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ASEN 5519 Homework 3

Exercise 1

a) $R(\alpha,\beta,\gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha)$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$*\ SX = \sin X$
$*\ CX = \cos X$

$X \in [\gamma, \beta, \alpha]$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1) $$R_y(\beta)R_z(\alpha) = \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix}\begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} C\beta C\alpha & -C\beta S\alpha & S\beta \\ S\alpha & C\alpha & 0 \\ -S\beta C\alpha & -S\alpha S\beta & C\beta \end{bmatrix}$$

2) $$R(\alpha,\beta,\gamma) = R_z(\gamma)R_1 =$$

$$= \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C\beta C\alpha & -C\beta S\alpha & S\beta \\ S\alpha & C\alpha & 0 \\ -S\beta C\alpha & -S\alpha S\beta & C\beta \end{bmatrix} =$$

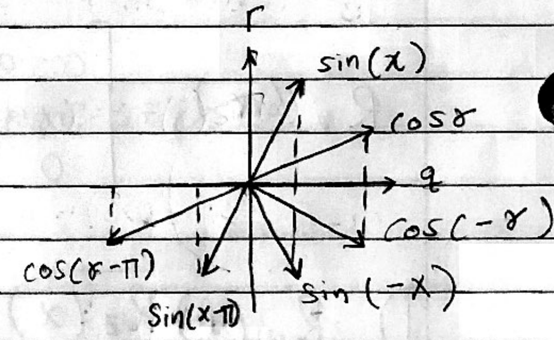
$$R(\alpha,\beta,\gamma) = \begin{bmatrix} c\gamma c\beta c\alpha - s\gamma s\alpha & -c\gamma c\beta s\alpha - s\gamma c\alpha & c\gamma s\beta \\ s\gamma c\beta c\alpha + c\gamma s\alpha & -s\gamma c\beta s\alpha + c\gamma c\alpha & s\gamma s\beta \\ -s\beta c\alpha & -s\alpha s\beta & c\beta \end{bmatrix}$$

$$\boxed{R(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & -\cos\gamma\cos\beta\sin\alpha - \sin\gamma\cos\alpha & \cos\gamma\sin\beta \\ \sin\gamma\cos\beta\cos\alpha + \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ -\sin\beta\cos\alpha & -\sin\alpha\sin\beta & \cos\beta \end{bmatrix}}$$

flip sign of cos terms

b) $R(\alpha-\pi, -\beta, \gamma-\pi) = R_-$

flip sign of all Sin terms (sin is odd)



$$R_- = \begin{bmatrix} (-\cos\gamma)(\cos\beta)(-\cos\alpha) - (-\sin\gamma\sin\alpha) & -(-\cos\gamma)(\cos\beta)(-\sin\alpha) - (-\sin\gamma)(-\cos\alpha) & (-\cos\gamma)(-\sin\beta) \\ (-\sin\gamma)(\cos\beta)(-\cos\alpha) + (-\cos\gamma)(-\sin\alpha) & -(-\sin\gamma)(\cos\beta)(-\sin\alpha) + (-\cos\gamma)(-\cos\gamma) & (-\sin\gamma)(-\sin\beta) \\ -(-\sin\beta)(-\cos\alpha) & -(-\sin\alpha)(-\sin\beta) & \cos\beta \end{bmatrix}$$

All signs cancel and we get

$$\boxed{R_- = R(\alpha,\beta,\gamma)}$$

c) $R' = \begin{bmatrix} R'_{11} & R'_{12} & R'_{13} \\ R'_{21} & R'_{22} & R'_{23} \\ R'_{31} & R'_{32} & R'_{33} \end{bmatrix}$

We get 9 nonlinear equations

$R'_{11} = \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha$

$R'_{12} = -\cos\gamma\cos\beta\sin\alpha - \sin\gamma\cos\alpha$

$R'_{13} = \cos\gamma\sin\beta$

R'_{21} $\sin\gamma\cos\beta\cos\alpha + \cos\gamma\sin\alpha$

$R'_{22} = -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha$

$R'_{23} = \sin\gamma\sin\beta$

$R'_{31} = -\sin\beta\cos\alpha$

$R'_{32} = -\sin\alpha\sin\beta$

(1) $\boxed{R'_{33} = \cos\beta}$ $\longrightarrow$ Replace $\cos\beta$ with R'_{33}

* $\cos^2\beta + \sin^2\beta = 1$

$R'^2_{33} + \sin^2\beta = 1 \Longrightarrow \boxed{\sin\beta = \pm\sqrt{1 - R'^2_{33}}}$ (2)

$R'_{11} = \cos\gamma\, R'_{33} \cos\alpha - \sin\gamma \sin\alpha$ ✓

$R'_{12} = -\cos\gamma\, R'_{33} \sin\alpha - \sin\gamma \cos\alpha$ ✓

$R'_{13} = \pm\cos\gamma \sqrt{1 - R'^2_{33}}$ ✓

$R'_{21} = \sin\gamma\, R'_{33} \cos\alpha + \cos\gamma \sin\alpha$ ✓

$R'_{22} = -\sin\gamma\, R'_{33} \sin\alpha + \cos\gamma \cos\alpha$

$R'_{23} = \pm\sin\gamma \sqrt{1 - R'^2_{33}}$ ✓

$R'_{31} = \mp\sqrt{1 - R'^2_{33}}\ \cos\alpha$ ✓

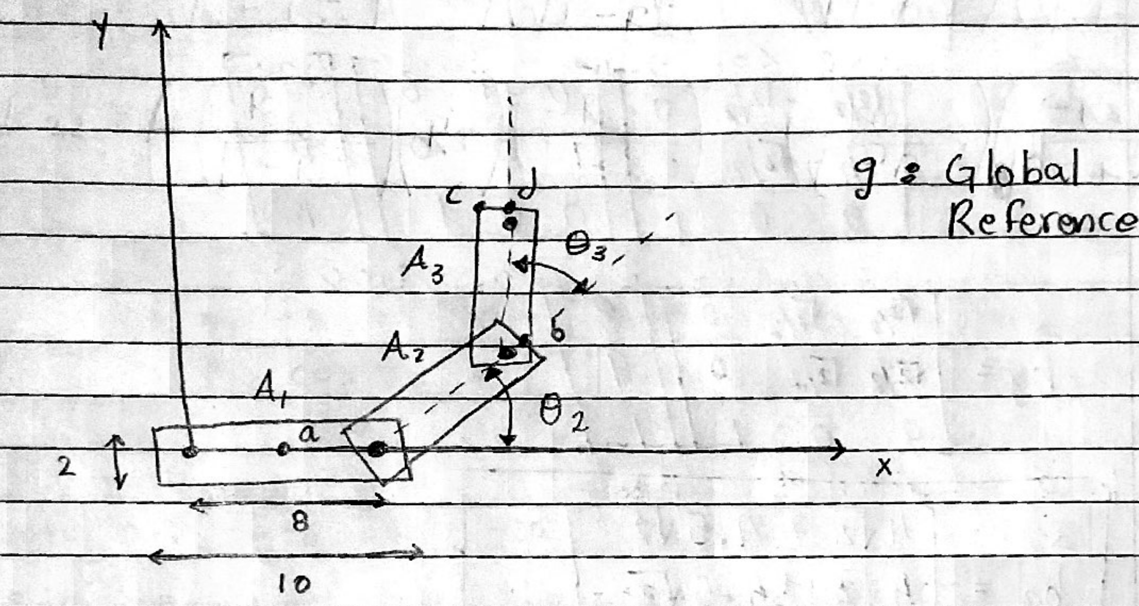
$R'_{32} = \mp\sin\alpha \sqrt{1 - R'^2_{33}}$ ✓

$$\cos\gamma = \frac{\pm R'_{13}}{\sqrt{1 - R'^2_{33}}} \qquad \sin\gamma = \frac{\pm R'_{23}}{\sqrt{1 - R'^2_{33}}}$$

$$\cos\alpha = \frac{\mp R'_{31}}{\sqrt{1 - R'^2_{33}}} \qquad \sin\alpha = \frac{\mp R'_{32}}{\sqrt{1 - R'^2_{33}}}$$

$$\cos\beta = R'_{33} \qquad \sin\beta = \pm\sqrt{1 - R'^2_{33}}$$

Exercise 2



a) $(\theta_1, \theta_2, \theta_3) = \left(\frac{\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{6}\right)$

$$\vec{a}_{A1} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \qquad T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{a}_g = T_1 \vec{a}_{A1} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{a}_g = \begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \\ 1 \end{bmatrix}}$$

$$\vec{b}_{A2} = \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 8 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{b}_g = T_1 T_2 \vec{b}_{A2}$$

$$T_2 = \begin{bmatrix} 0 & -1 & 8 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{b}_g = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 8 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{b}_g = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{b}_g = \begin{bmatrix} 4\sqrt{2} - 4.5\sqrt{2} \\ 4\sqrt{2} + 4.5\sqrt{2} \\ 1 \end{bmatrix}}$$

$$\vec{C}_{A_3} = \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{C}_g = T_1 T_2 T_3 \vec{C}_{A_3}$$

$$T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 8 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{C}_g = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 8 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 & 8 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix}$$

(Matlab)

$$\boxed{\vec{C}_g = \begin{bmatrix} -3.2953 \\ 19.782 \\ 1 \end{bmatrix}}$$

b) Find configurations

$$\vec{d}_g = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\vec{d}_{A3} = \begin{bmatrix} p \\ q \\ 1 \end{bmatrix} \qquad \vec{d}_g = T_1 T_2 T_3 \vec{d}_{A3} = R\,\vec{d}_{A3}$$

$$R = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & a_2 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & a_3 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 & a_2\cos\theta_1 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & a_2\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix} T_3$$

$$r_{11} = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\cos\theta_3 - (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\sin\theta_3$$

$$r_{12} = -(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_3 - (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\cos\theta_3$$

$$r_{13} = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)a_3 + a_2\cos\theta_1$$

$$r_{21} = (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\cos\theta_3 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_3$$

$$r_{22} = -(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\sin\theta_3 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\cos\theta_3$$

$$r_{23} = (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)a_3 + a_2\sin\theta_1$$

$r_{31} = 0 \qquad x = p\,r_{11} + q\,r_{12} + 1\,r_{13}$

$r_{32} = 0 \qquad y = p\,r_{21} + q\,r_{22} + 1\,r_{23}$

$r_{33} = 1 \qquad 1 = p\,r_{31} + q\,r_{32} + 1\,r_{33}$

- $\cos^2\theta_1 + \sin^2\theta_1 = 1$ • $\cos^2\theta_2 + \sin^2\theta_2 = 1$
- $\cos^2\theta_3 + \sin^2\theta_3 = 1$

$$x = p\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\cos\theta_3 - (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\sin\theta_3\right]$$

$$+ q\left[-(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_3 - (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\cos\theta_3\right]$$

$$+ (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)a_3 + a_2\cos\theta_1$$

$$\left[\begin{aligned} x = &\left[p\cos\theta_3 - q\sin\theta_3 + a_3\right](\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &- \left[p\sin\theta_3 + q\cos\theta_3\right](\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &+ a_2\cos\theta_1 \end{aligned}\right]$$

$$y = p\left[(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\cos\theta_3 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_3\right]$$

$$+ q\left[-(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\sin\theta_3 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\cos\theta_3\right]$$

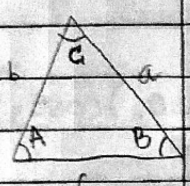
$$(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)a_3 + a_2\sin\theta_1$$

$$\left[\begin{aligned} y = &\left[p\cos\theta_3 - q\sin\theta_3 + a_3\right](\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &+ \left[p\sin\theta_3 + q\cos\theta_3\right](\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &+ a_2\sin\theta_1 \end{aligned}\right]$$

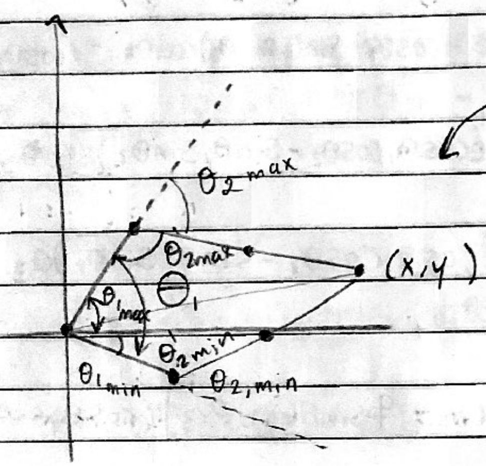
(angle + quadrant)

- We have 5 equations and 6 unknowns. This is because we have a 3 Joint Robot arm and a 2D work space

Law of Cosine



$c^2 = a^2 + b^2 - 2ab\cos G$



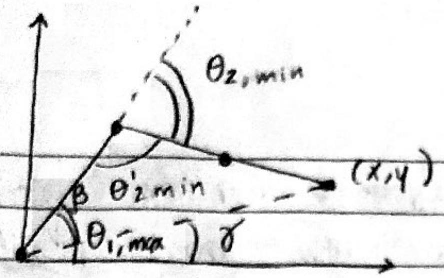
Like making it a two joint robot

θ_1 is the $\Delta\theta_1$ that A_1 can go through and still allow A_2 & A_3 to reach (X,Y). This means at the extremes of θ_1 (min θ_1 an max θ_1) the arms A_2 & A_3 are completely extended meaning $\theta_3 = 0°$. This gives us:

- $\cos^2\theta_1 + \sin^2\theta_1 = 1$ • $\cos^2\theta_2 + \sin^2\theta_2 = 1$

- $x = \boxed{P + a_3}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) - q(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) + a_2\cos\theta_1$

- $y = P(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) + q(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)a_3 + a_2\sin\theta_1$



$\theta_{1,max} = \beta + \gamma$

$\underbrace{a_3' = a_3 + 1 - 9}_{\text{using point } d}$

$x^2 + y^2 = (a_2 + a_3')^2 + (a_1)^2 - 2(a_2 + a_3)a_1 \cos(\theta'_{2,max})$

- $\cos(\theta'_{2,min}) = \dfrac{(a_2 + a_3')^2 + (a_1)^2 - x^2 - y^2}{2a_1(a_2 + a_3)}$

  - $\theta_{2,min} = \theta'_{2,min} - 180$
  - $\theta_{2,max} = 180 - \theta'_{2,min}$

- $\gamma = \text{arctan\_2}(x, y)$ ← need to know ~~quadrant~~

- $\cos(\beta) = \dfrac{a_1^2 + x^2 + y^2 - (a_2 + a_3')^2}{2a_1\sqrt{x^2 + y^2}}$

- $\theta_{1,max} = \beta + \gamma$

- $\theta_{1,min} = \gamma - \beta$

Can have $\theta_1 \in [\theta_{1,min}, \theta_{1,max}]$

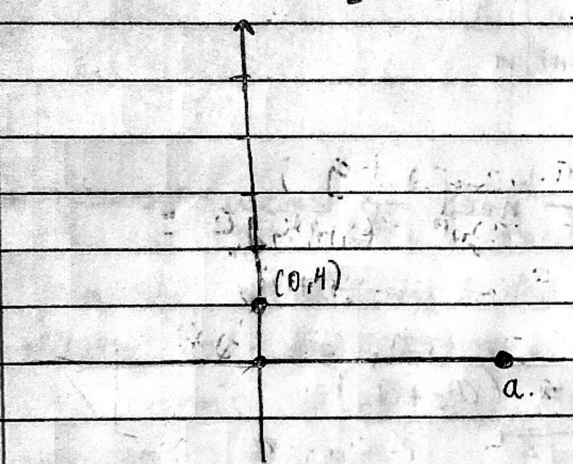
When $\theta_1 = \theta_{1,min} \Rightarrow \theta_2 = \theta_{2,max}$

$\theta_1 = \theta_{1,max} \Rightarrow \theta_2 = \theta_{2,min}$

Now, as long as we pick a $\theta_1 \in [\theta_{1,min}, \theta_{1,max}]$ we know we can reach the goal x, y with the the other two arms (with some assumptions).

* when implementing this I think having θ_1 move a mininum amount would be best:
if ($\theta_{1,min,new} \leq \theta_1 \leq \theta_{1,max,new}$) { dont move θ_1 }

Now, $\vec{d_g} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$



$\theta_{1,min} = \gamma - \beta \Rightarrow$ by inspection, $\gamma = 90^\circ$

$$\cos(\beta) = \frac{a_1^2 + x^2 + y^2 - (a_2 + a_3')^2}{2a_1\sqrt{x^2+y^2}}$$

$a_1 = 8$
$a_2 = 8$
$a_3' = 9$
$x = 0$
$y = 4$

$\cos(\beta) = -3.2625$

In this case, θ_1 can be any angle. Pick $\theta_1 = 0$

Assumed Theta 1 was 0 all the

Lots of math

way through ! :(In future will not do this

$\theta_1 = 0$, $\cos^2\theta_2 + \sin^2\theta_2 = 1$, $\cos^2\theta_3 + \sin^2\theta_3 = 1$

$$x = \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2 - \left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 + a_2$$

$$y = \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\sin\theta_2 + \left[P\sin\theta_3 + q\cos\theta_3\right]\cos\theta_2$$

Now I want $x^2 + y^2$

$$x^2 = \left(\left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2 - \left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 + a_2\right) \times$$
$$\left(\left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2 - \left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 + a_2\right)$$

$$= \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]^2\cos^2\theta_2 - \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\left[P\sin\theta_3 + q\cos\theta_3\right]\cos\theta_2\sin\theta_2$$
$$+ \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2 a_2 - \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\left[P\sin\theta_3 + q\cos\theta_3\right]\cos\theta_2\sin\theta_2$$
$$+ \left[P\sin\theta_3 + q\cos\theta_3\right]^2\sin^2\theta_2 - \left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 a_2$$
$$+ \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2 a_2 - \left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 a_2 + a_2^2$$

$$x^2 = \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]^2\cos^2\theta_2 - 2\left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\left[P\sin\theta_3 + q\sin\theta_3\right]\cos\theta_2\sin\theta_2$$
$$+ 2\left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2 a_2 + \left[P\sin\theta_3 + q\cos\theta_3\right]^2\sin^2\theta_2 - 2\left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 a_2$$
$$+ a_2^2$$

$$y^2 = \left(\left[p\cos\theta_3 - q\sin\theta_3 + a_3\right]\sin\theta_2 + \left[p\sin\theta_3 + q\cos\theta_3\right]\cos\theta_2\right)^2$$

$$y^2 = \left[p\cos\theta_3 - q\sin\theta_3 + a_3\right]^2\sin^2\theta_2 + 2\left[p\cos\theta_3 - q\sin\theta_3 + a_3\right]\sin\theta_2\left[p\sin\theta_3 + q\cos\theta_3\right]\cos\theta_2$$

$$+ \left[p\sin\theta_3 + q\cos\theta_3\right]^2\cos^2\theta_2$$

$$x^2 + y^2 = \left[p\cos\theta_3 - q\sin\theta_3 + a_3\right]^2\left(\sin^2\theta_2 + \cos^2\theta_2\right)$$

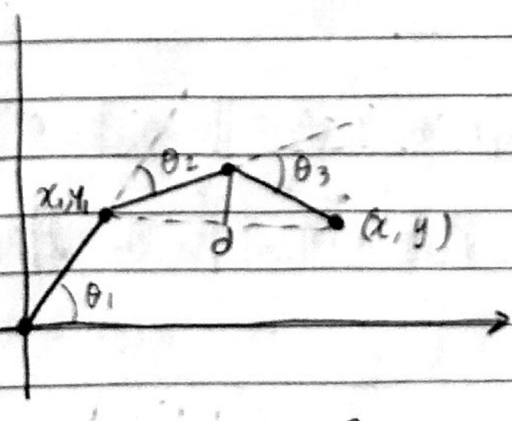
$$+ \left[p\sin\theta_3 + q\cos\theta_3\right]^2\left(\sin^2\theta_2 + \cos^2\theta_2\right)$$

$$+ 2\left[p\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2\, a_2$$

$$- 2\left[p\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2\, a_2$$

$$+ a_2^2$$

Some more geometry



$$d^2 = a_2^2 + a_3^2 + 2a_2a_3\cos\theta_3$$

$$x_1 = a_1\cos\theta_1$$

$$x_2 = a_1\sin\theta_1$$

$$d^2 = (x - x_1)^2 + (y - y_1)^2$$

$\left[* \ a'_3 = a_3 + \text{length to tip in } A_3 \right]$

$$\boxed{\begin{aligned} \bullet \cos\theta_3 &= \frac{d^2 - a_2^2 - a_3'^2}{2a_3' a_2} \\ \bullet \sin\theta_3 &= \pm\sqrt{1-\cos^2\theta_3} \end{aligned}}$$

$$x^2 + y^2 = \left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]$$

$$+ \left[P\sin\theta_3 + q\cos\theta_3\right]\left[P\sin\theta_3 + q\cos\theta_3\right]$$

$$+ 2a_2\left[P\cos\theta_3 - q\sin\theta_3 + a_3\right]\cos\theta_2$$

$$- 2a_2\left[P\sin\theta_3 + q\cos\theta_3\right]\sin\theta_2 + a_2^2$$

$$= \Big[\cancel{P^2\cos^2\theta_3} - \cancel{Pq\cos\theta_3\sin\theta_3} + \cancel{Pa_3\cos\theta_3} - \cancel{Pq\cos\theta_3\sin\theta_3} + \cancel{q^2\sin^2\theta_3}$$

$$- \cancel{qa_3\sin\theta_3} + \cancel{Pa_3\cos\theta_3} - \cancel{qa_3\sin\theta_3} + \cancel{a_3^2}\Big]$$

$$+ \left[\cancel{P^2\sin^2\theta_3} + \cancel{2Pq\sin\theta_3\cos\theta_3} + \cancel{q^2\cos^2\theta_3}\right]$$

$$+ \left[2a_2P\cos\theta_3\cos\theta_2 - 2a_2q\sin\theta_3\cos\theta_2 + 2a_2a_3\cos\theta_2\right]$$

$$- \left[2a_2P\sin\theta_3\sin\theta_2 + 2a_2q\cos\theta_3\sin\theta_2\right] + \cancel{a_2^2}$$

$$= P^2 + q^2 + 2Pa_3\cos\theta_3 - 2qa_3\sin\theta_3 + a_3^2 + a_2^2$$

$$+ 2a_2P\cos\theta_3\cos\theta_2 - 2a_2q\sin\theta_3\cos\theta_2 + 2a_2a_3\cos\theta_2$$

$$- 2a_2P\sin\theta_3\sin\theta_2 - 2a_2q\cos\theta_3\sin\theta_2$$

$$\underbrace{x^2+y^2-p^2-q^2-2pa_3\cos\theta_3+2qa_3\sin\theta_3-a_3^2-a_2^2}_{a}$$

$$= \underbrace{\left[2a_2p\cos\theta_3 - 2a_2q\sin\theta_3 + 2a_2a_3\right]}_{\beta}\cos\theta_2$$

$$- \underbrace{\left[2a_2p\sin\theta_3 + 2a_2q\cos\theta_3\right]}_{\gamma}\sin\theta_2$$

- $\cos^2\theta_2 + \sin^2\theta_2 = 1$

$$a = \beta\cos\theta_2 + \gamma\sin\theta_2$$

$$a = \beta\cos\theta_2 \pm \gamma\sqrt{1-\cos^2\theta_2}$$

$$\mp\gamma\sqrt{1-\cos^2\theta_2} = \beta\cos\theta_2 - a$$

$$\gamma^2(1-\cos^2\theta_2) = \beta^2\cos^2\theta_2 - 2a\beta\cos\theta_2 + a^2$$

$$\gamma^2 - \gamma^2\cos^2\theta_2 = \beta^2\cos^2\theta_2 - 2a\beta\cos\theta_2 + a^2$$

$$(\beta^2+\gamma^2)\cos^2\theta_2 - 2a\beta\cos\theta_2 + (a^2-\gamma^2) = 0$$

$$\cos\theta_2 = \frac{2a\beta \pm \sqrt{4a^2\beta^2 - 4(\beta^2+\gamma^2)(a^2-\gamma^2)}}{2(\beta^2+\gamma^2)}$$

$$\cos\theta_2 = \frac{a\beta}{(\beta^2+\gamma^2)} \pm \frac{\sqrt{a^2\beta^2-(\beta^2+\gamma^2)(a^2-\gamma^2)}}{(\beta^2+\gamma^2)}$$

$$\sin\theta_2 = \pm\sqrt{1-\cos^2\theta_2}$$

Now, we can solve for $\theta_1, \theta_2, \theta_3$ given

that $\vec{d_g} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

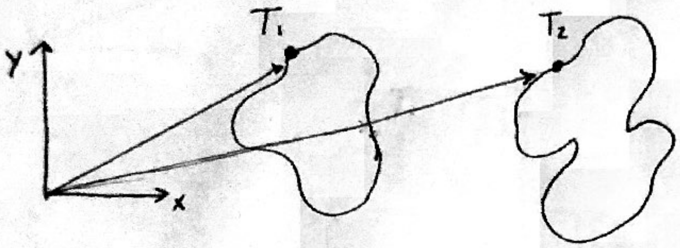
$\theta_1 = 0°$, $\theta_2 = 89.553°$, $\theta_3 = 116.8318°$

$\theta_1 = 0°$, $\theta_2 = -142.6837°$, $\theta_3 = -116.8318$

And infinite more! Due to the fact θ_1 can be a range of options.

Exercise 3

a) Two trains on two train tracks



Can describe the position of T_1 with (x_1, y_1) and T_2 with (x_2, y_2). They are not coupled so we need all parameters

Need to have two sets of points from $\mathbb{R}^2$,
So the C-space $C \subseteq \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$

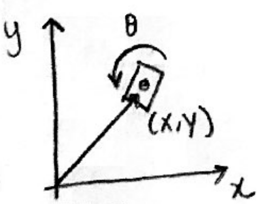
$\boxed{C \subset \mathbb{R}^4}$

$\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4\}$

* I think that we could also use $I^2 \times I^2$ for C-space *

$\boxed{\dim(C) = \text{\# of unique parameters} = 4}$

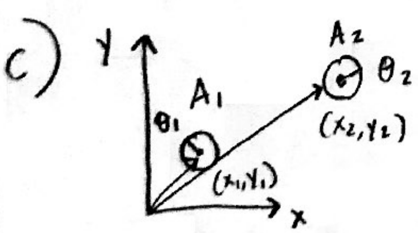
b) A spacecraft that can translate and rotate in 2D



Its position in 2D requires 2 coordinates $x, y \in \mathbb{R}$, and because we can only rotate along 1 axis (z out of page) one rotation angle $\theta \in S^1$ is needed

$\boxed{C = \mathbb{R}^2 \times S^1}$ $\boxed{\dim(C) = 3}$

c)

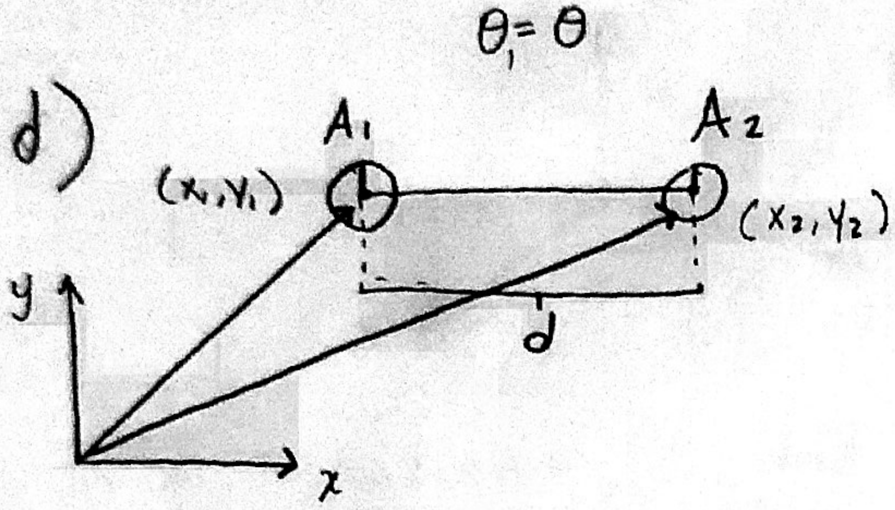


Because you cannot infer any thing about the position or orientation of A_1 by knowing (x_2, y_2, θ_2), and vice versa, you need $x \in \mathbb{R}$ $y \in \mathbb{R}$ and $\theta \in S^1$ for each robot

3 parameters per robot

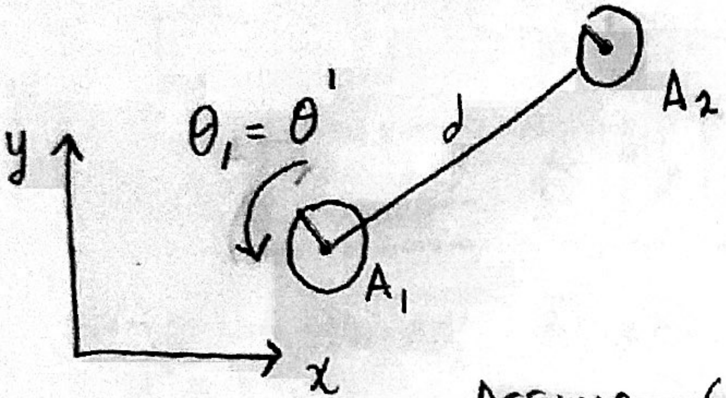
$\boxed{C = (\mathbb{R}^2 \times S^1) \times (\mathbb{R}^2 \times S^1)}$ $\boxed{\dim(C) = 6}$

d)



So if we know $x_1, y_1, d,$ and θ_1 we can find x_2, y_2, and θ_2. Because we know the robots are connected

$$\boxed{\theta_1 = \theta_2}$$



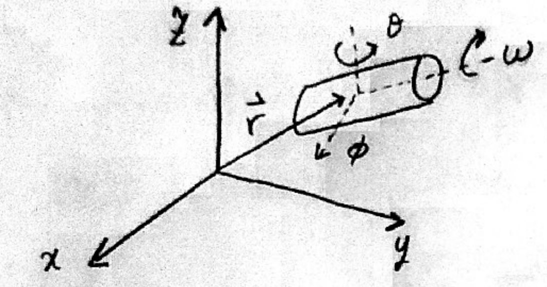
If we know d we can then find the othe robot by only knowing (x, y, θ) for the other

Assume (x_1, y_1, θ_1) are known, then by simple geometry we can see that

$$x_2 = d\cos\theta_2 \ , \quad y_2 = d\sin\theta_2 \ , \quad \theta_2 = \theta_1$$

If we know d, then $\boxed{\text{C-space} = \mathbb{R}^2 \times S^1}$ because we only need $(x_1, y_1) \in \mathbb{R}^2$ and $\theta_1 \in S^1$, 3 parameters, $\boxed{\dim(C) = 3}$

e)



We can translate along x, y, and z

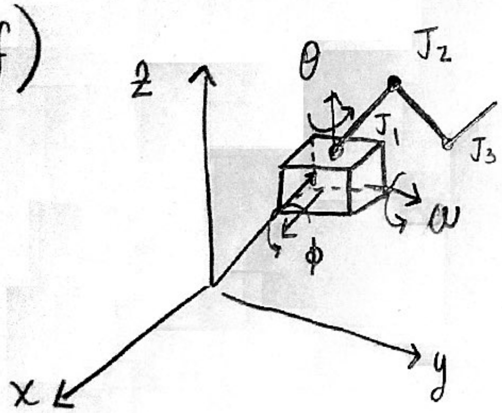
$(x, y, z) \in \mathbb{R}^3$

Only rotations θ and ϕ have an effect on the C-space. $\theta \in S^1$ and $\phi \in S^1 \rightarrow (\theta, \phi) \in S^2$

So $\boxed{C \subset \mathbb{R}^3 \times S^2}$, because the rotation ω does nothing detectable/visible. We only need x, y, z, θ, ϕ to describe the configuration of the cylinder (5 parameters)

$\boxed{\dim(C) = 5}$

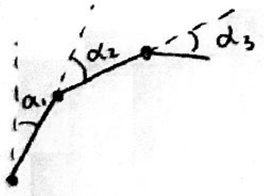
f)



To describe the position $(x, y, z) \in \mathbb{R}^3$, and orientation $(\omega, \phi, \theta) \in SO(3)$

We need a configuration in $\mathbb{R}^3 \times SO(3)$

but we must also include the three links of the robot arm. These require three distinct angles $(\alpha_1, \alpha_2, \alpha_3)$



$S^1 \times S^1 \times S^1 = T^3$ thus our

$\boxed{C\text{-space} = \mathbb{R}^3 \times SO(3) \times T^3}$

Need 9 parameters to describe configuration of the robot

$\boxed{\dim(C) = 9}$

9) The manipulator has 7 revolute Joints.

Each joint has some angle θ_i , $i \in \{1, 2, \ldots, 7\}$

Our configuration then requires 7 θ's where

$\theta_i \in S^1$, $(\theta_1, \theta_2, \ldots, \theta_7) \in S^1 \times S^1 \times \ldots \times S^1 = T^7$

$\boxed{C\text{-space} = T^7}$ $\boxed{\dim(C) = 7 = \#\text{ of revolute joints}}$

Exercise 4

$W \subseteq \mathbb{R}^n$ the obstacles $WO \subset W$ and a single obstacle $O \subset WO$, where O is convex.

$A \subset W$ and A is a convex robot.

We want to prove that C-space representation of O is also convex. We know that for a given obstacle and robot, the C-space can be found using their Minkowski sum (difference):

$$L = O \ominus A$$

Since O and A are convex, we know that for $t \in [0,1]$ and $x, y \in A$ $p, q \in O$ (from book pg 477)

$$(tx + (1-t)y) \in A \quad \text{and} \quad (tp + (1-t)q) \in O$$

Thus, $O \ominus A$ will contain points $p - x$ and $q - y$ which we can check for convexity

$$t(p-x) + (1-t)(q-y) = tp - tx + (1-t)q - (1-t)y$$

$$[\underbrace{tp + (1-t)q}_{\in O} - \underbrace{(tx + (1-t)y)}_{\in A}] \in L = O \ominus A$$

$$[t(p-x) + (1-t)(q-y)] \in L, \quad (p-x), (q-y) \in L$$

This shows that the set L is also a convex set!