Polygonal numbers

The triangle numbers:

$1, 3, 6, 10, 15, \ldots$ are given by the recurrence relation:

$T_1 = 1$ and $T_{n+1} = T_n + n + 1$

So:

$T_2 = T_1 + 2$ and $T_3 = T_2 + 3$ and $T_4 = T_3 + 4$ etc

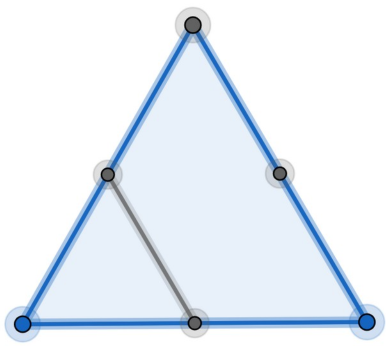
So:

$T_4 = T_3 + 4 = (T_2 + 3) + 4 = ((T_1 + 2) + 3) + 4 = 1 + 2 + 3 + 4$

In general:

$T_n = 1 + 2 + 3 + \ldots + n$

NEED BETTER DIAGRAM



Theorem:

$$T_n = \frac{1}{2} n(n+1)$$

Proof (by induction)

part 1:

If $n = 1$ then:

$LHS = T_1 = 1$ and $RHS = \frac{1}{2}(1)(2) = 1$ So the formula is true when $n = 1$

part 2:

If $T_n = \frac{1}{2} n(n+1)$ is true when $n = k$ then:

$T_k = \frac{1}{2} k(k+1)$ but $T_{k+1} = T_k + k + 1$

So:

$$T_{k+1}=\frac{1}{2}k(k+1)+k+1=...=\frac{1}{2}(k+1)(k+2)$$

So:

$T_n=\frac{1}{2}n(n+1)$ is true when $n=k+1$

The square numbers:

$1,4,9,16,25,\ldots$ are given by the recurrence relation:

$S_1=1$ and $S_{n+1}=S_n+2n+1$

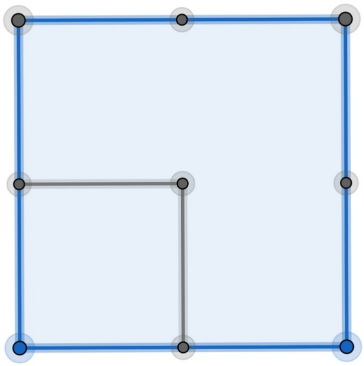
So:

$S_2=S_1+3$ and $S_3=S_2+5$ and $S_4=S_3+7$ etc

Show that:

$$S_n=1+3+5+...+(2n-1)$$

NEED BETTER DIAGRAM



Theorem:

$S_n=\frac{1}{2}n(2n)$ or if you prefer $S_n=n^2$

see Exercise 1

The pentagonal numbers:

$1,5,12,22,35,\ldots$ are given by the recurrence relation:

$P_1=1$ and $P_{n+1}=P_n+3n+1$

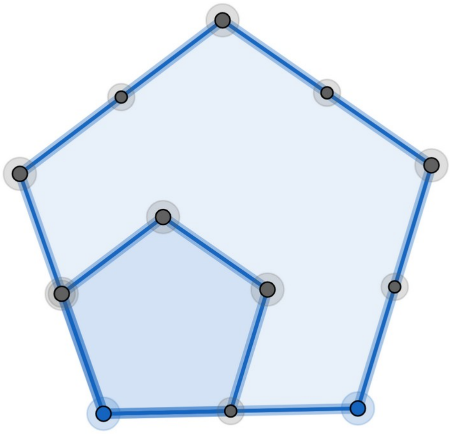
So:

$P_2=P_1+4$ and $P_3=P_2+7$ and $P_4=P_3+10$ etc

Show that:

$$P_n = 1 + 4 + 7 + \ldots + (3n - 2)$$

NEED BETTER DIAGRAM



Theorem:

$$P_n = \frac{1}{2} n (3n - 1)$$

The hexagonal numbers:

$1, 6, 15, 28, 45, \ldots$ are given by the recurrence relation:

$H_1 = 1$ and $H_{n+1} = H_n + 4n + 1$

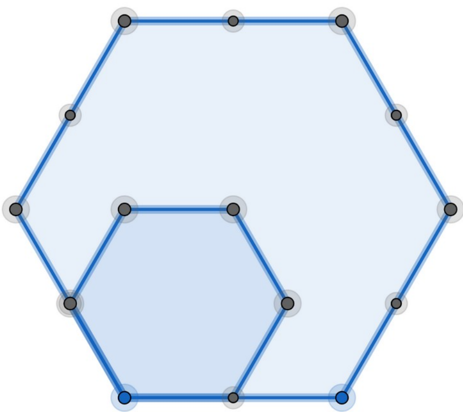
So:

$H_2 = H_1 + 5$ and $H_3 = H_2 + 9$ and $H_4 = H_3 + 13$ etc

Show that:

$$H_n = 1 + 5 + 9 + \ldots + (4n - 3)$$

NEED BETTER DIAGRAM



Theorem:

$$H_n = \frac{1}{2} n(4n-2)$$

The polygonal number theorem (difficult to prove)

Every positive integer can be written as the sum of:

3 (or fewer) triangle numbers

4 (or fewer) square numbers

5 (or fewer) pentagonal numbers

etc

see Exercise 2

EXERCISE 1

Square numbers:

Show that $S_n = \frac{1}{2} n(2n)$ Use proof by induction

EXERCISE 2

Prove the following:

1. The sum of two consecutive triangle numbers is a square number.
2. If T is a triangle number then $8T+1$ is a square number.
3. If T is a triangle number then $9T+1$ is a triangle number.
4. The difference of the squares of two consecutive triangle numbers is a cube.
5. $P_n = T_n + 2T_{n-1}$
6. $H_n = T_n + 3T_{n-1}$
7. The last digit of a triangle number can't be 2, 4, 7 or 9

SOLUTIONS 1

Proof (by induction)

part 1:

If $n=1$ then $LHS = S_1 = 1$ and $RHS = \frac{1}{2}(1)(2) = 1$ So the formula is true when $n=1$

part 2:

If $S_n=\frac{1}{2}n(2n)$ is true when $n=k$ then:

$S_k=\frac{1}{2}k(2k)$ but $S_{k+1}=S_k+2k+1$

So $S_{k+1}=\frac{1}{2}k(2k)+2k+1=...=\frac{1}{2}(k+1)2(k+1)$

So $S_n=\frac{1}{2}n(2n)$ is true when $n=k+1$

SOLUTIONS 2

1) $T_k+T_{k+1}=\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+2)=...=(k+1)^2$

Or:

In this diagram we have got: T_5 Ps and T_6 Qs

How many letters have we got?

Q	Q	Q	Q	Q	Q
P	Q	Q	Q	Q	Q
P	P	Q	Q	Q	Q
P	P	P	Q	Q	Q
P	P	P	P	Q	Q
P	P	P	P	P	Q

$T_5+T_6=6^2$

In general $T_k+T_{k+1}=(k+1)^2$

2) $8T_k+1=8\frac{1}{2}k(k+1)+1=...=(2k+1)^2$

3) $9T_k+1=9\frac{1}{2}k(k+1)+1=...=\frac{1}{2}(3k+1)(3k+2)$

4) $(T_{k+1})^2-(T_k)^2=(\frac{1}{2}(k+1)(k+2))^2-(\frac{1}{2}k(k+1))^2=...=(k+1)^3$

5) $P_n=\frac{1}{2}n(3n-1)$

$T_n+2T_{n-1}=\frac{1}{2}n(n+1)+2\frac{1}{2}(n-1)n=...=\frac{1}{2}n(3n-1)$

6) $H_n = \frac{1}{2} n(4n-2)$

$$T_n + 3T_{n-1} = \frac{1}{2} n(n+1) + 3\frac{1}{2}(n-1)n = \ldots = \frac{1}{2} n(4n-2)$$

7) mod 10:

n	0	1	2	3	4	5	6	7	8	9
$n+1$	1	2	3	4	5	6	7	8	9	0
$n(n+1)$	0	2	6	2	0	0	2	6	2	0

now $T = \frac{1}{2} n(n+1)$ so $2T = n(n+1)$ so the last digit of $2T$ is 0, 2 or 6

mod 10:

T	0	1		3		5	6		8	
$2T$	0	2		6		0	2		6	

If the last digit of $2T$ is 0, 2 or 6 then the last digit of T must be 0, 1, 3, 5, 6 or 8