Motives of Certain Hyperplane Sections of Milnor Hypersurfaces

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Abstract

We construct a hyperplane section Y of a Milnor hypersurface associated to a regular semisimple endomorphism φ. Exploiting its structure as a hyperplane section of a projective bundle and its natural torus action, we give a motivic decomposition of Y, which encodes both the cellular structure of Y and the arithmetic of the eigenvalues of φ. This decomposition is proven without using the "nilpotence principle", that is to say there are no "phantoms".

1 Introduction

Let V be a vector space of dimension $n+1$ over a field k. Consider the partial flag variety

$$E = \{W_1 \subseteq W_n \subseteq V : \dim W_i = i\}$$

Let $\varphi \in \mathrm{End}(V)$ be an endomorphism with distinct eigenvectors, so in particular the subalgebra $L := k[\varphi] \subseteq \mathrm{End}(V)$ is étale of dimension $n+1$ over k. Now define the hyperplane section of E

$$Y = \{W_1 \subseteq W_n \subseteq V : \dim W_i = i, \quad \varphi(W_1) \subseteq W_n\}$$

The main result of this paper is

Theorem 1.1. *If L is a finite product $\prod_i K$, for a fixed Galois extension K/k, and $n \geqslant 1$, then the motive of Y decomposes as*

$$\mathbf{M}(Y) = \bigoplus_{i=0}^{n-2} \mathbf{M}(\mathbb{P}^n_k)(i) \oplus \mathbf{M}(\mathrm{Spec}\, L)(n-1)$$

This gives a proof of a special case of the main theorem of [8] without "phantoms" – motives in the decomposition which become trivial after a field extension. In fact, this was the main inspiration for the present article and many of the results of [8] appear here, if only implicitly.

Our approach first considers the problem in the general setting of an $\mathscr{O}(1)$-type divisor of a projective bundle in Section 2, where we obtain a criterion (Theorem 2.2) for a decomposition as in Theorem 1.1 to hold. The criterion is then verified for Y in Section 3 using equivariant methods for torus actions. As a consequence, we obtain a natural proof showing that if φ satisfies the hypotheses of Theorem 1.1, L (or equivalently K) is an invariant of Y (Theorem 3.7).

Notation and conventions: A smooth variety over a field k is an equidimensional algebraic scheme which is smooth over k. For a vector space V over k and a k-scheme $s : X \to \mathrm{Spec}\, k$, $\underline{V}$ is the trivial vector bundle $s^*(\tilde{V})$ on X. $\mathbb{P}(V)$ is the projective space of one-dimensional quotients of V, and similarly $\mathbb{P}(\mathscr{E})$ is the projective bundle of rank one quotients of the vector bundle $\mathscr{E}$. A Cartier divisor linearly equivalent to the zero locus of a section of a line bundle $\mathscr{L}$ is said to be of $\mathscr{L}$-type. The i-th power of the Tate motive is written $\mathbb{Z}(i)$, and "twists" in the opposite direction of the Tate twist of ℓ-adic cohomology.

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2 Chow groups of $\mathscr{O}(1)$-type divisors on a projective bundle

Let k be an arbitrary field. We consider the following situation: X is a smooth projective variety over k with a vector bundle $\mathscr{E}$ of rank $r+1$ which is generated by global sections. Set $E = \mathbb{P}(\mathscr{E}) = \operatorname{Proj}(\operatorname{Sym}\mathscr{E})$ with projection map π. We will be concerned with sections $s \in H^0(X, \mathscr{E}) = H^0(E, \mathscr{O}(1))$ such that the zero locus Z of s is smooth of codimension $r+1$ in X and Y, the divisor corresponding to s, is smooth. Letting $U = X - Z$, $Y|_U := Y \times_X U$ is a projective bundle of rank $r-1$ over U (corresponding to the cokernel $\mathscr{F}$ of $\mathscr{O}_X \xrightarrow{s} \mathscr{E}$, restricted to U) and $Y|_Z = E|_Z$ is a projective bundle of rank r over Z (corresponding to $\mathscr{E} \otimes \mathscr{O}_Z$). The following commutative diagram summarises our notation for the inclusion maps:

$$\begin{tikzcd}
E|_Z \arrow[r, hook, "j'"'] \arrow[dr, hook, "j"] & E & E|_U \arrow[l, hook'] \\
& Y \arrow[u, hook, "i"] & Y|_U \arrow[l, hook'] \arrow[u, hook] \arrow[ul, hook, "i'"']
\end{tikzcd}$$

By the projective bundle theorem [4, Theorem 9.6], we have that $A^\bullet(E)$ is a free $A^\bullet(X)$-module generated by H_E^i, $H_E = c_1(\mathscr{O}(1))$ and $i = 0, \dots, r$. Analoguous statements hold for $A^\bullet(Y|_U)$, $A^\bullet(U)$ and $H_Y = i'^* H_E$ (rank r), and $A^\bullet(E|_Z)$, $A^\bullet(Z)$ and $H = j'^* H_E$ (rank $r+1$). The former requires some explanation. By construction, $Y|_U$, as a U-scheme, is $\mathbb{P}(\mathscr{F}|_U)$. The inclusion into $E|_U = \mathbb{P}(\mathscr{E}|_U)$ comes from the surjective homomorphism $\operatorname{Sym}\mathscr{E}|_U \twoheadrightarrow \operatorname{Sym}\mathscr{F}|_U$ induced by the quotient map $\mathscr{E} \twoheadrightarrow \mathscr{F}$. Hence the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{F}|_U)}(1)$ on $Y|_U$ is the pullback of $\mathscr{O}_{\mathbb{P}(\mathscr{E}|_U)}(1)$ by this inclusion (this follows from the local case as in [6, Proposition II.5.13.c]). Since $\mathscr{O}_{\mathbb{P}(\mathscr{E}|_U)}(1) = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|_U$, the identity $H_Y = i'^* H_E$ is the translation of the above for the first Chern classes of the line bundles.

Given a smooth projective variety S, we can take the data $X, \mathscr{E}, s$ and associate to it the data $X \times_k S, p_1^*\mathscr{E}, p_1^*(s)$. Then, applying the above constructions to $X \times_k S, p_1^*\mathscr{E}, p_1^*(s)$, we see that the varities E, Z, Y are obtained from those constructed from $X, \mathscr{E}, s$ by taking a product with S. The same holds for morphisms and classes in the Chow rings. All of these operations will simply be called "base change by S".

Remark 2.0.1. It is harmless to take Z_{red} in this setup instead of Z. Indeed, U remains the same, and all other data are unaffected. The description of the "base change" of Z will still work, since a smooth variety S is geometrically reduced, hence $Z_{\text{red}} \times_k S = (Z \times_k S)_{\text{red}}$. We will denote both by Z is the sequel.

Define a group homomorphism $\varphi : A^\bullet(E) \oplus A^\bullet(E|_Z) \to A^\bullet(Y)$ by $\varphi = (i^*, j_*)$

Proposition 2.1. *With the same notations as above:*

a. *φ is surjective*

b. *For every class $\gamma \in A^\bullet(Y)$, there exist $\alpha_0, \dots, \alpha_{r-1} \in A^\bullet(X)$ and $\beta \in A^\bullet(Z)$ such that*

$$\varphi\Big(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i, \pi|_Z^*\beta\Big) = \gamma$$

c. *If $j_* \circ (\pi|_Z)^*$ is injective, then such an element is unique.*

Proof. **a.** By the right exact sequence $A^\bullet(E|_Z) \xrightarrow{j_*} A^\bullet(Y) \to A^\bullet(Y|_U) \to 0$, we are reduced to showing that i'^* is surjective. $A^\bullet(Y|_U)$ is generated (as a ring) by $(\pi|_U)^* A^\bullet(U)$ and H_Y. Clearly H_Y is in the image of i'^* and the commutativity of

$$\begin{tikzcd}
A^\bullet(E) \arrow[r, "i'^*"] & A^\bullet(Y|_U) \\
A^\bullet(X) \arrow[u, "\pi^*"] \arrow[r] & A^\bullet(U) \arrow[u, "\pi|_U^*"']
\end{tikzcd}$$

and the surjectivity of the restriction $A^\bullet(X) \to A^\bullet(U)$ shows that $(\pi|_U)^* A^\bullet(U)$ is also in the image.

b. That we can eliminate positive powers of H follows from the equality $j_*(H \cdot \alpha) = i^*(j'_*(\alpha))$. This identity holds since $i^* \circ j'_* = (i^* \circ i_*) \circ j_*$ and since Y is a divisor we have $(i^* \circ i_*)(\beta) = i^*(H_E) \cdot \beta$ ([5, Proposition 2.6.c]). But by the projection formula, $i^* H_E \cdot j_*(\alpha) = j_*(j^*(i^*(H_E)) \cdot \alpha) = j_*(H \cdot \alpha)$.

Now let $\alpha \in A^\bullet(X)$, then $i'^*(\pi^*(\alpha) \cdot H_E^r) = \sum_{i=0}^{r-1}(\pi|_U)^*\gamma_i \cdot H_Y^i$ by the projective bundle theorem, hence by the proof of **a.** there are elements $\alpha_0, \dots, \alpha_{r-1}$ such that $i'^*(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i) = \sum_{i=0}^{r-1}(\pi|_U)^*\gamma_i \cdot H_Y^i$. Hence $i'^*(\pi^*\alpha \cdot H_E^r - \sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i) = 0$, so $i^*(\pi^*\alpha \cdot H_E^r) - i^*(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i) \in \operatorname{im} j_*$. So there are $\beta_0, \dots, \beta_r$ such that $\varphi(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i, \sum_{j=0}^{r}(\pi|_Z)^*\beta_j \cdot H^i) = i^*(\pi^*\alpha \cdot H_E^r)$. Eliminating the positive powers of H as above will then give an element of the desired form.

c. Elements of the form given in **b.** form a subgroup in $A^\bullet(E) \oplus A^\bullet(E|_Z)$, so we just need to prove $\ker \varphi$ meets this subgroup trivially. If $\varphi(x) = 0$, then $i_*(\varphi(x)) = 0$. We have $i_*(i^*(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i)) = \sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^{i+1}$. Let $\hat{j} : Z \hookrightarrow X$ denote the inclusion of Z in X. We have $i_*(j_*((\pi|_Z)^*\beta)) = j'_*((\pi|_Z^*\beta)) = \pi^*(\hat{j}_*\beta)$ since π is flat and

$$
\begin{tikzcd}
E|_Z \arrow[r, hook, "j'"] \arrow[d, "\pi|_Z"'] & E \arrow[d, "\pi"] \\
Z \arrow[r, hook, "\hat{j}"] & X
\end{tikzcd}
$$

is a fibre square (by definition!). Putting these two facts together, we find that

$$\left(\sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^i, (\pi|_Z)^*\beta\right) \in \ker \varphi \implies \pi^*(\hat{j}_*\beta) + \sum_{i=0}^{r-1} \pi^*\alpha_i \cdot H_E^{i+1} = 0$$

Since $1, H_E, \dots, H_E^r$ are a $A^\bullet(X)$ basis, this implies $\alpha_i = 0$ for $i = 0, \dots, r-1$. So we must have $j_*(\pi|_Z^*\beta) = 0$, hence by hypothesis $\beta = 0$, so the intersection with the kernel is trivial as desired. □

We will call this subgroup C, and note that **c.** just says that C is mapped isomorphically onto $A^\bullet(Y)$ by φ. Turning to the motive of Y, we use the category defined in [7]. In the case of the Chow ring, known in the literature as the *category of effective Chow motives*. We also follow the notation of [7, §3] concerning the "Identity principle".

Now consider the full subcategory of the category of effective Chow motives whose objects are finite direct sums of motives $\mathbf{M}(X)(i) = \mathbf{M}(X) \otimes \mathbb{Z}(i)$, with X a smooth projective variety . By means of direct sums, all hom-sets in this categoy can be obtained from correspondences in the graded groups $A^\bullet(X \times_k Y)$ for X, Y smooth projective varieties. So we get a variant of Yoneda's lemma for this subcategory, where we only need consider these correspondences. More precisely, we will use the following consequence: if M, N are such motives, and $\psi \in \operatorname{Hom}(M, N)$, then if for all smooth projective varieties S over k, $\psi_S : \operatorname{Hom}^\bullet(\mathbf{M}(S), M) \to \operatorname{Hom}^\bullet(\mathbf{M}(S), N)$ is an isomorphism, then $M \cong N$.

Let $x = i^*(H_E) \in A^1(Y)$, $f = \pi \circ i$ and $g = \pi|_Z$. Then we have correspondences:

$$\begin{aligned} c_x &\in \operatorname{Hom}^1(\mathbf{M}(Y), \mathbf{M}(Y)), & c_f &\in \operatorname{Hom}(\mathbf{M}(X), \mathbf{M}(Y)), \\ c_g &\in \operatorname{Hom}(\mathbf{M}(Z), \mathbf{M}(E|_Z)), & c_j^t &\in \operatorname{Hom}^r(\mathbf{M}(E|_Z), \mathbf{M}(Y)) \end{aligned}$$

and we define for $0 \leqslant i \leqslant r-1$ correspondences

$$f_i = c_x^{(i)} \circ c_f \in \operatorname{Hom}(\mathbf{M}(X)(i), \mathbf{M}(Y)), \qquad f' = c_j^t \circ c_g \in \operatorname{Hom}(\mathbf{M}(Z)(r), \mathbf{M}(Y))$$

and a morphism

$$\psi : M := \bigoplus_{i=0}^{r-1} \mathbf{M}(X)(i) \oplus \mathbf{M}(Z)(r) \to \mathbf{M}(Y), \qquad \psi = (f_0, \dots, f_{r-1}, f')$$

For any smooth projective variety S, we can factor ψ_S through the φ of Theorem 2.1 applied to varieties, bundle, section, etc "base changed" by S, such that $\operatorname{Hom}^\bullet(\mathbf{M}(S), M)$ maps isomorphically onto the distinguished subgroup $C \subseteq A^\bullet(E \times_k S) \oplus A^\bullet(E|_Z \times_k S)$. Using Theorem 2.1 **c.** we obtain

Corollary 2.2. *If $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ is injective for all smooth projective varieties S, then ψ is an isomorphism.*

3 Application to the hyperplane section

Let k be a field and V a vector space over k of dimension $n+1$, $n \geqslant 1$. We being by describing more precisely the varieties from the introduction:

The natural pairing $V \times V^* \to k$ defines a hypersurface $E \subseteq \mathbb{P}(V^*) \times_k \mathbb{P}(V)$, called a Milnor hypersurface. Concretely, fixing a basis $y_0, \ldots, y_n$ of V and corresponding dual basis $x_0, \ldots, x_n$ of V^*, E is defined by the equation $\sum_{i=0}^n x_i y_i = 0$, or what is the same, E is the divisor given by the section $\sum_{i=0}^n x_i \otimes y_i \in H^0(\mathbb{P}(V) \times_k \mathbb{P}(V^*), p_1^* \mathscr{O}_{\mathbb{P}(V^*)}(1) \otimes p_2^* \mathscr{O}_{\mathbb{P}(V)}(1))$. Restricting the first projection map p_1 to $E \hookrightarrow \mathbb{P}(V^*) \times_k \mathbb{P}(V)$, we obtain a projective bundle $\pi : E \to \mathbb{P}(V^*)$ (the restriction of p_2 will be denoted π'). Indeed, one sees $E \cong \mathbb{P}(\underline{V}/\mathscr{O}_{\mathbb{P}(V^*)}(-1))$, with $\mathscr{O}(1) = \pi'^* \mathscr{O}_{\mathbb{P}(V)}(1)$, via the inclusion $E \hookrightarrow \mathbb{P}(V^*) \times_k \mathbb{P}(V)$. Consequently, with $\mathscr{E} = (\underline{V}/\mathscr{O}_{\mathbb{P}(V^*)}(-1)) \otimes \mathscr{O}_{\mathbb{P}(V^*)}(1)$, we also have $E \cong \mathbb{P}(\mathscr{E})$ over $\mathbb{P}(V^*)$ and $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = \pi^* \mathscr{O}(1)_{\mathbb{P}(V^*)} \otimes \pi'^* \mathscr{O}(1)_{\mathbb{P}(V)}$ (see [6, Lemma II.7.9]).

Y is then defined by additionally imposing the equation coming from the twisted pairing $V \times V^* \to k$, $(v, f) \mapsto f(\varphi(v))$, so in particular, intersecting with a divisor of type $\mathscr{O}(1,1) = p_1^* \mathscr{O}_{\mathbb{P}(V^*)}(1) \otimes p_2^* \mathscr{O}_{\mathbb{P}(V)}(1)$.

Lemma 3.1. *Y is a smooth effective divisor in E corresponding to a global section $s \in H^0(E, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$.*

Note that this implies Y is very ample, hence the terminology "hyperplane section".

Proof. By the construction of Y, if it is a divisor on E, then it is of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$-type. Let $\alpha_0, \ldots, \alpha_n$ be the $n+1$ distinct eigenvectors of φ. We may assume k algebraically closed, so in particular we have $\alpha_i \in k$. Choosing a basis $y_0, \ldots, y_n \in V$ diagonalizing φ, and letting $x_0, \ldots, x_n \in V^*$ be the dual basis of the y_i, we see that Y is cut out by the polynomials

$$\sum_{i=0}^n x_i y_i = 0, \qquad \sum_{i=0}^n \alpha_i x_i y_i = 0$$

in $P = \mathbb{P}(V^*) \times_k \mathbb{P}(V)$.

We proceed by applying the Jacobian criterion to $Y \subseteq P$ away from $x_i y_j = 0$. For $0 \leqslant i \neq j \leqslant n$, let $U_{ij} \subseteq P$ denote the open set of points where $x_i y_j \neq 0$. It is affine since $\mathscr{O}(1,1)$ is very ample. Thus $U_{ij} \cong \mathbb{A}^{2n-2}$, with coordinate ring $k[x_0', \ldots, x_n', y_0', \ldots, y_n']$, $x_l' = \frac{x_l}{x_i}, y_l' = \frac{y_l}{y_j}$. The U_{ij} cover Y since if there is only one i with x_i or y_i non-zero at a point of Y, then $\sum_{l=0}^n x_l y_l \neq 0$. $Y \cap U_{ij}$ is given by the equations $\sum_{l=0}^n x_l' y_l'$ and $\sum_{l=0}^n \alpha_i x_l' y_l'$ so the matrix

$$\begin{pmatrix} y_j' & x_i' \\ \alpha_j y_j' & \alpha_i x_i' \end{pmatrix}$$

appears as a 2×2 submatrix of the Jacobian matrix, (with $y_j' \neq 0$ and $x_i' \neq 0$ by definition). Distinctness of the α_l shows that this matrix is non-singular. Thus the rank of the Jacobian matrix is 2, so Y is a smooth divisor on E. $\square$

Lemma 3.2. *The reduced zero locus Z of the s of the previous lemma in $\mathbb{P}(V^*)$ is isomorphic to* $\operatorname{Spec} L$.

Proof. Over an algebraic closure $\bar{k}/k$, the closed points of the zero locus of s are those which have a fibre of dimension $n-1$. With homogeneous coordinates x_i, y_j as before, these are the points $[c_0, \ldots, c_n]$ such that $\sum_{i=0}^n c_i y_i$ and $\sum_{i=0}^n \alpha_i c_i y_i$ are linearly dependent. Since the α_i are distinct, this happens precisely when $c_i = 0$ for all but one value of i. In coordinate-free terms, these correspond to the eigenspaces of $\varphi \otimes 1$ in $V \otimes_k \bar{k}$, which are already defined in $\mathbb{P}(V^*)(K)$. Since K is separable over k and we assume Z reduced, $Z \times_k K$ is reduced, hence $Z \times_k K \cong \coprod_{0 \leqslant i \leqslant n} \operatorname{Spec} K$ by the above description of $Z(\bar{k})$. This implies in turn that $Z \cong \operatorname{Spec} B$, where B is a reduced finite algebra over k with $\dim_k B = n+1$ and that B is a direct product of field extensions of k, $\prod_{1 \leqslant i \leqslant m} K_i$, where all K_i have an inclusion into K. By definition of K, each eigenvalue of φ has $\dim_k K$ conjugates. Since eigenspaces of conjugate eigenvalues are conjugate, this implies the orbit of any point of $Z(K)$ under the action of $\mathrm{Gal}(K/k)$ has cardinality $\dim_k K$. Hence, $\dim_k K_i = \dim_k K$ for all $1 \leqslant i \leqslant m$, so $K_i \cong K$. Now since $\dim_k B = \dim_k L = n+1$, we see that $B \cong L$, as desired. $\square$

If the eigenvalues of φ are in k, then V decomposes into one-dimensional eigenspaces V_i, $0 \leqslant i \leqslant n$. This gives a torus $T \subseteq GL(V)$ consisting of the elements which send the V_i into themselves, which acts on $\mathbb{P}(V^*)$ and $\mathbb{P}(V)$ via the trivial and dual representations, respectively. These actions are then such that E is T-stable under the induced T-action on $\mathbb{P}(V^*) \times_k \mathbb{P}(V)$, and so is Y since $\varphi \otimes 1$ commutes with the elements of $T(\bar{k})$. From Theorem 3.2, we see that $E|_Z$ consists of $n+1$ copies of $\mathbb{P}_k^{n-1}$, each of which is T-stable. These are precisely the fibres $E_i = \pi^{-1}([V_i])$, $0 \leqslant i \leqslant n \subseteq Y$.

Proposition 3.3. *For* $0 \leqslant i, j \leqslant n$, *let* $\gamma_i = [E_i] \in A^{n-1}(Y)$. *Then* $\deg(\gamma_i \cdot \gamma_j) = \delta_{ij}(-1)^{n-1}$.

Since the degree of a class in the zeroth Chow group of a proper variety is invariant under change of base field, we may assume that k is algebraically closed. This allows for the use of *localisation* ([2, Corollary 2.3.2]) for T-equivariant Chow groups to prove the proposition. To this end, we first gather some facts about the T-action on Y.

For each $0 \leqslant i \leqslant n$, we have a homomorphism $t_i : T \to GL(V_i) = \mathbb{G}_m$. The t_i generate the character group M of T, and we write χ_{ij} for $t_j - t_i \in M$. We let $R = \mathrm{Sym}_{\mathbb{Z}} M = \mathbb{Z}[t_0, \ldots, t_n]$ and Q be the field of fractions of R. For $0 \leqslant i \neq j \leqslant n$, let $z_{ij} = ([V_i], [\bigoplus_{0 \leqslant l \neq j \leqslant n} V_l]) \in E$. These are also contained in Y and $E|_Z$ and are the T-fixed points of these varieties.

Lemma 3.4. *The weights of the* T*-module* $\mathrm{Tan}_{z_{ij}}(Y)$ *are* χ_{lj} *and* χ_{il} *for* $0 \leqslant l \leqslant n$, $l \neq i, j$. *and the submodule* $\mathrm{Tan}_{z_{ij}}(E|_Z) \subseteq \mathrm{Tan}_{z_{ij}}(Y)$ *is spanned by the weight spaces of* χ_{lj}, $0 \leqslant l \leqslant n, l \neq i, j$.

Proof. For fixed $i \neq j$, for any $l \neq i, j$, the codimension 2 subspace $\bigoplus_{0 \leqslant s \leqslant n, s \neq j, l} V_s \subseteq V$ corresponds to a T-stable line $L_{lj} \subseteq \mathbb{P}(V)$. Clearly $C_{lj} = \{[V_i]\} \times L_{lj} \subseteq Y$ is T-stable and it is an easy computation that T acts on $\mathrm{Tan}_{z_{ij}}(C_{lj})$ by χ_{lj}. Similarly, one defines a line $L_{il} \subseteq \mathbb{P}(V^*)$ corresponding to $V_i \oplus V_l$, and sets $C_{il} = L_{il} \times \{[\bigoplus_{0 \leqslant s \neq j \leqslant n} V_s]\} \subseteq Y$.[1] Once again, it is easily verified that T acts on $\mathrm{Tan}_{z_{ij}}(C_{il})$ by χ_{il}. These are all weights of $\mathrm{Tan}_{z_{ij}}(Y)$ by the canonical inclusions of the tangent spaces of the T-stable curves, and they make up all weights since $\dim_k \mathrm{Tan}_{z_{ij}}(Y) = 2n - 2$. The characterisation of $\mathrm{Tan}_{z_{ij}}(E|_Z)$ follows since $\dim_k \mathrm{Tan}_{z_{ij}}(E|_Z) = n - 1$ and each of the C_{lj} is contained in $E|_Z$. $\square$

We denote the T-equivariant Chow ring of a smooth T-variety X by $A_T^\bullet(X)$ (for a general reference on equivariant intersection theory, see [3]). We write $\bar{\alpha}$ for the image of an element α under the forgetful map $A_T^\bullet(X) \to A^\bullet(X)$. If X is proper over k with structure morphism p, we have the equivariant Poincaré pairing $\langle \cdot, \cdot \rangle_T : A_T^\bullet(X) \times A_T^\bullet(X) \to A_T^\bullet(\mathrm{Spec}\, k) = R$ defined by $(\alpha, \beta) \mapsto p_*(\alpha \cdot \beta)$. Notice that $\overline{\langle \alpha, \beta \rangle_T} = \deg(\bar{\alpha} \cdot \bar{\beta})$ by the naturality of the forgetful map (we extend deg to all of $A^\bullet(X)$ by setting it to 0 for cycles of dimension greater than 0. This extension is of course just p_*).

Let $x \in X$ be a T-fixed point such the weights $\chi_1, \ldots, \chi_m$ of $\mathrm{Tan}_x(X)$ are non-zero. Following [2, Theorem 4.2], we define the *equivariant multiplicity* of a cycle $\alpha \in A_T^\bullet(X)$ $e_{x,X}(\alpha)$ to be the image of α by the unique R-linear map $e_{x,X} : A_T^\bullet(X) \to Q$ such that $e_{x,X}([x]) = 1$ and $e_{x,X}([X']) = 0$ for any T-invariant subvariety $X' \subseteq X$ which does not contain x. The smoothness of X implies that $e_{x,X}([X]) = (\prod_{1 \leqslant i \leqslant m} \chi_i)^{-1}$. Moreover, for smooth X', $e_{x,X}([X']) = e_{x,X'}([X'])$.

Lemma 3.5. *For* $\alpha \in A_T^\bullet(Y)$ *and* $0 \leqslant i \neq j \leqslant n$, *let* α_{ij} *be the pullback of* α *by the inclusion* $\{z_{ij}\} \hookrightarrow Y$. *We have the following identities:*

$$e_{z_{ij},Y}(\alpha) = \frac{\alpha_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \tag{1}$$

$$\langle \alpha, \beta \rangle_T = \sum_{0 \leqslant i \neq j \leqslant n} \frac{\alpha_{ij}\beta_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \tag{2}$$

Proof. For (1), let $\iota_{ij} : \{z_{ij}\} \hookrightarrow Y$, $\iota : Y^T \hookrightarrow Y$ be the obvious inclusion maps. By [2, Corollary 4.2] and Theorem 3.4,

$$[Y] = \sum_{0 \leqslant i \neq j \leqslant n} \frac{1}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} [z_{ij}], \qquad \alpha = \sum_{0 \leqslant i \neq j \leqslant n} e_{z_{ij},Y}(\alpha)[z_{ij}]$$

in $A_T^\bullet(Y) \otimes_R Q$. Using the identification $A_T^\bullet(Y^T) \otimes_R Q = \bigoplus_{0 \leqslant i \neq j \leqslant n} Q$ coming from the inclusion of each fixed point into Y^T, we can rewrite these equalities as

$$[Y] = \iota_* \left(\frac{1}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \right)_{ij}, \qquad \alpha = \iota_* \left(e_{z_{ij},Y}(\alpha) \right)_{ij}$$

But $\alpha = \alpha \cdot [Y]$, so by the projection formula we have

$$\iota_* \left(\frac{\alpha_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \right)_{ij} = \iota_* \left(e_{z_{ij},Y}(\alpha) \right)_{ij}$$

[1] The T-stable curves used in the proof are given in [1, §3.1] in the case where E is any adjoint variety.

By [2, Corollary 2.2], ι_* is an isomorphism after tensoring with Q. Since $A^\bullet_T(Y^T)$ is free as an R-module, the desired equality follows.

For (2), since $R \subseteq Q$, it is enough to compute after localising. By (1),

$$\alpha\beta = \sum_{0 \leqslant i \neq j \leqslant n} \frac{\alpha_{ij}\beta_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}} \iota_{ij*}(1)$$

Now, $p_* \circ \iota_{ij*}$ is the identity on Q since $p \circ \iota_{ij}$ is a map of a point to itself. Thus, by linearity we have

$$\langle \alpha, \beta \rangle_T = p_*(\alpha\beta) = \sum_{0 \leqslant i \neq j \leqslant n} \frac{\alpha_{ij}\beta_{ij}}{\prod_{l \neq i,j} \chi_{il}\chi_{lj}}$$

□

Proof of Proposition 3.3. It is enough to show that for $0 \leqslant i, j \leqslant n$, $\langle [E_i], [E_j] \rangle_T = \delta_{ij}(-1)^{n-1}$. By Theorem 3.4, $e_{z_{ij},Y}([E_i]) = (\prod_{l \neq i,j} \chi_{lj})^{-1}$. Hence by Theorem 3.5, $\langle [E_i], [E_j] \rangle_T = 0$ when $i \neq j$ (since E_i and E_j share no fixed points) and

$$\langle [E_i], [E_i] \rangle_T = \sum_{s \neq i} \frac{\prod_{l \neq i,s} \chi_{il}}{\prod_{l \neq i,s} \chi_{ls}}$$

This is seen to be $(-1)^{n-1}$ by the following observation in [8, Lemma 4.2]: treating R as a polynomial ring in t_i over $\mathbb{Z}[t_0, \ldots, \hat{t}_i, \ldots, t_n]$, by Lagrange interpolation it is enough to show that the polynomial

$$f(t_i) = \sum_{s \neq i} \frac{\prod_{l \neq i,s} \chi_{il}}{\prod_{l \neq i,s} \chi_{ls}}$$

of degree at most $n-1$ evaluated at t_j for each $j \neq i$ is $(-1)^{n-1}$. Clearly, $\prod_{l \neq i,s} \chi_{il}$ evaluated at t_j is 0 if $j \neq s$, thus

$$f(t_j) = \frac{\prod_{l \neq i,j} \chi_{jl}}{\prod_{l \neq i,j} \chi_{lj}} = (-1)^{n-1}$$

for $j \neq i$. □

Proposition 3.6. *If all of the eigenvalues of φ are in k, the map $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ is injective for any smooth projective variety S over k.*

Proof. We have that $A^\bullet(Z \times_k S) = \bigoplus_{0 \leqslant i \leqslant n} A^\bullet(S)$, with the images of the classes of the irreducible components of $Z \times_k S$ under $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$ being the classes $E|_i \times_k S$ in $A^\bullet(Y \times_k S)$, i.e. $\gamma_i \times 1_S$ ($1_S = [S]$), $0 \leqslant i \leqslant n$. Note that the homomorphism is $A^\bullet(S)$-linear, so it suffices to show that the γ_i are $A^\bullet(S)$-linearly independent in $A^\bullet(Y \times_k S)$. We define a relative Poincaré pairing $A^\bullet(Y \times_k S) \times A^\bullet(Y \times_k S) \to A^\bullet(S)$ by $\langle \alpha, \beta \rangle_S = (p \times \mathrm{id}_S)_*(\alpha \cdot \beta)$, where $p : Y \to \operatorname{Spec} k$ is the structure morphism. Note that this is $A^\bullet(S)$ bilinear. Then for $0 \leqslant i, j \leqslant n$, $\langle \gamma_i \times 1_S, \gamma_j \times 1_S \rangle_S = \deg(\gamma_i \cdot \gamma_j) \cdot 1_S = \delta_{ij}(-1_S)^{n-1}$ by Theorem 3.3. Linear independence follows. □

Proof of Theorem 1.1. Applying base change by K/k, we obtain the commutative diagram of Cartesian squares:

$$\begin{tikzcd}
(Z \times_k K) \times_K (S \times_k K) \arrow[d] & (E|_Z \times_k K) \times_K (S \times_k K) \arrow[l] \arrow[r] \arrow[d] & (Y \times_k K) \times_K (S \times_k K) \arrow[d] \\
Z \times_k S & E|_Z \times_k S \arrow[l, "\pi|_Z \times \mathrm{id}_S"'] \arrow[r, "j \times \mathrm{id}_S"] & Y \times_k S
\end{tikzcd}$$

which induces the commutative diagram on Chow rings

$$\begin{tikzcd}
A^\bullet(Z_K \times_K S_K) \arrow[r] & A^\bullet((E|_Z)_K \times_K S_K) \arrow[r] & A^\bullet(Y_K \times_K S_K) \\
A^\bullet(Z \times_k S) \arrow[u] \arrow[r, "(\pi|_Z \times \mathrm{id}_S)^*"] & A^\bullet(E|_Z \times_k S) \arrow[u] \arrow[r, "(j \times \mathrm{id}_S)_*"] & A^\bullet(Y \times_k S) \arrow[u]
\end{tikzcd}$$

where we write S_K for $S \times_k K$. The lefthand vertical map is injective. Indeed, by Theorem 3.2, $Z \cong \operatorname{Spec} L$. Since $K \otimes_k L \cong K \otimes_k (\prod_{1 \leqslant i \leqslant m} K) \cong \prod_{0 \leqslant i \leqslant n} K$, we need only check that the obvious map $\bigoplus_{1 \leqslant i \leqslant m} A^\bullet(S_K) \to \bigoplus_{0 \leqslant i \leqslant n} A^\bullet(S_K)$ is injective, which is clear. Thus, injectivity of the composite of the top row implies injectivity of the composite of the bottom row, i.e. $(j \times \mathrm{id}_S)_* \circ (\pi|_Z \times \mathrm{id}_S)^*$. The injectivity of the top row is exactly Theorem 3.6, in the case of $V \otimes_k K$ and $\varphi \otimes 1 \in \operatorname{End}(V \otimes_k K)$, since by definition the eigenvalues of $\varphi \otimes 1$ are in K. The theorem then follows from Theorem 2.2.

□

Corollary 3.7. *Suppose* $\varphi, \varphi' \in \operatorname{End}(V)$ *satisfy the hypotheses of Theorem 1.1. If the associated varieties* Y *and* Y' *are isomorphic, then* $L = k[\varphi]$ *and* $L' = k[\varphi']$ *are isomorphic* k*-algebras.*

Proof. By hypothesis, $L = \prod_i K$ and $L' = \prod_j K'$, K'/k Galois. Since $\dim_k L = \dim_k L'$ it is the same to show $Y \cong Y' \iff K \cong K'$. First, assume $Y \cong Y'$. Then $\operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K')(n-1), \mathbf{M}(Y)) \cong \operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K'), \mathbf{M}(Y))$ as abelian groups. The motive $\mathbf{M}(\mathbb{P}^n_k)$ decomposes as $\bigoplus_{0 \leqslant i \leqslant n} \mathbb{Z}(i)$, so $\operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K)(n-1), \mathbf{M}(\mathbb{P}^n_k)(m)) = \operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K)(n-1), \mathbb{Z}(n-1)) = \mathbb{Z}$ for all $m \geqslant 0$. Hence by Theorem 1.1, $\mathbb{Z}^{n-1} \oplus \operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K'), \mathbf{M}(\operatorname{Spec} L)) \cong \mathbb{Z}^{n-1} \oplus \operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K'), \mathbf{M}(\operatorname{Spec} L'))$. By definition, $\operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K'), \mathbf{M}(\operatorname{Spec} L'))$ is $A^0(\operatorname{Spec} K' \times_k \operatorname{Spec} L') = A^0(\coprod_{0 \leqslant i \leqslant n} \operatorname{Spec} K') = \mathbb{Z}^{n+1}$. Thus, $\operatorname{Hom}(\mathbf{M}(\operatorname{Spec} K'), \mathbf{M}(\operatorname{Spec} L)) \cong \mathbb{Z}^{n+1}$. But this means $\operatorname{Spec} L \times_k K'$ has $n+1$ irreducible components, but since $\dim_k L = n+1$, this must mean $\operatorname{Spec} L$ has a K'-point, i.e. there is an embedding $K \hookrightarrow K'$. By reversing the roles of K and K', we see there is also an embedding $K' \hookrightarrow K$, whence $K \cong K'$. □

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