

## Probability Paradoxes

### Example 1

#### Condorcet Paradox

Dice A has faces numbered 3, 3, 3, 3, 3, 3

Dice B has faces numbered 1, 1, 5, 5, 5, 5

Dice C has faces numbered 2, 2, 2, 2, 6, 6

We each choose a dice. We each roll our dice and the highest score wins.

Being a wonderfully nice person, I will let you choose first. Which dice will you choose?

Let's work out the probabilities.

In a contest between dice A and dice B:

$$p(A \text{ wins}) = \frac{2}{6} \text{ and } p(B \text{ wins}) = \frac{4}{6}$$

In a contest between dice A and dice C:

$$p(A \text{ wins}) = \frac{4}{6} \text{ and } p(C \text{ wins}) = \frac{2}{6}$$

In a contest between dice B and dice C:

6	C	C	C	C	C	C
6	C	C	C	C	C	C
2	C	C	B	B	B	B
2	C	C	B	B	B	B
2	C	C	B	B	B	B
2	C	C	B	B	B	B
	1	1	5	5	5	5

$$p(B \text{ wins}) = \frac{16}{36} \text{ and } p(C \text{ wins}) = \frac{20}{36}$$

Which dice will you choose?

If you pick A then I'll pick B. If you pick B then I'll pick C. If you pick C then I'll pick A.

### Example 2

#### St. Petersburg Paradox

You spin a coin      If it is heads, I give you £1, and the game ends

                                 If it is tails, you spin again.

You spin again      If it is heads, I give you £2, and the game ends.

If it is tails, you spin again.

You spin again      If it is heads, I give you £4, and the game ends.

If it is tails, you spin again.

You spin again      If it is heads, I give you £8, and the game ends.

If it is tails, you spin again.

etc

Let  $X$  be your winnings. The probability distribution for  $X$  is:

$x$	$p(x)$	$xp(x)$
1	1/2	1/2
2	1/4	1/2
4	1/8	1/2
8	1/16	1/2
...	...	...

Your expected winnings are:

$$E(X) = \sum_1^{\infty} xp(x) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \quad \text{this is infinite.}$$

So if I charge you £1,000,000 to play this game, then you should play.

### Example 3

#### Simpson's Paradox

A university offers courses in Engineering and Medicine.

Engineering:

100 women apply and 40 are accepted. 600 men apply and 150 are accepted.

So the acceptance rate is higher for women.

Medicine:

600 women apply and 72 are accepted. 100 men apply and 10 are accepted.

So the acceptance rate is higher for women.

University:

700 women apply and 112 are accepted. 700 men apply and 160 are accepted.

So the acceptance rate is higher for men.