Products and Sums in the Simply Typed Lambda Calculus

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Summary

We present an encoding for product and sum types in the simply-typed lambda calculus over a single base type which uses only closed terms.

Introduction

Let $\mathcal{T} ::= \mathcal{T} \to \mathcal{T} \mid 0$. In [1], Grzegorczyk established the following result:

Theorem 0.1 (Grzegorczyk)**.** *Let $\sigma, \tau \in \mathcal{T}$. There is some $\sigma \underline{\times} \tau \in \mathcal{T}$ with terms $\mathsf{P} : \sigma \to \tau \to \sigma \underline{\times} \tau$, $\mathsf{P}_1 : \sigma \underline{\times} \tau \to \sigma$, and $\mathsf{P}_2 : \sigma \underline{\times} \tau \to \tau$ with free variable $z : 0$ such for all terms $s : \sigma$, $t : \tau$ we have that*

$$\mathsf{P}_1(\mathsf{P}st) \rightsquigarrow_{\beta\eta} s$$
$$\mathsf{P}_2(\mathsf{P}st) \rightsquigarrow_{\beta\eta} t.$$

Grzegorczyk was studying functionals over the natural numbers, i.e. where 0 denotes the type of the naturals. One can then imagine the free variable z as denoting some numeral such as $\mathbf{0}$. Of course, one does not always have this luxury, so we present a modest improvement which shows that the same result can be established without using any free variables.

First we recall a few basic results:

Lemma 0.2. *Let $\sigma \in \mathcal{T}$. Then either (i) there is a closed term $S : \sigma$ (in which case we will call σ a tautology) or (ii) there are closed terms $F : \sigma \to 0$, $G : 0 \to \sigma$ (in which case we will call σ a non-tautology).*

Lemma 0.3. *Let $\sigma, \tau \in \mathcal{T}$ be non-tautologies. Then $\sigma \to \tau$ is a tautology.*

Lemma 0.4. *Let $\sigma_1 \to \ldots \to \sigma_n \to 0$ be a tautology. Then there is some σ_i which is a non-tautology.*

Lemma 0.5. *Let $\sigma_1 \to \ldots \to \sigma_n \to 0$ be a non-tautology. Then every σ_i is a tautology.*

Products

We are now ready to prove the following result:

Theorem 0.6. *Let $\sigma, \tau \in \mathcal{T}$. There is some $\sigma \underline{\times} \tau \in \mathcal{T}$ with closed terms $\mathsf{P} : \sigma \to \tau \to \sigma \underline{\times} \tau$, $\mathsf{P}_1 : \sigma \underline{\times} \tau \to \sigma$, and $\mathsf{P}_2 : \sigma \underline{\times} \tau \to \tau$ such for all terms $s : \sigma$, $t : \tau$ we have that*

$$\begin{aligned} \mathsf{P}_1(\mathsf{P}st) &\rightsquigarrow_{\beta\eta} s \\ \mathsf{P}_2(\mathsf{P}st) &\rightsquigarrow_{\beta\eta} t. \end{aligned}$$

Proof. Let $\sigma = \sigma_1 \to \ldots \to \sigma_n \to 0$ and $\tau = \tau_1 \to \ldots \tau_m \to 0$. We consider three cases separately:

Case i: σ and τ are both non-tautologies.

Let $\sigma \underline{\times} \tau := (0 \to 0 \to 0) \to \sigma_1 \to \ldots \to \sigma_n \to \tau_1 \to \ldots \to \tau_m \to 0$.

By 0.5, $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m$ are all tautologies. Therefore, there are closed terms $s_1, \ldots, s_n, t_1 \ldots t_m$ of each such type. We then let

$$\begin{aligned} \mathsf{P} &:= \lambda f^{\sigma} g^{\tau} h^{0 \to 0 \to 0} x_1^{\sigma_1} \ldots x_n^{\sigma_n} y_1^{\tau_1} \ldots y_m^{\tau_m}.h(f x_1 \ldots x_n)(g y_1 \ldots y_m) \\ \mathsf{P}_1 &:= \lambda P^{\sigma \underline{\times} \tau} \lambda x_1^{\sigma_1} \ldots x_n^{\sigma_n}.P(\lambda x^0 y^0.x) x_1 \ldots x_n t_1 \ldots t_m \\ \mathsf{P}_2 &:= \lambda P^{\sigma \underline{\times} \tau} \lambda y_1^{\tau_1} \ldots y_m^{\tau_m}.P(\lambda x^0 y^0.y) s_1 \ldots s_n y_1 \ldots y_m. \end{aligned}$$

We then have that for any $s : \sigma, t : \tau$,

$$\begin{aligned} \mathsf{P}_1(\mathsf{P}st) &= \Big(\lambda P\bar{x}.P(\lambda xy.x)\bar{x}\bar{t}\Big)\Big((\lambda fgh\bar{x}\bar{y}.h(f\bar{x})(g\bar{y}))st\Big) \\ &\rightsquigarrow_\beta \Big(\lambda P\bar{x}.P(\lambda xy.x)\bar{x}\bar{t}\Big)\Big(\lambda h\bar{x}\bar{y}.h(s\bar{x})(t\bar{y})\Big) \\ &\rightsquigarrow_\beta \lambda\bar{x}.\Big(\lambda h\bar{x}\bar{y}.h(s\bar{x})(t\bar{y})\Big)(\lambda xy.x)\bar{x}\bar{t} \\ &\rightsquigarrow_\beta \lambda\bar{x}.(\lambda xy.x)(s\bar{x})(t\bar{y}) \\ &\rightsquigarrow_\beta \lambda\bar{x}.s\bar{x} \\ &\rightsquigarrow_\eta s \end{aligned}$$

and similarly in the case of P_2.

Case ii: σ and τ are both tautologies.

By 0.4, there are σ_i, τ_j which are non-tautologies. Thus, $\tau_j \to \sigma_1, \ldots, \tau_j \to \sigma_n, \sigma_i \to \tau_1, \ldots, \sigma_i \to \tau_m$ are all tautologies by 0.3, and so there are corresponding closed terms $s_1, \ldots, s_n, t_1, \ldots, t_m$ of each of these types. We define P as before, but the definitions for $\mathsf{P}_1, \mathsf{P}_2$ need slight modifications to ensure that types match:

$$\mathsf{P}_1 := \lambda P^{\sigma \underline{\times} \tau} \lambda x_1^{\sigma_1} \dots x_n^{\sigma_n}.P(\lambda x^0 y^0.x)x_1 \dots x_n(t_1 x_i) \dots (t_m x_i)$$
$$\mathsf{P}_2 := \lambda P^{\sigma \underline{\times} \tau} \lambda y_1^{\tau_1} \dots y_m^{\tau_m}.P(\lambda x^0 y^0.y)(s_1 y_j) \dots (s_n y_j) y_1 \dots y_m.$$

The proofs for correctness work much like before.

Case iii: σ is a tautology and τ is a non-tautology.

We see here that we must modify $\sigma \underline{\times} \tau$, since we will need it to be a non-tautology, and yet it would be a tautology if we were to use the previous definition. The issue is that at least one of the σ_i's is a non-tautology. The simplest way to turn all of these into tautologies is to simply precede them with a 0; thus, our product in this case will be

$$\sigma \underline{\times} \tau := (0 \to 0 \to 0) \to (0 \to \sigma_1) \to \dots \to (0 \to \sigma_n) \to \tau_1 \to \dots \to \tau_m \to 0.$$

Since $0 \to \sigma_1, \dots, 0 \to \sigma_n, \tau_1, \dots, \tau_m$ are all tautologies, fix closed terms $s_1, \dots, s_n$ and $t_1, \dots, t_m$ of corresponding type. We then let

$$\mathsf{P} := \lambda f^\sigma g^\tau h^{0 \to 0 \to 0} x_1^{0 \to \sigma_1} \dots x_n^{0 \to \sigma_n} y_1^{\tau_1} \dots y_m^{\tau_m}.h(f(x_1(g\bar{y}) \dots (x_n(g\bar{y}))(gy_1 \dots y_m)$$
$$\mathsf{P}_1 := \lambda P^{\sigma \underline{\times} \tau} \lambda x_1^{\sigma_1} \dots x_n^{\sigma_n}.P(\lambda x^0 y^0.x)(\lambda z^0.x_1) \dots (\lambda z^0.x_n) t_1 \dots t_m$$
$$\mathsf{P}_2 := \lambda P^{\sigma \underline{\times} \tau} \lambda y_1^{\tau_1} \dots y_m^{\tau_m}.P(\lambda x^0 y^0.y) s_1 \dots s_n y_1 \dots y_m.$$

It may seem strange that $g(\bar{y})$ is present in the arguments of f, but we only need it as something of type 0 that can fill in the holes left by the newly added abstractions before each σ_i. The proof of P_2's correctness works much as before, but let us show why P_1 works:

$$\begin{aligned}
&\mathsf{P}_1(\mathsf{P}st) \\
&= \Big(\lambda P\bar{x}.P(\lambda xy.x)(\lambda z.x_1) \dots (\lambda z.x_n)\bar{t}\Big)\Big((\lambda fgh\bar{x}\bar{y}.h(f(x_1(g\bar{y}) \dots (x_n(g\bar{y}))(g\bar{y}))st\Big) \\
&\rightsquigarrow_\beta \lambda\bar{x}.\Big((\lambda h\bar{x}\bar{y}.h(s(x_1(t\bar{y})) \dots (x_n(t\bar{y}))(t\bar{y}))\Big)(\lambda xy.x)(\lambda z.x_1) \dots (\lambda z.x_n)\bar{t} \\
&\rightsquigarrow_\beta \lambda\bar{x}.(\lambda xy.x)(s((\lambda z.x_1)(t\bar{y})) \dots ((\lambda z.x_n)(t\bar{y}))(t\bar{y}) \\
&\rightsquigarrow_\beta \lambda\bar{x}.sx_1 \dots x_n \\
&\rightsquigarrow_\eta s.
\end{aligned}$$

$\square$

Naturally, the pairing and projection operations can be extended to higher arities. We shall let P^n denote the n-ary pairing function and P_i^n the n-ary projection function onto the i^{th} coordinate.

Sums

Theorem 0.7. *Let $\sigma, \tau \in \mathcal{T}$. Then there is a type $\sigma \underline{+} \tau \in \mathcal{T}$, closed terms $\mathsf{I}_1 : \sigma \to \sigma \underline{+} \tau$, $\mathsf{I}_2 : \tau \to \sigma \underline{+} \tau$, and for any $\rho \in \mathcal{T}$ a closed term $\mathsf{C} : (\sigma \to \rho) \to (\tau \to \rho) \to \sigma \underline{+} \tau \to \rho$ such that for all $f : \sigma \to \rho, g : \tau \to \rho, s : \sigma, t : \tau$ we have that*

$$\mathsf{C} fg(\mathsf{I}_1 s) \rightsquigarrow_{\beta\eta} fs$$
$$\mathsf{C} fg(\mathsf{I}_2 t) \rightsquigarrow_{\beta\eta} gt.$$

Proof. As before, we have three cases, depending on the logical value of the types σ, τ.

Case i : Both σ and τ are tautologies.

In this case, fix closed terms $S : \sigma$ and $T : \tau$. We now define $\sigma \underline{+} \tau := (0 \to 0 \to 0) \underline{\times} \sigma \underline{\times} \tau$. The two inclusions are defined by

$$\mathsf{I}_1 := \lambda x^\sigma . \mathsf{P}^3(\lambda u^0 v^0 . u) x T$$
$$\mathsf{I}_2 := \lambda y^\tau . \mathsf{P}^3(\lambda u^0 v^0 . v) S y.$$

The idea is much like the one behind the standard set-theoretic construction of the disjoint union: The first coordinate contains a boolean value signifying whether or not our element is a left element or a right element. In the remaining two coordinates is the term in question plus some garbage (S or T) which will never be accessed and is only there so that the types fit.

Given $\rho = \rho_1 \to \ldots \to \rho_n \to 0$ we define the case operator as follows:

$$\mathsf{C} := \lambda f^{\sigma \to \rho} g^{\tau \to \rho} x^{\sigma \underline{+} \tau} y_1^{\rho_1} \ldots y_n^{\rho_n} . (\mathsf{P}^3_1 x)(f(\mathsf{P}^3_2 x)\bar{y})(g(\mathsf{P}^3_3 x)\bar{y}).$$

For arbitrary f, g, s we then have that

$$\begin{aligned} \mathsf{C} fg(\mathsf{I}_1 s) &\rightsquigarrow_\beta \lambda \bar{y} . (\mathsf{P}^3_1(\mathsf{I}_1 s))(f(\mathsf{P}^3_2(\mathsf{I}_1 s))\bar{y})(g(\mathsf{P}^3_3(\mathsf{I}_1 s))\bar{y}) \\ &\rightsquigarrow_{\beta\eta} \lambda \bar{y} . (\lambda uv.u)(fs\bar{y})(gT\bar{y}) \\ &\rightsquigarrow_\beta \lambda \bar{y} . fs\bar{y} \\ &\rightsquigarrow_\eta fs. \end{aligned}$$

and similarly in the case of I_2.

Case ii: Both σ and τ are non-tautologies.

In this case, both $\sigma \to \tau$ and $\tau \to \sigma$ are then tautologies. We can then fix closed terms $F : \sigma \to \tau$ and $G : \tau \to \sigma$. We define $\sigma \underline{+} \tau$ and C as before, but $\mathsf{I}_1, \mathsf{I}_2$ are given slight modifications so that the garbage terms are produced by F and G:

$$\begin{aligned}\mathsf{I}_1 &:= \lambda x^\sigma.\mathsf{P}^3(\lambda u^0 v^0.u)x(Fx)\\ \mathsf{I}_2 &:= \lambda y^\tau.\mathsf{P}^3(\lambda u^0 v^0.v)(Gy)y.\end{aligned}$$

The proof of correctness works much like before.

Case iii: σ is a tautology and τ is a non-tautology.

In this case, fix closed terms $S : \sigma$ and $T : 0 \to \tau$. Since we know that the join of a tautology with a non-tautology is still a tautology, we must modify $\sigma \underline{+} \tau$ from the previous cases so that it is indeed a tautology: $\sigma \underline{+} \tau := (0 \to 0 \to 0) \times \sigma \times (0 \to \tau)$. The inclusions are given by

$$\begin{aligned}\mathsf{I}_1 &:= \lambda x^\sigma.\mathsf{P}^3(\lambda u^0 v^0.u)xT\\ \mathsf{I}_2 &:= \lambda y^\tau.\mathsf{P}^3(\lambda u^0 v^0.v)S(\lambda w^0.y).\end{aligned}$$

We see that C must be defined and work in a similar manner as before, the only difficulty being that the added abstraction λw^0 prevents us from fully accessing the y. Thus, we only need to show that we can build a term of type 0 from bound variables to get past this small nuisance.

Let $\rho = \rho_1 \to \ldots \to \rho_n \to 0$. We now have two sub-cases depending on ρ:

Case a: ρ is a tautology. In this case, there is a ρ_i which is a non-tautology, and so there is a closed term $R : \rho_i \to 0$. We then define C to be

$$\mathsf{C} := \lambda f^{\sigma \to \rho} g^{\tau \to \rho} x^{\sigma \underline{+} \tau} y_1^{\rho_1} \ldots y_n^{\rho_n}.(\mathsf{P}_1^3 x)(f(\mathsf{P}_2^3 x)\bar{y})(g((\mathsf{P}_3^3 x)(Ry_i))\bar{y}).$$

Case b: ρ is a non-tautology. In this case, $\sigma \to \rho$ is then a non-tautology (since σ is a tautology) and so there is a closed term $F : (\sigma \to \rho) \to 0$. We then define C to be

$$\mathsf{C} := \lambda f^{\sigma \to \rho} g^{\tau \to \rho} x^{\sigma \underline{+} \tau} y_1^{\rho_1} \ldots y_n^{\rho_n}.(\mathsf{P}_1^3 x)(f(\mathsf{P}_2^3 x)\bar{y})(g((\mathsf{P}_3^3 x)(Ff))\bar{y}).$$

$\square$

References

[1] Grzegorczyk, A. *Recursive Functionals in All Finite Types.* Fund. Math., 54 (1964), 73-93.