Euclid's Algorithm

An algorithm is a set of precise instructions that will solve a problem. Euclid's algorithm will solve the problem of finding the highest common factor of two positive integers.

Here is Euclid's algorithm for finding $HCF(3458, 651)$

$d = HCF(3458, 651)$

We divide 3458 by 651

$3458 = (5)(651) + 203$

If 3458 and 651 are both multiples of d then 651 and 203 are both multiples of d

We now repeat the procedure:

$651 = (3)(203) + 42$

If 651 and 203 are both multiples of d then 203 and 42 are both multiples of d

$203 = (4)(42) + 35$

If 203 and 42 are both multiples of d then 42 and 35 are both multiples of d

$42 = (1)(35) + 7$

If 42 and 35 are both multiples of d then 35 and 7 are both multiples of d

$35 = (5)(7) + 0$ STOP

So $d = 7$

See Exercise 1

We can now write HCF(3458,651) in the form $3458n + 65m$ for integers n and m

Working back up the page:

$7 = (42) + (-1)(35)$ But $35 = 203 + (-4)(42)$

$7 = (-1)(203) + (5)(42)$ But $42 = 651 + (-3)(203)$

$7 = (5)(651) + (-16)(203)$ But $203 = 3458 + (-5)(651)$

$7 = (-16)(3458) + (85)(651)$

We can use Euclid's algorithm to solve some Diophantine equations. A Diophantine equation requires integer solutions.

Example

Solve $3458x + 651y = 47894$ where x, y are integers

Run Euclid's algorithm to find $HCF(3458, 651)$

We have just done this and we found:

$HCF(3458, 651) = 7$

Then we found:

$7 = (-16)(3458) + (85)(651)$ so $(3458)(-16) + (651)(85) = 7$

Now $\frac{47894}{7} = 6842$ so we multiply both sides by 6842

So:

$(3458)(-109472) + (651)(581570) = 47894$

We have a solution to our equation: $x = -109472$ $y = 581570$

There are more solutions. The general solution is:

$x = -109472 + 93t$ and $y = 581570 - 494t$ for any integer t Can you see why?

See exercise 2

EXERCISE 1

Find the highest common factor of 41325 and 5814

SOLUTION 1

$41325 = (7)(5814) + (627)$

$5814 = (9)(627) + (171)$

$627 = (3)(171) + (114)$

$171 = (1)(114) + (57)$

$114 = (2)(57) + (0)$ STOP

$d = 57$

EXERCISE 2

1) Oranges cost 23p and apples cost 17p. I buy some and the cost is 549p

How many oranges and how many apples did I buy?

Hint: If I buy x oranges and y apples then $23x + 17y = 549$

2) In the following equations, we are looking for solutions where x, y are integers.

Why won't we find any?

a) $7x = 43$ b) $(x-3)^2 = 10$ c) $4x = 2y + 1$ d) $2^x = 3^y$

e) $6^x = 10^y$

SOLUTIONS 2

1) $23=(1)(17)+(6)$

$17=(2)(6)+(5)$

$6=(1)(5)+(1)$

$5=(5)(1)+0$ STOP

$d=1$ (this was obvious as 23 and 17 are primes)

Working back up the page

$1=(6)+(-1)(5)$ But $(5)=(17)+(-2)(6)$

$1=(-1)(17)+(3)(6)$ But $(6)=(23)+(-1)(17)$

$1=(3)(23)+(-4)(17)$

So $(23)(3)+(17)(-4)=1$ multiplying by 549 gives

$(23)(1647)+(17)(-2196)=549$

We have a solution to our equation $x=1647$ and $y=-2196$

So I buy 1647 oranges and -2196 apples

This is not a very practical solution.

The general solution is: $x=1647-17t$ and $y=-2196+23t$

We want $1647-17t\geq 0$ and $-2196+23t\geq 0$

So $t\leq 96.9$ and $t\geq 95.5$ and remember t is an integer, so $t=96$

This gives $x=15$ and $y=12$

2)

a) LHS is a multiple of 7 but RHS is not a multiple of 7

b) No integer squared is equal to 10

c) LHS is even but RHS is odd

d) LHS is even but RHS is odd

e) LHS is a multiple of 3 but the RHS is not a multiple of 3