SGN – Assignment #1

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1 Periodic orbit

Exercise 1

Consider the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$.

1) Find the x-coordinate of the Lagrange point L_1 in the rotating, adimensional reference frame with at least 10-digit accuracy.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.08892819445324 \\ y_0 &= 0 \\ z_0 &= 0.0591799623455459 \\ v_{x0} &= 0 \\ v_{y0} &= 0.257888699435051 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit that passes through z_0; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM either approximated through finite differences or achieved by integrating the variational equation.

The periodic orbits in the CRTBP exist in families. These can be computed by continuing the orbits along one coordinate, e.g., z_0. This is an iterative process in which one component of the state is varied, while the other components are taken from the solution of the previous iteration.

3) By gradually decreasing z_0 and using numerical continuation, compute the families of halo orbits until $z_0 = 0.034$.

(8 points)

1.0.1 Point 1

Given the Circular Rescticted Two-Body Problem (CRTBP), introduced the state

$$X = (x, y, z, v_x, v_y, v_z)$$

the dynamincs can written in first order form as

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{z} = v_z$$

$$\dot{v}_x = 2v_y + U_x$$

$$\dot{v}_y = -2v_x + U_y$$
$$\dot{v}_z = U_z$$

where

$$U_x = \frac{\partial U}{\partial x} = x - \frac{(1-\mu)}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x+\mu-1)$$
$$U_y = \frac{\partial U}{\partial y} = y - \frac{(1-\mu)}{r_1^3}y - \frac{\mu}{r_2^3}y$$
$$U_z = \frac{\partial U}{\partial z} = -\frac{(1-\mu)}{r_1^3}z - \frac{\mu}{r_2^3}z$$
$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$
$$r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}$$

In order to have and equilibrium point, we have to enforce the equilibrium condition by imposing

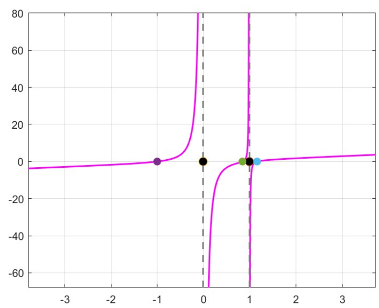
$$\dot{x} = \dot{y} = \dot{z} = 0$$
$$\dot{v}_x = \dot{v}_y = \dot{v}_z = 0$$

And so

$$U_x = U_y = U_z = 0$$

Now it is clear to see that `z` different from `0` cannot lead to equilibrium, therefore the equilibrium points, if existing, will be located on the `z = 0` plane. In order to find the point `L1`, appartaing to the Lagrange collinear libration points the condition that `y = 0` it is also to be imposed. Leading to and equation in x, solvable numerically. For `L1` the coordinate x=0.8369180073. Plotting also `L2` and `L3`, as we as the two-bodies it is possible to show that the equilibrium point `L1` is the requested one, located in between the two objects.

Figure 1: Collinear Lagrange Points



1.0.2 Point 2

It is possible to implement a differential correction by exploiting the definition of State Transition Matrix and creating the following matrix equation.
In particular, the first term from the left represents the State Transition Matrix (STM), propagated from the initial to the event time, corresponding in this case to the `x` axis equal to zero. Later will appear that it is not necessary to compute the entire STM, but only some coefficient from it.
The STM is multiplied for the term representing the correction to be applied to the orbit at the first time instant, in order to comply with the needed conditions `x` and `Vy`, have to be varied. The right side of the equation contains instead the difference between the final conditions after

the correction and the final conditions before the correction. In order to have a closed, periodic orbit the final conditions, described in the following equation have to be satisfied.

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} * \begin{bmatrix} \Delta x_0 \\ 0 \\ 0 \\ 0 \\ \Delta v_{y0} \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{x}_f \\ 0 \\ 0 \\ 0 \\ \tilde{v}_{yf} \\ 0 \end{bmatrix} - \begin{bmatrix} x_f \\ 0 \\ z_f \\ v_{xf} \\ v_{yf} \\ v_{zf} \end{bmatrix}$$

From the above expression can be compute the correction to be applied

$$\phi_{4,1} * \Delta x_0 + \phi_{4,5} * \Delta y_0 = -v_{xf}$$

$$\phi_{6,1} * \Delta x_0 + \phi_{6,5} * \Delta y_0 = -v_{zf}$$

Resulting in

$$\Delta y_0 = \frac{-v_{zf} + \dfrac{\phi_{6,1} * v_{xf}}{\phi_{4,1}}}{\phi_{6,5} - \frac{\phi_{6,1} * \phi_{4,5}}{\phi_{4,1}}}$$

$$\Delta x_0 = -\frac{\phi_{4,5} * \Delta y_0 + v_{xf}}{\phi_{4,1}}$$

Implementing a pseudo-newton method it is possible to obtain the new, corrected initial conditions for the orbit propagation:
`{1.0902780529, 0, 0.05917996234, 0, 0.260349391891, 0}`
It is also possible to retrive the final errors
Position error [km]: `{-2.487e-08, +3.136e-08, -7.104e-09}`
Velocity error [km]: `{-7.748e-08, +5.801e-08, -5.770e-08}`
Relative error [km]: `{2.281e-08, 1.000e+00, 1.200e-07, 1.000e+00, +2.228e-07, +1.000e+00}`

Figure 2: Corrected Orbit Forward-Backward Propagated

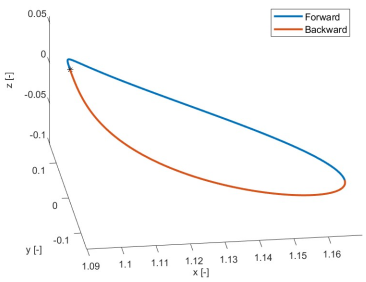
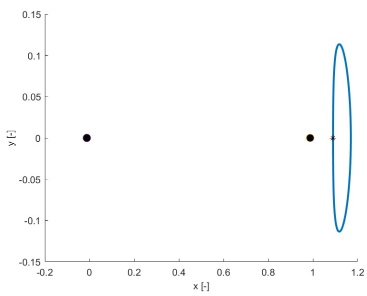


Figure 3: Corrected Orbit Propagated with Attractors



1.0.3 Point 3

By means of numerical continuation it is possible to create a linear space with the $\mathbf{z}$ coordinate ranging in the wanted interval. Moreover it is possible to implement two nested loop cycles, the external referring to numerical continuation of the $\mathbf{z}$ axis, and the internal computing the correction as before. Obtaining

Figure 4: Propagated Orbit Family

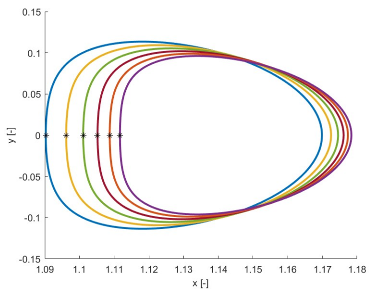
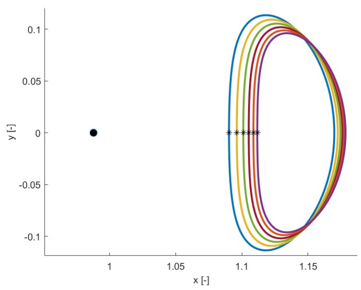


Figure 5: Orbit Family with Attractor



2 Impulsive guidance

Exercise 2

The Aphophis close encounter with Earth will occur on April 2029. You shall design a planetary protection guidance solution aimed at reducing the risk of impact with the Earth.

The mission shall be performed with an impactor spacecraft, capable of imparting a $\Delta\mathbf{v} = 0.00005\,\mathbf{v}(t_{\text{imp}})$, where $\mathbf{v}$ is the spacecraft velocity and t_{imp} is the impact time. The spacecraft is equipped with a chemical propulsion system that can perform impulsive manoeuvres up to a total Δv of 5 km/s.

The objective of the mission is to maximize the distance from the Earth at the time of the closest approach. The launch shall be performed between 2024-10-01 (LWO, Launch Window Open) and 2025-02-01 (LWC, Launch Window Close), while the impact with Apophis shall occur between 2028-08-01 and 2029-02-28. An additional Deep-Space Maneuver (DSM) can be performed between LWO+6 and LWC+18 months.

1) Analyse the close encounter conditions reading the SPK kernel and plotting in the time window [2029-01-01; 2029-07-31] the following quantities:

    a) The distance between Apophis and the Sun, the Moon and the Earth respectively.

    b) The evolution of the angle Earth-Apophis-Sun

c) The ground-track of Apophis for a time-window of 12 hours centered around the time of closest approach (TCA).

2) Formalize an unambiguous statement of the problem specifying the considered optimization variables, objective function, the linear and non-linear equality and inequality constraints, starting from the description provided above. Consider a multiple-shooting problem with $N = 3$ points (or equivalently 2 segments) from t_0 to t_{imp}.

3) Solve the problem with multiple shooting. Propagate the dynamics of the spacecraft considering only the gravitational attraction of the Sun; propagate the post-impact orbit of Apophis using a full n-body integrator. Use an event function to stop the integration at TCA to compute the objective function; read the position of the Earth at t_0 and that of Apophis at t_{imp} from the SPK kernels. Provide the optimization solution, that is, the optimized departure date, DSM execution epoch and the corresponding $\Delta\mathbf{v}$'s, the spacecraft impact epoch, and time and Distance of Closest Approach (DCA) in Earth radii. Suggestion: try different initial conditions.

(11 points)

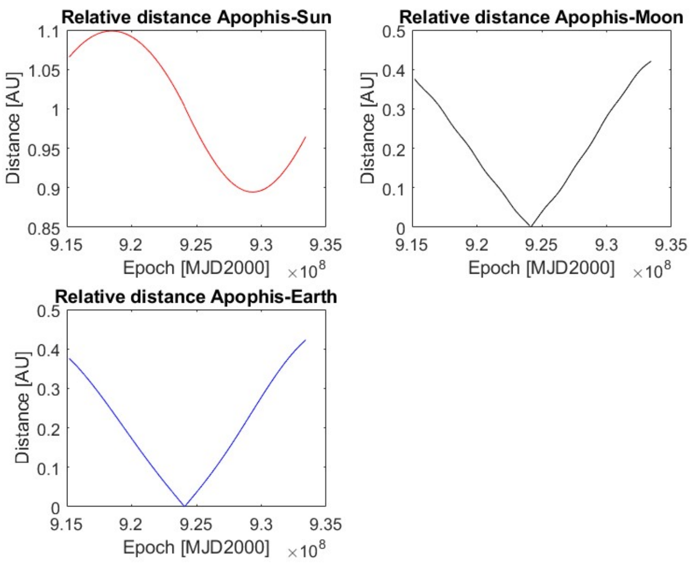
2.0.1 Point 1.a

Using the spice function *cspice spkezr* and *cspice spkpos*, is computed the position of and velocity of Apophis with respect to the other objects in the solar system, directly or using a vector equation as it follows, where the right side distances are computed with respect to the Solar System Baricenter (SSB):

$$\vec{r}_{Sun-Apophis} = \vec{r}_{Apophis-SSB} - \vec{r}_{Sun-SSB}$$

Then from the position vector is computed the distance.

Figure 6: Evolution in time of the distances between Apophis and Sun, Moon, Earth



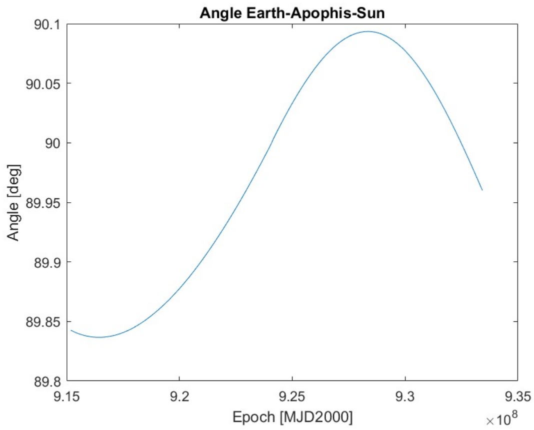
2.0.2 Point 1.b

Using the distance vector the angle is computed instead using the formula:

$$\theta = \arccos(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|})$$

Obtaining:

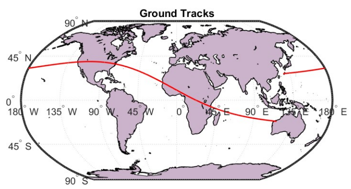
Figure 7: Evolution in time of the angle Earth-Apophis-Sun



2.0.3 Point 1.c

In order to plot the groundtracks, firstly the function *cspice pxform* used to retrive the rotation matrix from the inertial ecliptic J2000 frame to the Earth body-fixed rotating frame (IAU EARTH). Then

Figure 8: Ground Tracks of closest approach



2.0.4 Point 2

- A time grid with total number of points N=3 is selected, where $t_1 = t_{initial}$ and $t_3 = t_{final}$.
- Leading to the total number of variable of 20.

$$\underline{y} = (\underline{x}_1, \underline{x}_2, \underline{x}_3, t_1, t_3)$$

- The goal is to maximise the closest approach distance between Earth and Apophis. The distance can be expressed as

$$d(t_{min}) = \|\mathbf{r}_{Earth}(t_{min}) - \mathbf{r}_{Apophis}(t_{min})\|$$

Where the position vector of Earth and Apophis are computed with respect to the solar system baricenter. The position of Apophis can be further developed taking into account the perturbation caused by the impact, dividing thiss term in the sum of the unperturbed orbit of the asteroid plus the perturbation.

$$\mathbf{r}_{Apophis}(t_{min}) = \mathbf{r}_{Apophis-NotPerturbed}(t_{min}) + \delta\mathbf{r}_{Apophis}(t_{min})$$

The perturbation is computed considering the variation in velocity and position at impact time. Multiplying velocity and position with the corresponding state transition matrix terms, correctly propagated, (STM) it is possible to obtain their contributions at closest approach time. Accounting only for the position at final time:

$$\delta \mathbf{r}_{Apophis}(t_{min}) = \Phi_{rr}\delta\mathbf{r} + \Phi_{rv}\delta\mathbf{v}$$

It is also clear that at impact time the variation in position will be negligible, moreover the change in velocity is equal to $\Delta\mathbf{v} = 0.00005\,\mathbf{v}(t_{\text{imp}})$, where $\mathbf{v}$ is the spacecraft velocity and t_{imp} is the impact time. This considerations lead to:

$$d(t_{min}) = \|\mathbf{r}_{Earth}(t_{min}) - \mathbf{r}_{Apophis}(t_{min}) - \Phi_{rv}0.00005\mathbf{v}_{Spacecraft}(t_{\text{imp}})\|$$

Considering the use of the matlab function as minimizer, the final expression is

$$d(t_{min}) = -\|\mathbf{r}_{Earth}(t_{min}) - \mathbf{r}_{Apophis}(t_{min}) - \Phi_{rv}0.00005\mathbf{v}_{Spacecraft}(t_{\text{imp}})\|$$

- The constraints can be split in equality and inequality constraints:
  - **Equality Constraints** The equality constraints can be expressed in the form $\underline{c}(\underline{y}) = 0$, it is possible to further split this category into boundary conditions, imposing three constraints each

$$\psi_i := \underline{r}_1 - \underline{r}_i(t_1)$$
$$\psi_f := \underline{r}_3 - \underline{r}_f(t_3)$$

    and integration defects, contributing for six constraints each,

$$\underline{z}_1 := \varphi(\underline{x}_1, t_1, t_2)(1:3) - \underline{r}_2$$
$$\underline{z}_2 := \varphi(\underline{x}_2, t_2, t_3) - \underline{x}_3$$

  - **Inequality Constraints** One kind of the inequality constraints are expressed in the form $\underline{y}^L \leq \underline{y} \leq \underline{y}^U$ where L and U represents respectively the lower and upper bound

$$t_1^L \leq t_1 \leq t_1^U$$
$$t_2^L \leq t_2 \leq t_2^U$$
$$t_3^L \leq t_3 \leq t_3^U$$

    which can further developed, resulting in six constraints

$$t_1 - t_1^U \leq 0$$
$$t_1^L - t_1 \leq 0$$
$$t_2 - t_2^U \leq 0$$
$$t_2^L - t_2 \leq 0$$
$$t_3 - t_3^U \leq 0$$
$$t_3^L - t_3 \leq 0$$

    Moreover, it has to be imposed that $t_{n+1} \leq t_n$ where n is the time-step, written in the form $Ay \leq b$, these result in three constraints

$$t_2 - t_1 \geq 0$$
$$t_3 - t_2 \geq 0$$
$$t_3 - t_1 \geq 0$$

- **Problem Specific Constraints** Moreover a set of problem specific equality and inequality constraints is needed, related to the time of the deep space maneuver

$$t_2^U = t_1 + 18 Months$$

$$t_2^L = t_1 + 6 Months$$

and the maximum delta velocity, concerning the departure from Earth (E), the arrival to the asteroid (A), as well as the deep space maneuver

$$\underline{\Delta V_1} = \underline{v_1} - \underline{v_E}$$

$$\underline{\Delta V_2} = \underline{v_3} - \underline{v_2}$$

$$\underline{\Delta V_3} = \underline{v_A} - \underline{v_3}$$

The sum must be less equal than the total ΔV obtainable

$$\underline{\Delta V_1} + \underline{\Delta V_2} + \underline{\Delta V_3} \leq \underline{\Delta V_{tot}}$$

- Lastly the final expression to be minimized is then

$$\min_{\underline{y}} \|\mathbf{r}_{Earth}(t_{min}) - \mathbf{r}_{Apophis}(t_{min}) - \Phi_{rv} 0.00005 \mathbf{v}_{Spacecraft}(t_{\text{imp}})\|$$

2.0.5 Point 3

Launch	YYYY-MM-DD-HH:MM:SS.sss UTC
DSM	YYYY-MM-DD-HH:MM:SS.sss UTC
Impact	YYYY-MM-DD-HH:MM:SS.sss UTC
TCA	YYYY-MM-DD-HH:MM:SS.sss UTC
$\Delta\mathbf{v}_L$ [km/s]	±0000.0000 ±0000.0000 ±0000.0000
$\Delta\mathbf{v}_{DSM}$ [km/s]	±0000.0000 ±0000.0000 ±0000.0000
DCA [Re]	±0000.0000

Table 1: Guidance solution for the impactor mission.

3 Continuous guidance

Exercise 3

A low-thrust option is being considered for an Earth-Venus transfer. Provide a *time-optimal* solution under the following assumptions: the spacecraft moves in the heliocentric two-body problem, Venus instantaneous acceleration is determined only by the Sun gravitational attraction, the departure date is fixed, and the spacecraft initial and final states are coincident with those of the Earth and Venus, respectively.

1) Using the PMP, write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition.

2) Adimensionalize the problem using as reference length $LU = 1$ AU* and reference mass $MU = m_0$, imposing that $\mu = 1$. Report all the adimensionalized parameters.

3) Solve the problem considering the following data:

   - Launch date: `2023-05-28-14:13:09.000 UTC`
   - Spacecraft mass: $m_0 = 1000$ kg
   - Electric propulsion properties: $T_{\max} = 800$ mN, $I_{sp} = 3120$ s

   To obtain an initial guess for the costate, generate random numbers such that $\lambda_{0,i} \in [-20; +20]$, while $t_f < 2\pi$. Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Assess your results exploiting the properties of the Hamiltonian in problems that are not time-dependent and time-optimal solutions.

4) Solve the problem for a lower thrust level $T_{\max} = [500]$ mN. Tip: exploit numerical continuation.

(11 points)

3.0.1 Point 1

The general statement of the problem for free final time and terminal constraints, considering the Pontriagin Maximum Principle, is the following

$$\min_{\underline{u}(t)\in\mho} J(\underline{u}, t_f) := \int_{t_0}^{t_f} l(\underline{x}, \underline{u}, t)dt$$

subject to the constraints

$$\begin{cases} \underline{\dot{x}} = f(\underline{x}, \underline{u}, t) \\ \underline{x}(t_0) = \underline{x}_0 \\ \underline{x}(t_f) = \underline{\psi}(t_f) \end{cases} \tag{1}$$

The first constraint represents the system dynamics, referred to the state $\underline{x} = (\underline{r}, \underline{v}, m)$, and so, the spacecraft equations of motions, expressed as

$$\begin{cases} \underline{\dot{r}} = \underline{v} \\ \underline{\dot{v}} = -\frac{\mu}{r^3}\underline{r} + \mu\frac{T_{max}}{m}\underline{\hat{\alpha}} \\ \dot{m} = -\mu\frac{T_{max}}{Ispg_0} \end{cases} \tag{2}$$

*Read the value from SPICE

In case of a time-optimal problem the objective function becomes

$$\min_{\underline{u}(t)} J(\underline{u}) := \int_{t_0}^{t_f} 1dt$$

$$\underline{x}(t_0) = x_{0Earth}$$

$$\underline{x}(t_f) = x_{FVenus}$$

The Hamiltonian $\underline{\lambda} = (\underline{\lambda}_r, \underline{\lambda}_v, \lambda_m)^T$ can be developed as follows

$$H = l + \underline{\lambda}\underline{f} = 1 + \begin{pmatrix} \underline{\lambda}_r \\ \underline{\lambda}_v \\ \underline{\lambda}_m \end{pmatrix} \begin{pmatrix} \underline{v} \\ -\frac{\mu}{r^3}\underline{r} + \mu\frac{T_{max}}{m}\underline{\hat{\alpha}} \\ -\mu\frac{T_{max}}{Ispg_0} \end{pmatrix}$$

After some manipulations it becomes

$$H = 1 + \underline{\lambda}_r\underline{v} - \frac{\mu}{r^3}\underline{r}\underline{\lambda}_v + \mu\frac{T_{max}}{m}u(-\frac{\underline{\lambda}_v}{m}Ispg_0 - \underline{\lambda}_m)$$

By means of the Pontriagin Maximum Principle, is chosen, from the admissible set, the value of $\underline{\hat{\alpha}}^*$ minimizing the Hamiltonian. It is possible to consider $\underline{\hat{\alpha}}^* = -\frac{\underline{\lambda}_v}{\lambda_v}$.
With this considerations, from the Hamiltonian can be extracted the state and co-state dynamics, called also Euler-Lagrange Equations.

$$\underline{\dot{x}} = \frac{dH}{d\underline{\lambda}} = \begin{cases} \underline{\dot{r}} = \underline{v} \\ \underline{\dot{v}} = -\frac{\mu}{r^3}\underline{r} + u^*(m, \lambda_v, \lambda_m)\frac{T_{max}}{m}\frac{\underline{\lambda}_v}{\lambda_v} \\ \dot{m} = -\mu\frac{T_{max}}{Ispg_0} \end{cases} \tag{3}$$

$$\underline{\dot{\lambda}} = -\frac{dH}{d\underline{x}} = \begin{cases} \underline{\dot{\lambda}}_r = -\frac{3\mu}{r^5}(\underline{r}\underline{\lambda}_v)\underline{r} + \frac{\mu}{r^3}\underline{\lambda}_v \\ \underline{\dot{\lambda}}_v = -\underline{\lambda}_r \\ \underline{\dot{\lambda}}_m = -u^*(m, \lambda_v, \lambda_m)\frac{\lambda_v T_{max}}{m^2} \end{cases} \tag{4}$$

$$0 = \frac{dH}{d\underline{u}} \tag{5}$$

Moreover for a time-optimal travel the switching function is always positive, leading to keep the thrust always on. Leading to $u^*(m, \lambda_v, \lambda_m) = 1$. Finally adding to the system of E-L Equations the Transversality Condition are obtained the necessary conditions

$$\begin{cases} \underline{\dot{x}} = \frac{dH}{d\underline{\lambda}} & \underline{x}(t_0) = \underline{x}_0 \\ \underline{\dot{\lambda}} = -\frac{dH}{d\underline{x}} & \underline{x}(t_f) = \underline{\psi}(t_f) \\ 0 = \frac{dH}{d\underline{u}} \\ H(t_f) - \underline{\lambda}(t_f)\underline{\dot{\psi}}(t_f) = 0 \end{cases} \tag{6}$$

3.0.2 Point 2

The parameters to be used in the adimensionalization are the following: $MU = m_0$, $LU = 1AU$ (retrived from using function *cspice convert*), $t_{ref} = \sqrt{\frac{LU^3}{GM}}$. The last equation deriving from imposing the adimensionalized μ equal to one. Is obtained the following expression, to be inverted

$$\mu[\frac{m^3}{s^2}\frac{TU^2}{LU^3} = 1]$$

The adimensionalization is carried out in the following way

$$Thrust\ Adimensionalized = \frac{Thrust\ t^2{}_{ref}}{MULU}$$

$$Specific\ Impulse\ Adimensionalized = \frac{Specific\ Impulse}{T_{ref}}$$

$$\mu\ Adimensionalized = 1$$

$$g_0\ Adimensionalized = \frac{g_0\ {t^2}_{ref}}{LU}$$

$\mathbf{r}_0$	-0.4097	-0.9318	0.0003
$\mathbf{v}_0$	0.9025	-0.3997	0.0000
m_0	1		
I_{sp}	6.2119e-04		
T_{max}	134.9054		
g_0	1.6537e+06		
GM	1		

Table 2: Adimensionalized quantities ($T_{\text{max}} = 800$ mN).

3.0.3 Point 3

$\boldsymbol{\lambda}_{0,r}$	0.4458	-12.5706	0.0853
$\boldsymbol{\lambda}_{0,v}$	5.0638	-9.5668	1.3379
$\lambda_{0,m}$	1.7250		
t_f	`2023-OCT-20-12:47:57.4330 UTC`		
TOF [days]	144.9408		

Table 3: Time-optimal Earth-Venus transfer solution ($T_{\text{max}} = 800$ mN).

$\lvert\lvert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\rvert\rvert$	[km]	0.0754
$\lvert\lvert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\rvert\rvert$	[m/s]	2.9133e-08

Table 4: Final state error with respect to Venus' center ($T_{\text{max}} = 800$ mN).

3.0.4 Point 4

$\boldsymbol{\lambda}_{0,r}$	-0.7196	-19.4660	-0.3856
$\boldsymbol{\lambda}_{0,v}$	11.8208	-15.1062	1.4425
$\lambda_{0,m}$	2.3656		
t_f	`2024-JAN-10-21:44:34.5223 UTC`		
TOF [days]	227.3135		

Table 5: Time-optimal Earth-Venus transfer solution ($T_{\max} = 500$ mN).

$\vert\vert\mathbf{r}_f(t_f) - \mathbf{r}_V(t_f)\vert\vert$	[km]	0.0046
$\vert\vert\mathbf{v}_f(t_f) - \mathbf{v}_V(t_f)\vert\vert$	[m/s]	1.4148e-08

Table 6: Final state error with respect to Venus' center ($T_{\max} = 500$ mN).