SGN – Assignment #2

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Disclaimer: The story plot contained in the following three exercises is entirely fictional.

Exercise 1: Uncertainty propagation

The Prototype Research Instruments and Space Mission Technology Advancement (PRISMA) is a technology in-orbit test-bed mission for demonstrating Formation Flying (FF) and rendezvous technologies, as well as flight testing of new sensors and actuator equipment. It was launched on June 15, 2010, and it involves two satellites: Mango (Satellite 1, ID 36599), the chaser, and Tango (Satellite 2, ID 36827), the target.

You have been provided with an estimate of the states of Satellites 1 and 2 at the separation epoch t_{sep} = `2010-08-12T05:27:39.114` (UTC) in terms of mean and covariance, as reported in Table 1. Assume Keplerian motion can be used to model the spacecraft dynamics.

1. Propagate the initial mean and covariance for both satellites within a time grid going from t_{sep} to $t_{sep} + N\,T_1$, with a step equal to T_1, where T_1 is the orbital period of satellite 1 and $N = 10$, using both a Linearized Approach (LinCov) and the Unscented Transform (UT). We suggest to use $\alpha = 0.1$ and $\beta = 2$ for tuning the UT in this case.

2. Considering that the two satellites are in close formation, you have to guarantee a sufficient accuracy about the knowledge of their state over time to monitor potential risky situations. For this reason, at each revolution, you shall compute:

   - the norm of the relative position (Δr), and
   - the sum of the two covariances associated to the position elements of the states of the two satellites (P_{sum})

   The critical conditions which triggers a collision warning is defined by the following relationship:

   $$\Delta r < 3\sqrt{\max(\lambda_i(P_{\text{sum}}))}$$

   where $\lambda_i(P_{\text{sum}})$ are the eigenvalues of P_{sum}. Identify the revolution N_c at which this condition occurs and elaborate on the results and the differences between the two approaches (UT and LinCov).

3. Perform the same uncertainty propagation process on the same time grid using a Monte Carlo (MC) simulation *. Compute the sample mean and sample covariance and compare them with the estimates obtained at Point 1). Provide the plots of:

   - the time evolution for all three approaches (MC, LinCov, and UT) of $3\sqrt{\max(\lambda_i(P_{r,\text{i}}))}$ and $3\sqrt{\max(\lambda_i(P_{v,\text{i}}))}$, where $i = 1, 2$ is the satellite number and P_r and P_v are the 3x3 position and velocity covariance submatrices.
   - the propagated samples of the MC simulation, together with the mean and covariance obtained with all methods, projected on the orbital plane.

   Compare the results and discuss on the validity of the linear and Guassian assumption for uncertainty propagation.

*Use at least 100 samples drawn from the initial covariance

Table 1: Estimate of Satellite 1 and Satellite 2 states at t_0 provided in ECI J2000.

Parameter	Value
Ref. epoch t_{sep} [UTC]	2010-08-12T05:27:39.114
Mean state $\hat{\boldsymbol{x}}_{0,\text{sat1}}$ [km, km/s]	$\hat{\boldsymbol{r}}_{0,\text{sat1}}$ = [4622.232026629, 5399.3369588058, -0.0212138165769957] $\hat{\boldsymbol{v}}_{0,\text{sat1}}$ = [0.812221125483763, -0.721512914578826, 7.42665302729053]
Mean state $\hat{\boldsymbol{x}}_{0,\text{sat2}}$ [km, km/s]	$\hat{\boldsymbol{r}}_{0,\text{sat2}}$ = [4621.69343340281, 5399.26386352847, -3.09039248714313] $\hat{\boldsymbol{v}}_{0,\text{sat2}}$ = [0.813960847513811, -0.719449862738607, 7.42706066911294]
Covariance P_0 [km^2, km^2/s, km^2/s^2]	$\begin{bmatrix} +5.6e-7 & +3.5e-7 & -7.1e-8 & 0 & 0 & 0 \\ +3.5e-7 & +9.7e-7 & +7.6e-8 & 0 & 0 & 0 \\ -7.1e-8 & +7.6e-8 & +8.1e-8 & 0 & 0 & 0 \\ 0 & 0 & 0 & +2.8e-11 & 0 & 0 \\ 0 & 0 & 0 & 0 & +2.7e-11 & 0 \\ 0 & 0 & 0 & 0 & 0 & +9.6e-12 \end{bmatrix}$

1.1 Uncertainty Propagation with Linear Covariance and Uscent Transform

From the given state of Satellite 1, Mango, T1, its orbital period is computed as follows

$$T1 = 2\pi\sqrt{\frac{a^3}{\mu}}$$

where a is the Semi-Major Axis of the orbit, computed as

$$a = -\frac{1}{2}\frac{GM}{0.5v^2 - GM * \rho}$$

Resulting in $T1 = 6004[s]$.

1.1.1 Linear Covariance

The motion is propagated using as ΔT the period T1, computed previously, for each time-step the initial mean state is propagated assuming Keplerian Motion without disturbances, while the variance (P) is assumed to be linear and propagated using the State Transition Matrix as

$$P_{t_{final}} = \Phi_f P_{t_{initial}} \Phi_f^T$$

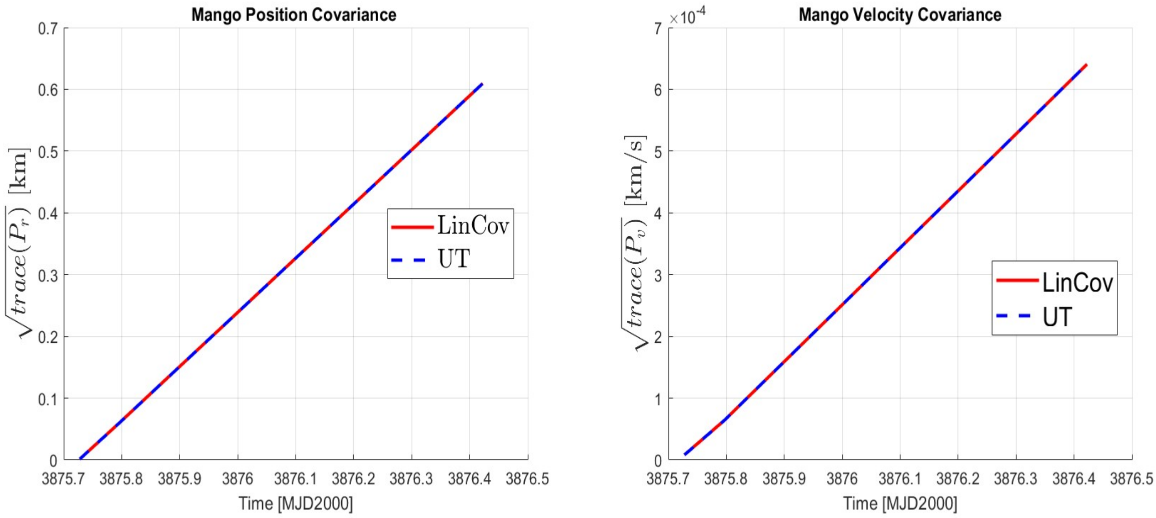
where Φ_f is the State Transition Matrix (STM), computed from $t_{initial}$ to t_{final}, $\Phi = \Phi(t, t{+}\Delta T)$

1.1.2 Unscent Transform

At each timestep, the uncertainties and the mean state computed at the previous time-step, and the tuning parameters of $\alpha = 0.1$ and $\beta = 2$ are given. $2N + 1$ sigma points are generated from the current mean state, they are subsequently propagated assuming Keplerian Motion, and used, weighted, to compute the mean and variance of the distribution at time $t_{final} = t_{initial} + \Delta T$.

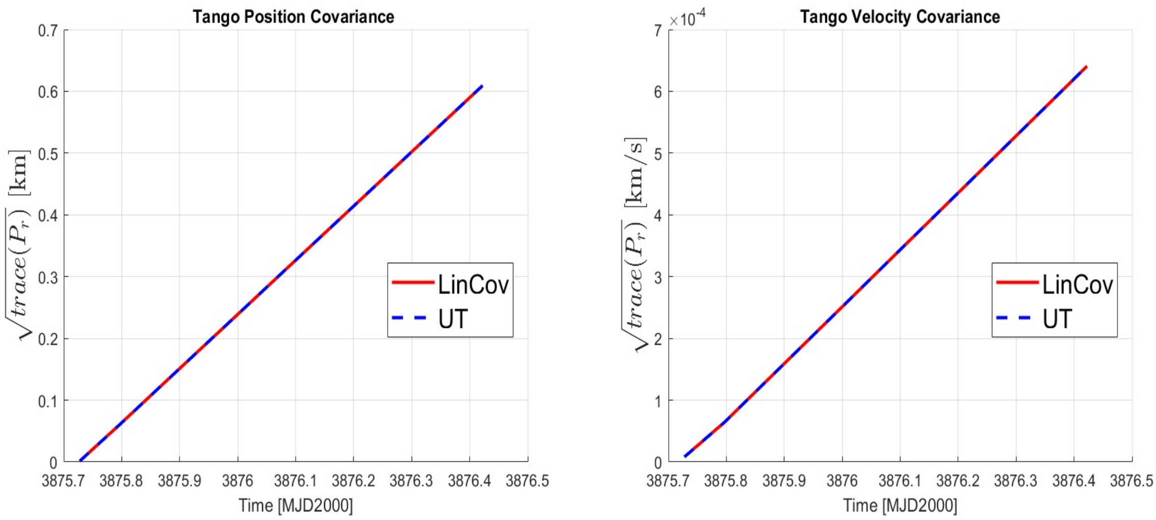
1.1.3 Discussion of the Results

Plots of the evolution in time of the square root of the trace of position and velocity part of the covariance are shown. It is possible to appreciate how the two approaches are converging to the same results, moreover the level of accuracy is granted from the values of the Mean Square Error, in the order of 10^{-6} for the position and 10^{-9} for the velocity, for both satellites.



(a) Propagation Position Covariance **(b)** Propagation Velocity Covariance

Figure 1: For Satellite 1, Mango



(a) Propagation Position Covariance **(b)** Propagation Velocity Covariance

Figure 2: For Satellite 2, Tango

1.2 Collision Warning

The two satellites are orbiting on a similar orbit, critical conditions triggering a collision warning are defined as follows

$$\Delta r < 3\sqrt{\max\left(\lambda(P_{sum})\right)}$$

Where $(\lambda(P_{sum})$ are the eigenvalues of the the sum of the two covariances associated to the position elements of the states of the two satellites, while

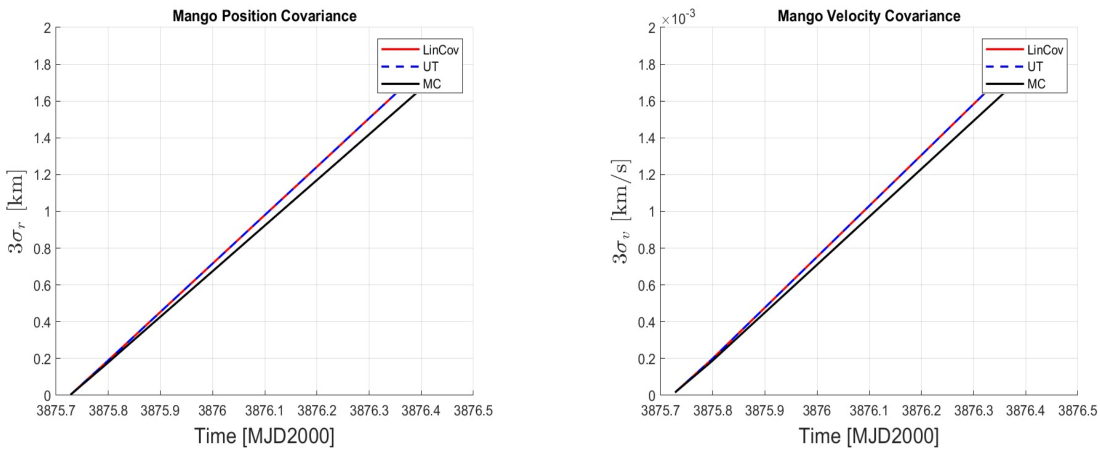
$$\Delta r = \|r_{Sat1} - r_{Sat2}\|$$

. Conditions, for both propagations, are meet in the fourth revolution, between $t = t_{initial} + 3\Delta T1$ and $t = t_{initial} + 4\Delta T1$.

1.3 Monte Carlo Analysis

The Monte Carlo simulation was carried out firstly with the minimum number of 100 samples in order to have a first glance at the problem saving computation time then a more precise 1000 samples estimate was carried out. The samples are propagated with the previously described modalities both for dynamics and time-step. Sample Mean and Covariance are computed from the propagated population. The Mean Absolute Error between Linear Covariance and Unscented Transform, and between this two approaches and Monte Carlo Method are depicted in the following tables:

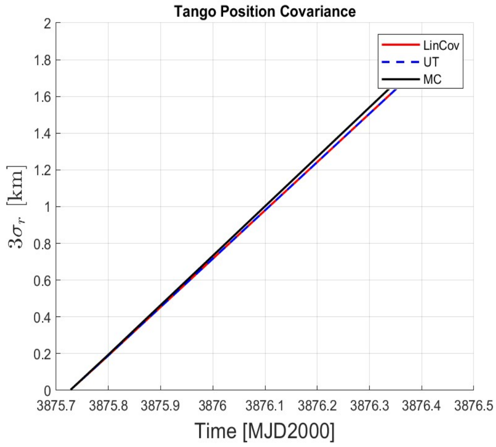
MAE LinCov and UT		
	Mango	Tango
Position	$2.5544e-06$	$2.5347e-06$
Velocity	$2.6849e-09$	$2.6660e-09$
Monte Carlo		
	Mango	Tango
Position	0.05307	0.0209
Velocity	$5.5030e-05$	$2.2971e-05$



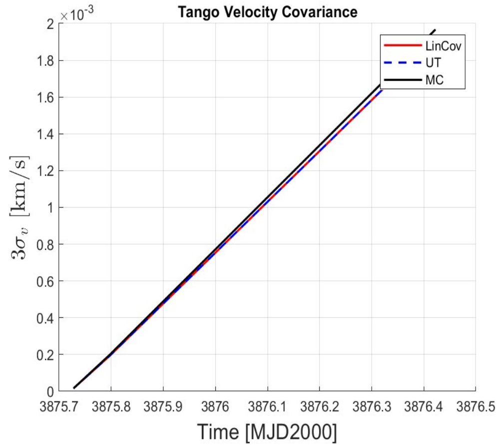
(a) Propagation Position Monte Carlo **(b)** Propagation Velocity Monte Carlo

Figure 3: For Satellite 1, Mango

Plotting the results after a projection in the orbital plane grants a complete picture of all the approaches used before. For each number of revolutions and for each method, its sample mean and covariance are plotted in the local vertical local horizontal plane or orbit plane. From this plots it is possible to notice how the Unscented Trasform and the Linear Covariance share a similar distribution, while the Monte Carlo is far away from these two. A further 1000 samples analysis was carried out in order to double check the previous results, leading to the same conclusions. This fact could mean that the satellites are far from the pericenter, the point on the orbit where the behaviour is the most distant from the linear one. Finally, from the last two plots belonging to the 1000 samples simulation, is is possible to see that the gaussian



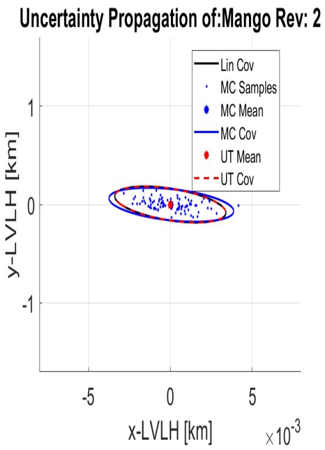
(a) Propagation Position Monte Carlo



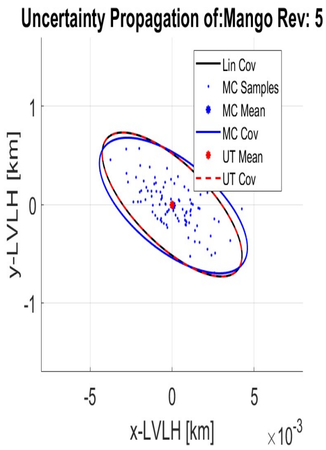
(b) Propagation Velocity Monte Carlo

Figure 4: For Satellite 2, Tango

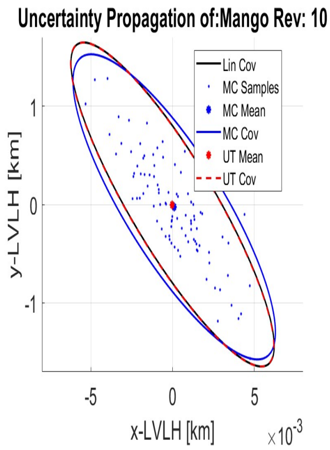
assumptions is satisfied, with the vast majority of the points belonging to the ellipses. There is no evidence of the so called "Banana-Shapes", that usually highlight non gaussianity of the distribution.



(a) Uncertainty in LVLH, Rev: 2

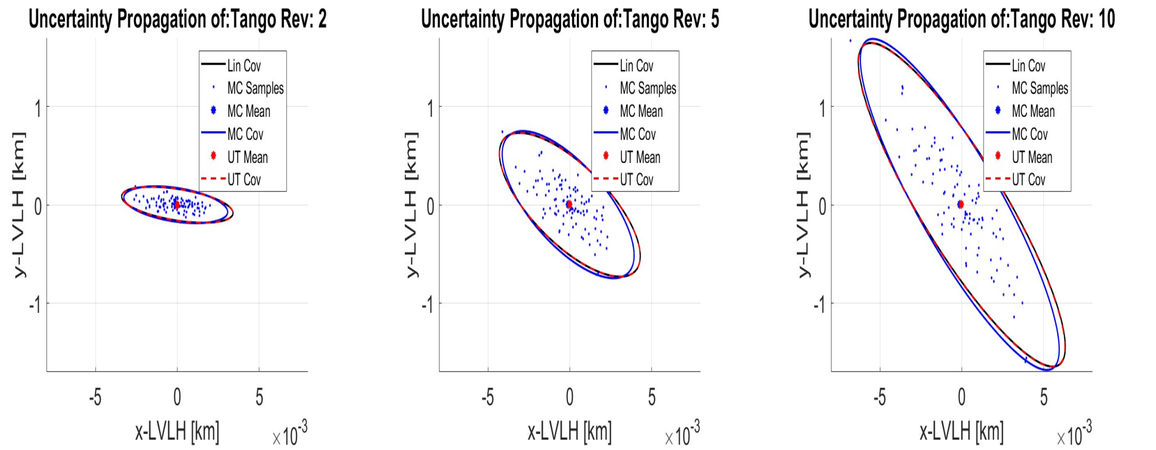


(b) Uncertainty in LVLH, Rev: 5



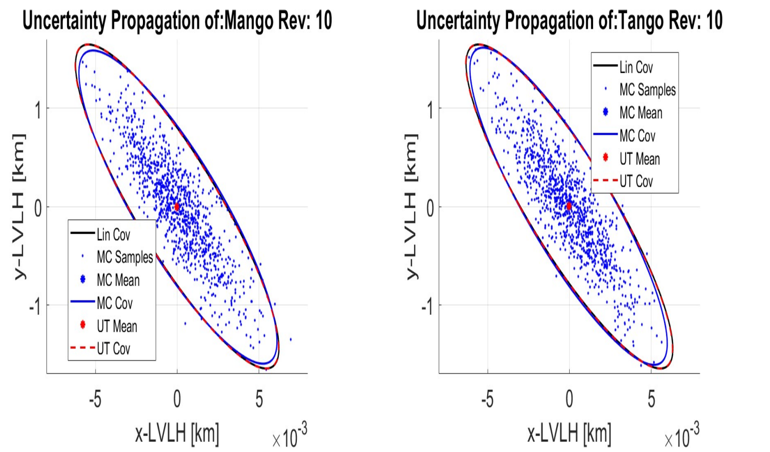
(c) Uncertainty in LVLH, Rev: 10

Figure 5: For Satellite 1, Mango, 100 Samples



(a) Uncertainty in LVLH, Rev: 2 **(b)** Uncertainty in LVLH, Rev: 5 **(c)** Uncertainty in LVLH, Rev: 10

Figure 6: For Satellite 2, Tango, 100 Samples



(a) Mango, Uncertainty in LVLH, Rev: 10 **(b)** Tango, Uncertainty in LVLH, Rev: 10

Figure 7: Uncertainty Propagation, Mango and Tango, 1000 Samples

Exercise 2: Batch filters

You have been asked to track Mango to improve the accuracy of its state estimate. To this aim, you shall schedule the observations from the two ground stations reported in Table 2.

1. *Compute visibility windows.* By using the mean state reported in Table 1 and by assuming Keplerian motion, predict the trajectory of the satellite over a uniform time grid (with a time step of 60 seconds) and compute all the visibility time windows from the available stations in the time interval from t_0 = `2010-08-12T05:30:00.000` (UTC) to t_f = `2010-08-12T11:00:00.000` (UTC). Plot the resulting predicted Azimuth and Elevation profiles in the visibility windows.

2. *Simulate measurements.* The Two-Line Elements (TLE) set of Mango are reported in Table 3 (and in WeBeep as 36599.3le). Use SGP4 and the provided TLEs to simulate the measurements acquired by the sensor network in Table 2 by:
   (a) Computing the spacecraft position over the visibility windows identified in Point 1 and deriving the associated expected measurements.
   (b) Simulating the measurements by adding a random error to the expected measurements (assume a Gaussian model to generate the random error, with noise provided in Table 2). Discard any measurements (i.e., after applying the noise) that does not fulfill the visibility condition for the considered station.

3. *Solve the navigation problem.* Using the measurements simulated at the previous point:
   (a) Find the least squares (minimum variance) solution to the navigation problem without a priori information using
   - the epoch t_0 as reference epoch;
   - the reference state as the state derived from the TLE set in Table 3 at the reference epoch;
   - the simulated measurements obtained for the KOROU ground station only;
   - pure Keplerian motion to model the spacecraft dynamics.

   (b) Repeat step 3a by using all simulated measurements from both ground stations.
   (c) Repeat step 3b by using J2-perturbed motion to model the spacecraft dynamics.

4. Provide the obtained navigation solutions and elaborate on the results, comparing the different solutions.

5. Select the best combination of dynamical model and ground stations and perform the orbit determination for the other satellite.

2.1 Visibility Windows Computation

Starting from the initial mean state position and velocity, of the spacecraft are propagated assuming pure keplerian dynamics without disturbances, in the given time epoch ranging from $t_0 = 2010 - 08 - 12T05 : 30 : 00.000$ to $t_f = 2010 - 08 - 12T11 : 00 : 00.000$. The state vector is then transformed from the initial ECI reference frame to the topocentric reference frame associated to each station. Consequently the positions were filtered to account from the visible, and thus feasible, measurements, accounting for the minimum elevation detectable from each station being respectively $El_{Kourou} = 10°$ and $El_{Svalbard} = 5°$. The transformation from one reference frame to the other was performed using the function *cspice sxform* and the *SPK CSpice Kernel*. The results of this analysis are showed below, it is possible to compare the feasible measurenments of the two ground stations, highlighting a greater number of possible measurenments from the Svalbard ground station.

Table 2: Sensor network to track Mango and Tango: list of stations, including their features.

Station name	KOUROU	SVALBARD
Coordinates	LAT = 5.25144° LON = -52.80466° ALT = -14.67 m	LAT = 78.229772° LON = 15.407786° ALT = 458 m
Type	Radar (monostatic)	Radar (monostatic)
Provided measurements	Az, El [deg] Range (one-way) [km]	Az, El [deg] Range (one-way) [km]
Measurements noise (diagonal noise matrix R)	$\sigma_{Az,El}$ = 100 mdeg σ_{range} = 0.01 km	$\sigma_{Az,El}$ = 125 mdeg σ_{range} = 0.01 km
Minimum elevation	10 deg	5 deg

Table 3: TLE of Mango.

```
1␣36599U␣10028B␣␣␣10224.22752732␣-.00000576␣␣00000-0␣-16475-3␣0␣␣9998
2␣36599␣098.2803␣049.5758␣0043871␣021.7908␣338.5082␣14.40871350␣␣8293
```

Table 4: TLE of Tango.

```
1␣36827U␣10028F␣␣␣10224.22753605␣␣.00278492␣␣00000-0␣␣82287-1␣0␣␣9996
2␣36827␣098.2797␣049.5751␣0044602␣022.4408␣337.8871␣14.40890217␣␣␣␣55
```

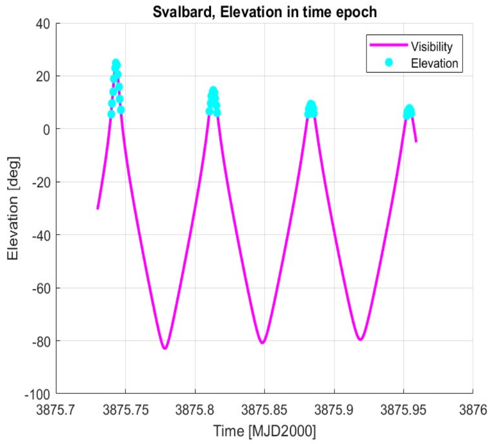
2.2 Measurements Simulation

Now the goal was shifted to the simulation of true measurements, for this purpose the trajectory of the spacecraft was propagated in the before-mentioned time epoch, but this time the SPG4 was exploited to derive the satellite states. SPG4 represents an analytical tool, granting faster computation and more accurate representation of the orbit, when compared to the integral way. Starting from the TLE string of the spacecraft, after having adjusted the nutation parameters, the propagation is performed. Subsequently the states are converted, the noise is added considering the noise matrix of the two ground station and filtered as already described.

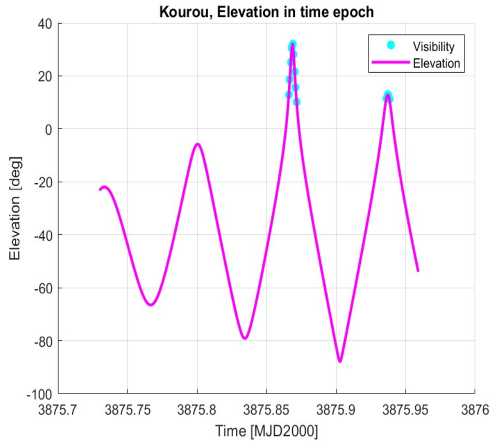
2.3 Solving the navigation problem

The objective of this section was to find the state at t_0 using the measurements reproduced as stated before, no other information was used. The final solution is compared with the corresponding SPG4 state. Three cases were considered for Satellite 1, Mango. The problem itself lies in the implementation of a non linear least squares method, based on the *Matlab* function *lsqnonlin*. More specifically for each one of the three cases a function assembling the residual was to be implemented. In each one of the three cases the following two assumptions where made:

1. The weight matrix is the inverse of the noise matrix of the considered stations, the more one measurement is uncertain, the less it will be taken into account.
2. The initial guess was produced considering the state predicted by the SPG4 propagation at t0.

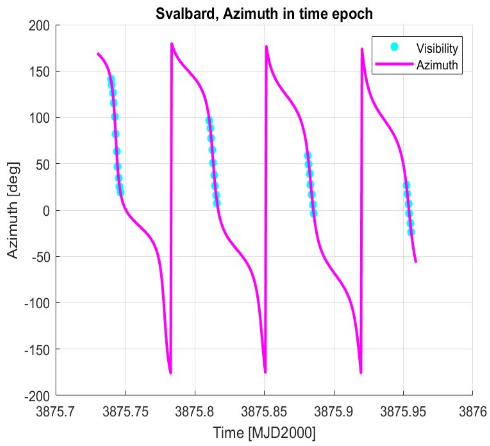


(a) Elevation in time with respect to Svalbard

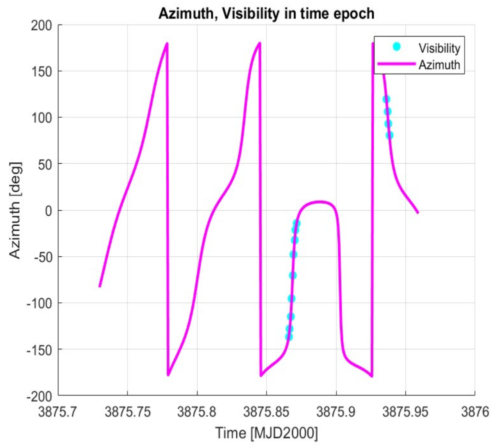


(b) Elevation in time with respect to Kourou

Figure 8: Elevation in time with respect to both stations, visible measurements



(a) Azimuth in time with respect to Svalbard



(b) Azimuth in time with respect to Kourou

Figure 9: Azimuth in time with respect to both stations, visible measurements

2.3.1 Solving the navigation problem: One Station

The residual was computed using the measurements of Kourou only, the guess state was propagated assuming pure keplerian motion, and measured. Followingly the difference between the "real" measurements and the ones coming from the propagation of the n-guess taken at the same time instant, is computed and weighted, assembling the residual.

$Error = x_{Real} - x_{Estimated}$			
	x	y	z
Position	22.2829	-15.5632	27.3885
Velocity	-0.0067	-0.0479	-0.0153

2.3.2 Solving the navigation problem: Two Stations

In this case the procedure differs from the before mentioned one only for the aspects related to the number of samples taken into accounted for. The n-guess is propagated for the time instants of the visible measurements of the two stations and measured. The matrix assembled for the residual is then larger than the previous case, leading to a finer least squares fitting.

$Error = x_{Real} - x_{Estimated}$			
	x	y	z
Position	19.0139	-10.3441	2.9284
Velocity	-0.01046	-0.0087	-0.0057

2.3.3 Solving the navigation problem: Two Stations and J2

At this stage the contribution of the acceleration coming from J2 is taken into account. The J2 acceleration is computed as

$$a_{J2} = \frac{3}{2}\mu J2 \frac{\mathbf{r}}{r^3}\left(\frac{R_{earth}}{r}\right)^2 \left(5\left(\frac{z}{r}\right)^2 - \begin{pmatrix}1\\2\\3\end{pmatrix}\right)$$

The used value for $J2 = 0.0010826269$.
This simulation should lead to even better results, given that the J2 acceleration is modeled by the analytical propagation. The results confirm this last point.

$Error = x_{Real} - x_{Estimated}$			
	x	y	z
Position	-0.0916	0.0005	-0.1288
Velocity	0.0001	0.0001	0.0001

2.3.4 Solving the navigation problem: Satellite 2, Tango

Given that, more measurements should lead to better results and the J2 perturbation is modelled by the SPG4 model, a simulation comprehending J2 effect and two station is chosen, as it should lead to the better solution, also as proven before.

$Error = x_{Real} - x_{Estimated}$			
	x	y	z
Position	0.1004	0.0254	0.7098
Velocity	-0.00044	-0.00037	-0.00019

Exercise 3: Sequential filters

According to the Formation Flying In Orbit Ranging Demonstration experiment (FFIORD), PRISMA's primary objectives include testing and validation of GNC hardware, software, and algorithms for autonomous formation flying, proximity operations, and final approach and recede operations. The cornerstone of FFIORD is a Formation Flying Radio Frequency (FFRF) metrology subsystem designed for future outer space formation flying missions.

FFRF subsystem is in charge of the relative positioning of 2 to 4 satellites flying in formation. Each spacecraft produces relative position, velocity and line-of-sight (LOS) of all its companions.

You have been asked to track Mango to improve the accuracy of the estimate of its absolute state and then, according to the objectives of the PRISMA mission, validate the autonomous formation flying navigation operations by estimating the relative state between Mango and Tango by exploiting the relative measurements acquired by the FFRF subsystem. The Two-Line Elements (TLE) set of Mango and Tango satellites are reported in Tables 3 and 4 (and in WeBeep as 36599.3le, and 36827.3le).

The relative motion between the two satellites can be modelled through the linear, Clohessy-Wiltshire (CW) equations[†]

$$\begin{aligned} \ddot{x} &= 3n^2x + 2n\dot{y} \\ \ddot{y} &= -2n\dot{x} \\ \ddot{z} &= -n^2 z \end{aligned} \tag{1}$$

where x, y,and z are the relative position components expressed in the LVLH frame, whereas n is the mean motion of Mango, which is assumed to be constant and equal to:

$$n = \sqrt{\frac{GM}{R^3}} \tag{2}$$

where R is the position of Mango at t_0.

The unit vectors of the LVLH reference frame are defined as follows:

$$\hat{\boldsymbol{i}} = \frac{\boldsymbol{r}}{r}, \quad \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}}, \quad \hat{\boldsymbol{k}} = \frac{\boldsymbol{h}}{h} = \frac{\boldsymbol{r} \times \boldsymbol{v}}{\|\boldsymbol{r} \times \boldsymbol{v}\|} \tag{3}$$

To perform the requested tasks you should:

1. *Estimate Mango absolute state.* You are asked to develop a sequential filter to narrow down the uncertainty on the knowledge of Mango absolute state vector. To this aim, you shall schedule the observations from the SVALBARD ground station[‡] reported in Table 2, and then proceed with the state estimation procedure by following these steps:

   (a) By using the mean state reported in Table 1 and by assuming Keplerian motion, predict the trajectory of the satellite over a uniform time grid (with a time step of 5 seconds) and compute the first visibility time window from the SVALBARD station in the time interval from t_0 = `2010-08-12T05:30:00.000` (UTC) to t_f = `2010-08-12T06:30:00.000` (UTC).

   (b) Use SGP4 and the provided TLE to simulate the measurements acquired by the SVALBARD station for the Mango satellite only. For doing it, compute the spacecraft position over the visibility window using a time-step of 5 seconds, and derive the associated expected measurements. Finally, simulate the measurements by adding a random error (assume a Gaussian model to generate the random error, with noise provided in Table 2).

[†]Notice that the system is linear, therefore it has an analytic solution of the state transition matrix $\boldsymbol{\Phi}$

[‡]Note that these are the same ones computed in Exercise 2

(c) Using an Unscented Kalman Filter (UKF), provide an estimate of the spacecraft state (in terms of mean and covariance) by sequentially processing the acquired measurements in chronological order. Plot the time evolution of the error estimate together with the 3σ of the estimated covariance for both position and velocity.

2. *Estimate the relative state.* To validate the formation flying operations, you are also asked to develop a sequential filter to narrow down the uncertainty on the knowledge of the relative state vector. To this aim, you can exploit the relative azimuth, elevation, and range measurements obtained by the FFRF subsystem, whose features are reported in Table 5, and then proceed with the state estimation procedure by following these steps:

    (a) Use SGP4 and the provided TLEs to propagate the states of both satellites at epoch t_0 in order to compute the relative state in LVLH frame at that specific epoch.

    (b) Use the relative state as initial condition to integrate the CW equations over the time grid defined in Point 1a. Finally, simulate the relative measurements acquired by the Mango satellite through its FFRF subsystem by adding a random error to the expected measurements. Assume a Gaussian model to generate the random error, with noise provided in Table 5.

    (c) Consider a time interval of 20 minutes starting from the first epoch after the visibility window (always with a time step of 5 seconds). Use an UKF to provide an estimate of the spacecraft relative state in the LVLH reference frame (in terms of mean and covariance) by sequentially processing the measurements acquired during those time instants in chronological order. Plot the time evolution of the error estimate together with the 3σ of the estimated covariance for both relative position and velocity.

3. *Reconstruct Tango absolute covariance.* Starting from the knowledge of the estimated covariance of the absolute state of Mango, computed in Point 1, and the estimated covariance of the relative state in the LVLH frame, you are asked to provide an estimate of the covariance of the absolute state of Tango. You can perform this operation as follows:

    (a) Pick the estimated covariance of the absolute state of Mango at the last epoch of the visibility window, and propagate it within the time grid defined in Point 2c.

    (b) Rotate the estimated covariance of the relative state from the LVLH reference frame to the ECI one within the same time grid.

    (c) Sum the two to obtain an estimate of the covariance of the absolute state of Tango. Plot the time evolution of the 3σ for both position and velocity and elaborate on the results.

Table 5: Parameters of FFRF.

Parameter	Value
Measurements noise (diagonal noise matrix R)	$\sigma_{Az,El} = 1$ deg $\sigma_{range} = 1$ cm

3.1 Estimation of the absolute state of Satellite 1

The quest was to estimate the absolute state of the satellite Mango, with the exploitation of a sequential filter.
An Unsceted Kalman Filter (UKF) is firstly implemented. The filter works on the given measurements from the first to the last, in a chronological order. Below a more in dept description of the procedure:

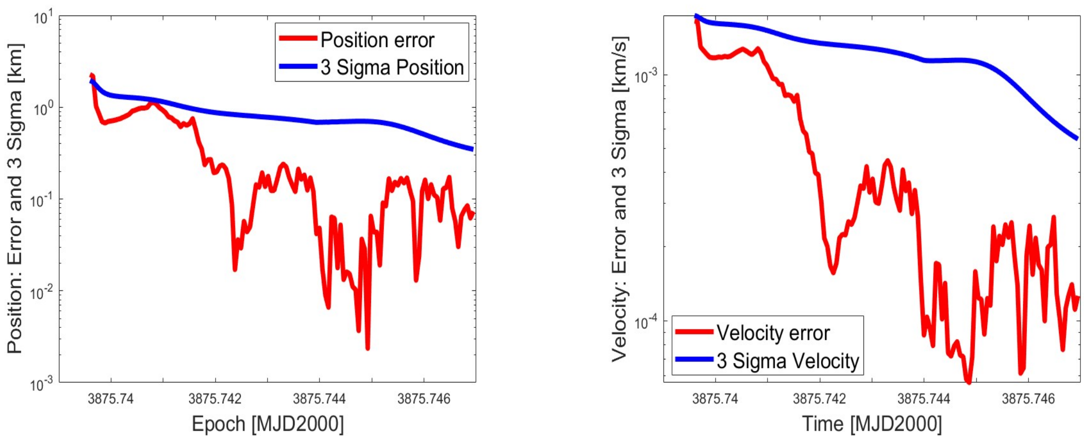
- An initial Keplerian Propagation in the given time window (from t_0 to t_f) is used only to compute the Ephemeris Time of the feasible measurements.
  The starting point is the initial mean state of Mango and its covariance (*Tab.1*). The propagated trajectory in ECI-2000 Reference Frame is used to find the corresponding states in the Topocentric Reference Frame associated to the Svalbard station. The computation exploits the *CSPice* function *CSpice Sxform*. Then using the function *CSpice Reclat* the actual measurements: Range, Azimuth and Elevation, were found. Lastly a time window is found, identifying the feasible measurenments from Svalbard, as the ones satisfying the condition on the minimum elevation.

- While the actual measurements, to be assumed as real and to be feed into the UKF, are created using an SGP4 propagation starting from the provided Two Line Element to the states corresponding to the Ephemeris Time of the previously found feasible measurements. These states are then measured as before, and the noise, characterizing the Svalbard station is added, introducing further uncertainty.

The metrics, showed below, characterizing the performances of the filter are:

- The normed difference between the state estimated by the filter and the one coming from the SGP4 propagation, treat separately for position and velocity.

- The square root of the trace of the covariance matrix estimated by filter, multiplied by three, treat separately for position and velocity.



(a) Position Error, Estimated Covariance Position **(b)** Velocity Error, Estimated Covariance Velocity

Figure 10: Satellite 1, Covariance and State Error evolution in time

3.2 Estimation of the relative state between Satellite 1 and Satellite 2

Starting from the given Two Line Elements for the satellites Tango and Mango, the states of the spacecrafts are propagated by means of SGP4 propagation until t_0, subsequently the relative state between the satellites is computed at t_0 in ECI-2000 Reference Frame as:

$$\mathbf{r}_{Rel} = \mathbf{r}_{Sat2} - \mathbf{r}_{Sat1}$$

In order to obtain a relative state describing the position of Satellite 2 with respect to Satellite 1 in the LVLH Frame centered at Satellite 1, this vector has to be rotated in LVLH, Local Vertical Local Horizontal, reference frame. To this purpose the rotation matrix is assembled in the following way:

- The LVLH reference frame is characterized starting from the mean state. The mean position identify the new reference frame origin, the axes, defining the orbital plane, $\hat{\imath}$ and $\hat{\jmath}$ are derived by the whole mean state in the following way:

$$\hat{\imath} = \frac{\mathbf{r}}{r}$$

$$\hat{\jmath} = \hat{k} \times \hat{\imath}$$

$$\hat{k} = \frac{\mathbf{h}}{h} = \frac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r} \times \mathbf{v}\|}$$

  Where r and h are the norms of $\mathbf{r}$ and $\mathbf{h}$ respectively The rotation matrix for the position is then assembled as

$$\mathbf{R} = \begin{pmatrix} \hat{\imath}_x & \hat{\imath}_y & \hat{\imath}_z \\ \hat{\jmath}_x & \hat{\jmath}_y & \hat{\jmath}_z \\ \hat{k}_x & \hat{k}_y & \hat{k}_z \end{pmatrix}$$

  So

$$\mathbf{r}_{LVLH} = \mathbf{R} \cdot (\mathbf{r}_{ECI} - r_{Origins})$$

  Where $r_{Origins}$ is the distance between the origins of the two reference frames.

- Then the time derivative of R has to be computed. The time derivatives of vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{k}}$ and $\hat{\mathbf{j}}$ are:

$$\frac{d\hat{\mathbf{i}}}{dt} = \frac{1}{r}\left[\mathbf{v} - \left(\hat{\mathbf{i}} \cdot \mathbf{v}\right)\hat{\mathbf{i}}\right]$$

$$\frac{d\hat{\mathbf{k}}}{dt} = \frac{1}{h}\left[\dot{\mathbf{h}} - \left(\hat{\mathbf{h}} \cdot \dot{\mathbf{h}}\right)\hat{\mathbf{h}}\right]$$

$$\frac{d\hat{\mathbf{j}}}{dt} = \left[\frac{d\hat{\mathbf{k}}}{dt} \times \hat{\mathbf{i}}\right] + \left[\hat{\mathbf{k}} \times \frac{d\hat{\mathbf{i}}}{dt}\right]$$

  Where $\dot{\mathbf{h}} = \mathbf{r} \times \mathbf{a}$

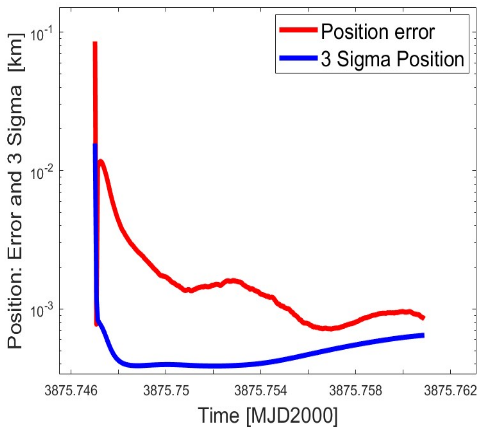
- Finally the matrix for the rotation of position and velocity is assembled as

$$\mathbf{R}_{New} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \dot{\mathbf{R}} & \mathbf{R} \end{pmatrix}$$

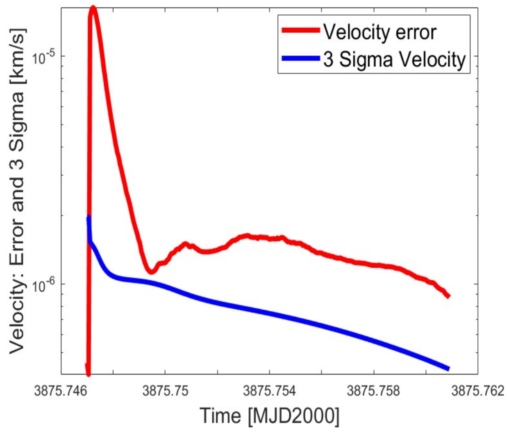
  To be used as

$$\mathbf{rv}_{LVLH} = \mathbf{R}_{New} \cdot \mathbf{rv}_{ECI}$$

The relative state now in LVLH Reference Frame, computed at t_0, is then propagated in time exploiting the Clohessy-Wiltshire dynamics up to t_f. With the purpose of reducing the uncertainty and estimate the relative state an Unscented Kalman Filter is applied in an estimation window lasting for 20 minutes starting from the first epoch after the visibility window, working from the last measurement of the visibility window itself. The produced results are obtained using two different initial covariance matrix Regarding this last measurement while its mean state was not modified two matrix for the initial covariance were taken into account. The first being the true matrix the latter a modified version with the goal of producing satifying results. This matrix was a trade off between an appropriate result for the estimation and the true values of the covariance itself. The obtained error is computed confronting the Kalman Filter State with the state coming from the CW dynamics. The evolution of the uncertainty was computed as the variation in time of trace of the matrices of position and velocity multiplied by three.

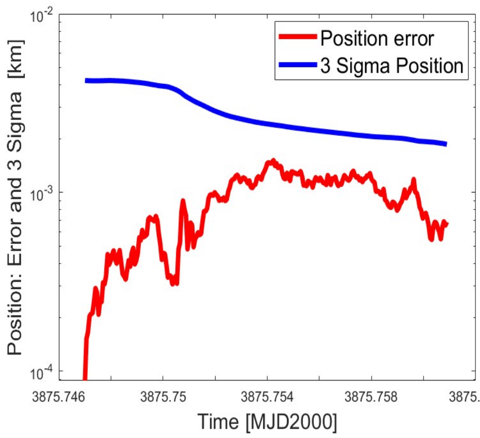


(a) Position Error, Estimated Covariance Position

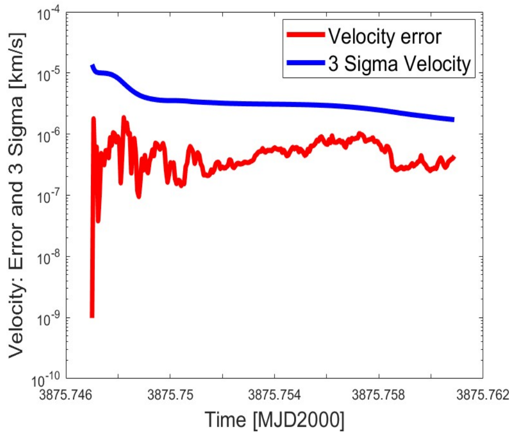


(b) Velocity Error, Estimated Covariance Velocity

Figure 11: Covariance and State Error evolution in time, Relative State, not modified Initial Covariance Matrix



(a) Position Error, Estimated Covariance of Relative State for Position



(b) Velocity Error, Estimated Covariance Velocity

Figure 12: Covariance and State Error evolution in time, Relative State, modified Initial Covariance Matrix

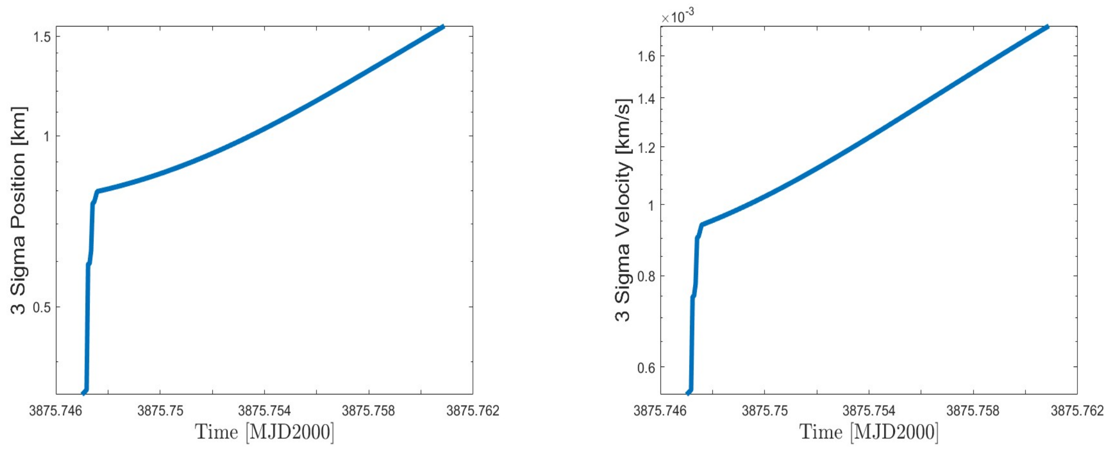
3.3 Estimation of the absolute covariance of Satellite 2

The estimate covariance of the absolute state of Satellite 1 and the relative states, the given in ECI-2000 and LVLH Reference Frames respectively, were exploited to compute the covariance of the absolute state of Satellite 2 in ECI-2000 Reference Frame. A more in dept description of the used procedure:

- In ECI-2000, the Unscent Transform was used to propagate the covariance of the absolute state of Satellite 1, during the before mentioned estimation interval. This was achieved exploiting the estimated covariance of the absolute state of Satellite 1 at the last epoch of the visibility window.
- Starting from the LVLH Frame the relative state was transformed into ECI-2000 at each time step of the estimation interval. This was achieved exploiting the previously described rotation matrix for the whole state $\mathbf{R}$, as

$$\sigma_{\mathbf{ECI},\mathbf{Rel}} = \mathbf{R}^{-1}\sigma_{\mathbf{LVLH},\mathbf{Rel}}(\mathbf{R}^{-1})^\top$$

  In this procedure $\mathbf{R}$ was computed using the SGP4 propagation for the computation of the absolute states of Satellite 1.
- The sum of the two previously computed covariances in the given period of time produces the covariance of the absolute state of Tango. The plots below show that, the covariance, divided for position and velocity, increases in time. The explanation for this is that the absolute state of Satellite 1 was estimated instead of propagated, with a consequent increase of uncertainty.



(a) Position Error, Estimated Covariance Position

(b) Estimated Covariance Velocity

Figure 13: Velocity Error, Absolute Covariance of Satellite 2, Evolution in Time