Lyon PhD Course Actuarial Science

Chapter 4 - Introduction to dependence

Professor : Etienne Marceau

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Contents

1 Introduction

Since the mid 1990s, we observe in actuarial science and in quantitative risk management an increasing interest in modeling the dependence between the risks.

It becomes essential for the actuary to be familiar with multivariate models and dependence modeling.

The dependence between risks has an impact on the risk pooling.

The actuary also needs to develop aggregation methods for dependent risks.

2 Fréchet Space

2.1 Definition

Let $F_{X_1}, ..., F_{X_n}$ be fixed univariate cfds.

Definition 1 *We define the Fréchet space, denoted by* $\Gamma\left(F_{X_1}, ..., F_{X_n}\right)$*, as the set of all multivariate cdfs with marginals (univariate cdfs)* F_{X_1}, ..., F_{X_n}.

2.2 Fréchet Bounds - bivariate case

We need the following lemma.

Lemma 2 *Let A and B be two events. Then, we have*

$$\max\left(\Pr\left(A\right)+\Pr\left(B\right)-1;0\right)\leq\Pr\left(A\cap B\right)\leq\min\left(\Pr\left(A\right);\Pr\left(B\right)\right). \quad (1)$$

Preuve. Proof of

$$\Pr\left(A\cap B\right)\leq\min\left(\Pr\left(A\right);\Pr\left(B\right)\right).$$

The two following inequalities must be satisfied :

$$\Pr\left(A\cap B\right)\leq P\left(A\right) \text{ and } \Pr\left(A\cap B\right)\leq\Pr\left(B\right),$$

which implies

$$\Pr(A \cap B) \leq \min(\Pr(A); \Pr(B)).$$

Proof of

$$\max(\Pr(A) + \Pr(B) - 1; 0) \leq \Pr(A \cap B)$$

We know that

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B). \tag{2}$$

Since

$$\Pr(A \cup B) \leq 1$$

we have

$$\Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1.$$

But, we also know that

$$\Pr(A \cap B) \geq 0$$

then (2) becomes

$$\Pr(A \cap B) \geq \max(\Pr(A) + \Pr(B) - 1; 0).$$

■

From (1), we find the following result

Proposition 3 Fréchet Bounds. *Consider the Fréchet space* $\Gamma\left(F_{X_1}, F_{X_2}\right)$ *generated from* F_{X_1} *and* F_{X_2}*. Then, we have*

$$F^{-}_{X_1,X_2}(x_1,x_2) \leq F_{X_1,X_2}(x_1,x_2) \leq F^{+}_{X_1,X_2}(x_1,x_2), \tag{3}$$

where

$$\begin{aligned} F^{-}_{X_1,X_2}(x_1,x_2) &= \max\left(F_{X_1}(x_1) + F_{X_2}(x_2) - 1; 0\right) \\ F^{+}_{X_1,X_2}(x_1,x_2) &= \min\left(F_{X_1}(x_1); F_{X_2}(x_2)\right). \end{aligned}$$

Both bounds are only functions of marginals F_{X_1} and F_{X_2}.

Important ! Whatever the dependence structure between the rvs X_1 and X_2, the bivariate cdf of (X_1, X_2) always satisfies (3).

Note that

$$F^{-}_{X_1,X_2}(x_1,x_2)$$

and

$$F^{+}_{X_1,X_2}(x_1, x_2)$$

are bivariate cdfs.

2.3 Fréchet's bounds - multivariate case

Theorem 4 *Let* $F_{X_1,...,X_n} \in \Gamma\left(F_{X_1},...,F_{X_n}\right)$*. Then, we have*

$$F^-_{X_1,...,X_n}(x_1,...,x_n) \leq F_{X_1,...,X_n}(x_1,...,x_n) \leq F^+_{X_1,...,X_n}(x_1,...,x_n),$$

where

$$F^-_{X_1,...,X_n}(\underline{x}) = \max\left(\sum_{i=1}^{n} F_{X_i}(x_i) - (n-1); 0\right)$$

and

$$F^+_{X_1,...,X_n}(\underline{x}) = \min\left(F_{X_1}(x_1); ...; F_{X_n}(x_n)\right).$$

Preuve. Proof of

$$F_{X_1,\ldots,X_n}(\underline{x}) = \Pr\left(\bigcap_{i=1}^{n}\{X_i \leq x_i\}\right).$$

Since

$$\bigcap_{i=1}^{n}\{X_i \leq x_i\} \subset \{X_i \leq x_i\}$$

for all $i = 1, 2, ..., n$, then

$$F_{X_1,\ldots,X_n}(\underline{x}) = \Pr\left(\bigcap_{i=1}^{n}\{X_i \leq x_i\}\right) \leq \Pr(X_i \leq x_i) = F_{X_i}(x_i), i = 1, 2, ..., n,$$

which leads to the desired result.

Proof of

$$F^{-}_{X_1,...,X_n}(x_1,...,x_n) \leq F_{X_1,...,X_n}(x_1,...,x_n).$$

The result follows from

$$\begin{aligned} \Pr\left(\bigcap_{i=1}^{n}\{X_i \leq x_i\}\right) &= 1 - \Pr\left(\bigcup_{i=1}^{n}\{X_i > x_i\}\right) \\ &\geq 1 - \sum_{i=1}^{n}\Pr(X_i > x_i) = 1 - n + \sum_{i=1}^{n} F_{X_i}(x_i). \end{aligned}$$

■

Notes about $F^{+}_{X_1,...,X_n}$:

- $F^{+}_{X_1,...,X_n}$ corresponds to the Fréchet upper bound

- $F^{+}_{X_1,\ldots,X_n}$ is a multivariate cdf

Notes about $F^{-}_{X_1,\ldots,X_n}$:

- $F^{-}_{X_1,\ldots,X_n}$ is called the Fréchet lower bound
- For $n = 2$, $F^{-}_{X_1,X_2}$ is a cdf
- For $n > 2$, $F^{-}_{X_1,\ldots,X_n}$ is not a cdf (see [**?**]).

3 Comonotonicity

3.1 Definition

Comonotonicity is specific dependent relation.

Since the mid nineties, it is frequently used in actuariual science.

Definition 5 *A vector of rvs* $\underline{X} = (X_1, ..., X_n)$ *is comonotonic if and only if it exists a rv* Z *and non decreasing functions* ϕ_1,...,ϕ_n *such that*

$$(X_1, ..., X_n) =_d \ (\phi_1 (Z), ..., \phi_n (Z)).$$

3.2 Example

We define a vector of rvs $\underline{X} = (X_1, ..., X_n)$ where

$$X_i = F_{X_i}^{-1}(U), \tag{4}$$

for $i = 1, 2, ..., n$ and $U \sim Unif(0, 1)$.

Then, by Definition 5, the components of $\underline{X}$ are comonotonic.

The relation in (4) is important for the risk aggregation when the risks are assumed to be comonotonic.

3.3 Comonotonicity and simulation

Let $\underline{X} = (X_1, .., X_n)$ be a vector of comonotonic rvs.

Algorithm 6 Simulation of samples of $\underline{X} = (X_1, ..., X_n)$

1. *We simulate a sample $U^{(j)}$ of the rv $U \sim Unif\,(0, 1)$.*
2. *We calculate $X_1^{(j)} = F_{X_1}^{-1}\left(U^{(j)}\right)$, ..., $X_n^{(j)} = F_{X_n}^{-1}\left(U^{(j)}\right)$.*

Let $\underline{X} = (X_1, .., X_n)$ be a vector of independent rvs.

Algorithm 7 Simulation of samples of $\underline{X} = (X_1, ..., X_n)$

1. *We simulate a sample* $\left(U_1^{(j)}, ..., U_n^{(j)}\right)$ *of the vector of iid rvs* $(U_1, ..., U_n)$ *where* $U_i \sim Unif\,(0,1)$, $i = 1, 2, ..., n$.
2. *We calculate* $X_1^{(j)} = F_{X_1}^{-1}\left(U_1^{(j)}\right)$, ..., $X_n^{(j)} = F_{X_n}^{-1}\left(U_n^{(j)}\right)$.

3.4 Comonotonicity and upper Fréchet bound

The link between comonotonocity and upper Fréchet bound is established in the following proposition.

Proposition 8 *The components of* $\underline{X} = (X_1, ..., X_n)$ *are comonotonic if and only if its multivariate cdf is the upper Fréchet bound* F^+*.*

Preuve. Let $\underline{X}$ be defined as in (4). Then, we have

$$\begin{aligned} F_{\underline{X}}(x_1, ..., x_n) &= \Pr\left(F_{X_1}^{-1}(U) \leq x_1, ..., F_{X_n}^{-1}(U) \leq x_n\right) \\ &= \Pr\left(U \leq F_{X_1}(x_1), ..., U \leq F_{X_n}(x_n)\right) \\ &= \Pr\left(U \leq \min\left(F_{X_1}(x_1); ...; F_{X_n}(x_n)\right)\right) \\ &= F_{\underline{X}}^{+}(\underline{x}), \end{aligned}$$

where

$$F^{+}_{X_1,\ldots,X_n}(\underline{x}) = \min\left(F_1(x_1);\ldots;F_n(x_n)\right).$$

■

Comonotonicity corresponds to the perfect dependence.

3.5 The components of a comonotonic random vector depend on one rv

Proposition 9 *Let $\underline{X}$ be a vector of continuous rvs with marginals F_{X_1}, ..., F_{X_n}. The components of $\underline{X}$ are comonotonic if, and only if,*

$$X_i = F^{-1}_{X_i}\left(F_{X_j}\left(X_j\right)\right), \tag{5}$$

for $j \in \{1, 2, ..., n\}$ *and all* $i \in \{1, 2, ..., n\}$ *with* $i \neq j$.

Preuve. To simplify the presentation, we fix $i = 1$ without loss of generalization.

If the components are comonotonic, we use (4) jointly with Theorem **??**

$$X_i = F_{X_i}^{-1}(U) = F_{X_i}^{-1}\left(F_{X_1}(X_1)\right)$$

for $i = 2, ..., n$. The reverse can be similarly proven. ■

3.6 Comonotonicity and aggregation

Let $\underline{X} = (X_1, ..., X_n)$ be vector of comonotonic rvs.

We define the rv $S = \sum_{i=1}^{n} X_i$.

From (4), we have

$$S = \sum_{i=1}^{n} F_{X_i}^{-1}(U). \tag{6}$$

Proposition 10 (Additivity of the VaR and the TVaR).

- *Let* $\underline{X} = (X_1, ..., X_n)$ *be vector of comonotonic rvs.*

- *We define the rv* $S = \sum_{i=1}^{n} X_i$.
- *Then, we have*

$$VaR_{\kappa}(S) = \sum_{i=1}^{n} VaR_{\kappa}(X_i). \tag{7}$$

and

$$TVaR_{\kappa}(S) = \sum_{i=1}^{n} TVaR_{\kappa}(X_i). \tag{8}$$

Preuve. We start with the case when the rvs are continuous.

From (6), we have

$$S = \sum_{i=1}^{n} F_{X_i}^{-1}(U) = \varphi(U).$$

Then, by Proposition **??**, we have

$$VaR_\kappa(S) = VaR_\kappa(\varphi(U)) = \varphi(VaR_\kappa(U))$$

which becomes

$$VaR_\kappa(S) = \varphi(\kappa) = \sum_{i=1}^{n} F_{X_i}^{-1}(\kappa) = \sum_{i=1}^{n} VaR_\kappa(X_i).$$

We replace (7) in the basic definition of $TVaR_\kappa(S)$ and we have

$$\begin{aligned} TVaR_\kappa(S) &= \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(S)\,\mathrm{d}u \\ &= \frac{1}{1-\kappa}\int_\kappa^1 \sum_{i=1}^{n} VaR_u(X_i)\,\mathrm{d}u \\ &= \sum_{i=1}^{n} TVaR_\kappa(X_i). \end{aligned}$$

The case when the rvs are not necessarily continuous is treated in [**?**]. ■

3.7 Pooling comonotonic risks

The capital is measured with the TVaR.

Then, the benefit of risk pooling is given by

$$B_{\kappa}^{TVaR}(X_1, ..., X_n) = \sum_{i=1}^{n} TVaR_{\kappa}(X_i) - TVaR_{\kappa}(S).$$

If the rvs $X_1, ..., X_n$ are comonotonic, then

$$B_{\kappa}^{TVaR}(X_1, ..., X_n) = 0.$$

3.8 Comonotonicity and aggregation – continous rvs

3.8.1 Preliminaries

We consider sum of comonotonic continuous rvs.

Recall that

$$F^{+}_{X_1,\ldots,X_n}(\underline{x}) = \min\left(F_1(x_1); \ldots; F_n(x_n)\right).$$

We show that it easy to find the expressions of $VaR_\kappa(S)$ and $TVaR_\kappa(S)$ with Proposition 10.

For specific case, we can derive the expression of F_S from the expression of $VaR_\kappa(S)$.

Otherwise the value of $F_S(x)$ can be found by numerical optimization.

3.8.2 General case for continuous rvs

Let $\underline{X} = (X_1, ..., X_n)$ be vector of continuous comonotonic rvs.

We define $S = X_1 + ... + X_n$.

We know:

- $VaR_\kappa(S) = \sum_{i=1}^{n} VaR_\kappa(X_i)$.
- $TVaR_\kappa(S) = \sum_{i=1}^{n} TVaR_\kappa(X_i)$.

In fact, we have

$$S = \varphi(U)$$

where $U \sim Unif(0,1)$.

Objective:

- We fix x.
- We want to compute $F_S(x)$.

Clearly, we have

$$\begin{aligned} F_S(x) &= \Pr(S \leq x) \\ &= \Pr(\varphi(U) \leq x) \\ &= \Pr\left(U \leq \varphi^{-1}(x)\right) \\ &= \varphi^{-1}(x), \end{aligned}$$

since $U \sim Unif\,(0,1)$ for the last line.

Then, we need to find the inverse of the function φ i.e. find the solution u to the equation of

$$x = \varphi(u).$$

Sometime we can find an analytic expression for φ^{-1}. Otherwise, we need to use optimization procedure.

3.8.3 Example #1

Here is an example where we can find the inverse of $\varphi(u) = \sum_{i=1}^{n} F_{X_i}^{-1}(u)$.

Example 11 *Let* $(X_1, ..., X_n)$ *be a vector of comonotonic rvs where* $X_i \sim Exp(\beta_i)$ *with* $\overline{F}_{X_i}(x) = \exp(-\beta_i x)$ *(*$i = 1, 2, ..., n$*).*

For $S = \sum_{i=1}^{n} X_i$*, we have*

$$\begin{aligned} S &= \left(-\frac{1}{\beta_1}\ln(1-U)\right) + ... + \left(-\frac{1}{\beta_n}\ln(1-U)\right) \\ &= \varphi(U) \end{aligned}$$

where

$$\varphi(u) = -\left(\frac{1}{\beta_1} + ... + \frac{1}{\beta_n}\right)\ln(1-u).$$

It implies that

$$S \sim Exp\left(\frac{1}{\frac{1}{\beta_1}+...+\frac{1}{\beta_n}}\right).$$

Indeed, we have

$$\begin{aligned} F_S(x) &= \Pr(S \leq x) \\ &= \varphi^{-1}(x) \end{aligned}$$

where $\varphi^{-1}(x)$ *is the solution to*

$$\varphi(u) = -\left(\frac{1}{\beta_1}+...+\frac{1}{\beta_n}\right)\ln(1-u) = x.$$

We find

$$\varphi^{-1}(x) = 1 - \exp\left(-\frac{1}{\frac{1}{\beta_1}+...+\frac{1}{\beta_n}}x\right).$$

$\square$

3.8.4 Example #2

Here is another example where we can find the inverse of $\varphi(u) = \sum_{i=1}^{n} F_{X_i}^{-1}(u)$.

Example 12 *Let $(X_1, ..., X_n)$ be a vector of comonotonic rvs where $X_i \sim Pa(\alpha, \lambda_i)$ with $F_{X_i}(x) = 1 - \left(\frac{\lambda_i}{\lambda_i + x}\right)^{\alpha}$ and $F_{X_i}^{-1}(u) = \frac{\lambda_i}{(1-u)^{\frac{1}{\alpha}}} - \lambda_i$ $(i = 1, 2, ..., n)$.*

For $S = \sum_{i=1}^{n} X_i$, we have

$$
\begin{aligned}
S &= \sum_{i=1}^{n} X_i = \left(\frac{\lambda_1}{(1-U)^{\frac{1}{\alpha}}} - \lambda_1\right) + ... + \left(\frac{\lambda_n}{(1-U)^{\frac{1}{\alpha}}} - \lambda_n\right) \\
&= \frac{\lambda^*}{(1-U)^{\frac{1}{\alpha}}} - \lambda^* = \varphi(U).
\end{aligned}
$$

Clearly,

$$\varphi^{-1}(x) = 1 - \left(\frac{\lambda^*}{\lambda^* + x}\right)^{\alpha}.$$

It implies that $S \sim Pa(\alpha, \lambda^*)$ *where*

$$\lambda^* = \lambda_1 + ... + \lambda_n.$$

$\square$

3.8.5 Example #3

Example 13 *Let* (X_1, X_2, X_3) *be a vector of comonotonic rvs where* $X_1 \sim U(0, 200)$, $X_2 \sim Pa(3, 200)$ *and* $X_3 \sim Exp\left(\frac{1}{100}\right)$.

- *We define* $S = X_1 + X_2 + X_3$.
- *Since* $E[X_i] = 100$, $i = 1, 2, 3$, *then* $E[S] = 300$.
- *We obtain*

$$VaR_\kappa(S) = 200\kappa + 200\left((1-\kappa)^{-\frac{1}{3}} - 1\right) + (-100)\ln(1-\kappa)$$

and

$$\begin{aligned} TVaR_\kappa(S) &= 200\frac{(1+\kappa)}{2} + (-100)(\ln(1-\kappa) + 1) \\ &\quad + 200\left(\frac{3}{3-1}(1-\kappa)^{-\frac{1}{3}} - 1\right), \end{aligned}$$

for $\kappa \in (0, 1)$.

- *Note that* $TVaR_0(S) = E[S]$.
- *Here, we have*

$$\varphi(u) = 200u + 200\left((1-u)^{-\frac{1}{3}} - 1\right) + (-100)\ln(1-u).$$

- *We cannot find a closed-form expression for* $\varphi^{-1}(x)$.
- *The value of* $F_S(x) = \varphi^{-1}(x)$ *can be found by numerical optimization.* $\square$

3.8.6 Example #4

Example 14 *Let* $(X_1, ..., X_n)$ *be a vector of comonotonic rvs. Assume also that the rvs are identically distributed i.e.* $F_{X_i} = F_X$ *and* $X_i \sim X$ *for* $i = 1, 2, ..., n$. *From (4), we have*

$$\begin{aligned} S &= \sum_{i=1}^{n} X_i = F_X^{-1}(U) + ... + F_X^{-1}(U) \\ &= nF_X^{-1}(U) = nX. \end{aligned}$$

Then,

$$\begin{aligned} F_S(x) &= \Pr(S \leq x) \\ &= \Pr(nX \leq x) \\ &= F_X\left(\frac{x}{n}\right). \end{aligned}$$

According to the assumption of the present example, it means that the rvs take all the same value. $\square$

3.9 Comonotonicity and aggregation - discrete rvs

3.9.1 Preliminaries

We consider sum of comonotonic continuous rvs.

Recall that

$$F^{+}_{X_1,\dots,X_n}(\underline{x}) = \min\left(F_1(x_1);\dots;F_n(x_n)\right).$$

We show that it easy to find the expressions of $VaR_\kappa(S)$ and $TVaR_\kappa(S)$ with Proposition 10.

For specific case, we can derive the expression of F_S from the expression of $VaR_\kappa(S)$.

3.9.2 Examples

Example 15 *We consider a portefolio of 10 comonotonic risks X_1, ..., X_{10}, with $X_i = b_i I_i$ where $I_i \sim Bern(q_i)$ and $b_i \in \mathbb{R}^+$ ($i = 1, 2, ..., 10$).*

The values of q_i and b_i are provided in the following table :

i	q_i	b_i	i	q_i	b_i
1	0.091	100	6	0.031	300
2	0.064	100	7	0.014	400
3	0.049	200	8	0.023	400
4	0.019	200	9	0.058	500
5	0.027	300	10	0.065	500

- *The rvs I_1,..., I_{10} are comonotonic.*
- *The rv S is defined by $S = \sum_{i=1}^{10} X_i$.*
- *We have $S = \sum_{i=1}^{10} b_i F_{I_i}^{-1}(U)$ where $U \sim U(0,1)$.*

- We obtain :

i	q_i	$1-q_i$	b_i	ordre de $1-q_i$
1	0.091	0.809	100	1
2	0.064	0.936	100	3
3	0.049	0.951	200	5
4	0.019	0.981	200	9
5	0.027	0.973	300	7
6	0.031	0.969	300	6
7	0.014	0.986	400	10
8	0.023	0.977	400	8
9	0.058	0.942	500	4
10	0.065	0.935	500	2

and

x	$\Pr(S = x)$	$\Pr(S \leq x)$
0	0.809	0.809
100	0.126	0.935
600	0.001	0.936
700	0.006	0.942
1200	0.009	0.951
1400	0.018	0.969
1700	0.004	0.973
2000	0.004	0.977
2400	0.004	0.981
2600	0.005	0.986
3000	0.014	1.000

□

4 Countermonotonicity

4.1 Definition

The countermonotonicity corresponds to the perfect negative dependence.

The countermonotonicity is only defined for couples of rvs.

Definition 16 *A couple of rvs $\underline{X} = (X_1, X_2)$ is countermonotonic if, and only if, it exists a rv Z, an increasing function ϕ_1 and a decreasing function ϕ_2 such that*

$$(X_1, X_2) =_d \ (\phi_1(Z), \phi_2(Z)).$$

Let $\underline{X} = (X_1, X_2)$ be a couple of rvs with

$$X_1 = F_{X_1}^{-1}(U) \text{ et } X_2 = F_{X_2}^{-1}(1-U). \tag{9}$$

Then, by Definition 5, $\underline{X} = (X_1, X_2)$ is countermonotonic.

4.2 Countermonotonicity and lower Fréchet bound

The bivariate cdf for a pair of countermontonic rvs $\underline{X} = (X_1, X_2)$ is provided in the following proposition.

Proposition 17 *The pair of rvs $\underline{X} = (X_1, X_2)$ is countermonotonic if, and only if, its bivariate cdf corresponds to the lower Fréchet bound $F^{-}_{X_1,X_2}$.*

Preuve. For $\underline{X}$ defined as (9), we have

$$\begin{aligned} F_{\underline{X}}(x_1, x_2) &= \Pr\left(F^{-1}_{X_1}(U) \le x_1, F^{-1}_{X_2}(1-U) \le x_2\right) \\ &= \Pr\left(U \le F_{X_1}(x_1), 1-U \le F_{X_2}(x_2)\right) \\ &= \Pr\left(U \le F_{X_1}(x_1), U > 1-F_{X_2}(x_2)\right) \end{aligned}$$

Then, we observe

$$\Pr\Big(U \leq F_{X_1}(x_1), 1-U \leq F_{X_2}(x_2)\Big) = \Pr\Big(U \leq F_{X_1}(x_1), U > 1-F_{X_2}(x_2)\Big).$$

If $F_{X_1}(x_1) \geq 1-F_{X_2}(x_2)$,

$$\begin{aligned} \Pr\Big(U \leq F_{X_1}(x_1), U > 1-F_{X_2}(x_2)\Big) &= F_{X_1}(x_1) - \Big(1-F_{X_2}(x_2)\Big) \\ &= F_{X_1}(x_1) + F_{X_2}(x_2) - 1. \end{aligned}$$

If $F_X(x_1) < 1-F_{X_2}(x_2)$,

$$\Pr\Big(U \leq F_{X_1}(x_1), U > 1-F_{X_2}(x_2)\Big) = 0.$$

Then, we conclude

$$\begin{aligned} \Pr\Big(U \leq F_{X_1}(x_1), 1-U \leq F_{X_2}(x_2)\Big) &= \max\Big(F_{X_1}(x_1) + F_{X_2}(x_2) - 1; 0\Big) \\ &= F^{-}_{X_1,X_2}(x_1, x_2) \end{aligned}$$

as defined in Theorem 4.

■

4.3 Countermonotonicity and simulation

Let (X_1, X_2) be a couple of countermonotonic rvs.

Algorithm 18 Simulation of samples of (X_1, X_2)

1. *We simulate a sample $U^{(j)}$ of the rv $U \sim Unif\,(0,1)$.*
2. *We calculate $X_1^{(j)} = F_{X_1}^{-1}\left(U^{(j)}\right)$, $X_2^{(j)} = F_{X_2}^{-1}\left(1 - U^{(j)}\right)$.*

4.4 Countermonotonicity and another result

Let $\underline{X} = (X_1, X_2)$ be a pair of continuous rvs with marginals F_{X_1} and F_{X_2}.

Then, $\underline{X}$ is countermonotonic if, and only if,

$$(X_1, X_2) =_d \left(X_1, F_{X_2}^{-1} \left(\overline{F}_{X_1} (X_1) \right) \right).$$

4.5 Countermonotonicity and risk aggregation

4.5.1 Preliminaries

Let $\underline{X} = (X_1, X_2)$ be a pair of countermonotonic rvs with marginals F_{X_1} and F_{X_2}.

We define $S = X_1 + X_2$.

Then, by (9), we have

$$S = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(1-U). \tag{10}$$

Generally, there is no "nice" general expression for the VaR and the TVaR of the rv S.

However, we may find close expressions for specific examples.

We need to be careful.

4.5.2 Example #1

Let $\underline{X} = (X_1, X_2)$ be a pair of rvs with

$$X_1 \sim Unif\,(0,1) \text{ and } X_2 \sim Unif\,(0,1)\,.$$

We define $S = 10X_1 + 6X_2$.

- Assume that the rvs X_1 are X_2 comonotonic.
  - Then, we have

$$\begin{aligned} S &= 10F_{X_1}^{-1}(U) + 6F_{X_2}^{-1}(U) \\ &= 10U + 6U \\ &= 16U. \end{aligned}$$

– The expression of $F_S(x)$ is

$$\begin{aligned} F_S(x) &= \Pr(16U \leq x) \\ &= \frac{x}{16},\ x \in [0, 16]. \end{aligned}$$

- Assume that the rvs X_1 are X_2 countermonotonic.
  – Then, by (10), we have

$$\begin{aligned} S &= 10F_{X_1}^{-1}(U) + 6F_{X_2}^{-1}(1-U) \\ &= 10U + 6(1-U) \\ &= 6 + 4U. \end{aligned}$$

  – It means that

$$S \in [6, 10].$$

– Then, we find

$$\begin{aligned} F_S(x) &= \Pr(6 + 4U \leq x) \\ &= \frac{x-6}{4},\ x \in [6, 10]. \end{aligned}$$

4.5.3 Example #2

Let us formalize the result of the previous example.

Let $\underline{X} = (X_1, X_2)$ be a pair of countermonotonic rvs with

$$X_1 \sim Unif\,(0, a) \text{ and } X_2 \sim Unif\,(0, b)\,,$$

with $a > b > 0$.

Then, we have

$$\begin{aligned} S &= aF_{X_1}^{-1}(U) + bF_{X_2}^{-1}(1-U) \\ &= aU + b(1-U) \\ &= b + (a-b)\,U \end{aligned}$$

We find

$$F_S(x) = \frac{x-b}{(a-b)},\ x \in [b,a]. \ \square$$

4.5.4 Example #3

Let $\underline{X} = (X_1, X_2)$ be a pair of countermonotonic rvs where

$$X_1 \sim N\left(\mu_1, \sigma_1^2\right)$$

and

$$X_2 \sim N\left(\mu_2, \sigma_2^2\right),$$

with $\sigma_1 > \sigma_2$.

Then, we have

$$\begin{aligned} S &= F_{X_1}^{-1}(U) + F_{X_2}^{-1}(1-U) \\ &= \mu_1 + \sigma_1 \Phi^{-1}(U) + \mu_2 + \sigma_2 \Phi^{-1}(1-U) \end{aligned}$$

We define the rv $Z \sim N(0,1)$.

Then we have

$$\Phi^{-1}(U) = Z$$

and

$$\Phi^{-1}(1-U) = -Z$$

We conclude

$$\begin{aligned} S &= F_{X_1}^{-1}(U) + F_{X_2}^{-1}(1-U) \\ &= \mu_1 + \sigma_1 Z + \mu_2 - \sigma_2 Z \\ &= \mu_1 + \mu_2 + (\sigma_1 - \sigma_2) Z. \end{aligned}$$

We find

$$F_S(x) = \Phi\left(\frac{x - (\mu_1 + \mu_2)}{(\sigma_1 - \sigma_2)}\right), \ x \in \mathbb{R}.$$

Note :

- If $\sigma_1 = \sigma_2$, then the distribution of the rv S is degenerated at $(\mu_1 + \mu_2)$.
- i.e $\Pr(S = (\mu_1 + \mu_2)) = 1$.

4.6 Countermonotonicity and risk aggregation - id case

4.6.1 Let's go

We apply (10) for a couple of countermonotonic continuous rvs (X_1, X_2) with identical marginals.

It means that

$$X_1 \sim X_2 \sim X$$

and

$$F_{X_1} = F_{X_2} = F_X.$$

The function

$$S = \varphi(u) = F_X^{-1}(u) + F_X^{-1}(1-u)$$

is not monotonic.

In most cases, the function $\varphi(u)$ is convex.

Let us assume that $\varphi(u)$ is convex (see following examples).

The minimum of $\varphi(u)$ is at $u = \frac{1}{2}$.

Then, for $u_1 \in [0, 0.5]$ and

$$u_2 = 1 - u_1,$$

we have

$$VaR_\kappa(S) = F_X^{-1}(u_1) + F_X^{-1}(u_2)$$

where

$$u_2 - u_1 = \kappa.$$

It means that

$$1 - u_1 - u_1 = \kappa.$$

We find

$$\begin{aligned} u_1 &= \frac{1-\kappa}{2} \\ &= \frac{1}{2} - \frac{\kappa}{2} \end{aligned}$$

and

$$\begin{aligned} u_2 &= 1 - \frac{1-\kappa}{2} \\ &= \frac{1}{2} + \frac{\kappa}{2}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} VaR_{\kappa}(S) &= F_X^{-1}\left(\frac{1}{2}-\frac{\kappa}{2}\right)+F_X^{-1}\left(\frac{1}{2}+\frac{\kappa}{2}\right) \\ &= VaR_{\frac{1}{2}-\frac{\kappa}{2}}(X)+VaR_{\frac{1}{2}+\frac{\kappa}{2}}(X). \end{aligned}$$

If the rvs X_1 and X_2 are positive, then the rvs

$$S = X_1 + X_2 \geq F_X^{-1}\left(\frac{1}{2}\right)+F_X^{-1}\left(\frac{1}{2}\right) = 2F_X^{-1}\left(\frac{1}{2}\right) > 0.$$

Then, we have

$$F_S\left(2F_X^{-1}\left(\frac{1}{2}\right)\right) = 0.$$

For specific distribution of the rv X, we can find a closed expression for F_S.

Otherwise, the value of $F_S(x)$ can be obtained by numerical optimization.

4.6.2 Example #1

Let (X_1, X_2) be a couple of countermonotonic continuous rvs with

$$X_1 \sim X_2 \sim Exp(\beta).$$

The function

$$S = \varphi(u) = F_X^{-1}(u) + F_X^{-1}(1-u)$$

is convex.

Then, we have

$$\begin{aligned} VaR_\kappa(S) &= \left(-\frac{1}{\beta}\right)\ln\left(1-\frac{1-\kappa}{2}\right) + \left(-\frac{1}{\beta}\right)\ln\left(1-\frac{1+\kappa}{2}\right) \\ &= \left(-\frac{1}{\beta}\right)\ln\left(\frac{1+\kappa}{2}\right) + \left(-\frac{1}{\beta}\right)\ln\left(\frac{1-\kappa}{2}\right). \end{aligned}$$

We find a closed expression for F_S :

$$F_S(x) = \sqrt{1 - 4e^{-\beta x}},\ x \geq \frac{1}{\beta}\ln(4).$$

It means

$$F_S\left(\frac{1}{\beta}\ln(4)\right) = 0.$$

The expression for the TVaR of the rv S is

$$
\begin{aligned}
TVaR_{\kappa}(S) &= \frac{1}{1-\kappa}\int_{\kappa}^{1} VaR_{u}(S)\,\mathrm{d}u \\
&= 2\left(-\frac{1}{\beta}\right) + \left(-\frac{1}{\beta}\right)\ln\left(\frac{1-\kappa}{2}\right) \\
&\quad + \left(-\frac{1}{\beta}\right)\ln\left(1-\frac{1-\kappa}{2}\right) \\
&\quad + \frac{2}{1-\kappa}\left(\frac{1}{\beta}\right)\ln\left(1-\frac{1-\kappa}{2}\right).
\end{aligned}
$$

4.6.3 Example #2

Let (X_1, X_2) be a couple of countermonotonic continuous rvs with

$$X_1 \sim X_2 \sim Pareto\,(\alpha, \lambda)\,.$$

The function

$$S = \varphi\,(u) = F_X^{-1}\,(u) + F_X^{-1}\,(1 - u)$$

is convex.

Then, we have

$$
\begin{aligned}
VaR_{\kappa}(S) &= \lambda\left(\left(1-\frac{1-\kappa}{2}\right)^{-\frac{1}{\alpha}}-1\right)+\lambda\left(\left(1-\frac{1+\kappa}{2}\right)^{-\frac{1}{\alpha}}-1\right) \\
&= \lambda\left(\left(\frac{1+\kappa}{2}\right)^{-\frac{1}{\alpha}}-1\right)+\lambda\left(\left(\frac{1-\kappa}{2}\right)^{-\frac{1}{\alpha}}-1\right).
\end{aligned}
$$

Remarks :

- $F_S\left(2\lambda\left(\left(\frac{1}{2}\right)^{-\frac{1}{\alpha}}-1\right)\right)=0$ i.e. $S\geq 2\lambda\left(\left(\frac{1}{2}\right)^{-\frac{1}{\alpha}}-1\right)$;
- values of $F_S(x)$ can be found by numerical inversion.

5 Challenges for research in actuarial science

- Find the distribution of the sum of countermonotic risks
- Comonotonic risks = extreme positive dependence
- Countermonotonic = extreme negative dependence in bivariate case
- Find the structure of dependence structure leading to extreme negative dependence
- Aggregation of dependent risk with unknown joint distribution or partial information on the joint distribution.
- Etc.