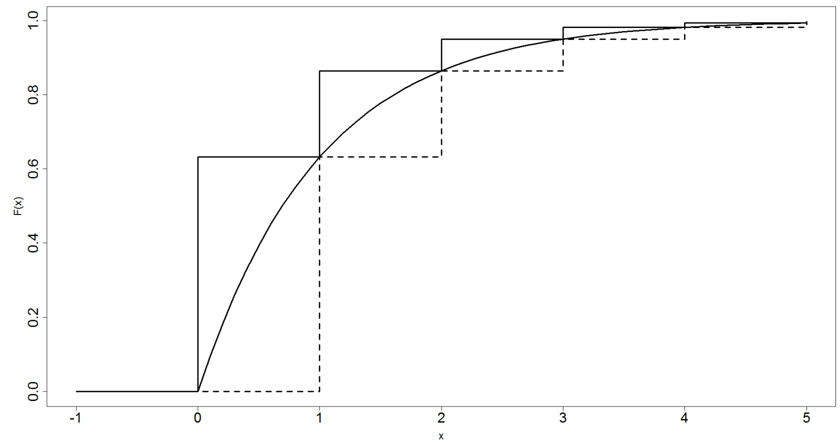

Figure 1: Values of $F_B(x)$ where $B \sim Exp(1)$, values of $F_{\widetilde{B}(up)}(x)$ (step continuous line) and values of $F_{\widetilde{B}(low)}(x)$ ((step dashed line) .

We observe

$$F_{\widetilde{B}(low)}(x) \leq F_B(x) \leq F_{\widetilde{B}(up)}(x), \; x \geq 0.$$

7.4.4 Numerical illustration 1 - Exact values vs approximated values

We apply the Panjer's algorithm jointly with the upper and lower methods of discretization.

Let

$$X \sim BNComp(r, q; F_B),$$

with $r = 1$, $q = 0.5$ and

$$B \sim Exp(0.2).$$

For this specific example, we know the closed form expression of the cdf of the X i.e.

$$F_X(x) = 1 - (1 - q)\,\mathrm{e}^{-\beta q x}, \qquad x \geq 0,$$

Then, we can compare exact and approximated values of F_X

x	Low $h=1$	Low $h=\frac{1}{4}$	Low $h=\frac{1}{16}$	Exacte	Up $h=\frac{1}{16}$	Up $h=\frac{1}{4}$	Up $h=1$
0	0.50000	0.50000	0.500000	0.5	0.50313	0.51250	0.54983
1	0.54532	0.54702	0.54744	0.54758	0.55055	0.55944	0.59470
2	0.58653	0.58961	0.59038	0.59064	0.59344	0.60186	0.63510
3	0.62400	0.62820	0.62924	0.62959	0.63225	0.64020	0.67147
4	0.65808	0.66316	0.66442	0.66484	0.66735	0.67485	0.70421
5	0.68907	0.69483	0.69626	0.69674	0.69910	0.70616	0.73369
10	0.78737	0.81375	0.81549	0.81606	0.81778	0.82289	0.84246
20	0.92523	0.93062	0.93191	0.93233	0.93317	0.93565	0.94487
30	0.97109	0.97416	0.97487	0.97511	0.97549	0.97662	0.98071
40	0.98882	0.99037	0.99073	0.99084	0.99101	0.99151	0.99325
50	0.99568	0.99641	0.99658	0.99663	0.99670	0.99691	0.99764

The difference between values obtained with *upper* and *lower* methods decreases (goes to zero) when h goes to 0.

The value of the chosen discretization span h depends on the range of "posslbie" values of the rv B.

In the following table, we provide exact and approximated values of $VaR_{\kappa}(X)$:

κ	Up $_{h=1}$	Up $_{h=\frac{1}{4}}$	Up $_{h=\frac{1}{16}}$	Exacte	Low $_{h=\frac{1}{16}}$	Low $_{h=\frac{1}{4}}$	Low $_{h=1}$
0.5	0	0	0	0	0	0	0
0.95	21	22.5	22.9375	23.02585	23.125	23.5	25
0.995	43	45.875	45.875	46.05167	46.25	46.75	49

□

The approximated values of F_X obtained with the upper (lower) method of discretization are always greater (smaller) than the exact values of F_X.

This relation can be shown using the stochastic orders.

7.4.5 Numericall illustration #2 – Approximated values only

We consider a portfolio of 100 independent car insurance contracts.

Let X_1,..., X_{100} be iid rvs with X_i (aggregate claim amount for the contract i) $\sim X$, for $i = 1, 2, ..., 100$.

Let $X \sim PComp\,(\lambda, F_B)$ with $\lambda = 0.025$ and $B \sim Pa\,(3, 10)$.

We want to evaluate F_S where

$$S = \sum_{i=1}^{100} X_i$$

We know that

$$S \sim PComp\,(100\lambda, F_B)\,.$$

In the following table, we provide the approximate values of $F_S\,(x)$ using the Panjer's algorithm jointly with the upper and lower methods of discretiztion (for $h = 1$ and $h = \frac{1}{4}$) :

x	$F_{\widetilde{S}^{low,h=1}}\,(x)$	$F_{\widetilde{S}^{low,h=\frac{1}{4}}}\,(x)$	$F_{\widetilde{S}^{up,h=\frac{1}{4}}}\,(x)$	$F_{\widetilde{S}^{up,h=1}}\,(x)$
0	0.0820850	0.0820850	0.09812643	0.1528517
1	0.1331183	0.1403239	0.16071324	0.2188115
5	0.3320781	0.3545721	0.38149450	0.4391453
10	0.5364597	0.5616138	0.58571454	0.6310597
20	0.7836771	0.7998287	0.81308686	0.8355891
30	0.8962240	0.9045299	0.91096430	0.9214718
40	0.9472100	0.9513226	0.95443381	0.9594453
50	0.9712884	0.9733614	0.97491838	0.9774225

Note that the exact value of $F_S\,(0)$ is 0.0820850.

We observe that the difference

$$F_{\widetilde{S}^{lup,h}}(x) - F_{\widetilde{S}^{low,h}}(x)$$

decreases as $h \downarrow 0$.

In the following table, we provide the values of the approximations for $VaR_\kappa(S)$:

κ	0.5	0.95	0.995
$VaR_\kappa\left(\widetilde{S}^{up,h=1}\right)$	7	37	84
$VaR_\kappa\left(\widetilde{S}^{up,h=\frac{1}{4}}\right)$	7.75	38.75	85.25
$VaR_\kappa\left(\widetilde{S}^{low,h=\frac{1}{4}}\right)$	8.5	39.75	86.25
$VaR_\kappa\left(\widetilde{S}^{low,h=1}\right)$	9	41	88

$\square$

7.4.6 Important relations

The upper and lower methods allow to construct bounds to the exact values for cdfs, VaRs, TVaRs and stop-loss premiums.

It is a clear advantage compared to the Monte Carlo simulation methods.

When $S = B_1 + ... + B_n$, it can be proven that

$$F_{\widetilde{S}(low,h')}(x) \leq F_{\widetilde{S}(low,h)}(x) \leq F_S(x) \leq F_{\widetilde{S}(up,h)}(x) \leq F_{\widetilde{S}(up,h')}(x)$$

for $x \geq 0$ and $h \leq h'$.

When

$$X = \begin{cases} \sum_{j=1}^{M} B_j, M > 0 \\ 0, M > 0 \end{cases},$$

it can be proven that

$$F_{\widetilde{X}(low,h')}(x) \leq F_{\widetilde{X}(low,h)}(x) \leq F_X(x) \leq F_{\widetilde{X}(up,h)}(x) \leq F_{\widetilde{X}(up,h')}(x)$$

for $x \geq 0$ and $h \leq h'$.

Those relations are shown in e.g. Denuit et al (2005), Marceau (2013, chapter 7), Muller and Stoyan (2002).

8 Fast Fourier Transform

It is important aggregation method.

Very efficient, even more efficient than Panjer's algorithm

9 Risk pooling and maximal claim

9.1 Result

The choice of the claim distribution may have a huge impact of the distribution of the aggregate claim amount S of the entire portfolio.

We examine the combined effect of risk pooling and the choice of the claim distribution on the maximal claim for a portfolio.

To this end, we consider two distributions:

- exponential distribution

- Pareto distribution.

Let $S_n = \sum_{i=1}^{n} X_i$ where X_1, X_2 ,..., X_n are iid rvs with

$$X_i \sim X \sim PComp\,(\gamma; F_B)$$

for $i = 1, 2, ..., n$, where n is the number of contracts within an insurance portfolio.

It implies that

$$S_n \sim CompPois\,(n\gamma; F_C)$$

which means that

$$S_n = \begin{cases} \sum_{k=1}^{N_n} C_k, & , N_n > 0 \\ 0, & , N_n = 0 \end{cases}$$

with the usual assumptions.

Note that the rv

$$N_n \sim Pois\left(\gamma n\right),$$

corresponds to the number of claims of the portfolio.

Furthermore, C_1, C_2, ... forms a sequence of iid rvs with $C_k \sim C \sim B$, $k = 1, 2, \ldots$.

Proposition 9 *The maximal amount of claim among the all the claims generated by the portfolio is defined by the rv $C_{\mathsf{max},n}$.*

Then, the expression of $F_{C_{\mathsf{max},n}}$ is given by

$$F_{C_{\mathsf{max},n}}(x) = P_{N_n}(F_B(x)) = \exp(n\gamma(F_B(x) - 1))$$

More specifically, for $x \geq 0$, we have

$$F_{C_{\mathsf{max},n}}(x) = \begin{cases} \exp\left(-n\gamma\left(\frac{\lambda}{\lambda+x}\right)^{\alpha}\right), & \text{, if } B \sim Pa(\alpha,\lambda) \\ \exp\left(-n\gamma \mathsf{e}^{-\beta\alpha}\right), & \text{, if } B \sim Exp(\beta) \end{cases}.$$

Proof. By conditioning on the rv $N_n \sim Pois(\gamma n)$, $F_{C_{\mathsf{max},n}}$ is given by

$$\begin{aligned}
F_{C_{\mathsf{max},n}}(x) &= f_{N_n}(0) + \sum_{k=1}^{\infty} f_{N_n}(k) \Pr\left(C_{\mathsf{max},n} \leq x | N_n = k\right) \\
&= f_{N_n}(0) + \sum_{k=1}^{\infty} f_{N_n}(k) \Pr\left(\max\left(C_1; ...; C_k\right) \leq x\right) \\
&= f_{N_n}(0) + \sum_{k=1}^{\infty} f_{N_n}(k) \left(F_C(x)\right)^k \\
&= f_{N_n}(0) + \sum_{k=1}^{\infty} f_{N_n}(k) \left(F_B(x)\right)^k
\end{aligned}$$

which become

$$\begin{aligned} F_{C_{\mathsf{max},n}}(x) &= \sum_{k=0}^{\infty} f_{N_n}(k)\,(F_B(x))^k \\ &= P_{N_n}(F_B(x)) \\ &= \exp\left(n\gamma\left(F_B(x)-1\right)\right). \end{aligned}$$

■

9.2 Numerical illustration

Assume that conditions of Proposition 9 are satisfied.

Let γ be fixed at 1.

In the following table, we provide the values of $\overline{F}_{C_{\max,n}}(x) = 1 - F_{C_{\max,n}}(x)$, for different values of n and for different values of the parameter α for the Pareto distribution.

The parameters are chosen such that

$$E\left[N_n\right] = n$$

and

$$E\left[S_n\right] = n \times 10.$$

$x\ (n=1)$	$Exp\left(\frac{1}{10}\right)$	$Pa\,(1.1;1)$	$Pa\,(1.5;5)$	$Pa\,(2.1;11)$
$1 \times E\left[S_n\right]$	0.30780	0.06903	0.17506	0.22678
$1.5 \times E\left[S_n\right]$	0.19999	0.04626	0.11750	0.15146
$2 \times E\left[S_n\right]$	0.12658	0.03451	0.08556	0.10731
$2.5 \times E\left[S_n\right]$	0.07881	0.02739	0.06578	0.07958
$3 \times E\left[S_n\right]$	0.04857	0.02262	0.05256	0.06116

$x\ (n=10)$	$Exp\left(\frac{1}{10}\right)$	$Pa\,(1.1;1)$	$Pa\,(1.5;5)$	$Pa\,(2.1;11)$
$1 \times E\left[S_n\right]$	0.000456	0.06500	0.09870	0.07498
$1.5 \times E\left[S_n\right]$	0	0.03930	0.05629	0.03506
$2 \times E\left[S_n\right]$	0	0.02885	0.03737	0.02002
$2.5 \times E\left[S_n\right]$	0	0.02267	0.02708	0.01286
$3 \times E\left[S_n\right]$	0	0.01860	0.02077	0.00892

x $(n = 100)$	$Exp\left(\frac{1}{10}\right)$	$Pa\,(1.1; 1)$	$Pa\,(1.5; 5)$	$Pa\,(2.1; 11)$
$1 \times E[S_n]$	0	0.04883	0.03488	0.00750
$1.5 \times E[S_n]$	0	0.03155	0.01987	0.00322
$2 \times E[S_n]$	0	0.02310	0.01238	0.00178
$2.5 \times E[S_n]$	0	0.01812	0.00888	0.00111
$3 \times E[S_n]$	0	0.01485	0.00676	0.00076

x $(n = 1000)$	$Exp\left(\frac{1}{10}\right)$	$Pa\,(1.1; 1)$	$Pa\,(1.5; 5)$	$Pa\,(2.1; 11)$
$1 \times E[S_n]$	0	0.03902	0.01111	0.00061
$1.5 \times E[S_n]$	0	0.02516	0.00606	0.00026
$2 \times E[S_n]$	0	0.01840	0.00394	0.00014
$2.5 \times E[S_n]$	0	0.01442	0.00282	0.00009
$3 \times E[S_n]$	0	0.01182	0.00215	0.00006

To interpret these results, we consider x as the premium income or the amount of money that the insurer has to meet its obligations over the next period.

It is found that the likelihood that the highest loss amount exceeds twice the hope of costs remains high even if the expected number of claims increases (assuming that the amount of a claim follows a Pareto distribution).

Thus, the probability is high that a large single claim (i.e the maximal amount of claim) can jeopardize the solvency of a company.

10 Challenges for research in actuarial science

- A lot of research has been done before mid 1990s.
- Now actuaries focus on aggregation of dependent risks.
- Impact of the heavy tailed distributions for claim amounts.
- Etc.