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SPACE-TIME FINITE ELEMENT METHODS FOR ELASTODYNAMICS: FORMULATIONS AND ERROR ESTIMATES*

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Space-time finite element methods are developed for classical elastodynamics. The approach employs the discontinuous Galerkin method in time and incorporates stabilizing terms of least-squares type. These enable a general convergence theorem to be proved in a norm stronger than the energy norm. Optimal error estimates are predicted, and confirmed numerically, for arbitrary combinations of displacement and velocity interpolations. The procedures developed are easily generalized to structural dynamics and a wide class of second-order hyperbolic problems.

1. Introduction

Most finite element procedures for time-dependent phenomena are based upon semidiscretizations: finite elements are used in space to reduce to a system of ordinary differential equations in time which are in turn discretized by traditional finite difference methods for ordinary differential equations. Procedures of this kind are widely used in practice and well understood numerically. It is frequently argued that finite elements represent a superior methodology to finite differences. Thus, it is not surprising that efforts have been made to exploit finite elements in time [1, 7, 26, 29–32]. In many of these works, however, the semidiscrete equations are simply multiplied by weighting functions and integrated over time intervals. Many traditional ordinary differential equation algorithms have been rederived in this manner. Because the discretization is performed first in space and then independently in time, the space-time mesh is inevitably *structured*, each element domain being the Cartesian product of a spatial element and a time interval. This negates one of the most powerful features of the finite element method: *unstructured meshes*. Others have speculated on the use of space-time methods in which continuity in time is assumed. This permits unstructured meshes, but leads to a coupled matrix system in which variables at all time levels need to be solved for simultaneously. Whatever their virtues, methods of this kind are prohibitively demanding of storage and computer time compared with existing techniques.

Another approach has evolved over the years. The idea is based on the *discontinuous Galerkin method* [13, 16, 18, 20], which was originally developed for hyperbolic equations in which information propagates in the direction of characteristics. The discontinuous Galerkin method has many attributes in this problem class and, in particular, can lead to stable,

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higher-order accurate finite element methods. Typically, L_2 error estimates are of order $k + \frac{1}{2}$, where k is the order of the finite element polynomial. See [16, 18] for recent developments. The discontinuous Galerkin method places emphasis on 'upwind' information, but there are no 'artificial viscosities', or 'tuneable' parameters present, in contrast with classical upwind finite difference methods. In the context of multidimensional advective-diffusive systems, there are, however, some drawbacks (see [8] for elaboration).

In physical, time-dependent problems, information flows in the direction of positive time. Solutions are said to be 'causal' [27], in that they depend upon the past but not on the future. Typical finite difference ordinary differential equation solvers are causal in that the solution at the current step depends only upon previous steps. Likewise, the discontinuous Galerkin method applied over space-time slabs of thickness Δt leads to a causal system in which the solution throughout the current slab ($\Omega \times [t_n, t_{n+1}]$, where Ω is the spatial domain and n the time step number) depends only upon the solution at t_n^-. It was first shown in [3, 15, 20] that the discontinuous Galerkin method in time leads to A-stable, higher-order accurate ordinary differential equation solvers. By Dahlquist's theorem [2], A-stable methods of accuracy greater than second-order cannot be linear multistep methods. Consequently, some additional computational complexity is inherent. Nevertheless, when applied to partial differential equations, time-discontinuous Galerkin methods seem to possess considerable potential not present in the semidiscrete approach. For example, the meshes in time slabs can be completely unstructured. This is useful for the development of various types of space-time adaptive schemes (see Fig. 1 for a schematic illustration). A particular application, which has never had a truly satisfactory development within the semidiscrete approach, is 'subcycling' in which different time steps are employed within different spatial elements (see [9, 24] for some analytical results on subcycling). Unstructured space-time meshes seem to provide a natural setting for the development and analysis of such schemes (see Fig. 2 for a schematic illustration). Furthermore, the time-discontinuous framework seems conducive to the establishment of rigorous convergence proofs and error estimates (see [4, 5, 10, 16–20, 25] for progress so far on parabolic and first-order hyperbolic problems). In fact, the more one thinks about it, the more apparent it becomes that the ubiquitous semidiscrete approach is conceptually confining, even schizophrenic. If indeed finite elements have advantages in space, they should also have advantages in space-time. This is the supposition underlying the present work.

Classical elastodynamics can be converted to first-order symmetric hyperbolic form, which has proved useful in theoretical studies [12]. Finite element methods for first-order symmetric hyperbolic systems are thus immediately applicable [10, 16]. However, there seems to be several disadvantages: in symmetric hyperbolic form the state vector consists of displacements, velocities, and stresses which is computationally uneconomical; and the generalization to nonlinear elastodynamics seems possible only in special circumstances [14]. For these reasons we have directly attacked the problem from a more natural second-order hyperbolic viewpoint. We allow independent displacement and velocity interpolations, although velocity can be eliminated if desired. Continuity of displacement and velocity between space-time slabs is enforced weakly. A novel, and apparently essential feature, is to enforce displacement continuity by way of the strain energy inner product. In order to establish a general convergence theorem, we need to introduce 'least-squares' terms which increase the stability of the Galerkin formulation. These terms vanish identically at the exact solution. Similar ideas

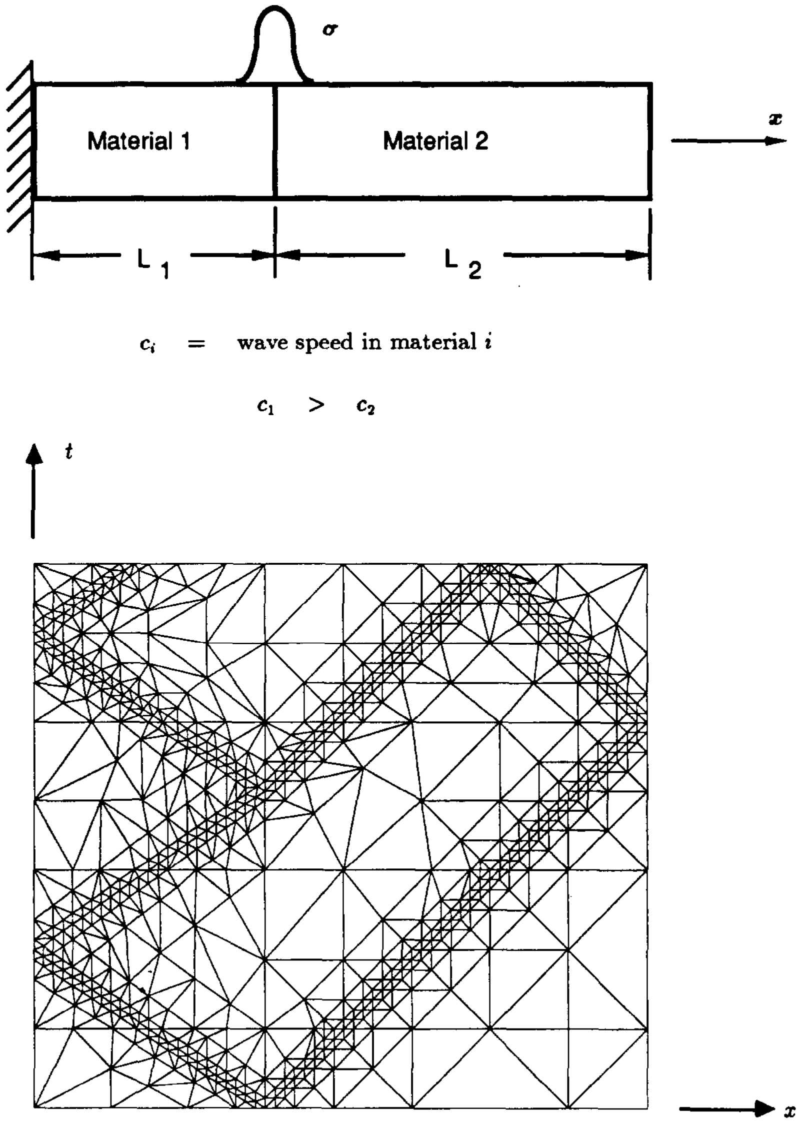

Fig. 1. Space-time mesh for a two-material elastic rod problem.

have been exploited by the senior author and his colleagues in other circumstances [6, 21–23]. However, one term which appears herein, involving the spatial continuity of traction, is atypical. An interpretation of the additional terms is that they are sophisticated 'artificial viscosities'. There is a well-deserved negative connotation associated with the concept of artificial viscosity, however, there are salient differences between classical ad hoc viscosities and the present ones, namely, the latter are dictated by the mathematical convergence proof; they are form-invariant with respect to the order of the element interpolations; and the resulting methods are higher-order accurate. Omitting one or more of these terms may be possible in specific circumstances. We have explored this numerically with some success.

An outline of the paper follows. In Section 2 we review the equations of elastodynamics. In Section 3 we establish the general form of our method and in Section 4 we prove convergence

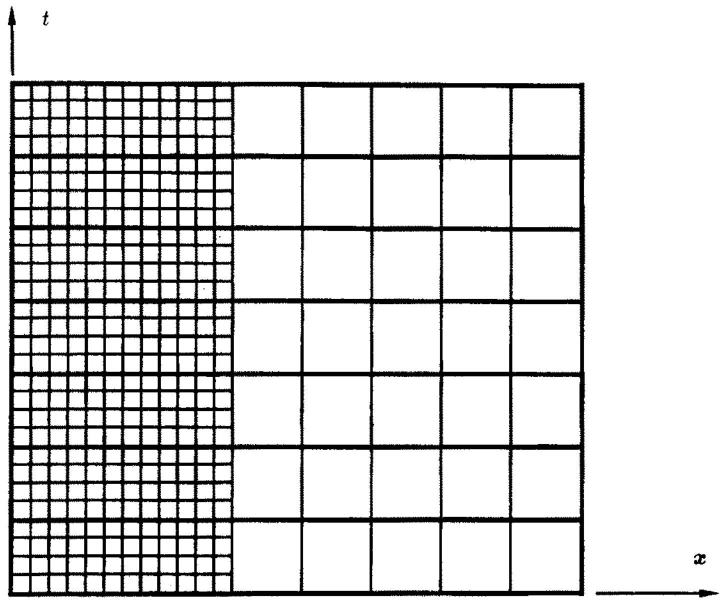

Fig. 2. Subcycling.

and obtain error estimates. We discuss some simplifications in Section 5 and present numerical results in Section 6. Finally, conclusions are drawn in Section 7.

We believe the efforts reported on herein open up new possibilities for the solution of a wide class of second-order hyperbolic problems, and provide additional evidence in contradiction of the tired old adage that somehow finite elements are inappropriate for hyperbolic problems.

2. Classical linear elastodynamics

Consider an elastic body occupying an open, bounded region $\Omega \subset \mathbb{R}^d$, where d is the number of space dimensions. The boundary of Ω is denoted by Γ. Let Γ_g and Γ_h denote nonoverlapping subregions of Γ such that

$$\Gamma = \overline{\Gamma_g \cup \Gamma_h} . \tag{1}$$

The displacement vector is denoted by $\boldsymbol{u}(\boldsymbol{x}, t)$, where $\boldsymbol{x} \in \bar{\Omega}$ and $t \in [0, T]$, the time interval of length $T > 0$. The stress is determined by the generalized Hooke's law:

$$\boldsymbol{\sigma}(\nabla \boldsymbol{u}) = \boldsymbol{c} \cdot \nabla \boldsymbol{u} , \tag{2}$$

or, in components,

$$\sigma_{ij} = c_{ijkl} u_{k,l} , \tag{3}$$

where $1 \leqslant i, j, k, l \leqslant d$, $u_{k,l} = \partial u_k / \partial x_l$, and summation over repeated indices is implied. The elastic coefficients $c_{ijkl} = c_{ijkl}(\boldsymbol{x})$ are assumed to satisfy the following conditions:

$$c_{ijkl} = c_{jikl} = c_{ijlk} \qquad \text{(minor symmetries)}\,, \tag{4}$$

$$c_{ijkl} = c_{klij} \qquad \text{(major symmetry)}\,, \tag{5}$$

$$c_{ijkl}\psi_{ij}\psi_{kl} > 0 \quad \forall \psi_{ij} = \psi_{ji} \neq 0 \quad \text{(positive-definiteness)}\,. \tag{6}$$

By the minor symmetries, $\boldsymbol{\sigma}$ is symmetric and depends only upon the symmetric part of $\nabla \boldsymbol{u}$. The minor symmetries play no role in the formulation presented in subsequent sections, however, the major symmetry and positive-definiteness are important.

The equations of the initial/boundary-value problem of elastodynamics are:

$$\rho \ddot{\boldsymbol{u}} = \nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}) + \boldsymbol{f} \quad \text{on } Q \equiv \Omega \times]0, T[\,, \tag{7}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } P_g \equiv \Gamma_g \times]0, T[\,, \tag{8}$$

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}) = \boldsymbol{h} \quad \text{on } P_h \equiv \Gamma_h \times]0, T[\,, \tag{9}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega\,, \tag{10}$$

$$\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \Omega\,, \tag{11}$$

where $\rho = \rho(\boldsymbol{x}) > 0$ is the density, a superposed dot indicates partial differentiation with respect to t, $\boldsymbol{f}$ is the body force, $\boldsymbol{g}$ is the prescribed boundary displacement, $\boldsymbol{h}$ is the prescribed boundary traction, $\boldsymbol{u}_0$ is the initial displacement, $\boldsymbol{v}_0$ is the initial velocity, and $\boldsymbol{n}$ is the unit outward normal to Γ. In components, $\nabla \cdot \boldsymbol{\sigma}$ and $\boldsymbol{n} \cdot \boldsymbol{\sigma}$ are $\sigma_{ij,j}$ and $\sigma_{ij}n_j$, respectively. The objective is to find a $\boldsymbol{u}$ that satisfies (7)–(11) for given ρ, $\boldsymbol{c}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{u}_0$, and $\boldsymbol{v}_0$.

Remark

The equations considered are prototypes of general second-order hyperbolic systems. Consequently, the results obtained have wider applicability than elastodynamics.

3. A time-discontinuous Galerkin least-squares formulation

3.1. Preliminaries

Consider a partition of $]0, T[$ having the form $0 = t_0 < t_1 < \cdots < t_N = T$. Let $I_n =]t_n, t_{n+1}[$. The following notations are employed:

$$Q_n = \Omega \times I_n\,, \tag{12}$$

$$P_n = \Gamma \times I_n\,, \tag{13}$$

$$(P_g)_n = \Gamma_g \times I_n\,, \tag{14}$$

$$(P_h)_n = \Gamma_h \times I_n\,. \tag{15}$$

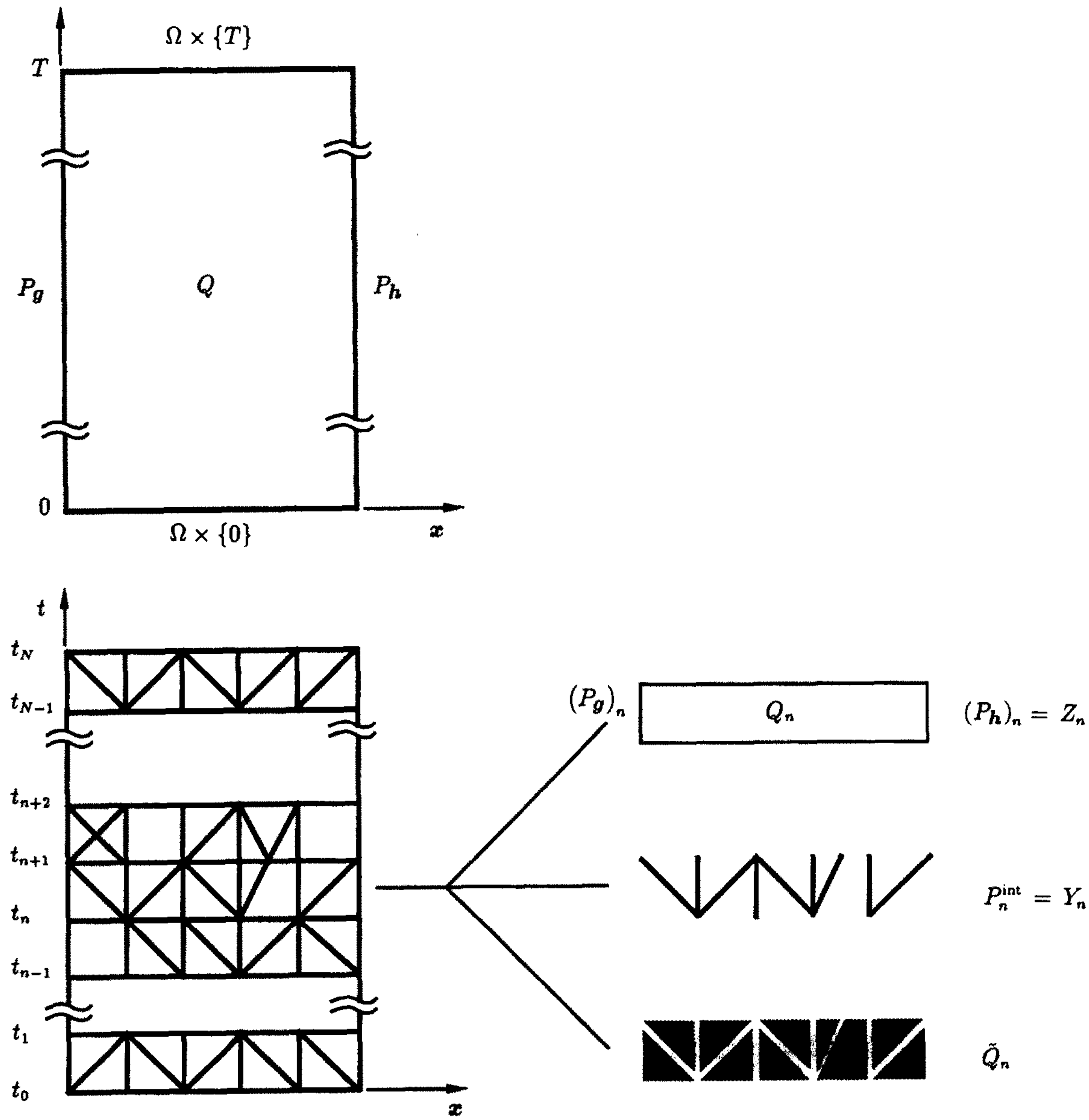

Fig. 3. Illustration of space-time finite element notation.

Let $(n_{\text{el}})_n$ denote the number of space-time elements in Q_n. Let $Q_n^e \subset Q_n$ denote the interior of the eth element, and let P_n^e denote its boundary. Let

$$\tilde{Q}_n = \bigcup_{e=1}^{(n_{\text{el}})_n} Q_n^e \qquad \text{(element interiors)}\,, \tag{16}$$

$$P_n^{\text{int}} = \bigcup_{e=1}^{(n_{\text{el}})_n} P_n^e - P_n \quad \text{(interior boundary)}\,. \tag{17}$$

Due to their frequent subsequent use, these simpler notations are introduced:

$$Y_n = P_n^{\text{int}} , \tag{18}$$

$$Z_n = (P_h)_n . \tag{19}$$

The setup is schematically illustrated in Fig. 3.

3.1.1. Jump operator in space

Consider a typical space-time element with domain Q^e and boundary P^e. Let $\boldsymbol{n}$ denote the outward unit normal vector to $P^e \cap \{t\}$ in the spatial hyperplane $Q^e \cap \{t\}$. If Q^e equals the product of Ω^e and a time interval, then $\boldsymbol{n}$ is the usual spatial unit outward normal vector to Γ^e.

Consider two adjacent space-time elements. Designate one element by + and the other by −, and let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ denote the spatial unit outward normal vectors along the interface (see Fig. 4). To simplify the notation, the argument t is suppressed. The *spatial jump operator* is defined by,

$$[\![\boldsymbol{w}(\boldsymbol{x})]\!] = \boldsymbol{w}(\boldsymbol{x}^+) - \boldsymbol{w}(\boldsymbol{x}^-) , \tag{20}$$

where

$$\boldsymbol{w}(\boldsymbol{x}^+) = \lim_{\varepsilon \to 0^-} \boldsymbol{w}(\boldsymbol{x} + \varepsilon \boldsymbol{n}) , \tag{21}$$

$$\boldsymbol{w}(\boldsymbol{x}^-) = \lim_{\varepsilon \to 0^+} \boldsymbol{w}(\boldsymbol{x} + \varepsilon \boldsymbol{n}) , \tag{22}$$

$$\boldsymbol{n} = \boldsymbol{n}^+ = -\boldsymbol{n}^- . \tag{23}$$

Consider the stress computed from the displacement field $\boldsymbol{w}$, namely $\boldsymbol{\sigma}(\nabla \boldsymbol{w})$. Then,

$$\begin{aligned} \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w})(\boldsymbol{x})]\!] &= \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w})(\boldsymbol{x}^+) - \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w})(\boldsymbol{x}^-) \\ &= \boldsymbol{n}^+ \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w})(\boldsymbol{x}^+) + \boldsymbol{n}^- \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w})(\boldsymbol{x}^-) , \end{aligned} \tag{24}$$

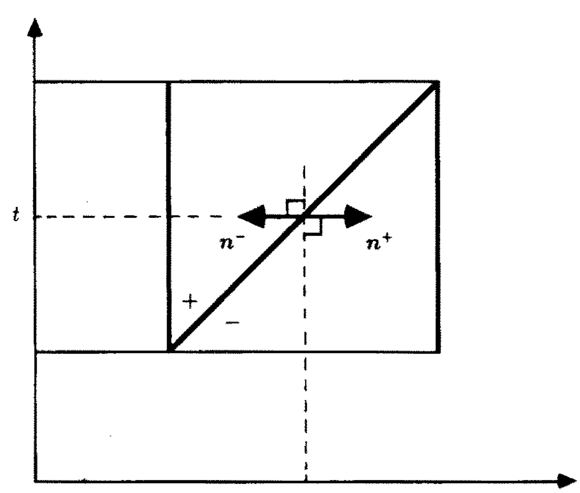

Fig. 4. Illustration of spatial outward normal vectors.

which demonstrates the invariance of $\boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w})(\boldsymbol{x})]\!]$ with respect to interchanging the + and − designators. Expressions like this frequently arise in the sequel.

3.1.2. Jump operator in time

Suppress the argument $\boldsymbol{x}$. The temporal jump operator is defined by

$$[\![\boldsymbol{w}(t)]\!] = \boldsymbol{w}(t^+) - \boldsymbol{w}(t^-) , \tag{25}$$

where

$$\boldsymbol{w}(t^+) = \lim_{\varepsilon \to 0^+} \boldsymbol{w}(t + \varepsilon) , \tag{26}$$

$$\boldsymbol{w}(t^-) = \lim_{\varepsilon \to 0^-} \boldsymbol{w}(t + \varepsilon) . \tag{27}$$

Some additional frequently-used notations are:

$$a(\boldsymbol{w}, \boldsymbol{u})_\Omega = \int_\Omega \nabla \boldsymbol{w} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}) \, \mathrm{d}\Omega \quad \text{(strain energy inner product)} , \tag{28}$$

$$a(\boldsymbol{w}, \boldsymbol{u})_{Q_n} = \int_{Q_n} \nabla \boldsymbol{w} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}) \, \mathrm{d}Q , \tag{29}$$

$$(\boldsymbol{w}, \boldsymbol{u})_{\tilde{Q}_n} = \int_{\tilde{Q}_n} \boldsymbol{w} \cdot \boldsymbol{u} \, \mathrm{d}Q , \tag{30}$$

$$(\boldsymbol{w}, \boldsymbol{u})_{Z_n} = \int_{Z_n} \boldsymbol{w} \cdot \boldsymbol{u} \, \mathrm{d}Z , \tag{31}$$

where

$$\int_{Q_n} \cdots \mathrm{d}Q = \int_{I_n} \int_\Omega \cdots \mathrm{d}\Omega \, \mathrm{d}t , \tag{32}$$

$$\int_{\tilde{Q}_n} \cdots \mathrm{d}Q = \sum_{e=1}^{(n_{\mathrm{el}})_n} \int_{Q_n^e} \cdots \mathrm{d}Q , \tag{33}$$

$$\int_{Z_n} \cdots \mathrm{d}Z = \int_{(P_h)_n} \cdots \mathrm{d}P = \int_{I_n} \int_{\Gamma_h} \cdots \mathrm{d}\Gamma \, \mathrm{d}t . \tag{34}$$

The meaning of other similar notations may be inferred from these. Note that (5) and (28) imply that the strain energy inner product is a symmetric bilinear form.

3.2. Variational equations

We need to develop an appropriate space-time mesh parameter. Consider hypercylinders in space-time with generating axes parallel to the time axis. Let Δx_n^e denote the diameter of the smallest such cylinder which contains element domain Q_n^e. Let

$$\Delta x = \max_{n=0, 1, \ldots, N-1} \Big(\max_{e=1, 2, \ldots, (n_{\mathrm{el}})_n} (\Delta x_n^e) \Big) , \tag{35}$$

$$\Delta t = \max_{n=0,1,\ldots,N-1} (t_{n+1} - t_n)\,, \tag{36}$$

and

$$h = \max\{c\Delta t, \Delta x\} \quad \text{(space-time mesh parameter)}\,, \tag{37}$$

where c is the dilatational wave velocity. For example, in the isotropic case,

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}\,, \tag{38}$$

where λ and μ are the Lamé parameters. This expression applies to the plane strain, axisymmetric, and three-dimensional cases. In plane stress,

$$c = \sqrt{\frac{4\mu(\lambda+\mu)}{\rho(\lambda+2\mu)}}\,. \tag{39}$$

Let $\mathscr{P}^m$ denote the space of mth-order polynomials, and let $\mathscr{C}^0$ denote the space of continuous functions. Consider the following collections of finite element interpolation functions:

Trial displacements,

$$\mathscr{S}_1^h = \left\{ \boldsymbol{u}_1^h \,\middle|\, \boldsymbol{u}_1^h \in \left(\mathscr{C}^0\left(\bigcup_{n=0}^{N-1} Q_n\right)\right)^d,\ \boldsymbol{u}_1^h\Big|_{Q_n^e} \in (\mathscr{P}^k(Q_n^e))^d,\ \boldsymbol{u}_1^h = \boldsymbol{g} \text{ on } P_g \right\}. \tag{40}$$

Trial velocities,

$$\mathscr{S}_2^h = \left\{ \boldsymbol{u}_2^h \,\middle|\, \boldsymbol{u}_2^h \in \left(\mathscr{C}^0\left(\bigcup_{n=0}^{N-1} Q_n\right)\right)^d,\ \boldsymbol{u}_2^h\Big|_{Q_n^e} \in (\mathscr{P}^l(Q_n^e))^d,\ \boldsymbol{u}_2^h = \dot{\boldsymbol{g}} \text{ on } P_g \right\}. \tag{41}$$

Displacement weighting functions,

$$\mathscr{V}_1^h = \left\{ \boldsymbol{w}_1^h \,\middle|\, \boldsymbol{w}_1^h \in \left(\mathscr{C}^0\left(\bigcup_{n=0}^{N-1} Q_n\right)\right)^d,\ \boldsymbol{w}_1^h\Big|_{Q_n^e} \in (\mathscr{P}^k(Q_n^e))^d,\ \boldsymbol{w}_1^h = \boldsymbol{0} \text{ on } P_g \right\}. \tag{42}$$

Velocity weighting functions,

$$\mathscr{V}_2^h = \left\{ \boldsymbol{w}_2^h \,\middle|\, \boldsymbol{w}_2^h \in \left(\mathscr{C}^0\left(\bigcup_{n=0}^{N-1} Q_n\right)\right)^d,\ \boldsymbol{w}_2^h\Big|_{Q_n^e} \in (\mathscr{P}^l(Q_n^e))^d,\ \boldsymbol{w}_2^h = \boldsymbol{0} \text{ on } P_g \right\}. \tag{43}$$

Let $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\boldsymbol{s}$ be given $d \times d$ positive-definite matrices. The matrices $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ have dimensions of time and $\boldsymbol{s}$ has dimensions of velocity inverse. We refer to $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ as *intrinsic time-scale matrices*, and $\boldsymbol{s}$ as the *slowness matrix*.

The objective is to find $\boldsymbol{U}^h = \{\boldsymbol{u}_1^h, \boldsymbol{u}_2^h\} \in \mathscr{S}_1^h \times \mathscr{S}_2^h$ such that for all $\boldsymbol{W}^h = \{\boldsymbol{w}_1^h, \boldsymbol{w}_2^h\} \in \mathscr{V}_1^h \times \mathscr{V}_2^h$,

$$B_n(\boldsymbol{W}^h, \boldsymbol{U}^h) = L_n(\boldsymbol{W}^h)\,, \quad n = 0, 1, \ldots, N-1\,, \tag{44}$$

where

$$\begin{aligned}
B_n(\boldsymbol{W}^h, \boldsymbol{U}^h) = {} & (\boldsymbol{w}_2^h, \rho \dot{\boldsymbol{u}}_2^h)_{Q_n} + a(\boldsymbol{w}_2^h, \boldsymbol{u}_1^h)_{Q_n} + a(\boldsymbol{w}_1^h, \mathscr{L}_1 \boldsymbol{U}^h)_{\tilde{Q}_n} \\
& + (\rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \mathscr{L}_2 \boldsymbol{U}^h)_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{W}^h, \mathscr{L}_1 \boldsymbol{U}^h)_{\tilde{Q}_n} \\
& + (\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h)(\boldsymbol{x})]\!])_{y_n} \\
& + (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h))_{Z_n} \\
& + (\boldsymbol{w}_2^h(t_n^+), \rho \boldsymbol{u}_2^h(t_n^+))_\Omega + a(\boldsymbol{w}_1^h(t_n^+), \boldsymbol{u}_1^h(t_n^+))_\Omega \,, \quad n = 0, 1, \ldots, N-1 \,,
\end{aligned} \tag{45}$$

$$\begin{aligned}
L_n(\boldsymbol{W}^h) = {} & (\boldsymbol{w}_2^h, \boldsymbol{f})_{Q_n} + (\boldsymbol{w}_2^h, \boldsymbol{h})_{Z_n} \\
& + (\rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \boldsymbol{f})_{\tilde{Q}_n} + (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h), \boldsymbol{h})_{Z_n} \\
& + (\boldsymbol{w}_2^h(t_n^+), \rho \boldsymbol{u}_2^h(t_n^-))_\Omega + a(\boldsymbol{w}_1^h(t_n^+), \boldsymbol{u}_1^h(t_n^-))_\Omega \,, \quad n = 1, 2, \ldots, N-1 \,,
\end{aligned} \tag{46}$$

$$\begin{aligned}
L_0(\boldsymbol{W}^h) = {} & (\boldsymbol{w}_2^h, \boldsymbol{f})_{Q_0} + (\boldsymbol{w}_2^h, \boldsymbol{h})_{Z_0} \\
& + (\rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \boldsymbol{f})_{\tilde{Q}_0} + (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h), \boldsymbol{h})_{Z_0} \\
& + (\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{v}_0)_\Omega + a(\boldsymbol{w}_1^h(0^+), \boldsymbol{u}_0)_\Omega \,,
\end{aligned} \tag{47}$$

in which

$$\mathscr{L}_1 \boldsymbol{U}^h = \dot{\boldsymbol{u}}_1^h - \boldsymbol{u}_2^h \,, \tag{48}$$

$$\mathscr{L}_2 \boldsymbol{U}^h = \rho \dot{\boldsymbol{u}}_2^h - \nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h) \,. \tag{49}$$

Note that L_0 is obtained from the general expression for L_n by setting $n = 0$ and replacing $\boldsymbol{u}_1^h(0^-)$ by $\boldsymbol{u}_0$, and $\boldsymbol{u}_2^h(0^-)$ by $\boldsymbol{v}_0$. B_n is a bilinear form and L_n is a linear form. The solution is constructed by solving the problems in order. The matrices $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s provide additional stability without compromising consistency. This may be inferred from the Euler–Lagrange form of the variational equation:

$$\begin{aligned}
0 &= B_n(\boldsymbol{W}^h, \boldsymbol{U}^h) - L_n(\boldsymbol{W}^h) \\
&= (\boldsymbol{w}_2^h + \rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \mathscr{L}_2 \boldsymbol{U}^h - \boldsymbol{f})_{\tilde{Q}_n} && \text{(equation of motion)} \\
&\quad + (\boldsymbol{w}_2^h + \rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h)(\boldsymbol{x})]\!])_{Y_n} \\
& && \text{(traction continuity in space)} \\
&\quad + (\boldsymbol{w}_2^h + \rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h) - \boldsymbol{h})_{Z_n} \\
& && \text{(traction boundary condition)} \\
&\quad + a(\boldsymbol{w}_1^h + \boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{W}^h, \mathscr{L}_1 \boldsymbol{U}^h)_{\tilde{Q}_n} && \text{(definition of velocity)}
\end{aligned}$$

$$+ (\boldsymbol{w}_2^h(t_n^+), \rho[\![\boldsymbol{u}_2^h(t_n)]\!])_\Omega \qquad \text{(velocity continuity in time)}$$

$$+ a(\boldsymbol{w}_1^h(t_n^+), [\![\boldsymbol{u}_1^h(t_n)]\!])_\Omega \qquad \text{(displacement continuity in time)}\,, \tag{50}$$

where we have used the integration-by-parts formula:

$$a(\boldsymbol{w}_2^h, \boldsymbol{u}_1^h)_{Q_n} = -(\boldsymbol{w}_2^h, \nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h))_{\tilde{Q}_n} + (\boldsymbol{w}_2^h, \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h)(\boldsymbol{x})]\!])_{Y_n}$$
$$+ (\boldsymbol{w}_2^h, \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h))_{Z_n}\,. \tag{51}$$

It follows from (50) that a sufficiently smooth exact solution of the initial/boundary-value problem, $\boldsymbol{U} = \{\boldsymbol{u}, \dot{\boldsymbol{u}}\}$, satisfies

$$B_n(\boldsymbol{W}^h, \boldsymbol{U}) = L_n(\boldsymbol{W}^h) \tag{52}$$

for all $\boldsymbol{W}^h \subset \mathcal{V}_1^h \times \mathcal{V}_2^h$ and $n = 0, 1, \ldots, N-1$.

Remarks

(1) Note, from (50), that the continuity of displacement in time is enforced through the strain energy inner product.

(2) The τ_1, τ_2, and s terms represent least-squares contributions added to the Galerkin variational equation. The traction continuity term requires a nonstandard assembly algorithm in that elements must take account of their (spatial) neighbors. The formulation may also be viewed as a Petrov–Galerkin method, although currently we do not favor this terminology (see [8] for elaboration).

(3) Velocities and displacements are governed by independent interpolations.

(4) The τ_2 terms involve second derivatives of the displacement interpolation functions. If linear simplex elements are employed, these terms drop out. If, in addition, the velocity is assumed constant in time, then $\mathcal{L}_2\boldsymbol{U}^h = \mathcal{L}_2\boldsymbol{W}^h = \boldsymbol{0}$.

(5) In the nonlinear case the meaning of $\boldsymbol{\sigma}(\nabla\mathcal{L}_1\boldsymbol{U}^h)$ is unclear. However, if we select the velocity space such that

$$\mathcal{L}_1\boldsymbol{U}^h = \dot{\boldsymbol{u}}_1^h - \boldsymbol{u}_2^h = \boldsymbol{0}\,, \qquad \mathcal{L}_1\boldsymbol{W}^h = \dot{\boldsymbol{w}}_1^h - \boldsymbol{w}_2^h = \boldsymbol{0}\,, \tag{53}$$

then the formalism is readily extendable.

4. Convergence analysis

For purposes of analysis it is convenient to sum (44) over the time slabs. After rearranging terms, we obtain

$$B(\boldsymbol{W}^h, \boldsymbol{U}^h) = L(\boldsymbol{W}^h)\,, \tag{54}$$

where

$$
\begin{aligned}
B(\boldsymbol{W}^h, \boldsymbol{U}^h) = \sum_{n=0}^{N-1} \{ & (\boldsymbol{w}_2^h, \rho \dot{\boldsymbol{u}}_2^h)_{Q_n} + a(\boldsymbol{w}_2^h, \boldsymbol{u}_1^h)_{Q_n} + a(\boldsymbol{w}_1^h, \mathscr{L}_1 \boldsymbol{U}^h)_{\tilde{Q}_n} \\
& + (\rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \mathscr{L}_2 \boldsymbol{U}^h)_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{W}^h, \mathscr{L}_1 \boldsymbol{U}^h)_{\tilde{Q}_n} \\
& + (\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h)(\boldsymbol{x})]\!])_{Y_n} \\
& + (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_1^h))_{Z_n} \} \\
& + \sum_{n=1}^{N-1} \{ (\boldsymbol{w}_2^h(t_n^+), \rho [\![\boldsymbol{u}_2^h(t_n)]\!])_\Omega + a(\boldsymbol{w}_1^h(t_n^+), [\![\boldsymbol{u}_1^h(t_n)]\!])_\Omega \} \\
& + (\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{u}_2^h(0^+))_\Omega + a(\boldsymbol{w}_1^h(0^+), \boldsymbol{u}_1^h(0^+))_\Omega ,
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
L(\boldsymbol{W}^h) = \sum_{n=0}^{N-1} \{ & (\boldsymbol{w}_2^h, \boldsymbol{f})_{Q_n} + (\boldsymbol{w}_2^h, \boldsymbol{h})_{Z_n} \\
& + (\rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \boldsymbol{f})_{\tilde{Q}_n} + (\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h)(\boldsymbol{x})]\!], \boldsymbol{h})_{Z_n} \} \\
& + (\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{v}_0)_\Omega + a(\boldsymbol{w}_1^h(0^+), \boldsymbol{u}_0)_\Omega .
\end{aligned}
\tag{56}
$$

By (50), an exact solution also satisfies the variational equation:

$$
B(\boldsymbol{W}^h, \boldsymbol{U}) = L(\boldsymbol{W}^h) . \tag{57}
$$

Consequently,

$$
B(\boldsymbol{W}^h, \boldsymbol{E}) = 0 \quad \forall \boldsymbol{W}^h \in \mathscr{V}_1^h \times \mathscr{V}_2^h . \tag{58}
$$

This is the *consistency condition*.

The *total energy* is given by:

$$
\mathscr{E}(\boldsymbol{W}^h) = \tfrac{1}{2} (\boldsymbol{w}_2^h, \rho \boldsymbol{w}_2^h)_\Omega + \tfrac{1}{2} a(\boldsymbol{w}_1^h, \boldsymbol{w}_1^h)_\Omega . \tag{59}
$$

LEMMA 4.1.

$$
\begin{aligned}
& \mathscr{E}(\boldsymbol{W}^h(T^-)) + \sum_{n=1}^{N-1} \mathscr{E}([\![\boldsymbol{W}^h(t_n)]\!]) + \mathscr{E}(\boldsymbol{W}^h(0^+)) \\
& = \sum_{n=0}^{N-1} (\boldsymbol{w}_2^h, \rho \dot{\boldsymbol{w}}_2^h)_{Q_n} + \sum_{n=1}^{N-1} (\boldsymbol{w}_2^h(t_n^+), \rho [\![\boldsymbol{w}_2^h(t_n)]\!])_\Omega \\
& \quad + (\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{w}_2^h(0^+))_\Omega + \sum_{n=0}^{N-1} a(\boldsymbol{w}_1^h, \dot{\boldsymbol{w}}_1^h)_{Q_n} \\
& \quad + \sum_{n=1}^{N-1} a(\boldsymbol{w}_1^h(t_n^+), [\![\boldsymbol{w}_1^h(t_n)]\!])_\Omega + a(\boldsymbol{w}_1^h(0^+), \boldsymbol{w}_1^h(0^+))_\Omega .
\end{aligned}
\tag{60}
$$

PROOF.

$$\sum_{n=0}^{N-1} (\boldsymbol{w}_2^h, \rho \dot{\boldsymbol{w}}_2^h)_{Q_n} = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (\boldsymbol{w}_2^h, \rho \dot{\boldsymbol{w}}_2^h)_\Omega \, \mathrm{d}t$$

$$= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{w}_2^h, \rho \boldsymbol{w}_2^h)_\Omega \, \mathrm{d}t$$

$$= \tfrac{1}{2}(\boldsymbol{w}_2^h(T^-), \rho \boldsymbol{w}_2^h(T^-))_\Omega - \tfrac{1}{2}(\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{w}_2^h(0^+))_\Omega$$

$$- \tfrac{1}{2} \sum_{n=1}^{N-1} \{(\boldsymbol{w}_2^h(t_n^+), \rho \boldsymbol{w}_2^h(t_n^+))_\Omega - (\boldsymbol{w}_2^h(t_n^-), \rho \boldsymbol{w}_2^h(t_n^-))_\Omega\} . \tag{61}$$

Therefore,

$$\sum_{n=0}^{N-1} (\boldsymbol{w}_2^h, \rho \dot{\boldsymbol{w}}_2^h)_{Q_n} + \sum_{n=1}^{N-1} (\boldsymbol{w}_2^h(t_n^+), \rho [\![\boldsymbol{w}_2^h(t_n)]\!])_\Omega + (\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{w}_2^h(0^+))_\Omega$$

$$= \tfrac{1}{2}(\boldsymbol{w}_2^h(T^-), \rho \boldsymbol{w}_2^h(T^-))_\Omega + \tfrac{1}{2}(\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{w}_2^h(0^+))_\Omega$$

$$+ \sum_{n=1}^{N-1} \{(\boldsymbol{w}_2^h(t_n^+), \rho \boldsymbol{w}_2^h(t_n^+))_\Omega - (\boldsymbol{w}_2^h(t_n^+), \rho \boldsymbol{w}_2^h(t_n^-))_\Omega - \tfrac{1}{2}(\boldsymbol{w}_2^h(t_n^+), \rho \boldsymbol{w}_2^h(t_n^+))_\Omega$$

$$+ \tfrac{1}{2}(\boldsymbol{w}_2^h(t_n^-), \rho \boldsymbol{w}_2^h(t_n^-))_\Omega\}$$

$$= \tfrac{1}{2}(\boldsymbol{w}_2^h(T^-), \rho \boldsymbol{w}_2^h(T^-))_\Omega + \tfrac{1}{2}(\boldsymbol{w}_2^h(0^+), \rho \boldsymbol{w}_2^h(0^+))_\Omega + \tfrac{1}{2} \sum_{n=1}^{N-1} ([\![\boldsymbol{w}_2^h(t_n)]\!], \rho [\![\boldsymbol{w}_2^h(t_n)]\!])_\Omega . \tag{62}$$

Proceeding in similar fashion,

$$\sum_{n=0}^{N-1} a(\boldsymbol{w}_1^h, \dot{\boldsymbol{w}}_1^h)_{Q_n} + \sum_{n=1}^{N-1} a(\boldsymbol{w}_1^h(t_n^+), [\![\boldsymbol{w}_1^h(t_n)]\!])_\Omega + a(\boldsymbol{w}_1^h(0^+), \boldsymbol{w}_1^h(0^+))_\Omega$$

$$= \tfrac{1}{2} a(\boldsymbol{w}_1^h(T^-), \boldsymbol{w}_1^h(T^-))_\Omega + \tfrac{1}{2} a(\boldsymbol{w}_1^h(0^+), \boldsymbol{w}_1^h(0^+))_\Omega + \tfrac{1}{2} \sum_{n=1}^{N-1} a([\![\boldsymbol{w}_1^h(t_n)]\!], [\![\boldsymbol{w}_1^h(t_n)]\!])_\Omega . \tag{63}$$

Combining (62) and (63) completes the proof. □

The norm in which we will prove convergence is defined by:

$$|||\boldsymbol{W}^h|||^2 = \mathscr{E}(\boldsymbol{W}^h(T^-)) + \mathscr{E}(\boldsymbol{W}^h(0^+)) + \sum_{n=1}^{N-1} \mathscr{E}([\![\boldsymbol{W}^h(t_n)]\!])$$

$$+ \sum_{n=0}^{N-1} \{(\rho^{-1} \boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{W}^h, \mathscr{L}_2 \boldsymbol{W}^h)_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{W}^h, \mathscr{L}_1 \boldsymbol{W}^h)_{\tilde{Q}_n}$$

$$+ (\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h)(\boldsymbol{x})]\!])_{Y_n}$$

$$+ (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^n), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{w}_1^h))_{Z_n}\} . \tag{64}$$

LEMMA 4.2.

$$B(\boldsymbol{W}^h, \boldsymbol{W}^h) = |||\boldsymbol{W}^h|||^2 . \tag{65}$$

This is the stability condition.

PROOF. The result follows from (55) and Lemma 4.1. □

Let $\tilde{\boldsymbol{U}}^h \in \mathcal{S}_1^h \times \mathcal{S}_2^h$ denote an interpolant of $\boldsymbol{U}$. Then,

$$\boldsymbol{E} = \boldsymbol{E}^h + \boldsymbol{H} , \tag{66}$$

where

$$\boldsymbol{E}^h = \boldsymbol{U}^h - \tilde{\boldsymbol{U}}^h \in \mathcal{V}_1^h \times \mathcal{V}_2^h , \tag{67}$$

$$\boldsymbol{H} = \tilde{\boldsymbol{U}}^h - \boldsymbol{U} \quad \text{(interpolation error)}. \tag{68}$$

In components,

$$\boldsymbol{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\} , \tag{69}$$

$$\boldsymbol{E}^h = \{\boldsymbol{e}_1^h, \boldsymbol{e}_2^h\} , \tag{70}$$

$$\boldsymbol{H} = \{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2\} . \tag{71}$$

LEMMA 4.3.

$$\begin{aligned}&\sum_{n=0}^{N-1} (\boldsymbol{e}_2^h, \rho\dot{\boldsymbol{\eta}}_2)_{Q_n} + \sum_{n=1}^{N-1} (\boldsymbol{e}_2^h(t_n^+), \rho[\![\boldsymbol{\eta}_2(t_n)]\!])_\Omega + (\boldsymbol{e}_2^h(0^+), \rho\boldsymbol{\eta}_2(0^+))_\Omega \\ &\quad = -\sum_{n=0}^{N-1} (\dot{\boldsymbol{e}}_2^h, \rho\boldsymbol{\eta}_2)_{Q_n} + (\boldsymbol{e}_2^h(T^-), \rho\boldsymbol{\eta}_2(T^-))_\Omega - \sum_{n=1}^{N-1} ([\![\boldsymbol{e}_2^h(t_n)]\!], \rho\boldsymbol{\eta}_2(t_n^-))_\Omega .\end{aligned} \tag{72}$$

PROOF.

$$\begin{aligned}(\boldsymbol{e}_2^h, \rho\dot{\boldsymbol{\eta}}_2)_{Q_n} + (\boldsymbol{e}_2^h(t_n^+), \rho[\![\boldsymbol{\eta}_2(t_n)]\!])_\Omega = &-(\dot{\boldsymbol{e}}_2^h, \rho\boldsymbol{\eta}_2)_{Q_n} + (\boldsymbol{e}_2^h(t_{n+1}^-), \rho\boldsymbol{\eta}_2(t_{n+1}^-))_\Omega \\ &- (\boldsymbol{e}_2^h(t_n^+), \rho\boldsymbol{\eta}_2(t_n^+))_\Omega + (\boldsymbol{e}_2^h(t_n^+), \rho\boldsymbol{\eta}_2(t_n^+))_\Omega \\ &- (\boldsymbol{e}_2^h(t_n^+), \rho\boldsymbol{\eta}_2(t_n^-))_\Omega ,\end{aligned} \tag{73}$$

$$\begin{aligned}&\sum_{n=0}^{N-1} (\boldsymbol{e}_2^h, \rho\dot{\boldsymbol{\eta}}_2)_{Q_n} + \sum_{n=1}^{N-1} (\boldsymbol{e}_2^h(t_n^+), \rho[\![\boldsymbol{\eta}_2(t_n)]\!])_\Omega + (\boldsymbol{e}_2^h(0^+), \rho\boldsymbol{\eta}_2(0^+))_\Omega \\ &\quad = \sum_{n=1}^{N-1} \{(\boldsymbol{e}_2^h, \rho\dot{\boldsymbol{\eta}}_2)_{Q_n} + (\boldsymbol{e}_2^h(t_n^+), \rho[\![\boldsymbol{\eta}_2(t_n)]\!])_\Omega\} + (\boldsymbol{e}_2^h, \rho\dot{\boldsymbol{\eta}}_2)_{Q_0} + (\boldsymbol{e}_2^h(0^+), \rho\boldsymbol{\eta}_2(0^+))_\Omega\end{aligned}$$

$$= \sum_{n=1}^{N-1} \{-(\dot{e}_2^h, \rho\boldsymbol{\eta}_2)_{Q_n} + (e_2^h(t_{n+1}^-), \rho\boldsymbol{\eta}_2(t_{n+1}^-))_\Omega - (e_2^h(t_n^+), \rho\boldsymbol{\eta}_2(t_n^-))_\Omega\} - (\dot{e}_2^h, \rho\boldsymbol{\eta}_2)_{Q_0}$$
$$+ (e_2^h(t_1^-), \rho\boldsymbol{\eta}_2(t_1^-))_\Omega - (e_2^h(0^+), \rho\boldsymbol{\eta}_2(0^+))_\Omega + (e_2^h(0^+), \rho\boldsymbol{\eta}_2(0^+))_\Omega$$
$$= -\sum_{n=0}^{N-1} (\dot{e}_2^h, \rho\boldsymbol{\eta}_2)_{Q_n} + (e_2^h(T^-), \rho\boldsymbol{\eta}_2(T^-))_\Omega - \sum_{n=1}^{N-1} ([\![e_2^h(t_n)]\!], \rho\boldsymbol{\eta}_2(t_n^-))_\Omega\,. \quad \square \tag{74}$$

LEMMA 4.4.

$$\sum_{n=0}^{N-1} a(e_1^h, \dot{\boldsymbol{\eta}}_1)_{Q_n} + \sum_{n=1}^{N-1} a(e_1^h(t_n^+), [\![\boldsymbol{\eta}_1(t_n)]\!])_\Omega + a(e_1^h(0^+), \boldsymbol{\eta}_1(0^+))_\Omega$$
$$= -\sum_{n=0}^{N-1} a(\dot{e}_1^h, \boldsymbol{\eta}_1)_{Q_n} + a(e_1^h(T^-), \boldsymbol{\eta}_1(T^-))_\Omega - \sum_{n=1}^{N-1} a([\![e_1^h(t_n)]\!], \boldsymbol{\eta}_1(t_n^-))_\Omega\,. \tag{75}$$

PROOF. The steps are identical to those for Lemma 4.3, and so are omitted. □

LEMMA 4.5.

$$(\dot{e}_2^h, \rho\boldsymbol{\eta}_2)_{Q_n} + a(e_1^h, \boldsymbol{\eta}_2)_{Q_n}$$
$$= (\mathscr{L}_2\boldsymbol{E}^h, \boldsymbol{\eta}_2)_{\tilde{Q}_n} + (\boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla e_1^h)(\boldsymbol{x})]\!], \boldsymbol{\eta}_2)_{Y_n} + (\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla e_1^h), \boldsymbol{\eta}_2)_{Z_n}\,. \tag{76}$$

PROOF. By integration-by-parts and the divergence theorem,

$$a(e_1^h, \boldsymbol{\eta}_2)_{Q_n} = -(\nabla\cdot\boldsymbol{\sigma}(\nabla e_1^h), \boldsymbol{\eta}_2)_{\tilde{Q}_n} + (\boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla e_1^h)(\boldsymbol{x})]\!], \boldsymbol{\eta}_2)_{Y_n} + (\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla e_1^h), \boldsymbol{\eta}_2)_{Z_n}\,. \tag{77}$$

Thus

$$(\dot{e}_2^h, \rho\boldsymbol{\eta}_2)_{Q_n} + a(e_1^h, \boldsymbol{\eta}_2)_{Q_n} = (\rho\dot{e}_2^h - \nabla\cdot\boldsymbol{\sigma}(\nabla e_1^h), \boldsymbol{\eta}_2)_{\tilde{Q}_n} + (\boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla e_1^h)(\boldsymbol{x})]\!], \boldsymbol{\eta}_2)_{Y_n}$$
$$+ (\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla e_1^h), \boldsymbol{\eta}_2)_{Z_n}\,. \tag{78}$$

Employing (49) completes the proof. □

LEMMA 4.6.

$$\sum_{n=1}^{N-1} \{-([\![e_2^h(t_n)]\!], \rho\boldsymbol{\eta}_2(t_n^-))_\Omega - a([\![e_1^h(t_n)]\!], \boldsymbol{\eta}_1(t_n^-))_\Omega\} + (e_2^h(T^-), \rho\boldsymbol{\eta}_2(T^-))_\Omega$$
$$+ a(e_1^h(T^-), \boldsymbol{\eta}_1(T^-))_\Omega$$
$$\leq \tfrac{1}{2}\Big[\sum_{n=1}^{N-1} \{\mathscr{E}([\![\boldsymbol{E}^h(t_n)]\!]) + 4\mathscr{E}(\boldsymbol{H}(t_n^-))\} + \mathscr{E}(\boldsymbol{E}^h(T^-)) + 4\mathscr{E}(\boldsymbol{H}(T^-))\Big]\,. \tag{79}$$

PROOF.

$$\begin{aligned}
&\sum_{n=1}^{N-1} \{-([\![e_2^h(t_n)]\!], \rho\boldsymbol{\eta}_2(t_n^-))_\Omega - a([\![e_1^h(t_n)]\!], \boldsymbol{\eta}_1(t_n^-))_\Omega\} + (e_2^h(T^-), \rho\boldsymbol{\eta}_2(T^-))_\Omega \\
&\quad + a(e_1^h(T^-), \boldsymbol{\eta}_1(T^-))_\Omega \\
&\quad \leq \tfrac{1}{2}\Big[\sum_{n=1}^{N-1} \{\tfrac{1}{2}([\![e_2^h(t_n)]\!], \rho[\![e_2^h(t_n)]\!])_\Omega + 2(\boldsymbol{\eta}_2(t_n^-), \rho\boldsymbol{\eta}_2(t_n^-))_\Omega \\
&\qquad\qquad + \tfrac{1}{2}a([\![e_1^h(t_n)]\!], [\![e_1^h(t_n)]\!])_\Omega + 2a(\boldsymbol{\eta}_1(t_n^-), \boldsymbol{\eta}_1(t_n^-))_\Omega\} \\
&\qquad + \tfrac{1}{2}(e_2^h(T^-), \rho e_2^h(T^-))_\Omega + 2(\boldsymbol{\eta}_2(T^-), \rho\boldsymbol{\eta}_2(T^-))_\Omega \\
&\qquad + \tfrac{1}{2}a(e_1^h(T^-), e_1^h(T^-))_\Omega + 2a(\boldsymbol{\eta}_1(T^-), \boldsymbol{\eta}_1(T^-))_\Omega \Big] \\
&\quad = \tfrac{1}{2}\Big[\sum_{n=1}^{N-1} \{\mathscr{E}([\![\boldsymbol{E}^h(t_n)]\!]) + 4\mathscr{E}(\boldsymbol{H}(t_n^-))\} + \mathscr{E}(\boldsymbol{E}^h(T^-)) + 4\mathscr{E}(\boldsymbol{H}(T^-)) \Big]. \quad \square \qquad (80)
\end{aligned}$$

4.1. Interpolation estimates

Let $|\cdot|$ denote any matrix norm. We assume $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s satisfy

$$c_1 h^\alpha \leq |\boldsymbol{\tau}_1| \leq c_2 h^\alpha\ , \tag{81}$$

$$c_1 h^\beta \leq |\boldsymbol{\tau}_2| \leq c_2 h^\beta\ , \tag{82}$$

$$c_3 h^\gamma \leq |s| \leq c_4 h^\gamma\ , \tag{83}$$

where c_1, c_2, c_3, and c_4 are positive constants.

Let $m = \max\{k, l\}$, and let $H^{m+1}(Q)$ denote the Sobolev space of functions that possesses $m+1$ square-integrable generalized derivatives. If $\boldsymbol{U} \in (H^{m+1}(Q))^{2d}$, then $\boldsymbol{H} = \{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ satisfies the following estimates:

$$\sum_{n=0}^{N-1} (\rho\boldsymbol{\tau}_2^{-1}\boldsymbol{\eta}_2, \boldsymbol{\eta}_2)_{Q_n} \leq C(\boldsymbol{U}) h^{2l+2-\beta}\ , \tag{84}$$

$$\sum_{n=0}^{N-1} a(\boldsymbol{\tau}_1^{-1}\boldsymbol{\eta}_1, \boldsymbol{\eta}_1)_{Q_n} \leq C(\boldsymbol{U}) h^{2k-\alpha}\ , \tag{85}$$

$$\sum_{n=0}^{N-1} (\rho^{-1}\boldsymbol{\tau}_2\mathscr{L}_2\boldsymbol{H}, \mathscr{L}_2\boldsymbol{H})_{\tilde{Q}_n} \leq C(\boldsymbol{U}) h^{\min(2k-2+\beta, 2l+\beta)}\ , \tag{86}$$

$$\sum_{n=0}^{N-1} a(\rho^{-1}\boldsymbol{\tau}_1\mathscr{L}_1\boldsymbol{H}, \mathscr{L}_1\boldsymbol{H})_{\tilde{Q}_n} \leq C(\boldsymbol{U}) h^{\min\{2k-2+\alpha, 2l+\alpha\}}\ , \tag{87}$$

$$\sum_{n=0}^{N-1} (\rho s^{-1}\boldsymbol{\eta}_2, \boldsymbol{\eta}_2)_{Y_n \cup Z_n} \leq C(\boldsymbol{U})h^{2l+1-\gamma}, \tag{88}$$

$$\sum_{n=0}^{N-1} \{(\rho^{-1}s\boldsymbol{n}\cdot [\![\boldsymbol{\sigma}(\nabla\boldsymbol{\eta}_1)(\boldsymbol{x})]\!], \boldsymbol{n}\cdot [\![\boldsymbol{\sigma}(\nabla\boldsymbol{\eta}_1)(\boldsymbol{x})]\!])_{Y_n} + (\rho^{-1}s\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{\eta}_1), \boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{\eta}_1))_{Z_n}\}$$
$$\leq C(\boldsymbol{U})h^{2k-1+\gamma}, \tag{89}$$

$$\mathscr{E}(\boldsymbol{H}(T^-)) + \sum_{n=1}^{N-1} \{\mathscr{E}(\boldsymbol{H}(t_n^-)) + \mathscr{E}(\boldsymbol{H}(t_n^+))\} + \mathscr{E}(\boldsymbol{H}(0^+)) \leq C(\boldsymbol{U})h^{\min\{2k-1,2l+1\}}, \tag{90}$$

where $C(\boldsymbol{U})$ is independent of h and may take on different values in the preceding and subsequent inequalities.

THEOREM 4.7. Assume $\alpha = \beta = 1$, *and* $\gamma = 0$. *Then,*

$$|||\boldsymbol{E}|||^2 \leq C(\boldsymbol{U})h^{\min\{2k-1,2l+1\}}. \tag{91}$$

PROOF.

$$\begin{aligned}
|||\boldsymbol{E}^h|||^2 &= B(\boldsymbol{E}^h, \boldsymbol{E}^h) \qquad \text{(by the stability condition, Lemma 4.2)} \\
&= B(\boldsymbol{E}^h, \boldsymbol{E} - \boldsymbol{H}) \quad \text{(by (66))} \\
&= -B(\boldsymbol{E}^h, \boldsymbol{H}) \qquad \text{(by the consistency condition, (58))} \\
&\leq |B(\boldsymbol{E}^h, \boldsymbol{H})| \\
&= \Bigg| \sum_{n=0}^{N-1} \{(\boldsymbol{e}_2^h, \rho\dot{\boldsymbol{\eta}}_2)_{Q_n} + a(\boldsymbol{e}_2^h, \boldsymbol{\eta}_1)_{Q_n} + a(\boldsymbol{e}_1^h, \mathscr{L}_1\boldsymbol{H})_{\tilde{Q}_n} \\
&\qquad + (\rho^{-1}\boldsymbol{\tau}_2\mathscr{L}_2\boldsymbol{E}^h, \mathscr{L}_2\boldsymbol{H})_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1\mathscr{L}_1\boldsymbol{E}^h, \mathscr{L}_1\boldsymbol{H})_{\tilde{Q}_n} \\
&\qquad + (\rho^{-1}s\boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla\boldsymbol{e}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla\boldsymbol{\eta}_1)(\boldsymbol{x})]\!])_{Y_n} \\
&\qquad + (\rho^{-1}s\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{e}_1^h), \boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{\eta}_1))_{Z_n}\} \\
&\quad + \sum_{n=1}^{N-1} \{(\boldsymbol{e}_2^h(t_n^+), \rho[\![\boldsymbol{\eta}_2(t_n)]\!])_\Omega + a(\boldsymbol{e}_1^h(t_n^+), [\![\boldsymbol{\eta}_1(t_n)]\!])_\Omega\} \\
&\quad + (\boldsymbol{e}_2^h(0^+), \rho\boldsymbol{\eta}_2(0^+))_\Omega + a(\boldsymbol{e}_1^h(0^+), \boldsymbol{\eta}_1(0^+))_\Omega \Bigg| \\
&\qquad\qquad\qquad\qquad \text{(by definition of } \boldsymbol{B}\text{, (55))} \\
&= \Bigg| \sum_{n=0}^{N-1} \{-(\mathscr{L}_2\boldsymbol{E}^h, \boldsymbol{\eta}_2)_{\tilde{Q}_n} - a(\mathscr{L}_1\boldsymbol{E}^h, \boldsymbol{\eta}_1)_{\tilde{Q}_n}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + (\rho^{-1}\boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{E}^h, \mathscr{L}_2 \boldsymbol{H})_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{E}^h, \mathscr{L}_1 \boldsymbol{H})_{\tilde{Q}_n} \\
&\qquad - (\boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h)(\boldsymbol{x})]\!], \boldsymbol{\eta}_2)_{Y_n} - (\boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h), \boldsymbol{\eta}_2)_{Z_n} \\
&\qquad + (\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{\eta}_1)(\boldsymbol{x})]\!])_{Y_n} \\
&\qquad + (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{\eta}_1))_{Z_n} \} \\
&\quad + \sum_{n=1}^{N-1} \{ -([\![\boldsymbol{e}_2^h(t_n)]\!], \rho \boldsymbol{\eta}_2(t_n^-))_\Omega - a([\![\boldsymbol{e}_1^h(t_n)]\!], \boldsymbol{\eta}_1(t_n^-))_\Omega \} \\
&\quad + (\boldsymbol{e}_2^h(T^-), \rho \boldsymbol{\eta}_2(T^-))_\Omega + a(\boldsymbol{e}_1^h(T^-), \boldsymbol{\eta}_1(T^-))_\Omega \Big| \\
&\hspace{20em} \text{(by Lemmas 4.3, 4.4, and 4.5)} \\
&\leqslant \sum_{n=0}^{N-1} \{ \tfrac{1}{4}(\rho^{-1}\boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{E}^h, \mathscr{L}_2 \boldsymbol{E}^h)_{\tilde{Q}_n} + (\rho \boldsymbol{\tau}_2^{-1} \boldsymbol{\eta}_2, \boldsymbol{\eta}_2)_{\tilde{Q}_n} \\
&\qquad + \tfrac{1}{4} a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{E}^h, \mathscr{L}_1 \boldsymbol{E}^h)_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1^{-1} \boldsymbol{\eta}_1, \boldsymbol{\eta}_1)_{\tilde{Q}_n} \\
&\qquad + \tfrac{1}{4}(\rho^{-1}\boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{E}^h, \mathscr{L}_2 \boldsymbol{E}^h)_{\tilde{Q}_n} + (\rho^{-1}\boldsymbol{\tau}_2 \mathscr{L}_2 \boldsymbol{H}, \mathscr{L}_2 \boldsymbol{H})_{\tilde{Q}_n} \\
&\qquad + \tfrac{1}{4} a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{E}^h, \mathscr{L}_1 \boldsymbol{E}^h)_{\tilde{Q}_n} + a(\boldsymbol{\tau}_1 \mathscr{L}_1 \boldsymbol{H}, \mathscr{L}_1 \boldsymbol{H})_{\tilde{Q}_n} \\
&\qquad + \tfrac{1}{4}(\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h)(\boldsymbol{x})]\!])_{Y_n} + (\rho s^{-1} \boldsymbol{\eta}_2, \boldsymbol{\eta}_2)_{Y_n} \\
&\qquad + \tfrac{1}{4}(\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h))_{Z_n} + (\rho s^{-1} \boldsymbol{\eta}_2, \boldsymbol{\eta}_2)_{Z_n} \\
&\qquad + \tfrac{1}{4}(\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h)(\boldsymbol{x})]\!])_{Y_n} \\
&\qquad + (\rho^{-1} s \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{\eta}_1)(\boldsymbol{x})]\!], \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\nabla \boldsymbol{\eta}_1)(\boldsymbol{x})]\!])_{Y_n} \\
&\qquad + \tfrac{1}{4}(\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{e}_1^h))_{Z_n} \\
&\qquad + (\rho^{-1} s \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{\eta}_1), \boldsymbol{n} \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{\eta}_1))_{Z_n} \} \\
&\quad + \tfrac{1}{2} \sum_{n=1}^{N-1} \mathscr{E}([\![\boldsymbol{E}^h(t_n)]\!]) + 2 \sum_{n=1}^{N-1} \mathscr{E}(\boldsymbol{H}(t_n^-)) + \tfrac{1}{2}\mathscr{E}(\boldsymbol{E}^h(T^-)) + 2\mathscr{E}(\boldsymbol{H}(T^-)) \\
&\hspace{20em} \text{(by Lemma 4.6)}\,. \qquad (92)
\end{aligned}
$$

The terms involving $\boldsymbol{E}^h$ may be subsumed by the left-hand side. The interpolation estimates with $\alpha = \beta = 1$ and $\gamma = 0$ then yield

$$
|||\boldsymbol{E}^h|||^2 \leqslant C(U) h^{\min\{2k-1, 2l+1\}}\,. \tag{93}
$$

Likewise,

$$|||\boldsymbol{H}|||^2 \leqslant C(\boldsymbol{U})h^{\min\{2k-1,2l+1\}} . \tag{94}$$

By the triangle inequality,

$$|||\boldsymbol{E}|||^2 \leqslant 2(|||\boldsymbol{E}^h|||^2 + |||\boldsymbol{H}|||^2) , \tag{95}$$

which completes the proof. □

Remarks

(1) The convergence rate is optimal for the $|||\cdot|||$-norm.
(2) We conjecture that, due to the presence of the $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s terms, 'localization results' should be able to be proved for the proposed methods (see [16, 25]).
(3) In the parlance of ordinary differential equation algorithms, the methods proposed are unconditionally stable.
(4) The analysis of semidiscrete formulations for classical elastodynamics is considered in [11, 28]. The latter reference addresses the two-field case.
(5) A generalization of the formulation to spatial differential operators of order $2m$, leads to error estimates of the form (91) in which k is replaced by $k+1-m$.
(6) Two practical definitions of the $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s matrices, that satisfy requirements (81)–(83), are:

(a) $$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \Delta x s = \tfrac{1}{2}\,\frac{\Delta x}{c}\,\boldsymbol{I} , \tag{96}$$

(b) $$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \Delta x s = \tfrac{1}{2}\,\Delta t\boldsymbol{I} , \tag{97}$$

where $\boldsymbol{I}$ is the $d \times d$ identity matrix. We prefer the first choice because, unlike the second, solutions are independent of Δt. For variable element size, Δx and Δt in (96) and (97) are to be interpreted as *local values*.

5. Simplified formulations

The general formulation is considerably more complex than existing semidiscrete algorithms. Therefore it is of interest to consider various simplifications.

Single-field formulation

Assume

$$\mathscr{L}_1\boldsymbol{U}^h = \dot{\boldsymbol{u}}_1^h - \boldsymbol{u}_2^h = \boldsymbol{0} , \qquad \mathscr{L}_1\boldsymbol{W}^h = \dot{\boldsymbol{w}}_1^h - \boldsymbol{w}_2^h = \boldsymbol{0} , \tag{98}$$

and let

$$\boldsymbol{\tau} = \boldsymbol{\tau}_2 , \tag{99}$$

$$\mathscr{L}\boldsymbol{u}^h = \rho\ddot{\boldsymbol{u}}^h - \nabla\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{u}^h) . \tag{100}$$

With these, (45) and (46) become:

$$
\begin{aligned}
b_n(\boldsymbol{w}^h, \boldsymbol{u}^h) = {} & (\dot{\boldsymbol{w}}^h, \rho\ddot{\boldsymbol{u}}^h)_{Q_n} + a(\dot{\boldsymbol{w}}^h, \boldsymbol{u}^h)_{Q_n} + (\rho^{-1}\boldsymbol{\tau}\mathscr{L}\boldsymbol{w}^h, \mathscr{L}\boldsymbol{u}^h)_{\tilde{Q}_n} \\
& + (\rho^{-1}s\boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h)(\boldsymbol{x})]\!], \boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\nabla\boldsymbol{u}^h)(\boldsymbol{x})]\!])_{Y_n} \\
& + (\rho^{-1}s\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h), \boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{u}^h))_{Z_n} \\
& + (\dot{\boldsymbol{w}}^h(t_n^+), \rho\dot{\boldsymbol{u}}^h(t_n^+))_\Omega + a(\boldsymbol{w}^h(t_n^+), \boldsymbol{u}^h(t_n^+))_\Omega ,
\end{aligned} \tag{101}
$$

$$
\begin{aligned}
l_n(\boldsymbol{w}^h) = {} & (\dot{\boldsymbol{w}}^h, \boldsymbol{f})_{Q_n} + (\dot{\boldsymbol{w}}^h, \boldsymbol{h})_{Z_n} + (\rho^{-1}\boldsymbol{\tau}\mathscr{L}\boldsymbol{w}^h, \boldsymbol{f})_{\tilde{Q}_n} \\
& + (\rho^{-1}s\boldsymbol{n}\cdot\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h)(\boldsymbol{x}), \boldsymbol{h})_{Z_n} \\
& + (\dot{\boldsymbol{w}}^h(t_n^+), \rho\dot{\boldsymbol{u}}^h(t_n^-))_\Omega + a(\boldsymbol{w}^h(t_n^+), \boldsymbol{u}^h(t_n^-))_\Omega .
\end{aligned} \tag{102}
$$

The convergence theorem with $l = k - 1$ applies to this formulation. The least-squares terms complicate implementation compared with in-place methodology. If we omit all these terms we obtain time-discontinuous Galerkin formulations:

Two-field Galerkin formulation

$$
\begin{aligned}
B_n(\boldsymbol{W}^h, \boldsymbol{U}^h) = {} & (\boldsymbol{w}_2^h, \rho\dot{\boldsymbol{u}}_2^h)_{Q_n} + a(\boldsymbol{w}_2^h, \boldsymbol{u}_1^h)_{Q_n} + a(\boldsymbol{w}_1^h, \mathscr{L}_1\boldsymbol{U}^h)_{\tilde{Q}_n} \\
& + (\boldsymbol{w}_2^h(t_n^+), \rho\boldsymbol{u}_2^h(t_n^+))_\Omega + a(\boldsymbol{w}_1^h(t_n^+), \boldsymbol{u}_1^h(t_n^+))_\Omega ,
\end{aligned} \tag{103}
$$

$$
L_n(\boldsymbol{W}^h) = (\boldsymbol{w}_2^h, \boldsymbol{f})_\Omega + (\boldsymbol{w}_2^h, \boldsymbol{h})_{Z_n} + (\boldsymbol{w}_2^h(t_n^+), \rho\boldsymbol{u}_2^h(t_n^-))_\Omega + a(\boldsymbol{w}_1^h(t_n^+), \boldsymbol{u}_1^h(t_n^-))_\Omega . \tag{104}
$$

Single-field Galerkin formulation

$$
\begin{aligned}
b_n(\boldsymbol{w}^h, \boldsymbol{u}^h) = {} & (\dot{\boldsymbol{w}}^h, \rho\ddot{\boldsymbol{u}}^h)_{Q_n} + a(\dot{\boldsymbol{w}}^h, \boldsymbol{u}^h)_{Q_n} \\
& + (\dot{\boldsymbol{w}}^h(t_n^+), \rho\dot{\boldsymbol{u}}^h(t_n^+))_\Omega + a(\boldsymbol{w}^h(t_n^+), \boldsymbol{u}^h(t_n^+))_\Omega ,
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
l_n(\boldsymbol{w}^h) = {} & (\dot{\boldsymbol{w}}^h, \boldsymbol{f})_{Q_n} + (\dot{\boldsymbol{w}}^h, \boldsymbol{h})_{Z_n} \\
& + (\dot{\boldsymbol{w}}^h(t_n^+), \rho\dot{\boldsymbol{u}}^h(t_n^-))_\Omega + a(\boldsymbol{w}^h(t_n^+), \boldsymbol{u}^h(t_n^-))_\Omega .
\end{aligned} \tag{106}
$$

The Galerkin formulations are not encompassed by the convergence theorem of the previous section. This does not mean necessarily that they are not convergent. In fact, we have had a number of successful experiences with the single-field formulation, and with equal-order interpolation in conjunction with the two-field formulation. On the other hand, with $l > k$, we have experienced divergence of the two-field formulation. These results suggest that it may be possible to prove convergence of the Galerkin formulations for *particular* interpolations.

Strongly enforcing continuity in time eliminates stability provided by the weak continuity terms. Nevertheless, we have examined some difference equations arising from strong enforcement and found them to correspond to consistent, unconditionally stable semidiscrete schemes. For example, if displacement and velocity are each assumed to be linear and continuous in time, the following member of the Newmark family of algorithms ensues: $\beta = \frac{4}{9}$, $\gamma = \frac{5}{6}$. Other assumptions may lead to conditionally stable algorithms. Clearly there are many possibilities.

6. Numerical results

To test the formulation, the response of a one-dimensional elastic rod was calculated. Both ends of the rod were fixed, no external loads were applied, the initial velocity was zero, and the initial displacement was taken proportional to the first harmonic. The element Courant number (i.e., $c\Delta t/\Delta x$) was fixed at 1.2. Quadrilateral elements of type $\mathrm{Q}k - \mathrm{Q}l$, $1 \leq k, l \leq 2$, were employed in the analysis; Q1 and Q2 are the standard bilinear and biquadratic interpolations.

The effect of omitting or including the $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s terms was investigated. Whenever these terms were employed, they were defined by (96). Errors were measured in three different norms:

$$|||\cdot|||_{\tau_1,\tau_2,s}, \qquad |||\cdot|||_{\tau_1,\tau_2,\mathbf{0}} \quad \text{and} \quad |||\cdot|||_{\mathbf{0},\mathbf{0},\mathbf{0}}, \tag{107}$$

where the presence or absence of the $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s terms in each norm is indicated by the subscript.

Figures 5–7 compare the convergence rates of the Q1Q1 element for different formulations. Note from Fig. 5 that the formulation with all terms present achieves the predicted convergence rate in the $|||\cdot|||_{\tau_1,\tau_2,s}$ norm. Omitting the $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s terms in the formulation does not affect the convergence rate. The same conclusions can be drawn for the $|||\cdot|||_{\tau_1,\tau_2,\mathbf{0}}$ norm as may be seen from Fig. 6. The rate of convergence of the Galerkin formulation (i.e. $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and s omitted) is improved in the $|||\cdot|||_{\mathbf{0},\mathbf{0},\mathbf{0}}$ norm as may be seen from Fig. 7. The effect of the s term in this problem is seen to be of little consequence. Due to the additional

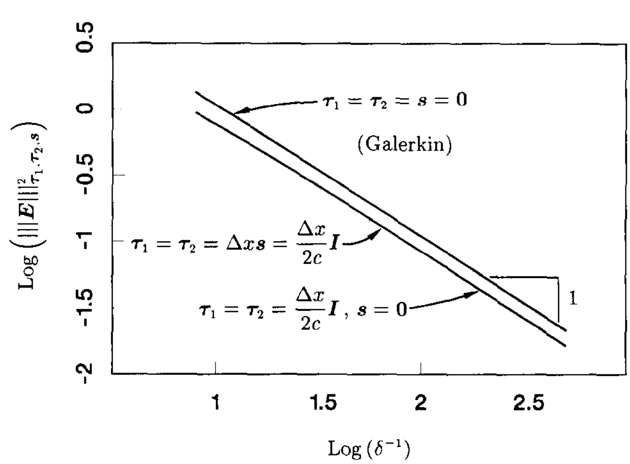

Fig. 5. Comparison of numerical error for different formulations employing the Q1Q1 element: error measured in the $|||\cdot|||_{\tau_1,\tau_2,s}$ norm, δ is the distance between adjacent nodes.

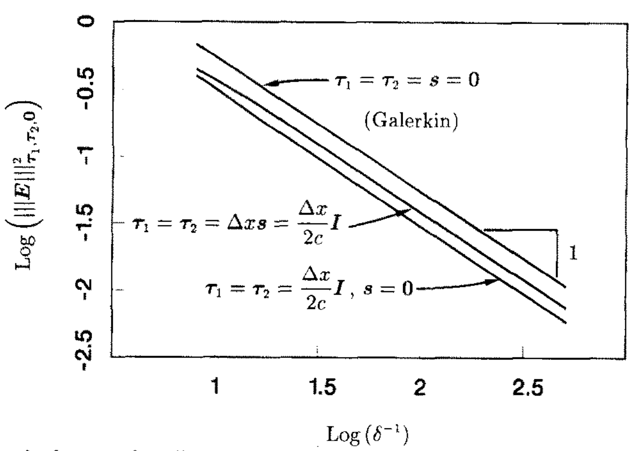

Fig. 6. Comparison of numerical error for different formulations employing the Q1Q1 element: error measured in the $|||\cdot|||_{\tau_1,\tau_2,0}$ norm, δ is the distance between adjacent nodes.

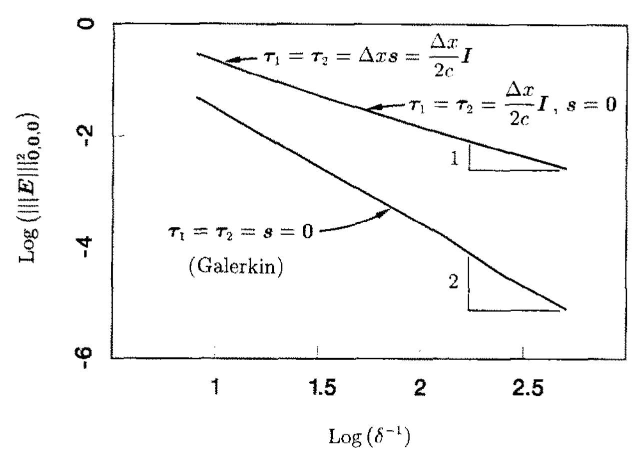

Fig. 7. Comparison of numerical error for different formulations employing the Q1Q1 element: error measured in the $|||\cdot|||_{0,0,0}$ norm, δ is the distance between adjacent nodes.

complications involved with its assembly, there is strong motivation to omit it in practice. The results presented in Figs. 5–7 suggest that this may be possible.

In Fig. 8 the case $s = 0$ is further studied. The additional velocity degrees of freedom of Q1Q2, compared with Q1Q1, are seen to slightly degrade the results. The results for Q2Q1 and Q2Q2 are almost identical. In the two-field formulation, the implementational simplicity of equal-order interpolations makes it the natural choice. The convergence rates observed are the same as those predicted for the formulation in which s is present. The calculations for the four elements were repeated for the Galerkin formulation. The same rates of convergence were attained, except for Q1Q2 which diverged (results not shown).

Calculations performed on problems involving propagating discontinuity surfaces have revealed the salubrious smoothing behavior of the τ_1, τ_2, and s terms. These results will be reported in future work.

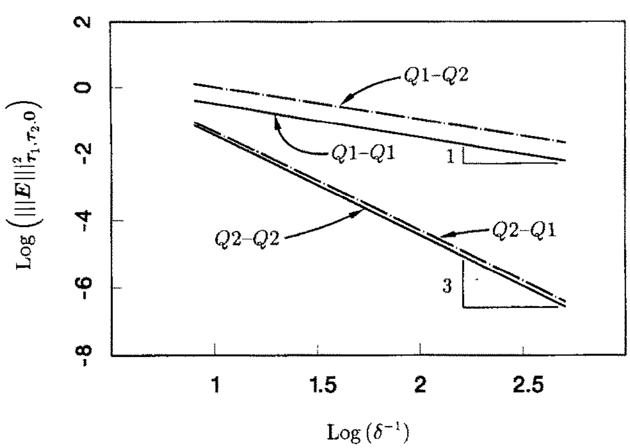

Fig. 8. Comparison of numerical error, δ is the distance between adjacent nodes.

7. Conclusions

We have developed a new space-time formulation of classical elastodynamics. The method is based upon the time-discontinuous Galerkin method and employs additional least-squares (also known as Petrov–Galerkin) terms which enhance stability. A mathematical analysis was performed and the method was proved to converge at the optimal rate in a norm stronger than the total energy norm. Numerical results confirmed the predicted convergence rates, and suggest that some simplifications may be possible.

The methods presented are implicit and unconditionally stable. It would be desirable to develop corresponding explicit and implicit-explicit methods which retain the unstructured space-time mesh features of the methods presented herein. These would undoubtably be useful for the development of space-time adaptive schemes, and, in particular, for the design and analysis of subcycling algorithms in which different time steps are used in different elements. Considerable further effort needs to be devoted to implementational aspects, as the new methods are rather different from classical procedures. Additional work also needs to be performed on simplifying the methods in order that they attain economic competitiveness with existing techniques. The attributes of the new method not possessed by in-place procedures indicate to us that there is considerable potential for future applications in elastodynamics, structural dynamics, and second-order hyperbolic systems in general.

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