Magic Squares

Example 1

Here is a 4×4 magic square:

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

The numbers in each column, each row and both diagonals add up to the same total.

Now $1+2+3+...+16=136$ Our sixteen numbers are arranged in four columns so the numbers in each column (and each row and both diagonals) must add up to $136/4=34$

Example 2

We want to arrange the numbers $1,2,3,...9$ into a 3×3 magic square.

Now $1+2+3+4+5+6+7+8+9=45$ Our nine numbers are arranged in three columns so the numbers in each column (and each row and both diagonals) must add up to $45/3=15$

Theorem

5 must go in the middle cell.

Proof

A	B	C
D	E	F
G	H	I

$A+E+I=15$ $C+E+G=15$ $B+E+H=15$ $D+E+F=15$

So $A+E+I+C+E+G+B+E+H+D+E+F=60$

But $A+B+C+D+E+F+G+H+I=45$ so $E=5$

Show that we cannot put 9 in the same column/row/diagonal as 8 or 7 or 6 or 3.

Put 9 in a corner. Now 1 must go in the opposite corner. What numbers can go in the other corners?

Show that 9 cannot go in a corner. So 9 must go in the middle of a side.

Put 3 in a corner. Show that 3 cannot go in a corner. So 3 must go in the middle of a side.

Where can we put 8?

Now complete the magic square.

I got:

8	3	4
1	5	9
6	7	2

And let's say you got:

4	3	8
9	5	1
2	7	6

We would say these are the same. Two magic squares are the same if we can change the first square into the second square by rotating the first square about its centre or by reflecting the first square about any mirror line passing through its centre.

A magic square will remain magic if:

we add k to all the numbers in the square, for any number k

we multiply all the numbers in the square by k for any number k

we swap 2 rows that are equidistant from the centre

we swap 2 columns that are equidistant from the centre

Try it!

Back to my 3×3 magic square.

If we add 9 to each number in my 3×3 magic square then we get a magic square with the numbers: 10 … 18

17	12	13
10	14	18
15	16	11

If we add another 9 to each number then we get a magic square with the numbers: 19 … 27

If we add another 9 to each number then we get a magic square with the numbers: 28 … 36

...

If we add another 9 to each number then we get a magic square with the numbers: 73 … 81

We can now assemble these nine magic squares into a 9×9 magic square:

(look carefully to see how I've arranged the nine magic squares)

71	66	67	26	21	22	35	30	31
64	68	72	19	23	27	28	32	36
69	70	65	24	25	20	33	34	29
8	3	4	44	39	40	80	75	76
1	5	9	37	41	45	73	77	81
6	7	2	42	43	38	78	79	74
53	48	49	62	57	58	17	12	13
46	50	54	55	59	63	10	14	18
51	52	47	60	61	56	15	16	11

Here is a method to find a $N\times N$ magic square if N is a multiple of 4.

Start with this square:

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Divide this square into four 4×4 squares.

Look at the numbers on the diagonals of these 4×4 squares:

1, 10, 19, 28 and 4, 11, 18, 25

5, 14, 23, 32 and 8, 15, 22, 29

33, 42, 51, 60 and 36, 43, 50, 57

37, 46, 55, 64 and 40, 47, 54, 61

Swap any two of these numbers that add up to 65.

swap: 1 and 64 swap: 10 and 55 swap: 19 and 46 etc

This gives us the magic square:

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

There are many other methods to find magic squares. Look them up.

Here is my favourite magic square:

7	53	41	27	2	52	48	30
12	58	38	24	13	63	35	17
51	1	29	47	54	8	28	42
64	14	18	36	57	11	23	37
25	43	55	5	32	46	50	4
22	40	60	10	19	33	61	15
45	31	3	49	44	26	6	56
34	20	16	62	39	21	9	59

It remains magic if all the numbers are squared!

EXERCISE

Arrange the numbers $1,2,3,...16$ into a 4×4 magic square.

SOLUTION

Start with the square:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

do the swaps to get:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1