If … Then

Example

p : Today is Friday q : I go to yoga today

We can combine two propositions using IF … THEN

$p \Rightarrow q$:If today is Friday then I go to yoga today

What does this tell you?

If today is Friday then it tells you that I go to yoga today.

If today is not Friday then it tells you nothing.

If I go to yoga today then it tells you nothing.

If I do not go to yoga today then it tells you that today is not Friday.

The only way

If today is Friday then I go to yoga today

can be false, is if:

today is Friday, is true

and I go to yoga today, is false.

The only way $p \Rightarrow q$ can be false is if p is true and q is false.

So we have this truth table:

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

$p \Rightarrow q$ can be read as:

if p then q

p implies q

q if p

p only if q

q is necessary for p

p is sufficient for q

$p \Rightarrow q$ is not the same as $p' \Rightarrow q'$

Compare:

If today is Friday then I go to yoga today

If today is not Friday then I do not go to yoga today

$p \Rightarrow q$ is not the same as $q \Rightarrow p$

Compare:

If today is Friday then I go to yoga today

If I go to yoga today then today is Friday

Note: $q \Rightarrow p$ is called the converse of $p \Rightarrow q$

$p \Rightarrow q$ is the same as $q' \Rightarrow p'$

Compare:

If today is Friday then I go to yoga today

If I do not go to yoga today then today is not Friday

Note: $q' \Rightarrow p'$ is called the contrapositive of $p \Rightarrow q$

Note: A common error or fudge is to prove $p \Rightarrow q$ and then pretend you have proved $q \Rightarrow p$

For example, in the chapter Euler Tours, we proved:

If a closed Euler tour exits then every vertex is even.

We then pretended to have proved:

If every vertex is even then a closed Euler tour exists.

This is very naughty.

see Exercise 1

We can combine two propositions using IF AND ONLY IF

$p \Leftrightarrow q$:Today is Friday if and only if I go to yoga today

What does this tell you?

If today is Friday then it tells you that I go to yoga today.

If today is not Friday then it tells you that I do not go to yoga today.

If I go to yoga today then it tells you that today is Friday.

If I do not go to yoga today then it tells you that today is not Friday.

So $p \Leftrightarrow q$ is true if p and q are both true or both false, otherwise it is false.

Truth table:

p	q	$p\Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

$p\Leftrightarrow q$ can be read as:

p if and only if q

p is necessary and sufficient for q

To prove $p\Leftrightarrow q$ we have to prove $p\Rightarrow q$ and we have to prove $q\Rightarrow p$

For example, in the chapter Rationals and Irrationals, we proved:

x is rational $\Leftrightarrow$ x is a terminating or recurring decimal

See Exercise 2

EXERCISE 1

1.

Use a truth table to show that:

$p\Rightarrow q$ is not the same as $p'\Rightarrow q'$

$p\Rightarrow q$ is not the same as $q\Rightarrow p$

$p\Rightarrow q$ is the same as $q'\Rightarrow p'$

2.Use a truth table to show that:

$(p\Rightarrow q)=(p'\vee q)$ and $(p\Rightarrow q)=(p\wedge q')'$

3.

Fill in the truth table

p	q	r	$p\Rightarrow q$	$q\Rightarrow r$	$(p\Rightarrow q)\wedge(q\Rightarrow r)$	$p\Rightarrow r$	$((p\Rightarrow q)\wedge(q\Rightarrow r))\Rightarrow(p\Rightarrow r)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					

1	1	0					
1	1	1					

SOLUTIONS 1

1.

p	q	$p \Rightarrow q$	p'	q'	$p' \Rightarrow q'$	$q \Rightarrow p$	$q' \Rightarrow p'$
0	0	1	1	1	1	1	1
0	1	1	1	0	0	0	1
1	0	0	0	1	1	1	0
1	1	1	0	0	1	1	1

2.

p	q	$p \Rightarrow q$	p'	$p' \vee q$	q'	$p \wedge q'$	$(p \wedge q')'$
0	0	1	1	1	1	0	1
0	1	1	1	1	0	0	1
1	0	0	0	0	1	1	0
1	1	1	0	1	0	0	1

3.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

So $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ is always true, regardless of the truth of p, q, r We call this a tautology.

EXERCISE 2

For each of the following, does $p \Rightarrow q$ or $q \Rightarrow p$ or $p \Leftrightarrow q$?

	p	q
a)	$x=4$	$2x=8$
b)	n is a multiple of 5	n is a multiple of 15
c)	n is not a multiple of 10	n is a prime
d)	$ABCD$ is a parallelogram	$ABCD$ is a square
e)	$x^2-6x+8=0$	$x=2$
f)	$x^2-4x+4=0$	$x=2$
g)	$x=4$	$x^2=16$
h)	$x>7$	$x>4$
i)	x is an integer	x is rational
j)	$x>2$	$x^2>4$
k)	$x<4$	$x^2<16$

SOLUTIONS 2

a) $p \Leftrightarrow q$ b) $q \Rightarrow p$ c) $q \Rightarrow p$ d) $q \Rightarrow p$ e) $q \Rightarrow p$ f) $p \Leftrightarrow q$

g) $p \Rightarrow q$ h) $p \Rightarrow q$ i) $p \Rightarrow q$ j) $p \Rightarrow q$ k) $q \Rightarrow p$