Lyon PhD Course Actuarial Science

Chapter 2 - Basic Risk Models

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Contents

1 Introduction

The main objective of this chaptre is to present the basic models used in non-life actuarial science.

We define a rv X.

There are many contexts of applications:

- Non-Life Insurance (Car Insurance, Home Insurance, etc.)
- Group Insurance
- Health Insurance
- Etc

2 Stochastic Model for X

2.1 General Model

Définition 1 *We define the rv X as a random sum*

$$X = \begin{cases} \sum_{k=1}^{M} B_k, & M > 0 \\ 0, & M = 0 \end{cases},$$

with

- *discrete rv M : counting (or frequency) rv which corresponds to the number of claims*
- *positive rv B_k: amount of k-claim (often called severity rv).*

The assumptions of the model are:

- $\left\{B_k, k \in \mathbb{N}^+\right\}$ forms a sequence of iid rvs :

$$B_k \sim B, \text{ for } k \in \mathbb{N}^+;$$

- rv M and $\left\{B_k, k \in \mathbb{N}^+\right\}$ are independent.

Interpretation :

- no claim: if $M = 0$, $X = 0$
- 1 claim: if $M = 1$, $X = B_1$
- 2 claims: if $M = 2$, $X = B_1 + B_2$
- 3 claims: if $M = 3$, $X = B_1 + B_2 + B_3$
- etc.

The pmf and the pgf are denoted by f_M and P_M.

The cdf of B and its mgf (if it exists) are denoted by F_B and M_B.

Interpertation of the assumptions :

- the claim amounts are independent ;
- the claim amounts have the same stochastic behavior ;
- the number of claims has no influence on the claim amounts;
- the claim amounts have no influence on the number of claims.

Remarks :

- rv M : counting or frequency rv $\longrightarrow$ counting or frequency (discrete) distributions

- rv B : severity rv $\longrightarrow$ severity (generally continuous) distribution
- rv X : the rv is said to follow a compound distribution.
- name of the risk model : frequency-severity risk model (also called "collective risk model")
- the choice of the distributions for M and B is very important.

2.2 Expectation of X

The expectation of X is obtained by conditionning on the rv M (the number of claims) and by using the total expectation formula

$$E[X] = E_M[E[X|M]]. \tag{1}$$

Observations:

- if $M = 0$, $E[X|M = 0] = 0$
- if $M = 1$, $E[X|M = 1] = E[B_1] = E[B]$
- if $M = 2$, $E[X|M = 2] = E[B_1 + B_2] = E[B_1] + E[B_2] = 2E[B]$
- if $M = 3$, $E[X|M = 3] = E[B_1 + B_2 + B_3] = E[B_1] + E[B_2] + E[B_3] = 3E[B]$

- etc.

Then, we have

$$E[X|M=k] = E[B_1 + ... + B_k] = k \times E[B] \ \ (k \in \mathbb{N}^+).$$

It follows that

$$E[X|M] = M \times E[B]. \tag{2}$$

We replace (2) in (1) and the expression for $E[X]$ is given by

$$E[X] = E[E[X|M]] = E[M \times E[B]] = E[M] E[B]. \tag{3}$$

The interpretation of (3) is the following :

Components		**Symbol**
Expected number of claims		$E[M]$
Expected amount for one claim	$\times$	$E[B]$
Expected cost for one risk	$=$	$E[X]$

In actuarial science and in insurance, $E[X]$ corresponds to the pure premium.

2.3 Variance of X

We condition on the rv M and we apply the formula (rule) of total variance

$$\mathrm{Var}(X) = E_M[\mathrm{Var}(X|M)] + \mathrm{Var}_M(E[X|M]). \tag{4}$$

Observations:

- if $M = 0$, $\mathrm{Var}(X|M=0) = 0$
- if $M = 1$, $\mathrm{Var}(X|M=1) = \mathrm{Var}(B_1) = Var(B)$
- if $M = 2$, $\mathrm{Var}(X|M=2) = \mathrm{Var}(B_1 + B_2) = 2Var(B)$
- if $M = 3$, $\mathrm{Var}(X|M=3) = \mathrm{Var}(B_1 + B_2 + B_3) = 3Var(B)$
- etc.

It means that

$$\mathsf{Var}\left(X|M=k\right)=\mathsf{Var}\left(\sum_{j=1}^{k}B_j\right)=\sum_{j=1}^{k}\mathsf{Var}\left(B_j\right)=k\times\mathsf{Var}\left(B\right)\ \ (k\in\mathbb{N}^{+}).$$

Then, we have

$$\mathsf{Var}\left(X|M\right)=M\times\mathsf{Var}\left(B\right). \tag{5}$$

We replace (5) and (2) in (4)

$$\begin{aligned}\mathsf{Var}\left(X\right) &= E\left[M\times\mathsf{Var}\left(B\right)\right]+\mathsf{Var}\left(ME\left[B\right]\right)\\ &= E\left[M\right]\mathsf{Var}\left(B\right)+\mathsf{Var}\left(M\right)\left(E\left[B\right]\right)^{2}.\end{aligned} \tag{6}$$

The interpretation of (4) or (6) is the following :

Sources of the variance of X ...		**Symbol**
explained by the claim amounts		$E\left[\mathsf{Var}\left(X\vert M\right)\right] = E\left[M\right]\mathsf{Var}\left(B\right)$
explained by the number of claims	$+$	$\mathsf{Var}\left(E\left[X\vert M\right]\right) = \mathsf{Var}\left(M\right)\left(E\left[B\right]\right)^2$
Variance of the costs for one risk	$=$	$Var\left(X\right)$

2.4 Cumulative distribution function of X

Let $x \geq 0$.

The cdf of X is obtained by conditioning on the rv M:

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) = \sum_{k=0}^{\infty} \Pr(M = k) \Pr(X \leq x | M = k) \\ &= f_M(0) + \sum_{k=1}^{\infty} f_M(k) \Pr(B_1 + ... + B_k \leq x) \\ &= f_M(0) + \sum_{k=1}^{\infty} f_M(k) F_{B_1+...+B_k}(x). \end{aligned} \tag{7}$$

Observations:

- if $M = 0$, $\Pr(X \leq x|M = 0) = 1_{\{x \geq 0\}}$
- if $M = 1$, $\Pr(X \leq x|M = 1) = \Pr(B_1 \leq x) = F_{B_1}(x)$
- if $M = 2$, $\Pr(X \leq x|M = 2) = \Pr(B_1 + B_2 \leq x) = F_{B_1+B_2}(x)$
- if $M = 3$, $\Pr(X \leq x|M = 3) = \Pr(B_1 + B_2 + B_3 \leq x) = F_{B_1+B_2+B_3}(x)$
- etc.

Remarks:

- If $\Pr(M = 0) > 0$, it means that there is a probability mass at 0.
- The probability $\Pr(M = 0)$ is large for individual non-life insurance contracts (ex : car insurance, home insurance).
- The expression of F_X in (7) is interesting if a closed form expression exists for

$$F_{B_1+...+B_n}.$$

- In this case, we need to use optimization in order to compute $VaR_\kappa(X)$.
- Later, we present methods which are useful in order to approximate F_X

2.5 Truncated expectation of X

Conditioning on the rv M, the expression of $E\left[X \times 1_{\{X \leq b\}}\right]$ becomes

$$\begin{aligned} E\left[X \times 1_{\{X \leq b\}}\right] &= \sum_{k=0}^{\infty} \Pr(M = k)\, E\left[X \times 1_{\{X \leq b\}} | M = k\right] \\ &= \sum_{k=1}^{\infty} f_M(k)\, E\left[(B_1 + ... + B_k) \times 1_{\{B_1 + ... + B_k \leq b\}}\right]. \end{aligned}$$

We obtain

$$E\left[X \times 1_{\{X > b\}}\right] = \sum_{k=1}^{\infty} f_M(k)\, E\left[(B_1 + ... + B_k) \times 1_{\{B_1 + ... + B_k > b\}}\right]. \tag{8}$$

Remarks:

- We can evaluate (8) when we have a closed form for

$$E\left[(B_1 + ... + B_k) \times 1_{\{B_1+...+B_k>b\}}\right].$$

- Later, methods will be proposed in order to approximate

$$E\left[X \times 1_{\{X>b\}}\right].$$

2.6 TVaR risk measure

The expression for the TVaR is obtained by replacing (7) and (8) in (**??**):

$$
\begin{aligned}
& TVaR_{\kappa}(X) \\
= \; & \frac{\sum_{k=1}^{\infty} f_M(k) E\left[(B_1 + ... + B_k) \times 1_{\{B_1+...+B_k > VaR_{\kappa}(X)\}}\right]}{1-\kappa} \\
& + \frac{VaR_{\kappa}(X)\left(F_X(VaR_{\kappa}(X)) - \kappa\right)}{1-\kappa},
\end{aligned} \tag{9}
$$

Remarks:

- We can evaluate (8) when we can evaluate explicitely

  $$E\left[(B_1 + ... + B_k) \times 1_{\{B_1+...+B_k > b\}}\right]$$

  and $F_X(VaR_{\kappa}(X))$.

- Later, methods will be proposed in order to approximate $F_X\left(VaR_\kappa\left(X\right)\right)$.

Let B be a continuous and strictly positive rv (i.e. with no probability mass at 0):

- The distribution of X has a probability mass at 0 and a continuous part for $x > 0$.
- If $\kappa < \Pr\left(M = 0\right)$ then $VaR_\kappa\left(X\right) = 0$.
- When $\kappa > \Pr\left(M = 0\right)$, it implies that $VaR_\kappa\left(X\right) > 0$ such that

$$F_X\left(VaR_\kappa\left(X\right)\right) = \kappa.$$

- In both cases, we have

$$VaR_\kappa\left(X\right)\left(F_X\left(VaR_\kappa\left(X\right)\right) - \kappa\right) = 0.$$

- Then, the expression for TVaR becomes

$$TVaR_{\kappa}(X) = \frac{\sum_{k=1}^{\infty} f_M(k)\, E\left[(B_1 + ... + B_k) \times 1_{\{B_1+...+B_k > VaR_{\kappa}(X)\}}\right]}{1-\kappa}. \tag{10}$$

Remarks:

- The expression in (10) is interesting the distribution of B belongs to a family of distributions closed on convolution (e.g. gamma).

2.7 Moment generating function

We assume that the mgf of the rv B and the pgf of the rv M exist.

It implies that the mgf also exists.

We condition on the rv M

$$E\left[\mathrm{e}^{tX}\right] = E_M\left[E\left[\mathrm{e}^{tX}\,|M\right]\right]. \tag{11}$$

Observations:

- if $M = 0$, $E\left[\mathrm{e}^{tX}\,|M = 0\right] = 1$
- if $M = 1$, $E\left[\mathrm{e}^{tX}\,|M = 1\right] = E\left[\mathrm{e}^{tB_1}\right] = E\left[\mathrm{e}^{tB}\right]$

- if $M = 2$, $E\left[\mathrm{e}^{tX} | M = 2\right] = E\left[\mathrm{e}^{t(B_1+B_2)}\right] = E\left[\mathrm{e}^{tB_1}\right] E\left[\mathrm{e}^{tB_2}\right] = E\left[\mathrm{e}^{tB}\right]^2$
- if $M = 3$, $E\left[\mathrm{e}^{tX} | M = 3\right] = E\left[\mathrm{e}^{t(B_1+B_2+B_3)}\right] = E\left[\mathrm{e}^{tB_1}\right] E\left[\mathrm{e}^{tB_2}\right] E\left[\mathrm{e}^{tB_3}\right] = E\left[\mathrm{e}^{tB}\right]^3$
- etc.

Then, for $M \in \mathbb{N}^+$, we have

$$E\left[\mathrm{e}^{tX} | M = k\right] = E\left[\mathrm{e}^{t(B_1+...+B_k)}\right] = E\left[\mathrm{e}^{tB_1}\right] \times ... \times E\left[\mathrm{e}^{tB_k}\right]$$

since the rvs B_1, ..., B_k are independent.

Since they are id, it follows that

$$E\left[\mathrm{e}^{tX} | M = k\right] = E\left[\mathrm{e}^{tB_1}\right] \times ... \times E\left[\mathrm{e}^{tB_k}\right] = E\left[\mathrm{e}^{tB}\right]^k.$$

Then, it means that

$$E\left[\mathrm{e}^{tX}\,|M\right] = M_B(t)^M. \tag{12}$$

We replace (12) in (11)

$$E\left[\mathrm{e}^{tX}\right] = E_M\left[M_B(t)^M\right] = P_M\left(M_B(t)\right), \tag{13}$$

where P_M is the pgf of the rv M.

2.8 Simulation of samples of a rv defined by a random sum

Let X be a rv defined by

$$X = \left[\begin{array}{l} \sum_{k=1}^{M} B_k, \; M > 0 \\ 0, \; M = 0 \end{array} \right. ,$$

with the usual assumptions and where

- M is discrete rv
- $B_1, B_2, ...$ forms a sequence of iid rvs $(B_k \sim B)$
- $\{B_1, B_2, ...\}$ is independent of the rv M.

Algorithme 2 Simulation of samples of a rv defined by a random sum

1. **Step 1**. *Simulate a sample $M^{(j)}$ of the rv M.*
2. **Step 2**. *If $M^{(j)} = 0$, then $X^{(j)} = 0$. Otherwise, go to Step 3.*
3. **Step 3**. *Simulate $M^{(j)}$ iid samples of the rv B and*

$$X^{(j)} = B_1^{(j)} + B_2^{(j)} + ... + B_{M^{(j)}}^{(j)}.$$

Repeat Steps 1-3 for $j = 1, 2, ..., m$.

The sampling method is illustrated in the following example.

Exemple 3 *Let*

$$X \sim BNComp\left(r = 2, q = 0.5; F_B\right)$$

with

$$M \sim NegBin\left(2, 0.5\right)$$

and with

$$B \sim LN\left(4, 3^2\right).$$

We have the following samples of $U \sim Unif\left(0,1\right)$:

j	1	2	3	4
$U^{(j)}$	0.0371	0.6234	0.3469	0.8976

We find $M^{(1)} = F_M^{-1}\left(U^{(1)}\right) = 0$, which implies $X^{(1)} = 0$.

Then, since $M^{(2)} = F_M^{-1}\left(U^{(2)}\right) = 2$*, it follows that*

$$\begin{aligned} X^{(2)} &= B_1^{(2)} + B_2^{(2)} = F_B^{-1}\left(U^{(3)}\right) + F_B^{-1}\left(U^{(4)}\right) \\ &= 16.758 + 2450.342 = 2467.100. \end{aligned}$$

□

3 Counting distributions

In actuarial science, the main discrete distributions for the counting rv M are Poisson, binomial and negative binomial.

Distribution for the rv M		Distribution for the rv X
Poisson	$\Rightarrow$	Compound Poisson
Binomial	$\Rightarrow$	Compound Binomial
Negative binomial	$\Rightarrow$	Compound Negative binomial

We use the following notation :

Distribution for the rv M		Distribution for the rv X
$M \sim Pois(\lambda)$	$\Rightarrow$	$X \sim CPois(\lambda; F_B)$
$M \sim Bin(r, q)$	$\Rightarrow$	$X \sim CBin(r, q; F_B)$
$M \sim NBin(r, q)$	$\Rightarrow$	$X \sim CNBin(r, q; F_B)$

The Poisson distribution is fundamental in actuarial science, especially in non-life insurance.

For that reason, we are also interested to extensions of the Poisson distributions, obtained notably by mixing and modification of the probability at 0.

3.1 Poisson Distribution

3.1.1 Key features

The Poisson distribution (or its extensions) is frequently used in actuarial science for modelling the number of claims, expecially in non-life insurance.

The Poisson distribution is equi-dispersed i.e. its variance is equal to its expectation

$$E[M] = \mathsf{Var}(M) = \lambda.$$

We summarize the key features of the distribution of the rv X in the following table:

Key feature		Symbol
expectation	$\Rightarrow$	$E[X] = \lambda E[B]$
variance	$\Rightarrow$	$\mathrm{Var}(X) = \lambda \mathrm{Var}(B) + \lambda E[B]^2 = \lambda E\left[B^2\right]$
mgf	$\Rightarrow$	$M_X(t) = \mathrm{e}^{\lambda(M_B(t)-1)}$

3.2 Binomial Distribution

3.2.1 Key features

We summarize the key features of the distribution of X in the following table :

Key feature		Symbol
expectation	$\Rightarrow$	$E[X] = nqE\,[B]$
variance	$\Rightarrow$	$\mathsf{Var}\,(X) = nq\mathsf{Var}\,(B) + nq\,(1-q)\,E\,[B]^2$
mgf	$\Rightarrow$	$M_X\,(t) = (1 - q + qM_B\,(t))^n$

The binomial distribution is under-dispersed i.e. its variance is lower than its expectation

$$E\,[M] = nq \geq nq\,(1-q) = \mathsf{Var}\,(M)\,.$$

3.2.2 Poisson vs Binomial Distribution

Let $M_{(n,q)} \sim Bin(n, q)$ and $M_{(\lambda)} \sim Pois(\lambda)$.

We fix the parameter q such that $nq = \lambda$ i.e. $q = \frac{\lambda}{n}$.

Then, $M_{\left(n,q=\frac{\lambda}{n}\right)}$ converges in distribution to $M_{(\lambda)}$ as $n \to \infty$.

Indeed, we have

$$\lim_{n\to\infty} P_{M_{\left(n,q=\frac{\lambda}{n}\right)}}(t) = \lim_{n\to\infty}\left(1 + \frac{\lambda(t-1)}{n}\right)^n = \mathrm{e}^{\lambda(t-1)} = P_{M_{(\lambda)}}(t),$$

which corresponds to the pgf of the Poisson distribution

3.3 Negative Binomial Distribution

3.3.1 Key features

We summarize the key features about the distribution of X in the following table :

Key feature		Symbol
expectation	$\Rightarrow$	$E[X] = r\frac{1-q}{q}E[B]$
variance	$\Rightarrow$	$\mathsf{Var}\,(X) = r\frac{1-q}{q^2}E[B]^2 + r\frac{1-q}{q}\mathsf{Var}\,(B)$
mgf	$\Rightarrow$	$M_X\,(t) = \left(1 - \frac{1-q}{q}(M_B(t) - 1)\right)^{-r}$

The negative binomial distribution is over-dispersed i.e. its variance is greater than its expectation

$$E[M] = r\frac{(1-q)}{q} \leq r\frac{(1-q)}{q^2} = \mathsf{Var}(M).$$

4 Claim amout distributions

The choice of the distribution for the claim amount is crucial.

Usually, positive continuous distributions are considered, especially in non-life actuarial science.

In almost practical applications (non-life insurance, health insurance, etc.), the claim amount distributions have a positive asymetry.

The principal distributions used for the claim amount are the following ones :

- Exponential distribution
- Gamma distribution

- Erlang distribution
- Lognormal distribution
- Weibull distribution
- Pareto distribution
- Burr distribution
- Mixtures of exponentials distribution
- Mixtures of Erlang distribution
- Etc.

4.1 Exponential distribution

4.1.1 Key features

The exponential distribution is fundamental in actuarial science, but also in many other areas (probability, statistic, queueing theory, etc.).

Key features	**Symbol**
Notation	$B \sim Exp(\beta)$
Parameter	$\beta > 0$
Support	$x \in \mathbb{R}^{+}$
Pdf	$f(x) = \beta \mathrm{e}^{-\beta x}$
Cdf	$F(x) = 1 - \mathrm{e}^{-\beta x}$
Expectation	$E[X] = \frac{1}{\beta}$
Variance	$\mathrm{Var}(X) = \frac{1}{\beta^2}$
Mgf	$M_X(t) = \left(\frac{\beta}{\beta - t}\right),\ t < \beta$
Moments of order k :	$E\left[X^k\right] = \frac{k!}{\beta^k}$
VaR	$VaR_\kappa(X) = -\frac{1}{\beta}\ln(1-\kappa)$
TVaR	$TVaR_\kappa(X) = VaR_\kappa(X) + E[X]$

4.1.2 Comments

The exponential distribution serves as a reference for modeling the claim amount distributions.

It has one parameter.

Its mode is at 0.

Its asymetrical coefficient is 2.

When $B \sim Exp(\beta)$, we may derive analytical expression for F_X, $TVaR_\kappa(X)$ and $\pi_X(d)$.

4.1.3 Cdf, mgf, and TVaR

The exponential distribution is special case of the gamma distribution.

This aspect is treated in the next subsection.

4.2 Gamma distribution

4.2.1 Key features

Key features	Symbol
Notation	$B \sim Ga(\alpha, \beta)$
Parameters	$\alpha, \beta > 0$
Support	$x \in \mathbb{R}^+$
Pdf	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x}$
Cdf	denoted $H(x; \alpha, \beta)$, no-closed for $\alpha \notin \mathbb{N}^+$
Sf	denoted $\overline{H}(x; \alpha, \beta)$, no-closed for $\alpha \notin \mathbb{N}^+$
Expectation	$E[B] = \frac{\alpha}{\beta}$
Variance	$\mathrm{Var}(B) = \frac{\alpha}{\beta^2}$
Mgf	$M_B(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$, $t < \beta$

Key features	Symbol
Truncated Expectation	$E\left[B \times 1_{\{X \leq d\}}\right] = \frac{\alpha}{\beta} H\left(d; \alpha+1, \beta\right)$
VaR	no closed form, if $\alpha \neq 1$
TVaR	$TVaR_{\kappa}\left(B\right) = \frac{1}{1-\kappa}\frac{\alpha}{\beta}\overline{H}\left(VaR_{\kappa}\left(B\right); \alpha+1, \beta\right.$
Special case: Exponential	$\alpha = 1$
Special case: Erlang	$\alpha \in \mathbb{N}^{+}$
Special case: Khi-square($\nu \in \mathbb{N}^{+}$)	$\alpha = \frac{\nu}{2}$, $\beta = 2$

4.2.2 Comments

Having two parameters, tthe gamma distribution is a generalisation of the exponential distribution.

With two parameters, α et β, it offers more flexibility.

- If $\alpha > 1$, its mode is greater than 0
- If $0 < \alpha < 1$, its mode is at 0.
- It $\alpha = 1$, it corresponds to the exponential distribution.

4.2.3 Illustration

On Figure 1, we depict the pdf of the gamma distribution for $\alpha = 0.5$, 1, 5, and 100 and $\beta = \frac{\alpha}{5}$ such that $E[B] = 5$.

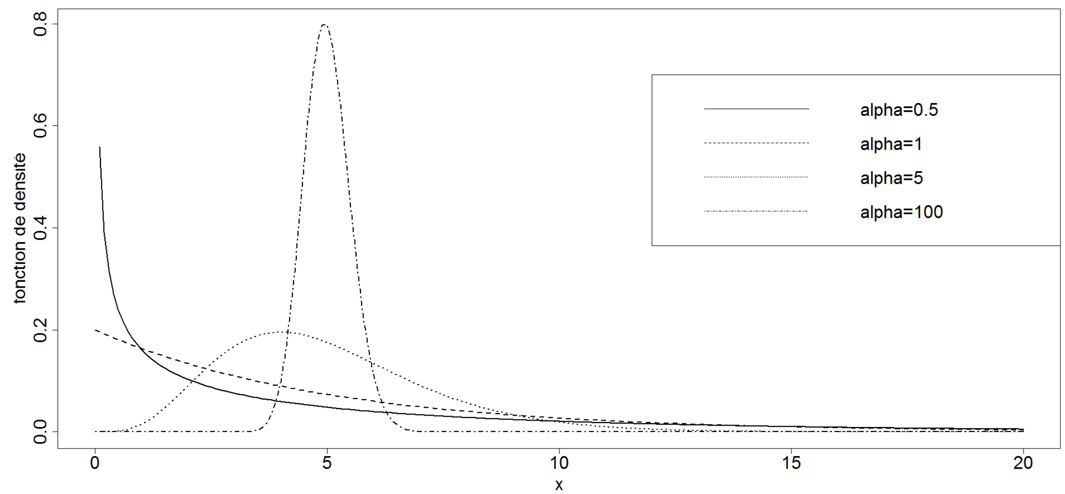

Figure 1: Values of $f_B(x)$ where $B \sim Ga\left(\alpha, \frac{\alpha}{5}\right)$, for $\alpha = 0.5$, 1, 5 and 100.

4.2.4 Cdf, mgf, and TVaR

Proposition 4 *Let $B \sim Ga(\alpha, \beta)$. Then, we find*

$$F_X(x) = f_M(0) + \sum_{k=1}^{\infty} f_M(k) H(x; \alpha k, \beta), \tag{14}$$

$$E\left[X \times 1_{\{X>b\}}\right] = \sum_{k=1}^{\infty} f_M(k) \frac{k\alpha}{\beta} \overline{H}(b; \alpha k + 1, \beta) \tag{15}$$

and

$$TVaR_\kappa(X) = \frac{1}{1-\kappa} \sum_{k=1}^{\infty} f_M(k) \frac{k\alpha}{\beta} \overline{H}(VaR_\kappa(X); \alpha k + 1, \beta). \tag{16}$$

Preuve. When $B \sim Ga(\alpha, \beta)$ with $F_B(x) = H(x; \alpha, \beta)$, we know that $B_1 + ... + B_k \sim Ga(\alpha k, \beta)$.

Then, we replace $F_{B_1+...+B_k}(x)$ by $H(x;\alpha k,\beta)$ in (7) and we obtain (14).

Also, replacing

$$E\left[(B_1+...+B_k)\times 1_{\{B_1+...+B_k>b\}}\right]$$

by

$$\frac{k\alpha}{\beta}\overline{H}(b;\alpha k+1,\beta)$$

in (8) and (10), we obtain (15) et (16). ■

Remarque 5 *The value of $VaR_\kappa(X)$ is compted with (14) using an optimization tool (e.g.* `solver` *in* `Excel`® *;* `optimize` *or* `uniroot` *in* `R`*).*

Exemple 6 *Let* $X \sim PComp(\lambda, F_B)$ *where* $\lambda = 1.4$ *and* $B \sim Ga\Big(\alpha = 1.8, \beta = 100$ *with* $E[X] = 2520$*. We obtain the following values for* $VaR_\kappa(X)$ *and* $TVaR_\kappa(X)$ *:*

κ	$VaR_\kappa(X)$	$TVaR_\kappa(X)$
0	0	2520
0.5	1834.662	4521.468
0.95	7767.176	9872.831
0.99	11 175.341	13 127.725
0.995	12 558.726	14 464.324

$\square$

4.3 Lognormal distribution

4.3.1 Key features

- Notation : $B \sim LN(\mu, \sigma^2)$
- Parameters : $-\infty < \mu < \infty, \sigma^2 > 0$
- Support : $x \in \mathbb{R}^+$
- Pdf : $f(x) = \frac{1}{x\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$
- Cdf : $F(x) = \Phi(\frac{\ln(x)-\mu}{\sigma})$
- Expectation : $E[B] = \mathrm{e}^{\mu + \frac{\sigma^2}{2}}$
- Variance : $\mathrm{Var}(B) = \mathrm{e}^{2\mu+\sigma^2}\left(\mathrm{e}^{\sigma^2} - 1\right)$
- Mgf : not analytic
- Moments of order k : $E[B^k] = \mathrm{e}^{k\mu + k^2\frac{\sigma^2}{2}}$

- Truncated Expectation : $E\left[B \times 1_{\{X \leq d\}}\right] = \exp(\mu+\sigma^2/2)\Phi(\frac{\ln d-\mu-\sigma^2}{\sigma})$
- Risk measure VaR : $VaR_\kappa(B) = \exp(\mu + \sigma VaR_\kappa(Z))$
- Risk measure $TVaR$:

$$TVaR_\kappa(B) = \frac{1}{1-\kappa}e^{\mu+\sigma^2/2}(1 - \Phi(VaR_\kappa(B) - \sigma))$$

4.3.2 Comments

The lognormal distribution is frequently used in actuarial science.

Notably, is is applied in the context of catastrophe, car insurance, health insurance, etc.

Its mode is greater than supérieur à 0.

It has a positive asymetric coefficient.

Its mean excess function is increasing.

4.3.3 Illustration

In Figure 2, we show graphs of pdf and VaR of the lognormal distribution for three pairs of parameters (μ, σ).

For the three cases. the parametes are fixed such the three expectations are identical.

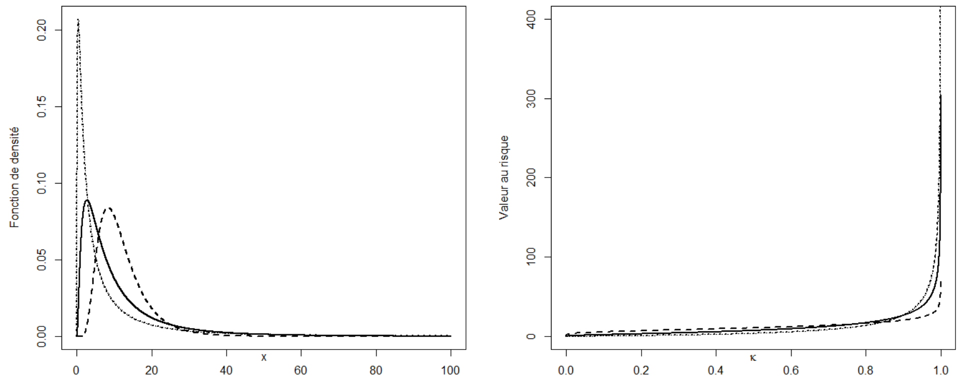

Figure 2: Valeurs de la la fonction de densité et de la VaR de $X \sim LN(\mu, \sigma)$ pour les couples suivants de (μ, σ) : $\left(2.375, \frac{1}{2}\right)$ (ligne avec petits traits), $(2, 1)$ (ligne continue) et $(0.875, 1.5)$ (ligne pointillée). Pour les 3 couples, l'espérance de X est 12.1825.

4.3.4 Cdf, mgf, and TVaR

If $B \sim LNorm(\mu, \sigma)$, then it is impossible to obtain analytical expressions for F_X, $E\left[X \times 1_{\{X>b\}}\right]$ and $TVaR_\kappa(X)$.

However, methods will be proposed in order to approximate those quantitities (chapters **??** and **??**).

4.4 Pareto and Burr distributions

4.4.1 Key features

Pareto distribution

- Notation : $B \sim Pa(\alpha, \lambda)$
- Parameters : $\alpha > 0$, $\lambda > 0$
- Support : $x \in \mathbb{R}^+$
- Pdf : $f(x) = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}$
- Cdf : $F(x) = 1 - \left(\frac{\lambda}{\lambda+x}\right)^\alpha$
- Survival function : $\overline{F}(x) = \left(\frac{\lambda}{\lambda+x}\right)^\alpha$
- Expectation (for $\alpha > 1$) : $E[B] = \frac{\lambda}{\alpha-1}$

- Variance (for $\alpha > 2$) : $\mathrm{Var}(B) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}$
- Mgf : does not exist
- Moments of order k (for $\alpha > k \in \mathbb{N}^+$): $E\left[B^k\right] = \frac{\lambda^k k!}{\prod_{i=1}^{k}(\alpha-i)}$
- Moments of order k : $E\left[B^k\right] = \frac{\lambda^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}$, si $-1 < k < \alpha$
- Truncated Expectation (for $\alpha > 1$) :

$$E\left[B \times 1_{\{X \leq d\}}\right] = \frac{\lambda}{\alpha - 1}\left(1 - \frac{\lambda^{\alpha-1}}{(\lambda + d)^{\alpha-1}}\right) - d\left(\frac{\lambda}{\lambda + d}\right)^{\alpha}$$

- Risk measure VaR : $VaR_\kappa(B) = \lambda\left((1-\kappa)^{-\frac{1}{\alpha}} - 1\right)$
- Risk measure $TVaR$ (for $\alpha > 1$) : $TVaR_\kappa(B) = \lambda\left(\frac{\alpha}{\alpha-1}(1-\kappa)^{-\frac{1}{\alpha}} - 1\right)$

Burr distribution

- Notation : $B \sim Burr\left(\alpha, \lambda, \tau\right)$
- Parameters : $\alpha > 0$, $\lambda > 0$, $\tau > 0$
- Support : $x \in \mathbb{R}^{+}$
- Pdf : $f_X\left(x\right) = \frac{\alpha\tau\lambda^{\alpha}x^{\tau-1}}{(\lambda+x^{\tau})^{\alpha+1}}$
- Cdf : $F\left(x\right) = 1 - \left(\frac{\lambda}{\lambda+x^{\tau}}\right)^{\alpha}$
- Expectation : $E\left[B\right] = \frac{1}{\Gamma(\alpha)}\lambda^{1/\tau}\Gamma(1+\frac{1}{\tau})\Gamma(\alpha-\frac{1}{\tau})$
- Variance : $\mathrm{Var}\left(B\right) = \frac{\lambda^{2/\tau}}{\Gamma(\alpha)}\left(\Gamma(1+\frac{2}{\tau})\Gamma(\alpha-\frac{2}{\tau}) - \frac{(\Gamma(1+\frac{1}{\tau})\Gamma(\alpha-\frac{1}{\tau}))^2}{\Gamma(\alpha)}\right)$
- Mgf : does not exist
- Moments of order k : $E\left[B^k\right] = \frac{1}{\Gamma(\alpha)}\lambda^{k/\tau}\Gamma(1+\frac{k}{\tau})\Gamma(\alpha-\frac{k}{\tau})$, $-\tau < k < \alpha\tau$

- Truncated Expectation :

$$
\begin{aligned}
&E\left[B \times 1_{\{X \leq d\}}\right] \\
= \;& \frac{1}{\Gamma(\alpha)} \lambda^{1/\tau} \Gamma(1+\frac{1}{\tau}) \Gamma(\alpha-\frac{1}{\tau}) B\left(\frac{d^\tau}{\lambda+d^\tau}; 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)
\end{aligned}
$$

- Risk measure VaR :

$$
VaR_\kappa(B) = (\lambda\left\{(1-\kappa)^{-1/\alpha}-1\right\})^{1/\tau}
$$

- Risk measure $TVaR$:

$$
\begin{aligned}
&TVaR_\kappa(B) \\
= \;& \frac{\lambda^{1/\tau}\Gamma(1+\frac{1}{\tau})\Gamma(\alpha-\frac{1}{\tau})}{(1-\kappa)\Gamma(\alpha)} \overline{B}\left(\frac{VaR_\kappa(B)^\tau}{\lambda+VaR_\kappa(B)^\tau}; 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)
\end{aligned}
$$

- Special case : the Pareto distribution is a special case of the Burr distribution of the Burr distribution with $\tau = 1$.

4.4.2 Comments

The Pareto distribution is very important in actuarial science, for the modelization of the claim amounts.

It has two parameters.

It is frequently consider for modeling the distribution of claims with very high amounts.

It has a mode at 0.

Its expectation exists if $\alpha > 1$.

Its variance exists if $\alpha > 2$.

Later, in section **??**, we show that the Pareto distribution can be seen as a mixture of exponentials, where the mixing distribution is gamma.

The Burr distribution is an extension of the Pareto distribution

It has three parameters.

It can be seen as a mixture of Weibull distributions where the mixing distribution is gamma.

The Pareto and the Burr distributions belong to the family of heavy tailed distibutions.

4.4.3 Cdf, mgf, and TVaR

If $B \sim Pa(\alpha, \beta)$ or $B \sim Burr\left(\alpha, \lambda, \tau\right)$, then it is impossible to obtain analytical expressions for F_X, $E\left[X \times 1_{\{X>b\}}\right]$ and $TVaR_{\kappa}\left(X\right)$.

5 Light and heavy tailed distributions

In actuarial science, claim distributions are classified in terms of light tailed distributions and heavy tailed distributions.

Heavy tailed distributions are important when one models large claim amounts, notably in reinsurance.

Well-known members of the class of heavy tailed distributions:

- Pareto
- Lognormal
- Weibull with parameter $\tau \in]0,1[$
- Burr

- Log-logistic.

Well-known members of the class of light tailed distributions:

- Exponential
- Gamma
- Weibull with parameter $\tau \geq 1$.

There are different criteria are used for classification (heavy vs light tailed distributions)

5.1 Existence of the mgf

The existence of the mgf may serve as a classification criteria:

- if the mgf of a rv X exists, then its distribution is light tailed;
- if the mgf of a rv X does not exist, then its distribution is heavy tailed.

Let X be a rv with cdf F_X.

The cdf F_X is said to belong to the class of heavy tailed distribution if we have

$$\lim_{x \to \infty} \mathrm{e}^{rx} \overline{F}_X(x) \to \infty,$$

for all $r > 0$.

Otherwise, F_X belong to the class light tailed distributions if we have

$$\lim \mathrm{e}^{rx}\overline{F}_X(x) < \infty,$$

for all $r > 0$.

According to this criteria, here are some heavy tailed distributions :

- Pareto
- Lognormal
- Weibull with parameter $\tau \in]0,1[$
- Burr
- Log-logistic.

According to this criteria, here are some light tailed distributions :

- Exponential
- Gamma
- Weibull with parameter $\tau \geq 1$.

5.2 Sub-exponential distribution

The distribution of X is said to be sub-exponential if, for all $n \geq 2$, we have

$$\Pr\left(X_1 + ... + X_n > x\right) \sim n\overline{F}_X\left(x\right), \text{ when } x \to \infty, \tag{17}$$

where the rvs $X_1, ..., X_n$ are iid with $X_i \sim X$ for $i = 1, ..., n$.

This criteria is equivalent to

$$\Pr\left(X_1 + ... + X_n > x\right) \sim \Pr\left(\max\left(X_1; ...; X_n\right) > x\right), \text{ when } x \to \infty.$$

Interpretation :

- the claim with the largest amount explain most of the stochastic behavior of the aggreggate claim amount for the whole portfolio.

According to this criteria, some heavy tailed distributions :

- Pareto
- Lognormal
- Weibull with parameter $\tau \in]0,1[$
- Burr
- Log-logistic.

Let X be a rv having a compound disitribution defined in terms of the claim number rv M and the claim amount rv B .

If the distribution of the rv B is sub-exponential, then it implies that

$$\overline{F}_X(x) = \sum_{k=1}^{\infty} f_M(k) \overline{F}_{B_1+\ldots+B_k}(x) \sim \sum_{k=1}^{\infty} f_M(k) k\overline{F}_B(x) = E[M] \overline{F}_B(x), \tag{18}$$

when $x \longrightarrow \infty$.

This asymptotic result means that the stochastic behavior fo the rv X for large claim amount is mostly explained by the largest amount.

6 Extensions ot the main counting distributions

There are various ways to extend the three main claim number distributions

- mixing,
- compounding, and
- modication of the mass probability at 0.

6.1 Mixed Poisson distributions

6.1.1 Definition and basic properties

Often, in practical applications, the Poisson distribution does not offer good fit to the data.

For instance, one may observe that $Var\left(M\right) > E\left[M\right]$.

Then, mixed Poisson distributions can play an interesting role.

Let Θ be a positive rv, called the mixing rv, such that $E\left[\Theta\right] = 1$, $\mathrm{Var}\left(\Theta\right) < \infty$ and $M_{\Theta}\left(t\right)$ exists.

Given $\Theta = \theta$, the conditional distribution of M is Poisson i.e.

$$(M|\Theta = \theta) \sim Pois\,(\lambda\theta)$$

with $\lambda > 0$.

It implies that

$$E\,[M|\Theta] = \Theta\lambda,$$

$$\mathrm{Var}\,(M|\Theta) = \Theta\lambda$$

and

$$P_{M|\Theta}\,(t) = E\left[t^M|\Theta\right] = \mathrm{e}^{\Theta\lambda(t-1)}.$$

We observe that

$$E\,[M] = E_{\Theta}\,[E\,[M|\Theta]] = E\,[\Theta\lambda] = \lambda \times 1 = \lambda \tag{19}$$

and

$$\begin{aligned} \mathrm{Var}(M) &= E_{\Theta}[\mathrm{Var}(M|\Theta)] + \mathrm{Var}_{\Theta}(E[M|\Theta]) \\ &= E[\Theta\lambda] + \mathrm{Var}(\Theta\lambda) \\ &= \lambda + \lambda^2 \mathrm{Var}(\Theta) \\ &= \lambda(1 + \lambda \mathrm{Var}(\Theta)) \geq \lambda. \end{aligned} \tag{20}$$

An interpretation :

- rv $M =$ number of claims for a car driver for a car insurance contract ;
- the insurer does not know the driving habits of the insured;
- then, the rv Θ represents the incertainty about his/her driving habits compared to the the habits of the insureds of his/her risk class;
- this uncertainty does not affect the expected number of claim;
- this uncertainty introduces overdispersion i.e. $Var(M) > E[M] = \lambda$.

The pgf is given by

$$P_M(t) = E\left[t^M\right] = E_\Theta\left[E\left[t^M|\Theta\right]\right] = E\left[\mathrm{e}^{\Theta\lambda(t-1)}\right] = M_\Theta(\lambda(t-1)). \tag{21}$$

The rv Θ can be discrete or continuous.

- If Θ is a continuous and positive rv with pdf f_Θ, then the pmf of M is

$$\Pr(M = k) = \int_0^\infty \mathrm{e}^{-\lambda\theta}\frac{(\lambda\theta)^k}{k!}f_\Theta(\theta)\,\mathrm{d}\theta,\ \ k \in \mathbb{N}^+.$$

- If Θ is a positive discrete rv which take finite or countably infinite number of values, then the pmf of M is

$$\Pr(M = k) = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda\theta_j}\frac{\left(\lambda\theta_j\right)^k}{k!}\Pr\left(\Theta = \theta_j\right),\ \ k \in \mathbb{N}^+.$$

The choice of the distribution for Θ has a significant impact on the stochastic behabior of the rv M.

We consider three cases:

- mixed Poisson-Gamma distribution, also called Negative Binomial distribution;
- mixed Poisson-Inverse Gaussian distribution
- mixed Poisson-Lognormal distribution.

6.1.2 Poisson-Gamma distribution or Negative Binomial distribution

Let $\Theta \sim Ga\left(\alpha = r, \beta = r\right)$ such that

$$E\left[\Theta\right] = \frac{r}{r} = 1,$$

$$\mathrm{Var}\left(\Theta\right) = \frac{r}{r^2} = \frac{1}{r}$$

and

$$M_{\Theta}\left(t\right) = \left(\frac{r}{r-t}\right)^{r}.$$

Then, the rv M follows a mixed Poisson-Gamma distribution, denoted by $M \sim P - Ga\left(\lambda, r\right)$.

In fact, the mixed Poisson-Gamma distribution corresponds to the Negative Binomial distribution.

From (19), (20) and (21), we obtain

$$E\left[M\right] = \lambda,$$

$$\mathsf{Var}\left(M\right) = \lambda + \frac{\lambda^2}{r}$$

and

$$P_M\left(t\right) = \left(\frac{r}{r-s}\right)^r = \left(\frac{1}{1-\frac{\lambda}{r}\left(t-1\right)}\right)^r. \tag{22}$$

Let $\frac{\lambda}{r}=\frac{1-q}{q}$ which implies that

$$q=\frac{1}{1+\frac{\lambda}{r}}.$$

Then, (22) becomes

$$P_M\left(t\right)=\left(\frac{1}{1-\frac{1-q}{q}\left(t-1\right)}\right)^r=\left(\frac{q}{1-\left(1-q\right)t}\right)^r,$$

with

$$E\left[M\right]=\lambda=r\frac{1-q}{q}$$

and

$$\mathrm{Var}\left(M\right)=\lambda+\frac{\lambda^2}{r}=r\frac{1-q}{q^2}.$$

In the following example, we illustrate the impact of the parameters r and q on the distribution of the aggregate claim amount for an insurance contract.

Exemple 7 *Let $X \sim BNComp\left(r, q; F_B\right)$, with claim amount $B \sim Exp\left(\beta = 1\right)$.*

The parameters r and q of the distribution for the counting rv M are fixed such that $E\left[M\right] = 200$.

We consider 4 pairs of values for $\left(r, q\right)$: $\left(1, \frac{1}{201}\right)$, $\left(2, \frac{1}{101}\right)$, $\left(5, \frac{1}{41}\right)$ and $\left(25, \frac{1}{9}\right)$.

In Figure 3, we depict the cdf $F_X\left(x\right)$ of X for the four pairs.

We observe the significant impact of the parameters of the the Negative Binomial distribution on the stochastic behavior of of the rv X.

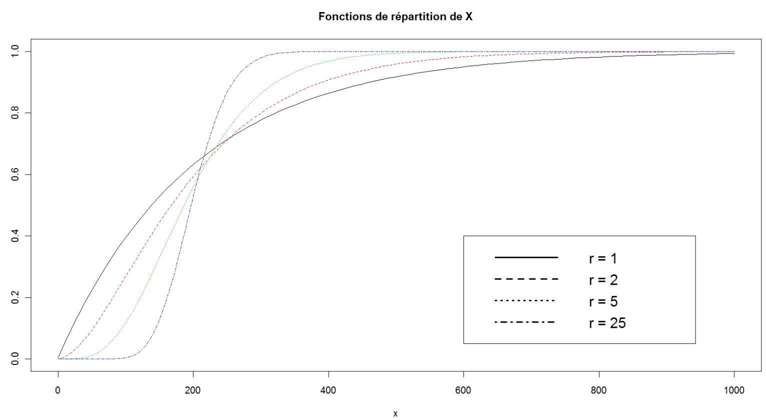

Figure 3: Curves of $F_X(x)$ where $X \sim BNComp(r, q; F_B)$, with $B \sim Exp(1)$ and 4 pairs of values for (r, q) : $\left(1, \frac{1}{201}\right)$, $\left(2, \frac{1}{101}\right)$, $\left(5, \frac{1}{41}\right)$ and $\left(25, \frac{1}{9}\right)$.

In the following table, we provide values of $VaR_{0.5}(X)$, $VaR_{0.995}(X)$, $TVaR_{0.5}(X)$ and $TVaR_{0.995}(X)$:

r	q	$VaR_{0.5}(X)$	$VaR_{0.995}(X)$	$TVaR_{0.5}(X)$	$TVaR_{0.995}(X)$
1	$\frac{1}{201}$	138.320	1063.959	339.320	1264.959
2	$\frac{1}{101}$	167.509	748.435	306.217	861.415
5	$\frac{1}{41}$	186.499	511.316	271.108	567.148
25	$\frac{1}{9}$	196.973	332.139	235.481	352.004

In Figure 4, we depict values of $VaR_{\kappa}(X)$ for the four pairs of (r, q).

The curves of $TVaR_{\kappa}(X)$ for the four pairs (r, q) are depicted in Figure 5.

For a fixed value of κ, we observe that $TVaR_{\kappa}(X)$ increases as the parameter r decreases (such that the expectation of l'espérance du nombre de sinistres reste identique). This phenomena is explained in Chapter **??**. $\square$

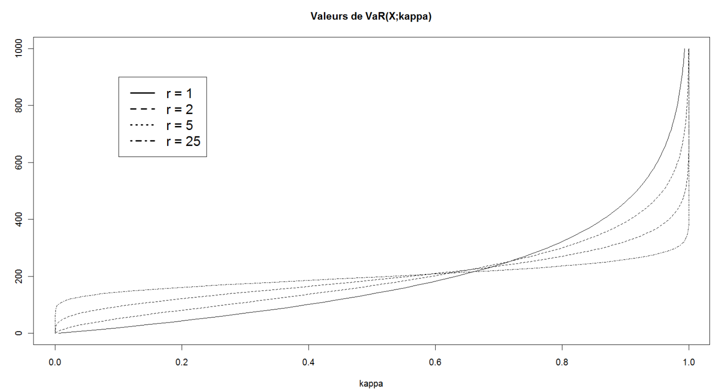

Figure 4: Curves of $VaR_\kappa(X)$ where $X \sim BNComp(r, q; F_B)$, with $B \sim Exp(1)$ and 4 pairs of values for (r, q) : $\left(1, \frac{1}{201}\right)$, $\left(2, \frac{1}{101}\right)$, $\left(5, \frac{1}{41}\right)$ and $\left(25, \frac{1}{9}\right)$.

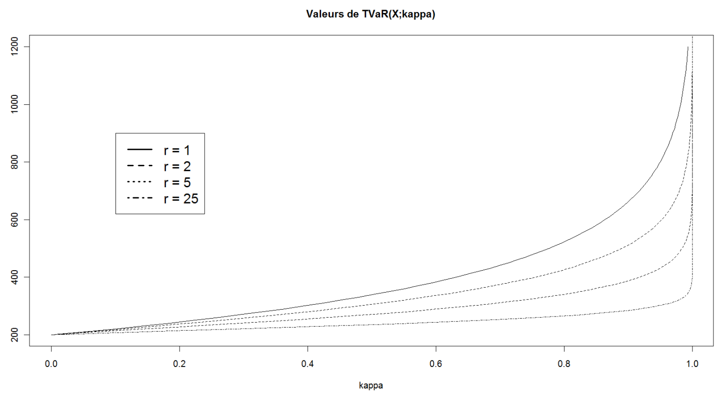

Figure 5: Curves of $TVaR_\kappa(X)$ where $X \sim BNComp(r, q; F_B)$, with $B \sim Exp(1)$ and 4 pairs of values for (r, q) : $\left(1, \frac{1}{201}\right)$, $\left(2, \frac{1}{101}\right)$, $\left(5, \frac{1}{41}\right)$ and $\left(25, \frac{1}{9}\right)$.

We also observe

$$
\begin{aligned}
\lim_{r\rightarrow\infty} E[M] &= \lambda \\
\lim_{r\rightarrow\infty} \mathrm{Var}(M) &= \lim_{r\rightarrow\infty} \lambda + \frac{\lambda^2}{r} = \lambda \\
\lim_{r\rightarrow\infty} P_M(t) &= \lim_{r\rightarrow\infty} \left(\frac{1}{1-\frac{\lambda}{r}(t-1)}\right)^r = \mathrm{e}^{\lambda(t-1)}.
\end{aligned}
\tag{23}
$$

Consequently, from (23), if $r \rightarrow \infty$ such that $E[M] = \lambda$, then the variance of M tends to λ and the distribution of M tends to the distribution of the Poisson distribution with parameter λ.

Exemple 8 *Let the rvs M_1, ..., M_5 where*

$$
M_i \sim BN\left(r = r_i, q = \left(1+\frac{2}{r_i}\right)^{-1}\right),\ i = 1, 2, 3, 4,
$$

with $r_1 = 0.5$, $r_2 = 1$, $r_3 = 2$, $r_4 = 100$, *and* $M_5 \sim Pois\,(\lambda = 2)$ *such that* $E\,[M_i] = 2$, $i = 1, 2, ..., 5$.

Within the following table, we provide the values of f_{M_i} *for* $i = 1, 2, ..., 5$ *:*

k	$f_{M_1}(k)$	$f_{M_2}(k)$	$f_{M_3}(k)$	$f_{M_4}(k)$	$f_{M_5}(k)$
0	0.447214	0.333333	0.250000	0.138033	0.135335
1	0.178885	0.222222	0.250000	0.270653	0.270671
2	0.107331	0.148148	0.187500	0.267999	0.270671
3	0.071554	0.098765	0.125000	0.178668	0.180447
4	0.050088	0.065844	0.078125	0.090209	0.090224
5	0.036063	0.043896	0.046875	0.036791	0.036089
10	0.008461	0.005781	0.002686	0.000049	0.000038
15	0.002273	0.000761	0.000122	0.000000	0.000000
20	0.000646	0.000100	0.000005	0.000000	0.000000

As the parameter r *increases, the values of the pmf get closer to the values of the pmf of the Poisson distribution* $\square$

6.1.3 Poisson-Inverse Gaussian

Let $\Theta \sim IG(1, \beta)$ such that $E[\Theta] = 1$ and $\mathrm{Var}(\Theta) = \beta$.

The characteristic of the Inverse-Gaussian Distribution are given below :

- Notation : $X \sim IG(\mu, \beta)$
- Parameters : $\mu,\ \beta \in \mathbb{R}^+$
- Support : $x \in \mathbb{R}^+$
- Pdf : $f_X(x) = \frac{\mu}{\sqrt{2\pi\beta x^3}} \exp\left(-\frac{1}{2\beta x}(x-\mu)^2\right)$
- Cdf :

$$F_X(x) = \Phi\left(\sqrt{\frac{1}{\beta x}}(x-\mu)\right) + \mathrm{e}^{\frac{2\mu}{\beta}} \Phi\left(-\sqrt{\frac{1}{\beta x}}(x+\mu)\right)$$

- Expectation : $E[X] = \mu$

- Variance : $\mathrm{Var}(X) = \mu\beta$
- Mgf : $M_X(t) = e^{\frac{\mu}{\beta}\left(1-\sqrt{(1-2\beta t)}\right)}$
- Truncated Expectation :

$$
\begin{aligned}
E\left[X \times 1_{\{X \leq d\}}\right] &= d - (2d-\mu)\Phi\left((d-\mu)\sqrt{\frac{1}{\beta d}}\right) \\
&\quad -(2d+\mu)\mathrm{e}^{\frac{2\mu}{\beta}}\Phi\left(-(d+\mu)\sqrt{\frac{1}{\beta d}}\right)
\end{aligned}
$$

- Risk measure VaR : numerical optimization
- Risk measure $TVaR$:

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa}\left(\mu - d + (2d+\mu)\mathrm{e}^{\frac{2\mu}{\beta}}\right) \\
&\quad + \frac{1}{1-\kappa}\left((2d-\mu)\Phi\left((d-\mu)\sqrt{\frac{1}{\beta d}}\right)\right),
\end{aligned}
$$

with $d = VaR_\kappa(X)$

- Stop-loss function :

$$\begin{aligned}\pi_d(X) &= (\mu - d)\left(1 - \Phi\left((d-\mu)\sqrt{\frac{1}{\beta d}}\right)\right) \\ &\quad +(d+\mu)\mathrm{e}^{\frac{2\mu}{\beta}}\Phi\left(-(d+\mu)\sqrt{\frac{1}{\beta d}}\right)\end{aligned}$$

- Excess-of-loss function :

$$e_d(X) = \frac{(\mu - d)\left(1 - \Phi\left((d-\mu)\sqrt{\frac{1}{\beta d}}\right)\right)}{1-\left(\Phi\left(\sqrt{\frac{1}{\beta x}}(d-\mu)\right)+\mathrm{e}^{\frac{2\mu}{\beta}}\Phi\left(-\sqrt{\frac{1}{\beta x}}(d+\mu)\right)\right)} + \frac{+(d+\mu)\mathrm{e}^{\frac{2\mu}{\beta}}\Phi\left(-(d+\mu)\sqrt{\frac{1}{\beta d}}\right)}{1-\left(\Phi\left(\sqrt{\frac{1}{\beta x}}(d-\mu)\right)+\mathrm{e}^{\frac{2\mu}{\beta}}\Phi\left(-\sqrt{\frac{1}{\beta x}}(d+\mu)\right)\right)}$$

- Limited expectation :

$$\begin{aligned} E\left[\min(X;d)\right] &= d-(d-\mu)\Phi\left((d-\mu)\sqrt{\frac{1}{\beta d}}\right) \\ &\quad -(d+\mu)e^{\frac{2\mu}{\beta}}\Phi\left(-(d+\mu)\sqrt{\frac{1}{\beta d}}\right) \end{aligned}$$

Then, the rv M follows a Poisson-Inverse Gaussian distribution, denoted by $M \sim P - IG(\lambda, \beta)$.

From (19), (20) and (21), we find

$$E[M] = \lambda,$$

$$\mathrm{Var}(M) = \lambda + \lambda^2\beta$$

and

$$P_M(t) = \exp\left(\frac{1}{\beta}\left(1-\sqrt{1-2\beta\lambda(t-1)}\right)\right).$$

There is no closed form expression for f_M because we can not find an explicit solution for

$$f_M(k) = \int_0^\infty \mathrm{e}^{-\lambda\theta}\frac{(\lambda\theta)^k}{k!}\frac{1}{\sqrt{2\pi\beta\theta^3}}\exp\left(-\frac{1}{2\beta\theta}(\theta-1)^2\right)\mathrm{d}\theta.$$

We propose an alternative option.

The values of $f_M(k)$ can be recursively computed

- Starting points :

$$f_M(0) = P_M(0) = \exp\left(\frac{1}{\beta}\left(1 - \sqrt{1+2\beta\lambda}\right)\right)$$

and

$$\begin{aligned} f_M(1) &= \left.\frac{\mathrm{d}P_M(t)}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\lambda}{\sqrt{1-2\beta\lambda(t-1)}} P_M(t)\right|_{t=0} \\ &= \frac{1}{\sqrt{1+2\beta\lambda}} f_M(0). \end{aligned}$$

- Recursive relation :

$$\begin{aligned} f_M(k) &= \frac{2\lambda\beta}{1+2\lambda\beta}\left(1 - \frac{3}{2k}\right) f_M(k-1) \\ &\quad + \frac{\lambda^2}{(1+2\lambda\beta)\,k\,(k+1)} f_M(k-2), \end{aligned}$$

for $k = 2, 3, \ldots$.

The Poisson-Inverse Gaussian distribution can be considered as an alternative to the Poisson-Gamma (Negative Binomial) distribution.

6.1.4 Poisson-Lognormal distribution

Let $\Theta \sim LN\left(\mu, \sigma^2\right)$ where $\mu = -\frac{1}{2}\sigma^2$ such that

$$E[\Theta] = \exp\left(\mu + \frac{1}{2}\sigma^2\right) = 1$$

and

$$\mathrm{Var}(\Theta) = \mathrm{e}^{\left(2\mu+\sigma^2\right)}\left(\mathrm{e}^{\sigma^2} - 1\right) = \left(\mathrm{e}^{\sigma^2} - 1\right).$$

Then, the rv M follows a Poisson-Lognormal distribution with parameters λ and σ, denoted

$$M \sim P - LN\left(\lambda, \sigma^2\right).$$

From (19) and (20), we obtain

$$E[M] = \lambda$$

et

$$\mathrm{Var}(M) = \lambda + \lambda^2 \left(\mathrm{e}^{\sigma^2} - 1\right).$$

There is no closed expression for f_M.

We may use numerical integration methods to compute

$$f_M(k) = \int_0^\infty \mathrm{e}^{-\lambda\theta} \frac{(\lambda\theta)^k}{k!} f_\Theta(\theta)\, \mathrm{d}\theta = \int_0^\infty \mathrm{e}^{-\lambda\theta} \frac{(\lambda\theta)^k}{k!} \frac{1}{\theta\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(\ln\theta + \frac{\sigma^2}{2})^2}{2\sigma^2}} \mathrm{d}\theta.$$

In the following example, we compare the values of the pmfs for the Poisson-Gamma distribution and the Poisson-Lognormal distribution.

Exemple 9 *Let*

$$M_1 \sim P - LN\left(\lambda = 2, \sigma^2 = 1\right)$$

and

$$M_2 \sim BN\left(r, q\right)$$

where $r = \left(e^{\sigma^2} - 1\right) = 0.5820$ *and* $q = \left(1 + \frac{2}{r}\right)^{-1} = 0.2254$.

The parameters of both distributions are fixed such that

$$E\left[M_1\right] = E\left[M_2\right] = 2$$

and

$$\mathrm{Var}\left(M_1\right) = \mathrm{Var}\left(M_2\right) = 8.8731.$$

The values of $f_{M_1}(k)$ *and* $f_{M_2}(k)$*, for* $k = 0, 1, ..., 5$*, are provided in the following table :*

k	$f_{M_1}(k)$	$f_{M_2}(k)$
0	0.3325	0.4202
1	0.2506	0.1894
2	0.1526	0.1161
3	0.0910	0.0774
4	0.0556	0.0537
5	0.0352	0.0381

$\square$

6.2 Zero-modified distribution

In certain contexts, the Poisson distribution, Binomial Distribution, Negative Binomial distribution, or the Mixed Poisson distributions do not fit appropriately the probability mass that can be observed at 0.

Then, it is possible to make the following modification.

Let M' be a discrete rv with pmf $f_{M'}$ and $I \sim Bern(q)$.

We define the rv M with a pmf given by

$$f_M(k) = \begin{cases} qf_{M'}(k), k \in \mathbb{N}^+ & , k \in \mathbb{N}^+ \\ (1-q) + qf_{M'}(0) & , k = 0 \end{cases} .$$

It follows that

$$E[M] = qE\left[M'\right]$$

$$\mathsf{Var}(M) = qE\left[M'^2\right] - q^2 E[M]^2$$

and

$$P_M(t) = 1 - q + qP_{M'}(t).$$

7 Challenges for research in actuarial science

- Advances statistical methods
  - Find the appropriate distribution of the counting rv
  - Find the appropriate distribution of the claim rv
  - Taking into account explanatory variables
  - Regression models for counting data (discrete data)
  - Regression models for claim amount data (non-gaussian)
- Auto insurance and property insurance :
  - How many explanatory variables should one take into account in the definitions of the distribution for the counting rv ?
  - How many explanatory variables should one take into account in the definitions of the distribution for the counting rv ?
  - Large amount of data.
  - Hidden data

  - Heterogeneity
  - Big Data and data analytics.
  - Important revolution ahead in property insurance.
- Heavy-tailed distributions
- Catastrophic Risks
- Numerical and Monte-Carlo Simulation Methods
- Etc.