Image Denoising Elías Masquil

Course 1 assignments

Ex 4.1 $X \sim \mathcal{P}(\lambda)$

$$E[X] = \sum_{n=0} \frac{n \lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n \geq 1} \frac{\lambda^n}{(n-1)!} = e^{-\lambda} \lambda \sum_{n \geq 1} \frac{\lambda^{n-1}}{(n-1)!}$$

Taylor series of e^x

$$= e^{-\lambda} \lambda e^{\lambda} = \lambda$$

$$\sigma^2(X) = \sum_{n=0} n^2 \frac{\lambda^n e^{-\lambda}}{n!} - \lambda^2 = e^{-\lambda} \lambda \sum_{n \geq 1} n \frac{\lambda^{n-1}}{(n-1)!} - \lambda^2$$

$$= e^{-\lambda} \lambda \left[\sum_{n \geq 1} \frac{(n-1)\lambda^{n-1}}{(n-1)!} + \sum_{n \geq 1} \frac{\lambda^{n-1}}{(n-1)!} \right] - \lambda^2$$

$$= e^{-\lambda} \lambda \left[\sum_{n \geq 0} \frac{n \lambda^n}{n!} + \sum_{n \geq 1} \frac{\lambda^{n-1}}{(n-1)!} \right] - \lambda^2$$

Previous result

$$= e^{-\lambda} \lambda \left[e^{\lambda} \lambda + e^{\lambda} \right] - \lambda^2$$

$$= \lambda$$

Ex 4.2 $X_1, \dots X_n \overset{iid}{\sim} \mathcal{P}(\lambda)$

$Y = \sum X_i \sim \mathcal{P}(\Sigma \lambda)$

Let's prove that if $\left.\begin{array}{l} X \sim \mathcal{P}(\lambda_x) \\ Y \sim \mathcal{P}(\lambda_y) \end{array}\right\} \Rightarrow X+Y \; \mathcal{P}(\lambda_x + \lambda_y)$

If this holds, then $\sum_n X_n \sim \mathcal{P}(\Sigma \lambda_n)$

$$\left[\underbrace{\left(\underbrace{X_1 + X_2}_{\mathcal{P}} + \cdots \underbrace{X_{n-1} + X_n}_{\mathcal{P}} \right)}_{\mathcal{P}} \right]$$

$$P(X+Y=k) = \sum_{t=0}^{k} P(X=t) P(Y=k-t) = \sum_{t=0}^{k} \frac{\lambda_x^t e^{-\lambda_x}}{t!} \frac{\lambda_y^{k-t} e^{-\lambda_y}}{(k-t)!}$$

$$= \sum_{t=0}^{k} \frac{\lambda_x^t \lambda_y^{k-t} e^{-\lambda_x - \lambda_y}}{t!(k-t)!} = \sum_{t=0}^{k} \frac{\lambda_x^t \lambda_y^{k-t} k! \, e^{-\lambda_x - \lambda_y}}{t!(k-t)! \, k!}$$

$$= \frac{\sum \lambda_x^t \lambda_y^{k-t} \binom{k}{t} e^{-\lambda_x - \lambda_y}}{k!} = \frac{e^{-(\lambda_x + \lambda_y)} (\lambda_x + \lambda_y)^k}{k!}$$

Binomial Identity

Ex 4.3

We have $\tilde{U} \approx U + g(U)\, n \qquad n \sim \mathcal{N}(0,1)$

We want f smooth, such that $f(\tilde{U})$ gets uniform std independent of U

$$f(\tilde{U}) \simeq f(U) + f'(U)(\tilde{U}-U)$$
$$\simeq f(U) + f'(U)\, g(U)\, n$$

we want $\dfrac{f'(U)}{g(U)} = C$; in our case $g(U) = \sqrt{U}$

$$\Rightarrow f(U) = \int_0^U \frac{C\,dt}{\sqrt{t}} = 2C\, t^{1/2}\Big|_0^U = 2C\sqrt{U}$$

Ex 4.5

$$D_{inf} = \arg\min_D \mathbb{E}\{\|U - D\tilde{U}\|^2\}$$

$$\arg\min_D \mathbb{E}\{\|U - \sum_{i=1}^M \alpha(i)\langle \tilde{U}, b_i\rangle b_i\|^2\}$$

$$\arg\min_D \mathbb{E}\{\|\sum_{i=1}^M \langle U, b_i\rangle b_i - \sum_{i=1}^M \alpha(i)\langle \tilde{U}, b_i\rangle b_i\|^2\}$$

$$\arg\min_D \mathbb{E}\{\|\sum_{i=1}^M \langle U, b_i\rangle b_i - \sum_{i=1}^M \alpha(i)\langle U + N, b_i\rangle b_i\|^2\}$$

$$\arg\min_D \mathbb{E}\{\|\sum_i \langle U - \alpha(i)U, b_i\rangle b_i - \sum_i \alpha(i)\langle N, b_i\rangle b_i\|^2\}$$

$$\arg\min_D \mathbb{E}\{\|\sum_i (1-\alpha(i))\langle U, b_i\rangle b_i - \sum_i \alpha(i)\langle N, b_i\rangle b_i\|^2\}$$

The cross terms between U and N are 0 in expectation since N is indep of U and $\mathbb{E}(N)=0$

$$\arg\min_D \mathbb{E}\{\|\sum_i (1-\alpha(i))\langle U, b_i\rangle b_i\|^2\} + \mathbb{E}\{\|\sum_i \alpha(i)\langle N, b_i\rangle b_i\|^2\}$$

U is not random

$$\arg\min_D \left\langle \sum_i (1-\alpha(i))\langle U, b_i\rangle b_i , \sum_j (1-\alpha(j))\langle U, b_j\rangle b_j \right\rangle$$
$$+ \mathbb{E}\left\{ \left\langle \sum_i \alpha(i)\langle N, b_i\rangle b_i , \sum_j \alpha(j)\langle N, b_j\rangle b_j \right\rangle \right\}$$

b_i is an orthonormal basis

$$\arg\min_D \sum_i (1-\alpha(i))^2 \langle U, b_i\rangle^2 + \mathbb{E}\{\sum_i \alpha(i)^2 \langle N, b_i\rangle^2\}$$

Noise remains uncorelated in the new basis

$$\arg\min_D \sum_i (1-\alpha(i))^2 \langle U, b_i\rangle^2 + \sum_i \alpha(i)^2 \sigma^2$$

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Ex 4 5 The objective is convex in $\alpha(i)$, so:

Deriving wrt $\alpha(i)$ and equal to 0

$$-2\,(1-\alpha(i)\,\langle U,b_i\rangle^2 + 2\,\alpha(i)\,\sigma^2 = 0$$

$$\alpha(i)\left[2\sigma^2 + 2\langle U,b_i\rangle^2 = 2\langle U,b_i\rangle^2\right.$$

$$\boxed{\alpha(i) = \frac{\langle U,b_i\rangle^2}{\sigma^2 + \langle U,b_i\rangle^2}}$$

Evaluating the MSE on $\alpha(i)^*$:

$$\sum_i \alpha_i^2\left[\sigma^2 + \langle U,b_i\rangle^2\right] - 2\alpha_i\langle U,b_i\rangle^2 + \langle U,b_i\rangle^2$$

$$\sum_i \frac{\langle U,b_i\rangle^4}{(\sigma^2+\langle U,b_i\rangle^2)^2}\left(\sigma^2 + \cancel{\langle U,b_i\rangle^2}\right) - \frac{2\langle U,b_i\rangle^4}{\sigma^2+\langle U,b_i\rangle^2} + \langle U,b_i\rangle^2$$

$$\sum_i \frac{\langle U,b_i\rangle^4(-1)}{\sigma^2+\langle U,b_i\rangle^2} + \langle U,b_i\rangle^2$$

$$\boxed{\sum_i \frac{\langle U,b_i\rangle^2\,\sigma^2}{\sigma^2+\langle U,b_i\rangle^2}} \quad \underline{\text{MSE}}$$

Ex 4 6

Now $\alpha(i)^* = \begin{cases} 1 & \text{if } |\langle U,b_i\rangle|^2 \geq c\sigma^2 \\ 0 & \text{otherwise}\end{cases}$ for some $c > 1$

Let's define $S = \{ i \,/\, \langle U,b_i\rangle^2 \geq c\sigma \}$

$$\Rightarrow \text{MSE} = \sum_{i\in S}\sigma^2 + \sum_{i\notin S}\langle U,b_i\rangle^2 \leq \sum_{i\in S} c\sigma^2 + \sum_{i\notin S}\langle U,b_i\rangle^2$$

Using the formula from 4.5

for $c \geq 1$

Then $\sum_{i\in S} c\sigma^2 + \sum_{i\notin S}\langle U,b_i\rangle^2$ is $\sum_i \min\left(\langle U,b_i\rangle^2, c\sigma^2\right)$

So, if using $\alpha^*(i) = \begin{cases} 1 & \text{if } |\langle U,b_i\rangle^2| \geq c\sigma \\ 0 & \text{otherwise}\end{cases}$

$\text{MSE} \leq \sum_i \min(\quad)$ and the equality holds when $c = 1$

Ex 4.7 DCT

$X, Y \in \mathbb{R}^N$

$$f(X)_K = \alpha_K 2 \sum_{j=0}^{N-1} X_j \cos\left(\pi \left(j+\frac{1}{2}\right)\frac{K}{N}\right) = \langle X, V_K \rangle$$

$$f(Y)_K = \alpha_K 2 \sum_{j=0}^{N-1} Y_j \cos\left(\pi \left(j+\frac{1}{2}\right)\frac{K}{N}\right) = \langle Y, V_K \rangle$$

Let's call

$$V_K = 2\alpha_K \begin{pmatrix} \cos\left[\pi(0+1/2)\frac{K}{N}\right] \\ \vdots \\ \cos\left[\pi(N-1/2)\frac{K}{N}\right] \end{pmatrix}$$

We'll prove that the V_K form an orthogonal base of $\mathbb{R}^n$ (so DCT is an isometry)

$$\langle V_K, V_S \rangle = \sum_{j=0}^{N-1} 4\alpha_K \alpha_S \cos\left[\pi(j+1/2)\frac{K}{N}\right] \cos\left[\pi(j+1/2)\frac{S}{N}\right]$$

$$\langle V_K, V_S \rangle = \sum_{j=0}^{N-1} 2\alpha_K\alpha_S \left[\cos\left[\frac{\pi}{N}(j+1/2)(K+S)\right] + \cos\left[\frac{\pi}{N}(j+1/2)(K-S)\right]\right]$$

$$\langle V_K, V_S \rangle = \sum_{j=0}^{N-1} 2\alpha_K\alpha_S \left[\frac{e^{i\left[\frac{\pi}{N}(j+1/2)(K+S)\right]} + e^{i\left[\frac{\pi}{N}(j+1/2)(-K-S)\right]} + e^{i\left[\frac{\pi}{N}(j+1/2)(K-S)\right]} + e^{i\left[\frac{\pi}{N}(j+\frac{1}{2})(-K+S)\right]}}{2}\right]$$

$$\langle V_K, V_S \rangle = \alpha_K\alpha_S \left[e^{\frac{i\pi(K+S)}{2N}} \left(\sum_{j}^{N-1} e^{\frac{i\pi(K+S)j}{N}} + \sum_{j=0}^{N-1} e^{\frac{-i\pi(K+S)(j+1)}{N}} \right) + e^{\frac{i\pi(K-S)}{2N}} \left(\sum_{j=0}^{N-1} e^{\frac{i\pi(K-S)j}{N}} + \sum_{j=0}^{N-1} e^{\frac{-i\pi(K-S)(j+1)}{N}} \right) \right]$$

$$\langle V_K, V_S \rangle = \alpha_K\alpha_S \left[e^{\frac{i\pi(K+S)}{2N}} \underbrace{\sum_{j=-N}^{N-1} e^{\frac{i\pi(K+S)j}{N}}} + e^{\frac{i\pi(K-S)}{2N}} \underbrace{\sum_{j=-N}^{N-1} e^{\frac{+i\pi(K-S)j}{N}}} \right]$$

$$\langle V_K, V_S \rangle = \alpha_K\alpha_S \left[e^{\frac{i\pi(K+S)}{2N}} \left(\frac{e^{-i\pi(K+S)} - e^{i\pi(K+S)}}{1 - e^{\frac{i\pi(K+S)}{N}}} \right) + e^{\frac{i\pi(K-S)}{2N}} \left(\frac{e^{-i\pi(K-S)} - e^{i\pi(K-S)}}{1 - e^{\frac{i\pi(K-S)}{N}}} \right) \right]$$

If $K \neq S$: $\langle V_K, V_S \rangle = 0$ because the substracted exponentials are either all 1 or all -1.

If $K = S = 0$ $\quad \langle V_K, V_S \rangle = \frac{1}{4N} \sum_{j=-N}^{N-1} 2 = \frac{4N}{4N} = 1$

If $K = S \neq 0$

$$\langle V_K, V_S \rangle = \frac{1}{2N} \sum_{j=-N}^{N+1} 1 = \frac{2N}{2N} = 1$$

Ex 4.7 IDCT

The IDCT has the exact same basis as the DCT
It's clear that's also an isometry.

DCT - IDCT

Let's call V to the DCT matrix and Q to the IDCT. Since both matrices are orthogonal

$$V^{-1} = V^T$$

$$Q^{-1} = Q^T$$

$$V_{ij} = 2\alpha_i \cos\left[\pi\left(j+1/2\right)\frac{i}{N}\right]$$

$$Q_{ij} = 2\beta_j \cos\left(\pi\left(i+1/2\right)\frac{j}{N}\right];$$ considering $\beta_j = \begin{cases} \sqrt{1/4N} & j=0 \\ \sqrt{1/2N} & \text{otherwise} \end{cases}$

then $\beta_i = \alpha_i$

Then, $(V^T)_{ij} = V_{ji} = Q_{ij}$

$\Rightarrow V^T = V^{-1} = Q$

and

$Q^T = Q^{-1} = V$

$\Rightarrow$ DCT and IDCT are inverse transformations

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Ex 4.8

$$\arg\min_{\sum \alpha_K = 1} \sum_K \alpha_K^2 \sigma_K^2$$

$$\mathcal{L} = \sum_K \alpha_K^2 \sigma_K^2 + \lambda\left(\sum \alpha_K - 1\right)$$

$$\frac{\partial \mathcal{L}}{\partial \alpha_K} = 2\alpha_K \sigma_K^2 + \lambda = 0 \iff \boxed{\lambda = -2\alpha_K \sigma_K^2} \ \forall K$$

Ex 4.9

The optimal α_K : $\alpha_K = \dfrac{\sigma_K^{-2}}{\sum_K \sigma_K^{-2}}$ $\forall K$

Proove that for wiene filter

$$\alpha_K = \frac{\|P_{PK}\|^{-2}}{\sum_j \|P_{Pj}\|^{-2}}$$

Now for each pixel, we have K^2 estimators P_K

$\Rightarrow$ σ_K^2 is the variance of the K-th estimator for a pixel

Each estimator of the patch is atenuated by $F_K(w)$ (on each freq)

The variance is due to the residual noise

On the DCT domain we have

$$\sigma^2 \underbrace{\begin{pmatrix} P_P(0) \\ P_P(1) \\ \vdots \\ P_P(s-1) \end{pmatrix}}_{P_P(K)}$$

which is noise across all frequencies (atenuated)

By parseval

$\sigma_K^2 = \|\sigma^2 P_P(K)\|^2 \Rightarrow \boxed{\sigma_K^2 = \sigma^2 \|P_{PK}\|^2}$