Introduction

This book is aimed at students studying A level or IB mathematics in their last year or two at school, teachers looking for ideas to stretch their bright students and anyone who enjoys mathematics and is looking for something interesting to read.

This is not a text book. It will not help you pass exams. The topics covered are not usually taught at school. But it will show you some of the breadth of mathematics from the first chapter where we see the difficulties in designing voting systems to the last chapter where we see that some infinite numbers are bigger than others.

Each chapter is short. It is just a starting point. If you find a chapter interesting then an internet search will give you lots more information about that topic. See Appendix 3.

You will need to think hard when reading this book. There are exercises (with solutions) for you to do. You will need a pencil, some paper and a waste-paper-basket.

The chapters cover different topics and you don’t need to read them in order. Roughly speaking, the easier chapters are nearer the front of the book. If there is something you don’t understand then just skip over it.

A Nice Proof

Theorem

$$1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$$

Proof

There are 64 competitors in a knock-out tennis tournament. How many matches will there be during this tournament?

Method 1

First round	32 matches	Second round	16 matches	Third round	8 matches
Fourth round	4 matches	Fifth round	2 matches	Sixth round	1 match

Answer: $1+2+4+8+16+32$

Method 2

Each match knocks-out one competitor. By the end, 63 competitors have been knocked-out

Answer: 63

Comparing our answers we have: $1+2+4+8+16+32=63$

In general:

$$1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$$

A useful result and nothing to do with tennis.

A Nice Sum

E	D	C	B	A
D	E	D	C	B
C	D	E	D	C
B	C	D	E	D
A	B	C	D	E

In the above table we have:

1 letter A, two letter B, three letter C, four letter D, five letter E, four letter D, three letter C, two letter B and one letter A.

How many letters have we got?

We have got $1+2+3+4+5+4+3+2+1$ letters. But we have got 5^2 letters.

So $1+2+3+4+5+4+3+2+1=5^2$

In general:

$1+2+3+...+n+...+3+2+1=n^2$

Now let's add up all the numbers in this table:

1×1	1×2	1×3	1×4
2×1	2×2	2×3	2×4
3×1	3×2	3×3	3×4
4×1	4×2	4×3	4×4

First method:

The numbers in the table add up to $(1+2+3+4)^2$ (check by multiplying out the brackets)

Second method:

W	X	Y	Z
X	X	Y	Z
Y	Y	Y	Z
Z	Z	Z	Z

The number in the W cell:

$$1\times1=1$$

The numbers in the X cells:

$$(2\times1)+(2\times2)+(1\times2)=2(1+2+1)=2(2^2)=2^3$$

The numbers in the Y cells:

$$(3\times1)+(3\times2)+(3\times3)+(2\times3)+(1\times3)=3(1+2+3+2+1)=3(3^2)=3^3$$

The numbers in the Z cells:

$$(4\times1)+(4\times2)+(4\times3)+(4\times4)+(3\times4)+(2\times4)+(1\times4)=4(1+2+3+4+3+2+1)=4(4^2)=4^3$$

So the numbers in the table add up to $1^3+2^3+3^3+4^3$

So comparing our results using the first method and the second method:

$$(1+2+3+4)^2=1^3+2^3+3^3+4^3$$

In general:

$$(1+2+3+\ldots+n)^2=1^3+2^3+3^3+\ldots+n^3$$

Appendix 1

A Few Short Programs – written in Python

1) Print the first 10 Fibonacci numbers

Program:

```
x=1

y=1

for j in range(1,9):

    z=x+y

    print(z)

    x=y

    y=z
```

Notes:

a) for j in range(1,9):

This means

Put j=1 and perform the indented statements

Put j=2 and perform the indented statements

…

Put j=8 and perform the indented statements

Confusingly, for j in range(1,9) means j=1, 2, … 8 but not j=9

b) x=y

This means put x equal to the current value of y. So, for example

x=3

y=7

x=y

We now have x=7 and y=7

y=x This means put y equal to the current value of x. So, for example

x=3

y=7

y=x

We now have x=3 and y=3

So x=y and y=x do not mean the same thing!

c) We can run this program by keeping track of the values of j, z, x, y and see what gets printed.

j	z	print	x	y
			1	1
1	2	2	1	2
2	3	3	2	3
3	5	5	3	5
4	8	8	5	8
5	13	13	8	13
...				

This program will print the 2nd, 3rd, 4th, … 10th Fibonacci numbers which is almost what we wanted!

2) Use Euclid’s algorithm to find the highest common factor of 3458 and 651

Program:

```
r=1
a=3458
b=651
while r!=0:
    r=a%b
    a=b
    b=r
print(a)
```

Notes:

a) while r!=0:

This means while r does not equal 0

Perform the indented statements

Perform the indented statements

… until r=0

Then continue with the rest of the program

When we start this while loop r has to have a value so we gave it a value in line 1

b) $r = a\%b$

This means put r equal to the remainder when a is divided by b So, for example

a=17

b=5

$r = a\%b$

We now have $r = 2$

c) We can run this program

r	a	b	print
1	3458	651	
203	651	203	
42	203	42	
35	42	35	
7	35	7	
0	7	0	7

d) We can change lines 2 and 3 to find the highest common factor of any two positive integers.

3) Test if 28 a perfect number

Program:

```
n=28
c=0
for j in range(1,n):
    if n%j==0:
        c=c+j
if c==n:
    print("perfect")
else:
    print("non-perfect")
```

Notes:

a) if n%j==0:

This means if the remainder when n is divided by j is zero

In other words, if j is a factor of n

We have to use == and not = in an if statement.

b) c=c+j

This means c adds up all the factors of n

c) if c==n:

This means if the sum of the factors of n (including 1 but excluding n) is equal to n

In other words, if n is a perfect number

4) Test if 17 is a prime number

Program:

```
n=17
prime=1
for j in range(2,n):
    if n%j==0:
        prime=0
        break
print(p)
```

Notes:

a) Once we have run the program, prime=1 if m is a prime number and prime=0 if m is not a prime number.

b) if n%j==0:

This means if j is a factor of n

In other words, if n is not a prime number.

c) break

This means jump out of the for loop and continue with the rest of the program

d) We can make this program part of a larger program to print out all the prime numbers less than, say 100

```
for n in range(3,100):
    prime=1
    for j in range(2,n):
        if n%j==0:
            prime=0
            break
    if prime==1:
        print(n)
```

5) Test if 21 is the sum of two squares

Program:

```
N=21
flag=0
for j in range(0,N+1):
    for k in range(j,N+1):
        if N==j*j+k*k:
            flag=1

            break
print(flag)
```

Notes:

a) We set flag=0 at the start. We only enact flag=1 if N is the sum of two squares.

b) print(flag)

This means we print 1 if N is the sum of two squares. Otherwise we print 0.

c) try running this program by hand.

6) Calculate $f(17)$ where $f(x)=2x+7$

Program:

```
def f(x):
    return (2*x)+7
y=f(17)
print(y)
```

Notes:

Lines 1 and 2 define the function f(x). We can put this at the start of the program and then refer to it later in the program.

7) Calculate u_{12} in the sequence $u_1=3$ $u_{n+1}=2u_n+3n$

Program:

```
def u(n):
    if n==1:
        return 3
    else:
        return 2*u(n-1)+3(n-1)
t=u(12)
print(t)
```

Notes:

a) Line 3 tells us that $u(1)=3$

Line 5 tells us that $u(n)=2u(n-1)+3(n-1)$

$u_{n+1}=2u_n+3n$ and $u_n=2u_{n-1}+3(n-1)$ mean the same thing.

For example, if we put n=7 in the first version we get $u_8=2u_7+21$ and if we put n=8 in the second version we get $u_8=2u_7+21$

8) Calculate the 10th derangement number using the recurrence relation

Remember $D_1=0$ and $D_2=1$ and $D_{n+2}=(n+1)D_n+(n+1)D_{n+1}$

Program:

```
def D(n):
    if n==1:
        return 0
    if n==2:
        return 1
    else:
        return (n-1)*D(n-2)+(n-1)*D(n-1)
h=D(10)
print(h)
```

9) Toss a coin 5 times and count the number of heads and tails

Program:

```
import random
h=0
t=0
for j in range(1,6):
    c=random.randint(0,1)
    if c==0:
        h=h+1
    else:
        t=t+1
print(h,t)
```

Notes:

a) import random

This means the program can generate random numbers

b) c=random.randint(0,1)

This means c is randomly set to 0 or 1.

Think of c=0 as tossing a coin and getting a head and c=1 as getting a tail.

10) Play the game Chuck – a – Luck a million times and find the average winnings per game

Program:

```
import random
T=0
for j in range(1,1000001):
    D=0
    dice1=random.randint(1,6)
    dice2=random.randint(1,6)
    dice3=random.randint(1,6)
    if dice1==6:
        D=D+1
    if dice2==6:
        D=D+1
    if dice3==6:
        D=D+1
    if D==0:
        W=-1
    else:
        W=D
    T=T+W
print(T/1000000)
```

Notes:

a) dice 1 is the score on the first dice, etc

b) D is the total number of sixes on the three dice in one game

c) W is how much I win in one game.

d) T is my total winnings in 1,000,000 games

e) If you work it out exactly, you will find that:

my average winnings, per game, in the long run, is $£-17/216$

Most(?) people are surprised by this as they would guess my average winnings, per game, in the long run would be positive. This, therefore gives you the opportunity, to extort money from people who have not read this book!

11) Estimate $\frac{\pi}{4}$ using random numbers see chapter: Pi CHECK

```
import random

N=1000000

T=0

for j in range(1,N+1):

   x=random.uniform(0,1)

   y=random.uniform(0,1)

   if x*x+y*y<1:

      T=T+1

print(T/N)
```

Notes:

a) N is the number of points we will generate in the unit square.

b) T is the number of these points that are inside the quarter circle.

c) x=random.uniform(0,1)

This means x is a random number between 0 and 1

d) x*x+y*y<1

This is true if the point (x, y) is inside the quarter circle.

NOTE

To run these programs, go online and find an online IDE. Choose Python as the language, type in the program and run it.

This is not a course in programming. These examples are here to entice you into learning some programming if you have not done any before. These programs could be improved. For example, in program (4) we do not need the range of the for loop to be (2,m) Why not?

Program (4) Test if 17 is a prime number. We can easily adapt this program to test any positive integer to see if it is a prime number. Or we could print out a list of the first 100 prime numbers. Or ...

Appendix 2

1) Arithmetic Sequences

Here is an arithmetic sequence: $7, 10, 13, 16, 19, 22, 25, \dots$

The first term is 7 and the difference between consecutive terms is 3

$u_1=7 \quad u_2=7+(3)=10 \quad u_3=7+(2\times3)=13 \quad u_4=7+(3\times3)=16 \quad \dots \quad u_n=7+(n-1)3$

In general:

Arithmetic sequence: $a+(a+d)+(a+2d)+(a+3d)+(a+4d)+\dots$

The nth term is $u_n=a+(n-1)d$

2) Geometric Sequences

Here is a geometric sequence:

$2, 6, 18, 54, 162, 486, 1458, \dots$

The first term is 2 and the ratio of consecutive terms is 3

$u_1=2 \quad u_2=2\times(3)=6 \quad u_3=2\times(3^2)=18 \quad u_4=2\times(3^3)=54 \quad \dots \quad u_n=2(3^{n-1})$

In general:

Geometric sequence:

$a, ar, ar^2, ar^3, ar^4, \dots$

The nth term is:

(ar^{n-1})

Summing an infinite geometric series:

$S=a+ar+ar^2+ar^3+ar^4+\dots$

So:

$rS=ar+ar^2+ar^3+ar^4+ar^5+\dots$

So:

$S-rS=(a+ar+ar^2+ar^3+ar^4+\dots)-(ar+ar^2+ar^3+ar^4+ar^5+..)=a$

So:

$S(1-r)=a$

So:

$S=\frac{a}{1-r}$ this result is only valid if $-1<r<1$

3) Indices

examples	in general
$3^2\times3^4=(3\times3)\times(3\times3\times3\times3)=3^6$	$(x^m)(x^n)=x^{m+n}$
$\frac{3^6}{3^2}=\frac{3\times3\times3\times3\times3\times3}{3\times3}=3^4$	$\frac{x^m}{x^n}=x^{m-n}$
$\frac{3^6}{3^5}=\frac{3\times3\times3\times3\times3\times3}{3\times3\times3\times3\times3}=3$ but $\frac{3^6}{3^5}=3^1$	$x^1=x$
$\frac{3^6}{3^6}=\frac{3\times3\times3\times3\times3\times3}{3\times3\times3\times3\times3\times3}=1$ but $\frac{3^6}{3^6}=3^0$	$x^0=1$
$(3^4)^2=(3\times3\times3\times3)\times(3\times3\times3\times3)=3^8$	$(x^m)^n=x^{mn}$
$3^{-4}\times3^4=3^0=1$ so $3^{-4}=\frac{1}{3^4}$	$x^{-m}=\frac{1}{x^m}$
$3^{1/2}\times3^{1/2}=3^1=3$ so $3^{1/2}=\sqrt{3}$	$x^{1/2}=\sqrt{x}$
$3^{\frac{5}{2}}=\left(3^{\frac{1}{2}}\right)^5$	$x^{\frac{m}{n}}=\left(x^{\frac{1}{n}}\right)^m$

4) Logarithms

If $125=5^3$ then $\log_5 125=3$

In general:

If $c=a^b$ then $\log_a c=b$

Now

$8=2^3$ so $\log_2 8=3$ and $32=2^5$ so $\log_2 32=5$

examples	in general
$8\times32=2^{3+5}$ so $\log_2(8\times32)=\log_2 8+\log_2 32$	$\log_a(xy)=\log_a(x)+\log_a(y)$

$\frac{32}{8} = 2^{5-3}$ so $\log_2\left(\frac{32}{8}\right) = \log_2 32 - \log_2 8$ $\qquad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

$32^4 = (2^5)^4 = 2^{4\times5}$ so $\log_2(32^4) = 4\log_2 32$ $\qquad \log_a(x^n) = n\log_a(x)$

5) Factor theorem

example

$f(x) = x^2 - 5x + 6$

So:

$f(2) = 2^2 - (5\times2) + 6 = 0$

The factor theorem tells us that if $f(2)=0$ then $(x-2)$ is a factor of $f(x)$

So:

$f(x) = (x-2)(\ldots)$

In general:

If $f(a)=0$ then $(x-a)$ is a factor of $f(x)$

6) Factorials

$1! = 1$

$2! = 1\times2$

$3! = 1\times2\times3$

$4! = 1\times2\times3\times4$ etc

7) Binomial theorem for multiplying out brackets

$(1+x)^1 = 1+x$

$(1+x)^2 = 1+2x+x^2$

$(1+x)^3 = 1+3x+3x^2+x^3$

$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$

In general: if n is a positive integer

$(1+x)^n = (nC0) + (nC1)x + (nC2)x^2 + (nC3)x^3 + \ldots + (nCn)x^n$

Appendix 3 – Where to find out more.

1) Online resources

a) A good place to start is the Plus Magazine website. Here you will find lots of interesting articles, written at roughly the same level as this book.

b) If you are interested in the history of mathematics, then I would recommend the MacTutor website.

Biographies - Alphabetical Index	this will take you to articles about mathematicians.
History Topics - Alphabetical List	this will take you to articles about mathematical topics.

c) Search YouTube for:

Numberfile

Mathologer

d) Online searches

Penrose tiles

M. C. Escher

2) Books

Number Theory: A Very Short Introduction	R. Wilson
Combinatorics: A Very Short Introduction	R. Wilson
Mathematics: A Very Short Introduction	T. Gowers
Cryptography: A Very Short Introduction	F. Piper, S. Murphy
The Code Book	S. Singh
The Annotated Alice	M. Gardner
Lewis Carroll in Numberland	R. Wilson
How Music Works	J. Powell
Golden Ratio	M. Livio
The Code Book	S. Singh
Enigma, The Battle For The Code	H. Sebag-Montefiore

Taking Chances	J. Haigh
The Drunkard’s Walk	L. Mlodinow
Euler’s Gem	D. S. Richeson
The Annotated Alice	M. Gardner
Logic And Its Limits	P. Shaw

Arrangements and Selections

Rule 1

In how many ways can you arrange the people Alice, Bill and Carol, in a line?

You have 3 choices for the first person in the line:

A	B	C

For each of these choices, you have 2 choices for the second person in the line:

AB	AC	BA	BC	CA	CB

For each of these choices, you have 1 choice for the third person in the line:

ABC	ACB	BAC	BCA	CAB	CBA

Answer: $3\times2\times1$

We can write $3\times2\times1$ as $3!$ (See Appendix 2 - Factorials)

In general:

In how many ways can you arrange n different items in a line?

Answer: $n!$

example

In how many ways can you arrange 10 people in a line?

Answer: $10!$

Rule 2

In how many ways can you arrange 3 people chosen from Alice, Bill, Carol, David, Eric, in a line?

You have 5 choices for the first person in the line.

For each of these choices, you have 4 choices for the second person in the line.

For each of these choices, you have 3 choice for the third person in the line.

Answer: $5\times4\times3$

We can write $5\times4\times3$ as $\frac{5\times4\times3\times2\times1}{2\times1}=\frac{5!}{2!}$

We write this as $(5P3)$ The P stands for permutation.

In general:

In how many ways can you arrange r items chosen from n different items, in a line?

Answer: $(nPr)=\frac{n!}{(n-r)!}$

example

In how many ways can you arrange 3 people in a line if there are 17 people to choose from?

Answer: $(17P3)=4080$

Rule 3

In how many ways can you select 3 people chosen from Alice, Bill, Carol, David and Eric?

You have 5 choices for the first person.

For each of these choices, you have 4 choices for the second person.

For each of these choices, you have 3 choice for the third person.

Answer: $5\times4\times3$ No! $5\times4\times3$ is the answer to the question:

In how many ways can you arrange 3 people chosen from Alice, Bill, Carol, David, Eric, in a line?

For each selection of 3 people there are $3!$ arrangements.

BCE is one selection.

BCE, BEC, CBE, CEB, EBC, ECB are the $3!$ possible arrangements.

The answer $5\times4\times3$ has counted each selection $3!$ times.

Answer: $\frac{5\times4\times3}{3!}$

We can write $\frac{5\times4\times3}{3!}$ as $\frac{5\times4\times3\times2\times1}{3!\times2\times1}=\frac{5!}{3!2!}$

We write this as $(5C3)$ The C stands for combination.

In general:

In how many ways can you select r items chosen from n different items?

Answer: $(nCr)=\frac{n!}{r!(n-r)!}$

example

In how many ways can you select 3 people from a group of 17 people?

Answer: $(17C3)=680$

Rule 4

You start each day with a cup of tea or a cup of coffee. In how many ways can you select your drinks for a week?

Note: if you choose tea on the first day then you are allowed to choose tea on the second day – so repetitions are allowed.

You have 2 choices of drink on the first day.

You have 2 choices of drink on the second day

…

You have 2 choices of drink on the seventh day.

Answer: $2^7=128$

In general:

In how many ways can you arrange r items chosen from n different items if repetitions are allowed?

Answer: n^r

example

In how many ways can you make a 4 letter word?

Answer: 26^4

of course, most of these won't be real words

Rule 5

A sweet shop sells 5 varieties of sweets. The varieties are called A, B, C, D, E.

For £1 you can buy any 12 sweets. In how many ways can you make your selection?

You can record a selection like this:

sweet	A	B	C	D	E
number	√ √	√ √ √	√ √		√ √ √ √ √

This selection is, two As, three Bs, two Cs, no Ds and five Es

You could record this selection as:

√ √ | √ √ √ | √ √ | | √ √ √ √ √

So the question becomes:

In how many ways can you arrange 12 √ symbols and 4 | symbols in a line?

There are 16 places in the line and we need to choose 12 of these places for the √ symbols.

Answer: $(16C12)$

In general:

In how many ways can you select r items chosen from n different items if repetitions are allowed?

Answer: $((n+r-1)Cr)$

example

A shop sells 6 varieties of bread rolls. In how many ways can I select 20 rolls?

Answer: $(25C20)$

Note: independent choices and multiplication

There are 5 men and 4 women in a room. In how many ways can you select 2 men and 3 women?

There are $(5C2)=10$ ways to select the men. There are $(4C3)=4$ ways to select the women.

For each of the 10 ways you can select the men there are 4 ways you can select the women. The choice of men is independent of the choice of women. Under these circumstances:

Answer: $(5C2)\times(4C3)$

Example 1

In how many ways can you arrange 7 boys and 3 girls in a line if:

a) the girls are at the front?

b) the girls stand next to each other?

c) no two girls stand next to each other?

Solutions

a) There are $3!$ ways to arrange the girls and there are $7!$ ways to arrange the boys.

Answer: $3!\times7!$

b) We have 8 items to arrange in a line. A block of girls and 7 boys. We can do this in $8!$ ways. But for each of these arrangements, we can shuffle the girls in $3!$ ways.

Answer: $8!\times3!$

c) We can arrange the boys in $7!$ ways. Then we add the girls. There are 8 places where we can put the first girl. At the front, at the back or between two boys. There are 7 places we can put the next girl and there are 6 places we can put the third girl.

Answer: $7!\times8\times7\times6$

Example 2

In how many ways can you arrange 10 Physics, 4 French and 7 Biology books in a line if books of the same subject must be kept together?

Solution

We can arrange the 3 subjects in $3!$ ways. We can then shuffle the Physics books in $10!$ ways, the French books in $4!$ ways and the Biology books in $7!$ ways.

Answer: $3!\times10!\times4!\times7!$

Example 3

There are 8 boys and 5 girls in a class. In how many ways can you arrange 4 boys and 3 girls in a line?

Solution

We can select the pupils in $(8C4)\times(5C3)$ ways.

Having selected the pupils, we can arrange them in $7!$ ways.

Answer: $(8C4)\times(5C3)\times7!$

Example 4

There are 10 boys and 12 girls in a class. In how many ways can you select 5 pupils if you must include at least one boy and at least one girl?

Solution

There are $(22C5)$ ways to select 5 pupils. But some selections are no good.

There are $(10C5)$ selections which are all boys and $(12C5)$ selections that are all girls.

Answer: $(22C5)-(10C5)-(12C5)$

Example 5

In how many ways can you arrange the letters A, A, A, B, B, B, B, B, in a line?

Solution

There are 8 places and we need to choose 3 of these places for the A’s

Answer: $(8C3)$

Or

There are 8 places and we need to choose 5 of these places for the B’s

Answer: $(8C5)$

Example 6

A pack of cards has 52 cards. Each card has a suit (spade, heart, diamond or club) and a rank (ace, two, three, … king). In a game of poker, a hand consists of 5 cards. How many hands are:

a) Straight-flush	5 consecutive cards all in the same suit
b) 4 of a kind	4 cards of one rank and 1 other card
c) Full-house	3 cards of one rank and 2 cards of another rank
d) Flush	5 non-consecutive cards all in the same suit
e) Straight	5 consecutive cards not all in the same suit
f) 3 of a kind	3 cards of one rank and 2 other cards of different ranks

Solutions

a) Straight-flush:

There are 10 ways you can get 5 consecutive cards:

ace, 2, 3, 4, 5

2, 3, 4, 5, 6

3, 4, 5, 6, 7

…

10, jack, queen, king, ace

There are 4 choices for the suit.

Answer: $10\times4=40$

b) 4 of a kind:

There are 13 choices for the rank of the 4 cards and 48 choices for the 1 other card.

Answer: $13\times48=624$

c) Full-house:

There are 13 choices for the rank of the 3 cards and $(4C3)$ choices for the 3 cards of that rank.

There are 12 choices for the rank of the 2 cards and $(4C2)$ choices for the 2 cards of that rank.

Answer: $13\times(4C3)\times12\times(4C2)$

d) Flush:

There are 4 choices for the suit of the 5 cards and $(13C5)$ choices for the 5 cards of that suit.

But we have included the 40 straight-flushes.

Answer: $4\times(13C5)-40$

e) Straight:

There are 10 ways you can get 5 consecutive cards:

For each card in the straight there are 4 choices for its suit.

But we have included the 40 straight-flushes.

Answer: $10\times4\times4\times4\times4\times4-40$

f) 3 of a kind:

There are 13 choices for the rank of the 3 cards and $(4C3)$ choices for the 3 cards of that rank.

There are $(12C2)$ ways to choose the ranks of the other 2 cards.

For each of the other 2 cards there are 4 choices for their suits

Answer: $13\times(4C3)\times(12C2)\times4\times4$

Example 7

There are 20 people at a party. Everyone shakes hands once with every-one else. Have many hand-shakes take place?

It can be fun to think of different ways to answer the same question.

Solution 1

There are 20 people and everyone has 19 hand-shakes.

Answer: 20×19 No! We have counted every handshake twice. Can you see why?

Answer: $\dfrac{20\times19}{2}$

Solution 2

Alice does 19 hand-shakes and then goes home.

Bill then does 18 hand-shakes and goes home.

Jane then does 17 hand-shakes and goes home.

etc

Answer: $19+18+17+\ldots+1$

Solution 3

There are $(20C2)$ ways to select a pair of people. For every pair of people there is a hand-shake.

Answer: $(20C2)$

see EXERCISE

EXERCISE

1) I have 12 pens, 8 pencils and 4 crayons. In how many ways can you select one of each?

2) On a restaurant menu there are 3 choices for the first course, 10 choices for the second course and 5 choices for the third course. In how many ways can you select a meal?

3) Each day I buy a coffee, a tea or a beer at my local cafe. In how many ways can I select my drinks for a week?

4) In how many ways can you arrange the letters A, B, C, D, E?

5) In how many ways can you arrange 20 people in a line?

6) In how many ways can you arrange 3 of the letters A, B, C, D, E, F, G, H?

7) In how many ways can I arrange 7 ornaments in a line if I have18 ornaments to choose from?

8) In how many ways can you arrange the digits 1, 2, 3, 4, 5, 6, 7 to form an odd number?

9) There are 10 runners in a race. In how many ways can the gold, silver and bronze medals be awarded if there are no dead-heats?

10) In how many ways can you select 3 of the letters A, B, C, D, E, F, G, H?

11) In how many ways can you select 13 cards from a pack of 52 cards?

12) In a lottery, you have to select 6 numbers from 1, 2, 3, …, 49

In how many ways can you select your lottery numbers?

13) In a class of 25 pupils, everyone shakes hands exactly once with everyone-else. How many hand-shakes take place?

14) 10 points are marked around a circle. A line is drawn between every pair of points. How many lines will be drawn?

15) In how many ways can you split a class of 28 pupils into a group of 20 and a group of 8?

16) In how many ways can you split a class of 28 pupils into a group of 18, a group of 6 and a group of 4?

17) In how many ways can you split a class of $(x+y+z)$ pupils into a group of x pupils, a group of y pupils and a group of z pupils if x and y and z are different numbers?

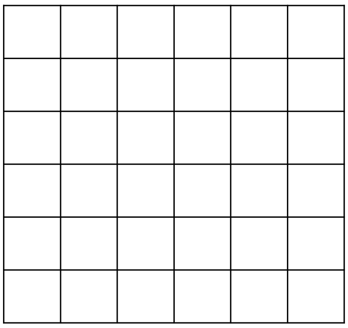
18) There are 16 boys and 18 girls in a class. In how many ways can you select 6 boys and 9 girls?

19) I have 30 letters and 30 envelopes. In how many ways can I place one letter in each envelope?

20) In how many ways can you select some (or none) people from a group of 7 people?

21) How many hands of 13 cards have 4 spades, 4 hearts, 4 diamonds and 1 club?

22) Here is the street plan of a city. The lines represent the streets.



All roads run north-south or east-west. I want to walk from the bottom left-hand corner to the top right-hand corner. I only want to walk north or east. In how many ways can I select my route?

23) Brag

In a game of Brag, a hand consists of 3 cards. How many hands are:

a) 3 of a kind	3 cards of one rank
b) Straight-flush	3 consecutive cards all in the same suit
c) Straight	3 consecutive cards not all in the same suit suit
d) Flush	3 non-consecutive cards all in the same suit
e) 2 of a kind	2 cards of one rank and 1 other card

SOLUTIONS

1) Answer: $12 \times 8 \times 4$

2) Answer: $3 \times 10 \times 5$

3) Answer: $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

4) Answer: $5!$

5) Answer: $20!$

6) Answer: $(8P3)$

7) Answer: $(18P7)$

8) There are 4 choices for the last digit because the last digit must be odd. The other 6 digits can be arranged in $6!$ ways.

Answer: $4 \times 6!$

9) We want to arrange 3 of the 10 runners on the podium.

Answer: $(10P3)$

10) Answer: $(8C3)$

11) Answer: $(52C13)$

12) Answer: $(49C6)$

13) For each way we can choose two pupils there is a hand-shake.

Answer: $(25C2)$

14) For each way we can choose two points there is a line.

Answer: $(10C2)$

15) Once you have chosen 20 pupils from 28 pupils there is nothing-else to do.

Answer: $(28C20)$

16) Once you have chosen 18 pupils from 28 pupils and then chosen 6 pupils from the remaining 10 pupils there is nothing-else to do.

Answer: $(28C18)\times(10C6)$

17) Once you have chosen x pupils from (x+y+z)pupils and then chosen y pupils from the remaining(y+z)pupils there is nothing-else to do.

Answer: $((x+y+z)Cx)\times((y+z)Cy)$

This answer simplifies to: $\frac{(x+y+z)!}{x!y!z!}$

18) Answer: $(16C6)\times(18C9)$

19) Answer: $30!$

20) For each person you have a choice of 2 options – select that person or don’t select that person.

Answer: $2\times2\times2\times2\times2\times2\times2$

21) You have to choose 4 of the 13 spades and …

Answer: $(13C4)\times(13C4)\times(13C4)\times(13C1)$

22) I need to walk 6 blocks north and 6 blocks east.

A possible route is: N, N, E, E, E, N, E, E, N, N, N, E

There are 12 letters in the line and we need to choose 6 positions for the N letters.

Answer: $12C6$

23)

a) 3 of a kind:

There are 13 choices for the rank of the 3 cards and $(4C3)$ choices for the 3 cards of that rank.

Answer: $13\times(4C3)$

b) Straight-flush:

There are 12 ways you can get 3 consecutive cards:

ace, 2, 3

2, 3, 4

3, 4, 5

…

queen, king, ace

There are 4 choices for the suit.

Answer: $12 \times 4 = 48$

c) Straight:

There are 12 ways you can get 3 consecutive cards:

For each card in the straight there are 4 choices for its suit.

But we have included the 48 straight-flushes.

Answer: $12 \times 4 \times 4 \times 4 - 48$

d) Flush:

There are 4 choices for the suit of the 3 cards and $(13C3)$ choices for the 3 cards of that suit.

But we have included the 48 straight-flushes.

Answer: $4 \times (13C3) - 48$

e) 2 of a kind:

There are 13 choices for the rank of the 2 cards and $(4C2)$ choices for the 2 cards of that rank.

There are 48 choices for the 1 other card.

Answer: $13 \times (4C2) \times 48$

Choosing a Pub

Three students, Bill, Alice and Jane decide to go to the pub for a drink. There are four pubs to choose from; The Crown, The Bear, The Cross Keys, and The Eight Bells. Each student has put the pubs in order of preference:

Bill: The Cross Keys, The Bear, The Crown, The Eight Bells

Alice: The Bear, The Crown, The Eight Bells, The Cross Keys

Jane: The Crown, The Eight Bells, The Cross Keys, The Bear

Jane says: Let's go to The Crown, it's my favourite pub.

However two students prefer The Bear to The Crown so they decide to go to The Bear.

However two students prefer The Cross Keys to The Bear so they decide to go to The Cross Keys.

However two students prefer The Eight Bells to The Cross Keys so they decide to go to The Eight Bells.

So off they go to The Eight Bells.

After a few pints, Bill says: I prefer The Crown to The Eight Bells and the other two both agree.

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Dodgy Algebra

Example 1

$3>2$

$3\log(\frac{1}{2})>2\log(\frac{1}{2})$

$\log(\frac{1}{2})^3>\log(\frac{1}{2})^2$

$\log(\frac{1}{8})>\log(\frac{1}{4})$

$\frac{1}{8}>\frac{1}{4}$

Example 2

$x>3$

$3x>9$

$3x-x^2>9-x^2$

$x(3-x)>(3+x)(3-x)$

$x>3+x$

$0>3$

Example 3

$x=3$

$x^2=3x$

$x^2-9=3x-9$

$(x+3)(x-3)=3(x-3)$

$(x+3)=3$

$x=0$

$3=0$

Example 4

$x+y=2$

$(x+y)(x-y)=2(x-y)$

$x^2-y^2=2x-2y$

$x^2-y^2+(y^2-2x+1)=2x-2y+(y^2-2x+1)$

$x^2-2x+1=y^2-2y+1$

$$(x-1)^2=(y-1)^2$$
$$x-1=y-1$$
$$x=y$$

Example 5

$$\sin 70° = \sin 110°$$
$$70° = 110°$$

Example 6

$$3-\frac{x+4}{x-2}=\frac{2x-10}{x-3}$$
$$\frac{3(x-2)-(x+4)}{x-2}=\frac{2x-10}{x-3}$$
$$\frac{2x-10}{x-2}=\frac{2x-10}{x-3}$$
$$2=3$$

EXERCISE

So where did it all go wrong?

SOLUTIONS

1)

We multiplied both sides of an inequality by $\log(1/2)$

But $\log(1/2)$ is negative so we should reverse the inequality sign.

2)

We divided both sides of an inequality by $(3-x)$

But $(3-x)$ is negative so we should reverse the inequality sign.

3)

We divided both sides of an equation by $(x-3)$

But $(x-3)=0$

4)

If $(x-1)^2=(y-1)^2$ then either $(x-1)=(y-1)$ or $(x-1)=-(y-1)$

5)

Look at the graph $y=sinx$

6)

$$\frac{2x-10}{x-2}=\frac{2x-10}{x-3}$$

So:

$(2x-10)(x-3)=(2x-10)(x-2)$ provided $x\neq 3$ and $x\neq 2$

Either:

$(2x-10)=0$ so $x=5$

Or:

$(x-3)=(x-2)$ which has no solution

Dodgy Integrals if you know about integration ...

1. Find the area under the curve:

$y=\frac{1}{x^2}$ between $x=-1$ and $x=1$

DIAGRAM?

The area is above the x axis, so the integral will be positive.

Show that:

$$\int_{-1}^{1}\frac{1}{x^2}dx=...=-2$$

2. Now:

$\int(sinxcosx)dx=\frac{1}{2}\sin^2 x$ check by differentiation

And:

$\int(sinxcosx)dx=-\frac{1}{2}\cos^2 x$ check by differentiation

So:

$$\frac{1}{2}\sin^2 x=-\frac{1}{2}\cos^2 x$$

So:

$$\sin^2 x+\cos^2 x=0$$

3. Now:

$\int\left(\frac{1}{7x}\right)dx=\frac{1}{7}lnx$ check by differentiation

And:

$\int\left(\frac{1}{7x}\right)dx=\frac{1}{7}\ln 7x$ check by differentiation

So:

$$lnx=\ln 7x$$

4. Using integration by parts, show that:

$$\int\left(\frac{1}{x}\right)dx=\int 1\times\left(\frac{1}{x}\right)dx=x\frac{1}{x}+\int\left(\frac{1}{x}\right)dx$$

So:

$$\int \left(\frac{1}{x}\right) dx = 1 + \int \left(\frac{1}{x}\right) dx$$

So:

$$0 = 1$$

5. Now:

$$\int 2\sin 2x dx = -\cos 2x$$ check by differentiation

And:

$$\int 2\sin 2x dx = 2\sin^2 x$$ check by differentiation

So:

$$-\cos 2x = 2\sin^2 x$$

6. Let:

$$I = \int_0^{\pi} \cos^2 x dx$$

So:

$$I = \int_0^{\pi} cosxcosxdx = \int_0^{\pi} \sqrt{1-\sin^2 x}\, cosxdx$$

Use the substitution:

$$sinx = t$$

Show:

$$cosxdx = dt$$

Show that:

I= $\int_0^0 \sqrt{1-t^2} dt = 0$

7. Now:

$$sec^2 x \geq 0$$

So:

$$\int_0^{\pi} sec^2 xdx > 0$$

But:

$$\int_0^{\pi} sec^2 xdx = [tanx]_0^{\pi} = 0$$

Encryption

Part 1.

Alice wants to send a secret message to Bill. Eve wants to read this message. Alice writes the message on a piece of paper and puts the piece of paper in a box. Then she puts a padlock on the box and sends it to Bill. Eve might be able to intercept the box but she can't open it as she does not have a key to the padlock. Bill receives the box and unlocks the padlock with his key. Alice and Bill will have to make arrangements, in advance, so that they each have a copy of the padlock key.

This example is a bit like private-key encryption.

Private-key encryption

Alice wants to send a secret message to Bill using email. Eve wants to read this message. Alice decides that she can't prevent Eve intercepting the email but she can prevent Eve reading the message. She can use encryption.

We will look at two private-key encryption methods - substitution and addition. In both methods, each letter in the message is replaced by a different (or perhaps the same) letter.

1. Substitution

Example 1

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
G	C	R	K	Q	W	P	F	I	T	A	J	X	R	B	Z	N	Y	O	V	H	D	M	E	U	L

Method: replace a letter in the top row by the corresponding letter in the second row

Key: the above table (we could use any rearrangement of the alphabet)

We will assume that Eve knows the method being used but she does not have the key.

The message Alice wants to send is: meet me tonight

message	M	E	E	T	M	E	T	O	...
encryption	X	Q	Q	V	X	Q	V	B	...

EXERCISE 1

Alice encrypts a message using the above key and sends it to Bill.

Bill receives: VFQXBRQUIOIRVFQOFQK

Can you decrypt this message for Bill?

Alice and Bill will have to make arrangements, in advance, to use substitution encryption and to agree on the key. How are they going to do this?

There are $26!-1$ different ways we can rearrange the letters of the alphabet, so there are $26!-1$ possible keys. Eve might not be able to try all these keys. However this method of encryption has a serious flaw. Every time there is an A in the message it is replaced by a G in the encryption. Every time there is an B in the message it is replaced by a C in the encryption …etc

This means that Eve can use frequency analysis.

In a piece of text, written in English, some letters will appear more often than others.

The list of letters in order of how often they appear is roughly:

E, T, A, O, I, N, S, H, R, D, L, C, U, M, W, F, G, X, P, B, V, K, J, Q, Y, Z

So Eve can count how many times each letter appears in the encryption. The letter with the highest frequency will probably correspond to an E in the message. The letter with the second highest frequency might correspond to a T or perhaps an A … etc

This technique works well for long messages.

2. Addition

How do you add two letters? What is $P+U$?

We give each letter a number:

$A=0$ $B=1$ $C=2$ $D=3$ … $Z=25$

and then add the numbers, mod 26 (see chapter, Modulo Arithmetic)

So $P+U=15+20=35=9=J$

Or we can use a Vigenere square (see footnote)

We can use this idea for encryption.

Method: add the letters of the message to the letters of the key

Key: see examples below

We will assume that Eve knows the method being used but she does not have the key.

The message Alice wants to send is: meet me tonight

Example 2

key: D (we could use any letter)

message	M	E	E	T	M	E	T	O	...
	12	4	4	19	12	4	19	14	...
key	D	D	D	D	D	D	D	D	...
	3	3	3	3	3	3	3	3	...
addition	15	7	7	22	15	7	22	17	...
encryption	P	H	H	W	P	H	W	R	...

Example 3

key: ERIC (we could use any word)

message	M	E	E	T	M	E	T	O	...
	12	4	4	19	12	4	19	14	...
key	E	R	I	C	E	R	I	C	...
	4	17	8	2	4	17	8	2	...
addition	16	21	12	21	16	21	27	16	...
encryption	Q	V	M	V	Q	V	B	Q	...

Example 4

key: GEUKAQPTY (we could use any random list of letters)

message	M	E	E	T	M	E	T	O	...
	12	4	4	19	12	4	19	14	...
key	G	E	U	K	A	Q	P	T	...
	6	4	20	10	0	16	15	19	...
addition	18	8	24	29	12	20	34	33	...
encryption	S	I	Y	D	M	U	I	H	...

EXERCISE 2

Alice encrypts a message using the key: DRMPKZTQDF and sends it to Bill.

Bill receives: IIQSSRTISD

Can you decrypt this message for Bill?

Alice and Bill will have to make arrangements, in advance, to use addition encryption and to agree on the key. How are they going to do this?

Some comments:

Example (2) is known as Caesar encryption.

There are only 25 possible keys. Eve could easily try them all, so this method is a bit rubbish.

Example (3) is known as key-word encryption.

The first E in the message is replaced by a V in the encryption but the second E in the message is replaced by an M in the encryption. This gets around the problem of frequency analysis.

Unfortunately, there are clever statistical techniques that Eve can use to find the length of the key-word. If Eve has discovered that the key-word is four letters long then:

1st, 5th, 9th, 13th, … letters of the message have all been added to the same letter (in my example, E)

2nd, 6th, 10th, 14th, ...letters of the message have all been added to the same letter (in my example, R)

etc

So Eve can now do a frequency analysis on the 1st, 5th, 9th, 13th, … letters of the encryption.

Then Eve can do a frequency analysis on the 2nd, 6th, 10th, 14th, ...letters of the encryption.

etc

Example (4) is known as one time-pad-encryption.

Imagine a note-pad. On each page is a random list of letters. You use the letters on page one of the note-pad as the key for your first message. You use the letters on page two of the note-pad as the key for your second message. etc Every time you encrypt a new message, you use a new page of random letters. As the lists of letters in the note-pad are random then the string of letters in the encryption is random and the encryption is uncrackable.

Part 2.

Alice wants to send a secret message to Bill. Eve wants to read this message. Alice writes the message on a piece of paper and puts the piece of paper in a box. Bill has lots of identical padlocks which he makes available to anyone who wants to send him a message. There is just one key that fits all these padlocks and Bill has got it. Alice gets one of these padlocks and puts it on the box and sends it to Bill. Eve might be able to intercept the box but she can’t open it. Bill receives the box and unlocks it with his key. Alice and Bill do not have to make arrangements, in advance. Anyone can send a message to Bill.

This example is a bit like public-key encryption.

Public-key encryption

Example 5

RSA encryption

Bill picks p and q where p and q are two different prime numbers.

Bill picks e where e and $(p-1)(q-1)$ have no common factor.

Bill solves $ed=1, mod(p-1)(q-1)$ to find d

Bill publishes the numbers e and pq in a public directory for all to see, but he keeps the number d secret.

for example:

Bill picks $p=5$ $q=11$ as 5 and 11 are two different prime numbers.

Bill picks $e=7$ as 7 and 40 have no common factor.

Bill solves $7d=1, mod\,40$ to get $d=23$

Bill publishes the numbers 7 and 55 in a public directory for all to see, but he keeps the number 23 secret.

Alice wants to send a message to Bill. She looks-up the numbers e and pq in the public directory. She uses these numbers to encrypt her message.

(don’t worry about how she does this)

When he receives the message, Bill uses the number d to decrypt it.

(don’t worry about how he does this)

Eve intercepts the message. She knows the numbers e and pq because she (like anyone-else) can look them-up in the public directory.

But she needs to know p and q so she can solve $ed=1, mod(p-1)(q-1)$ to find d

So Alice’s message appears to be safe, unless Eve can factorise pq to find p and q

If p and q are large enough then factorising pq will be difficult.

Now don’t worry about the details! Here are some important points:

a) With private-key encryption, Alice and Bill need a way to get together, in advance, and agree upon a key. With public key encryption, anyone can send an encrypted message to Bill. Even someone Bill has never met.

b) The security of RSA encryption relies on the difficulty of factorising large numbers.

c) RSA encryption enables you to buy stuff online. I type-in my credit card number and this is encrypted. Anyone can do it.

d) How does Bill know that the message really came from Alice?

e) My example with $p=5$ and $q=11$ is to illustrate the method. In practise Eve will be able to factorize 55. But if p and q are hundreds of digits long then Eve has a problem.

Encryption is used …

when you withdraw money from a cash machine
when you send messages over the internet or a mobile phone
to protect business and military secrets
in banking
for data storage
for data transmission
for on-line shopping etc

A lot has been written about the use of encryption during the Second World War. In particular, the use of the enigma machine. Read the books. Watch the films. Visit Bletchley Park.

SOLUTION 1
THE MONEY IS IN THE SHED

SOLUTION 2
FRED IS A SPY

footnote

Vigenere square

To find P+U, find the intersection of the P row and the U column. P+U=J

+	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
X	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
Y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
Z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

e

Here are some nice formulas for e – there are lots more:

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$$

$$\frac{1}{e}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\dots$$

If $n\to\infty$ then $\left(1+\frac{1}{n}\right)^n\to e$

If $n\to\infty$ then $\left(1-\frac{1}{n}\right)^n\to\frac{1}{e}$

If $n\to\infty$ then $\frac{n}{(n!)^{1/n}}\to e$

Example 1

e and compound interest

I invest $£1$ for one year. How much is my investment worth if ...

a) the interest rate is 100% per year

answer $£(1+1)^1$

b) the interest rate is 50% per $1/2$ year

answer $£\left(1+\frac{1}{2}\right)^2$

c) the interest rate is 10% per $1/10$ year

answer $£\left(1+\frac{1}{10}\right)^{10}$

d) the interest rate is 5% per $1/20$ year

answer $£\left(1+\frac{1}{20}\right)^{20}$

e) the interest rate is $(100/n)\%$ per $1/n$ year

answer $£\left(1+\frac{1}{n}\right)^n$

Note:

as $n\to\infty$ so $answer\to£e$

Example 2

e and arranging ornaments

I have 3 ornaments. In how many ways can I arrange some (or none) ornaments in a line on my mantelpiece? see chapter: Arrangements and Selections

There are $(3P0)$ arrangements with no ornaments.

There are $(3P1)$ arrangements with one ornament.

There are $(3P2)$ arrangements with two ornaments.

There are $(3P3)$ arrangements with three ornaments.

The total number of arrangements is:

$$(3P0)+(3P1)+(3P2)+(3P3)=\frac{3!}{3!}+\frac{3!}{2!}+\frac{3!}{1!}+3!=3!\left(\frac{1}{3!}+\frac{1}{2!}+\frac{1}{1!}+1\right)=3!\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\right)$$

In general:

If I have n ornaments then the number of arrangements is:

$$n!\left(1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}\right)$$

If n is large then the number of arrangements is about $n!e$

Example 3

Theorem

$e<3$

Proof:

$$e=1+\left(1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...\right)<1+\left(1+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...\right)=1+\frac{1}{1-1/2}=3$$

Example 4

Theorem

e is irrational

Note:

$$\frac{7!}{8!}+\frac{7!}{9!}+\frac{7!}{10!}+...=\left(\frac{1}{8}\right)+\left(\frac{1}{8\times 9}\right)+\left(\frac{1}{8\times 9\times 10}\right)+...<\frac{1}{8^1}+\frac{1}{8^2}+\frac{1}{8^3}+...=\frac{\frac{1}{8}}{\left(1+\frac{1}{8}\right)}=\frac{1}{7}$$

So:

$$\frac{7!}{8!}+\frac{7!}{9!}+\frac{7!}{10!}+...<\frac{1}{7}$$

Proof (by contradiction)

Assume e is rational, say $e = \frac{19}{7}$

Now:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

So:

$$\frac{19}{7} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

So:

$$\frac{19}{7} \times 7! = \left(7! + \frac{7!}{1!} + \frac{7!}{2!} + \frac{7!}{3!} + \dots + \frac{7!}{7!}\right) + \left(\frac{7!}{8!} + \frac{7!}{9!} + \frac{7!}{10!} + \dots\right)$$

The LHS is an integer and the first bracket on the RHS is an integer but the second bracket isn't – see above.

Contradiction.

If we assume:

$e = \frac{p}{q}$ where p and q are any positive integers, then we can repeat the above argument and again get a contradiction. So e must be irrational.

It has been proved that e π and e^{π} are irrational

We do not know about $\pi + e$ πe π^{π} e^{e} π^{e}

Example 5 if you know about differentiation ...

Investigation

$2^4 = 4^2$ Can you think of another pair of positive integers with this property?

Hint:

If:

$$p^q = q^p$$

show that:

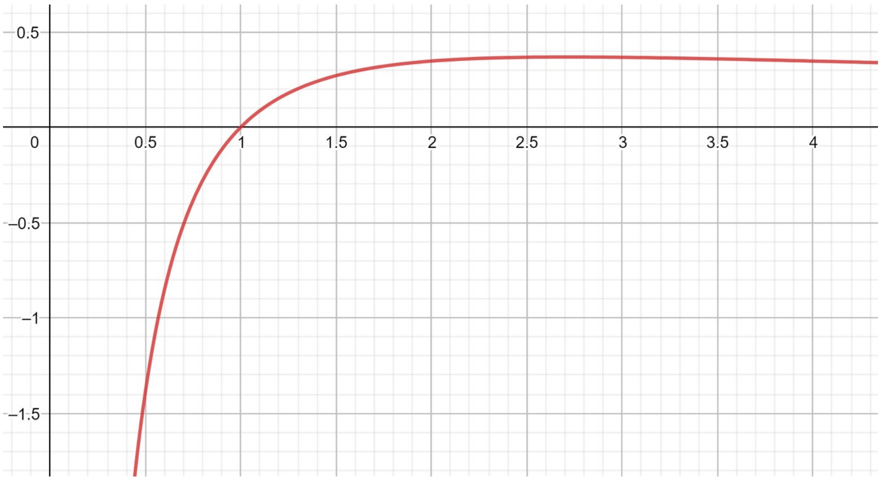
$$q \ln p = p \ln q$$

so:

$$\frac{\ln p}{p} = \frac{\ln q}{q}$$

Here is the graph:

$$y = \frac{\ln x}{x}$$



We want two different x values, call them p and q with the same y value.

So we want to be able to draw a horizontal line that cuts the graph twice.

Use differentiation to show that the maximum point on the graph occurs at $x=e$

Hence show that $2^4=4^2$ is the only solution of $p^q=q^p$ if p and q are positive integers.

Also

The maximum on the graph occurs at $x=e$

So:

$\frac{\ln e}{e}>\frac{\ln x}{x}$ for any x value, in particular $\frac{\ln e}{e}>\frac{\ln \pi}{\pi}$

Show:

$\pi \ln e > e \ln \pi$

Show:

$e^{\pi}>\pi^{e}$

Grid Puzzles

EXERCISE

1)

There are 3 people, Alice, Bill and Jane. One lives in Redland, one lives in Knowle and one lives in Easton. Alice does not live in Easton. Bill lives in Redland. Who lives where?

We put this information onto a grid:

	Redland	Knowle	Easton
Alice			X
Bill	√		
Jane			

Each row and each column has one √ and two X. Can you complete the grid?

2)

There are 3 people, Alice, Bill and Jane. One lives in Redland, one lives in Knowle and one lives in Easton. Alice does not live in Redland. Jane lives in Knowle. Who lives where?

3)

There are 3 people, Alice, Bill and Jane. One lives in Redland, one lives in Knowle and one lives in Easton. One drives a Volvo, one drives a Ford and one drives a Honda. Alice does not live in Easton. The Honda driver does not live in Easton. The Ford driver lives in Redland. Bill drives the Honda. Who lives where and who drives what?

We put this information onto a more complicated grid:

	Redland	Knowle	Easton	Volvo	Ford	Honda
Alice			X			
Bill						√
Jane						
Volvo				*	*	*
Ford	√			*	*	*
Honda			X	*	*	*

Can you complete the grid? It’s not so easy. Think of it as three 3×3 grids where each row and each column has one √ and two X. Ignore the * squares.

4)

There are 3 people, Alice, Bill and Jane. One lives in Redland, one lives in Knowle and one lives in Easton. One drives a Volvo, one drives a Ford and one drives a Honda. Alice drives the Volvo. The Honda driver does not live in Knowle. Jane does not live in Easton. Bill lives in Redland. Who lives where and who drives what?

SOLUTIONS

1)

	Redland	Knowle	Easton
Alice	X	√	X
Bill	√	X	X
Jane	X	X	√

2)

	Redland	Knowle	Easton
Alice	X	X	√
Bill	√	X	X
Jane	X	√	X

3)

	Redland	Knowle	Easton	Volvo	Ford	Honda
Alice			X			X
Bill				X	X	√
Jane						X
Volvo	X	X	√	*	*	*
Ford	√	X	X	*	*	*
Honda	X	√	X	*	*	*

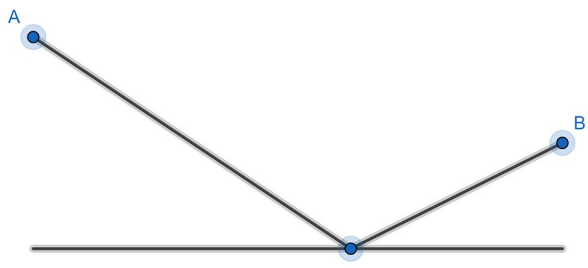
You might be stuck at this point. But, Bill drives the Honda and the Honda lives in Knowle so Bill lives in Knowle. Now can you complete the grid?

4)

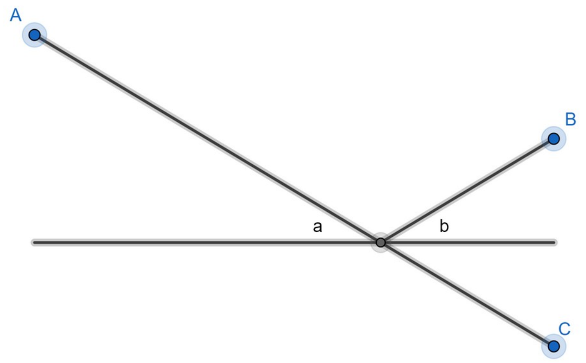
	Redland	Knowle	Easton	Volvo	Ford	Honda
Alice	X	X	√	√	X	X
Bill	√	X	X	X	X	√
Jane	X	√	X	X	√	X
Volvo	X	X	√	*	*	*
Ford	X	√	X	*	*	*
Honda	√	X	X	*	*	*

Heron’s Theorem

We want to find the shortest path from point A to the line and back to point B.



Reflect point B in the line to get point C



The distance from point A to the line and back to point B, is the same as the distance from point A to the line and on to point C. The shortest path from point A to the line and on to point C is a straight line. So the shortest path from point A to the line and back to point B is where angle a is equal to angle b This is Heron’s theorem.

Incidently, if light travelled from point A to the line (which acts as a mirror) and back to point B then it would take this path (remember, angle of incidence equals angle of reflection)

How To Tune A Piano

When you press a key on a piano, a hammer hits a string and the string vibrates. The frequency of this vibration determines the pitch of the note. For example, a string vibrating 261.6 times per second will produce the note middle C. We write 261.6 times per second as 261.6 Hz

The interval between two notes is the ratio of their frequencies.
The interval between 800 Hz and 600 Hz is $800/600=4/3$

Look at this sequence of notes:

200 Hz, 300 Hz, 450 Hz, 675 Hz, …

Now

$$\frac{300}{200}=\frac{450}{300}=\frac{675}{450}=...=\frac{3}{2}$$

So the intervals between consecutive notes are the same. The frequencies of the notes form a geometric sequence with common ratio $3/2$ (see Appendix 2: Geometric Sequence)

If one note has twice the frequency of another note then the interval between these notes is called an octave. In western European music, the octave is divided into 12 intervals. So if one note on the piano has frequency 440 Hz then the note, one octave above, has frequency 880 Hz and we need to put another eleven notes in-between. What are the frequencies of these other notes?
If we want equal intervals between consecutive notes then the frequencies must form a geometric sequence:

$$440 \quad 440r^1 \quad 440r^2 \quad 440r^3 \quad \ldots \quad 440r^{11} \quad 440r^{12}$$

We want:

$$440r^{12}=880 \quad \text{so} \quad r^{12}=2 \quad \text{so} \quad r=2^{1/12}$$

This gives the following frequencies:

440 A	466.2 A#	493.9 B	523.3 C	554.4 C#	587.3 D	622.3 D#	659.3 E	698.5 F	740.0 F#	784.0 G	830.6 G#	880 A

The names of the notes are given below the frequencies.
Notes one octave apart are given the same name.

Once we know the frequencies of all the notes in one octave we can calculate the frequencies of all the notes on the piano.
To find the frequencies of the notes in the octave above we just double these frequencies.
To find the frequencies of the notes in the octave below we just halve these frequencies.
etc
This method for tuning a piano is called equal-tempered tuning because there are equal intervals between the consecutive notes.

Why do we want equal intervals between consecutive notes on the piano?
Here are the first seven notes of "Twinkle Twinkle Little Star":
523.3 Hz   523.3 Hz   784.0 Hz   784.0 Hz   880 Hz   880 Hz   784.0 Hz
We can play this on the piano. But what if someone says they can't sing notes that low. Can we play the same tune only higher?
If we multiply the frequencies of all these notes by say r^3 then we get:
622.3 Hz   622.3 Hz   932.3 Hz   932.3 Hz   1046.5 Hz   1046.5 Hz   932.3 Hz
All these notes are on the piano. This version of the song has all the same intervals as the first version. It will sound just the same. Only higher.

Musicologists (including Pythagoras) have claimed that two notes sound nice when played together if the interval between them is a simple ratio. So 600 Hz and 400 Hz sound nice when played together because $600/400=3/2$ is a simple ratio.
Unfortunately, if we use equal-tempered tuning then the interval between any two notes (apart from octaves) is not a simple ratio. It is irrational.
Fortunately, some interval are nearly simple ratios.
The interval between say C and G is $2^{7/12}$ which is nearly $3/2$
The interval between say C and F is $2^{5/12}$ which is nearly $4/3$
The interval between say C and E is $2^{4/12}$ which is nearly $5/4$
Musicologists (including Pythagoras) have devised tuning schemes with lots of simple ratios. But these do not have equal intervals between consecutive notes. So we have a problem.
We want equal intervals. We want simple ratios. We can't have both.

A piano creates a sound with a vibrating string. A bugle creates a sound with a vibrating column of air. A skilled player can produce different notes on a bugle by altering the way their lips vibrate. Physics tells us that if the lowest note you can get on a bugle is 110Hz then the other notes you can get are: 220 Hz, 330 Hz, 440 Hz, 550 Hz, 660 Hz, ...

110 Hz is an A

220 Hz is an A

330 Hz is nearly an E (329.6Hz)

440 Hz is an A

550 Hz is nearly a C# (554.4Hz)

660 Hz is nearly an E (659.3Hz)

So the bugle notes don’t quite match up with the piano notes.

“The Last Post” is played on a bugle with the notes:

220 Hz, 330 Hz, 440 Hz, 550 Hz, 660 Hz

It will sound slightly different if you play it on a piano.

J.S.Bach wrote a piece of music called The Well-Tempered Clavier to demonstrate the advantages of equal-tempered tuning. Check it out.

Infinite Numbers

I walk into a classroom and see that every student is sitting on a chair and every chair has a student sitting on it. We can pair-up the students with the chairs, so there must be the same number of each.

Example 1

We can pair-up the positive integers with the even positive integers:

1	2	3	4	...
2	4	6	8	...

So there are the same number of positive integers and even positive integers.

Example 2

We can pair-up the positive integers with the squares:

1	2	3	4	...
1	4	9	16	...

So there are the same number of positive integers and squares.

Example 3

We can pair-up the positive integers with all the integers:

1	2	3	4	5	6	7	8	...
0	1	-1	2	-2	3	-3	4	...

So there are the same number of positive integers and integers.

Example 4

Can we pair-up the positive integers with the positive rational numbers?

We cannot write a list of all the positive rational numbers in numerical order, because between any two positive rational numbers we can always find another positive rational number.

for example: half way between $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{2bd}$

However, we can write a list of all the positive rational numbers. Look at this table:

$1/1$					
$1/2$	$2/1$				
$1/3$	$2/2$	$3/1$			
$1/4$	$2/3$	$3/2$	$4/1$		

$1/5$	$2/4$	$3/3$	$4/2$	$5/1$	
...	...	...	...	...	...

We can read-off the positive rational numbers along the rows of the table:

$1/1 \quad 1/2 \quad 2/1 \quad 1/3 \quad 3/1 \quad 1/4 \quad 2/3 \quad 3/2 \quad 4/1 \quad \ldots$

This is a list of all the positive rational numbers.

note: we omitted $2/2$ because we have already had $1/1$ etc

We can now pair-up the positive integers with the positive rational numbers:

1	2	3	4	5	6	7	8	...
$1/1$	$1/2$	$2/1$	$1/3$	$3/1$	$1/4$	$2/3$	$3/2$	...

So there are the same number of positive integers and positive rational numbers. Even though, between any two consecutive integers there are an infinite number of rational numbers.

Example 5

Can we pair-up the positive integers with the real numbers, between 0 and 1?

We cannot write a list of all the real numbers between 0 and 1, in numerical order, because between any two real numbers we can always find another real number.

for example: half way between x and y is $\frac{x+y}{2}$

We thought of a clever trick so we could write a list of all the positive rational numbers, so perhaps we can think of another clever trick so we could write a list of all the real numbers between 0 and 1.

Say, here is our list of all the real numbers, between 0 and 1

0.475922… , 0.887885… , 0.490035… , 0.186792… , 0.676764… , ...

We can now pair-up the positive integers with the real numbers between 0 and 1:

1	2	3	4	5	...
0.475922...	0.887885...	0.490035...	0.186792...	0.676764…	...

Now this won’t do. I can always find a real number between 0 and 1, that is not on the list.

For example 0.59187… is not on the list. How do we know this?

It is not the first number on the list because it has a different first decimal place. It is not the second number on the list because it has a different second decimal place. It is not the third number on the list because it has a different third decimal place, etc

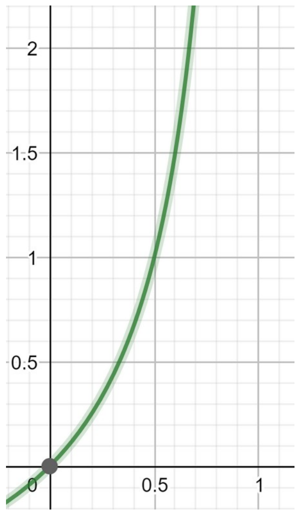
We cannot pair-up the positive integers with the real numbers between 0 and 1.

There are more real numbers between 0 and 1, than positive integers.

Example 6

We can pair-up the real numbers between 0 and 1 with the positive real numbers.

Look at the graph $y=\frac{x}{1-x}$ for $0\leq x<1$ and $0\leq y<\infty$



We can pair-up each x value with each y value.

There are the same number of real numbers between 0 and 1, and positive real numbers.

Subsets:

The finite set F has 3 members: $\{p,q,r\}$

A subset of F is a selection of none, some or all of the members of F

To form a subset, we have 2 choices, include or exclude p For each of these choices we have 2 choices, include or exclude q For each of these choices we have 2 choices, include or exclude r So there are $2\times2\times2=8$ choices and therefore 8 possible subsets.

The subsets of F are: $(),(p),(q),(r),(p,q),(p,r),(q,r),(p,q,r)$

In general:

A set with n members has 2^n subsets

Example 7

$A=\{1,2,3,...\}$ is the set of positive integers.

Can we pair-up the members of A with the subsets of A ?

Let’s say we have thought of a clever trick and we can pair-up the members of A with the subsets of A like this:

1	2	3	4	5	...
$\{3,19,47\}$	$\{3,4,6,...\}$	$\{1,3\}$	$\{2,5,...\}$	$\{3,5,7,...\}$	...
X	X	$\surd$	X	$\surd$	...

$1 \leftrightarrow (3,19,47)$ we put a X below because 1 is not a member of this subset.

$2 \leftrightarrow (3,4,6,...)$ we put a X below because 2 is not a member of this subset.

$3 \leftrightarrow (1,3)$ we put a $\surd$ below because 3 is a member of this subset.

etc

One of the subsets of A is $C=(1,2,4,...)$ The set of all the integers with a X

As C is a subset of A it must be paired-up with a member of A

Let’s say N is the positive integer that we pair-up with C

Is N a $\surd$ or a X integer?

If N is a $\surd$ integer then N is a member of C so N is a X integer

If N is a X integer then N is a member of C so N is a $\surd$ integer

Contradiction.

We cannot pair-up the members of A with the subsets of A

There are more subsets of A than members of A

One of the subsets of A is $(1,4,5,7,\ldots)$

We could use an alternative notation to denote this subset.

We could write it as $[1,0,0,1,1,0,1,...]$

The 1 in the 1st position denotes: include the integer 1

The 0 in the 2nd position denotes: exclude the integer 2

The 0 in the 3rd position denotes: exclude the integer 3

The 1 in the 4th position denotes: include the integer 4

etc

We can think of $[1,0,0,1,1,0,1,...]$ as representing $0.1001101\ldots$

We can think of $0.1001101\ldots$ as a real number (written in binary) between 0 and 1

So we can pair-up any subset of A with a real number between 0 and 1

There are the same number of subsets of A and real numbers between 0 and 1

Let ∞_1 be the number of positive integers.

Let ∞_2 be the number of positive real numbers between 0 and 1.

We know that $\infty_2 = 2^{\infty_1}$ and $\infty_2 > \infty_1$

Cantor's Theorem

A set with ∞ members will have 2^{∞} subsets and $2^{\infty} > \infty$

So we can generate an unending sequence of bigger and bigger infinite numbers:

$\infty_1 \quad \infty_2 = 2^{\infty_1} \quad \infty_3 = 2^{\infty_2} \quad \infty_4 = 2^{\infty_3}$ ...

EXERCISE

The rooms in Hilbert's Hotel hotel are numbered 1, 2, 3, 4, 5 … (this hotel has an infinite number of rooms). All the rooms are taken. A new guest arrives and asks for a room.

"No problem" says the owner "we can fit you in"

a) how can this be done?

Later that day, an infinite number of new guests arrive and each one asks for a room.

"No problem" says the owner "we can fit you all in"

b) how can this be done?

SOLUTION

a) The owner moves:

the person in room 1, to room 2

the person in room 2, to room 3

the person in room 3, to room 4 etc

The new guest is given room 1.

b)The owner moves:

the person in room 1, to room 2

the person in room 2, to room 4

the person in room 3, to room 6 etc

The new guests are given rooms 1, 3, 5, …

Infinite series

Example 1

We say the infinite series $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots$ converges. What does this mean?

As we add up more and more terms ...

$$\frac{1}{2^1}=\frac{1}{2} \quad \frac{1}{2^1}+\frac{1}{2^2}=\frac{3}{4} \quad \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}=\frac{7}{8} \quad \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=\frac{15}{16} \quad \ldots$$

the total gets bigger and bigger but never exceeds a certain number (in this case, 1)

Example 2

We say the infinite series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges. What does this mean?

As we add up more and more terms ...

$$\frac{1}{1}=1 \quad \frac{1}{1}+\frac{1}{2}=\frac{3}{2} \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}=\frac{11}{6} \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12} \quad \ldots$$

the total gets bigger and bigger and will eventually exceed any number.

Proof

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots=\left(\frac{1}{1}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots$$

The value of each bracket on the RHS is greater than $1/2$

So:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots>\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\ldots$$

So the series will eventually exceed any number.

Here is another proof (by contradiction)

Assume the series converges to some finite number S

So: $S=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$

So: $S>\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\ldots$

So: $S>\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\ldots$

So: $S>\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots$

So: $S>S$

Contradiction.

The series:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+...$$

is called the harmonic series. It diverges but very slowly. The sum of the first billion terms is only about 21.3

Example 3

Compare the series: $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\dots$ with the series: $\frac{1}{2^1}+\frac{1}{3^2}+\frac{1}{4^3}+\frac{1}{5^4}+...$

The terms in the second series are equal to or smaller than the corresponding terms in the first series. We know the first series converges so the second series must converge.

Example 4

Compare the series: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$ with the series: $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+...$

The terms in the second series are equal to or larger than the corresponding terms in the first series. We know the first series diverges so the second series must diverge.

Example 5

Look at this series: $\sum_{1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...$

Show that: $\frac{1}{k^2}<\frac{2}{k(k+1)}$ for any $k>1$

Show that: $\frac{2}{k}-\frac{2}{k+1}=\frac{2}{k(k+1)}$

Show that: $\sum_{1}^{\infty}\frac{1}{k^2}<\sum_{1}^{\infty}\left(\frac{2}{k}-\frac{2}{k+1}\right)=\left(\frac{2}{1}-\frac{2}{2}\right)+\left(\frac{2}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\dots$

Show that: $\sum_{1}^{\infty}\frac{1}{k^2}<2$

So the series converges.

If you study maths at a higher level you will learn how to manage infinite series properly.

In the meantime we can have some fun ...

Example 6

Evaluate: $S=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$

Now: $S=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\dots$

So: $S>0$

But: $S=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+...\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...\right)=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...\right)$

So: $S=0$

Example 7

Evaluate: $S=1-1+1-1+1-1+\ldots$

1st attempt: $S=(1-1)+(1-1)+(1-1)+\ldots$ $\qquad S=0$

2nd attempt: $S=1-(1-1)-(1-1)-(1-1)-\ldots$ $\qquad S=1$

3rd attempt: $S=1-(1-1+1-1+1-1+...)=1-S$ $\qquad S=1/2$

Example 8

Evaluate: $S=1+2+4+8+...$

Now: $2S=2+4+8+16+\ldots$

So: $2S-S=(2+4+8+16+...)-(1+2+4+8+16+...)$

So: $S=-1$

Example 9

Evaluate: $S=1-2+4-8+\ldots$

1st attempt: $S=1+(-2+4)+(-8+16)+\ldots$ $\qquad S\rightarrow\infty$

2nd attempt: $S=(1-2)+(4-8)+\ldots$ $\qquad S\rightarrow-\infty$

3rd attempt: $S=1-2(1-2+4-8+...)=1-2S$ $\qquad S=1/3$

Example 10

$U=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...$ and $V=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+...$

So: $U+V=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...=2V$

So: $U=V$

But: $U-V=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots$

So: $U>V$

Example 11

Now: $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots$ $\qquad$ it's a geometric series

So if we sub in $x=2$ we get: $1-2+4-8+16-...=\frac{1}{3}$

And if we sub in $x=-2$ we get: $1+2+4+8+16+...=-1$

Example 12

Now: $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots$ $\qquad$ it's a geometric series

If we differentiate we get: $(1-x)^{-2}=1+2x+3x^2+4x^3+...$

So if we sub in $x=-1$ we get: $1-2+3-4+...=\frac{1}{4}$

Example 13

Now: $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots$ it's a geometric series

So if we sub in $x=1$ we get: $1-1+1-1+...=\frac{1}{2}$

Example 14

We know: $a=1-1+1-1+1-1+...=\frac{1}{2}$ see example 14

We know: $b=1-2+3-4+5-6+\ldots=\frac{1}{4}$ see example 13

Let: $c=1+2+3+4+5+6+\ldots$

So: $b-c=(1-2+3-4+5-6+...)-(1+2+3+4+5+6+...)$

So: $b-c=(1-1)+(-2-2)+(3-3)+(-4-4)+(5-5)+(-6-6)+...$

So: $b-c=0-4+0-8+0-12+...=-4-8-12-...=-4(1+2+3+...)=-4c$

So: $b=-3c$ but $b=\frac{1}{4}$ so $c=-\frac{1}{12}$

Hence: $1+2+3+4+5+6+\ldots=-\frac{1}{12}$

Example 15

Consider the series: $a_1+a_2+a_3+a_4+...$

$a_1=a_1$

$a_2=(a_1+a_2)-a_1$

$a_3=(a_1+a_2+a_3)-(a_1+a_2)$

$a_4=(a_1+a_2+a_3+a_4)-(a_1+a_2+a_3)$ etc

The terms on the LHS sum to:

$a_1+a_2+a_3+a_4+...$

The terms on the RHS all cancel.

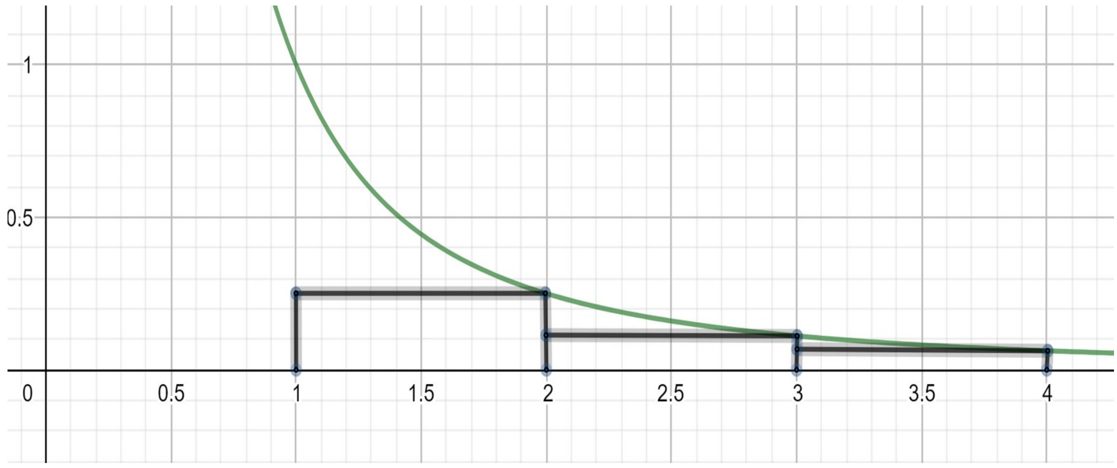
So $a_1+a_2+a_3+a_4+...=0$

Every infinite series sums to zero. We should have done this example first!

If you know about integration ...

Example

Here is the graph $y=\frac{1}{x^2}$



The diagram shows blocks between $x=1$ and $x=4$

The area of the blocks is: $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$

The area under the graph is: $\int_1^\infty \frac{1}{x^2}\,dx=1$

The area of the blocks is less than the area under the graph.

So: $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots<1$

So: $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$ converges.

Example

Here is the graph $y=\frac{1}{x}$

The area of the blocks is: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

The area under the graph is: $\int_1^\infty \frac{1}{x}\,dx \rightarrow \infty$

The area of the blocks is greater than the area under the graph.

So: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges.

Knight Tours

Example 1

Here is a knight tour on a 6×6 board:

1	28	15	12	3	34
16	11	2	35	22	13
27	36	29	14	33	4
10	17	8	23	30	21
7	26	19	32	5	24
18	9	6	25	20	31

The knight must visit each square once (and only once). The squares are numbered in the order they are visited by the knight. The knight starts on square 1, moves to square 2, moves to square 3, ... and ends on square 36.

This is a closed tour because the end square (36) is a knight move from the start square (1)

Any tour can be reversed. The knight could move: 36, 35, 34, … 3, 2, 1

We can think of a closed tour as starting on any square. We can think of the above tour as starting on square 17. The knight could move: 17, 18, 19, … 36, 1, 2, 3, … 16

Example 2

Here is another knight tour:

1	20	25	8	3	18
26	9	2	19	32	7
21	24	33	6	17	4
10	27	22	35	14	31
23	34	29	12	5	16
28	11	36	15	30	13

This is an open tour because the end square (36) is not a knight move from the start square (1)

This tour could be reversed. This tour could not start on any square.

Theorem

There is no closed tour on any $N \times N$ board where N is odd.

Proof

We can colour the squares on a board, black and white. Like a chess board. A knight on a black square can only move to a white square. A knight on a white square can only move to a black square. If the start square of a closed tour is black then the end square must be white, because the end square is a knight move from the start square. So there must be the same number of black and white squares. This won't happen on an $N \times N$ board if N is odd.

Theorem

There are no closed tours on a 2×2 board.

Proof

Just think about it.

Theorem

There are no closed tours on a $4x4$ board.

Proof

We can colour the board, black and white.

W	B	W	B
B	W	B	W
W	B	W	B
B	W	B	W

The knight must alternate between black and white squares.

We can colour the board, red and green.

R	R	R	R
G	G	G	G
G	G	G	G
R	R	R	R

A knight on a red square can only move to a green square. We can think of a closed tour as starting on any square. If the start square is red then the end square must be green, because the end square is a knight move from the start square. The knight must visit the same number of red and green squares. So the knight must alternate between red and green squares.

If the knight starts on the bottom left hand corner square, which is a black/red square then it can only move to a white/green square then to a black/red square etc. It isn't ever going to visit the black/green squares or the white/red squares. So there is no closed tour on a $4x4$ board.

Schwenk's Closed Tour Theorem

a) There is no closed tour on any $N \times N$ board where N is odd.

b) There is a closed tour on every $N \times N$ board where N is even and greater than 4.

Have we proved Schwenk's theorem?

We have proven part (a)

What about part (b)?

Schwenk says: if N is even and greater than 4 then there is a closed tour

We have proved: if N is even and isn't greater than 4 then there isn't a closed tour

Not the same thing at all.

Example 3

Here is a tour on a 5×5 board:

3	10	21	16	5
20	15	4	11	22
9	2		6	17
14	19	8	23	12
1	24	13	18	7

We can finish by going into the middle square.

Example 4

Here is a tour on a 9×9 board: CHECK THIS

5	20	47	34	7	22	49	36	9
46	33	6	21	48	35	8	23	50
19	4						10	37
32	45						51	24
3	18						38	11
44	31						25	52
17	2						12	39
30	42	16	55	28	41	14	53	26

1	56	29	42	15	54	27	40	13

We can finish by going into the middle 5×5 square.

Try this method on a 13×13 board.

See EXERCISE 1

Example 5

Here is a tour on a chess-board:

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

Amusingly, the numbers in each column and each row add up to the same total.

We call this a semi-magic square.

There are many methods to find knight tours. Look them up.

See Exercise

EXERCISE

1)

I am looking for a tour on this board:

		A		

Why will I fail if I start at square A?

2)

Look at this 6×6 board:

B	D	A	C	B	D
A	C	B	D	A	C
D	B			D	B
C	A			C	A
B	D	A	C	B	D
A	C	B	D	A	C

The knight can tour all the A squares. The knight can tour all the B squares, etc.

Can you find a tour by linking up these cycles, using the middle squares?

3)

Look at this 8×8 board:

D	B	C	A	D	B	C	A
C	A	D	B	C	A	D	B
B	D	A	C	B	D	A	C
A	C	B	D	A	C	B	D
D	B	C	A	D	B	C	A
C	A	D	B	C	A	D	B
B	D	A	C	B	D	A	C
A	C	B	D	A	C	B	D

The knight can tour all the A squares. The knight can tour all the B squares, etc.

Can you find a tour by linking up these cycles?

4)

Divide and conquer methods involve dividing the board into parts and touring each part separately.

Here is a 10×10 board:

5	16	21	12	7					
22	11	6	15	20					

17	4	13	8	25					
10	23	2	19	14					
3	18	9	24	1					

We divide the board into four 5×5 quarter boards and tour each quarter board separately.
A tour of the top left hand corner quarter board is shown.
Rotate this $90°$ clockwise and place in the top right hand corner. Rotate again for the bottom right hand corner. Rotate again for the bottom left hand corner. Now join up these four quarter board tours. Unfortunately, we cannot use this method on an 8×8 board because there are no tours of a 4×4 board.

SOLUTIONS

1)

Let's colour the board black and white, with the bottom left-hand corner black. There are 13 black squares and 12 white squares. The knight must alternate between black and white squares. So this will only work if the tour goes B, W, B, W, … B. Unfortunately, the start square is white.

2)

16	23	4	31	10	25
3	30	17	24	5	32
22	15	36	9	26	11
29	2	27	18	33	6
14	21	8	35	12	19
1	28	13	20	7	34

3)

38	55	22	13	36	51	18	11
23	14	37	54	17	12	35	50
56	39	16	21	52	33	10	19
15	24	53	40	9	20	49	34
42	57	28	1	48	61	32	7

25	2	41	60	29	8	47	62
58	43	4	27	64	45	6	31
3	26	59	44	5	30	63	46

4)

5	16	21	12	7	28	35	42	47	30
22	11	6	15	20	43	49	29	36	41
17	4	13	8	25	34	27	38	31	46
10	23	2	19	14	49	44	33	40	37
3	18	9	24	1	26	39	50	45	32
82	95	100	89	76	51	74	59	67	53
87	90	83	94	99	64	69	52	73	60
96	81	88	77	84	75	58	63	54	67
91	86	79	98	93	70	65	56	61	72
80	97	92	85	78	57	62	71	66	55

Lanchester model if you know about differential equations …

In a tank battle, one army has x tanks and the other army has y tanks. We are going to assume that the rate at which one army's tanks are destroyed is proportional to the number of tanks in the opposing army.

So:

$\frac{dx}{dt}=-k_1 y$ and $\frac{dy}{dt}=-k_2 x$

Let's also assume that each army is equally good at aiming so that:

$k_1=k_2$

Dividing these equations gives:

$\frac{dy}{dx}=\frac{x}{y}$

So $\int ydy \int xdx$ So $\frac{1}{2}y^2=\frac{1}{2}x^2+c$ So $y^2=x^2+2c$

So as the battle proceeds y^2-x^2 will remain constant.

Example

At the start of the battle:

$x=24$ and $y=25$

Throughout the battle:

$y^2-x^2=49$

At the end of the battle:

$x=0$ and $y=7$

The strength of a tank army is not proportional to the number of tanks but to the square of the number of tanks. This means weird stuff can happen.

Example

I start with 30 tanks and you start with 42 tanks. In one big battle with all the tanks, I am going to lose. But what if I could arrange some small skirmishes.

If 5 of my tanks engage with 3 of your tanks, then at the end of this skirmish, I'll have 4 tanks and you will have no tanks. If I keep doing this, then I'll soon have more tanks than you!

Lewis Carroll

Charles Dodgson (1832 – 1898) was a mathematics lecturer at Oxford University. He was also the author of several books for children, which he wrote under the pen-name, Lewis Carroll. His two most famous books are:

Alice’s Adventures In Wonderland and Through The Looking Glass.

These books have always been popular with mathematicians and you should read them. It might be best to read The Annotated Alice by Martin Gardner as this book explains the jokes and the logic which are easy to miss. Here are some quotes from these books:

Alice laughed, "There's no use trying" she said, "one can't believe impossible things."

"I daresay you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

“Take some more tea” the March Hare said to Alice very earnestly.

“I’ve had nothing yet” Alice replied in an offended tone “so I can’t take more.”

“You mean you can’t take less” said the Hatter “it’s very easy to take more than nothing”

"Then you should say what you mean," the March Hare went on.

"I do, " Alice hastily replied, "at least I mean what I say, that's the same thing, you know."

"Not the same thing a bit!" said the Hatter. "Why, you might just as well say that "I see what I eat" is the same thing as "I eat what I see!"

"It's very good jam," said the Queen.

"Well, I don't want any to-day, at any rate."

"You couldn't have it if you did want it," the Queen said. "The rule is jam tomorrow and jam yesterday but never jam to-day."

"It must come sometimes to "jam to-day,"Alice objected.

"No it can't," said the Queen. "It's jam every other day; to-day isn't any other day, you know."

"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean, neither more nor less."

"The question is," said Alice, "whether you can make words mean so many different things."

"The question is," said Humpty Dumpty, "which is to be master, that's all."

Maclaurin Series if you know about differentiation and integration ...

Sometimes we can write a function $f(x)$ as a power series:

$f(x)=a+bx+cx^2+dx^3+ex^4+\ldots$

Put $x=0$

$f(0)=a$

Differentiate

$f'(x)=b+2cx+3dx^2+4ex^3+\ldots$

Put $x=0$

$f'(0)=b$

Differentiate

$f''(x)=2c+(3\times2)dx+(4\times3)ex^2+\ldots$

Put $x=0$

$f''(0)=2c$

Differentiate

$f'''(x)=(3\times2)d+(4\times3\times2)ex+\ldots$

Put $x=0$

$f'''(0)=(3\times2)d$

etc

So $f(x)=f(0)+f'(0)x+\dfrac{f''(0)}{2!}x^2+\dfrac{f'''(0)}{3!}x^3+\dfrac{f''''(0)}{4!}x^4+\ldots$

Example 1

Lets try this out with $f(x)=sinx$

$f(x)=sinx$	$f(0)=0$
$f'(x)=cosx$	$f'(0)=1$
$f''(x)=-sinx$	$f''(0)=0$
$f'''(x)=-cosx$	$f'''(0)=-1$
$f''''(x)=sinx$	$f''''(0)=0$

etc

So $sinx=x-\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5+\ldots$

Find a graph plotter online and plot the following graphs:

$y=x$

$$y = x - \frac{1}{3!}x^3$$

$$sinx = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

etc

And see how these graphs get more and more like $y = sinx$

Example 2

Show that:

$$cosx = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$$

Example 3

Show that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

Example 4

There is a sneaky way to find the Maclaurin series for $\ln(1+x)$

We start with:

$1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$ it is a geometric series

So:

$$\int \frac{1}{1+x}dx = \int (1 - x + x^2 - x^3 + \ldots)dx$$

So:

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ this is only valid for $-1 < x \leq 1$

Example 5

There is also a sneaky way to find the Maclaurin series for $\tan^{-1}x$

We start with:

$1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1+x^2}$ it is a geometric series

So:

$$\int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \ldots)dx$$

So:

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$ this is only valid for $-1 \leq x \leq 1$

As a bonus:

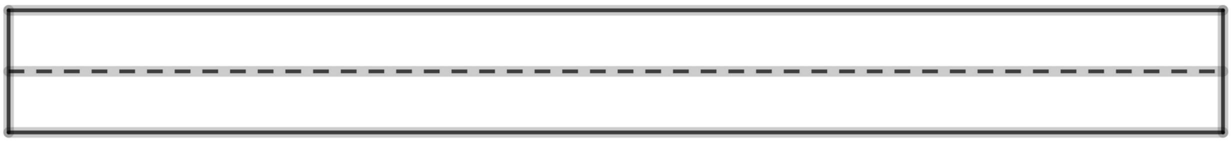
show that:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

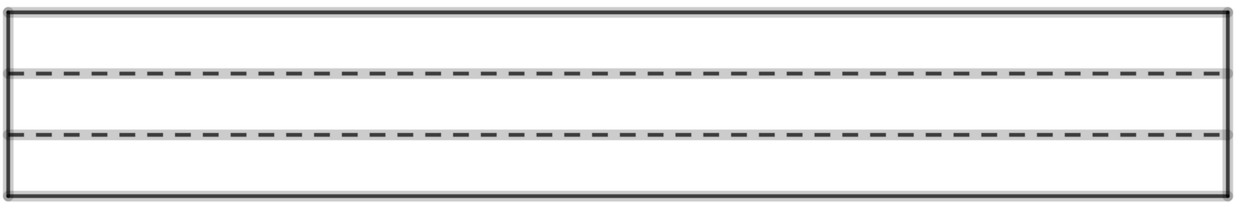
Möbius Strip DIAGRAMS ???

1) Take a long, thin strip of paper. Put one twist in it. Glue the two ends together. This is a Möbius strip. Try painting one side of the Möbius strip red and the other side blue.

2) Take a long, thin strip of paper and draw a dotted line along the length. Put one twist in it. Glue the two ends together. Now cut along the dotted line.



3) Take a long, thin strip of paper and draw two dotted lines along the length. Put one twist in it. Glue the two ends together. Now cut along the dotted line.



4) Play around. Try putting two twist in the strip of paper. Try drawing three dotted lines. etc

Multiples:

rule of 2: 36478 is a multiple of 2 because 8 (the last digit) is a multiple of 2

$36478=36470+8$ and 36470 is a multiple of 2

rule of 3: 264 is a multiple of 3 because $2+6+4$ is a multiple of 3

$264=2(100)+6(10)+4=2(99)+6(9)+(2+6+4)$ and $2(99)+6(9)$ is a multiple of 3

rule of 4: 94524 is a multiple of 4 because 24 (last two digits) is a multiple of 4

$94524=94500+24$ and 94500 is a multiple of 4

rule of 5: 743665 is a multiple of 5 because 5 (last digit) is a multiple of 5

$74365=74360+5$ and 74360 is a multiple of 5

rule of 6: 29622 is a multiple of 6 because 29622 is a multiple of 2 and a multiple of 3

rule of 8: 59136 is a multiple of 8 because 136 (last three digits) is a multiple of 8

$59136=59000+136$ and 59000 is a multiple of 8

rule of 9: 648 is a multiple of 9 because $6+4+8$ is a multiple of 9

$648=6(100)+4(10)+8=6(99)+4(9)+(6+4+8)$ and $6(99)+4(9)$ is a multiple of 9

rule of 10: 89210 is a multiple of 10 because the last digit is 0

rule of 11: 836 is a multiple of 11 because $8-3+6$ is a multiple of 11

$836=8(100)+3(10)+6=8(99)+3(11)+(8-3+6)$ and $8(99)+3(11)$ is a multiple of 11

EXERCISE

1)

Is 36470587624275 a multiple of 3?

2)

Is 47385900738828 a multiple of 8?

3)

Is 49775883661205 a multiple of 11?

4)

Show that every palindrome with an even number of digits (like 637736) is a multiple of 11

5)

Show that any 3-digit-repeater (like 726726) is a multiple of 7 and 11 and 13

6)

If n and x are positive integers, prove the following using the factor theorem:

a) x^n+1 is a multiple of $x+1$ if n is odd

b) x^n-1 is a multiple of $x+1$ if n is even

c) x^n-1 is a multiple of $x-1$

SOLUTIONS

1)

Yes. Because $3+6+4+7+0+5+8+7+6+2+4+2+7+5=66$ a multiple of 3

2)

No. Because 828 is not multiple of 8

3)

No. Because $4-9+7-7+5-8+8-3+6-6+1-2+0-5=-9$ not a multiple of 11

4)

$6-3+7-7+3-6=0$ and 0 is a multiple of 11

5)

$7\times11\times13=1001$ and $726726=726\times1001$

6)

a) if n is odd:

$f(x)=x^n+1$ so $f(-1)=0$ so $(x+1)$ is a factor of $f(x)$

b) if n is even:

$f(x)=x^n-1$ so $f(-1)=0$ so $(x+1)$ is a factor of $f(x)$

c) $f(x)=x^n-1$ so $f(1)=0$ so $(x-1)$ is a factor of $f(x)$

Paint Pot if you know about integration …

Take the curve $y=\frac{1}{x}$ between $x=1$ and $x=\infty$ and rotate it $360°$ around the x axis to form a long, funnel shaped paint pot. DIAGRAM???

The volume of the paint pot is:

$$\int_1^\infty \pi y^2 dx = \int_1^\infty \frac{1}{x^2} dx = ... = \pi$$

The surface area of the paint pot is:

$$\int_1^\infty 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx = \int_1^\infty 2\pi \frac{1}{x} \sqrt{1+\left(\frac{1}{x^4}\right)} dx$$

Now this integral is too difficult for me but …

$$2\pi \frac{1}{x} \sqrt{1+\left(\frac{1}{x^4}\right)} > 2\pi \frac{1}{x}$$

So the surface area of the paint pot is greater than:

$$\int_1^\infty 2\pi \frac{1}{x} dx = ... = \infty$$

So the paint pot has a finite volume but an infinite surface area.

If you fill the pot with paint then you won’t have enough paint to cover the surface of the pot!

Paradoxes

Example 1

Achilles and the tortoise decide to run a race. Because Achilles can run 10 times as fast as the tortoise, the tortoise is given a head start of 100m. The tortoise starts at point A.

By the time Achilles reaches point A, the tortoise has moved on by 10m, to point B.

By the time Achilles reaches point B, the tortoise has moved on by 1m, to point C.

By the time Achilles reaches point C, the tortoise has moved on by 0.1m, to point D.

Each time Achilles reaches the point where the tortoise was, the tortoise has moved on.

So Achilles can never catch up with the tortoise.

Example 2

In the first half of the cricket season:

Fred faced 475 deliveries and scored 170 runs. John faced 717 deliveries and scored 250 runs.

So Fred had the better batting average.

In the second half of the cricket season:

Fred faced 725 deliveries and scored 185 runs. John faced 483 deliveries and scored 112 runs.

So Fred had the better batting average.

Over the whole season:

Fred faced 1200 deliveries and scored 355 runs. John faced 1200 deliveries and scored 362 runs.

So John had the better batting average.

Example 3

I claim that all ravens are black. You wish to investigate my claim, so you look at lots of ravens.

Every time you see a raven and it turns out to be black, your confidence in my claim increases.

All ravens are black, means the same as, all non-black things are non-ravens.

So every time you see a non-black thing and it turns out to be a non-raven, your confidence in my claim increases. Suppose you see a yellow thing and it turns out to be a banana. This should increase your confidence in my claim.

Example 4

There are two envelopes on the table. One envelope contains twice as much money as the other envelope. You can keep one of these envelopes and the money inside. But you are only allowed to look inside one envelope before making your decision. What should you do?

You look inside one envelope. It contains £100. So the other envelope must contain £50 or £200. If you choose to keep the other envelope, you could lose £50 but you are just as likely to gain £100.

So the best plan is to keep the other envelope. You come to this conclusion however much money is in the first envelope. So to save time, choose an envelope, don't bother to look inside it, and keep the other one.

Example 5

Teacher Alice sets her pupils a test and their mean score is 60%. Teacher Bill sets his pupils the same test and their mean score is 50%. Susan is in Alice's class. Susan scored 54%. If Susan is moved from Alice's class to Bill's class then the mean score for both classes would increase.

Example 6

A naughty girl did not complete her maths homework, so she is to be punished. She is allowed to make one statement. If the statement is true, she will have to clean the board. If the statement is false, she will have to pick-up litter. The girl makes the statement: "I shall have to pick-up litter".
So what happens?

Example 7

I teach my maths class every Monday, Tuesday, Wednesday, Thursday and Friday. I tell them that they are going to have a test next week. But to add to their misery, they will not know, at the start of each day, if they are getting the test that day. Then one of my brighter students says:
"We can't have the test on Friday, because if we haven't had the test by then, we will know at the start of Friday that we are getting the test that day"
So the test has to take place on Monday, Tuesday, Wednesday or Thursday.
The student then argues that the test can't take place on Thursday or Wednesday or Tuesday or Monday.
The student therefore concludes that I can't give them an unexpected test.
Imagine their surprise when they get the test on Tuesday!

Pascal's Triangle

The triangle is usually set out like an isosceles triangle but I have set it out slightly differently:

	Col 0	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9
Row 0	1									
Row 1	1	1								
Row 2	1	2	1							
Row 3	1	3	3	1						
Row 4	1	4	6	4	1					
Row 5	1	5	10	10	5	1				
Row 6	1	6	15	20	15	6	1			
Row 7	1	7	21	35	35	21	7	1		
Row 8	1	8	28	56	70	56	28	8	1	
Row 9	1	9	36	84	126	126	84	36	9	1

1) The numbers in the triangle are selection numbers. (see chapter: Arrangements and Selections)

For example, the number in row 9 and column 3 is $(9C3)$

2) We can generate each row of the triangle from the row above. To generate row 10:

$(10C0)=1$

$(10C1)=(9C0)+(9C1)=1+9=10$

$(10C2)=(9C1)+(9C2)=9+36=45$

$(10C3)=(9C2)+(9C3)=36+84=120$ etc

3) Look at the numbers in row 7 of Pascal's triangle: $1,7,21,35,35,21,7,1$

A typical number in this row is $(7C3)=\dfrac{7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)(4\times3\times2\times1)}$

After lots of cancelling, we are left with a positive integer. The 7 on the top of the fraction can't be cancelled out by numbers on the bottom of the fraction because 7 is prime. So $(7C3)$ must be a multiple of 7

In general:

If p is prime then all the numbers in line p of Pascal's triangle will be a multiple of p

(apart from the 1 at each end)

4) Binomial theorem for multiplying out brackets

$(1+x)^1=1+x$

$(1+x)^2=1+2x+x^2$

$(1+x)^3=1+3x+3x^2+x^3$

$(1+x)^4=1+4x+6x^2+4x^3+x^4$

In general: if n is a positive integer

$(1+x)^n=(nC0)+(nC1)x+(nC2)x^2+(nC3)x^3+...+(nCn)x^n$

sub in $x=1$ $(nC0)+(nC1)+(nC2)+...+(nCn)=2^n$

sub in $x=-1$ $(nC0)-(nC1)+(nC2)-(nC3)+...=0$

sub in $x=2$ $(nC0)+2(nC1)+2^2(nC2)+...+2^n(nCn)=3^n$

etc

By adding or subtracting the first two results we get:

$(nC0)+(nC2)+(nC4)+...=2^{n-1}$

$(nC1)+(nC3)+(nC5)+...=2^{n-1}$

EXERCISE

Write down row 10 of Pascal's triangle.

SOLUTION

1, 1+9=10, 9+36=45, 36+84=120, 84+126=210, 126+126=252, 126+84=210, 84+36=120, 36+9=45, 9+1=10, 1

Pigeon-hole Principle

I have 10 boxes and 14 pigeons. I put each pigeon in a box. Obviously one (or more) box must end up with two (or more) pigeons.

How can something so trivial be of any use? Let’s find out.

EXERCISE

1) There are 10 people in a room. Everyone shouts out an integer between 1 and 9

Why must two (or more) people shout out the same integer?

2) There are 40 people at a party. Everyone shakes hands with at least one other person. At the end of the party, everyone is asked: “How many people did you shake hands with?”

Why must two (or more) people give the same reply?

3) Pick 11 different positive integers.

Why must two (or more) of these integers have a difference that is a multiple of ten.

4) Pick 6 different integers from 1, 2, 3, … 10

Why must two (or more) of these integers add up to 11?

SOLUTIONS

1) I have 9 boxes, labelled 1, 2, … 9. I put each person in a box.

If a person shouts out 8 then I put them in the box with 8 on the label. etc

There are 9 boxes and 10 people. One (or more) box must contain two (or more) people.

2) I have 39 boxes, labelled 1, 2, … 39. I put each person in a box.

If a person shakes hands with 17 people, then I put them in the box with 17 on the label. etc

There are 39 boxes and 40 people. One (or more) box must contain two (or more) people.

Note: each person shakes hands with 1 or 2 or 3 … or 39 people.

Note: this argument will apply however many people are at the party.

3) I have 10 boxes, labelled 0, 1, 2, … 9. I put each integer in a box.

If an integer is 3768 then I put it in the box with 8 on the label because 8 is its last digit. etc

There are 10 boxes and 11 integers. One (or more) box must contain two (or more) integers.

Note: integers in the same box have the same last digit so their difference is a multiple of 10

4) I have 5 boxes, labelled A, B, C, D, E. I put each integer in a box.

1 and 10 go in box A.

2 and 9 go in box B.

3 and 8 go in box C.

4 and 7 go in box D.

5 and 6 go in box E.

There are 5 boxes and 6 integers. One (or more) box must contain two (or more) integers.

Note: integers in the same box add up to 11

Pi

Here are some nice formulas for π – there are lots more.

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

$$\frac{\pi}{2}=\frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\ldots$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

$$\frac{\pi^2}{6}=\left(\frac{1}{1-1/2^2}\right)\left(\frac{1}{1-1/3^2}\right)\left(\frac{1}{1-1/5^2}\right)\left(\frac{1}{1-1/7^2}\right)\left(\frac{1}{1-1/11^2}\right)\ldots$$

$\frac{\pi}{4}=\frac{3}{4}\times\frac{5}{4}\times\frac{7}{8}\times\frac{11}{12}\times\frac{13}{12}\ldots$ where the numerators are the primes (not including2) and each denominator is the multiple of 4 nearest to the corresponding numerator.

$$\frac{\pi}{4}=4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right)$$

$$\frac{\pi\sqrt{2}}{4}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots$$

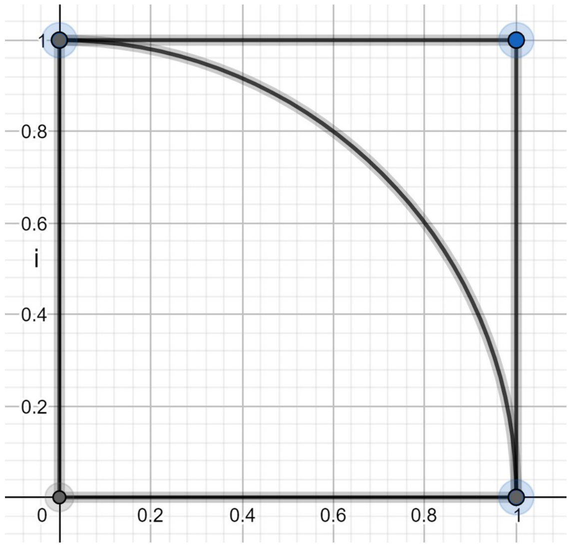
$$\frac{\pi-3}{4}=\frac{1}{2\times3\times4}-\frac{1}{4\times5\times6}+\frac{1}{6\times7\times8}-\ldots$$

$$\frac{\pi}{2}=1+\frac{1}{3}+\frac{1\times2}{3\times5}+\frac{1\times2\times3}{3\times5\times7}+\ldots$$

$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\ldots$ where $1/n$ is preceded by a $-$ sign if and only if n has an odd number of prime factors of the form $4k+1$

$$\frac{\pi}{2}-1=\frac{(2^1)(1!)^2}{3!}+\frac{(2^2)(2!)^2}{5!}+\frac{(2^3)(3!)^2}{7!}+\frac{(2^4)(4!)^2}{9!}+\ldots$$

We can estimate π using random numbers



Get two random numbers x and y where $0 \leq x \leq 1$ and $0 \leq y \leq 1$

We can think of (x, y) as the co-ordinates of a point inside the unit square. This point will be inside the quarter circle if $x^2 + y^2 < 1$ Now get lots of points.

The number of points inside the quarter circle divided by the total number of points will be approximately equal to the area of the quarter circle divided by the area of the unit square.

Show that this is $\pi/4$

So if we pick 1000 points and 763 of these points are inside the quarter circle then:

$$\frac{763}{1000} \approx \frac{\pi}{4} \quad \text{giving} \quad \pi \approx 3.05$$

To get a better approximation we need to take more points. See Appendix 1 for a computer program to estimate π

Viete's formula for π if you know about trigonometry … (all angles in radians)

a) We know:

$\cos 2\theta = 2\cos^2\theta - 1$ the double angle formula

Show that:

$cos\theta = \sqrt{\left(\frac{1+\cos 2\theta}{2}\right)}$ the half angle formula

We know:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Use the half angle formula to show that:

$$\cos\frac{\pi}{8}=\sqrt{\left(\frac{1+\cos\pi/4}{2}\right)}=\sqrt{\left(\frac{1+\sqrt{2}/2}{2}\right)}=...=\frac{\sqrt{2+\sqrt{2}}}{2}$$

Use the half angle formula to show that:

$$\cos\frac{\pi}{16}=\sqrt{\left(\frac{1+\cos\pi/8}{2}\right)}=...=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

etc

b) We know:

$\sin 2\theta=2\,cos\theta sin\theta$ the double angle formula

If we repeatedly use the double angle formula we get:

$$sin\theta=2\cos\frac{\theta}{2}\sin\frac{\theta}{2}=2\cos\frac{\theta}{2}\left(2\cos\frac{\theta}{4}\sin\frac{\theta}{4}\right)=2\cos\frac{\theta}{2}2\cos\frac{\theta}{4}\left(2\cos\frac{\theta}{8}\sin\frac{\theta}{8}\right)\quad \text{etc}$$

So:

$$sin\theta=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cos\frac{\theta}{16}...\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$$

So:

$$\frac{sin\theta}{\left(\frac{\theta}{2^n}\right)}=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}...\cos\frac{\theta}{2^n}\left(\frac{\sin\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}\right)$$

So:

$$\frac{sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}...\cos\frac{\theta}{2^n}\left(\frac{\sin\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}\right)$$

We know:

if $x\rightarrow 0$ then $\frac{sinx}{x}\rightarrow 1$

So:

if $n\rightarrow\infty$ then $\left(\frac{\sin\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}\right)\rightarrow 1$

So:

$$\frac{sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}...$$

Put:

$$\theta = \frac{\pi}{2}$$

and show that:

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\ldots$$

c) Use part (a) and part (b) to obtain Viete’s formula:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\ldots$$

Quotes

You can easily find lots of mathematics quotes on the internet. Here are a few:

1)

To call in the statistician after the experiment is done may be no more than asking him to perform a post-mortem examination. He may be able to say what the experiment died of.

R. A. Fisher

2)

Everything should be made as simple as possible, but not simpler.

A. Einstein

3)

A mathematician is a machine for turning coffee into theorems.

P. Erdos

4)

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

G. H. Hardy

5)

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

D. Hilbert

6)

Logic is the art of going wrong with confidence.

M. Kline

7

In mathematics you don't understand things, you just get used to them.

J. von Neumann

8)

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for re-election.

H. Rossi

9)

I remember once going to see Ramanujan when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

G. H. Hardy

10)

It is more important to have beauty in one's equations that to have them fit experiment.

P. A. M. Dirac

11)

Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.

H. Poincaré

12)

Ordinary language is totally unsuited for expressing what Physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say.

B. Russell

13)

The propositions of mathematics have, therefore, the same unquestionable certainty which is typical of such propositions as "All bachelors are unmarried," but they also share the complete lack of empirical content which is associated with that certainty. The propositions of mathematics are devoid of all factual content; they convey no information whatever on any empirical subject matter.

C. Hempel

14)

The universe cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.

G. Galileo

15)

[Criticized for using formal mathematical manipulations, without understanding how they worked] Should I refuse a good dinner simply because I do not understand the process of digestion?

O. Heaviside

16)

If triangles invented a god, they would make him three-sided.

Montesquieu

17)

If I have seen further than others, it is by standing upon the shoulders of giants.

I. Newton

18)

Mathematics is the art of giving the same name to different things.

H. Poincare

19)

Thus Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

B. Russell

20)

In science, you want to say something that nobody knew before, in words which everyone can understand. In poetry you want to say something that everyone knows already in words that nobody can understand.

P. A. M. Dirac

21)

One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.

H. Hertz

22)

Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess play: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy

23)

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

B. Russell

24)

Read Euler, read Euler, he is the master of us all.

P. S. Laplace

25)

There is no branch of mathematics, however abstract, which may not some day be applied to the phenomena of the real world

N. Lobatchevsky

26)

He uses statistics as a drunken man uses lamp-posts. For support rather than illumination.

A. Lang

Rationals and Irrationals

$\frac{13}{7}$ is a rational number because it is an integer divided by another integer.

Theorem

x is a rational number if and only if x is a terminating or recurring decimal.

This is really two theorems:

Theorem 1

If x is a rational number then x is a terminating or recurring decimal.

Proof

Say $x=\frac{13}{7}$

$\frac{13}{7}=1+\frac{6}{7}$ $\frac{13}{7}$ equals 1 remainder 6

$\frac{6}{7}=\frac{1}{10}\left(\frac{60}{7}\right)=\frac{1}{10}\left(8+\frac{4}{7}\right)$ $\frac{60}{7}$ equals 8 remainder 4

$\frac{4}{7}=\frac{1}{10}\left(\frac{40}{7}\right)=\frac{1}{10}\left(5+\frac{5}{7}\right)$ $\frac{40}{7}$ equals 5 remainder 5

$\frac{5}{7}=\frac{1}{10}\left(\frac{50}{7}\right)=\frac{1}{10}\left(7+\frac{1}{7}\right)$ $\frac{50}{7}$ equals 7 remainder1

etc

So $\frac{13}{7}=1+\frac{8}{10}+\frac{5}{100}+\frac{7}{1000}+...=1.857...$

The remainders can only be $0,1,2,3,4,5,6$

Either we will get a remainder of 0, in which case the decimal terminates.

Or we will get a remainder we have had before, in which case the decimal recurs.

Either way x is a terminating decimal or a recurring decimal

Theorem 2

If x is a terminating or recurring decimal then x is a rational number.

Proof

If:

x is a terminating decimal, say $x=0.123$

then:

$x = \frac{123}{1000}$ so x is a rational number.

If:

x is a recurring decimal, say $x = 0.123123123\ldots$

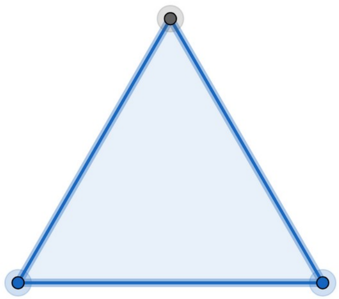
then:

$1000x = 123.123123123\ldots$ so $999x = 123$ so $x = \frac{123}{999}$ so x is a rational number.

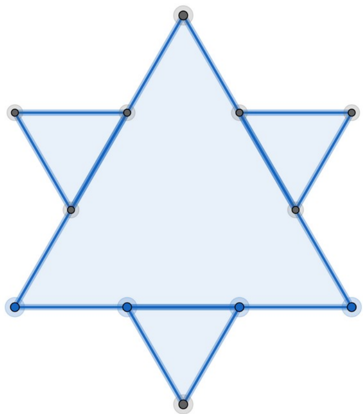
Either way x is a rational number.

Snow-flake Curve

We start with an equilateral triangle:



Then we add a smaller equilateral triangle to each of the three sides:



Then we add an even smaller equilateral triangle to each of the twelve sides.

We repeat this an infinite number of times to get the snowflake curve. We want to find the perimeter and area of the this curve.

Perimeter

At the start, we have 3 sides of length L so the perimeter is $3L$

The first iteration replaces each side by 4 sides of length $\frac{L}{3}$ so the perimeter is $\frac{4L}{3}$

Each iteration increases the perimeter by a factor of $\frac{4}{3}$

So after an infinite number of iterations, the perimeter is infinite.

Area

At the start, we have a triangle with area 1

The first iteration adds 3 triangles, each of area $\frac{1}{9}$

The second iteration adds 3×4 triangles, each of area $\left(\frac{1}{9}\right)^2$

The third iteration adds $3 \times 4 \times 4$ triangles, each of area $\left(\frac{1}{9}\right)^3$

etc

So after an infinite number of iterations, the area is:

$$1+\left(3 \times \frac{1}{9}\right)+\left(3 \times 4 \times \frac{1}{9^2}\right)+\left(3 \times 4^2 \times \frac{1}{9^3}\right)+\ldots$$

Now:

$$\left(3 \times \frac{1}{9}\right)+\left(3 \times 4 \times \frac{1}{9^2}\right)+\left(3 \times 4^2 \times \frac{1}{9^3}\right)+\ldots=\frac{3/9}{1-4/9}=\frac{3}{5} \quad \text{(see Appendix 2: Geometric Sequence)}$$

So the final area is:

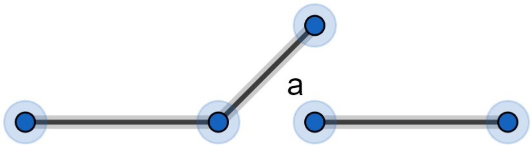
$$1+\frac{3}{5}=\frac{8}{5}$$

So the snow-flake curve has a finite area but an infinite perimeter.

Switching Circuits

Example 1

Here is a switch called a . It can be closed or open.



If a is closed then electric current can flow. We say $a=1$

If a is open then electric current cannot flow. We say $a=0$

Example 2

Here are two switches in series:

$a.b$ denotes switches a and b in series (this is not multiplication!)

If a is closed and b is closed then electric current can flow.

So if $a=1$ and $b=1$ then $a.b=1$ So $1.1=1$

If a is open or b is open (or both) then electric current cannot flow.

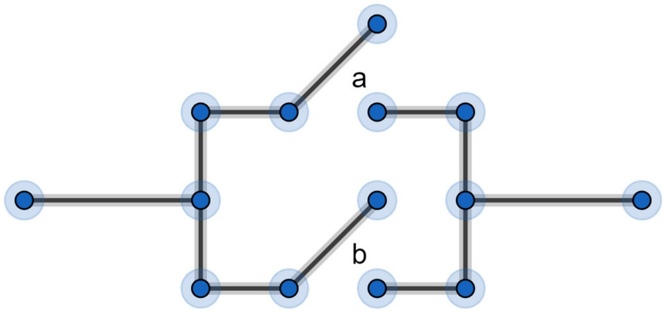
So if $a=0$ or $b=0$ (or both) then $a.b=0$ So $0.1=0$ $1.0=0$ $0.0=0$

We can set this out in a table:

a	b	$a.b$
0	0	0
0	1	0
1	0	0
1	1	1

Example 3

Here are two switches in parallel:



$a+b$ denotes switches a and b in parallel (this is not addition!)

a is closed or b is closed (or both) then electric current can flow.

So if $a=1$ or $b=1$ (or both) then $a+b=1$ So $1+0=1$ $0+1=1$ $1+1=1$

If a is open and b is open then electric current cannot flow.

Soif $a=0$ and $b=0$ then $a+b=0$ So $0+0=0$

We can set this out in a table:

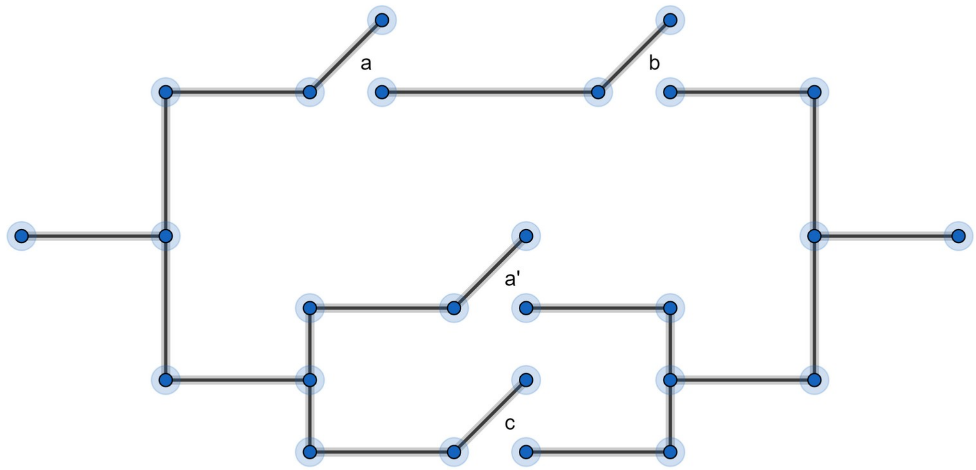
a	b	$a+b$
0	0	0
0	1	1
1	0	1
1	1	1

We can have switches that are linked to each other. If two switches are both called a then they are always in the same state, either both open or both closed. If one switch is called a and another switch is called a' then they are always in opposite states, one open and the other one closed.

If $a=1$ then $a'=0$ If $a=0$ then $a'=1$

Example 4

Here is a circuit:



The mathematical expression for this circuit is: $(a.b)+(a'+c)$ Think about it.

The table for this circuit is:

a	b	c	$a.b$	a'	$a'+c$	$(a.b)+(a'+c)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	1	1
0	1	1	0	1	1	1
1	0	0	0	0	0	0
1	0	1	0	0	1	1
1	1	0	1	0	0	1
1	1	1	1	0	1	1

see EXERCISE 1

Look at Exercise 1 questions (4) and (5)

You should have found that the columns for $(a.b)+(a.c)$ and $a.(b+c)$ are the same.

We say $(a.b)+(a.c)=a.(b+c)$

The circuit in (5) does the same thing as the circuit in (4) but uses fewer switches.

$a.1$ denotes switch a in series with a closed switch

electric current can flow if a is closed, so $a.1=1$ if $a=1$

electric current cannot flow if a is open, so $a.1=0$ if $a=0$

So $a.1=a$

$a+1$ denotes switch a in parallel with a closed switch. Electric current can always flow.

So $a+1=1$

$a.0$ denotes switch a in series with an open switch. Electric current can never flow.

So $a.0=0$

$a+0$ denotes switch a in parallel with an open switch

electric current can flow if a is closed, so $a+0=1$ if $a=1$

electric current cannot flow if a is open, so $a+0=0$ if $a=0$

So $a+0=a$

Use tables to prove the following rules: (no need to do them all)

$(a')'=a$

$a.a=a$	$a+a=a$
$a.a'=0$	$a+a'=1$
$a.b=b.a$	$a+b=b+a$
$(a.b).c=a.(b.c)$	$(a+b)+c=a+(b+c)$
$a.(b+c)=(a.b)+(a.c)$	$a+(b.c)=(a+b).(a+c)$
$(a.b)'=a'+b'$	$(a+b)'=a'.b'$
$a.(a+b)=a$	$a+(a.b)=a$

EXERCISE

1) Draw the circuit and fill in the table for $(a+b)+(a.b)$

a	b	$a+b$	$a.b$	$(a+b)+(a.b)$
0	0			
0	1			
1	0			
1	1			

2) Draw the circuit and fill in the table for $a.(a'+b)$

a	b	a'	$a'+b$	$a.(a'+b)$
0	0			
0	1			
1	0			
1	1			

3) Draw the circuit and fill in the table for $(a.b)+c$

a	b	c	$a.b$	$(a.b)+c$
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

4) Draw the circuit and fill in the table for $(a.b)+(a.c)$

a	b	c	$a.b$	$a.c$	$(a.b)+(a.c)$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

5) Draw the circuit and fill in the table for $a.(b+c)$

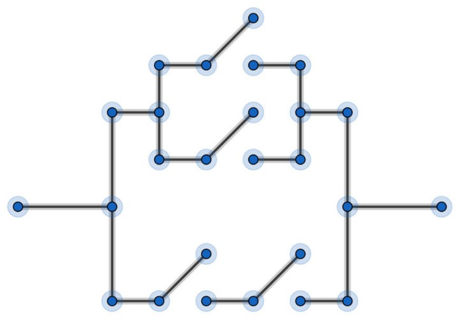
a	b	c	$b+c$	$a.(b+c)$
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

SOLUTIONS

1)

a	b	$a+b$	$a.b$	$(a+b)+(a.b)$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	1

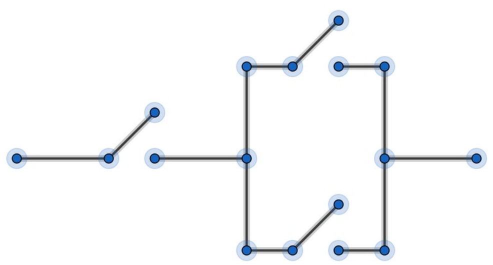
Here is the circuit. Can you label the switches?



2)

a	b	a'	$a'+b$	$a.(a'+b)$
0	0	1	1	0
0	1	1	1	0
1	0	0	0	0
1	1	0	1	1

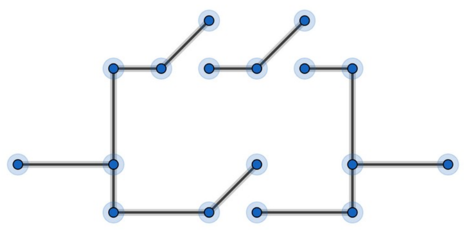
Here is the circuit. Can you label the switches?



3)

a	b	c	$a.b$	$(a.b)+c$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

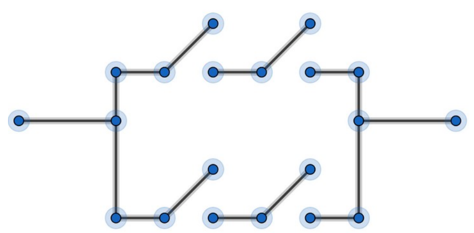
Here is the circuit. Can you label the switches?



4)

a	b	c	$a.b$	$a.c$	$(a.b)+(a.c)$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

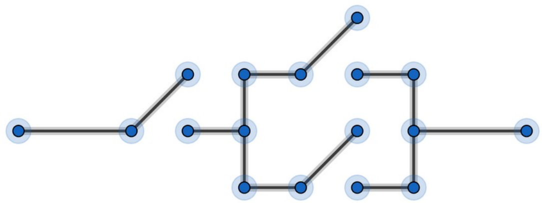
Here is the circuit. Can you label the switches?



5)

a	b	c	$b+c$	$a.(b+c)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Here is the circuit. Can you label the switches?



Three Games

1)

Nim – a game for two players

There is a pile of 18 counters on the table. Players take turns to remove counters. Each player can remove 1, 2, 3 or 4 counters on each turn. The player who removes the last counter is the winner.

For example:

Eric takes 4 counters, Bill takes 2, Eric takes 3, Bill takes 4, Eric takes 3, Bill takes 2. Bill wins.

Jane has a good strategy to play this game. If Eric takes n counters then Jane next takes $5-n$ counters so they have taken 5 counters between them.

We start with 18 counters. Jane goes first and takes 3 counters, leaving 15 counters.

15 is a multiple of 5. Now it is Eric's go. Can you see why Jane will win?

Note: You can play this game with any number of counters at the start.

We start with 97 counters. Jane goes first and takes 2 counters, leaving 95 counters.

95 is a multiple of 5. Now it is Eric's go. Can you see why Jane will win?

We start with 30 counters. Jane lets Eric go first. How nice of her. Can you see why Jane will win?

2)

Fifteen – a game for two players

We have nine cards numbered 1, 2, … 9. Players take turns to take a card. The first player who has taken three cards that add up to fifteen is the winner.

For example:

Eric takes the 4, Bill takes the 9, Eric takes the 6, Bill takes the 5, Eric takes the 3, Bill takes the 8, Eric takes the 2, Bill takes the 1. Bill holds the 9, 5 and 1. So Bill wins.

Jane has a good strategy to play this game. She has a magic square:

8	3	4
1	5	9
6	7	2

How will this help?

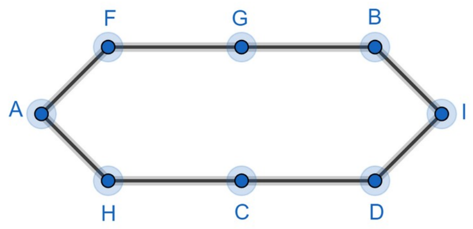
3)

A game for one player

A	B	C
D	E	F
G	H	I

Place white knights on squares A and C. Place black knights on squares G and I. Move the knights (in any order) so that the white knights end up on the squares G and I and the black knights end up on the squares A and C.

The diagram below shows which squares are connected by knight moves. So, for example, a knight can move from square F to square A or to square G. We could play the game on this diagram. It would be much easier. Can you see why?



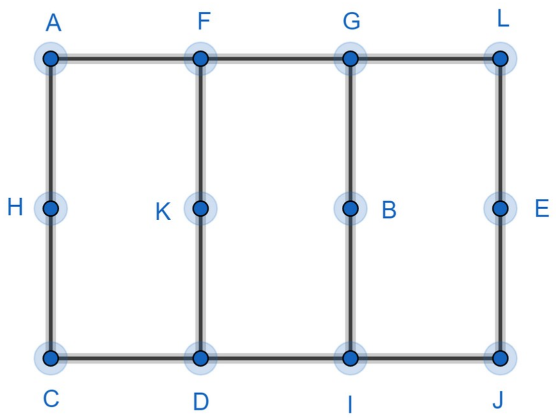
4)

A game for one player

A	B	C
D	E	F
G	H	I
J	K	L

Place white knights on squares A, B and C. Place black knights on squares J, K and L. Move the knights (in any order) so that the white knights end up on the squares J, K and L and the black knights end up on the squares A, B and C.

The diagram below shows which squares are connected by knight moves ...



Tiling

Example 1

I want to cover my 8×8 chess-board with thirty-two 2×1 tiles. Try it. It's easy.

Example 2

Someone has removed two squares from my chess-board. The bottom left-hand corner square and the top right-hand corner square. Can I cover my mutilated chess-board with thirty-one 2×1 tiles?

Each 2×1 tile covers one black square and one white square. So however I arrange the tiles, I will always cover the same number of black squares and white squares. The bottom left-hand corner square and the top right-hand corner are both black. So the mutilated chess-board has 30 black squares and 32 white squares. So I cannot cover my mutilated chess-board with thirty-one 2×1 tiles.

Example 3

Someone has removed two squares from my chess-board. Can I cover my mutilated chess-board with thirty-one 2×1 tiles?

The answer will depend on which squares have been removed. We know, from the previous example that if two black squares have been removed or if two white squares have been removed then the answer is: No!

Gomory's Theorem

I can cover my mutilated chess-board with thirty-one 2×1 tiles if any black square and any white square have been removed.

Proof

Here is a chess-board:

1	64	63	62	61	60	59	58
2	51	52	53	54	55	56	57
3	50	49	48	47	46	45	44
4	37	38	39	40	41	42	43
5	36	35	34	33	32	31	30
6	23	24	25	26	27	28	29
7	22	21	20	19	18	17	16
8	9	10	11	12	13	14	15

Look at the way I have numbered the squares. I can go for a walk around the board, visiting every square in the order 1, 2, 3, … 64

Say, the black square 24 and the white square 41 have been removed on my mutilated chess-board. I can cover the board as follows:

Put the first tile on squares 25 and 26, the next tile on squares 27 and 28, … on squares 39 and 40, the next tile on squares 42 and 43, the next tile on squares 44 and 45, … on squares 22 and 23.

This method will work whichever black square and whichever white square have been removed.

Investigation

Someone has removed two black squares and two white squares from my chess-board. Can I cover my mutilated chess-board with thirty 2×1 tiles?

Example 4

Can I cover a chess-board with 3×1 tiles?

Each tile covers 3 squares. There are 64 squares on the board. So the answer is: No!

Example 5

Someone has removed one square from my chess-board. Can I cover my mutilated chess-board with twenty-one 3×1 tiles?

The answer will depend on which square has been removed. Let's colour the squares white, red and black:

W	R	B	W	R	B	W	R
B	W	R	B	W	R	B	W
R	B	W	R	B	W	R	B
W	R	B	W	R	B	W	R
B	W	R	B	W	R	B	W
R	B	W	R	B	W	R	B
W	R	B	W	R	B	W	R
B	W	R	B	W	R	B	W

Each 3×1 tile covers one white square, one red square and one black square. So however I arrange the tiles, I will always cover the same number of white squares, red squares and black squares. The chess-board (before the square has been removed) has 22 white squares, 21 red squares and 21 black squares. So the removed square must be white. But can it be any white square?

If I could cover the mutilated chess-board with the bottom right-hand corner square removed then I could rotate the board $90°$ clockwise and I would have covered the mutilated chess-board with the bottom left-hand corner removed. But we know that this is not possible.

To have any hope of covering my mutilated chess-board, the removed square must be one of the * squares:

		*			*		
		*			*		

Can I can cover my mutilated chess-board if the removed square is one of the * squares?

I don’t know. Try it.

Example 6

This final example is a bit different and involves no mutilation.

I have a 100×101 board.

I have lots of 2×2 tiles 4×4 tiles 6×6 tiles and 13×13 tiles available.

Can I cover my board with tiles?

We are going to colour the squares black and white, but in an unusual way:

B	B	B	B	B	B	...
W	W	W	W	W	W	...
B	B	B	B	B	B	...
W	W	W	W	W	W	...
B	B	B	B	B	B	...
W	W	W	W	W	W	...
...	...	...	...	...	...	...

Each 2×2 tile covers 2 black squares and 2 white squares.

Each 4×4 tile covers 8 black squares and 8 white squares.

Each 6×6 tile covers 18 black squares and 18 white squares.

Each 13×13 tile covers 91 black squares and 78 white squares or 78 black squares and 91 white squares. So the difference between the number of black squares I can cover and the number of white squares I can cover must be a multiple of 13. But the board has 5100 black squares and 5000 white squares. So the answer is: No!

Triangle Problem.

Theorem

Mark six points (A, B, C, D, E, F) on a piece of paper so that no three points are in a straight line.

Join each pair of points with a straight line. Colour each line red or green.

However you choose to colour the lines, there will always be a triangle with 3 red lines or a triangle with 3 green lines.

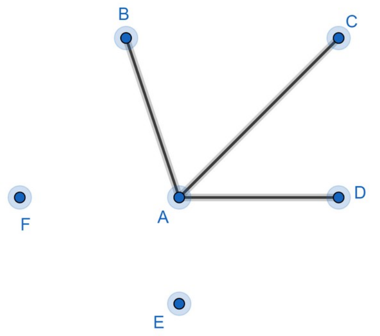
Proof

A is connected to 5 lines and each line is either red or green.

either: A is connected to 3 (or more) red lines

or: A is connected to 3 (or more) green lines

Consider the case where A is connected to 3 (or more) red lines, say AB, AC and AD.



If line BC is red then triangle ABC has 3 red lines

If line CD is red then triangle ACD has 3 red lines

If line BD is red then triangle ABD has 3 red lines

If lines BC, CD, BD are all green then triangle BCD has 3 green lines

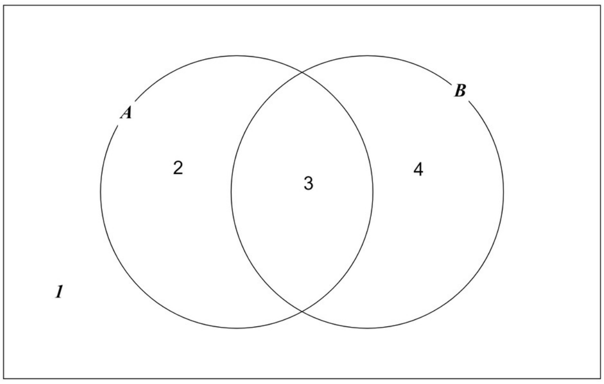
We can repeat this argument if A connected to 3 (or more) green lines.

Either way, we have the required triangle.

Venn Diagrams and Tables

Venn Diagrams

Two circle diagrams:



Let's talk about regions inside this rectangle.

a is the region inside the A circle — regions 2 and 3

a' is the region not inside the A circle — regions 1 and 4

$a \cap b$ is the region inside both the A circle and the B circle — region 3

$a \cup b$ is the region inside either the A circle or the B circle (or both) — regions 2, 3 and 4

etc

Example 1

a	regions 2 and 3
b'	regions 1 and 2
$a \cap b'$	region 2

Example 2

a'	regions 1 and 4
b	regions 3 and 4
$a' \cup b$	regions 1,3 and 4

Example 3

$a \cap b$	region 3
$(a \cap b)'$	regions 1, 2 and 4

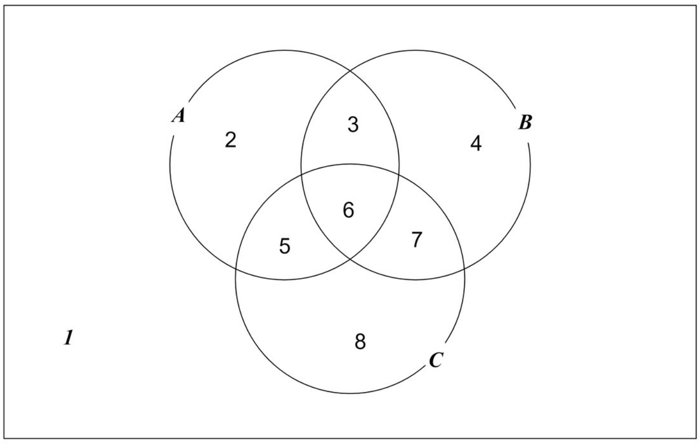
Example 4

a'	regions 1 and 4
b'	regions 1 and 2
$a' \cup b'$	regions 1,2 and 4

Now $(a \cap b)'$ and $a' \cup b'$ are both regions 1,2 and 4

We say $(a \cap b)' = a' \cup b'$

Three circle diagrams:



Let's talk about regions inside this rectangle.

Example 5

$a \cap b$	regions 3 and 6
c	regions 5, 6, 7, 8
$(a \cap b) \cap c$	region 6

Example 6

$a \cup b$	regions 2, 3, 4, 5, 6, 7
$a \cup c$	regions 2, 3, 5, 6, 7, 8
$(a \cup b) \cap (a \cup c)$	regions 2,3,5,6, 7

Example 7

$a \cap b$	regions 3, 6
$a \cap c$	regions 5, 6
$(a \cap b) \cup (a \cap c)$	regions 3, 5, 6

Example 8

$b \cup c$	regions 3, 4, 5, 6, 7, 8
a	regions 2, 3, 5, 6
$a \cap (b \cup c)$	regions 3, 5, 6

Now

$(a \cap b) \cup (a \cap c)$ and $a \cap (b \cup c)$ are both regions 3, 5, 6

We say $(a \cap b) \cup (a \cap c) = a \cap (b \cup c)$

Let 1 denote the whole region inside the rectangle, that is regions 1, 2, 3, 4, 5, 6, 7, 8

Let 0 denote no region. So:

$a \cap 1$	regions 2, 3, 5, 6	so $a \cap 1 = a$
$a \cup 1$	regions 1, 2, 3, 4, 5, 6, 7, 8	so $a \cup 1 = 1$
$a \cap 0$	no region	so $a \cap 0 = 0$
$a \cup 0$	regions 2, 3, 5, 6	so $a \cup 0 = a$

Use Venn diagrams to prove the following rules: (no need to do them all)

$(a')' = a$

$a \cap a = a$	$a \cup a = a$
$a \cap a' = 0$	$a \cup a' = 1$
$a \cap b = b \cap a$	$a \cup b = b \cup a$
$(a \cap b) \cap c = a \cap (b \cap c)$	$(a \cup b) \cup c = a \cup (b \cup c)$
$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$	$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$
$(a \cap b)' = a' \cup b'$	$(a \cup b)' = a' \cap b'$
$a \cap (a \cup b) = a$	$a \cup (a \cap b) = a$

You will have noticed that we have the same rules for Propositions, Switching Circuits and Venn Diagrams (with slightly different notation). This means that we can use Venn diagrams to simplify expressions arising from propositions or switching circuits.

Propositions

We can use a Venn diagram to show that: $(a \cap b) \cup (a \cap c) = a \cap (b \cup c)$

And this shows that: $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$

Switching circuits

We can use a Venn diagram to show that: $(a \cap b) \cup (a \cap c) = a \cap (b \cup c)$

And this shows that: $(a.b) + (a.c) = a.(b + c)$

see EXERCISE 1

Tables

We can also use tables to simplify expressions arising from propositions or switching circuits.

Let's use the notation we had for switching circuits. (We could equally well use the notation we had for propositions)

Here is a table for two variables. The cell marked * represents $a'.b$

	a	a'
b		*
b'		

Here is another table. The cells marked * together represent $(a.b)+(a.b')$

	a	a'
b	*	
b'	*	

But:

$$(a.b)+(a.b')=a.(b+b')=a.1=a$$

So the cells marked * together represent a

etc

Example 1

Simplify: $(a.b)+(a'.b)$

We mark the cells

	a	a'
b	*	*
b'		

The * occupy all the b cells.

So:

$$(a.b)+(a'.b)=b$$

We could do this using the rules:

$$(a.b)+(a'.b)=(a+a').b=1.b=b$$

Example 2

Simplify: $(a'.b)+(a'.b')+(a.b')$

We mark the cells

	a	a'
b		*
b'	*	*

The * occupy all the a' cells and all the b' cells.

So:

$$(a'.b)+(a'.b')+(a.b')=a'+b'$$

Wait a minute. Haven't we counted the $a'.b'$ cell twice?

Yes we have and it's OK.

You will recall that:

$a+a=a$ so $a'.b'=(a'.b')+(a'.b')$

So:

$$\begin{aligned}(a'.b)+(a'.b')+(a.b')&=(a'.b)+(a'.b')+(a.b')+(a'.b')\\&=a'.(b+b')+(a+a').b'\\&=(a'.1)+(1.b')\\&=a'+b'\end{aligned}$$

Here is a table for three variables. The cell marked * represents $a.b'.c'$

	a	a	a'	a'
b				
b'	*			
	c'	c	c	c'

etc

Example 3

Simplify $(a'.b.c)+(a'.b'.c)$

We mark the cells

	a	a	a'	a'
b			*	
b'			*	
	c'	c	c	c'

The * occupy all the $a'.c$ cells

So:

$$(a'.b.c)+(a'.b'.c)=a'.c$$

Example 4

Simplify $(a.b.c)+(a'.b.c)+(a'.b'.c)$

We mark the cells

	a	a	a'	a'
b		*	*	
b'			*	
	c'	c	c	c'

The * occupy all the $a'.c$ cells and all the $b.c$ cells

So:

$$(a.b.c)+(a'.b.c)+(a'.b'.c)=(a'.c)+(b.c)$$

We can further simplify this to:

$$(a'+b).c$$

Example 5

Simplify $(a.b.c)+(a'.b.c)+(a.b.c')+(a'.b.c')$

We mark the cells

	a	a	a'	a'
b	*	*	*	*
b'				
	c'	c	c	c'

The * occupy all the b cells

So:

$$(a.b.c)+(a'.b.c)+(a.b.c')+(a'.b.c')=b$$

Example 6

Simplify $(a.b.c)+(a.b'.c)+(a'.b.c)+(a'.b'.c)+(a.b.c')+(a'.b.c')$

We mark the cells

	a	a	a'	a'
b	*	*	*	*
b'		*	*	
	c'	c	c	c'

The * occupy all the b cells and all the c cells.

So:

$$(a.b.c)+(a.b'.c)+(a'.b.c)+(a'.b'.c)+(a.b.c')+(a'.b.c')=b+c$$

Example 7

Simplify $(a.b.c)+(a.b'.c)+(a'.b.c)+(a'.b'.c)+(a.b.c')+(a.b'.c')$

We mark the cells

	a	a	a'	a'
b	*	*	*	
b'	*	*	*	
	c'	c	c	c'

The * occupy all the a cells and all the c cells.

So:

$$(a.b.c)+(a.b'.c)+(a'.b.c)+(a'.b'.c)+(a.b.c')+(a.b'.c')=a+c$$

Example 8

Simplify $(a.b.c.d)+(a.b'.c.d)+(a.b'.c.d')+(a'.b.c.d)+(a'.b'.c.d)+(a'.b'.c.d')$

We mark the cells

	a	a	a'	a'	
b					d'
b		*	*		d
b'		*	*		d
b'		*	*		d'
	c'	c	c	c'	

The * occupy all the $b'.c$ cells and all the $c.d$ cells

So:

$$(a.b.c.d)+(a.b'.c.d)+(a.b'.c.d')+(a'.b.c.d)+(a'.b'.c.d)+(a'.b'.c.d')=(b'.c)+(c.d)$$

We can further simplify this to:

$c.(b'+d)$

We have used Venn diagrams and tables to simplify expressions. Instead we could bash through the algebra.

Example

$$
\begin{aligned}
(a.b.c)+(a'.b.c)+(a.b'.c)+(a'.b'.c) &= ((a.b)+(a'.b)+(a.b')+(a'.b')).c \\
&= ((a.b)+(a.b')+(a'.b)+(a'.b')).c \\
&= (a.(b+b')+a'(b+b')).c \\
&= ((a.1)+(a'.1)).c \\
&= (a+a').c \\
&= 1.c \\
&= c
\end{aligned}
$$

I think I prefer to use Venn diagrams and tables.

EXERCISE 1

1)

Use a Venn diagram to show:

$(a\cap b)\cup(a\cap b')\cup(a'\cap b')=a\cup b'$

So the circuit:

$(a.b)+(a.b')+(a'.b')$

is equivalent to circuit:

$a+b'$

We have a circuit. We write down a mathematical expression to describe this circuit. We simplify this expression using a Venn diagram. We redesign our circuit using fewer switches.

How cool is that?

2)

Use a Venn diagram to show:

$(a\cap b\cap c)+(a\cap b\cap c')+(a\cap b'\cap c)+(a\cap b'\cap c')=a$

So circuit:

$(a.b.c)+(a.b.c')+(a.b'.c)+(a.b'.c')$

is equivalent to a single switch

3)

Re-do question (1) using a table.

4)

Re-do question (2) using a table.

EXERCISE 2

Simplify the following, using tables.

1) $(a.b')+(a'.b')$

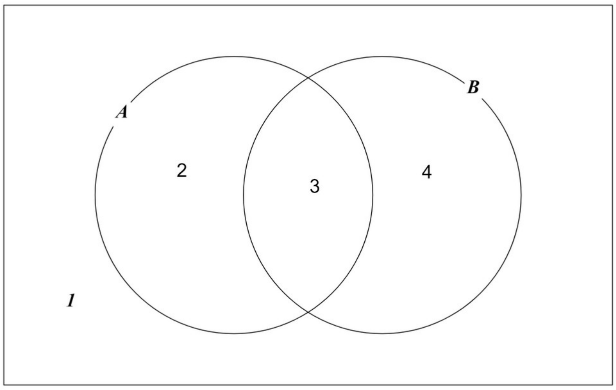
2) $(a.b)+(a.b')+(a'.b')$

3) $(a.b.c)+(a.b'.c)+(a'.b.c)+(a'.b'.c)$

4) $(a'.b.c)+(a'.b'.c)+(a.b.c')+(a.b'.c')+(a'.b.c')$

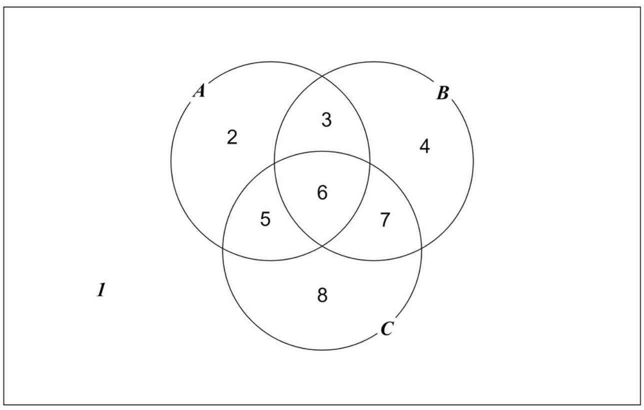
SOLUTIONS 1

1)



$a \cap b$	region 3
$a \cap b'$	region 2
$a' \cap b'$	region 1
$(a \cap b)+(a \cap b')+(a' \cap b')$	regions 1, 2, 3

and



$a \cup b'$ regions 1, 2, 3

2)

$a \cap b \cap c$ region 6

$a \cap b \cap c'$ region 3

$a \cap b' \cap c$ region 5

$a \cap b' \cap c'$ region 2

$(a \cap b \cap c) + (a \cap b \cap c') + (a \cap b' \cap c) + (a \cap b' \cap c')$ regions 2, 3, 5, 6

and

a regions 2, 3, 5, 6

3) $(a.b)+(a.b')+(a'.b')$

	a	a'
b	*	
b'	*	*

$(a.b)+(a.b')+(a'.b') = a+b'$

4) $(a.b.c)+(a.b.c')+(a.b'.c)+(a.b'.c')$

	a	a	a'	a'
b	*	*		
b'	*	*		
	c'	c	c	c'

$(a.b.c)+(a.b.c')+(a.b'.c)+(a.b'.c') = a$

SOLUTIONS 2

1)

	a	a'
b		
b'	*	*

Simplifies to b'

2)

	a	a'
b	*	
b'	*	*

Simplifies to $a+b'$

3)

	a	a	a'	a'
b		*	*	
b'		*	*	
	c'	c	c	c'

Simplifies to c

4)

	a	a	a'	a'
b	*		*	*
b'	*		*	
	c'	c	c	c'

Simplifies to

$(a'.c)+(a.c')+(a'.b)$

Which further simplifies to:

$a'.(b+c)+(a.c')$

Voting Systems

Example 1

There are 3 candidates A, B and C. We need to elect one of them. There are 62 voters and each voter has put the candidates in order of preference:

Order of preference	Number of voters
ABC	18
ACB	15
BAC	5
BCA	14
CAB	8
CBA	2

This means:

18 voters put A as their first choice, B as their second choice and C as their third choice, etc.

Let's look at three different voting systems:

a) First-Past-The-Post

If a voter puts a candidate first choice, then that candidate gets 1 point.

A gets $18+15=33$ points, B gets $5+14=19$ points and C gets $8+2=10$ points.

The winner is the candidate with the most points.

So A is the winner.

b) Alternative-Vote

If a voter puts a candidate first choice, then that candidate gets 1 point.

A gets $18+15=33$ points, B gets $5+14=19$ points and C gets $8+2=10$ points.

The candidate with the fewest points is eliminated. So C is eliminated.

The 8 voters whose order of preference was CAB now put A as their first choice and B as their second choice.

The 2 voters whose order of preference was CBA now put B as their first choice and A as their second choice.

A now gets $18+15+8=41$ points and B now gets $5+14+2=21$ points.

So A is the winner.

c) Most-Popular

In an election between just A and B:

$18+15+8=41$ voters prefer A to B and $5+14+2=21$ voters prefer B to A

So A is more popular than B.

In an election between just A and C:

$18+15+5=38$ voters prefer A to C and $8+2+14=24$ voters prefer C to A

So A is more popular than C.

In an election between just B and C:

$5+14+18=37$ voters prefer B to C and $8+2+15=25$ voters prefer C to B

So B is more popular than C.

A is more popular than B and A is more popular than C.

So A is the winner.

See EXERCISE

This all seems straight forward, but ...

Example 2

Order of preference	Number of voters
ABC	40
ACB	10
BAC	5
BCA	30
CAB	8
CBA	40

B is more popular than A and C is more popular than A.

But with First-Past-The-Post, A is the winner.

Example 3

Order of preference	Number of voters
ABC	6
ACB	15
BAC	5
BCA	15
CAB	8
CBA	10

C is more popular than A and C is more popular than B.

But with Alternative-Vote, C is eliminated.

Example 4

Order of preference	Number of voters
ABC	14
ACB	7
BAC	8
BCA	12
CAB	12
CBA	6

A is more popular than B and B is more popular than C and C is more popular than A.

So with Most-Popular, there is no winner.

Example 5

Order of preference	Number of voters
ABC	20
ACB	20
BAC	5
BCA	24
CAB	17
CBA	14

With Alternative-Vote:

A gets 40 points, B gets 29 points and C gets 31 points. So B is eliminated.

A now gets 45 points and C now gets 55 points. So C is the winner.

If 3 of the voters whose order of preference was ABC, had voted tactically and voted BAC then:

Order of preference	Number of voters
ABC	17
ACB	20
BAC	8
BCA	24
CAB	17
CBA	14

A gets 37 points, B gets 32 points and C gets 31 points. So C is eliminated.

A now gets 54 points and B now gets 46 points. So A is the winner.

So tactical voting paid off. But you need to be careful ...

If 10 of the voters whose order of preference was ABC, had voted tactically and voted BAC then:

Order of preference	Number of voters
ABC	10
ACB	20
BAC	15
BCA	24
CAB	17
CBA	14

A gets 30 points, B gets 39 points and C gets 31 points. A is eliminated. Whoops!

Example 6

Order of preference	Number of governors
ABC	5
ACB	4
BAC	5
BCA	3
CAB	1
CBA	3

There are 3 candidates for a job at a school. Each of the governors has put the candidates in order of preference.

With First-Past-The-Post:

A gets 9 points, B gets 8 points, C gets 4 points. So the governors decide to appoint A.

However, just before the Principal announces the result, C gets a call on her phone, offering her a job at a different school, which she accepts. “Never mind” says the Principal “we were not going to give her the job anyway”. Not so fast! If C is no longer available:

A gets 10 points, B gets 11 points.

There are many other voting systems for electing one candidate. However ...

Arrow’s Theorem:

There is no perfect voting system. Arrow wrote a list of the features you would certainly want in any voting system. Arrow’s theorem proves that no voting system can have all these features.

Gibbard–Satterthwaite theorem:

We would like a voting system where there is no benefit in tactical voting. The Gibbard–Satterthwaite theorem proves that this is not possible.

EXERCISE

1)

Order of preference	Number of voters
ABC	5
ACB	7
BAC	1
BCA	9
CAB	2
CBA	7

a) Who wins with First-Past-The-Post?

b) Who wins with Alternative-Vote?

c) Who wins with Most-Popular?

2)

Another voting system is Borda score.

If a voter puts a candidate first choice, then the candidate gets 3 points.

If a voter puts a candidate second choice, then the candidate gets 2 points.

If a voter puts a candidate third choice, then the candidate gets 1 point.

The winner is the candidate with the most points.

Look at Example 1 at the start of this chapter. Who is the winner with Borda score?

SOLUTIONS

1)

a) A gets $5+7=12$ B gets $1+9=10$ C gets $2+7=9$

A is the winner.

b) A gets $5+7=12$ B gets $1+9=10$ C gets $2+7=9$ C is eliminated.

A now gets $5+7+2=14$ B now gets $1+9+7=17$

B is the winner.

c) A against B A gets $5+7+2=14$ B gets $1+9+7=17$

A against C A gets $5+7+1=13$ C gets $2+7+9=18$

B against C B gets $1+9+5=15$ C gets $2+7+7=16$

C is the winner.

2)

A gets $(18+15)\times 3+(5+8)\times 2+(14+2)\times 1=141$

B gets $(5+14)\times 3+(18+2)\times 2+(15+8)\times 1=120$

C gets $(8+2)\times 3+(15+14)\times 2+(18+5)\times 1=111$

A is the winner.

Wason Test

EXERCISE

1) I have four cards. Each card has a letter on one side and an integer on the other side.

I put the cards down on a table, so you can only see one side of each card.

You can see: A, M, 4, 7

Which cards do you have to turn over to test the rule:

If the letter is a vowel then the integer is even?

2) I have four cards. Each card has a town on one side and a mode of transport on the other side.

I put the cards down on a table, so you can only see one side of each card.

You can see: Leeds, Manchester, Car, Train

Which cards do you have to turn over to test the rule:

If the town is Manchester then the mode of transport is Train?

3) I have four cards. Each card has a person's age on one side and a drink on the other side.

I put the cards down on a table, so you can only see one side of each card.

You can see: 16 years old, 24 years old, Beer, Lemonade

Which cards do you have to turn over to test the rule:

If the drink is Beer then the person's age must be over 18 years old?

SOLUTIONS

We need to look for potential rule breakers.

1) A rule-breaker has got a vowel on one side and an odd integer on the other side.

We need to turn over A in case the other side is an odd integer.

We need to turn over 7 in case the other side is a vowel.

2) A rule-breaker has got Manchester on one side and not Train on the other side.

We need to turn over Manchester in case the other side is not Train.

We need to turn over Car in case the other side is Manchester.

3) A rule breaker has got Beer on one side and not over 18 years old on the other side.

We need to turn over Beer in case the other side is not over 18 years old.

We need to turn over 16 years old in case the other side is Beer.

Note: These three questions are logically equivalent. However research has shown that nearly every-one gets example (1) incorrect but nearly every-one gets example (3) correct.

There is no research available on example (2) because I just made it up.