Knight Tours

Example 1

Here is a knight tour on a 6×6 board:

1	28	15	12	3	34
16	11	2	35	22	13
27	36	29	14	33	4
10	17	8	23	30	21
7	26	19	32	5	24
18	9	6	25	20	31

The knight must visit each square once (and only once). The squares are numbered in the order they are visited by the knight. The knight starts on square 1, moves to square 2, moves to square 3, ... and ends on square 36.

This is a closed tour because the end square (36) is a knight move from the start square (1)

Any tour can be reversed. The knight could move: 36, 35, 34, … 3, 2, 1

We can think of a closed tour as starting on any square. We can think of the above tour as starting on square 17. The knight could move: 17, 18, 19, … 36, 1, 2, 3, … 16

Example 2

Here is another knight tour:

1	20	25	8	3	18
26	9	2	19	32	7
21	24	33	6	17	4
10	27	22	35	14	31
23	34	29	12	5	16
28	11	36	15	30	13

This is an open tour because the end square (36) is not a knight move from the start square (1)

This tour could be reversed. This tour could not start on any square.

Theorem

There is no closed tour on any $N \times N$ board where N is odd.

Proof

We can colour the squares on a board, black and white. Like a chess board. A knight on a black square can only move to a white square. A knight on a white square can only move to a black square. If the start square of a closed tour is black then the end square must be white, because the end square is a knight move from the start square. So there must be the same number of black and white squares. This won't happen on an $N \times N$ board if N is odd.

Theorem

There are no closed tours on a 2×2 board.

Proof

Just think about it.

Theorem

There are no closed tours on a $4 x 4$ board.

Proof

We can colour the board, black and white.

W	B	W	B
B	W	B	W
W	B	W	B
B	W	B	W

The knight must alternate between black and white squares.

We can colour the board, red and green.

R	R	R	R
G	G	G	G
G	G	G	G
R	R	R	R

A knight on a red square can only move to a green square. We can think of a closed tour as starting on any square. If the start square is red then the end square must be green, because the end square is a knight move from the start square. The knight must visit the same number of red and green squares. So the knight must alternate between red and green squares.

If the knight starts on the bottom left hand corner square, which is a black/red square then it can only move to a white/green square then to a black/red square etc. It isn't ever going to visit the black/green squares or the white/red squares. So there is no closed tour on a $4 x 4$ board.

Schwenk's Closed Tour Theorem

a) There is no closed tour on any $N \times N$ board where N is odd.

b) There is a closed tour on every $N \times N$ board where N is even and greater than 4.

Have we proved Schwenk's theorem?

We have proven part (a)

What about part (b)?

Schwenk says: if N is even and greater than 4 then there is a closed tour

We have proved: if N is even and isn't greater than 4 then there isn't a closed tour

Not the same thing at all.

Example 3

Here is a tour on a 5×5 board:

3	10	21	16	5
20	15	4	11	22
9	2		6	17
14	19	8	23	12
1	24	13	18	7

We can finish by going into the middle square.

Example 4

Here is a tour on a 9×9 board: CHECK THIS

5	20	47	34	7	22	49	36	9
46	33	6	21	48	35	8	23	50
19	4						10	37
32	45						51	24
3	18						38	11
44	31						25	52
17	2						12	39
30	42	16	55	28	41	14	53	26

1	56	29	42	15	54	27	40	13

We can finish by going into the middle 5×5 square.

Try this method on a 13×13 board.

See EXERCISE 1

Example 5

Here is a tour on a chess-board:

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

Amusingly, the numbers in each column and each row add up to the same total.

We call this a semi-magic square.

There are many methods to find knight tours. Look them up.

See Exercise

EXERCISE

1)

I am looking for a tour on this board:

		A		

Why will I fail if I start at square A?

2)

Look at this 6×6 board:

B	D	A	C	B	D
A	C	B	D	A	C
D	B			D	B
C	A			C	A
B	D	A	C	B	D
A	C	B	D	A	C

The knight can tour all the A squares. The knight can tour all the B squares, etc.

Can you find a tour by linking up these cycles, using the middle squares?

3)

Look at this 8×8 board:

D	B	C	A	D	B	C	A
C	A	D	B	C	A	D	B
B	D	A	C	B	D	A	C
A	C	B	D	A	C	B	D
D	B	C	A	D	B	C	A
C	A	D	B	C	A	D	B
B	D	A	C	B	D	A	C
A	C	B	D	A	C	B	D

The knight can tour all the A squares. The knight can tour all the B squares, etc.

Can you find a tour by linking up these cycles?

4)

Divide and conquer methods involve dividing the board into parts and touring each part separately.

Here is a 10×10 board:

5	16	21	12	7					
22	11	6	15	20					

17	4	13	8	25					
10	23	2	19	14					
3	18	9	24	1					

We divide the board into four 5×5 quarter boards and tour each quarter board separately.
A tour of the top left hand corner quarter board is shown.
Rotate this $90°$ clockwise and place in the top right hand corner. Rotate again for the bottom right hand corner. Rotate again for the bottom left hand corner. Now join up these four quarter board tours. Unfortunately, we cannot use this method on an 8×8 board because there are no tours of a 4×4 board.

SOLUTIONS

1)

Let's colour the board black and white, with the bottom left-hand corner black. There are 13 black squares and 12 white squares. The knight must alternate between black and white squares. So this will only work if the tour goes B, W, B, W, … B. Unfortunately, the start square is white.

2)

16	23	4	31	10	25
3	30	17	24	5	32
22	15	36	9	26	11
29	2	27	18	33	6
14	21	8	35	12	19
1	28	13	20	7	34

3)

38	55	22	13	36	51	18	11
23	14	37	54	17	12	35	50
56	39	16	21	52	33	10	19
15	24	53	40	9	20	49	34
42	57	28	1	48	61	32	7

25	2	41	60	29	8	47	62
58	43	4	27	64	45	6	31
3	26	59	44	5	30	63	46

4)

5	16	21	12	7	28	35	42	47	30
22	11	6	15	20	43	49	29	36	41
17	4	13	8	25	34	27	38	31	46
10	23	2	19	14	49	44	33	40	37
3	18	9	24	1	26	39	50	45	32
82	95	100	89	76	51	74	59	67	53
87	90	83	94	99	64	69	52	73	60
96	81	88	77	84	75	58	63	54	67
91	86	79	98	93	70	65	56	61	72
80	97	92	85	78	57	62	71	66	55