SF2568: Parallel Computations for Large-Scale Problems

Lecture 6: Image Reconstruction and Poisson's Equation

January 30, 2024

Acknowledgements

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Introduction

Question

What does image processing and the solution of partial differential equations have in common?

Digital Images

Definition

A (digital) **image** is a $M \times N$ matrix of pixel values, the *pixmap*

- We will assume that each pixel is represented by its gray level. Thus, we assume the image to be black and white
- A coloured image consists of a collection of pixmaps
- We assume that the type of pixel is `double`. In practice, most often 8-bit values are used (`unsigned char`)

Smoothing, Sharpening, Noise Reduction

- **Smoothing**: suppresses large fluctuations in intensity over the image
- **Sharpening**: accentuates transitions and enhances the details
- **Noise reduction**: suppresses a noise signal present in an image

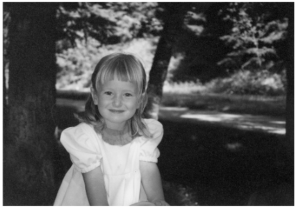
Smoothing By Local Filtering

Idea: Replace each pixel value $\tilde{u}_{mn}$ by the mean of the surrounding pixels:

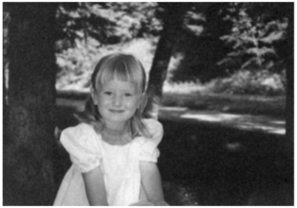
$$\begin{aligned}\tilde{u}_{mn} = \frac{1}{9}\Big(& u_{m-1,n-1} && + u_{m-1,n} && + u_{m-1,n+1} \\ + & u_{m,n-1} && + u_{m,n} && + u_{m,n+1} \\ + & u_{m+1,n-1} && + u_{m+1,n} && + u_{m+1,n+1}\Big)\end{aligned}$$

Smoothing – Example

Original



Result



Weighted Masks

The mean value can conveniently be described by a 3×3 matrix W,

$$W = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Application:

$$\begin{aligned} \tilde{u}_{mn} = \quad & w_{-1,-1}u_{m-1,n-1} + w_{-1,0}u_{m-1,n} + w_{-1,1}u_{m-1,n+1} \\ & + w_{0,-1}u_{m,n-1} + w_{0,0}u_{m,n} + w_{0,1}u_{m,n+1} \\ & + w_{1,-1}u_{m+1,n-1} + w_{1,0}u_{m+1,n} + w_{1,1}u_{m+1,n+1} \end{aligned}$$

Mathematically: Convolution

Noise Reduction

$$W = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

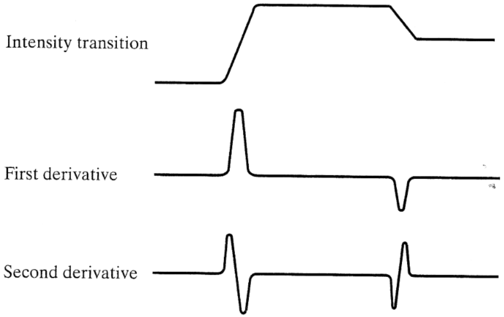
Noisy image

denoised

Edge Detection

- **Edge detection** is the highlighting of the edges of an object, where an edge is a significant change in the grey-level intensity
- **Basic idea**: The rate of change of a quantity can be measured by the **magnitude of its derivative(s)**

Example


Intensity transition
First derivative
Second derivative

The Laplace Operator

Definition

For any function u defined on some two-dimensional domain, the Laplacian Δu of u is defined as

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Approximating The Laplacian

- Approximate the derivatives,

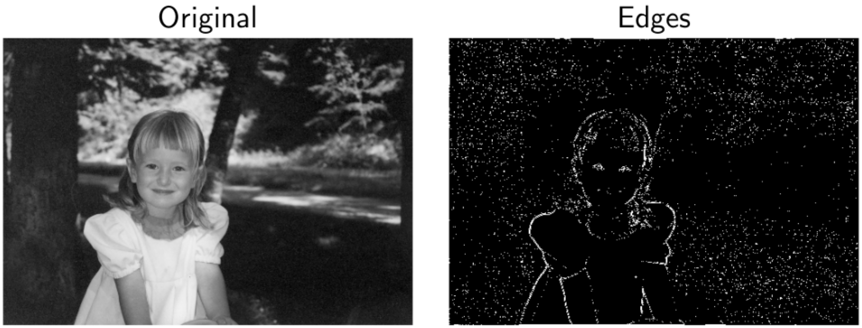
$$\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{1}{h^2}\left(u(x-h,y) - 2u(x,y) + u(x+h,y)\right), \qquad h > 0$$

- We obtain the weight matrix,

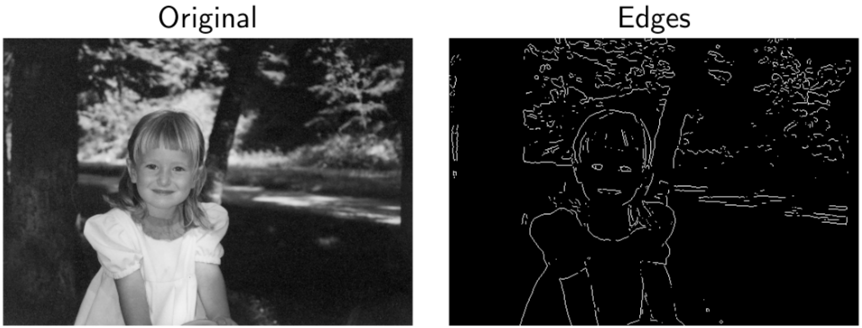
$$W = \frac{1}{h^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- *This fits exactly in our framework!*

Edge Detection By The Discrete Laplacian



Using First Order Derivatives – Sobel Operator



Poisson Equation

- Ubiquitous equation
  - Fluid flow, electromagnetics, gravitational interaction,. . .
- In two dimensions, Poisson's equation reads:
  - Solve $\Delta u = f(x, y)$ for $(x, y \in \Omega)$
  - Subject to the boundary condition $u(x, y) = g(x, y)$ for $(x, y) \in \partial\Omega$
- For simplicity, consider only $\Omega = (0, 1) \times (0, 1)$
- Generalizations to other dimensions are obvious

Discrete Approximation

- Define a *mesh* (or grid): For a given N, let

$$h = 1/(N-1), \qquad x_m = mh, \qquad y_n = nh$$

- Let $u_{mn} \approx u(x_m, y_n)$, $f_{mn} = f(x_m, y_n)$
- Using the Laplace approximation from above, we obtain a system of equations

$$\frac{1}{h^2}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n}\right) = f_{mn},$$

$$0 < m, n < N-1$$

- In the context of PDE's, the matrix W is usually called a **stencil**

Jacobi Iteration

- **Basic idea**: Rewrite the equation as

$$u_{mn} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn}\right)$$

- For some starting guess (e.g., $u_{mn} = 0$), iterate this equation,

```
while ( not_done )
  for (m,n) in 1:N-2 x 1:N-2
    ut(m,n) =( u(m-1,n) + u(m+1,n)
             +  u(m,n-1) + u(m,n+1)
             -  h^2 f(m,n)          )/4;
  end
  u = ut;
end
```

Accuracy

- How do we know that the answer is "good enough"?
    - When the computed solution has reached a reasonable approximation to the exact solution
    - When we can validate the computed solution in the field
- But often we do not know the exact solution, and must estimate the error, e.g.,
    - Stop when the residual is small enough, $r = Au - f$
    - Stop when the change $u - u'$ in u is small
    - Both approaches must be designed carefully!

Boundary Condition

- Evaluating the stencil is not possible near the boundary
- For Poisson's equation $\rightarrow$ invoke the boundary condition
- In image processing, there are two possibilities:
    1. Discard the boundary (the new image is 2 pixels smaller in both dimensions)
    2. Modify the weight matrices such that only existing neighbours are used

The Common Denominator

Conclusion

The methods considered use a **uniform mesh** for their data

- Such methods are very common in applications
- They can easily be adapted to problems of any (spatial) dimension

Observations

Observations

- The computations for each point u_{ij} are completely decoupled
- The number of operations per data point is constant
- The new value at each data point depends only on its nearest neighbours

Conclusion

- A good parallelisation strategy is data partitioning

To keep things as simple as possible, consider only a one-dimensional array (a vector)

$$u = (u_0, \ldots, u_{M-1})^T$$

Data Distribution

Definition

Assume that we have P processes (enumerated $0, \dots, P-1$). A **P-fold data distribution** of the index set $\mathcal{M} = \{0, \dots, \mathcal{M}-1\}$ is a bijective mapping μ which maps each **global index** $m \in \mathcal{M}$ to a pair of indices (p, i) where p is the process identifier and i the **local index**.

Notes

- This definition allows for the fact that the number of elements on each process varies with p. Of course, this is necessary if P does not divide M evenly
- Technically, we assume that the local index set on each process is a set of consecutive integers, often (**but not always!**) $0 \leq i < I_p$

Example – Linear Data Distribution

Idea

Split the vector into equal chunks and allocate the p-th chunk to process p

- At $p = 0:\ u_0, u_1, \ldots, u_{l_0 - 1}$
- At $p = 1:\ u_{l_0}, \ldots, u_{l_0 + l_1 - 1}$
- In general, $p:\ u_{l_{p-1}}, \ldots, u_{l_{p-1} + l_p - 1}$
- If P does not divide M evenly, distribute the remaining R elements to the first few processes

Example – Linear Data Distribution

- The load-balanced linear data distribution is:

$$
\begin{aligned}
L &= \left\lfloor \frac{M}{P} \right\rfloor \\
R &= M \text{ mod } P \\
\mu(m) &= (p, i) \text{ where } \begin{cases} p = \max\left(\left\lfloor \frac{m}{L+1} \right\rfloor, \left\lfloor \frac{m-R}{L} \right\rfloor\right) \\ i = m - pL - \min(p, R) \end{cases} \\
I_p &= \left\lfloor \frac{M+P-p-1}{P} \right\rfloor \\
\mu^{-1}(p, i) &= pL + \min(p, R) + i
\end{aligned}
$$

Example – Scatter Distribution

Idea

Allocate consecutive vector components to consecutive processes

- At $p = 0:\ u_0, u_P, u_{2P}, \dots$
- At $p = 1:\ u_1, u_{P+1}, \dots$
- In general, $p:\ u_p, u_{P+p}, \dots$

Example – Scatter Distribution

- The load-balanced scatter distribution is:

$$\mu(m) = (p, i) \text{ where } \begin{cases} p = m \text{ mod } P \\ i = \left\lfloor \dfrac{m}{P} \right\rfloor \end{cases}$$

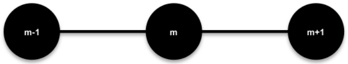
$$I_p = \left\lfloor \frac{M + P - p - 1}{P} \right\rfloor$$

$$\mu^{-1}(p, i) = iP + p$$

A Distributed Vector

- The one-dimensional version of the convolution formula reads

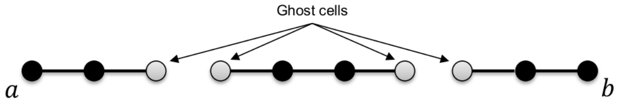
$$\tilde{u}_m = w_{-1}u_{m-1} + w_0u_m + w_1u_{m+1}$$



- Each evaluation needs its neighbours. Consequently, the **linear data distribution** is most appropriate
- Each process needs one element stored on the processes to the “left” and “right”
- This additional data is called ghost cells (or ghost points)

Ghost Cells

- Two adjacent processes p and $p+1$ need to share two data points. This is called the **overlap** between two processes, and is dependent on the **width** of the stencil
- *For Example:* Assume our convolution formula and the domain between a and b
- On a distributed memory machine we need to divide it into chunks
- The local array $u_i,\ 0 \leq i < I_p$ will be surrounded by two cells (with the exception of the first and the last processes)

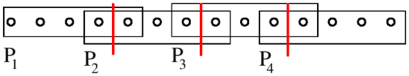


- This is conveniently done by **enlarging** the local vector

Fill The Ghost Cells – Communication

- Before we can start applying the stencil, the ghost cells must be filled
- Attempted erroneous solution (assume an overlap of two for simplicity)

```
receive(u[0],p-1);
send(u[1],p-1);
receive(u[Ip-1],p+1);
send(u[Ip-2],p+1);
```



Deadlock!

- Mismatch in communication. All processes waiting to receive
- Possible solutions:
    - Rewrite program so that calls to send and receive are matched
    - Use non-blocking communication

Communication – A New Attempt

- Exchange send and receive

```
if p > 0
  send(u[1],p-1);
  receive(u[0],p-1);
end
if p < P-1
  receive(u[Ip-1],p+1);
  send(u[Ip-2],p+1);
end
```

- Code works! But very inefficient!
  - Most processes are idle during communication
  - Possible solution: Use different communication pattern

An Efficient But Unreliable Solution

```
send(u[lp-2],p+1);
receive(u[0],p-1);
send(u[1],p-1);
receive(u[lp-1],p+1);
```

Properties:

+ communication time is optimal: $2(t_{startup} + 8t_{data})$
- Relies on the network to buffer the messages.
  This is not guaranteed by MPI!

The Safe Solution

- The idea is a red-black (checker-board) colouring:
  - Even p: assign red
  - Odd p: assign black

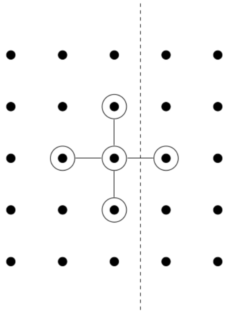
Communication appears in two steps: red/black and black/red

```
if mycolor == red
  send(u[Ip-2],p+1);
  receive(u[Ip-1],p+1);
  send(u[1],p-1);
  receive(u[0],p-1);
else
  receive(u[0],p-1);
  send(u[1],p-1);
  receive(u[Ip-1],p+1);
  send(u[Ip-2],p+1);
end
```

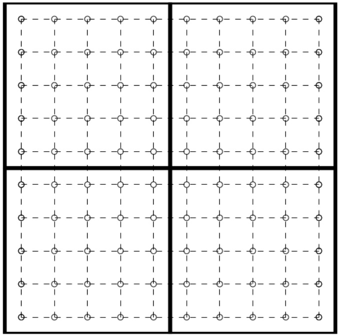
Communication time is only doubled compared to the previous

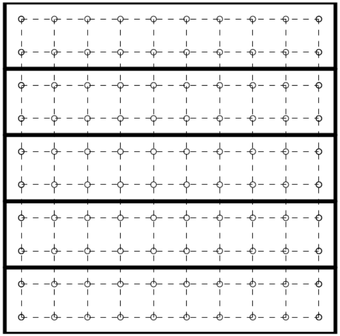
Generalizations To Two Dimensions

- Sample stencil (Poisson)
- Use an array of $R = P \times Q$ processes
- Distributed equal chunks of the pixmap/solution onto these processes
- Different partitions are called **process geometry** or **process topology**



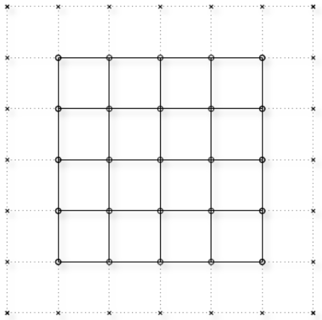
Process Topology


P=Q=2


P=1, Q=5

Ghost Cells

- Each process needs values found on neighbouring processes
- Use **ghost cells**



  - Circles: local grid points
  - Crosses: ghost points
- The data distribution is constructed individually for the x and y directions along the lines of the 1D example

Communication of Ghost Points

Question

How should the exchange of the ghost points corresponding to the inter-process boundaries be implemented?

The handling of the outer boundaries depends on the problem at hand (either ignore them or apply physical boundary conditions)

Some Notation

For a process with "coordinates" (p, q), the neighbours are defined as follows (if they exist)

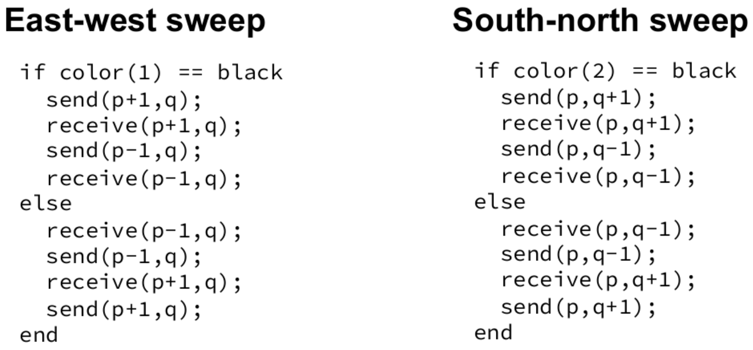
Neighbour	**Coordinates**
east	$(p+1, q)$
west	$(p-1, q)$
north	$(p, q+1)$
south	$(p, q-1)$

Non-Blocking Implementation

1. Initiate send (`MPI_Isend`) to east, west, north, and south neighbours (if present)
2. Initiate receive (`MPI_Irecv`) from west, east, south, and north neighbours
3. Evaluate the stencil away from the boundaries
4. Wait for communication to complete
5. Evaluate stencil near boundaries

Red-Black Communication

- Similar to red-black communication in 1D
- Associate each process with a color (red or black) in the p and q directions such that no neighbour has the same color



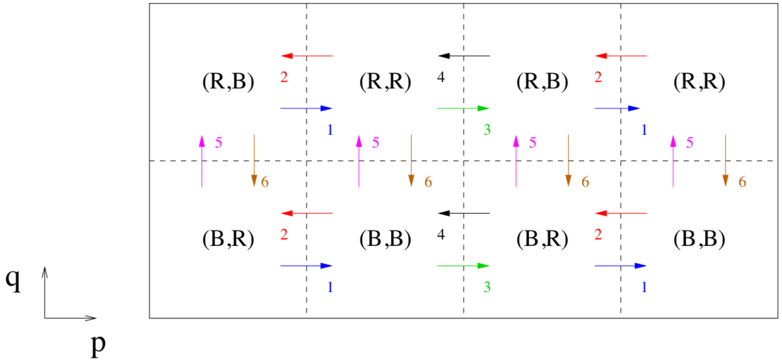
East-west sweep

```
if color(1) == black
  send(p+1,q);
  receive(p+1,q);
  send(p-1,q);
  receive(p-1,q);
else
  receive(p-1,q);
  send(p-1,q);
  receive(p+1,q);
  send(p+1,q);
end
```

South-north sweep

```
if color(2) == black
  send(p,q+1);
  receive(p,q+1);
  send(p,q-1);
  receive(p,q-1);
else
  receive(p,q-1);
  send(p,q-1);
  receive(p,q+1);
  send(p,q+1);
end
```

Red-Black Communication



Number and colours show the communication pattern process colour indicated by (q, p) (note the order)

Red-Black Communication Time

- We assume a perfectly load balanced (linear) distribution,

$$I_p \approx \frac{M}{P} \qquad J_q \approx \frac{N}{Q}$$

- East-west sweep:

$$t_{comm,1} = C(P)(t_{startup} + I_p t_{data})$$

where

$$C(P) = \begin{cases} 0, & \text{if } P = 1 \\ 2, & \text{if } P = 2 \\ 4, & \text{if } P \geq 3 \end{cases}$$

- Similarly, for the South-north sweep:

$$t_{comm,2} = C(Q)(t_{startup} + J_q t_{data})$$

- Total communication time

$$t_{comm} = (C(P) + C(Q))t_{startup} + \frac{t_{data}}{PQ}(C(P)QM + C(Q)PN)$$

Computation Time

- Assume a (compact) stencil

$$W = \begin{pmatrix} w_{-1,-1} & w_{0,-1} & w_{1,-1} \\ w_{-1,0} & w_{0,0} & w_{1,0} \\ w_{-1,1} & w_{0,1} & w_{1,1} \end{pmatrix}$$

- Let w be the number of nonzero entries in W. Then

$$t_{comm,pq} = \alpha w I_p J_q t_a \approx \alpha w \frac{MN}{PQ} t_a$$

  ($0 < \alpha$ is a small constant)

- Best sequential time

$$T_S^* = \alpha w M N t_a$$

Speedup

$$\begin{aligned} S_R = S_{PQ} &= \frac{T_S^*}{T_R} \\ &\geq R\frac{\alpha w M N t_a}{\alpha w M N t_a + 8Rt_{startup} + 4(QM + PN)t_{data}} \\ &\geq R\frac{1}{1 + \frac{8Rt_{startup}}{\alpha w M N t_a} + \frac{4}{\alpha w}\left(\frac{P}{M} + \frac{Q}{N}\right)\frac{t_{data}}{t_a}} \end{aligned}$$

Conclusion

- For constant R, the speedup reaches an optimal value if MN becomes large
- If MN is fixed, the speedup will eventually degrade if R gets larger
- The speedup becomes better if $(P/M + Q/N)$ attains a minimum for a given problem size and a given number of processes

Optimal Process Topology

- For a given problem size MN and a given number of processes R, find P and $Q = R/P$ such that

$$\phi(P) = \left(\frac{P}{M} + \frac{Q}{N}\right)$$

  becomes minimal

- A simple calculation gives

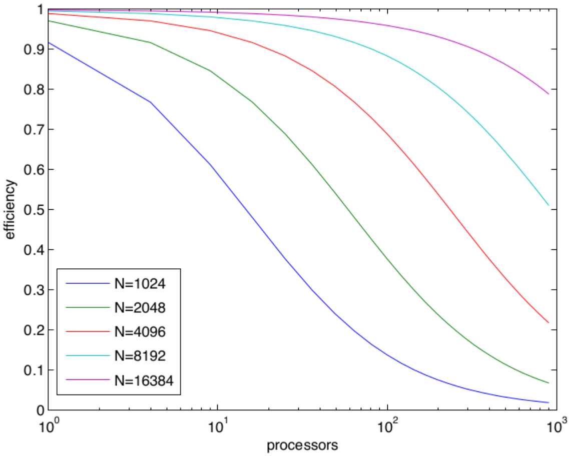
$$P = \sqrt{\frac{M}{N}R}$$

  (provided that these are integers)

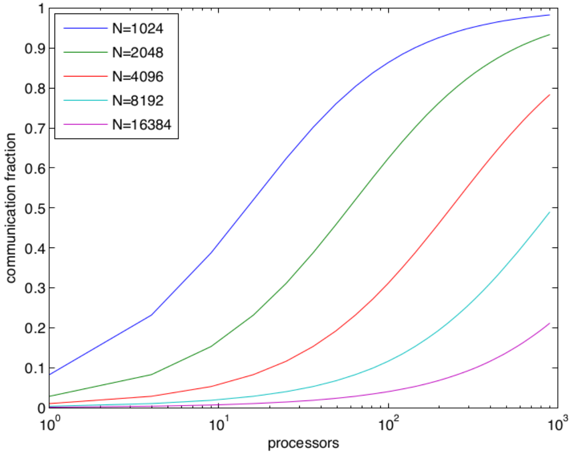
- In the case $M = N$ and R being a square, $P = \sqrt{R}$

Efficiency

For typical data on Lucidor (at PDC), this is the efficiency $E_R = S_R/R$



Communication Fraction



Surface to Volume Ratio

Observation:

- The computation time t_{comp} is proportional to the area $I_p \times J_q$ of the data
- The *communication time* t_{comm} is proportional to the **perimeter** $2(I_p + J_q)$

"Area-perimeter law"

The communication time is negligible if the number of data $M \times N$ is large compared to the number of processes

The Curse of Dimensionality

As we move to higher dimensional spaces, communication becomes relatively more costly,

- In 1D: $2/N$
- In 2D: $4N/N^2 = 4/N$
- In 3D: $6N^2/N^3 = 6/N$

Very important to find overlapping possibilities in 3D!

Virtual Topologies

Virtual Topologies

MPI includes a number of standard routines for defining and handling different process topologies. They are called **virtual topologies**. These routines lead to great simplification of the programming efforts needed

Jacobi Iteration

- **Basic idea**: Rewrite the equation as

$$u_{mn} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn}\right)$$

- For some starting guess (e.g., $u_{mn} = 0$), iterate this equation,

```
while ( not_done )
  for (m,n) in 1:N-2 x 1:N-2
    ut(m,n) =( u(m-1,n) + u(m+1,n)
             +  u(m,n-1) + u(m,n+1)
             -  h^2 f(m,n)          )/4;
  end
  u = ut;
end
```

Gauss-Seidel Iteration

Observation:

- The Jacobi iteration converges very slowly

$$u_{mn} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn}\right)$$

Idea:

- Use the new (better?) values as soon as they are available

Gauss-Seidel Iteration

```
while ( not_done )
  for (m,n) in 1:N-2 x 1:N-2
    u(m,n) =( u(m-1,n) + u(m+1,n)
            +  u(m,n-1) + u(m,n+1)
            -  h^2 f(m,n)          )/4;
  end
  % u = ut;
end
```

Observation: This iteration depends on the order of the unknown!

Lexicographic

Definition

The lexicographic order of the array u_{mn} is given by

$$u_{11}, u_{21}, u_{31}, \ldots, u_{M1}, u_{12}, u_{22}, \ldots, u_{MN}$$

Lexicographic order from the iteration is not safe

```
for n = 1:N
  for m = 1:M
    u(m,n) = ...
  end
end
```

Pipelined Computations

- Gauss-Seidel iterations are purely sequential
- Assume a $P \times Q$ process grid as before
- Process (p, q) cannot start computing before the values on processes $(p-1, q)$ and $(p, q-1)$ are available
- This leads to **pipelined computations**
- At every moment in time, only the processes along diagonals are active . . .

How to Parallelize?

Idea: Use red-black ordering!

- Black points: $m + n$ is even
- Red points : $m + n$ is odd
- Gauss-Seidel iteration

$$u_{mn} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn}\right)$$

- If u_{mn} is black, the values on the right hand side are all red and vice versa
- The "black sweep" and the "red sweep" can be parallelized independently
- Note: This is a different kind of iteration!

Final Remarks

- More efficient methods for solving Poisson's equation include multigrid methods (asymptotical optimal!)
- For a full 9-point stencil, four colours are needed
- Today, the most complex parallel circuit in a PC is the GPU (graphic processing unit)
- Not surprisingly, the GPU is used as a parallel solver unit even for PDEs
- MPI includes the possibility to define virtual topologies thus simplifying the design of the communication a lot

What did we learn?

- Evaluation of stencils for different purposes (image processing, solutions of partial differential equations)
- Data distributions, ghost points, practical aspects
- Efficient communication strategies
- Performance evaluation of the corresponding algorithms
- Pipelined computations (Gauss-Seidel iterations)
- Reformulation of recursive algorithms