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FINITE ELEMENTS IN TIME AND SPACE

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The idea of finite element discretisation is applied to time dependent phenomena. Hamilton's principle is used as a suitable variational statement which means that the time is discretised into a set of finite elements which are taken to be the same for all structural elements. The method is introduced for the unidimensional case and thereafter generalised for multidegree of freedom systems. The use of arbitrary time elements is outlined.

The method appears particularly suitable for the investigation of time dependent dynamic phenomena without prior deduction of the natural modes and frequencies. Linear as well as nonlinear phenomena may be analysed.

1. Introduction

The present paper seeks to apply the ideas of discretisation to time dependent phenomena. As a suitable variational statement we may use Hamilton's principle. In practise this means that the time is discretised into a set of finite elements which are taken to be the same for all structural elements. A finite element in time consists simply of a fixed time interval. In our present discussion we detail in particular the case when at the beginning and end of the time interval the generalized displacements and velocities are given. For dynamic problems this is the minimum of information required, but the technique may easily be extended to account for additional "timewise degrees of freedoms"; see Section 4. Introducing an appropriate interpolation procedure we may obtain the displacement and velocity at any instant of time. It is then possible to carry out in the variational statement the time integration explicitly and to obtain hence a system of linear equations. The method is extremely simple, since the time interpolation of all structural freedoms of an element in space is the same. We also demonstrate that the general case of a multi degree of freedoms system can be made to depend on the matrices which describe the unidimensional motion of a mass point.

The method appears particularly suitable for the investigation of time dependent dynamic phenomena without prior deduction of the natural modes and frequencies. Linear as well as nonlinear phenomena may be analysed. Lack of space precludes the discussion of unsteady heat flow.

A superior t denotes the transpose of a matrix.

2. Demonstration of the principle; the unidimensional case

The movement of a mass point m along the x-axis may be described by the extremum statement

$$A = \int_{t_0}^{t_l} \left(\tfrac{1}{2} m\dot{x}^2 + fx\right) \mathrm{d}t = \min \;, \tag{1}$$

in which $x(t_0)$ and $x(t_l)$ are not varied and the force f on the mass point is to be understood as a known function of the time t. The Eulerian equation of the variational problem (1) is the standard Newtonian expression

$$m\ddot{x} - f = 0 . \tag{2}$$

Eq. (1) is the simplest formulation of Hamilton's principle. The statement (1) may be discretised in time by subdividing the interval t_0 to t_l into a number of steps t_{j-1} to t_j. By prescribing a certain interpolation procedure for the time dependence, we may derive the values of x, $\dot{x}$ and higher derivatives at any instant t, where $t_{j-1} \leqslant t \leqslant t_j$ in terms of their values at the instants t_{j-1}, t_j. For the equations of motion, it is necessary to provide for the specification of x and $\dot{x}$ since otherwise the initial dynamic conditions cannot be selected simply. The lowest order interpolation set are hence the third order Hermitian polynomials. We restrict ourselves initially to these functions, which may be formed into the (1×4) matrix

$$\boldsymbol{\omega}(\tau) = [1 + 3\tau^2 + 2\tau^3 \quad \tau - 2\tau^2 + \tau^3 \quad 3\tau^2 - 2\tau^3 \quad -\tau^2 + \tau^3] , \tag{3}$$

where τ is a non-dimensional time parameter running from 0 to 1. We form with the displacements x and the velocities $\dot{x}$ at the moments t_{j-1} and t_j the (4×1) space-time vector

$$\hat{\mathbf{x}}_j = \{x(t_{j-1}) \quad \dot{x}(t_{j-1}) \quad x(t_j) \quad \dot{x}(t_j)\} = \{x_{j-1} \quad \dot{x}_{j-1} \quad x_j \quad \dot{x}_j\} , \tag{4}$$

which corresponds to the interpolation rule (3). The selected timewise interpolation in the interval $t_{j-1} \leqslant t \leqslant t_j$ may be written as the transformation

$$x(t) = \mathbf{w}_j(t)\hat{\mathbf{x}}_j , \tag{5}$$

where

$$\mathbf{w}_j(t) = \boldsymbol{\omega}(\tau)\,\mathbf{d}_j , \tag{6}$$

in which

$$\tau = \frac{t - t_{j-1}}{\Delta t_j} \quad \text{and} \quad \Delta t_j = t_j - t_{j-1} , \tag{7}$$

and

$$\mathbf{d}_j = \ulcorner\, 1 \quad \Delta t_j \quad 1 \quad \Delta t_j \,\lrcorner . \tag{8}$$

Similarly for the velocity $\dot{x}$ we obtain

$$\dot{x}(t) = \dot{\mathbf{w}}_j(t)\hat{\mathbf{x}}_j , \tag{9}$$

where

$$\dot{\mathbf{w}}_j = \frac{\mathrm{d}\mathbf{w}_j}{\mathrm{d}t} = \frac{1}{\Delta t_j}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}\tau}\,\mathbf{d}_j = \frac{1}{\Delta t_j}\,\boldsymbol{\omega}'\mathbf{d}_j , \tag{10}$$

with

$$\boldsymbol{\omega}' = [\,-6\tau + 6\tau^2 \quad 1 - 4\tau + 3\tau^2 \quad 6\tau - 6\tau^2 \quad -2\tau + 3\tau^2\,]\ . \tag{11}$$

It is now possible to evaluate the integral A of eq. (1) in the typical jth time interval. At the same time we may generalise f into $f - c\dot{x} - kx$, which includes linear damping and spring forces. We thus find

$$A_j = \int_{t_{j-1}}^{t_j} [\tfrac{1}{2} m\dot{x}^2 + x(f - c\dot{x} - kx)]\,\mathrm{d}t = \hat{\mathbf{x}}_j^t [\tfrac{1}{2} m\mathbf{h}_j^{11} - c\mathbf{h}_j^{01} - k\mathbf{h}_j^{00}]\,\hat{\mathbf{x}}_j + \hat{\mathbf{x}}_j^t \mathbf{h}_j^{00} \hat{\mathbf{f}}_j\ . \tag{12}$$

Here $\hat{\mathbf{f}}_j$ is the force vector

$$\hat{\mathbf{f}}_j = \{f_{j-1} \quad \dot{f}_{j-1} \quad f_j \quad \dot{f}_j\}\ , \tag{13}$$

which corresponds to $\hat{\mathbf{x}}_j$ of eq. (4). Note the interpolation rule

$$f(t) = \mathbf{w}_j(t)\hat{\mathbf{f}}_j\ . \tag{14}$$

We also use in eq. (12) the abbreviation

$$\mathbf{h}_j^{00} = \int_{t_{j-1}}^{t_j} \mathbf{w}_j^t \mathbf{w}_j\,\mathrm{d}t\ , \qquad \mathbf{h}_j^{01} = \int_{t_{j-1}}^{t_j} \mathbf{w}_j^t \dot{\mathbf{w}}_j\,\mathrm{d}t\ , \qquad \mathbf{h}_j^{11} = \int_{t_{j-1}}^{t_j} \dot{\mathbf{w}}_j^t \dot{\mathbf{w}}_j\,\mathrm{d}t\ . \tag{15}$$

Applying eqs. (6) and (10) we obtain, for example

$$\mathbf{h}_j^{00} = \Delta t_j \mathbf{d}_j \left[\int_0^1 \boldsymbol{\omega}^t \boldsymbol{\omega}\,\mathrm{d}\tau\right] \mathbf{d}_j = \Delta t_j \mathbf{d}_j \mathbf{h}^{00} \mathbf{d}_j\ , \tag{16}$$

where $\mathbf{h}^{00}$ is a purely numerical matrix. Similar transformations are valid for $\mathbf{h}_j^{01}$ and $\mathbf{h}_j^{11}$. We find

$$\mathbf{h}_j^{01} = \mathbf{d}_j \mathbf{h}^{01} \mathbf{d}_j \qquad \text{and} \qquad \mathbf{h}_j^{11} = \frac{1}{\Delta t_j} \mathbf{d}_j \mathbf{h}^{11} \mathbf{d}_j\ . \tag{17}$$

Here $\mathbf{h}^{00}$ and $\mathbf{h}^{11}$ are the symmetrical matrices

$$\mathbf{h}^{00} = \frac{1}{420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ \text{symm.} & & & 4 \end{bmatrix}, \qquad \mathbf{h}^{11} = \frac{1}{30} \begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ \text{symm.} & & & 4 \end{bmatrix}, \tag{18}$$

and $\mathbf{h}^{01}$ is the non-symmetrical matrix

$$\mathbf{h}^{01} = \frac{1}{60} \begin{bmatrix} -30 & 6 & 30 & -6 \\ -6 & 0 & 6 & -1 \\ -30 & -6 & 30 & 6 \\ 6 & 1 & -6 & 0 \end{bmatrix}. \tag{18a}$$

Assuming there are l time intervals we next form the $(2l + 2 \times 1)$ vector

$$\hat{\mathbf{x}} = \{x_0 \quad \dot{x}_0 \quad x_1 \quad \dot{x}_1 \cdots x_{j-1} \quad \dot{x}_{j-1} \quad x_j \quad \dot{x}_j \cdots x_e \quad \dot{x}_e\} . \tag{19}$$

A typical subvector $\hat{\mathbf{x}}_j$ is then selected by the Boolean transformation

$$\hat{\mathbf{x}}_j = \mathbf{a}_{Tj}\, \hat{\mathbf{x}} , \tag{20}$$

where

$$\mathbf{a}_{Tj} = [\mathbf{O}_{4,2j-2} \quad \mathbf{I}_4 \quad \mathbf{O}_{4,2l-2j}] \quad (4 \times 2l + 2) . \tag{21}$$

We also form the complete $\mathbf{a}_T$ matrix as

$$\mathbf{a}_T = \{\mathbf{a}_{T1} \quad \mathbf{a}_{T2} \cdots \mathbf{a}_{Tj} \cdots \mathbf{a}_{T2}\} \quad (4l \times 2l + 2) . \tag{22}$$

The summation of the contributions A_j of all l intervals yields the following expression for the integral (1)

$$A = \tfrac{1}{2}\, m\, \hat{\mathbf{x}}^t \mathbf{H}^{11}\, \hat{\mathbf{x}} - c\, \hat{\mathbf{x}}^t \mathbf{H}^{01}\, \underline{\hat{\mathbf{x}}} - k\, \hat{\mathbf{x}}^t \mathbf{H}^{00}\, \underline{\hat{\mathbf{x}}} - \hat{\mathbf{x}}^t \mathbf{H}^{00}\, \hat{\mathbf{f}} , \tag{23}$$

where

$$\mathbf{H}^{nm} = \mathbf{a}_T^t\, \mathbf{h}^{nm}\, \mathbf{a}_T , \tag{24}$$

with

$$\mathbf{h}^{nm} = \ulcorner \mathbf{h}_1^{nm} \quad \mathbf{h}_2^{nm} \cdots \mathbf{h}_j^{nm} \cdots \mathbf{h}_l^{nm} \lrcorner , \tag{25}$$

and $\hat{\mathbf{f}}$ corresponds to $\hat{\mathbf{x}}$ of eq. (19).

We may now apply the minimum condition to the expression (23) in the form

$$\frac{\partial A}{\partial \hat{\mathbf{x}}^t} = \mathbf{o} . \tag{26}$$

Note that the underlined vectors $\hat{\mathbf{x}}$ in eq. (23) must thereby be taken as constants since they represent a part of the force $f(t)$ in eqs. (1) and (2). We obtain

$$[m\mathbf{H}^{11} - c\mathbf{H}^{01} - k\mathbf{H}^{00}]\, \hat{\mathbf{x}} + \mathbf{H}^{00} \hat{\mathbf{f}} = \mathbf{o} . \tag{27}$$

The reader will notice that the introduction of the potential energy $\frac{1}{2}\, kx^2$ would have yielded the same term in eq. (27) without any restriction in the variation of $\hat{\mathbf{x}}$. In accordance with what we said initially the values of x at $t = 0$ and $t = t_i$ may *not* be varied. Hence the two equations in the system (27), which correspond to the derivatives of A with respect to x_0 and x_i, must be suppressed. On the other hand we may for example prescribe x_0 and $\dot{x}_0$ as initial values. It follows that the restricted set (27) represents $2l$ equations for the $2l$ unknowns

$$x_1, \dot{x}_1, x_2, \dot{x}_2, \ldots, x_l, \dot{x}_l .$$

Clearly, we can split the whole operation into a sequence of l_1, l_2 etc. intervals, which can be solved sequentially. The last two entries in the $\hat{\mathbf{x}}$ vector serve then as initial values for the subsequent interval. If the lengths of the intervals and the number of the coupled intervals are the same, we note the appearance of the same system of equations, in which only the right hand sides are different. This presumes, of course, a linearity of the system in which the spring constant k and the damping c do not change. No difficulty arises in allowing for non-linear effects. It is only necessary to know k and c as functions of time. This will lead in general to an iterative method of solution.

3. Generalisation for multidegree of freedoms systems

The above procedure may be easily generalised to encompass systems with n degrees of freedom. For compactness we use in our derivation the Kronecker product of matrices, which is defined by

$$\mathbf{A} \times \mathbf{B} = \mathbf{C} = [\mathbf{C}_{ij}] \ , \tag{28}$$

where the submatrices $\mathbf{C}_{ij}$ are given by

$$\mathbf{C}_{ij} = a_{ij} \mathbf{B} \ . \tag{29}$$

The following theorems enter in our subsequent argument:

$$(\mathbf{A} \times \mathbf{B})(\mathbf{C} \times \mathbf{D}) = (\mathbf{AC}) \times (\mathbf{BD}) \ , \tag{30}$$

$$\mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} = (\mathbf{A} + \mathbf{B}) \times \mathbf{C} \ , \tag{31}$$

$$(\mathbf{A} \times \mathbf{B})^t = \mathbf{A}^t \times \mathbf{B}^t \ , \tag{32}$$

$$\mathbf{A}(\mathbf{b} \times \mathbf{C}) = \mathbf{b} \times (\mathbf{AC}) \ , \tag{33}$$

where $\mathbf{b}$ is a row matrix. Denoting the $(n \times 1)$ displacement vector of the system row by $\mathbf{r}$, we form the $(n(2l + 2) \times 1)$ vector

$$\hat{\mathbf{r}} = \{\mathbf{r}_0 \quad \dot{\mathbf{r}}_0 \quad \mathbf{r}_1 \quad \dot{\mathbf{r}}_1 \cdots \mathbf{r}_{j-1} \quad \dot{\mathbf{r}}_{j-1} \quad \mathbf{r}_j \quad \dot{\mathbf{r}}_j \cdots \mathbf{r}_l \quad \dot{\mathbf{r}}_l\} \ , \tag{34}$$

where $\mathbf{r}_j$ stands for the displacements and $\dot{\mathbf{r}}_j$ for the corresponding velocities at the instant $t = t_j$.

For the jth time element we have

$$\hat{\mathbf{r}}_j = [\mathbf{a}_{Tj} \times \mathbf{I}_n] \hat{\mathbf{r}} \qquad (4n \times 1) \tag{35}$$

where $\mathbf{a}_{Tj}$ is once more the Boolean operator of eq. (21). Within this time interval the ith structural element has a space-time displacement vector

$$\hat{\boldsymbol{\rho}}_{ji} = \{\boldsymbol{\rho}_{j-1} \quad \dot{\boldsymbol{\rho}}_{j-1} \quad \boldsymbol{\rho}_j \quad \dot{\boldsymbol{\rho}}_j\}_i \ . \tag{36}$$

If we denote here the structural assembly matrix by

$$\mathbf{a}_S = \{\mathbf{a}_{S1} \quad \mathbf{a}_{S2} \cdots \mathbf{a}_{Si} \cdots \mathbf{a}_{Sr}\} \ , \tag{37}$$

we have

$$\hat{\boldsymbol{\rho}}_{ji} = [\mathbf{I}_4 \times \mathbf{a}_{Si}]\,\hat{\mathbf{r}}_j = [\mathbf{I}_4 \times \mathbf{a}_{Si}]\,[\mathbf{a}_{Tj} \times \mathbf{I}_n]\,\hat{\mathbf{r}}\,. \tag{38}$$

Using theorem (30) we deduce

$$\hat{\boldsymbol{\rho}}_{ji} = [\mathbf{a}_{Tj} \times \mathbf{a}_{Si}]\,\hat{\mathbf{r}}\,. \tag{39}$$

If the ith structural element has q degrees of freedom, we find for any instant t within $t_{j-1} \leqslant t \leqslant t_j$

$$\boldsymbol{\rho}_{ji}(t) = [\mathbf{w}_j \times \mathbf{I}_q]\,\hat{\boldsymbol{\rho}}_{ji} \tag{40}$$

and

$$\dot{\boldsymbol{\rho}}_{ji}(t) = [\dot{\mathbf{w}}_j \times \mathbf{I}_q]\,\hat{\boldsymbol{\rho}}_{ji}\,. \tag{41}$$

Here $\mathbf{w}_j$ and $\dot{\mathbf{w}}_j$ are the standard (1×4) interpolation matrices of eqs. (6) and (10).

It suffices to demonstrate the detials of the integration process on the kinetic energy. Let as assume that the ith structural element has the kinematically equivalent mass matrix $\mathbf{m}_i$. The contribution to the kinetic energy integral of this structural element is within the jth time interval

$$A_{ji} = \tfrac{1}{2}\int_{t_{j-1}}^{t_j} \dot{\boldsymbol{\rho}}_{ji}^t(t)\,\mathbf{m}_i\,\dot{\boldsymbol{\rho}}_{ji}(t)\,\mathrm{d}t = \tfrac{1}{2}\,\hat{\boldsymbol{\rho}}_{ji}^t\left[\int_{t_{j-1}}^{t_j} (\dot{\mathbf{w}}_j \times \mathbf{I}_q)^t\,\mathbf{m}_i(\dot{\mathbf{w}}_j \times \mathbf{I}_q)\,\mathrm{d}t\right]\hat{\boldsymbol{\rho}}_{ji}\,. \tag{42}$$

Applying theorems (32), (33) and (30) we confirm that

$$A_{ji} = \tfrac{1}{2}\,\hat{\boldsymbol{\rho}}_{ji}^t\,[\mathbf{h}_j^{11} \times \mathbf{m}_i]\,\hat{\boldsymbol{\rho}}_{ji}\,, \tag{43}$$

in which $\mathbf{h}_j^{11}$ is the integral matrix of eq. (17) as in the unidimensional case. Substituting eq. (39) into eq. (43) we obtain

$$A_{ji} = \tfrac{1}{2}\,\hat{\mathbf{r}}^t\,[\mathbf{a}_{Tj} \times \mathbf{a}_{Si}]^t\,[\mathbf{h}_j^{11} \times \mathbf{m}_i]\,[\mathbf{a}_{Tj} \times \mathbf{a}_{Si}]\,\hat{\mathbf{r}} = \tfrac{1}{2}\,\hat{\mathbf{r}}^t\,[(\mathbf{a}_{Tj}^t\mathbf{h}_j^{11}\mathbf{a}_{Tj}) \times (\mathbf{a}_{Si}^t\mathbf{m}_i\mathbf{a}_{Si})]\,\hat{\mathbf{r}}\,. \tag{44}$$

Using theorem (31) we find for the complete system by summation of the individual time and space elements

$$A_{\text{kin.}} = \tfrac{1}{2}\,\hat{\mathbf{r}}^t[\mathbf{H}^{11} \times \mathbf{M}]\,\hat{\mathbf{r}}\,, \tag{45}$$

where $\mathbf{H}^{11}$ is given in eq. (24) and $\mathbf{M} = \mathbf{a}_S^t\mathbf{m}\mathbf{a}_S$ is the standard mass matrix of the assembled system. It is now straightforward to generalise our integration procedure in the presence of a stiffness matrix $\mathbf{K}$ and a damping matrix $\mathbf{C}$ to deduce

$$[\mathbf{H}^{11} \times \mathbf{M} - \mathbf{H}^{01} \times \mathbf{C} - \mathbf{H}^{00} \times \mathbf{K}]\,\hat{\mathbf{r}} + [\mathbf{H}^{00} \times \mathbf{I}_n]\,\hat{\mathbf{R}} = \mathbf{o}\,, \tag{46}$$

where the force matrix $\hat{\mathbf{R}}$ corresponds to $\hat{\mathbf{r}}$ of eq. (34). We suppress as in Section 2 the $2n$ equations associated with $\mathbf{r}_0$ and $\mathbf{r}_l$. If we eliminate the corresponding rows in the $\mathbf{H}$-matrices the purification of the system (46) is carried out automatically. The reader will observe that the number of unknowns increases by $2n$ for every time interval added. Our comments at the end of Section 2 apply here too.

4. Arbitrary time elements

In Section 2 we illustrated the interpolation in time with Hermitian polynomials of the third order and two pivotal points. We now intend to generalise the procedure by allowing an arbitrary number p of pivotal points, in each of which a derivatives are considered. The presentation is restricted to the case of a single degree of freedom, since the general case may be easily deduced following Section 3. We write the time-space displacement vector of the element as

$$\hat{\mathbf{x}} = \{\hat{\mathbf{x}}_0 \;\; \hat{\mathbf{x}}_1 \; --- \; \hat{\mathbf{x}}_{p-1}\} \qquad (r \times 1) \tag{47}$$

where $r = pa$ and

$$\hat{\mathbf{x}}_i = \left\{x \;\; \dot{x} \;\; \ddot{x} \; --- \; \frac{\partial x^{a-1}}{\partial t^{a-1}}\right\}_i \qquad (a \times 1)\,. \tag{48}$$

For simplicity we number in what follows the components of $\hat{\mathbf{x}}$ consecutively in the form

$$\hat{\mathbf{x}} = \{x_1 \;\; x_2 \;\; x_3 \; --- \; x_r\}\,. \tag{49}$$

In the non-linear case we have to consider a further time dependent quantity $f(t)$, where f may stand for m, c and k; see eq. (9). For the interpolation of f we admit $\bar{p}$ pivotal points with $\bar{a}$ derivatives. Corresponding to $\hat{\mathbf{x}}$ we introduce the vector

$$\hat{\mathbf{f}} = \{f_1 \;\; f_2 \;\; f_3 \; --- \; f_{\bar{r}}\} \qquad (\bar{r} \times 1) \tag{50}$$

where $\bar{r} = \bar{p}\bar{a}$. Next the interpolation matrices

$$\mathbf{w}(t) = [w_1 \;\; w_2 \;\; w_3 \; --- \; w_r] \tag{51}$$

and

$$\overline{\mathbf{w}}(t) = [\bar{w}_1 \;\; \bar{w}_2 \;\; \bar{w}_3 \; --- \; \bar{w}_{\bar{r}}] \tag{52}$$

are defined, which yield

$$x(t) = \mathbf{w}\hat{\mathbf{x}} \qquad \text{and} \qquad f(t) = \overline{\mathbf{w}}\hat{\mathbf{f}}\,. \tag{53}$$

The matrices $\mathbf{w}$ and $\overline{\mathbf{w}}$ may be deduced by suitable transformations from

$$\pi = [\pi_1 \;\; \pi_2 \; --- \; \pi_r] \qquad \text{and} \qquad \bar{\pi} = [\pi_1 \;\; \pi_2 \; --- \; \pi_{\bar{r}}]\,, \tag{54}$$

which are defined within an interval 0 to 1 of the time measure τ by the relation

$$\pi_k = \tau^{k-1}\,. \tag{54a}$$

If the time element considered extends from t_l to $t_l + \Delta t$ we set

$$\tau = \frac{t - t_l}{\Delta t}\,. \tag{55}$$

There follows

$$\mathrm{d}t = \Delta t\, \mathrm{d}\tau \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\Delta t}\frac{\mathrm{d}}{\mathrm{d}\tau}\,. \tag{56}$$

We first reduce the matrices $\mathbf{w}$, $\bar{\mathbf{w}}$ to $\boldsymbol{\omega}$, $\bar{\boldsymbol{\omega}}$ within the interval $0 \leqslant \tau \leqslant 1$ using

$$w_i(t) = \omega_i(\tau)\,(\Delta t)^I \qquad \text{and} \qquad \bar{w}_i(t) = \bar{\omega}_i(\tau)\,(\Delta t)^{\bar{I}}\,. \tag{57}$$

Here

$$I = \operatorname{mod}(i-1, a)\,, \qquad \bar{I} = \operatorname{mod}(i-1, \bar{a})\,. \tag{58}$$

It is straightforward to establish equations of the type

$$\boldsymbol{\omega} = \boldsymbol{\pi}\mathbf{B}\,, \qquad \bar{\boldsymbol{\omega}} = \bar{\boldsymbol{\pi}}\bar{\mathbf{B}}\,, \tag{59}$$

or

$$\omega_j = \sum_{v=1}^{r} B_{vj}\pi_v\,, \qquad \omega_i = \sum_{u=1}^{\bar{r}} \bar{B}_{ui}\pi_u\,. \tag{59a}$$

This may be seen by writing the inverse relations

$$\boldsymbol{\pi} = \boldsymbol{\omega}\mathbf{B}^{-1} = \boldsymbol{\omega}\mathbf{A}\,, \qquad \bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\omega}}\bar{\mathbf{B}}^{-1} = \bar{\boldsymbol{\omega}}\bar{\mathbf{A}}\,. \tag{60}$$

The matrices $\mathbf{A}$ and $\bar{\mathbf{A}}$ are evidently set up with the elements

$$A_{ij} = \left.\frac{\partial^I \pi_j}{\partial \tau^I}\right|_{\tau=\tau_i} \qquad \text{and} \qquad \bar{A}_{ij} = \left.\frac{\partial^{\bar{I}} \pi_j}{\partial \tau^{\bar{I}}}\right|_{\tau=\bar{\tau}_i}\,. \tag{61}$$

For I and $\bar{I}$ we use relations (58). Also

$$\tau_i = \frac{i-I-1}{(p-1)a}\,, \qquad \bar{\tau}_i = \frac{i-\bar{I}-1}{(\bar{p}-1)\bar{a}}\,. \tag{62}$$

Following these introductory remarks, a typical element of an $\mathbf{h}$-matrix may be written in the general form

$$h_{i,jk}^{nm} = \int_{t_l}^{t_l+\Delta t} \bar{w}_i\, \frac{\mathrm{d}^n w_j}{\mathrm{d}t^n}\, \frac{\mathrm{d}^m w_k}{\mathrm{d}t^m}\,. \tag{63}$$

Here n and m take the values

n	m			
0	0		$f = k$	(stiffness)
0	1	for	$f = c$	(damping)
1	1		$f = m$	(mass) .

The lower index i runs from 1 to $\bar{r}$ and j, k denote the corresponding element in the matrix $\mathbf{h}_i^{nm}$. Applying the functions $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$ we can reduce the integration in (63) to the range 0 to 1. Thus,

$$h_{i,jk}^{nm} = (\Delta t)^{1+\bar{I}+J+K-n-m} \int_0^1 \bar{\omega}_i \frac{\mathrm{d}^n \omega_j}{\mathrm{d}\tau^n} \frac{\mathrm{d}^m \omega_k}{\mathrm{d}\tau^m} \,\mathrm{d}\tau \,. \tag{64}$$

J and K are, of course, computed from the first equation (58). Introducing eqs. (59a) in eq. (64) we find

$$h_{i,jk}^{nm} = (\Delta t)^{1+\bar{I}+J+K-n-m} \sum_{u=1}^{\bar{r}} \sum_{v,w=1}^{r} \bar{B}_{ui} B_{vj} B_{wk} I_{u,vw}^{nm} \,. \tag{65}$$

Here

$$I_{u,vw}^{nm} = \int_0^1 \pi_u \frac{\mathrm{d}^n \pi_v}{\mathrm{d}\tau^n} \frac{\mathrm{d}^m \pi_w}{\mathrm{d}\tau^m} \,\mathrm{d}\tau = (v-1)(v-2) \cdots (v-n)(w-1)(w-2) \cdots (w-m) \frac{1}{u+v+w-n-m-2} \tag{66}$$

and may be understood as the (v, w) element of an $(r \times r)$ matrix $\mathbf{I}_u^{nm}$. We have now deduced all the basic data and present finally a more attractive matrix formulation. Let us assume that the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{f}}$ of eqs. (49) and (50) are given and that we desire to find

$$A_i^{nm} = \int_{t_i}^{t_i+\Delta t} f(t) \frac{\mathrm{d}^n x}{\mathrm{d}t^n} \frac{\mathrm{d}^m x}{\mathrm{d}t^m} \,\mathrm{d}t \,. \tag{67}$$

We obtain

$$A_i^{nm} = \hat{\mathbf{x}}^t \left[\sum_{i=1}^{\bar{r}} f_i \mathbf{h}_i^{nm} \right] \hat{\mathbf{x}} \,, \tag{68}$$

where $\mathbf{h}_i^{nm}$ may be written as

$$\mathbf{h}_i^{nm} = (\Delta t)^{1+\bar{I}-n-m} \, \mathbf{d} \mathbf{g}_i^{nm} \mathbf{d} \,, \tag{69}$$

with

$$\mathbf{d} = \lceil d_1 \;\; d_2 \; \cdots \; d_j \; \cdots \; d_r \rfloor \,, \qquad d_j = (\Delta t)^J \,, \tag{70}$$

and

$$\mathbf{g}_i^{nm} = \sum_{u=1}^{\bar{r}} \bar{B}_{ui} \left[\mathbf{B}^t \mathbf{I}_u^{nm} \mathbf{B} \right] . \tag{71}$$

The present method is very simple to program and permits the inclusion of any derived interpolation technique.