Elías Masquil

Image Denoising Part 2 - Homework 4

Ex 3.1

$$F^{\lambda}(v)_x = \lambda F(v)_x + (1-\lambda) v_x$$

- $\mathbb{E}_v\{v|u\} = u$
- $v_x|u \perp\!\!\!\perp v_y|u$ for $x \neq y$

1) Show that

$$\mathbb{E}_{v,u}\{\|F^{\lambda}(v) - u\|^2\} = \lambda^2 \mathbb{E}_{v,u}\{\|F(v)-u\|^2\} + (1-\lambda)^2 \mathbb{E}_{v,u}\{\|v-u\|^2\} + 2\lambda(1-\lambda)\mathbb{E}_{v,u}\{\langle F(v)-u, v-u\rangle\}$$

Proof

$$\mathbb{E}_{v,u}\{\|F^{\lambda}(v)-u\|^2\} = \mathbb{E}_{v,u}\{\|\lambda F(v) + (1-\lambda)v - u\|^2\}$$

$$= \mathbb{E}_{v,u}\{\|\lambda F(v) - \lambda u + (1-\lambda)v - u + \lambda u\|^2\}$$

$$= \mathbb{E}_{v,u}\{\|\lambda(F(v)-u) + (1-\lambda)(v-u)\|^2\} = \lambda^2 \mathbb{E}_{v,u}\{\|F(v)-u\|^2\} + (1-\lambda)^2 \mathbb{E}_{v,u}\{\|v-u\|^2\} + 2\lambda(1-\lambda)\mathbb{E}_{v,u}\{\langle F(v)-u, v-u\rangle\}$$

2) Show that $\underbrace{\mathbb{E}_{v,u}\{\langle F(v)-u, v-u\rangle\}}_{*} = 0$

Proof

$$* = \mathbb{E}_u \mathbb{E}_v\{\langle F(v)-u, v-u\rangle | u\} = \mathbb{E}_u \mathbb{E}_v\{\langle F(v), v\rangle - \langle F(v), u\rangle - \langle u, v\rangle + \langle u,u\rangle | u\}$$

$$= \mathbb{E}_u \mathbb{E}_v\{\langle F(v), v\rangle | u\} - \mathbb{E}_u \mathbb{E}_v\{\langle F(v), u\rangle | u\} \underbrace{\underbrace{- \mathbb{E}_u \mathbb{E}_v\{\langle u, v\rangle | u\}}_{- \mathbb{E}_u\{\langle u, \mathbb{E}_v\{v|u\}\rangle\}}}_{\cancel{- \mathbb{E}_u\{\langle u, u\rangle\}}} + \underbrace{\mathbb{E}_u \mathbb{E}_v\{\langle u,u\rangle | u\}}_{\cancel{+ \mathbb{E}_u\{\langle u,u\rangle\}}}$$

$$* = \mathbb{E}_u \mathbb{E}_v\{\langle F(v), v-u\rangle | u\} = \mathbb{E}_u \mathbb{E}_v\{\sum_j F(v)_j (v_j - u_j) | u\}$$

Since F is J-invariant: $F(v)_j \perp\!\!\!\perp v_j$

$$* = \sum_j \mathbb{E}_u \mathbb{E}_v\{F(v)_j | u\} \underbrace{\mathbb{E}_u \mathbb{E}_v\{(v_j - u_j) | u\}}_{0} = 0$$

3) λ^* that minimizes MSE

$$\frac{\partial MSE}{\partial \lambda} = 2\lambda \, \mathbb{E}_{v,u}\{\|F(v)-u\|^2\} - 2(1-\lambda)\, \mathbb{E}_{v,u}\{\|v-u\|^2\} = 0$$

$$\lambda\left(\mathbb{E}_{v,u}\{\|F(v)-u\|^2\} + \mathbb{E}_{v,u}\{\|v-u\|^2\}\right) = \mathbb{E}_{v,u}\{\|v-u\|^2\}$$

$$\Rightarrow \lambda^* = \frac{\mathbb{E}_{v,u}\{\|v-u\|^2\}}{\mathbb{E}_{v,u}\{\|F(v)-u\|^2\} + \mathbb{E}_{v,u}\{\|v-u\|^2\}}$$

$$\lambda^* = \frac{\mathbb{E}_u\{\mathbb{V}(v|u)\}}{\mathbb{E}_{v,u}\{\|F(v)-u\|^2\} + \mathbb{E}_u\{\mathbb{V}(v|u)\}}$$

4) $R_{N2S} = R_{N2C} + \mathbb{E}_u\{\mathbb{V}(v|u)\}$

$$\Rightarrow R_{N2C} = R_{N2S} - \mathbb{E}_u\{\mathbb{V}(v|u)\}$$

$$\boxed{\lambda^* = \frac{d\sigma^2}{R_{N2S}(F)}}$$

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Ex 3.2

$\hat{u}(v)$ estimator of u

$$\mathbb{E}_v\{\|\hat{u}(v)-u\|^2|u\} = \mathbb{E}_v\{\|\mathbb{E}_v\{\hat{u}(v)|u\}-u + \hat{u}(v) - \mathbb{E}_v\{\hat{u}(v)|u\}\|^2|u\}$$

$$= \mathbb{E}_v\{\|\mathbb{E}_v\{\hat{u}(v)|u\}-u\|^2\} + \mathbb{E}_v\{\|\hat{u}(v) - \mathbb{E}_v\{\hat{u}(v)|u\}\|^2|u\}$$

$$+ 2\,\underbrace{\mathbb{E}_v\{[\mathbb{E}_v\{\hat{u}(v)|u\}-u|u][\hat{u}(v) - \mathbb{E}_v\{\hat{u}(v)|u\}]|u\}}_{[\mathbb{E}_v\{\hat{u}(v)|u\}-u][\cancelto{0}{\mathbb{E}_v\{\hat{u}(v)|u\} - \mathbb{E}_v\{\hat{u}(v)|u\}}]}$$

$$\Rightarrow \mathbb{E}_v\{\|\hat{u}(v)-u\|^2|u\} = \|\mathbb{E}_v\{\hat{u}(v)|u\}-u\|^2 + \mathbb{E}_v\{\|\hat{u}(v) - \mathbb{E}_v\{\hat{u}(v)|u\}\|^2|u\}$$