Rationals and Irrationals

$\dfrac{13}{7}$ is a rational number because it is an integer divided by another integer.

Theorem

x is a rational number if and only if x is a terminating or recurring decimal.

This is really two theorems:

Theorem 1

If x is a rational number then x is a terminating or recurring decimal.

Proof

Say $x = \dfrac{13}{7}$

$\dfrac{13}{7} = 1 + \dfrac{6}{7}$ $\qquad\qquad$ $\dfrac{13}{7}$ equals 1 remainder 6

$\dfrac{6}{7} = \dfrac{1}{10}\left(\dfrac{60}{7}\right) = \dfrac{1}{10}\left(8 + \dfrac{4}{7}\right)$ $\qquad$ $\dfrac{60}{7}$ equals 8 remainder 4

$\dfrac{4}{7} = \dfrac{1}{10}\left(\dfrac{40}{7}\right) = \dfrac{1}{10}\left(5 + \dfrac{5}{7}\right)$ $\qquad$ $\dfrac{40}{7}$ equals 5 remainder 5

$\dfrac{5}{7} = \dfrac{1}{10}\left(\dfrac{50}{7}\right) = \dfrac{1}{10}\left(7 + \dfrac{1}{7}\right)$ $\qquad$ $\dfrac{50}{7}$ equals 7 remainder 1

etc

So $\dfrac{13}{7} = 1 + \dfrac{8}{10} + \dfrac{5}{100} + \dfrac{7}{1000} + ... = 1.857...$

The remainders can only be $0, 1, 2, 3, 4, 5, 6$

Either we will get a remainder of 0, in which case the decimal terminates.

Or we will get a remainder we have had before, in which case the decimal recurs.

Either way x is a terminating decimal or a recurring decimal

Theorem 2

If x is a terminating or recurring decimal then x is a rational number.

Proof

If:

x is a terminating decimal, say $x = 0.123$

then:

$x = \dfrac{123}{1000}$ so x is a rational number.

If:

x is a recurring decimal, say $x = 0.123123123\ldots$

then:

$1000\,x = 123.123123123\ldots$ so $999\,x = 123$ so $x = \dfrac{123}{999}$ so x is a rational number.

Either way x is a rational number.