Arrangements and Selections

Rule 1

In how many ways can you arrange the people Alice, Bill and Carol, in a line?

You have 3 choices for the first person in the line:

A B C

For each of these choices, you have 2 choices for the second person in the line:

AB AC BA BC CA CB

For each of these choices, you have 1 choice for the third person in the line:

ABC ACB BAC BCA CAB CBA

Answer: $3\times2\times1$

We can write $3\times2\times1$ as $3!$ (See Appendix 2 - Factorials)

In general:

In how many ways can you arrange n different items in a line?

Answer: $n!$

example

In how many ways can you arrange 10 people in a line?

Answer: $10!$

Rule 2

In how many ways can you arrange 3 people chosen from Alice, Bill, Carol, David, Eric, in a line?

You have 5 choices for the first person in the line.

For each of these choices, you have 4 choices for the second person in the line.

For each of these choices, you have 3 choice for the third person in the line.

Answer: $5\times4\times3$

We can write $5\times4\times3$ as $\frac{5\times4\times3\times2\times1}{2\times1}=\frac{5!}{2!}$

We write this as $(5P3)$ The P stands for permutation.

In general:

In how many ways can you arrange r items chosen from n different items, in a line?

Answer: $(nPr)=\frac{n!}{(n-r)!}$

example

In how many ways can you arrange 3 people in a line if there are 17 people to choose from?

Answer: $(17P3)=4080$

Rule 3

In how many ways can you select 3 people chosen from Alice, Bill, Carol, David and Eric?

You have 5 choices for the first person.

For each of these choices, you have 4 choices for the second person.

For each of these choices, you have 3 choice for the third person.

Answer: $5\times4\times3$ No! $5\times4\times3$ is the answer to the question:

In how many ways can you arrange 3 people chosen from Alice, Bill, Carol, David, Eric, in a line?

For each selection of 3 people there are $3!$ arrangements.

BCE is one selection.

BCE, BEC, CBE, CEB, EBC, ECB are the $3!$ possible arrangements.

The answer $5\times4\times3$ has counted each selection $3!$ times.

Answer: $\frac{5\times4\times3}{3!}$

We can write $\frac{5\times4\times3}{3!}$ as $\frac{5\times4\times3\times2\times1}{3!\times2\times1}=\frac{5!}{3!2!}$

We write this as $(5C3)$ The C stands for combination.

In general:

In how many ways can you select r items chosen from n different items?

Answer: $(nCr)=\frac{n!}{r!(n-r)!}$

example

In how many ways can you select 3 people from a group of 17 people?

Answer: $(17C3)=680$

Rule 4

You start each day with a cup of tea or a cup of coffee. In how many ways can you select your drinks for a week?

Note: if you choose tea on the first day then you are allowed to choose tea on the second day – so repetitions are allowed.

You have 2 choices of drink on the first day.

You have 2 choices of drink on the second day

…

You have 2 choices of drink on the seventh day.

Answer: $2^7=128$

In general:

In how many ways can you arrange r items chosen from n different items if repetitions are allowed?

Answer: n^r

example

In how many ways can you make a 4 letter word?

Answer: 26^4

of course, most of these won't be real words

Rule 5

A sweet shop sells 5 varieties of sweets. The varieties are called A, B, C, D, E.

For £1 you can buy any 12 sweets. In how many ways can you make your selection?

You can record a selection like this:

sweet	A	B	C	D	E
number	√ √	√ √ √	√ √		√ √ √ √ √

This selection is, two As, three Bs, two Cs, no Ds and five Es

You could record this selection as:

√ √ | √ √ √ | √ √ | | √ √ √ √ √

So the question becomes:

In how many ways can you arrange 12 √ symbols and 4 | symbols in a line?

There are 16 places in the line and we need to choose 12 of these places for the √ symbols.

Answer: $(16C12)$

In general:

In how many ways can you select r items chosen from n different items if repetitions are allowed?

Answer: $((n+r-1)Cr)$

example

A shop sells 6 varieties of bread rolls. In how many ways can I select 20 rolls?

Answer: $(25C20)$

Note: independent choices and multiplication

There are 5 men and 4 women in a room. In how many ways can you select 2 men and 3 women?

There are $(5C2)=10$ ways to select the men. There are $(4C3)=4$ ways to select the women.

For each of the 10 ways you can select the men there are 4 ways you can select the women. The choice of men is independent of the choice of women. Under these circumstances:

Answer: $(5C2)\times(4C3)$

Example 1

In how many ways can you arrange 7 boys and 3 girls in a line if:

a) the girls are at the front?

b) the girls stand next to each other?

c) no two girls stand next to each other?

Solutions

a) There are $3!$ ways to arrange the girls and there are $7!$ ways to arrange the boys.

Answer: $3!\times7!$

b) We have 8 items to arrange in a line. A block of girls and 7 boys. We can do this in $8!$ ways. But for each of these arrangements, we can shuffle the girls in $3!$ ways.

Answer: $8!\times3!$

c) We can arrange the boys in $7!$ ways. Then we add the girls. There are 8 places where we can put the first girl. At the front, at the back or between two boys. There are 7 places we can put the next girl and there are 6 places we can put the third girl.

Answer: $7!\times8\times7\times6$

Example 2

In how many ways can you arrange 10 Physics, 4 French and 7 Biology books in a line if books of the same subject must be kept together?

Solution

We can arrange the 3 subjects in $3!$ ways. We can then shuffle the Physics books in $10!$ ways, the French books in $4!$ ways and the Biology books in $7!$ ways.

Answer: $3!\times10!\times4!\times7!$

Example 3

There are 8 boys and 5 girls in a class. In how many ways can you arrange 4 boys and 3 girls in a line?

Solution

We can select the pupils in $(8C4)\times(5C3)$ ways.

Having selected the pupils, we can arrange them in $7!$ ways.

Answer: $(8C4)\times(5C3)\times7!$

Example 4

There are 10 boys and 12 girls in a class. In how many ways can you select 5 pupils if you must include at least one boy and at least one girl?

Solution

There are $(22C5)$ ways to select 5 pupils. But some selections are no good.

There are $(10C5)$ selections which are all boys and $(12C5)$ selections that are all girls.

Answer: $(22C5)-(10C5)-(12C5)$

Example 5

In how many ways can you arrange the letters A, A, A, B, B, B, B, B, in a line?

Solution

There are 8 places and we need to choose 3 of these places for the A's

Answer: $(8C3)$

Or

There are 8 places and we need to choose 5 of these places for the B's

Answer: $(8C5)$

Example 6

A pack of cards has 52 cards. Each card has a suit (spade, heart, diamond or club) and a rank (ace, two, three, … king). In a game of poker, a hand consists of 5 cards. How many hands are:

a) Straight-flush	5 consecutive cards all in the same suit
b) 4 of a kind	4 cards of one rank and 1 other card
c) Full-house	3 cards of one rank and 2 cards of another rank
d) Flush	5 non-consecutive cards all in the same suit
e) Straight	5 consecutive cards not all in the same suit
f) 3 of a kind	3 cards of one rank and 2 other cards of different ranks

Solutions

a) Straight-flush:

There are 10 ways you can get 5 consecutive cards:

ace, 2, 3, 4, 5

2, 3, 4, 5, 6

3, 4, 5, 6, 7

…

10, jack, queen, king, ace

There are 4 choices for the suit.

Answer: $10 \times 4 = 40$

b) 4 of a kind:

There are 13 choices for the rank of the 4 cards and 48 choices for the 1 other card.

Answer: $13 \times 48 = 624$

c) Full-house:

There are 13 choices for the rank of the 3 cards and $(4C3)$ choices for the 3 cards of that rank.

There are 12 choices for the rank of the 2 cards and $(4C2)$ choices for the 2 cards of that rank.

Answer: $13 \times (4C3) \times 12 \times (4C2)$

d) Flush:

There are 4 choices for the suit of the 5 cards and $(13C5)$ choices for the 5 cards of that suit.

But we have included the 40 straight-flushes.

Answer: $4 \times (13C5) - 40$

e) Straight:

There are 10 ways you can get 5 consecutive cards:

For each card in the straight there are 4 choices for its suit.

But we have included the 40 straight-flushes.

Answer: $10 \times 4 \times 4 \times 4 \times 4 \times 4 - 40$

f) 3 of a kind:

There are 13 choices for the rank of the 3 cards and $(4C3)$ choices for the 3 cards of that rank.

There are $(12C2)$ ways to choose the ranks of the other 2 cards.

For each of the other 2 cards there are 4 choices for their suits

Answer: $13 \times (4C3) \times (12C2) \times 4 \times 4$

Example 7

There are 20 people at a party. Everyone shakes hands once with every-one else. Have many hand-shakes take place?

It can be fun to think of different ways to answer the same question.

Solution 1

There are 20 people and everyone has 19 hand-shakes.

Answer: 20×19 No! We have counted every handshake twice. Can you see why?

Answer: $\frac{20 \times 19}{2}$

Solution 2

Alice does 19 hand-shakes and then goes home.

Bill then does 18 hand-shakes and goes home.

Jane then does 17 hand-shakes and goes home.

etc

Answer: $19+18+17+\ldots+1$

Solution 3

There are $(20C2)$ ways to select a pair of people. For every pair of people there is a hand-shake.

Answer: $(20C2)$

see EXERCISE

EXERCISE

1) I have 12 pens, 8 pencils and 4 crayons. In how many ways can you select one of each?

2) On a restaurant menu there are 3 choices for the first course, 10 choices for the second course and 5 choices for the third course. In how many ways can you select a meal?

3) Each day I buy a coffee, a tea or a beer at my local cafe. In how many ways can I select my drinks for a week?

4) In how many ways can you arrange the letters A, B, C, D, E?

5) In how many ways can you arrange 20 people in a line?

6) In how many ways can you arrange 3 of the letters A, B, C, D, E, F, G, H?

7) In how many ways can I arrange 7 ornaments in a line if I have18 ornaments to choose from?

8) In how many ways can you arrange the digits 1, 2, 3, 4, 5, 6, 7 to form an odd number?

9) There are 10 runners in a race. In how many ways can the gold, silver and bronze medals be awarded if there are no dead-heats?

10) In how many ways can you select 3 of the letters A, B, C, D, E, F, G, H?

11) In how many ways can you select 13 cards from a pack of 52 cards?

12) In a lottery, you have to select 6 numbers from 1, 2, 3, …, 49

In how many ways can you select your lottery numbers?

13) In a class of 25 pupils, everyone shakes hands exactly once with everyone-else. How many hand-shakes take place?

14) 10 points are marked around a circle. A line is drawn between every pair of points. How many lines will be drawn?

15) In how many ways can you split a class of 28 pupils into a group of 20 and a group of 8?

16) In how many ways can you split a class of 28 pupils into a group of 18, a group of 6 and a group of 4?

17) In how many ways can you split a class of $(x+y+z)$ pupils into a group of x pupils, a group of y pupils and a group of z pupils if x and y and z are different numbers?

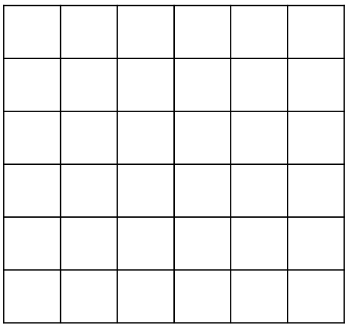
18) There are 16 boys and 18 girls in a class. In how many ways can you select 6 boys and 9 girls?

19) I have 30 letters and 30 envelopes. In how many ways can I place one letter in each envelope?

20) In how many ways can you select some (or none) people from a group of 7 people?

21) How many hands of 13 cards have 4 spades, 4 hearts, 4 diamonds and 1 club?

22) Here is the street plan of a city. The lines represent the streets.



All roads run north-south or east-west. I want to walk from the bottom left-hand corner to the top right-hand corner. I only want to walk north or east. In how many ways can I select my route?

23) Brag

In a game of Brag, a hand consists of 3 cards. How many hands are:

a) 3 of a kind	3 cards of one rank
b) Straight-flush	3 consecutive cards all in the same suit
c) Straight	3 consecutive cards not all in the same suit suit
d) Flush	3 non-consecutive cards all in the same suit
e) 2 of a kind	2 cards of one rank and 1 other card

SOLUTIONS

1) Answer: $12\times 8\times 4$

2) Answer: $3\times 10\times 5$

3) Answer: $3\times 3\times 3\times 3\times 3\times 3\times 3$

4) Answer: $5!$

5) Answer: $20!$

6) Answer: $(8P3)$

7) Answer: $(18P7)$

8) There are 4 choices for the last digit because the last digit must be odd. The other 6 digits can be arranged in $6!$ ways.

Answer: $4\times 6!$

9) We want to arrange 3 of the 10 runners on the podium.

Answer: $(10P3)$

10) Answer: $(8C3)$

11) Answer: $(52C13)$

12) Answer: $(49C6)$

13) For each way we can choose two pupils there is a hand-shake.

Answer: $(25C2)$

14) For each way we can choose two points there is a line.

Answer: $(10C2)$

15) Once you have chosen 20 pupils from 28 pupils there is nothing-else to do.

Answer: $(28C20)$

16) Once you have chosen 18 pupils from 28 pupils and then chosen 6 pupils from the remaining 10 pupils there is nothing-else to do.

Answer: $(28C18)\times(10C6)$

17) Once you have chosen x pupils from (x+y+z)pupils and then chosen y pupils from the remaining(y+z)pupils there is nothing-else to do.

Answer: $((x+y+z)Cx)\times((y+z)Cy)$

This answer simplifies to: $\frac{(x+y+z)!}{x!y!z!}$

18) Answer: $(16C6)\times(18C9)$

19) Answer: $30!$

20) For each person you have a choice of 2 options – select that person or don’t select that person.

Answer: $2\times2\times2\times2\times2\times2\times2$

21) You have to choose 4 of the 13 spades and …

Answer: $(13C4)\times(13C4)\times(13C4)\times(13C1)$

22) I need to walk 6 blocks north and 6 blocks east.

A possible route is: N, N, E, E, E, N, E, E, N, N, N, E

There are 12 letters in the line and we need to choose 6 positions for the N letters.

Answer: $12C6$

23)

a) 3 of a kind:

There are 13 choices for the rank of the 3 cards and $(4C3)$ choices for the 3 cards of that rank.

Answer: $13\times(4C3)$

b) Straight-flush:

There are 12 ways you can get 3 consecutive cards:

ace, 2, 3

2, 3, 4

3, 4, 5

…

queen, king, ace

There are 4 choices for the suit.

Answer: $12 \times 4 = 48$

c) Straight:

There are 12 ways you can get 3 consecutive cards:

For each card in the straight there are 4 choices for its suit.

But we have included the 48 straight-flushes.

Answer: $12 \times 4 \times 4 \times 4 - 48$

d) Flush:

There are 4 choices for the suit of the 3 cards and $(13C3)$ choices for the 3 cards of that suit.

But we have included the 48 straight-flushes.

Answer: $4 \times (13C3) - 48$

e) 2 of a kind:

There are 13 choices for the rank of the 2 cards and $(4C2)$ choices for the 2 cards of that rank.

There are 48 choices for the 1 other card.

Answer: $13 \times (4C2) \times 48$