Rationals and Irrationals

$\frac{13}{7}$ is a rational number because it is an integer divided by another integer.

Theorem

x is a rational number if and only if x is a terminating or recurring decimal.

This is really two theorems:

Theorem 1

If x is a rational number then x is a terminating or recurring decimal.

Proof

Say $x=\frac{13}{7}$

$\frac{13}{7}=1+\frac{6}{7}$ $\qquad$ $\frac{13}{7}$ equals 1 remainder 6

$\frac{6}{7}=\frac{1}{10}\left(\frac{60}{7}\right)=\frac{1}{10}\left(8+\frac{4}{7}\right)$ $\qquad$ $\frac{60}{7}$ equals 8 remainder 4

$\frac{4}{7}=\frac{1}{10}\left(\frac{40}{7}\right)=\frac{1}{10}\left(5+\frac{5}{7}\right)$ $\qquad$ $\frac{40}{7}$ equals 5 remainder 5

$\frac{5}{7}=\frac{1}{10}\left(\frac{50}{7}\right)=\frac{1}{10}\left(7+\frac{1}{7}\right)$ $\qquad$ $\frac{50}{7}$ equals 7 remainder1

etc

So $\frac{13}{7}=1+\frac{8}{10}+\frac{5}{100}+\frac{7}{1000}+...=1.857...$

The remainders can only be $0,1,2,3,4,5,6$

Either we will get a remainder of 0, in which case the decimal terminates.

Or we will get a remainder we have had before, in which case the decimal recurs.

Either way x is a terminating decimal or a recurring decimal

Theorem 2

If x is a terminating or recurring decimal then x is a rational number.

Proof

If:

x is a terminating decimal, say $x=0.123$

then:

$x = \frac{123}{1000}$ so x is a rational number.

If:

x is a recurring decimal, say $x = 0.123123123\ldots$

then:

$1000x = 123.123123123\ldots$ so $999x = 123$ so $x = \frac{123}{999}$ so x is a rational number.

Either way x is a rational number.