Euler’s Sine Formula

Reminder of the factor theorem

Example

$p(x)=x^4-17x^3+99x^2-223x+140$

Now:

$p(1)=0$ so $(x-1)$ is a factor of $p(x)$ - this is the factor theorem - see Appendix 2

$p(4)=0$ so $(x-4)$ is a factor

$p(5)=0$ so $(x-5)$ is a factor

$p(7)=0$ so $(x-7)$ is a factor

So:

$p(x)=c(x-1)(x-4)(x-5)(x-7)$ where c is some constant

Now:

$p(0)=140$ so $c=1$

Rearranging gives:

$$p(x)=(1-x)(4-x)(5-x)(7-x)=140\left(1-\frac{x}{1}\right)\left(1-\frac{x}{4}\right)\left(1-\frac{x}{5}\right)\left(1-\frac{x}{7}\right)$$

Now:

$\sin(0)=0$ so $(x-0)$ is a factor of $sinx$

$\sin(\pi)=0$ so $(x-\pi)$ is a factor

$\sin(-\pi)=0$ so $(x+\pi)$ is a factor

$\sin(2\pi)=0$ so $(x-2\pi)$ is a factor

$\sin(-2\pi)=0$ so $(x+2\pi)$ is a factor

$\sin(3\pi)=0$ so $(x-3\pi)$ is a factor

$\sin(-3\pi)=0$ so $(x+3\pi)$ is a factor etc

So:

$sinx=cx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$ where c is some constant

So:

$$\frac{sinx}{x}=c(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$$

Sub in $x=0$ see Footnote

$1=c(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\ldots$

So:

$$c=\frac{1}{(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\ldots}$$

So:

$$sinx=\frac{x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)...}{(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)...}$$

Fool around with this and show that:

$$sinx=x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\dots$$

Or if you prefer:

$$\sin(x)=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\dots$$ this is Euler's sine formula

We can have some fun with this.

1) We recall that:

$$sinx=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$$ this is the Maclaurin series

Equating Maclaurin's formula and Euler's formula we have:

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\dots$$

Equating coefficients of x^3 we (eventually) get:

$$-\frac{1}{3!}=-\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\dots$$

Rearranging gives:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

What a surprising answer. Why is π doing here?

As a bonus:

Equating coefficients of x^5 we (eventually) get:

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...=\frac{\pi^4}{90}$$

We can also find formulas for:

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+...$$ and $$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\dots$$ etc

What about?

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\dots$$ and $$\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\dots$$ etc

Well Euler failed to find a formula for these series.

2) If we put $x=\frac{\pi}{2}$ into Euler's sine formula we get:

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2 2^2}\right)\left(1-\frac{1}{2^2 3^2}\right)\ldots$$

Now:

$$1-\frac{1}{2^2 n^2}=\frac{(2n-1)(2n+1)}{2^2 n^2}$$

So:

$$1=\frac{\pi}{2}\left(\frac{1\times 3}{2^2}\right)\left(\frac{3\times 5}{2^2 2^2}\right)\left(\frac{5\times 7}{2^2 3^2}\right)...=\frac{\pi}{2}\left(\frac{1\times 3}{2\times 2}\right)\left(\frac{3\times 5}{4\times 4}\right)\left(\frac{5\times 7}{6\times 6}\right)..$$

Which gives us Wallis's formula:

$$\frac{\pi}{2}=\frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\ldots$$

3)

$$sinx=x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\ldots$$

So:

$$lnsinx=lnx+\ln\left(1-\frac{x}{\pi}\right)+\ln\left(1+\frac{x}{\pi}\right)+\ln\left(1-\frac{x}{2\pi}\right)+\ln\left(1+\frac{x}{2\pi}\right)+\ln\left(1-\frac{x}{3\pi}\right)+\ln\left(1+\frac{x}{3\pi}\right)...$$

Differentiate both sides and show that:

$$\frac{cosx}{sinx}=\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}-\frac{1}{3\pi-x}+\frac{1}{3\pi+x}-\ldots$$

Sub in $x=\frac{\pi}{4}$ and show that:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\ldots$$

Footnote:

Am I really saying?

$$\frac{\sin 0}{0}=1$$

Look at the Maclaurin series:

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\frac{sinx}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

$$x \to 0 \qquad \frac{sinx}{x} \to 1$$

Euler’s Zeta Function

Euler introduced the zeta function:

$$\zeta(x)=\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\frac{1}{5^x}+\frac{1}{6^x}+\frac{1}{7^x}+\frac{1}{8^x}+\ldots$$

where x is a real number and $x>1$ which guarantees the series converges.

Look at this infinite product of infinite series:

$$\left(1+\frac{1}{2^1}+\frac{1}{2^2}+\ldots\right)\left(1+\frac{1}{3^1}+\frac{1}{3^2}+\ldots\right)\left(1+\frac{1}{5^1}+\frac{1}{5^2}+\ldots\right)\left(1+\frac{1}{7^1}+\frac{1}{7^2}+\ldots\right)\left(1+\frac{1}{11^1}+\frac{1}{11^2}+\ldots\right)\ldots$$

where the denominators of the fractions are powers of the prime numbers.

First attempt:

Each bracket is a geometric series. So this infinite product is equal to:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)\ldots$$

Second attempt:

If we multiply out the brackets, we get a lot of fractions. All these fractions will have 1 as the numerator. No two fractions will have the same denominator. The denominator of each fraction will be a product of powers of primes. Every possible product of powers of primes will appear as a denominator. So this infinite product is equal to:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$$

Equating:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)\ldots=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$$

Now look at this infinite product of infinite series:

$$\left(1+\frac{1}{2^x}+\frac{1}{2^{2x}}+\ldots\right)\left(1+\frac{1}{3^x}+\frac{1}{3^{2x}}+\ldots\right)\left(1+\frac{1}{5^x}+\frac{1}{5^{2x}}+\ldots\right)\left(1+\frac{1}{7^x}+\frac{1}{7^{2x}}+\ldots\right)\left(1+\frac{1}{11^x}+\frac{1}{11^{2x}}+\ldots\right)\ldots$$

By repeating what we did above we (eventually) get:

$$\left(\frac{1}{1-\frac{1}{2^x}}\right)\left(\frac{1}{1-\frac{1}{3^x}}\right)\left(\frac{1}{1-\frac{1}{5}^{x}}\right)\left(\frac{1}{1-\frac{1}{7^x}}\right)\left(\frac{1}{1-\frac{1}{11^x}}\right)\ldots=\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\frac{1}{5^x}+\frac{1}{6^x}+\frac{1}{7^x}+\frac{1}{8^x}+\ldots \quad ***$$

Notice, the right-hand side is $\zeta(x)$

So we can write the zeta function in terms of primes:

$$\zeta(x)=\left(\frac{1}{1-\frac{1}{2^x}}\right)\left(\frac{1}{1-\frac{1}{3^x}}\right)\left(\frac{1}{1-\frac{1}{5^x}}\right)\left(\frac{1}{1-\frac{1}{7^x}}\right)\left(\frac{1}{1-\frac{1}{11^x}}\right)\ldots$$

Note:

In the chapter, Euler's Sine Formula, we got the result:

$$\zeta(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

So:

$$\frac{\pi^2}{6}=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{1}{1-\frac{1}{3^2}}\right)\left(\frac{1}{1-\frac{1}{5^2}}\right)\left(\frac{1}{1-\frac{1}{7^2}}\right)\left(\frac{1}{1-\frac{1}{11^2}}\right)\ldots$$

And we have a formula for π in terms of primes.

Note:

If we sub $x=1$ into ***

We know the RHS diverges, so the LHS diverges, so there must be an infinite number of primes!

Euler's Prime Sum

In the chapter, Euler's Zeta Function, we got the result:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)...=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+...$$

Taking logs of both sides:

$$\ln\left(\frac{1}{1-\frac{1}{2}}\right)+\ln\left(\frac{1}{1-\frac{1}{3}}\right)+\ln\left(\frac{1}{1-\frac{1}{5}}\right)+\ln\left(\frac{1}{1-\frac{1}{7}}\right)+...=\ln\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+...\right)$$

So:

$$\sum_{primes}\ln\left(\frac{1}{1-\frac{1}{p}}\right)=\ln\sum_{1}^{\infty}\frac{1}{n} \quad *$$

Now:

$$\left(\frac{1}{1-\frac{1}{p}}\right)=\left(1-\frac{1}{p}\right)^{-1} \quad \text{So} \quad \ln\left(\frac{1}{1-\frac{1}{p}}\right)=-\ln\left(1-\frac{1}{p}\right)$$

So we can write * as:

$$\sum_{primes}-\ln\left(1-\frac{1}{p}\right)=\ln\sum_{1}^{\infty}\frac{1}{n} \quad **$$

Now:

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+... \quad \text{this is the Maclaurin series}$$

Put $x=-\frac{1}{p}$

We get:

$$\ln\left(1-\frac{1}{p}\right)=-\frac{1}{p}-\frac{1}{2p^2}-\frac{1}{3p^3}-\frac{1}{4p^4}-... \quad \text{So} \quad -\ln\left(1-\frac{1}{p}\right)=\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...$$

So we can write ** as:

$$\sum_{primes}\left(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...\right)=\ln\sum_{1}^{\infty}\frac{1}{n}$$

So:

$$\sum_{primes}\frac{1}{p}+\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...\right)=\ln\sum_{1}^{\infty}\frac{1}{n} \quad ***$$

Consider:

$$\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...\right)$$

Now:

$$\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...<\frac{1}{p^2}+\frac{1}{p^3}+\frac{1}{p^4}+...=\frac{\left(\frac{1}{p^2}\right)}{\left(1-\frac{1}{p}\right)}=...=\frac{1}{p-1}-\frac{1}{p}$$

So:

$$\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...\right)<\sum_{primes}\left(\frac{1}{p-1}-\frac{1}{p}\right)<\sum_{2}^{\infty}\left(\frac{1}{p-1}-\frac{1}{p}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=...=1$$

Look at *** again:

$$\sum_{primes}\frac{1}{p}+\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...\right)=\ln\sum_{1}^{\infty}\frac{1}{n}$$

We know:

$\ln\sum_{1}^{\infty}\frac{1}{n}$ diverges.

We have just shown that:

$\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+...\right)$ converges.

So we can deduce that:

$\sum_{primes}\frac{1}{p}$ diverges.

So:

$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{23}+...$ diverges. Astonishing!

Euler's Constant

Here is the graph $y=\frac{1}{x}$ ADD SHADING

Look at the graph from $x=1$ to $x=4$

The area of the shaded bits is the area of the blocks minus the area under the graph.

The area of the blocks is:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}$$

The area under the graph is:

$$\int_1^4 \frac{1}{x}dx=\ln 4$$

So the area of the shaded bits is:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}-\ln 4$$

Imagine the graph went from $x=1$ to $x=n+1$

The area of the shaded bits is:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n+1)$$

Now let $n\to\infty$

The total area of all the shaded bits is called γ This is Euler's constant.

$$\gamma=limit(n\to\infty)\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n+1)$$

Imagine sliding all the shaded bits horizontally to the left. They will all fit inside the first block with room to spare. So $\gamma < 1$

Incidently, it is not known if γ is rational or irrational.