On the Motivic Decomposition of a Hyperplane Section of a Twisted Milnor Hypersurface

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Abstract

This note reports on recent work of Xiong and Zaynullin ([9]) on the motivic decomposition of a hyperplane section of a twisted Milnor hypersurface, clarifying some aspects. In particular, it is shown how to adapt the construction of the hyperplane section to an arbitrary central simple algebra and a cohomological criterion is given for when the monodromy action admits a simple description.

Introduction

Let k be a field and L/k a finite abelian extension of degree n. Consider a central simple algebra A over k of degree n containing L. Let

$$Y(A, L) = \{I \subseteq I' \subseteq A : I, I' \text{ right ideals}, \dim_k I = n, \dim_k I' = n(n-1), LI \subseteq I'\}$$

$Y(A, L)$ is a hyperplane section of a twisted form of a Milnor hypersurface (see §1). In §2, we define a cohomological condition on $L \subseteq A$. Pairs $L \subseteq A$ satisfying this condition are called *special*. Assuming that all $Y(A, L)$ satisfy Rost nilpotence (see Definition 3.7), we have the following result:

Main Theorem. *Let $L \subseteq A$ be special, then the (effective) Chow motive of $Y = Y(A, L)$ decomposes as*

$$\mathbf{M}(Y) = \bigoplus_{i=0}^{n-3} \mathbf{M}(\mathrm{SB}(A))(i) \oplus \mathbf{M}(\mathrm{Spec}\, L)(n-2)$$

where $\mathrm{SB}(A)$ is the Severi-Brauer variety associated to A.

The part of the decomposition consisting of motives of Severi-Brauer varieties (twisted by powers of the Tate motive) is considered in §1. In order to construct a cycle to identify the Artin motive $\mathbf{M}(\mathrm{Spec}\, L)(n-2)$ within the decomposition, the monodromy action is introduced in §2 and then applied in §3.

1 The Hyperplane Section

Generalities

Let V be a finite dimensional vector space over k of dimension N. Recall that the Grassmannian $\mathrm{Gr}(V, n)$ paramaterises n-dimensional subspaces of V in the following sense: for any k-scheme S, the S-points are

$$\begin{aligned} \mathrm{Gr}(V, n)(S) &= \{[\underline{V} \twoheadrightarrow \mathscr{Q}] : \mathscr{Q} \text{ locally free of rank } N-n\} \\ &= \{\mathscr{E} \subseteq \underline{V} : \mathscr{E} \text{ is locally a direct summand of rank } n\} \end{aligned}$$

where $\underline{V}$ is the pullback to S of the locally free sheaf $\tilde{V}$ on $\mathrm{Spec}\, k$. In the case of k-points, one gets the n-dimensional subspaces of V, as expected. General flag varieties are the k-schemes (non-singular projective varietes, in fact) representing the analoguous functor whose k-points are flags in V with the prescribed dimensions. From here on, we simply write the k-points and the maps on k-points for the scheme or morphism of schemes, with the extension to S-points always being made as above.

Let A be a finite dimensional (unital associative) k-algebra, then $\{I_{n_1} \subseteq I_{n_2} \subseteq \ldots \subseteq I_{n_k} \subseteq A : I_{n_i} \text{ right ideals}\}$ is a closed subvariety of $\mathrm{Fl}(A, n_1, n_2, \ldots, n_k)$. We refer to these as *ideal varieties*. In the special case $A = \mathrm{End}(V)$, V a vector space over k of dimension n, the right ideals are in one-to-one correspondence with subspaces of V. Indeed, the maps sending a right ideal $I \subseteq A$ to $\bigcup_{x \in I} \operatorname{im} x$ and a subspace $W \subseteq V$ to $\{x \in A : \operatorname{im} x \subseteq W\}$ are inverses. One sees easily that for $I \subseteq A$ a right ideal, $\dim_k I = n \dim_k W$, where W is its corresponding subspace, so in particular we have the natural identification

$$\{V_{i_1} \subseteq V_{i_2} \subseteq \ldots \subseteq V_{i_m} \subseteq V : \dim V_{i_j} = i_j\} = \{I_{i_1} \subseteq I_{i_2} \subseteq \ldots \subseteq I_{i_m} \subseteq \mathrm{End}(V) : \dim I_{i_j} = n i_j\}$$

Thus, since the correspondence is natural, ideal varieties over a matrix algebra are just flag varieties. If we take A a central simple algebra, then there is a separable extension L/k with $A \otimes_k L \cong \mathrm{End}(L^n)$, $n = \deg A$. Then any ideal variety X of A will be such that $X_L := X \times_{\operatorname{Spec} k} \operatorname{Spec} L$ is a flag variety, and hence is itself a projective non-singular variety.

Finally we recall the notion of a Grassmann bundle, where the "initial data" of vector space over k (i.e. a locally free sheaf over $\operatorname{Spec} k$) is replaced with an arbitrary locally free sheaf $\mathscr{E}$ of rank N on a noetherian k-scheme S. Precisely

$$\begin{aligned} \mathrm{Gr}(\mathscr{E}, n)(S') &= \{[f^*\mathscr{E} \twoheadrightarrow \mathscr{Q}] : \mathscr{Q} \text{ locally free of rank } N - n\} \\ &= \{\mathscr{F} \subseteq f^*\mathscr{E} : \mathscr{F} \text{ is locally a direct summand of rank } n\} \end{aligned}$$

for $f : S' \to S$. In the case $\mathscr{E} \cong \underline{V}$, one recognises this as the base-change by $S \to \operatorname{Spec} k$ of $\mathrm{Gr}(V, n)$. Using this result and gluing, we see that Grassmann bundles exist in general and are non-singular projective varieties over k when S is.

Let X denote one of the schemes defined above. They are all noetherian, and so we can restrict to considering their functor of points for locally noetherian schemes. In this case, the locally direct summand subsheaves of locally free sheaves from the definitions are themselves locally free. We will call these subvector bundles or simply subbundles. The identity map $\mathrm{id}_X : X \to X$ will give a subbundle $\mathscr{S}$ of the defining vector bundle, which we call the *universal* or *tautological* bundle by $\mathscr{S}$, and a quotient $\mathscr{Q}$, which we call the *quotient* bundle. Since $\mathscr{S}$ (resp. $\mathscr{Q}$) comes from the identity map, it is clear that all other bundles $\mathscr{E}$ of its form on other schemes determine a unique map to X which pulls back $\mathscr{S}$ to $\mathscr{E}$ (resp. the quotient map to $\mathscr{Q}$).

For later use, we quickly recall important examples of the schemes described above:

Example 1.0.1. $\mathrm{Gr}(V, 1)$ is just $\mathbb{P}(V)$ and $\mathrm{Gr}(V, \dim V - 1)$ is $\mathbb{P}(V^*)$. For $\mathscr{E}$ a vector bundle of rank n on a noetherian k-scheme S, $\mathrm{Gr}(\mathscr{E}, 1)$ is the projective bundle $\mathbb{P}(\mathscr{E})$ and $\mathrm{Gr}(\mathscr{E}, n-1)$ is the dual $\mathbb{P}(\mathscr{E}^*)$. We then have the identification $\mathbb{P}(\mathscr{E}) = \mathrm{Gr}(\mathscr{E}^*, n-1)$, and we use this to define the twisting sheaf $\mathscr{O}(1)$ of $\mathbb{P}(\mathscr{E})$ as the quotient bundle of $\mathrm{Gr}(\mathscr{E}^*, n-1)$.

Example 1.0.2. For a finite dimensional vector space V over k of dimension n, $\mathrm{Fl}(V, 1, n-1)$ has an explicit realization as a hypersurface X in $Z = \mathbb{P}(V) \times \mathbb{P}(V^*)$: the pairing $V \times V^* \to k$ gives an element of $H^0(Z, p_1^*(\mathscr{O}_{\mathbb{P}(V)}(1))) \otimes H^0(Z, p_2^*(\mathscr{O}_{\mathbb{P}(V^*)}(1))) = H^0(Z, \mathscr{O}_Z(1,1))$, where $\mathscr{O}_Z(1,1) := p_1^*(\mathscr{O}_{\mathbb{P}(V)}(1)) \otimes p_2^*(\mathscr{O}_{\mathbb{P}(V^*)}(1))$. The subvariety X given by the vanishing of this section clearly agrees with the incidence condition on the flag variety. We will refer to this variety as a *Milnor hypersurface*.

X can be re-interpreted as the projective bundle of the dual of the quotient bundle of $\mathbb{P}(V)$, i.e. $\mathbb{P}(\mathscr{Q}^*) = \mathbb{P}((\underline{V}/\mathscr{O}(-1))^*) = \mathrm{Gr}(\mathscr{Q}, n-2)$, with the inclusion $i : X \hookrightarrow Z = \mathrm{Gr}(\underline{V}, n-1)$ induced by the quotient map $\underline{V} \twoheadrightarrow \mathscr{Q}$. In this case, the twisting sheaf on $\mathbb{P}((\underline{V}/\mathscr{O}(-1))^*)$ is simply the pullback by the inclusion of the twisting sheaf on Z. It is easy to see that the twisting sheaf on Z is $\pi_2^*(\mathscr{O}(1))$, where π_2 is the projection onto $\mathbb{P}(V^*)$. Hence the twisting sheaf for X is $\mathscr{O}_X(0,1) = (\pi_2 \circ i)^*(\mathscr{O}_{\mathbb{P}(V^*)}(1))$.

Example 1.0.3. For a central simple algebra A of degree n, $\mathrm{SB}(A) := \{I \subseteq A : \dim I = n\}$ is the *Severi-Brauer variety* of A – these are the twisted forms of projective space. The variety $\{I \subseteq I' \subseteq A : \dim_k I = n, \dim_k I' = n(n-1)\}$ is a twisted form of $\mathrm{Fl}(V, 1, n-1)$, $\dim V = n$, or by the previous example, a *twisted Milnor hypersurface*.

Definition and first properties

Definition 1.1. Let A be a central simple algebra over a field k and $k \subseteq L \subseteq A$ a subfield separable over k with $[L : k] = \deg A$. We associate to this data the twisted Milnor hypersurface $X(A) = \{I \subseteq I' \subseteq A\}$ and a closed subscheme $Y(A, L) = \{I \subseteq I' \subseteq A : LI \subseteq I'\}$.

To see that this really gives a closed subscheme, notice that for each element of L one gets an endomorphism of A. Thinking of $X(A)$ as a flag variety of A the vector space, the condition that this endomorphism sends I into I' gives a bihomgeneous polynomial in the Plücker coordinates for I and I'. The following proposition explains the terminology "hyperplane section".

Proposition 1.2. *Let* $X := X(A)$ *and* $Y := Y(A, L)$ *be as above, with* $\deg A \geqslant 3$. *Then*
(i) Y *is a very ample divisor on* X.
(ii) Y *is a non-singular variety.*

Proof. L/k is a finite separable extension, so it is generated by a single element $\alpha \in L$. Since $k = Z(A)$, we then have that $LI \subseteq I'$ iff $\alpha I \subseteq I'$ for right ideals $I, I' \subseteq A$. To prove both (i) and (ii), it is enough to show it for $\overline{Y} := Y \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ and $\overline{X}$.

Fix an isomorphism $A \otimes_k \overline{k} \cong \operatorname{End}(V)$, V a vector space over $\overline{k}$ of dimension $n = \deg A = [L : k]$. Let $\varphi \in \operatorname{End}(V)$ be the image of $\alpha \otimes 1$ under this isomorphism. φ is semisimple since its minimal polynomial is the same as that of α, which has distinct roots since L/k is separable. Similarly to the Milnor hypersurface, $\overline{Y}$ will thus be the zero locus of the sections of $\mathscr{O}_Z(1,1)$ in $Z := \mathbb{P}(V) \times \mathbb{P}(V^*)$ given by the canonical pairing $V \times V^* \to \overline{k}$ and the pairing $(v, f) \mapsto f(\varphi v)$. Fixing an eigenbasis of φ $y_1, \ldots, y_n \in V$, and the corresponding dual basis $x_1, \ldots, x_n \in V^*$, $x_i \otimes y_j \in H^0(Z, \mathscr{O}_Z(1,1))$, $1 \leqslant i, j \leqslant n$, span the complete linear system and the two sections defining $\overline{Y} \subseteq Z$ are $\sum_{i=1}^n x_i \otimes y_i$ and $\sum_{i=1}^n \alpha_i x_i \otimes y_i$, where the α_i are the n distinct conjugates of α in $\overline{k}$. We can identify the zero locus of the first section with $\overline{X}$, then $\mathscr{O}(\overline{Y}) = \mathscr{O}_Z(1,1) \otimes_{\mathscr{O}_Z} \mathscr{O}_{\overline{X}} \in \operatorname{Pic} \overline{X}$, so $\overline{Y}$ is very ample as $\mathscr{O}_Z(1,1)$ gives the Segre embedding.

For (ii), the fact that $\overline{Y}$ is a very ample divisor of a normal projective variety $\overline{X}$ of dimension $\geqslant 2$ (since $n \geqslant 3$) implies that $\overline{Y}$ is connected by the lemma of Enriques-Severi-Zariski [5, Corollary III.7.8], so it is enough to show that $\overline{Y}$ is non-singular. We proceed by applying the Jacobian criterion to $\overline{Y} \subseteq Z$ away from $x_i \otimes y_j = 0$ for $1 \leqslant i, j \leqslant n$. Let $U_{ij} \subseteq Z$ denote the open set of points where $x_i \otimes y_j \neq 0$ (in the sense of the residue of its stalk, not the stalk itself). It is affine since $\mathscr{O}_Z(1,1)$ is very ample. $U_{ij} \cong \mathbb{A}^{2n-2}$, with coordinate ring $k[x'_1, \ldots, x'_n, y'_1, \ldots, y'_n]$, $x'_l = \frac{x_l}{x_i}, y'_l = \frac{y_l}{y_j}$, and $\overline{Y} \cap U_{ij}$ is given by the equations $\sum_{l=1}^n x'_l y'_l$ and $\sum_{l=1}^n \alpha_i x'_l y'_l$. Since $x'_i y'_j = 1$, for any point $P \in \overline{Y} \cap U_{ij}$ we must have $x'_j \neq 0$ or $y'_i \neq 0$ or $x'_l y'_l \neq 0$ for some $l \neq i, j$. If only one of the conditions holds for a single index or the first two hold but $i = j$, then we have that two of the α_l are equal, a contradiction. Thus for any point $P \in \overline{Y} \cap U_{ij}$ we can always find distinct indices $1 \leqslant l, m \leqslant n$ for which

$$\begin{pmatrix} y'_l & x'_m \\ \alpha_l y'_l & \alpha_m x'_m \end{pmatrix}$$

will be a 2×2 submatrix of the Jacobian matrix, with $y'_l \neq 0$ and $x'_m \neq 0$ at P. Distinctness of the α_l shows that this matrix is non-singular. Thus the rank of the Jacobian agrees with the codimension, so $\overline{Y}$ is non-singular. $\square$

Proposition 1.3. *Let* $\pi : X \to \mathrm{SB}(A)$ *be the morphism given by* $I \subseteq I' \mapsto I$. *The triple* $(X, \pi, \mathscr{O}(Y))$ *is a projective bundle on* $\mathrm{SB}(A)$.

First, we will need the following lemma:

Lemma 1.4. *Let* S *be a* k*-scheme, and* $\mathscr{I} \subseteq \underline{A}$ *a sheaf of right ideals which is a subvector bundle of rank* n. *Let* $\mathscr{E} = \underline{\operatorname{Hom}}_{\underline{A}}(\underline{A}/\mathscr{I}, \mathscr{I})$. $\mathscr{E}$ *is a vector bundle on* S *of rank* $n-1$ *and subvector bundles of* $\mathscr{E}$ *or rank 1 are in natural bijection with subvector bundles* $\mathscr{I}' \subseteq \underline{A}$ *of rank* $n(n-1)$ *which are right ideals and contain* $\mathscr{I}$.

Proof. The problem is local on S, and by standard argument we can reduce to proving it for local rings $\mathscr{O}_{S,x}$, $x \in S$. Let $R = \mathscr{O}_{S,x}$, $A_R = \underline{A}_x = A \otimes \mathscr{O}_{S,x}$ and $I = \mathscr{I}_x$. We have injective maps $A_R^{\mathrm{opp}} \to \operatorname{End}_R(I)$ and $R \to \operatorname{End}_{A_R}(I)$ defined by right multiplication. By Nakayama's lemma, we can check surjectivity after the base change $R \to \kappa(R)$. But then since $A \otimes_R \kappa(R)$ will have a right ideal of dimension n over $\kappa(R)$, it is split so both induced maps will be surjective. Thus the first map is an isomorphism which shows that A_R is split, and the second map shows that the A_R-automorphisms of I are the units of R. Moreover, any generator of a rank one R-direct summand of $\operatorname{Hom}_{A_R}(A_R/I, I)$ will give a surjective map after reduction, so by Nakayama's lemma is surjective. Since I is free, the kernel of this map is thus free of rank $n(n-1)$. The choice of generator clearly does not effect the kernel, so we have one direction of the association. Using the splitting of A_R, we have for any R-direct summand right ideal I' of rank $n(n-1)$ of A_R containing I, that $A_R/I' \cong I$, since $\operatorname{Aut}(I) = R^\times$, we get a well-defined rank 1 direct summand of $\operatorname{Hom}_{A_R}(A/I, I)$. Clearly these constructions are mutually inverse. $\square$

Proof of the proposition. First we show that $\pi : X \to \mathrm{SB}(A)$ is a projective bundle, agnostic to its $\mathscr{O}(1)$. This is the same as showing that given a morphism $f : S \to \mathrm{SB}(A)$, (equiv. a subvector bundle of right ideals of rank n $\mathscr{I} \subseteq \underline{A}$ on S), there is a vector bundle $\mathscr{E}$ on $\mathrm{SB}(A)$ such that $\mathrm{SB}(A)$-morphisms from S to X are in natural correspondence with rank 1 subvector bundles of $f^*\mathscr{E}$. By the lemma, $\mathscr{E} = \underline{\mathrm{Hom}}_{\underline{A}}(\underline{A}/\mathscr{S}, \mathscr{S})$ does this, since $f^*\mathscr{E} = \underline{\mathrm{Hom}}_{\underline{A}}(\underline{A}/\mathscr{I}, \mathscr{I})$.

Now let $\overline{\mathscr{E}}$ be the pullback of $\mathscr{E}$ to $\mathbb{P}(V) \cong \mathrm{SB}(A)_{\overline{k}} \to \mathrm{SB}(A)$. This is $\underline{\mathrm{Hom}}_{\underline{\mathrm{End}}(\underline{V})}(\underline{\mathrm{End}}(\underline{V})/\mathscr{J}, \mathscr{J})$, where $\mathscr{J}$ is the tautological bundle and This injects into $\underline{\mathrm{End}}_{\underline{\mathrm{End}}(\underline{V})}(\underline{\mathrm{End}}(\underline{V})) = \underline{\mathrm{End}}(\underline{V})$. Using the ideal-subspace correspondence, we see the image is $\underline{\mathrm{Hom}}_{\overline{X}}(\underline{V}/\mathscr{O}(-1), \mathscr{O}(-1))$, where $\mathscr{O}(-1) \subseteq \underline{V}$ is the tautological bundle for $\mathbb{P}(V)$. Hence $\overline{\pi} : \overline{X} \to \mathbb{P}(V)$ is isomorphic over $\mathbb{P}(V)$ to $\mathrm{Gr}(\overline{\mathscr{E}}^*, n-2)$. Since $\overline{\mathscr{E}}^* = \underline{V}/\mathscr{O}(-1) \otimes \mathscr{O}(1)$, this will give, as a scheme over $\mathbb{P}(V)$, the same bundle as $\mathbb{P}((\underline{V}/\mathscr{O}(-1))^*)$, which by Example 1.0.2 has twisting sheaf $\mathscr{O}_{\overline{X}}(0,1)$. The twisiting sheaf of $\mathbb{P}(\overline{\mathscr{E}})$ will thus be $\overline{\pi}^*\mathscr{O}(1) \otimes \mathscr{O}_{\overline{X}}(0,1) = \mathscr{O}_{\overline{X}}(1,1) \cong \mathscr{O}(\overline{Y})$. This line bundle is the pullback of the twisting sheaf for $\mathbb{P}(\mathscr{E})$, and since field extensions induce injective maps on Picard groups, the twisiting sheaf of $\mathbb{P}(\mathscr{E})$ must be $\mathscr{O}(Y)$. □

Severi-Brauer part of the decomposition

We now use these geomtric inputs to partially describe the motive of $Y(A, L)$. Let $\mathbf{V}(k)$ denote the category of non-singular projective varieties over k. We will say that a *correspondence* for $X, Y \in \mathbf{V}(k)$ is an element of the Chow ring $\mathrm{CH}(X \times Y)$. A *homogeneous correspondence* of degree $i \in \mathbb{Z}$ will be a correspondence in $\mathrm{CH}^{n+i}(X \times Y)$, where $n = \dim X$. As is described in [7, §2] there is an associative composition law of correspondences $\circ : \mathrm{CH}(X \times Y) \times \mathrm{CH}(Y \times Z) \to \mathrm{CH}(X \times Z)$, with identity $\Delta_X \in \mathrm{CH}(X \times X)$. For homogeneous correspondences, composition is additive on degree, and Δ_X is homogeneous of degree 0. In particular, taking the same objects as $\mathbf{V}(k)$ and degree 0 correspondences as morphisms, one obtains a category. As is shown in [7], this is an additive category, and there is a contravariant functor from $\mathbf{V}(k)$ to it, and hence to its idempotent completion. We will write $\mathbf{M}(X)$ for the object assigned to $X \in \mathbf{V}(k)$ by this functor, and $c(\varphi) \in \mathrm{Hom}(\mathbf{M}(X), \mathbf{M}(Y))$ for the morphism (i.e. correspondence) induced by $\varphi : X \to Y$. We shall also write $\mathrm{Hom}^i(\mathbf{M}(X), \mathbf{M}(Y))$ for correspondences of degree i.

Consider the projective bundle $\pi : X \to \mathrm{SB}(A)$, with twisting sheaf $\mathscr{O}(Y)$. In this setting, the projective bundle theorem says that $\mathrm{CH}(X)$ is a finite free module as an $\mathrm{SB}(A)$-algebra (the structure morphism is given by π^*) and that for $x = [Y] \in \mathrm{CH}^1(X)$, $1, x, \ldots, x^{n-2}$ gives a basis. Thus we can write an element $\gamma \in \mathrm{CH}(X)$ as $\gamma = \sum_{i=0}^{n-2} \pi^*(\gamma_i)x^i$ for unique $\gamma_i \in \mathrm{CH}(\mathrm{SB}(A))$. In this notation, we also have that $\pi_*(\gamma) = \gamma_{n-2}$.

As observed in [7, §7], this fact allows one to deduce a motivic decomposition for X. We define correspondences $f_i \in \mathrm{Hom}^{-i}(\mathbf{M}(X), \mathbf{M}(\mathrm{SB}(A)))$, $0 \leqslant i \leqslant n-2$, by downward induction. First, we set $f_{n-2} = c(\pi)^t$, the trasposed class of $c(\pi)$ (see [7, §2-3]). Having defined $f_{n-2}, \ldots, f_{i+1}$, we define

$$f_i = c(\pi)^t \circ c_{x^{n-2-i}} \circ \left(\Delta_X - \sum_{j=i+1}^{n-2} c_{x^j} \circ c(\pi) \circ f_j \right)$$

where $c_\gamma = \delta_{X*}(\gamma) \in \mathrm{Hom}(\mathbf{M}(X), \mathbf{M}(X))$ (see [7, §3]). We then define

$$g_i = c_{x^i} \circ c(\pi) \in \mathrm{Hom}^i(\mathbf{M}(\mathrm{SB}(A)), \mathbf{M}(X)), 0 \leqslant i \leqslant n-2$$

By downward induction on i, one sees that $f_i \circ \gamma = \gamma_i$, where we view γ as a correspondence of $\mathrm{Spec}\, k$ and $\mathrm{SB}(A)$. Moreover, for any $S \in \mathbf{V}(k)$, $\mathrm{id}_S \times \pi : S \times X \to S \times \mathrm{SB}(A)$ is also a projective bundle, with $[\mathscr{O}(1)] = 1_S \times x \in \mathrm{CH}(S \times X)$, hence $f_i \circ \gamma' = \gamma'_i$ for all $\gamma' \in \mathrm{CH}(S \times X)$. By Manin's identity principle [7, §3], we then have that $f_i \circ g_j = \delta_{ij}\Delta_{\mathrm{SB}(A)}$ and from this is follows that $p_i = g_i \circ f_i$ form a complete set of orthogonal idempotents. For any other projective bundle X' of rank $n-2$ over $\mathrm{SB}(A)$ with $[\mathscr{O}_{X'}(1)] = x'$, we construct analoguous f'_i, g'_i, p'_i. Then one checks similarly that $g'_i \circ f_i$ and $g_i \circ f'_i$ give mutually inverse isomorphisms of $(\mathbf{M}(X), p_i)$ and $(\mathbf{M}(X'), p'_i)$. For the trivial projective bundle $X' = \mathbb{P}^{n-2}_{\mathrm{SB}(A)}$, we have $(\mathbf{M}(X'), p'_i) \cong \mathbf{M}(\mathrm{SB}(A)) \otimes \mathbb{Z}(i) = \mathbf{M}(\mathrm{SB})(i)$, where $\mathbb{Z} = \mathbf{M}(\mathrm{Spec}\, k)$ and $M(i)$ is the i-th twist of a motive M by the Tate motive (see [7, §8]). Thus, we get the decomposition

$$\mathbf{M}(X) = \bigoplus_{i=0}^{n-2} \mathbf{M}(\mathrm{SB}(A))(i)$$

Let $i : Y \hookrightarrow X$ be the embedding of Y into X. Let $\overline{f}_i = (i \times \mathrm{id}_{\mathrm{SB}(A)})^*(f_i), \overline{g}_i = (\mathrm{id}_{\mathrm{SB}(A)} \times i)^*(g_i)$, $0 \leqslant i \leqslant n-2$. Observe that $\overline{f}_i \in \mathrm{Hom}^{1-i}(\mathbf{M}(Y), \mathbf{M}(\mathrm{SB}(A)))$ and $\overline{g}_i \in \mathrm{Hom}^i(\mathbf{M}(\mathrm{SB}(A)), \mathbf{M}(Y))$.

Lemma 1.5. *For* $0 \leqslant i, j \leqslant n-3$, $\overline{f}_{i+1} \circ \overline{g}_j = \delta_{i,j}\Delta_{SB(A)}$.

Proof. It is a direct computation using the fact that $i_*(1_Y) = x$. See [9, Lemma 2.4] $\square$

The lemma implies that $\overline{p}_i = \overline{g}_i \circ \overline{f}_{i+1}$ are orthogonal idempotents in $\mathrm{Hom}(\mathbf{M}(Y), \mathbf{M}(Y))$. Thus they can be completed by adding $\overline{p} = \Delta_Y - \sum_{i=0}^{n-3} \overline{p}_i$. Notice also that $g_i \circ \overline{f}_{i+1}$ and $\overline{g}_i \circ f_i$ give mutually inverse isomorphisms of $(\mathbf{M}(X), p_i)$ and $(\mathbf{M}(Y), \overline{p}_i)$ for $0 \leqslant i \leqslant n-3$. Thus we get the decomposition

$$\mathbf{M}(Y) = \bigoplus_{i=0}^{n-3} \mathbf{M}(SB(A))(i) \oplus (\mathbf{M}(Y), \overline{p})$$

2 Monodromy Action

Galois action and correspondences

Let $Z \in \mathbf{V}(k)$ and L/k a finite Galois extension with Galois group $\Gamma_L = \mathrm{Gal}(L/k)$. By the normal basis theorem, $L \otimes_k L \cong \prod_{\sigma \in \Gamma_L} L$, where the following diagram commutes:

$$\begin{tikzcd}
L \otimes_k L \arrow[r, "\simeq"] & \prod_{\sigma\in\Gamma_L} L \arrow[d, "\pi_\sigma"] \\
L \arrow[u, "1\otimes \mathrm{id}_L"] \arrow[r, "\sigma"'] & L
\end{tikzcd}$$

Taking the equivalent maps on the spectrum and base changing by Z, we obtain the following commutative diagram:

$$\begin{tikzcd}
Z_L \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L \arrow[d] & \coprod_{\sigma\in\Gamma_L} Z_L \arrow[l, "\simeq"'] \\
Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L & Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L \arrow[l, "\mathrm{id}_Z \times \sigma"'] \arrow[u, "i_\sigma"]
\end{tikzcd}$$

where $Z_L = Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$.

Let $\mathrm{res}_{L/k} : \mathrm{CH}(Z) \to \mathrm{CH}(Z_L)$ denote the pullback of $\pi : Z_L = Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L \to Z$, i.e. $\gamma \mapsto \gamma \times 1_{\mathrm{Spec}\, L}$. The above diagram implies that for Z_L, $\mathrm{res}_{L/k}(\gamma) = (\sigma\gamma)_{\sigma\in\Gamma_L}$ after making the identification $\mathrm{CH}(Z_L \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L) = \bigoplus_{\sigma\in\Gamma_L} \mathrm{CH}(Z_L)$, with the Galois action being the induced one from the diagrams. We also define a pairing $\langle \cdot\, , \cdot \rangle_{Z_L} : \mathrm{CH}(Z_L) \times \mathrm{CH}(Z_L) \to \mathrm{CH}(\mathrm{Spec}\, L)$ by $\langle x, y\rangle_L = p_*(x \cdot y)$, p the projection $Z_L \to \mathrm{Spec}\, L$.

Lemma 2.1. *Let* $\gamma_1 \in \mathrm{Hom}(\mathbf{M}(\mathrm{Spec}\, L), \mathbf{M}(Z))$ *and* $\gamma_2 \in \mathrm{Hom}(\mathbf{M}(Z), \mathbf{M}(\mathrm{Spec}\, L))$. *Then*

$$\mathrm{res}_{L/k}(\gamma_1 \circ \gamma_2) = \sum_{\sigma\in\Gamma_L} \sigma\gamma_1 \times \sigma\gamma_2$$

under the identification $\mathrm{CH}(Z \times_{\mathrm{Spec}\, k} Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L) = \mathrm{CH}(Z_L \times_{\mathrm{Spec}\, L} Z_L)$ *and*

$$\mathrm{res}_{L/k}(\gamma_2 \circ \gamma_1) = (\langle \sigma\gamma_1, \tau\gamma_2^t\rangle_{Z_L})_{\sigma,\tau\in\Gamma_L}$$

under the identification $\mathrm{CH}(\mathrm{Spec}\, L \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L) = \bigoplus_{\sigma,\tau\in\Gamma_L} \mathrm{CH}(\mathrm{Spec}\, L)$.

Proof. We remark that $\mathrm{res}_{L/k}(\gamma_i \circ \gamma_j) = \mathrm{res}_{L/k}(\gamma_i) \circ \mathrm{res}_{L/k}(\gamma_j)$, where the second composition is for correspondences over L.

For $\mathrm{res}_{L/k}(\gamma_2 \circ \gamma_1)$ we have $\mathrm{CH}(Z_L \times_{\mathrm{Spec}\, L} (\mathrm{Spec}\, L)_L \times_{\mathrm{Spec}\, L} Z_L) \cong \bigoplus_{\sigma\in\Gamma_L} \mathrm{CH}(Z_L \times_{\mathrm{Spec}_L} Z_L)$, with projection onto the product of the outer factors on the left hand side corresponding to summing the terms of the right hand side. Thus using the fact that $\mathrm{res}_{L/k}(\gamma_i) = (\sigma\gamma_i)_{\sigma\in\Gamma_L}$ $(i = 1, 2)$ gives the desired equation.

For $\mathrm{res}_{L/k}(\gamma_1 \circ \gamma_2)$, we have $\mathrm{CH}((\mathrm{Spec}\, L)_L \times_{\mathrm{Spec}\, L} Z_L \times_{\mathrm{Spec}\, L} (\mathrm{Spec}\, L)_L) \cong \bigoplus_{\sigma,\tau\in\Gamma_L} \mathrm{CH}(Z_L)$ and $\mathrm{CH}((\mathrm{Spec}\, L)_L \times_{\mathrm{Spec}\, L} (\mathrm{Spec}\, L)_L) \cong \bigoplus_{\sigma,\tau\in\Gamma_L} \mathrm{CH}(\mathrm{Spec}\, L)$. The projection map corresponds to p_* as above, so the equation now follows directly from the definition of composing correspondences and the computation of $\mathrm{res}_{L/k}(\gamma_i)$. $\square$

With the same notation, we have

Corollary 2.2. $\gamma_2 \circ \gamma_1 = \Delta_{\operatorname{Spec} L}$ *if and only if* $\langle \sigma\gamma_1, \gamma_2^t \rangle_L = \delta_{1\sigma} 1_{\operatorname{Spec} L}$

Proof. $\operatorname{res}_{L/k}$ is clearly injective, so the condition $\gamma_2 \circ \gamma_1 = \Delta_{\operatorname{Spec} L}$ is equivalent to $\operatorname{res}_{L/k}(\gamma_2 \circ \gamma_1) = (\sigma\Delta_{\operatorname{Spec} L})_{\sigma \in \Gamma_L}$. The isomorphism $\operatorname{CH}(\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L) \cong \bigoplus_{\tau \in \Gamma_L} \operatorname{CH}(\operatorname{Spec} L)$ maps $\sigma\Delta_{\operatorname{Spec} L}$ to $(\delta_{\sigma\tau} 1_{\operatorname{Spec} L})_{\tau \in \Gamma_L}$. Considering $\operatorname{CH}((\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L)_L)$ as $\bigoplus_{\sigma, \tau \in \Gamma_L} \operatorname{CH}(\operatorname{Spec} L)$, Lemma 2.1 completes the proof. □

Cocycle considerations

We will need the following description of the isomorphism $H^2(\Gamma_{k^s}, (k^s)^\times) \cong \operatorname{Br}(k)$, following [8, §IV.3]. Let $\mathscr{A}(L/k)$ be the class of central simple algebras A/k containing the Galois extension L/k with $\deg A = [L : k]$. For any such algebra A, one can produce a right L-basis of A consisting of elements e_σ, $\sigma \in \Gamma_L$, where the inner automorphism given by e_σ acts as σ on $L \subseteq A$. Then necessarily $e_\sigma e_\tau = \varphi_{\sigma,\tau} e_{\sigma\tau}$, with $\varphi_{\sigma,\tau} \in L^\times$. Since A is associative, this defines a 2-cocycle $\varphi : \Gamma_L \times \Gamma_L \to L^\times$, and for any other choice of the e_σ, the 2-cocycle φ' obtained from them is cohomologous to φ. This gives a well-defined map $f : \mathscr{A}(L/k)/_\sim \to H^2(\Gamma_L, L^\times)$. There is a construction $A(\varphi)$, $\varphi \in Z^2(\Gamma_L, L^\times)$ called the crossed-product algebra of φ, which induces an inverse to f. Let $\operatorname{Br}(L/k)$ denote the kernel of $\operatorname{res}_{L/k} : \operatorname{Br}(k) \to \operatorname{Br}(L)$. Then every element of $\operatorname{Br}(L/k)$ has a unique representative in $\mathscr{A}(L/k)$, which induces an isomorphism of groups $H^2(\Gamma_L, L^\times) \cong \operatorname{Br}(L/k)$. This isomorphism is also natural, in the sense that for a finite Galois extension $L \subseteq K$

$$\begin{tikzcd}
\operatorname{Br}(L/k) \arrow[r, hook] & \operatorname{Br}(K/k) \\
H^2(\Gamma_L, L^\times) \arrow[u, "\cong"] \arrow[r, hook, "\inf"'] & H^2(\Gamma_K, K^\times) \arrow[u, "\cong"']
\end{tikzcd}$$

commutes, where the bottom horizontal map is the usual inflation map. Thus taking unions over all finite Galois extensions, we get an isomorphism $H^2(\Gamma_{k^s}, (k^s)^\times) \cong \operatorname{Br}(k)$.

Definition 2.3. Let L/k be finite Galois. A class in $H^2(\Gamma_L, L^\times)$ is special iff it has a representative φ with values in $k^\times$ and it is symmetric, i.e. for any $\sigma, \tau \in \Gamma_L$, $\varphi(\sigma, \tau) = \varphi(\tau, \sigma) \in k^\times$. The special classes form a subgroup, which we denote by S_L.

The inflation map clearly sends S_L into S_K for Galois extensions K/L, and thus so too for the isomorphic images of S_L and S_K in $\operatorname{Br}(L/k)$ and $\operatorname{Br}(K/k)$ respectively.

For $A \in \mathscr{A}(L/k)$, choose a set of e_σ as above, or equivalently, an isomorphism $A \cong A(\varphi)$ for φ a representative of class in $H^2(\Gamma_L, L^\times)$ associated to A. Fix an order $\sigma_1, \ldots, \sigma_n$ of the elements of Γ_L. Then to an element $a \in A$, we get an element $\rho(a) \in M_n(L)$, defined as the matrix associated to the L-linear map $A \to A, x \mapsto ax$ with respect to the ordered basis $e_{\sigma_1}, \ldots e_{\sigma_n}$. This gives a map of k-algebras $\rho : A \to M_n(L)$.

Lemma 2.4. *The induced map* $\rho_L : A \otimes_k L \to M_n(L)$ *is an isomorphism.*

Proof. It is enough to show that $\rho(A)$ contains n^2 elements which are linearly independent over L. Let β_{σ_i}, $1 \leqslant i \leqslant n$, be a normal basis for L/k. We claim that the elements $\rho(e_{\sigma_i} \beta_{\sigma_j})$ $1 \leqslant i, j \leqslant n$ are linearly independent over L. Notice that the first column of $\rho(e_{\sigma_i} \beta_{\sigma_j})$, $1 \leqslant i, j \leqslant n$ has one non-zero entry, with the row depending only on i, so we reduce to showing that $\rho(e_\sigma \beta_{\sigma_i})$, $1 \leqslant i \leqslant n$, $\sigma \in \Gamma_L$ are linearly independent. Since $\beta_{\sigma_i} e_{\sigma_j} = e_{\sigma_j} \beta_{\sigma_j^{-1}\sigma_i}$, this reduces to showing that the matrix $(\beta_{\sigma_j^{-1}\sigma_i})_{1 \leqslant i,j \leqslant n}$ is invertible, and one can always choose such a normal basis (see [6, Theorem 13.1]). □

Such a ρ_L is not in general Γ_L-equivariant (the actions being the one from base change on $A \otimes_k L$ and the entry-wise action on $M_n(L)$). Indeed, it is well known that the obstruction is a 1-cocyle with values in $\operatorname{PGL}_n(L)$. More precisely, let $a_\sigma = \sigma \circ \rho_L \circ \sigma^{-1} \circ \rho_L^{-1}$. It is an L-algebra automorphism of $M_n(L)$, so comes from conjugation by a unique element of $\operatorname{PGL}_n(L)$.

Proposition 2.5. *Suppose* L/k *abelian. If* $[A] \in S_L \subseteq \operatorname{Br}(L/k)$*, then there is a choice of* e_{σ_i} *such that* $a_{\sigma_i} = \operatorname{Inn}(\rho_L(e_{\sigma_i}))$*. In particular,* $a_{\sigma_i} \circ a_{\sigma_j} = a_{\sigma_i \sigma_j}$.

Proof. The condition on the cocycle means we can choose e_{σ_i} such that

(1) $e_{\sigma_i}e_{\sigma_j} = \alpha e_{\sigma_i\sigma_j}$, $\alpha \in k^\times$ and,

(2) $e_{\sigma_i}e_{\sigma_j} = \alpha e_{\sigma_i\sigma_j} = e_{\sigma_j}e_{\sigma_i}$

Since $M_n(L)$ is generated as an L-algebra by the $\rho_L(e_{\sigma_i}), \rho_L(\beta_{\sigma_i})$, $1 \leqslant i \leqslant n$, it is enough to show the maps agree on these. $\mathfrak{a}_\sigma(\rho_L(e_{\sigma_i})) = \sigma(\rho_L(e_{\sigma_i}))$. All the entries of $\rho_L(e_{\sigma_i})$ are in k by condition (1), so this is just $\rho_L(e_{\sigma_i})$. By (2), $\mathrm{Inn}(\rho_L(e_\sigma))$ also fixes the $\rho_L(e_{\sigma_i})$. Similarly $\mathfrak{a}_\sigma(\rho_L(\beta_{\sigma_i})) = \sigma(\rho_L(\beta_{\sigma_i}))$. $\rho_L(\beta_{\sigma_i})$ is a diagonal matrix with jj-th entry $\beta_{\sigma_j^{-1}\sigma_i}$, $1 \leqslant j \leqslant n$. Since Γ_L is abelian, this implies that $\sigma(\rho_L(\beta_{\sigma_i})) = \rho_L(\beta_{\sigma\sigma_i})$, but by the defintion of e_σ, this agrees with $\mathrm{Inn}(\rho_L(e_\sigma))$. $\square$

We will often refer to the hypothesis on A and L of this lemma by calling the pair $L \subseteq A$ special.

Example 2.5.1. Let L/k be a cyclic field extension, i.e. Galois with Γ_L cyclic of order n. Choosing $\alpha \in k^\times$, one can define uniquely a central simple algebra A over k of degree n containing L such that A is generated by L and an element $u \notin L$ such that $u^n = \alpha$ and $au = u\tau(a), \forall a \in L$, τ a generator of Γ_L (see [4, Proposition 2.5.2]). It is clear that taking powers of u for the e_{σ_i} will give a special cocycle in $H^2(\Gamma_L, L^\times)$.

Example 2.5.2. Consider two quaternion algebras over k, Q_1 and Q_2, respectively containing splitting fields K_1 and K_2. If $A = Q_1 \otimes_k Q_2$ is non-split, then the compositum $L = K_1K_2$ is a maximal subfield of A with Galois group $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$. Moreover, $[A] = [Q_1] \cdot [Q_2]$ in $\mathrm{Br}(L/k)$. By the previous example, $[Q_i] \in S_{K_i}$, hence inflates to a class in S_L. Thus $L \subseteq A$ is special.

Remark 2.5.1. Unfortunately, the subgroup S generated by the S_L, L/k finite Galois, is not very useful since ultimately we are interested in whether a class is special for a specic Galois extension. However, the situation could be remedied if one could prove that in general $S_L = \inf^{-1}(S_K)$, K/L finite Galois (or even finite abelian, since we can only consider classes in $\mathrm{Br}(k^{ab}/k)$ to begin with). In this case, it would follow that $S \cap \mathrm{Br}(L/k) = S_L$. This (essentially cohomological statement) can be interpreted as roughly saying that going from A to $M_n(A)$ does not give you enough “wiggle room” to choose elements e_σ as in Proposition 2.5 if you could not already in A.

Action on equivariant Chow rings

We now always assume that L/k is abelian and $L \subseteq A$ is special. Let $X_0 = X(M_n(k))$, using the notation of Definition 1.1. The isomorphism $\rho_L : A \otimes_k L \to M_n(L)$ of Proposition 2.5 induces an isomorphism $\rho : X_L \to (X_0)_L$. For each $\sigma \in \Gamma_L$, $\mathfrak{a}_\sigma$ also induces an automorphism $\mathfrak{a}_\sigma : X_0 \to X_0$. Let T be the standard torus for GL_n, then X_0 has a natural T-action. A k-point $V \subseteq V' \subseteq k^n$ is fixed by this T-action iff it is of the form $(v_i) \subseteq (v_1, \ldots, \hat{v}_j, \ldots, v_n)$, $i \neq j$, where v_i, $1 \leqslant i \leqslant n$, is an eigenbasis for T. We refer to these points by x_{ij}. Clearly this is suitably functorial, so we get a fixed point subscheme X_0^T which is finite, reduced and where all points are rational over k, which we will also call x_{ij}. $\mathfrak{a}_\sigma$ maps x_{ij} to $x_{\sigma(i)\sigma(j)}$, where $\sigma\sigma_l = \sigma_{\sigma(l)}$, $1 \leqslant l \leqslant n$. Indeed, $\rho_L(e_\sigma)$ permutes the subspaces (v_i) in this fashion. These observations, together with the definition of $\mathfrak{a}_\sigma$, give us the following commutative diagram:

$$\begin{tikzcd} X_L \arrow[r, "\rho"] \arrow[d, "\sigma"'] & (X_0)_L \arrow[d, "\mathfrak{a}_{\sigma^{-1}} \times \sigma"] & (X_0^T)_L \arrow[l, hook'] \arrow[d, "\mathfrak{a}_{\sigma^{-1}} \times \sigma"] \\ X_L \arrow[r, "\rho"'] & (X_0)_L & (X_0^T)_L \arrow[l, hook'] \end{tikzcd}$$

The T-action on X_0 induces a T_L-action on $(X_0)_L$, with $(X_0)_L^{T_L} = (X_0^T)_L$, and one can pullback the T_L-action to X_L by ρ. By the definition of ρ, Y_L is T_L-stable and $(X_L)^{T_L} \subseteq Y_L$. It is important to note that the action of T on X_0 does not commute with $\mathfrak{a}_\sigma$. However, $\mathfrak{a}_\sigma$, the element of PGL_n, acts on T by conjugation, and we have that

$$\begin{tikzcd} T \times X_0 \arrow[r] \arrow[d, "\mathrm{Inn}(\mathfrak{a}_\sigma) \times \mathfrak{a}_\sigma"'] & X_0 \arrow[d, "\mathfrak{a}_\sigma"] \\ T \times X_0 \arrow[r] & X_0 \end{tikzcd}$$

commutes.

For a torus T over k, let $M(T)$ (or simply M) be the group of characters $\mathrm{Hom}_{k^s}(T_{k^s}, \mathbb{G}_{m,k^s})$. This comes with a natural action of the absolute Galois group by conjugation, and T is split iff this action is trivial. This also implies that if K/k is a Galois splitting field for T, the natural map $\mathrm{Hom}_K(T_K, \mathbb{G}_{m,K}) \to \mathrm{Hom}_{k^s}(T_{k^s}, \mathbb{G}_m)$ is an isomorphism, and hence the Galois action of Γ_{k^s} factors through Γ_K. For Z a non-singular variety over k with a torus action, we denote by $CH_T(Z)$ the T-equivariant Chow ring of [2]. This is naturally an $S(T) = \mathrm{Sym}\, M(T)$ module, with $CH_T(\mathrm{Spec}\, k) = S(T)$ (or simply S). The equivariant Chow ring has the usual functorial properties of the Chow ring for equivariant maps. By the localisation theorem of [2, §3], if Z is complete, Z^T is a non-singular variety and the map $i^* : CH_T(Z) \to CH_T(Z^T)$ induced by the inclusion $i : Z^T \hookrightarrow Z$ is injective. Combining all the observations of this section, we obtain

Proposition 2.6. *$CH_{T_L}((X_0)_L^{T_L}) \cong \bigoplus_{1\leqslant i\neq j\leqslant n} S(T_L)$. For $\sigma \in \Gamma_L$, let*

$$\hat{\sigma} : \bigoplus_{1\leqslant i\neq j\leqslant n} S(T_L) \to \bigoplus_{1\leqslant i\neq j\leqslant n} S(T_L), \quad (\varphi_{ij})_{ij} \mapsto (\sigma^{-1}\varphi_{\sigma(i)\sigma(j)})_{ij}$$

where the action of Γ_L on S is given by $t_i \mapsto t_{\sigma(i)}$, $t_i \in M$ the character given by the i-th factor of T_L. Then the following diagram commutes

$$\begin{tikzcd}
CH_{T_L}(Y_L) \arrow[d, "\sigma"] & CH_{T_L}(X_L) \arrow[l, "i^*"'] \arrow[r, "\rho^*"] \arrow[d, "\sigma"] & CH_{T_L}((X_0)_L) \arrow[r, hook] \arrow[d, "a_{\sigma^{-1}}\times\sigma"] & CH_{T_L}((X_0)_L^{T_L}) \arrow[r, "\cong"] \arrow[d, "a_{\sigma^{-1}}\times\sigma"] & \bigoplus_{1\leqslant i\neq j\leqslant n} S \arrow[d, "\hat{\sigma}"] \\
CH_{T_L}(Y_L) & CH_{T_L}(X_L) \arrow[l, "i^*"] \arrow[r, "\rho^*"'] & CH_{T_L}((X_0)_L) \arrow[r, hook] & CH_{T_L}((X_0)_L^{T_L}) \arrow[r, "\cong"] & \bigoplus_{1\leqslant i\neq j\leqslant n} S
\end{tikzcd}$$

where the T_L actions of the bottom row are conjugated by $a_{\sigma^{-1}}$.

Proof. The isomorphism comes from the description of the fixed point locus. The three leftmost squares commute by functoriality, so we need only check that $\hat{\sigma}$ computes the action on fixed points. The pre-image of x_{ij} under $a_{\sigma^{-1}}$ is $x_{\sigma(i)\sigma(j)}$, which gives the interchange of summands. The conjugation by $a_{\sigma^{-1}}$ on T_L gives in the pullback to the action of Γ_L defined on S. The Galois action on the individual fixed points corresponds in the pullback on equivariant Chow rings to the usual Galois action on M, which is trivial since T is split. Hence $\hat{\sigma}$ does agree with the map on $CH_{T_L}((X_0)_L^{T_L})$ induced by $a_{\sigma^{-1}} \times \sigma$. □

This gives actions of Γ_L on $CH_{T_L}(Y_L)$, $CH_{T_L}(X_L)$ and $\bigoplus_{1\leqslant i\neq j\leqslant n} S$. We refer to all of these as the *monodromy action*.

3 The Artin Motive

Using the monodromy action of the previous section, we can identify $\mathbf{M}(Y, \overline{p})$ as the Artin motive $\mathbf{M}(\mathrm{Spec}\, L)(n-2)$, up to "phantoms", that is motives in the decomposition which vanish after base change by a field extension. To begin, we collect some facts about Y_L.

The geometry of Y_L

In the previous section, we used the action of a torus $T \subseteq GL(V)$ on the Milnor hypersurface $X_0(V)$. Notice that the inclusion $X_0(V) \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(V^*)$ is $GL(V)$-equivariant. $\mathbb{P}(V) \times \mathbb{P}(V^*)$ has a clear interpretation as the unique closed orbit of the projectified representation of $V \otimes V^*$, where V is the standard representation of $GL(V)$. The Milnor hypersurface will then correspond to the restriction of this orbit to the projectification of a codimension 1 subrepresentation of $V \otimes V^*$ corresponding to the kernel of the pairing of V and V^*. However, under the identification $V \otimes V^* = \mathfrak{gl}(V)$, this corresponds to $\mathfrak{sl}(V)$. In other words, $X_0(V)$ (over the closure) is the closed orbit for the adjoint representation of $SL(V)$. This allows us to make use of results of [1].

Recall that for any flag variety $Z = GL(V)/B$, any T-stable curve is rational, and passes through exactly two T-fixed points. Moreover, these T-fixed points each correspond to roots α, β of G (the fact that it is the adjoint representation of course implies this is one-to-one) and determine the T-stable curve uniquely. We will denote such a curve by $C_{\alpha\beta}$ and say that it has *weight* $\alpha - \beta \in M(T)$. For details, see for example [3].

Lemma 3.1. *Consider the isomorphism* $\rho : X_L \to (X_0)_L$ *and* x_{ij}, $1 \leqslant i \neq j \leqslant n$, *the* T_L*-fixed points in* $(X_0)_L$. *The* T_L*-stable curves contained in* $\rho(Y_L)$ *connect* x_{ij} *to* x_{ik} *and* x_{ij} *to* x_{kj}, i, j, k *distinct.*

Proof. Suppose i, j, k distinct, and let $\alpha_{ij}, \alpha_{jk}, \ldots$ denote roots in M where $\alpha_{ij} = t_j - t_i$. $s_{\alpha_{jk}}(\alpha_{ij}) = \alpha_{ik}$ and we can similarly move from α_{ij} to α_{kj}. From [3], there are T_L-stable curves connecting the fixed points x_{ij} to x_{jk} and x_{ij} to x_{kj} in X_L. Proposition 3.8 of [1] implies that these curves are contained in Y_L, and are the only ones. □

Lemma 3.2. *The image of* $CH_{T_L}(Y_L)$ *in* $\bigoplus_{1 \leqslant i \neq j \leqslant n} S$ *(under the map induced by* $(X_L)^{T_L} \subseteq Y_L$, ρ *and the isomorphism of Proposition 2.6) is* $\{(\varphi_{ij})_{ij} : \varphi_{ij} \equiv \varphi_{kl} \mod \alpha, i = k \text{ and } \alpha = \alpha_{jl} \text{ or } j = l \text{ and } \alpha = \alpha_{ik}\}$.

Proof. Notice that all the weights for the T_L-stable curves of Y_L are primitive, so the description of the image follows from [2, Theorem 3.4] and Lemma 3.1. □

Lemma 3.3. *Define the (equivariant) pairing* $\langle \cdot , \cdot \rangle_{Y_L, T_L} : CH_{T_L}(Y_L) \to CH_{T_L}(\operatorname{Spec} L)$ *by* $\langle x, y \rangle_{Y_L, T_L} = s_*(x \cdot y)$, *where* $s : Y_L \to \operatorname{Spec} L$ *is the structure morphism. Then*

$$\langle \varphi, \psi \rangle_{Y_L, T_L} = \sum_{1 \leqslant i \neq j \leqslant n} \frac{\varphi_{ij}\psi_{ij}}{\prod_{k \neq i,j} \alpha_{ik}\alpha_{kj}}$$

where $(\varphi_{ij})_{ij}$ *and* $(\psi_{ij})_{ij}$ *are the images of* φ *and* ψ *in* $\bigoplus_{1 \leqslant i \neq j \leqslant n} S$

Proof. For the fixed point x_{ij}, the weight spaces $\operatorname{Tan}_{x_{ij}} Y_L$ coming from the T_L-stable curves have weights α_{ik} and α_{kj}, $k \neq i, j$. Since there are $\dim Y_L = 2n - 4$ of these, they span the entire tangent space. The equivariant multiplicity formula of [2, §4.2] then gives the result. □

Support and orthogonality of the Artin motive

For $1 \leqslant l \leqslant n$, let $\gamma_l = (\gamma^l_{ij}) \in \bigoplus_{1 \leqslant i \neq j \leqslant n} S$, with $\gamma^l_{ij} = \delta_{i,l} \prod_{s \neq i,j} \alpha_{is}$. They satisfy the following important properties:

Lemma 3.4. *For* $1 \leqslant k, l \leqslant n$ *and* $\sigma \in \Gamma_L$, *(i)* γ_l *is in the image of* $CH^{n-2}_{T_L}(Y_L)$, *(ii)* $\hat{\sigma}(\gamma_l) = \gamma_{\sigma^{-1}(l)}$, *(iii)* $\langle \gamma_k, \gamma_l \rangle_{Y_L, T_L} = (-1)^{n-2}\delta_{k,l}$.

We refer to the proof given in [9, §4.2]. Let $\overline{\gamma}_l$ be the image of γ_l under the natural map $CH_{T_L}(Y_L) \to CH(Y_L)$. Let $\epsilon : CH_{T_L}(\operatorname{Spec} L) \to CH(\operatorname{Spec} L)$ be said map for the case of $\operatorname{Spec} L$, then we have $\langle \overline{\gamma}_l, \overline{\gamma}_k \rangle_{Y_L} = \epsilon(\langle \gamma_l, \gamma_k \rangle_{Y_L, T_L})$.

Let $f = \overline{\gamma}_1 \in CH^{n-2}(Y \times_{\operatorname{Spec} k} \operatorname{Spec} L) = \operatorname{Hom}^{-(n-2)}(\mathbf{M}(Y), \mathbf{M}(\operatorname{Spec} L))$ and $g = (-1)^{n-2}\overline{\gamma}_1 \in CH^{n-2}(Y \times_{\operatorname{Spec} k} \operatorname{Spec} L) = \operatorname{Hom}^{n-2}(\mathbf{M}(\operatorname{Spec} L), \mathbf{M}(Y))$.

Proposition 3.5. $p = g \circ f$ *is an idempotent of* $CH(Y \times Y)$ *and* $(\mathbf{M}(Y), p) \cong \mathbf{M}(\operatorname{Spec} L)(n-2)$.

Proof. By Proposition 2.6, we have that the image of $\hat{\sigma}\gamma_l$ in $CH(Y \times \operatorname{Spec} L)$ is $\sigma\overline{\gamma}_l$ for all $\sigma \in \Gamma_L$. It then follows that for $\sigma \in \Gamma_L$, we have

$$\langle \overline{\gamma}_1, (-1)^{n-2}\sigma\overline{\gamma}_1 \rangle_{Y_L} = \epsilon(\langle \gamma_1, (-1)^{n-2}\gamma_{\sigma^{-1}(1)} \rangle_{Y_L, T_L}) = \delta_{1\sigma^{-1}(1)}$$

by Lemma 3.4. Corollary 2.2 then implies that $f \circ g = \Delta_{\operatorname{Spec} L}$, so $p = g \circ f$ is idempotent. The degrees of f, g imply that they are mutually inverse isomorphisms of $\mathbf{M}(Y, p)$ to $\mathbf{M}(\operatorname{Spec} L)(n-2)$. □

Proposition 3.6. p *is orthogonal to the* $\overline{p}_i$, $0 \leqslant i \leqslant n-3$, *and* $p = \overline{p}$ *over* L.

The proof is rather technical and subtle, so we refer the reader to [9, §4.3]. Let $p' = \overline{p} - p$. Then we have the motivic decomposition

$$\mathbf{M}(Y) = \bigoplus_{i=0}^{n-3} \mathbf{M}(SB(A))(i) \oplus \mathbf{M}(\operatorname{Spec} L)(n-2) \oplus (\mathbf{M}(Y), p')$$

where by Proposition 3.6 the motive $(\mathbf{M}(Y), p')$ is a phantom.

Definition 3.7. The Rost nilpotence principle holds for a variety Z over k if for any field extension K/k, the kernel of the homomorphism $\operatorname{End}(\mathbf{M}(Z)) \to \operatorname{End}(\mathbf{M}(Z_K))$ consists of nilpotent elements, where the second endomorphism ring has composition as correspondences over K.

This result implies that all phantoms are trivial, since nilpotent idempotents are 0. The main theorem from the introduction then holds assuming Y satisfies Rost nilpotence.

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