Elías Masquil

Image Denoising - Homework 3

Ex 8.1

From $\underset{\substack{\uparrow \\ \text{noisy} \\ \text{image}}}{\tilde{u}} = \underset{\substack{\uparrow \\ \text{noiseless} \\ \text{image}}}{u} + \underset{\substack{\nwarrow \\ \text{Gaussian} \\ \text{noise}}}{n}$ it follows that if

we restrict to a patch:

$$\tilde{p} = p + n_p$$

$$\mathrm{Cov}(\tilde{p}) = \mathbb{E}\left[\tilde{p}\,\tilde{p}^T\right] - \mathbb{E}[\tilde{p}]\,\mathbb{E}[\tilde{p}]^T$$

$\boxed{\mathbb{E}[\tilde{p}] = \mathbb{E}[p] = \bar{p}}$ since we assumed a gaussian model for the noiseless patch and $\mathbb{E}[n_p] = 0$.

$$\mathbb{E}\left[(p + n_p)(p^T + n_p^T)\right] = \mathbb{E}[pp^T] + 2\,\mathbb{E}[p n_p^T] + \mathbb{E}[n_p n_p^T]$$

$$2\,\mathbb{E}[p n_p^T] \underset{\text{independence}}{=} 2\,\mathbb{E}[p]\,\underbrace{\mathbb{E}[n_p^T]}_{\text{zero-mean noise}} = 0$$

$$\Rightarrow \mathrm{Cov}(\tilde{p}) = \mathbb{E}[pp^T] + \sigma^2 I - \mathbb{E}[p]\,\mathbb{E}[p]^T$$

Then, $\boxed{\mathrm{Cov}(\tilde{p}) = \mathrm{Cov}(p) + \sigma^2 I}$

Ex 8.3

eq 8.8: $\hat{p}_1 = \bar{p} + \left[C_{\tilde{p}} - \sigma^2 I\right] C_{\tilde{p}}^{-1} (\tilde{p} - \bar{p})$

eq 8.12: $\hat{p}_2 = \bar{p}^1 + C^1_p \left[C^1_p + \sigma^2 I\right]^{-1} (\tilde{p} - \bar{p}^1)$

eq 4.10: $D\tilde{U}_1 = \sum \alpha(i) \langle \tilde{u}, g_i \rangle g_i$

$$\alpha(i) = \max\left\{0, \frac{|\langle \tilde{u}, g_i \rangle|^2 - c\sigma^2}{|\langle \tilde{u}, g_i \rangle|^2}\right\}$$

eq 4.11 $D\tilde{U}_2 = \sum \alpha(i) \langle \tilde{u}, g_i \rangle g_i$

$$\alpha(i) = \frac{|\langle \hat{u}_1, g_i \rangle|^2}{|\langle \hat{u}_1, g_i \rangle|^2 + \sigma^2}$$

Diagonalizing $C_{\tilde{p}} = U \Lambda U^T$:

(8.8) $\hat{p}_1 - \bar{p} = (I - \sigma^2 I C_{\tilde{p}}^{-1}) (\tilde{p} - \bar{p})$

$\hat{p}_1 - \bar{p} = (I - \sigma^2 U \Lambda^{-1} U^T)(\tilde{p} - \bar{p})$

$$\hat{p}_1 - \bar{p} = \underbrace{\Lambda^{-1}(\Lambda - \sigma^2 I)}_{\underbrace{\mathrm{diag}\left(\frac{\lambda_k - \sigma^2}{\lambda_k}\right)}_{S}} (\tilde{p} - \bar{p})$$

λ_k eigen value of $C_{\tilde{p}}$

Note how the diagonal operator S is simillar to the first step of the wiener filter (empirical), when using as orthogonal basis, the basis formed by the directions of $C_{\tilde{P}}$, which are the principal components of the noisy image.

Then, for eq 8.12:

$$\hat{P}_2 - \bar{P}^1 = C_P^1 \left(C_P^1 + \sigma^2 I \right)^{-1} \left(\tilde{P} - \bar{P}^1 \right)$$

Diagonalizing $C_P^1 = V \Lambda V^T$

$$\hat{P}_2^1 - \bar{P}^1 = V \Lambda V^T \left[V \Lambda V^T + \sigma^2 I \right]^{-1} \left(\tilde{P} - \bar{P}^1 \right)$$

$$\hat{P}_2 - \bar{P}^1 = \underbrace{\Lambda \left(\Lambda + \sigma^2 I \right)^{-1}} \left(\tilde{P} - \bar{P}^1 \right)$$

$S' = \text{diag}\left(\frac{\lambda_k}{\lambda_k + \sigma^2} \right)$ where λ_k are the eigen values of the oracle image.

Again, the equation takes the form of a wiener filter, in this case, using the oracle image and as orthogonal basis, the principal components of the <u>oracle image</u>.

By oracle image, I refer to the image obtained after the first step of the denoising method (empirical wiener).

One difference between formulas is that the basis where the components shrinkage is performed is different between the empirical and oracle step. In the original Wiener derivation, the basis was the same in both steps (e.g. DCT). In the Bayesian case, first we use the principal components of the original image and then the oracle one.

Elías Masquil

Image Denoising - Homework 3

Ex 8.4

Def $MSE = \int \mathbb{P}(p) \int \mathbb{P}(\tilde{p}/p) \|p-\hat{p}\|^2 \, d\tilde{p} \, dp$

By Bayes'

$$MSE = \int \cancel{\mathbb{P}(p)} \int \frac{\mathbb{P}(p/\tilde{p})\,\mathbb{P}(\tilde{p})}{\cancel{\mathbb{P}(p)}} \|p-\hat{p}\|^2 \, d\tilde{p} \, dp$$

By Fubini - Tonelli's

$$MSE = \int \mathbb{P}(\tilde{p}) \int \mathbb{P}(p/\tilde{p}) \|p-\hat{p}\|^2 \, dp \, d\tilde{p}$$

Ex 8.5

(8.18) $MSE(\tilde{p}) = \int \mathbb{P}(p/\tilde{p}) (p-\hat{p})^2 \, dp$

$\big\downarrow \, d/d\hat{p}$

$$-2 \int \mathbb{P}(p/\tilde{p}) (p-\hat{p}) \, dp = 0$$

$$\int \mathbb{P}(p/\tilde{p}) \, \hat{p} \, dp = \int \mathbb{P}(p/\tilde{p}) \, p \, dp$$

$$\hat{p} = \int \mathbb{P}(p/\tilde{p}) \, p \, dp$$

$$\boxed{\hat{p} = \mathbb{E}[p/\tilde{p}]}$$