Lyon PhD Course Actuarial Science

Chapter 8 - Capital Allocation

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Introduction

We consider a portfolio of n risks on an insurance company, a bank or any financial firm

Let $\underline{X} = (X_1, ..., X_n)$ be a random vector where X_i is the aggregated losses for risk i, $i = 1, 2, ..., n$.

The aggregate claim amount for the entire portfolio is represented by the rv S where

$$S = \sum_{i=1}^{n} X_i.$$

Let ζ_κ be a risk measure with level of confidence $\kappa \in (0, 1)$.

The amount of capital for the entire portfolio is given by $\zeta_\kappa(S)$.

Question : how do we allocate to each risk its fair share of $\zeta_\kappa(S)$?

The share, called the contribution $C_\kappa(X_i)$, is determined according to a rule of allocation, also an "method of allocation".

Here are some naive rules of allocation:

- $C_\kappa(X_i) = \zeta_\kappa(S) \frac{E[X_i]}{\sum_{i=1}^n E[X_i]}$, $i = 1, 2, ..., n$;
- $C_\kappa(X_i) = \zeta_\kappa(S) \frac{Var(X_i)}{\sum_{i=1}^n Var(X_i)}$, $i = 1, 2, ..., n$;
- $C_\kappa(X_i) = \zeta_\kappa(S) \frac{\zeta_\kappa(X_i)}{\sum_{i=1}^n \zeta_\kappa(X_i)}$, $i = 1, 2, ..., n$.

However, these rules do not take into account the dependence structure among the components of $\underline{X}$, neither the marginal distributions of these components.

Here the proposed procedure for capital allocation :

- Establish the joint distribution for $\underline{X} = (X_1, ..., X_n)$.
- Choose the risk measure ζ_κ for the computation of amount of capital for the portfolio.
- Use a capital allocation method to find the contributions of each component of $\underline{X}$.

In this chapter, we mainly use the capital allocation method based on a result due to Euler. The method is called "Euler's rule of capital allocation" or "Euler's capital allocation method". Of course, Euler (1707-1783) did not have "capital allocation" in his mind when he derived his result ... ;-). Note that one of the teacher of was Johann Bernoulli (one member of the famous Bernoulli family).

1 Desirable properties

Let $\underline{X} = (X_1, ..., X_n)$.

We define

$$S = X_1 + ... + X_n \, .$$

Let ζ_κ be a risk measure with level of confidence κ, for $\kappa \in]0, 1[$.

The amount of capital is determined by $\zeta_\kappa (S)$.

The objective of a capital allocation method is to determine the amount of $\zeta_\kappa (S)$ that should be allocatted to the rv X_i (risk i) , denoted by $C_\kappa^\zeta (X_i)$.

Property 1 Complete allocation. $\zeta_\kappa(S) = \sum_{i=1}^{n} C_\kappa^\zeta(X_i)$, *for any* $\kappa \in (0,1)$.

Property 2 Positive Benefit of pooling. *For* $i = 1, 2, ..., n$ *and for any* $\kappa \in (0,1)$, *we have*

$$C_\kappa^\zeta(X_i) \leq \zeta_\kappa(X_i)$$

In this chapter, the allocation method is based on a result due to Euler, about (positive) homogeneous function

2 Euler's result about homogeneous function

2.1 Homogeneous function

Definition 3 *Let $\varphi(x_1, ..., x_n)$ be a function defined on $\mathbb{R}^n$ taking values in $\mathbb{R}$. The function φ is said to be homogeneous with degree m if*

$$\varphi(\lambda x_1, ..., \lambda x_n) = \lambda^m \varphi(x_1, ..., x_n),$$

for all $\lambda > 0$.

Example 4 *Let*

$$\varphi(x_1, ..., x_n) = a_1 x_1 + ... + a_n x_n$$

wiht $a_i \in \mathbb{R}$. Then, we have

$$\begin{aligned} \varphi\left(\lambda x_1, ..., \lambda x_n\right) &= a_1 \lambda x_1 + ... + a_n \lambda x_n \\ &= \lambda \varphi\left(x_1, ..., x_n\right), \end{aligned}$$

i.e. φ is homogeneous with degree 1. ■

Example 5 *Let*

$$\varphi\left(x_1, ..., x_n\right) = b \times x_1 \times ... \times x_n$$

with $b \in \mathbb{R}$. Then, we have

$$\begin{aligned} \varphi\left(\lambda x_1, ..., \lambda x_n\right) &= b \times \lambda x_1 \times ... \times a_n \lambda x_n \\ &= \lambda^n \varphi\left(x_1, ..., x_n\right), \end{aligned}$$

i.e. φ is homogeneous with degree n. ■

Example 6 *Let*

$$\varphi(x_1, ..., x_n) = a_1 x_1^m + ... + a_n x_n^m$$

with $a_i \in \mathbb{R}$. Then, we have

$$\begin{aligned} \varphi(\lambda x_1, ..., \lambda x_n) &= a_1 \lambda^m x_1^m + ... + a_n \lambda^m x_n^m \\ &= \lambda^m \varphi(x_1, ..., x_n), \end{aligned}$$

i.e. φ is homogeneous with degree m. ■

Example 7 *The functions*

$$\ln(x_1 + ... + x_n)$$

and

$$\exp(x_1 + ... + x_n)$$

are not homogeneous. ■

2.2 Euler's Theorem

Theorem 8 Euler's Theorem. *Let $\varphi(x_1, ..., x_n)$ be a function defined on $\mathbb{R}^n$ taking value in $\mathbb{R}$, which is also assumed to be derivable for all $(x_1, ..., x_n) \in \mathbb{R}^n$. If the function φ is (positively) homogeneous of degree m, then we have*

$$m\varphi(x_1, ..., x_n) = \sum_{i=1}^{n} x_i \frac{\partial \varphi}{\partial x_i}(x_1, ..., x_n) \tag{1}$$

for all $(x_1, ..., x_n) \in \mathbb{R}^n$.

Preuve. If $\varphi(x_1, ..., x_n)$ is homogeneous with degree m, it means that

$$\varphi(\lambda x_1, ..., \lambda x_n) = \lambda^m \varphi(x_1, ..., \lambda_n) \tag{2}$$

for all $\lambda > 0$.

We take the derivative on both sides of (2) and let $\lambda = 1$.

On the left-hand side of (2), we have

$$\begin{aligned}
\left.\frac{d\varphi\left(\lambda x_1, ..., \lambda x_n\right)}{d\lambda}\right|_{\lambda=1} &= \left.\sum_{i=1}^{n} \frac{\partial\varphi\left(\lambda x_1, ..., \lambda x_n\right)}{\partial\left(\lambda x_i\right)} \times \frac{\partial\left(\lambda x_i\right)}{\partial\lambda}\right|_{\lambda=1} \\
&= \left.\sum_{i=1}^{n} \frac{\partial\varphi\left(\lambda x_1, ..., \lambda x_n\right)}{\partial\left(\lambda x_i\right)} \times x_i\right|_{\lambda=1} \\
&= \sum_{i=1}^{n} \frac{\partial\varphi\left(x_1, ..., x_n\right)}{\partial x_i} \times x_i.
\end{aligned}$$

On the right-hand side of (2), we have

$$\begin{aligned}\left.\frac{d\left(\lambda^{m}\varphi\left(x_{1},...,x_{n}\right)\right)}{d\lambda}\right|_{\lambda=1} &= \left.m\lambda^{m-1}\varphi\left(x_{1},...,x_{n}\right)\right|_{\lambda=1} \\ &= m\varphi\left(x_{1},...,x_{n}\right).\end{aligned}$$

■

Remark 9 *For $m = 1$, (2) becomes*

$$\varphi\left(x_{1},...,x_{n}\right)=\sum_{i=1}^{n}x_{i}\frac{\partial\varphi\left(x_{1},...,x_{n}\right)}{\partial x_{i}}=\sum_{i=1}^{n}C_{i}\left(x_{1},...,x_{n}\right). \tag{3}$$

Then, φ correspond to the sum of the contributions of each variable x_i, given by

$$C_{i}\left(x_{1},...,x_{n}\right)=x_{i}\frac{\partial\varphi\left(x_{1},...,x_{n}\right)}{\partial x_{i}} \tag{4}$$

for $i = 1, 2, ..., n$.

Remark 10 *From (4), we have*

$$\begin{aligned} C_i(x_1, ..., x_n) &= x_i \frac{\partial \varphi(x_1, ..., x_n)}{\partial x_i} \\ &= \lim_{h \to 0} x_i \frac{\varphi(x_1, ..., x_i + h, ..., x_n) - \varphi(x_1, ..., x_i, ..., x_n)}{h} \end{aligned} \tag{5}$$

For $h = \varepsilon x_i$, (5) becomes

$$\begin{aligned} C_i(x_1, ..., x_n) &= \lim_{\varepsilon \to 0} x_i \frac{\varphi(x_1, ..., (1+\varepsilon) x_i, ..., x_n) - \varphi(x_1, ..., x_i, ..., x_n)}{(1+\varepsilon) x_i - x_i} \\ &= \lim_{\varepsilon \to 0} \frac{\varphi(x_1, ..., (1+\varepsilon) x_i, ..., x_n) - \varphi(x_1, ..., x_i, ..., x_n)}{\varepsilon} \\ &= \left. \frac{\partial \varphi(x_1, ..., \lambda_i x_i, ..., x_n)}{\partial \lambda_i} \right|_{\lambda_i = 1}, \end{aligned} \tag{6}$$

for $i = 1, 2, ..., n$.

From Remark 10, we find the following result.

Corollary 11 *Let φ be a homogeneous function of degree 1 which is also derivable. Then, (3) becomes*

$$\begin{aligned}\varphi(x_1,...,x_n) &= \sum_{i=1}^{n} x_i \frac{\partial\varphi(x_1,...,x_n)}{\partial x_i} \\ &= \sum_{i=1}^{n} \left.\frac{\partial\varphi(\lambda_1 x_1,...,\lambda_n x_n)}{\partial\lambda_i}\right|_{\lambda_1=...=\lambda_n=1} \\ &= \sum_{i=1}^{n} C_i(x_1,...,x_n),\end{aligned}$$

i.e. the contribution of x_i is

$$C_i(x_1,...,x_n) = \left.\frac{\partial\varphi(\lambda_1 x_1,...,\lambda_n x_n)}{\partial\lambda_i}\right|_{\lambda_1=...=\lambda_n=1}.$$

Remark 12 Approximation of the contribution. *If no-closed form of $C_i(x_1,...,x_n)$*

in (6), then the following approximation can be used

$$C_i(x_1, ..., x_n) \simeq \frac{\varphi(x_1, ..., (1+\varepsilon)x_i, ..., x_n) - \varphi(x_1, ..., x_i, ..., x_n)}{\varepsilon}$$

with a small ε *(e.g.* 10^{-3} *or* 10^{-4}*).*

3 Euler's method of allocation

Let $\underline{X} = (X_1, ..., X_n)$.

We define

$$S = X_1 + ... + X_n\,.$$

Let ζ_κ be a risk measure with level of confidence κ, for $\kappa \in]0,1[$.

The amount of capital is determined by

$$\zeta_\kappa(S) = \zeta_\kappa(X_1 + ... + X_n).$$

The following results follows from Corollary 11.

Proposition 13 (Euler's method of allocation). *Let ζ_κ be a positive homogeneous function of degree 1 i.e. it means let ζ_κ be a positive homogenous function of degree 1. Under Euler's method, the contribution of the (risk) rv X_i to the global risk $S = X_1 + ... + X_n$ of the portfolio is given by*

$$C_\kappa^\zeta(X_i) = \left.\frac{\partial \zeta_\kappa(\lambda_1 X_1, ..., \lambda_n X_n)}{\partial \lambda_i}\right|_{\lambda_1 = ... = \lambda_n = 1} \tag{7}$$

such that

$$\zeta_\kappa(S) = \zeta_\kappa(X_1 + ... + X_n) = \sum_{i=1}^{n} C_i(X_1, ..., X_n).$$

Example 14 *The risk measures*

$$TVaR_\kappa(X_1 + ... + X_n)$$

and

$$VaR_\kappa(X_1 + ... + X_n)$$

are positively homogeneous of degree 1.

Indeed, we have

$$VaR_\kappa(\lambda X_1 + ... + \lambda X_n) = \lambda VaR_\kappa(X_1 + ... + X_n)$$

and

$$TVaR_\kappa(\lambda X_1 + ... + \lambda X_n) = \lambda TVaR_\kappa(X_1 + ... + X_n)$$

for all $\lambda > 0$. ■

Example 15 *The risk measure*

$$\sqrt{Var(X_1 + ... + X_n)}$$

postively homogeneous of degree 1.

Indeed, we have

$$\begin{aligned}\sqrt{Var(\lambda X_1 + ... + \lambda X_n)} &= \sqrt{\lambda^2 Var(X_1 + ... + X_n)} \\ &= \lambda\sqrt{Var(X_1 + ... + X_n)}.\end{aligned}$$

Called "volatility", $\sqrt{Var(...)}$ *is frequently used in finance.* ■

Example 16 *The variance*

$$Var\left(X_1 + ... + X_n\right)$$

is positive homogeneous of degree 2, since

$$\begin{aligned} Var\left(X_1 + ... + X_n\right) &= Var\left(\lambda X_1 + ... + \lambda X_n\right) \\ &= \lambda^2 Var\left(X_1 + ... + X_n\right). \end{aligned}$$

It does not ■

3.1 Euler's method of allocation and standard deviation

Let $\zeta\left(S\right) = \sqrt{Var\left(S\right)}$.

Then, we have

$$
\begin{aligned}
Var(S) &= Var(X_1 + ... + X_n) \\
&= \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} Cov\left(X_i, X_j\right).
\end{aligned}
$$

Objective: find $C^{\zeta}(X_i)$ using (7) in Euler's method of allocation.

We have

$$
\begin{aligned}
C^{\sqrt{Var}}(X_i) &= \left.\frac{\partial \varphi(\lambda_1 X_1, ..., \lambda_n X_n)}{\partial \lambda_i}\right|_{\lambda_1=...=\lambda_n=1} \\
&= \left.\frac{\partial \sqrt{Var(\lambda_1 X_1 + ... + \lambda_n X_n)}}{\partial \lambda_i}\right|_{\lambda_1=...=\lambda_n=1} \\
&= \left.\frac{\partial \sqrt{\sum_{i=1}^n Var(\lambda_i X_i) + \sum_{i=1}^n \sum_{j=1, j\neq i}^n Cov\left(\lambda_i X_i, \lambda_j X_j\right)}}{\partial \lambda_i}\right|_{\lambda_1=...=} \\
&= \left.\frac{\partial \sqrt{\sum_{i=1}^n \lambda_i^2 Var(X_i) + \sum_{i=1}^n \sum_{j=1, j\neq i}^n \lambda_i \lambda_j Cov\left(X_i, X_j\right)}}{\partial \lambda_i}\right|_{\lambda_1=...=}
\end{aligned}
$$

It becomes

$$
\begin{aligned}
C^{\sqrt{Var}}(X_i) &= \left.\frac{1}{2}\frac{2\lambda_i Var(X_i) + 2\sum_{j=1,j\neq i}^{n}\lambda_j Cov\left(X_i, X_j\right)}{\sqrt{\sum_{i=1}^{n} Var(\lambda_i X_i) + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} Cov\left(\lambda_i X_i, \lambda_j X_j\right)}}\right|_{\lambda_1=\ldots=} \\
&= \frac{Var(X_i) + \sum_{j=1,j\neq i}^{n} Cov\left(X_i, X_j\right)}{\sqrt{Var(S)}} \\
&= \frac{Cov(X_i, S)}{\sqrt{Var(S)}}.
\end{aligned}
$$

Then,

$$
C^{\sqrt{Var}}(X_i) = \frac{Cov(X_i, S)}{\sqrt{Var(S)}}.
$$

As expected, we observe

$$\begin{aligned}\sum_{i=1}^{n} C^{\sqrt{Var}}(X_i) &= \sum_{i=1}^{n} \frac{Cov(X_i, S)}{\sqrt{Var(S)}} \\ &= \frac{\sum_{i=1}^{n} Cov(X_i, S)}{\sqrt{Var(S)}} \\ &= \frac{Var(S)}{\sqrt{Var(S)}} \\ &= \sqrt{Var(S)}.\end{aligned}$$

Corollary 17 Euler's method of allocation and standard deviation. *Let* $\zeta(S) = \sqrt{Var(S)}$, *with* $S = X_1 + ... + X_n$. *Then, we have*

$$C^{\sqrt{Var}}(X_i) = \frac{Cov(X_i, S)}{\sqrt{Var(S)}}.$$

3.2 Euler's method of allocation and VaR

We need the following lemma.

Lemma 18 *Let Y be a continuous rv. Then, we have*

$$b = \frac{E\left[Y \times 1_{\{Y=b\}}\right]}{f_Y(b)} = E\left[Y|Y=b\right].$$

Let $\zeta(S) = VaR_\kappa(S)$, where

$$VaR_\kappa(S) = VaR_\kappa(X_1 + ... + X_n).$$

To simplify the presentation, let $X_1..., X_n$ be continuous rvs.

From lemma 18, we observe that

$$VaR_\kappa(X_1 + ... + X_n) = E[X_1 + ... + X_n | X_1 + ... + X_n = VaR_\kappa(X_1 + ... + X$$

Objective: find $C^\zeta(X_i)$ using (7) in Euler's method of allocation.

With (7), we have

$$\begin{aligned} C_\kappa^{VaR}(X_i) &= \left.\frac{\partial E[\lambda_1 X_1 + ... + \lambda_n X_n | S = VaR_\kappa(S)]}{\partial \lambda_i}\right|_{\lambda_1 = ... = \lambda_n = 1} \\ &= E[\lambda_i X_i | S = VaR_\kappa(S)]|_{\lambda_1 = ... = \lambda_n = 1} \\ &= E[X_i | S = VaR_\kappa(S)] \\ &= \frac{E\left[X_i \times 1_{\{S = VaR_\kappa(S)\}}\right]}{f_S(VaR_\kappa(S))}. \end{aligned}$$

Then, we find

$$C_{\kappa}^{VaR}(X_i) = \frac{E\left[X_i \times \mathbf{1}_{\{S=VaR_{\kappa}(S)\}}\right]}{f_S(VaR_{\kappa}(S))}.$$

As expected, we observe

$$\begin{aligned}
\sum_{i=1}^{n} C_{\kappa}^{VaR}(X_i) &= \sum_{i=1}^{n} \frac{E\left[X_i \times \mathbf{1}_{\{S=VaR_{\kappa}(S)\}}\right]}{f_S(VaR_{\kappa}(S))} \\
&= \frac{E\left[\sum_{i=1}^{n} X_i \times \mathbf{1}_{\{S=VaR_{\kappa}(S)\}}\right]}{f_S(VaR_{\kappa}(S))} \\
&= \frac{E\left[S \times \mathbf{1}_{\{S=VaR_{\kappa}(S)\}}\right]}{f_S(VaR_{\kappa}(S))} \\
&= VaR_{\kappa}(S).
\end{aligned}$$

Corollary 19 Euler's method of allocation and VaR. *Let* $\zeta(S) = VaR_\kappa(S)$, *where* $S = X_1 + ... + X_n$. *Then, we have*

$$C_\kappa^{VaR}(X_i) = \frac{E\left[X_i \times 1_{\{S=VaR_\kappa(S)\}}\right]}{f_S(VaR_\kappa(S))}. \tag{8}$$

3.3 Euler's method of allocation and TVaR

Let $\zeta(S) = TVaR_\kappa(S)$, where

$$TVaR_\kappa(S) = TVaR_\kappa(X_1 + ... + X_n).$$

To simplify the presentation, let $X_1..., X_n$ be continuous rvs.

We know that

$$\begin{aligned} TVaR_\kappa(X_1+...+X_n) &= E\left[(X_1+...+X_n)\,|S>VaR_\kappa(S)\right] \\ &= \frac{E\left[(X_1+...+X_n)\times 1_{\{S>VaR_\kappa(S)\}}\right]}{1-\kappa} \\ &= \frac{1}{1-\kappa}\int_{VaR_\kappa(X_1+...+X_n)}^{\infty} E\left[(X_1+...+X_n)\times 1_{\{S= \end{aligned}$$

From (8), we find

$$\begin{aligned} C_\kappa^{TVaR}(X_i) &= \frac{1}{1-\kappa}\int_{VaR_\kappa(X_1+...+X_n)}^{\infty} E\left[X_i\times 1_{\{S=y\}}\right]dy \\ &= \frac{1}{1-\kappa}E\left[X_i\times 1_{\{S>VaR_\kappa(X_1+...+X_n)\}}\right]dy. \end{aligned}$$

Corollary 20 Euler's method of allocation and TVaR. *Let* $\zeta(S)=TVaR_\kappa(S)$, *where* $S = X_1 + ... + X_n$. *Then, we have*

$$C_{\kappa}^{TVaR}(X_i) = \frac{E\left[X_i \times 1_{\{S > VaR_{\kappa}(S)\}}\right]}{1-\kappa}. \tag{9}$$

3.4 Euler's contribution and subadditivy

Let ζ_{κ} be a subadditive risk measure. Then, we have

$$C_{\kappa}^{\zeta}(X_i) \leq \zeta_{\kappa}(X_i),$$

for all $i = 1, 2, ..., n$, and $\kappa \in (0, 1)$.

Recall that $TVaR_{\kappa}(S)$ is subadditive. Then, we have

$$C_{\kappa}^{TVaR}(X_i) \leq TVaR_{\kappa}(X_i),$$

for all $i = 1, 2, ..., n$, and all $\kappa \in (0, 1)$.

Recall that $VaR_\kappa(S)$ is not subadditive. Then,

$$C_\kappa^{VaR}(X_i) \leq VaR_\kappa(X_i),$$

is not satisfied for all $i = 1, 2, ..., n$, and all $\kappa \in (0, 1)$.

3.5 Approximation of the contribution

If there is no closed form for $C_\kappa^\zeta(X_i)$, there are various approches to approximate $C_\kappa^{VaR}(X_i)$, $i = 1, ..., n$.

Here is one of them.

From Remark 12, we have the following approximation :

$$C_{\kappa}^{\zeta}(X_i) \simeq \frac{\zeta_{\kappa}(X_1, ..., (1+\varepsilon) X_i, ..., X_n) - \zeta_{\kappa}(X_1, ..., X_i, ..., X_n)}{\varepsilon}$$

for a small ε (e.g. 10^{-3} or 10^{-4}).

3.6 Approximation of the contribution based MC simulation

Let $\underline{X}^{(j)} = \left(X_1^{(j)}, ..., X_n^{(j)}\right)$ be the sampled value of $\underline{X}$ and $S^{(j)} = \sum_{i=1}^{m} X_i^{(j)}$ be the sampled value of S, $j = 1, 2, ..., m$.

To ease the presentation, we assume the following conditions.

Condition 21 *In this section, we assume that* $\kappa \times m \in \mathbb{N}$.

Condition 22 *In this section, we assume that the rvs* $X_1, ..., X_n$ *continuous continues.*

Let $\left\{S^{[j]}, j = 1, 2, ..., m\right\}$ be the set of ordered sampled values of S.

We fix j_0 such that $F_m^{-1}(\kappa) = S^{[j_0]}$ where F_n is the empirical cdf, built from $\left\{S^{[j]}, j = 1, 2, ..., m\right\}$.

It means that $j_0 = \kappa m$.

The approximation to $VaR_\kappa(S)$ is $F_m^{-1}(\kappa)$ i.e.

$$VaR_\kappa(S) \simeq \widetilde{VaR}_\kappa(S) = F_m^{-1}(\kappa) = S^{[j_0]}.$$

The approximation of $TVaR_\kappa(S)$ is given by

$$
\begin{aligned}
\widetilde{TVaR}_\kappa(S) &\simeq \frac{1}{1-\kappa}\left(\frac{1}{m}\sum_{j=1}^{m} S^{(j)} \times 1_{\left\{S^{(j)}>\widehat{VaR}_\kappa(S)\right\}}\right) \\
&= \frac{1}{m-j_0}\sum_{j=j_0+1}^{m} S^{[j]}.
\end{aligned}
$$

Then, the approximations to $C_\kappa^{VaR}(X_i)$ and $C_\kappa^{TVaR}(X_i)$ are respectively given by

$$
C_\kappa^{VaR}(X_i) \simeq \widetilde{C}_\kappa^{VaR}(X_i) = \sum_{j=1}^{m} X_i^{(j)} \times 1_{\left\{S^{(j)}=S^{[j_0]}\right\}}
$$

and

$$\begin{aligned}
C_{\kappa}^{TVaR}(X_i) &\simeq \widetilde{C}_{\kappa}^{TVaR}(X_i) = \frac{1}{(1-\kappa)\,m}\sum_{j=1}^{m} X_i^{(j)} \times 1_{\left\{S^{(j)}>S^{[j_0]}\right\}} \\
&= \frac{1}{m-j_0}\sum_{j=1}^{m} X_i^{(j)} \times 1_{\left\{S^{(j)}>S^{[j_0]}\right\}},
\end{aligned}$$

for $i = 1, 2, ..., n$.

Example 23 *Let* $\underline{X} = (X_1, X_2, X_3)$ *be a vector of 3 continuous rvs. In the* *following table, we provide* $\underline{X}^{(j)}$*, for* $j = 1, 2, ...10$ *:*

j	$X_1^{(j)}$	$X_2^{(j)}$	$X_3^{(j)}$	$S^{(j)}$	*rank*
1	442	636	4159	5237	5
2	1545	1620	2436	5601	6
3	3733	1933	7860	13526	10
4	1915	1637	2147	5699	7
5	1197	1448	1363	4008	3
6	2503	195	265	2963	1
7	918	1185	1131	3234	2
8	959	672	2718	4349	4
9	1991	1770	4137	7898	9
10	2667	2505	639	5811	8

In the following table, we provide $\widetilde{VaR}$ *and* $\widetilde{C}_{\kappa}^{VaR}(X_i)$, $i = 1, 2, 3$:

κ	$\widetilde{C}_{\kappa}^{VaR}(X_1)$	$\widetilde{C}_{\kappa}^{VaR}(X_2)$	$\widetilde{C}_{\kappa}^{VaR}(X_3)$	$\widetilde{VaR}_{\kappa}(S)$
0.7	1915	1637	2147	5699
0.8	2667	2505	639	5811
0.9	1991	1770	4137	7898

In the following table, we provide $\widetilde{TVaR}$ *and* $\widetilde{C}_{\kappa}^{TVaR}(X_i)$, $i = 1, 2, 3$:

κ	$\widetilde{C}_{\kappa}^{TVaR}(X_1)$	$\widetilde{C}_{\kappa}^{TVaR}(X_2)$	$\widetilde{C}_{\kappa}^{TVaR}(X_3)$	$\widetilde{TVaR}_{\kappa}(S)$
0.7	2797	2069.33	4212	9078.33
0.8	2862	1851.5	5998.5	10712
0.9	3733	1933	7860	13526

The values are computed with only 10 sampled values. This explains the strange values for $\widetilde{C}_{\kappa}^{TVaR}(X_2)$.

In the following table, we illustrate that $\widetilde{C}_{\kappa}^{TVaR}(X_i) \leq \widetilde{TVaR_{\kappa}}(X_i)$, *for all* $i = 1, 2, 3$, *and all* $\kappa \in (0, 1)$. *:*

κ	$\widetilde{TVaR_{\kappa}}(X_1)$	$\widetilde{TVaR_{\kappa}}(X_2)$	$\widetilde{TVaR_{\kappa}}(X_3)$
0.7	2967.67	2069.33	5385.33
0.8	3200	2219	6009.5
0.9	3733	2505	7860

Observe that $\widetilde{C}_{0.8}^{VaR}(X_2) = 2505 > \widetilde{VaR}_{0.9}(X_2) = 1770$.

4 Euler's method of allocation with VaR et TVaR

Let $\underline{X} = (X_1, ..., X_n)$.

We define

$$S = X_1 + ... + X_n\,.$$

Let ζ_κ be a risk measure with level of confidence κ, for $\kappa \in]0, 1[$.

The amount of capital is determined by

$$\zeta\left(S\right) = \zeta\left(X_1 + ... + X_n\right).$$

Let ζ_κ be a positive homogeneous function of degree 1 une mesure positivement homogène d'ordre 1.

The following results follows from Corollary 11.

Corollary 24 *The contribution of the (risk) rv X_i to the global risk $S = X_1 + ... + X_n$ of the portfolio is given by*

$$C^{\zeta}(X_i) = \left.\frac{\partial \zeta(\lambda_1 X_1, ..., \lambda_n X_n)}{\partial \lambda_i}\right|_{\lambda_1 = ... = \lambda_n = 1}$$

such that

$$\zeta(S) = \zeta(X_1 + ... + X_n) = \sum_{i=1}^{n} C_i(X_1, ..., X_n).$$

The above result is valid for continuous or discrete (or mixed) rvs X_1, ..., X_n, dependent or independent.

Let $\zeta_\kappa = VaR_\kappa$. Then, we have

$$C_\kappa^{VaR}(X_i) = VaR_\kappa(X_i; S) = E[X_i | S = VaR_\kappa(S)]. \tag{10}$$

Observe that

$$\begin{aligned}\sum_{i=1}^{n} C_i &= \sum_{i=1}^{n} E\left[X_i | S = VaR_\kappa(S)\right] = E\left[\sum_{i=1}^{n} X_i | S = VaR_\kappa(S)\right] \\ &= E\left[S | S = VaR_\kappa(S)\right] = VaR_\kappa(S)\end{aligned}$$

is satisfied.

Let $\zeta_\kappa = TVaR_\kappa$. Then, we have

$$C_\kappa^{TVaR}(X_i) = TVaR_\kappa(X_i; S) = \frac{E\left[X_i \times 1_{\{S>VaR_\kappa(S)\}}\right] + E\left[X_i \times 1_{\{S=VaR_\kappa(S\right.}}{1-\kappa} \tag{11}$$

with

$$\beta = \begin{cases} \frac{(\Pr(S \leq VaR_\kappa(S)) - \kappa)}{\Pr(S = VaR_\kappa(S))}, \text{ if } \Pr(S = VaR_\kappa(S)) > 0 \\ 0, \text{ if } \Pr(S = VaR_\kappa(S)) = 0 \end{cases}.$$

Observe that

$$
\begin{aligned}
\sum_{i=1}^{n} TVaR_{\kappa}\left(X_i; S\right) &= \sum_{i=1}^{n} \frac{E\left[X_i \times 1_{\{S>VaR_{\kappa}(S)\}}\right] + E\left[X_i \times 1_{\{S=VaR_{\kappa}(S)\}}\right]\beta}{1-\kappa} \\
&= \frac{E\left[S \times 1_{\{S>VaR_{\kappa}(S)\}}\right] + E\left[S \times 1_{\{S=VaR_{\kappa}(S)\}}\right]\beta}{1-\kappa} \\
&= \frac{E\left[S \times 1_{\{S>VaR_{\kappa}(S)\}}\right] + VaR_{\kappa}\left(S\right)\Pr\left(S = VaR_{\kappa}\left(S\right)\right)\beta}{1-\kappa} \\
&= \frac{E\left[S \times 1_{\{S>VaR_{\kappa}(S)\}}\right] + VaR_{\kappa}\left(S\right)\left(\Pr\left(S \leq VaR_{\kappa}\left(S\right)\right)\right.}{1-\kappa} \\
&= TVaR_{\kappa}\left(S\right),
\end{aligned}
$$

for $\kappa \in (0,1)$.

We examine the computation of $C_{\kappa}^{VaR}\left(X_i\right)$ and $C_{\kappa}^{TVaR}\left(X_i\right)$, in various contexts.

5 Multivariate distributions with arithmetic support

Let $X_1, ..., X_n, S \in \{0, 1h, 2h, ...\}$.

Assume that $VaR_\kappa(S) = k_0 h$.

Then, we have

$$E[X_i|S = VaR_\kappa(S)] = \frac{E\left[X_1 \times 1_{\{S=k_0h\}}\right]}{\Pr(S = k_0 h)},$$

with

$$E\left[X_i \times 1_{\{S=k_0h\}}\right] = \sum_{j=0}^{k_0} jh \Pr\left(X_i = jh, \sum_{l=1, l\neq i}^{n} X_l = (k_0 - j) h\right),$$

Also, we have

$$TVaR_{\kappa}(X_i;S) = \frac{E\left[X_i \times 1_{\{S>k_0h\}}\right] + E\left[X_i \times 1_{\{S=k_0h\}}\right]\beta}{1-\kappa},$$

where

$$\beta = \begin{cases} \frac{(\Pr(S \leq k_0h) - \kappa)}{\Pr(S=k_0h)}, \text{ si } \Pr(S=k_0h) > 0 \\ 0, \text{ si } \Pr(S=k_0h) = 0 \end{cases}.$$

We find

$$E\left[X_i \times 1_{\{S>k_0h\}}\right] = \sum_{k=k_0+1}^{\infty} E\left[X_i \times 1_{\{S=kh\}}\right]$$

or

$$
\begin{aligned}
E\left[X_i \times 1_{\{S>k_0 h\}}\right] &= E[X_i] - E\left[X_i \times 1_{\{S\leq k_0 h\}}\right] \\
&= E[X_i] - \sum_{k=0}^{k_0} E\left[X_i \times 1_{\{S=kh\}}\right].
\end{aligned}
$$

In summary, the difficulty is to calculate $E\left[X_i \times 1_{\{S=k_0 h\}}\right]$.

Let $S_{-i} = \sum_{l=1,l\neq i}^{n} X_l$. We find

$$
E\left[X_i \times 1_{\{S=k_0 h\}}\right] = \sum_{j=0}^{k_0} jh \Pr\left(X_i = jh, S_{-i} = (k_0 - j)\, h\right). \tag{12}
$$

5.1 Simple example with dependent rvs

Let (X_1, X_2) be a pair of discrete rvs. The values of the joint pmf f_{X_1,X_2} are provide in the following table:

$k_1 \backslash k_2$	0	1	2
0	0.30	0.10	0.05
1	0.04	0.20	0.06
2	0.12	0.05	0.08

We find :

k	$f_{X_1}(k)$	$f_{X_2}(k)$
0	0.45	0.46
1	0.30	0.35
2	0.25	0.19

We find :

k	$f_S(k)$	$F_S(k)$
0	0.30	0.30
1	0.14	0.44
2	0.37	0.81
3	0.11	0.92
4	0.08	1.00

The value of $VaR_{0.9}(S)$ is 3.

We find

$$E\left[S \times 1_{\{S>3\}}\right] = 4 \times 0.08 = 0.32$$

Then, we have

$$\begin{aligned} TVaR_{0.9}(S) &= \frac{1}{1-\kappa}\left(E\left[S \times 1_{\{S>3\}}\right] + 3 \times (F_S(3) - \kappa)\right) \\ &= \frac{1}{1-0.9}(0.32 + 3 \times (0.92 - 0.9)) \\ &= 3.8. \end{aligned}$$

The values of $E\left[X_1 \times 1_{\{S=3\}}\right]$ and $E\left[X_2 \times 1_{\{S=3\}}\right]$ are

$$\begin{aligned} E\left[X_1 \times 1_{\{S=3\}}\right] &= \sum_{k=0}^{3} k f_{X_1,X_2}(k, 3-k) \\ &= 0 \times 0 + 1 \times 0.06 + 2 \times 0.05 + 3 \times 0 \\ &= 0.16 \end{aligned}$$

and

$$
\begin{aligned}
E\left[X_2 \times 1_{\{S=3\}}\right] &= \sum_{k=0}^{3} k f_{X_1,X_2}(3-k,k) \\
&= 0 \times 0 + 1 \times 0.05 + 2 \times 0.06 + 3 \times 0 \\
&= 0.17.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
C_{0.9}^{VaR}(X_1) &= E\left[X_1 | S=3\right] \\
&= \frac{E\left[X_1 \times 1_{\{S=3\}}\right]}{\Pr(S=3)} \\
&= \frac{0.16}{0.11} \\
&= 1\frac{5}{11}
\end{aligned}
$$

and

$$\begin{aligned} C_{0.9}^{VaR}(X_2) &= E\left[X_2|S=3\right] \\ &= \frac{E\left[X2 \times 1_{\{S=3\}}\right]}{\Pr(S=3)} \\ &= \frac{0.17}{0.11} \\ &= 1\frac{6}{11}. \end{aligned}$$

We observe that

$$C_{0.9}^{VaR}(X_1) + C_{0.9}^{VaR}(X_2) = \frac{16}{11} + \frac{17}{11} = 3.$$

The values of $E\left[X_1 \times 1_{\{S=4\}}\right]$ and $E\left[X_2 \times 1_{\{S=4\}}\right]$ are

$$\begin{aligned} E\left[X_1 \times 1_{\{S=4\}}\right] &= \sum_{k=0}^{4} k f_{X_1,X_2}(k, 4-k) \\ &= 2 \times 0.08 \\ &= 0.16 \end{aligned}$$

and

$$\begin{aligned} E\left[X_2 \times 1_{\{S=4\}}\right] &= \sum_{k=0}^{4} k f_{X_1,X_2}(4-k, k) \\ &= 2 \times 0.16 \\ &= 0.16. \end{aligned}$$

Then, we obtain

$$
\begin{aligned}
C_{0.9}^{TVaR}(X_1) &= \frac{E\left[X_1 \times 1_{\{S>3\}}\right] + E\left[X_1 \times 1_{\{S=3\}}\right] \frac{(F_S(3)-0.9)}{\Pr(S=3)}}{1-0.9} \\
&= \frac{E\left[X_1 \times 1_{\{S=4\}}\right] + E\left[X_1 \times 1_{\{S=3\}}\right] \frac{(F_S(3)-0.9)}{\Pr(S=3)}}{1-0.9} \\
&= \frac{0.16 + 0.16\frac{(0.92-0.9)}{0.11}}{1-0.9} \\
&= 1.8\overline{90}
\end{aligned}
$$

and

$$
\begin{aligned}
C_{0.9}^{TVaR}(X_1) &= \frac{E\left[X_2 \times 1_{\{S>3\}}\right] + E\left[X_2 \times 1_{\{S=3\}}\right] \frac{(F_S(3)-0.9)}{\Pr(S=3)}}{1-0.9} \\
&= \frac{E\left[X_2 \times 1_{\{S=4\}}\right] + E\left[X_2 \times 1_{\{S=3\}}\right] \frac{(F_S(3)-0.9)}{\Pr(S=3)}}{1-0.9} \\
&= \frac{0.16 + 0.17\frac{(0.92-0.9)}{0.11}}{1-0.9} \\
&= 1.\overline{90}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
C_{0.9}^{TVaR}(X_1) + C_{0.9}^{TVaR}(X_2) &= 1.8\overline{90} + 1.\overline{90} \\
&= 1.8909091 + 1.9090909 \\
&= 3.8.
\end{aligned}
$$

5.2 Independent rvs

Let the rvs $X_1, ..., X_n$ be independent. Then, (12) becomes

$$E\left[X_i \times 1_{\{S=k_0 h\}}\right] = \sum_{j=0}^{k_0} jh f_{X_i}(jh) f_{S_{-i}}((k_0 - j) h). \tag{13}$$

Observe that the expression (13) looks like a convolution product.

Let X_i' be an arithmetic rv with

$$f_{X_i'}(jh) = \frac{jh f_{X_i}(jh)}{E[X_i]}$$

for $k = 0, 1, ..$.

Then, we have

$$
\begin{aligned}
E\left[X_i \times 1_{\{S=k_0 h\}}\right] &= \sum_{j=0}^{k_0} jhf_{X_i}(jh) f_{S_{-i}}((k_0-j)h) \\
&= E[X_i] \sum_{j=0}^{k_0} \frac{jhf_{X_i}(jh)}{E[X_i]} f_{S_{-i}}((k_0-j)h) \\
&= E[X_i] \sum_{j=0}^{k_0} f_{X_i'}(jh) f_{S_{-i}}((k_0-j)h).
\end{aligned}
$$

Observe that

$$
E\left[X_i \times 1_{\{S=k_0 h\}}\right] = E[X_i] \sum_{j=0}^{k_0} f_{X_i'}(jh) f_{S_{-i}}((k_0-j)h) \le E[X_i].
$$

Values for

$$\sum_{j=0}^{k_0} f_{X_i'}(jh)\, f_{S_{-i}}((k_0 - j)\, h)$$

can be easily computed with FFT or other tools for convolution product.

5.3 Portfolio of independent Poisson distributed rvs

Let $X_1, ..., X_n$ be independent rvs where $X_i \sim Pois(\lambda_i)$ $i = 1, 2, ..., n$.

We define $S = \sum_{i=1}^{n} X_i$.

It implies that $S \sim Pois(\lambda_S)$ and $S_{-i} \sim Pois(\lambda_S - \lambda_i)$.

Then, with $h = 1$, (13) becomes

$$E\left[X_i \times 1_{\{S=k_0\}}\right] = \sum_{j=0}^{k_0} j \frac{\mathrm{e}^{-\lambda_i}\lambda_i^j}{j!} \frac{\mathrm{e}^{-(\lambda_S-\lambda_i)}\left(\lambda_S - \lambda_i\right)^{k_0-j}}{(k_0-j)!}.$$

Then, we obtain

$$
\begin{aligned}
E\left[X_i \times 1_{\{S=k_0\}}\right] &= \sum_{j=0}^{k_0} j \frac{\mathrm{e}^{-\lambda_i}\lambda_i^j}{j!} \frac{\lambda_S^{k_0}}{\lambda_S^{k_0}} \frac{k_0!}{k_0!} \frac{\mathrm{e}^{-(\lambda_S-\lambda_i)}\left(\lambda_S-\lambda_i\right)^{k_0-j}}{(k_0-j)!} \\
&= \frac{\mathrm{e}^{-\lambda_S}\lambda_S^{k_0}}{k_0!} \sum_{j=0}^{k_0} j \frac{k_0!}{j!\,(k_0-j)!} \frac{1}{\lambda_S^j} \frac{1}{\lambda_S^{k_0-j}} \left(\lambda_i\right)^j \left(\lambda_S-\lambda_i\right)^{k_0-j} \\
&= \frac{\mathrm{e}^{-\lambda_S}\lambda_S^{k_0}}{k_0!} \sum_{j=0}^{k_0} j \underbrace{\frac{k_0!}{j!\,(k_0-j)!} \left(\frac{\lambda_i}{\lambda_S}\right)^j \left(1-\frac{\lambda_i}{\lambda_S}\right)^{k_0-j}}_{\text{pmf for binomial distribution}} \\
&= \frac{\mathrm{e}^{-\lambda_S}\lambda_S^{k_0}}{k_0!} k_0 \left(\frac{\lambda_i}{\lambda_S}\right) \\
&= \Pr\left(S=k_0\right) k_0 \left(\frac{\lambda_i}{\lambda_S}\right).
\end{aligned}
$$

6 Multivariate continuous distributions

Let $X_1, ..., X_n$ be continuous rvs.

It implies that $S = \sum_{i=1}^{n} X_i$ is also continuous.

We define $S_{-i} = \sum_{l=1, l \neq i}^{n} X_l$.

Let $VaR_\kappa(S) = s_0$.

Then, we have

$$C_\kappa^{VaR}(X_i) = VaR_\kappa(X_i; S) = \frac{E\left[X_i \times 1_{\{S=s_0\}}\right]}{f_S(s_0)},$$

where

$$E\left[X_i \times 1_{\{S=s\}}\right] = \int_0^s x f_{X_i,S_{-i}}(x, s-x)\,\mathrm{d}x. \tag{14}$$

Then, we have

$$C_\kappa^{TVaR}(X_i) = TVaR_\kappa(X_i; S) = \frac{E\left[X_i \times 1_{\{S>s_0\}}\right]}{1-\kappa}, \tag{15}$$

where

$$E\left[X_i \times 1_{\{S>s_0\}}\right] = \int_{s_0}^{\infty} E\left[X_i \times 1_{\{S=s\}}\right]\mathrm{d}s.$$

Also,

$$E\left[X_i \times 1_{\{S>s_0\}}\right] = E[X_i] - E\left[X_i \times 1_{\{S\leq s_0\}}\right],$$

with

$$E\left[X_i \times 1_{\{S>s_0\}}\right] = \int_0^{s_0} E\left[X_i \times 1_{\{S=s\}}\right] \mathrm{d}s. \tag{16}$$

6.1 Portfolio of independent exponentially distributed rvs

Let $X_i \sim Exp(\beta_i)$, $i = 1, 2$, with $\beta_1 > \beta_2$, be independent.

We defined $S = X_1 + X_2$

The cdf of S is

$$F_S(x) = H(x; \beta_1, \beta_2) = \begin{cases} 1 - \mathrm{e}^{-\beta x} \sum_{j=0}^{2-1} \frac{(\beta x)^j}{j!}, & \beta_1 = \beta_2 = \beta \quad (\mathrm{I} \\ \sum_{i=1}^{2} \left(\prod_{j=1, j\neq i}^{2} \frac{\beta_j}{\beta_j - \beta_i} \right) \left(1 - \mathrm{e}^{-\beta_i x} \right), & \beta_1 \neq \beta_2 \quad (\mathrm{I} \end{cases} \tag{17}$$

The truncated expectation of S is

$$E\left[S \times 1_{\{S>b\}} \right] = \zeta(b; \beta_1, \beta_2) = \begin{cases} \frac{2}{\beta} \overline{H}(b; \beta, \beta) = \frac{2}{\beta} \left(\mathrm{e}^{-\beta b} \sum_{j=0}^{2} \frac{(\beta b)^j}{j!} \right), & \beta_1 \\ \sum_{i=1}^{2} \left(\prod_{j=1, j\neq i}^{2} \frac{\beta_j}{\beta_j - \beta_i} \right) \left(b\mathrm{e}^{-\beta_i b} + \frac{\mathrm{e}^{-\beta_i b}}{\beta_i} \right), & \beta_1 \end{cases} \tag{18}$$

The expression for (14) becomes

$$
\begin{aligned}
E\left[X_1 \times 1_{\{S=s\}}\right] &= \int_0^s x f_{X_1,X_2}(x, s-x)\,\mathrm{d}x \\
&= \int_0^s x f_{X_1}(x)\, f_{X_2}(s-x)\,\mathrm{d}x \text{ (indépendance)} \\
&= \int_0^s x \beta_1 \mathrm{e}^{-\beta_1 x} \beta_2 \mathrm{e}^{-\beta_2 (s-x)} \mathrm{d}x
\end{aligned}
$$

We obtain

$$
E\left[X_1 \times 1_{\{S=s\}}\right] = \begin{cases} \frac{1}{\beta} h(x; 3, \beta), & \beta_1 = \beta_2 = \\ \beta_1 \beta_2 \left(\frac{e^{-\beta_2 s}}{(\beta_1 - \beta_2)^2} - \frac{e^{-\beta_1 s}}{(\beta_1 - \beta_2)^2} - \frac{s}{(\beta_1 - \beta_2)} e^{-\beta_1 s} \right), & \beta_1 \neq \beta_2 \end{cases} \tag{19}
$$

Then, we replace (19) in (16)

$$
\begin{aligned}
E\left[X_1 \times 1_{\{S>b\}}\right] &= \xi_1\left(b;\beta_1,\beta_2\right) \\
&= \int_0^{s_0} E\left[X_i \times 1_{\{S=s\}}\right] \mathrm{d}s \\
&= \begin{cases} \frac{1}{\beta}\overline{H}\left(b;3,\beta\right), & \beta_1=\beta_2=\beta \\ \frac{\beta_2 e^{-\beta_1 b}\left(b+\frac{1}{\beta_1}\right)}{(\beta_2-\beta_1)} - \left(\frac{\beta_2 e^{-\beta_1 b}}{(\beta_1-\beta_2)^2} - \frac{\beta_1 e^{-\beta_2 b}}{(\beta_1-\beta_2)^2}\right), & \beta_1 \neq \beta_2 \end{cases}
\end{aligned}
$$

6.2 Portfolio of independent gamma distributed rvs

Let $X_1, ..., X_n$ be independent rvs with $X_i \sim Ga\left(\alpha_i, \beta\right)$.

Then, $S = \sum_{i=1}^{n} X_i \sim Ga\left(\alpha_S, \beta\right)$ with $\alpha_S = \alpha_1 + ... + \alpha_n$.

The expression for $E\left[X_i \times 1_{\{S=s\}}\right]$ is

$$\begin{aligned}
E\left[X_i \times 1_{\{S=s\}}\right] &= \int_0^s x f_{X_i}(x) f_{S_{-i}}(s-x)\,\mathrm{d}x \\
&= \int_0^s x h(x;\alpha_i,\beta)\, h(s-x;\alpha_S-\alpha_i,\beta)\,\mathrm{d}x \\
&= \int_0^s x \frac{e^{-\beta x}}{\Gamma(\alpha_i)}\beta^{\alpha_i}\left(x^{\alpha_i-1}\right)\frac{e^{-\beta(s-x)}}{\Gamma(\alpha_S-\alpha_i)}\beta^{\alpha_S-\alpha_i}\left((s-x)^{\alpha_S-\alpha_i-1}\right)\mathrm{d}x \\
&= \int_0^s \frac{e^{-\beta x}}{\Gamma(\alpha_i)}\frac{\alpha_i}{\alpha_i}\frac{\beta}{\beta}\beta^{\alpha_i}\left(x^{\alpha_i+1-1}\right)\frac{e^{-\beta(s-x)}}{\Gamma(\alpha_S-\alpha_i)}\beta^{\alpha_S-\alpha_i}\left((s-x)^{\alpha_S-\alpha_i-1}\right)\mathrm{d}x \\
&= \frac{\alpha_i}{\beta}\int_0^s \frac{e^{-\beta x}}{\Gamma(\alpha_i+1)}\beta^{\alpha_i+1}\left(x^{\alpha_i+1-1}\right)\frac{e^{-\beta(s-x)}}{\Gamma(\alpha_S-\alpha_i)}\beta^{\alpha_S-\alpha_i}\left((s-x)^{\alpha_S-\alpha_i-1}\right)\mathrm{d}x \\
&= \frac{\alpha_i}{\beta} h(s;\alpha_S+1,\beta).
\end{aligned}$$

Then, we find

$$\begin{aligned}
E\left[X_i \times 1_{\{S>b\}}\right] &= \int_b^\infty E\left[X_i \times 1_{\{S=s\}}\right]\mathrm{d}s \\
&= \int_b^\infty \frac{\alpha_i}{\beta} h(s;\alpha_S+1,\beta)\,\mathrm{d}s
\end{aligned}$$

which becomes

$$E\left[X_i \times 1_{\{S>b\}}\right] = \frac{\alpha_i}{\beta}\overline{H}\left(b;\alpha_S+1,\beta\right). \tag{20}$$

Replace (20) in (15) and we obtain

$$TVaR_\kappa(X_i;S) = \frac{\alpha_i}{\beta}\frac{\overline{H}\left(VaR_\kappa\left(S\right);\alpha_S+1,\beta\right)}{1-\kappa}. \tag{21}$$

6.3 Portfolio of dependent rvs and multivariate normal distribution

Let $(X_1,..,X_n)$ follows a multivariate normal distribution with

$$E\left[X_i\right] = \mu_i \text{ and } Var\left(X_i\right) = \sigma_i^2$$

Pearson's linear correlation coefficient of $\left(X_i, X_j\right)$ is

$$\rho_P\left(X_i, X_j\right) = \rho_{i,j},$$

for $i, j \in \{1, 2, ..., n\}$.

Then, we have

$$Cov\left(X_i, X_j\right) = \rho_{i,j} \times \sigma_i \times \sigma_j,$$

for $i, j \in \{1, 2, ..., n\}$.

Define $S = \sum_{i=1}^{n} X_i$.

Here, we directly apply Proposition 13 – **Euler's method of allocation** to find contributions $C^{VaR}(X_i)$ and $C^{TVaR}(X_i)$

We have shown

$$S \sim Norm\left(\mu_S, \sigma_S^2\right)$$

with

$$\mu_S = \sum_{i=1}^{n} \mu_i$$

and

$$\sigma_S^2 = \sum_{i=1}^{n} \sigma_i^2 + \sum_{i=1}^{n} \sum_{i=1, j \neq i}^{n} \rho_{i,j} \sigma_i \sigma_j .$$

Then, we have

$$
\begin{aligned}
VaR_{\kappa}(S) &= \mu_S + \sigma_S \times \Phi^{-1}(\kappa) \\
&= \sum_{i=1}^{n} \mu_i + \sqrt{\sum_{i=1}^{n} \sigma_i^2 + \sum_{i=1}^{n} \sum_{i=1, j\neq i}^{n} \rho_{i,j}\sigma_i\sigma_j} \times \Phi^{-1}(\kappa) \\
&= \sum_{i=1}^{n} E[X_i] + \sqrt{\sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{i=1, j\neq i}^{n} Cov(X_i, X_j)} \times \Phi^{-1}(\kappa).
\end{aligned}
$$

and

$$
\begin{aligned}
TVaR_{\kappa}(S) &= \mu_S + \sigma_S \times \frac{1}{1-\kappa}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \\
&= \sum_{i=1}^{n} \mu_i + \sqrt{\sum_{i=1}^{n} \sigma_i^2 + \sum_{i=1}^{n} \sum_{i=1, j\neq i}^{n} \rho_{i,j}\sigma_i\sigma_j} \times \frac{1}{1-\kappa}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \\
&= \sum_{i=1}^{n} E[X_i] + \sqrt{\sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{i=1, j\neq i}^{n} Cov(X_i, X_j)} \times \frac{1}{1-\kappa}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2}.
\end{aligned}
$$

The contributions $C^{VaR}(X_i)$ and $C^{TVaR}(X_i)$ are obtained directly from 11.

We have

$$\begin{aligned} C^{\zeta}(X_i) &= VaR_{\kappa}(X_i; S) \\ &= \left.\frac{\partial \zeta(\lambda_1 X_1, ..., \lambda_n X_n)}{\partial \lambda_i}\right|_{\lambda_1=...=\lambda_n=1} \\ &= E[X_i] + \frac{Var(X_i) + \sum_{j=1, j\neq i}^{n} Cov\left(X_i, X_j\right)}{\sqrt{Var(S)}} \times \Phi^{-1}(\kappa). \end{aligned}$$

We have

$$\begin{aligned} C^{\zeta}(X_i) &= TVaR_{\kappa}(X_i; S) \\ &= \left.\frac{\partial \zeta(\lambda_1 X_1, ..., \lambda_n X_n)}{\partial \lambda_i}\right|_{\lambda_1=...=\lambda_n=1} \\ &= E[X_i] \\ &\quad + \frac{Var(X_i) + \sum_{j=1, j\neq i}^{n} Cov(X_i, X_j)}{\sqrt{Var(S)}} \times \frac{1}{1-\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2}. \end{aligned}$$

For each rv X_i and with theTVaR, the benefit of risk pooling is

$$\begin{aligned}
& TVaR_{\kappa}\left(X_i\right) - TVaR_{\kappa}\left(X_i; S\right) \\
= \; & E\left[X_i\right] + \sqrt{Var\left(X_i\right)} \frac{1}{1-\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \\
& - \left(E\left[X_i\right] + \frac{Var\left(X_i\right) + \sum_{j=1, j\neq i}^{n} Cov\left(X_i, X_j\right)}{\sqrt{Var\left(S\right)}} \times \frac{1}{1-\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \right) \\
= \; & \left(\sqrt{Var\left(X_i\right)} - \frac{Var\left(X_i\right) + \sum_{j=1, j\neq i}^{n} Cov\left(X_i, X_j\right)}{\sqrt{Var\left(S\right)}} \right) \times \frac{1}{1-\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \\
= \; & \sqrt{Var\left(X_i\right)} \left(1 - \frac{\sqrt{Var\left(X_i\right)} + \sum_{j=1, j\neq i}^{n} \sqrt{Var\left(X_j\right)} \rho_{i,j}}{\sqrt{Var\left(S\right)}} \right) \times \frac{1}{1-\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2}.
\end{aligned}$$

We can show

$$\sqrt{Var\left(X_i\right)} \left(1 - \frac{\sqrt{Var\left(X_i\right)} + \sum_{j=1, j\neq i}^{n} \sqrt{Var\left(X_j\right)} \rho_{i,j}}{\sqrt{Var\left(S\right)}} \right) \times \frac{1}{1-\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \geq 0.$$

Also, we have

$$\sqrt{Var(X_i)}\left(1-\frac{\sqrt{Var(X_i)}+\sum_{j=1,j\neq i}^{n}\sqrt{Var(X_j)}\rho_{i,j}}{\sqrt{Var(S)}}\right)\times\frac{1}{1-\kappa}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2}=0$$

when

$$\rho_{i,j}=1$$

for all pairs (i,j).

For each risk X_i and using the VaR, the "benefit" of risk pooling is

$$\begin{aligned}
& VaR_\kappa(X_i)-VaR_\kappa(X_i;S) \\
= \; & E[X_i]+\sqrt{Var(X_i)}\times\Phi^{-1}(\kappa) \\
& -E[X_i]+\frac{Var(X_i)+\sum_{j=1,j\neq i}^{n}Cov(X_i,X_j)}{\sqrt{Var(S)}}\times\Phi^{-1}(\kappa) \\
= \; & \left(\sqrt{Var(X_i)}-\frac{Var(X_i)+\sum_{j=1,j\neq i}^{n}Cov(X_i,X_j)}{\sqrt{Var(S)}}\right)\times\Phi^{-1}(\kappa) \\
= \; & \sqrt{Var(X_i)}\left(1-\frac{\sqrt{Var(X_i)}+\sum_{j=1,j\neq i}^{n}\sqrt{Var(X_j)}\rho_{i,j}}{\sqrt{Var(S)}}\right)\times\Phi^{-1}(\kappa).
\end{aligned}$$

For $\kappa \in]0, 0.5[$, we have

$$\sqrt{Var(X_i)}\left(1-\frac{\sqrt{Var(X_i)}+\sum_{j=1,j\neq i}^{n}\sqrt{Var(X_j)}\rho_{i,j}}{\sqrt{Var(S)}}\right)\times\Phi^{-1}(\kappa)\leq 0$$

For $\kappa \in]0.5, 1[$, we have

$$\sqrt{Var(X_i)}\left(1-\frac{\sqrt{Var(X_i)}+\sum_{j=1,j\neq i}^{n}\sqrt{Var(X_j)}\rho_{i,j}}{\sqrt{Var(S)}}\right)\times\Phi^{-1}(\kappa)\geq 0$$

For $\kappa = 0.5$, we have

$$\sqrt{Var(X_i)}\left(1-\frac{\sqrt{Var(X_i)}+\sum_{j=1,j\neq i}^{n}\sqrt{Var(X_j)}\rho_{i,j}}{\sqrt{Var(S)}}\right)\times\Phi^{-1}(\kappa)=0.$$

6.4 Numerical approximation methods

We have already presented an approximation method based on Monte Carlo simulation. We can also used the numerical methods based on discretization, presented in the previous chapters.

7 Challenges for research in actuarial science

- Capital allocations for a portfolio of dependent risks.
- Capilal allocations with different risk measures.
- Sensibility of the capital allocation to the choice of the joint distribution.
- Etc.