Summary

Included here are some new results extending our work done in [1] on boolean combinatory clones. We show that the structure of Post's Lattice is replicated in the case of combinatory clones of self-dual, false- and true-preserving, and dual-preserving elements.

Completeness of the Flat Self-dual Elements

Recall that the **dualizer** $d_\sigma : \sigma \to \sigma$ is defined inductively:

$$\begin{aligned} d_0 &:= \neg \\ d_{\sigma\to\tau} &:= d_\tau \circ - \circ d_\sigma. \end{aligned}$$

We let SD denote the self-dual elements, and $\mathrm{SD}_{\mathrm{flat}}$ the self-dual elements of flat type. We let **Maj** denote the ternary majority function, and more generally $\mathbf{Maj}^k$ the k-ary majority function for odd k.

Claim 0.1. *d_σ is $\mathrm{SD}_{\mathrm{flat}}$-definable for each σ.*

Proof. Induction on σ. □

Let **F** denote the 5-ary connective defined by

$$\mathbf{F}(x_1, x_2, x_3, x_4, x_5) := \begin{cases} x_5, & \text{the arguments split } 4/1 \\ \mathbf{Maj}^5(x_1, x_2, x_3, x_4, x_5), & \text{otherwise.} \end{cases}$$

Claim 0.2. **F** *is self-dual.*

We will need to be able to encode certain logical operations using only $\mathrm{SD}_{\mathrm{flat}}$ functions. To do this, we will treat $\lambda x^0.x$ as true and $\lambda x^0.\bot, \lambda x^0.\top$ as false.

Claim 0.3. *There is an $\mathrm{SD}_{\mathrm{flat}}$-definable function* **If** $: (0 \to 0) \to 0 \to 0 \to 0$ *such that for all $a, b \in \mathbf{B}$,*

$$\begin{aligned} \mathbf{If}(\lambda x.x)ab &= a \\ \mathbf{If}(\lambda x.\bot)ab &= b \\ \mathbf{If}(\lambda x.\top)ab &= b. \end{aligned}$$

Proof. Let $\mathbf{If} := \lambda fxy.\mathbf{F}(fx)(f(\neg x))xxy$. We have that

$$\mathbf{If}(\lambda x.x) = \lambda xy.\mathbf{F}x(\neg x)xxy.$$

If y disagrees with x then there is a 3/2 split favoring x, meaning that x is the output. If y agrees with x then there is a 4/1 split, meaning that y, which equals x, is the output.

Finally, we have that for $b \in \mathbf{B}$,

$$\mathbf{If}(\lambda x.b) = \lambda xy.\mathbf{F}bbxxy.$$

If b agrees with x, then if y agrees with both, then y is output. If y disagrees with both, there there is a 4/1 split, in which case y is still output. If b disagrees with x, then there will be a 3/2 split in favor of y regardless, and so y will be output. $\square$

Claim 0.4. *For any σ there is an $\mathrm{SD}_{\mathrm{flat}}$-definable $\mathbf{If}_\sigma : (0 \to 0) \to \sigma \to \sigma \to \sigma$ such that for all $a, b : \sigma$,*

$$\begin{aligned}
\mathbf{If}_\sigma(\lambda x.x)ab &= a \\
\mathbf{If}_\sigma(\lambda x.\bot)ab &= b \\
\mathbf{If}_\sigma(\lambda x.\top)ab &= b.
\end{aligned}$$

Proof. Induction on σ. For 0, let $\mathbf{If}_0 := \mathbf{If}$. For $\sigma \to \tau$, let

$$\mathbf{If}_{\sigma\to\tau} := \lambda f^{0\to 0} g_1^{\sigma\to\tau} g_2^{\sigma\to\tau} x^\sigma.\mathbf{If}_\tau f(g_1 x)(g_2 x).$$

$\square$

Claim 0.5. *There is an $\mathrm{SD}_{\mathrm{flat}}$-definable function $\sqcap : (0 \to 0) \to (0 \to 0) \to 0 \to 0$ such that for all $f \in \mathbf{B}_{0\to 0}$ and $b \in \mathbf{B}$,*

$$\begin{aligned}
\sqcap(\lambda x.x)f &= f \\
\sqcap(\lambda x.b)f &= \lambda x.b.
\end{aligned}$$

Proof. Take $\sqcap := \lambda f^{0\to 0} g^{0\to 0}.\mathbf{If}_{0\to 0} fgf$. $\square$

Note that $\sqcap$ behaves like a conjunction with respect to our interpretation of $\mathbf{B}_{0\to 0}$ as truth values.

Claim 0.6. *For every σ there is some $\mathrm{SD}_{\mathrm{flat}}$-definable $\mathbf{Eq}_\sigma : \sigma \to \sigma \to 0 \to 0$ such that for all $s, s' : \sigma$,*

$$\mathbf{Eq}_\sigma ss' = \begin{cases} \lambda x.x & s = s' \\ \lambda x.\bot \textit{ or } \lambda x.\top & \textit{otherwise.} \end{cases}$$

Proof. Induction on σ. We can take $\mathbf{Eq}_0 := \lambda xy.\mathbf{Maj}x(\neg y)$. Let $\mathbf{B}_\sigma = \{s_1, \ldots, s_N\}$. Since the flat self-dual functions are functionally complete with the addition of a single boolean, there are $\mathrm{SD}_{\mathrm{flat}}$-definable $T_{s_1}, \ldots, T_{s_N} : 0 \to \sigma$ such that $T_{s_i}\bot = s_i$ for all i. Since each T_{s_i} is self-dual, we have that $T_{s_i}\top =$

$d_\sigma s_i$ as well.

We then take

$$\mathbf{Eq}_{\sigma\to\tau} := \lambda f^{\sigma\to\tau} g^{\sigma\to\tau} x^0 . \Big(\prod_{i=1}^{N} \mathbf{Eq}_\tau (f(T_{s_i} x))(g(T_{s_i} x)) \Big) x.$$

$\square$

Claim 0.7. *For each σ there is an $\mathrm{SD}_{\mathrm{flat}}$-definable* $\mathbf{isSD} : \sigma \to 0 \to 0$ *such that for all $s : \sigma$,*

$$\mathbf{isSD}_\sigma s = \begin{cases} \lambda x.x & s \in \mathrm{SD}_\sigma \\ \lambda x.\bot \textit{ or } \lambda x.\top & \textit{otherwise.} \end{cases}$$

Proof. Take $\mathbf{isSD}_\sigma := \lambda s^\sigma . \mathbf{Eq}_\sigma s(d_\sigma s)$. $\square$

Claim 0.8. *All* SD *elements are* $\mathrm{SD}_{\mathrm{flat}}$*-definable.*

Proof. Induction on the type. Of course there are no self-dual elements of type 0. Suppose the type in question is $\sigma \to \tau$. If $\sigma \to \tau$ is a non-tautology, there are no self-dual elements, so assume that $\sigma \to \tau$ is a tautology. We now consider two possibilities:

Case (i) σ is a non-tautology.

Since σ is a non-tautology, fix some lambda-definable $S : \sigma \to 0$. There are no self-dual elements of type σ, so suppose that $\mathbf{B}_\sigma = \{s_1, \dots, s_N, d_\sigma s_1, \dots, d_\sigma s_N\}$ without repetition. Furthermore, suppose that for each i that $\mathbf{isSD}_\sigma s_i = \lambda x.\bot$ (for each pair of dual elements, one will output $\lambda x.\bot$ and one will output $\lambda x.\top$, so this is justified).

Let $f : \sigma \to \tau$ be an arbitrary self-dual function. We may assume that f has the following form, as given by a table:

x	fx
s_1	t_1
$\vdots$	$\vdots$
s_N	t_N
$d_\sigma s_1$	$d_\tau t_1$
$\vdots$	$\vdots$
$d_\sigma s_N$	$d_\tau t_N$.

As before, fix $\mathrm{SD}_{\mathrm{flat}}$-definable $T_{s_1}, \dots, T_{s_N} : 0 \to \sigma$ and $T_{t_1}, \dots, T_{t_N} : 0 \to \tau$ such that $T_{s_i}\bot = s_i$ for each i and $T_{t_j}\bot = t_j$ for each j.

We now define f as follows:

$$
\begin{aligned}
f = \lambda s^{\sigma}.&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s(T_{s_1}(\mathbf{isSD}_{\sigma} s(Ss)))(T_{t_1}(\mathbf{isSD}_{\sigma} s(Ss)))\\
&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s(T_{s_2}(\mathbf{isSD}_{\sigma} s(Ss)))(T_{t_2}(\mathbf{isSD}_{\sigma} s(Ss)))\\
&\vdots\\
&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s(T_{s_{N-1}}(\mathbf{isSD}_{\sigma} s(Ss)))(T_{t_{N-1}}(\mathbf{isSD}_{\sigma} s(Ss)))\ \ (T_{t_N}(\mathbf{isSD}_{\sigma} s(Ss))).
\end{aligned}
$$

On input s_i, we have that

$$
\begin{aligned}
&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s_i(T_{s_j}(\mathbf{isSD}_{\sigma} s_i(Ss_i))))(T_{t_j}(\mathbf{isSD}_{\sigma} s_i(Ss_i)))M\\
=&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s_i(T_{s_j}((\lambda x.\bot)(Ss_i))))(T_{t_j}((\lambda x.\bot)(Ss_i)))M\\
=&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s_i(T_{s_j}\bot))(T_{t_j}\bot)M\\
=&\mathbf{If}_{\tau}(\mathbf{Eq}_{\sigma} s_i s_j)t_j M\\
=&\begin{cases} t_j & s_i = s_j \\ M & \text{otherwise.} \end{cases}
\end{aligned}
$$

Thus, our construction behaves correctly on inputs s_i. Since the construction defined is self-dual, we have that it behaves correctly on inputs $d_{\sigma} s_i$ as well, so the construction indeed defines f.

Case (ii) σ is a tautology.

Since $\sigma \to \tau$ is a tautology, τ is a tautology as well. We can assume that $\mathbf{B}_{\sigma} = \{s_1, \ldots, s_N, u_1, \ldots, u_M, d_{\sigma} u_1, \ldots, d_{\sigma} u_M\}$ with the s_i's self-dual and everything else non-self-dual.

Let $f : \sigma \to \tau$ be an arbitrary self-dual function, with f given by the table

x	fx
s_1	t_1
$\vdots$	$\vdots$
s_N	t_N
u_1	v_1
$\vdots$	$\vdots$
u_M	v_M
$d_{\sigma} u_1$	$d_{\tau} v_1$
$\vdots$	$\vdots$
$d_{\sigma} u_M$	$d_{\tau} v_M$.

By self-duality, each t_i is self-dual as well, and so by inductive hypothesis there are $\mathrm{SD}_{\mathrm{flat}}$ definitions for each s_i and t_i.

The flat self-duals along with the constant function $\lambda x.\bot$ generate all elements of every type which is a tautology. Thus, for every ρ a tautology and $r : \rho$ there is a $\mathrm{SD}_{\mathrm{flat}}$-definable $T_r : (0 \to 0) \to \rho$ such that $T_r(\lambda x.\bot) = r$. Thus, fix such terms $T_{u_1}, \ldots, T_{u_M} : (0 \to 0) \to \sigma$ and $T_{v_1}, \ldots, T_{v_M} : (0 \to 0) \to \tau$.
The construction used to define f works much like before:

$$
\begin{aligned}
f = \lambda s^\sigma.&\mathbf{If}_\tau(\mathbf{Eq}_\sigma s s_1)t_1 \\
&\mathbf{If}_\tau(\mathbf{Eq}_\sigma s s_2)t_2 \\
&\vdots \\
&\mathbf{If}_\tau(\mathbf{Eq}_\sigma s s_N)t_N \\
&\mathbf{If}_\tau(\mathbf{Eq}_\sigma s(T_{u_1}(\mathbf{isSD}_\sigma s)))(T_{v_1}(\mathbf{isSD}_\sigma s)) \\
&\mathbf{If}_\tau(\mathbf{Eq}_\sigma s(T_{u_2}(\mathbf{isSD}_\sigma s)))(T_{v_2}(\mathbf{isSD}_\sigma s)) \\
&\vdots \\
&\mathbf{If}_\tau(\mathbf{Eq}_\sigma s(T_{u_{M-1}}(\mathbf{isSD}_\sigma s)))(T_{v_{M-1}}(\mathbf{isSD}_\sigma s)) \;\; (T_{v_M}(\mathbf{isSD}_\sigma s)).
\end{aligned}
$$

□

Completeness of the Flat False- and True-Preservers

Let $\mathcal{G}(\mathbf{B})$ denote the full combinatory clone over $\mathbf{B}$ and $\mathcal{G}^{\mathrm{taut}}(\mathbf{B})$ the combinatory clone of those elements whose type is a tautology. More generally, let $\mathcal{G}^{\mathrm{taut}} := \mathcal{G} \cap \mathcal{G}^{\mathrm{taut}}(\mathbf{B})$.

Let FP (TP) denote the false- (true-) preserving elements and $\mathrm{FP}_{\mathrm{flat}}$ ($\mathrm{TP}_{\mathrm{flat}}$) the flat false- (true-) preservers. Furthermore, let $\mathrm{FTP} := \mathrm{FP} \cap \mathrm{TP}$ and $\mathrm{FTP}_{\mathrm{flat}} := \mathrm{FP}_{\mathrm{flat}} \cap \mathrm{TP}_{\mathrm{flat}}$.

Let $\mathbf{If} : 0 \to 0 \to 0 \to 0$ denote the standard if-then-else boolean operation. Post shows that $\{\mathbf{If}, \bot\}$, $\{\mathbf{If}, \top\}$, $\{\mathbf{If}\}$ serve as bases for $\mathrm{FP}_{\mathrm{flat}}$, $\mathrm{TP}_{\mathrm{flat}}$, and $\mathrm{FTP}_{\mathrm{flat}}$, respectively.

We recall the following results: FP, $\mathcal{G}^{\mathrm{taut}}(\mathbf{B})$, and TP are covered by $\mathcal{G}(\mathbf{B})$; $\mathrm{FP}^{\mathrm{taut}}$ is covered by FP and $\mathcal{G}^{\mathrm{taut}}(\mathbf{B})$; $\mathrm{TP}^{\mathrm{taut}}$ is covered by TP and $\mathcal{G}^{\mathrm{taut}}(\mathbf{B})$; and all six of the aforementioned combinatory clones are generated by flat elements (that is to say, they all correspond to clones from Post's Lattice). Our goal shall be to show that FTP is covered by $\mathrm{FP}^{\mathrm{taut}}$ and $\mathrm{TP}^{\mathrm{taut}}$, and that it is generated by $\mathrm{FTP}_{\mathrm{flat}}$.

Lemma 0.9. $\mathrm{FTP} \subseteq \mathrm{FP}^{\mathrm{taut}}$.

Proof. Let $s : \sigma$ be false-preserving and truth-preserving. It suffices to check that σ is a tautology. Suppose otherwise. Then there is some lambda-definable $S : \sigma \to 0$, and so $Ss : 0$ is both false-preserving and truth-preserving. Of course, this is not possible. $\square$

Before continuing, let us prove a quick lemma about true-preserving functions:

Lemma 0.10. *Let $f : \sigma_1 \to \ldots \to \sigma_N \to 0$. Then f is true-preserving if and only if for all $s_1 \in \mathrm{TP}_{\sigma_1}, \ldots, s_N \in \mathrm{TP}_{\sigma_N}$, $fs_1 \ldots s_N = \top$.*

Proof. Left to right is trivial. Right to left is proven by induction on N. If $N = 0$, the statement is trivial. Suppose then that for all $s_1, s_2, \ldots, s_N \in \mathrm{TP}$, $fs_1 s_2 \ldots s_N = \top$. Thus, fs_1 has the same property, so by inductive hypothesis, $fs_1 \in \mathrm{TP}$. Since $s_1 \in \mathrm{TP}$ was arbitrary, $f \in \mathrm{TP}$. $\square$

Lemma 0.11. FTP *is covered by* $\mathrm{FP}^{\mathrm{taut}}$.

Proof. Let $f : \sigma_1 \to \ldots \to \sigma_N \to 0 \in \mathrm{FP}^{\mathrm{taut}} \setminus \mathrm{FTP}$. Since $\{\mathbf{If}, \lambda x^0.\bot\}$ is a basis for $\mathrm{FP}^{\mathrm{taut}}$, it suffices to show that $\lambda x^0.\bot$ is definable from f and FTP. Since f is false-preserving and not in FTP, f cannot be true-preserving. Thus, by the previous lemma, there are TP elements $s_1, \ldots, s_N$ such that $fs_1 \ldots s_N = \bot$. Since each s_i is true-preserving, and $\{\mathbf{If}, \top\}$ is a basis for TP, there are lambda-definable elements S_i for each i such that $S_i \mathbf{If} \top = s_i$. We now consider the expression

$$T := \lambda x^0. f(S_1 \mathbf{If} x) \ldots (S_N \mathbf{If} x).$$

Since T is made from false-preserving elements, $T\bot = \bot$. We also have that

$$\begin{aligned} T\top &= f(S_1 \mathbf{If} \top) \ldots (S_N \mathbf{If} \top) \\ &= fs_1 \ldots s_N \\ &= \bot. \end{aligned}$$

Thus, $T = \lambda x.\bot$. $\square$

Lemma 0.12. FTP *is generated by* **If***.*

Proof. Let $f : \sigma_1 \to \ldots \to \sigma_N \to 0$ be both false- and true-preserving. There are then lambda definable M and N such that $f = M\mathbf{If}\bot$ and $f = N\mathbf{If}\top$. Since the type of f is a tautology, there is some σ_i which is a non-tautology, so pick some lambda definable $S : \sigma_i \to 0$. Consider the expression

$$T := \lambda x_1^{\sigma_1} \ldots x_N^{\sigma_N}. \mathbf{If}(Sx_i)(N\mathbf{If}(Sx_i)\bar{x})(M\mathbf{If}(Sx_i)\bar{x}).$$

For arbitrary $s_1, \ldots, s_N$ we then have that

$$
\begin{aligned}
Ts_1 \dots s_N &= \mathbf{If}(Ss_i)(N\mathbf{If}(Ss_i)\bar{s})(M\mathbf{If}(Ss_i)\bar{s}) \\
&= \begin{cases} M\mathbf{If}\bot\bar{s} & Ss_i = \bot \\ N\mathbf{If}\top\bar{s} & Ss_i = \top \end{cases} \\
&= \begin{cases} f\bar{s} & Ss_i = \bot \\ f\bar{s} & Ss_i = \top \end{cases} \\
&= f\bar{s}.
\end{aligned}
$$

Thus, $T = f$.

□

Completeness of the Flat Dual Preservers

Sitting directly below the self-duals and the false- and true-preservers in Post's lattice is their meet, the functions which are self-dual, false-preserving, and true-preserving, denoted by DP. Post gives them the basis $\{\mathbf{Maj}, \oplus_3\}$, where $\oplus_3$ denotes the ternary binary addition operation.

We will show that these results carry over in the context of combinatory clones as well: Take DP := FTP ∩ SD. Then DP is covered by FTP and SD, and DP is generated by its flat elements.

Lemma 0.13. *Let $s \in \mathrm{FP}_\sigma$. Then $d_\sigma s \in \mathrm{TP}_\sigma$ and vice-versa.*

Proof. Induction on type. For 0, the statement is obvious. Suppose $f : \sigma \to \tau$ is false-preserving. Let $s \in \mathrm{TP}_\sigma$ be arbitrary. We then have that $(d_{\sigma\to\tau} f)s = d_\tau(f(d_\sigma s))$. By inductive hypothesis, $d_\sigma s \in \mathrm{FP}$. Since $f \in \mathrm{FP}$, $f(d_\sigma s) \in \mathrm{FP}$. Finally, by inductive hypothesis again, $d_\tau(f(d_\sigma s)) \in \mathrm{TP}$. Thus, $d_{\sigma\to\tau} f \in \mathrm{TP}$.

The proof works similarly when f is assumed true-preserving.

□

Lemma 0.14. SD *covers* DP.

Proof. Let $f : \sigma_1 \to \dots \to \sigma_N \to 0 \in \mathrm{SD} \setminus \mathrm{DP}$ be arbitrary. Since $\{\mathbf{Maj}, \neg\}$ is a basis for SD and $\mathbf{Maj} \in \mathrm{DP}$, it suffices to show that $\neg$ can be defined from f and DP. Since f is self-dual, f must be either non-FP or non-TP. Suppose then that it is non-FP. There are then $s_1, \dots, s_N \in \mathrm{FP}$ such that $fs_1 \dots s_N = \top$.

For each i let $T_{s_i} : 0 \to \sigma_i$ be the function such that $T_{s_i}\bot = s_i$ and $T_{s_i}\top = d_{\sigma_i} s_i$. Observe that each such T_{s_i} is self-dual, false-preserving, and (as a consequence of the previous lemma) true-preserving. Thus, we are free to use them. Consider the following expression:

$$T := \lambda x^0. f(T_{s_1} x) \dots (T_{s_N} x).$$

We then have that

$$\begin{aligned} T\bot &= f(T_{s_1}\bot) \dots (T_{s_N}\bot) \\ &= f s_1 \dots s_N \\ &= \top. \end{aligned}$$

By self-duality, we then have that $T\top = \bot$. Thus, $T = \neg$.

The case where f is non-TP works in much the same way.

$\square$

Lemma 0.15. *The set* $\{\mathbf{Maj}, \oplus_3, \bot, \top\}$ *is functionally complete.*

Proof. Note that $\wedge = \mathbf{Maj}\bot$ and $\neg = \oplus_3\bot\top$. $\square$

Lemma 0.16. $\{\vee, \oplus_3\}$ *and* $\{\wedge, \oplus_3\}$ *are interdefinable.*

Proof. Note that $x \vee y = \oplus_3 xy(x \wedge y)$ and $x \wedge y = \oplus_3 xy(x \vee y)$. $\square$

Lemma 0.17. $\{\vee, \oplus_3, \mathbf{Maj}\}$ *and* $\{\wedge, \oplus_3, \mathbf{Maj}\}$ *are both bases for* FTP*.*

Proof. Since each of $\vee, \wedge$ can be recovered from the other by the previous lemma, it suffices to check that $\{\vee, \wedge, \oplus_3, \mathbf{Maj}\}$ is a basis for FTP. It has already been established that $\{\mathbf{If}\}$ is a basis for FTP, so we can merely give a definition of $\mathbf{If}$ in terms of those four functions:

$$\mathbf{If} xyz = \mathbf{Maj} yz((x \wedge y) \vee ((\oplus_3 xyz) \wedge z)).$$

The idea behind this formula: if y, z agree, then x doesn't matter; this is reflected in the first two arguments of $\mathbf{Maj}$. If they disagree, then the third argument breaks the tie. We see then that $\oplus_3 xyz = \neg x$, and so $(x \wedge y) \vee (\neg x \wedge z)$ is the output. $\square$

Lemma 0.18. *Let* $f : \sigma_1 \to \dots \to \sigma_N \to 0$*. Then*

$$d_{\sigma_1 \to \dots \to \sigma_N \to 0} f = \lambda x_1^{\sigma_1} \dots x_N^{\sigma_N}. \neg(f(d_{\sigma_1} x_1) \dots (d_{\sigma_N} x_N)).$$

Proof. Induction on N. For $N = 0$ it is obvious. For $N > 0$ we then have that

$$\begin{aligned} d_{\sigma_1 \to \dots \to \sigma_N \to 0} f &= \lambda x_1^{\sigma_1}. d_{\sigma_2 \to \dots \to \sigma_N \to 0}(f(d_{\sigma_1} x_1)) \\ &= \lambda x_1^{\sigma_1}. \lambda x_2^{\sigma_2} \dots x_N^{\sigma_N}. \neg((f(d_{\sigma_1} x_1))(d_{\sigma_2} x_2) \dots (d_{\sigma_N} x_N)) \\ &= \lambda x_1^{\sigma_1} \dots x_N^{\sigma_N}. \neg(f(d_{\sigma_1} x_1) \dots (d_{\sigma_N} x_N)). \end{aligned}$$

$\square$

Lemma 0.19. FTP *covers* DP.

Proof. Let $f : \sigma_1 \to \ldots \to \sigma_N \to 0 \in \mathrm{FTP} \setminus \mathrm{DP}$ be arbitrary. In light of lemma 0.5, and since $\oplus_3, \mathbf{Maj} \in \mathrm{DP}$, it suffices to show that one of $\wedge$ or $\vee$ are definable from f and DP. Since f is false- and true-preserving, f is not self-dual. By the previous lemma, there are then elements $s_1, \ldots, s_N$ such that $fs_1 \ldots s_N \neq \neg(f(d_{\sigma_1}s_1)\ldots(d_{\sigma_N}s_N))$. Thus, $fs_1 \ldots s_N = f(d_{\sigma_1}s_1)\ldots(d_{\sigma_N}s_N) = b$ for some $b \in \mathbf{B}$.

By lemma 0.3, for every $s : \sigma$, there is a DP-definable term $T_s : 0 \to 0 \to \sigma$ such that $T_s \bot \top = s$ (each **Maj** and $\oplus_3$ in a given definition of s are fine, since they are in DP, but each occurrence of $\bot$, $\top$ are abstracted out as arguments). By duality, we also have that $T_{s_i} \top \bot = d_\sigma s$.

Consider then the expression

$$T := \lambda x^0 y^0 . f(T_{s_1}xy)\ldots(T_{s_N}xy).$$

T is false- and true-preserving, so $T\bot\bot = \bot$ and $T\top\top = \top$. We also have that

$$\begin{aligned} T\bot\top &= f(T_{s_1}\bot\top)\ldots(T_{s_N}\bot\top) \\ &= fs_1 \ldots s_N \\ &= b \end{aligned}$$

and that

$$\begin{aligned} T\top\bot &= f(T_{s_1}\top\bot)\ldots(T_{s_N}\top\bot) \\ &= f(d_{\sigma_1}s_1)\ldots(d_{\sigma_N}s_N) \\ &= b. \end{aligned}$$

Thus, $T = \wedge$ if $b = \bot$, and $T = \vee$ if $b = \top$.

□

Lemma 0.20. DP *is generated by its flat elements.*

Proof. Let $f : \sigma_1 \to \ldots \to \sigma_N \to 0 \in \mathrm{DP}$. The type of f must be a tautology, so there is some non-tautology σ_i and lambda-definable $S : \sigma_i \to 0$.

Since f is false- and true-preserving, there is, by lemma 0.5, some lambda-definable M such that $f = M\mathbf{Maj} \oplus_3 \wedge$. Since $\mathbf{Maj}, \oplus_3 \in \mathrm{DP}$, we only need to worry about removing the $\wedge$. Consider the expression

$$T := \lambda x_1^{\sigma_1} \ldots x_N^{\sigma_N} . M\mathbf{Maj} \oplus_3 (\mathbf{Maj}(Sx_i))\bar{x}.$$

□

Let $s_1, \ldots, s_N$ be arbitrary. If $Ss_i = \bot$, we have that

$$\begin{aligned} T\bar{s} &= M\mathbf{Maj} \oplus_3 (\mathbf{Maj}(Ss_i))\bar{s} \\ &= M\mathbf{Maj} \oplus_3 (\mathbf{Maj}\bot)\bar{s} \\ &= M\mathbf{Maj} \oplus_3 \wedge\bar{s} \\ &= f\bar{s}. \end{aligned}$$

If $Ss_i = \top$, we exploit self-duality of both T and f:

$$\begin{aligned} T\bar{s} &= \neg(T(d\bar{s})) \\ &= \neg\,(M\mathbf{Maj} \oplus_3 (\mathbf{Maj}(S(d_{\sigma_i}s_i)))d\bar{s}) \\ &= \neg\,(M\mathbf{Maj} \oplus_3 (\mathbf{Maj}(\neg(Ss_i)))d\bar{s}) \\ &= \neg\,(M\mathbf{Maj} \oplus_3 (\mathbf{Maj}(\neg\top))d\bar{s}) \\ &= \neg\,(M\mathbf{Maj} \oplus_3 (\mathbf{Maj}\bot)d\bar{s}) \\ &= \neg\,(M\mathbf{Maj} \oplus_3 \wedge d\bar{s}) \\ &= \neg\,(fd\bar{s}) \\ &= f\bar{s}. \end{aligned}$$

Thus, $f = T$.

References

[1] Marzion, Evan. *Closed Sets of Higher-Order Functions.* MoL Thesis Series, 2016.

[2] Post, E.L. *The Two-Valued Iterative Systems of Mathematical Logic*, Annals of Mathematics Studies 5, 1941.