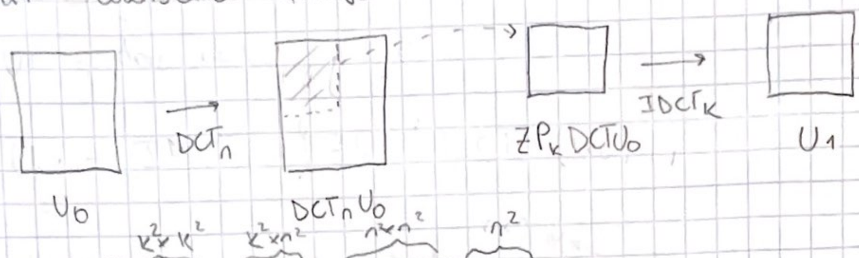
Elías Masquil

Image Denoising - Homework 2

Ex 5.1

n: initial image dimension
k: image dimension at lower scale
u_0: initial image
u_1: downscaled image



$$U_1 = \underbrace{IDCT_k}_{k^2 \times k^2} \underbrace{ZP_k}_{k^2 \times n^2} \underbrace{DCT_n}_{n^2 \times n^2} \underbrace{U_0}_{n^2}$$

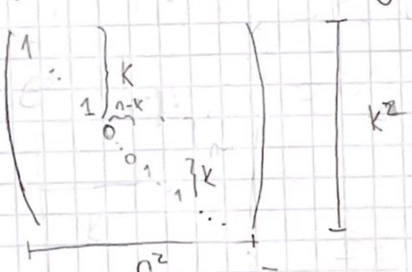
$$U_1 = \bar{U}_1 + \varepsilon' \quad , \quad \varepsilon' \sim N(0, \sigma'^2)$$

$$U_0 = \overline{U_0} + \varepsilon \quad , \quad \varepsilon \sim N(0, \sigma^2)$$

because DCT preseves noise distribution

$$\boxed{Var(U_0) = \sigma^2 I_{n^2} \quad (\text{dimensions } n^2 \times n^2)}$$

ZP_k has the following form



we repeat the 1 k and 0 n-k pattern until we fill the matrix

Note that $ZP_k \, ZP_k^T = I_{k^2}$. Ex: $(n=3, k=2)$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

$$Var(U_1) = DCT_k^T \, ZP_k \, DCT_n \, \sigma^2 I_{n^2} \, DCT_n^T \, ZP_k^T \, DCT_k$$

$$Var(U_1) = \sigma^2 DCT_k^T \, ZP_k \, I_{n^2} \, ZP_k^T DCT_k$$

$$Var(U_1) = \sigma^2 DCT_k^T \, I_{k^2} \, DCT_k$$

$$\boxed{Var(U_1) = \sigma^2 I_{k^2} \to k^2 \times k^2 \text{ dimensions}}$$

⊛ Since DCT is an isometry $DCT_n^T DCT_n = I_n$

Both covariance matrices have the same entries but the original one is bigger. Although this change on the dimensions seem to play a role on the noise STD, I don't see how it gets reduced by $\frac{k}{n}$

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Ex 6.1

$n(i)$ white noise $i \in \mathbb{Z}$

Discrete finite filter $a(i) \geq 0$ supported in $[-m, m]$

$$a * n(i) = \sum_{j=-m}^{m} n(i-j)\, a(j)$$

a) $Var(a * n(i))$?

(assuming white noise 0 mean)

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}[(a*n(i)] = \mathbb{E}\left[\sum_{j=-m}^{m} n(i-j)a(j)\right] = \sum \underbrace{\mathbb{E}[n(i-j)a(j)]}_{0 \ \forall i} = 0$$

$$\mathbb{E}([a*n(i)]^2] = \mathbb{E}\left[\sum_{j=-m}^{m} \sum_{k=-m}^{m} n(i-j)a(j)\, n(i-k)\, a(k)\right]$$

$$= \sum_k \sum_j a(j)a(k)\, \mathbb{E}[n(i-j)\, n(i-k)]$$

Since n is white noise, all samples are independent from each other. Then, $\mathbb{E}(n(i)n(j)) = \mathbb{E}(n(i))\,\mathbb{E}(n(j)) = 0 \cdot 0 = 0$ if $i \neq j$

$$\mathbb{E}\left[(a*n)(i)^2\right] = \sum_{t=-m}^{m} a^2(t)\, \mathbb{E}\left[n(t-m)^2\right] = \boxed{\sum_{-m}^{m} a^2(t)\, \sigma^2}$$

b) $\sum_i a(i) = 1$ and $a(i) \geq 0 \quad \forall i$

Then $a(i)^2 \leq a(i) \quad \forall i$

$$\sum a(i)^2 \leq \sum a(i) = 1$$

$$\Rightarrow Var[(a*n(i))] = \sigma^2 \sum a(i)^2 \leq \sigma^2$$

c) $\min \sum a_i^2$

st $\sum a_i - 1 = 0$

$$\mathcal{L}(a_i, \mu) = \sum a_i^2 + \mu\left(\sum a_i - 1\right)$$

$$\frac{\partial \mathcal{L}}{\partial a_i} = 2a_i + \mu = 0 \Rightarrow \mu = -2a_i$$

$$\sum a_i = 1 \iff -\frac{\sum \mu}{2} = 1 \Rightarrow \mu = \frac{-2}{2m+1}$$

$$\boxed{a_i = \frac{1}{2m+1}}$$

$a_i \geq 0$ ✓