

## Euler's Sine Formula

### Reminder of the factor theorem

#### Example

$$p(x) = x^4 - 17x^3 + 99x^2 - 223x + 140$$

Now:

$$p(1)=0 \text{ so } (x-1) \text{ is a factor of } p(x) \text{ - this is the factor theorem - see Appendix 2}$$

$$p(4)=0 \text{ so } (x-4) \text{ is a factor}$$

$$p(5)=0 \text{ so } (x-5) \text{ is a factor}$$

$$p(7)=0 \text{ so } (x-7) \text{ is a factor}$$

So:

$$p(x) = c(x-1)(x-4)(x-5)(x-7) \text{ where } c \text{ is some constant}$$

Now:

$$p(0)=140 \text{ so } c=1$$

Rearranging gives:

$$p(x) = (1-x)(4-x)(5-x)(7-x) = 140 \left(1 - \frac{x}{1}\right) \left(1 - \frac{x}{4}\right) \left(1 - \frac{x}{5}\right) \left(1 - \frac{x}{7}\right)$$

Now:

$$\sin(0)=0 \text{ so } (x-0) \text{ is a factor of } \sin x$$

$$\sin(\pi)=0 \text{ so } (x-\pi) \text{ is a factor}$$

$$\sin(-\pi)=0 \text{ so } (x+\pi) \text{ is a factor}$$

$$\sin(2\pi)=0 \text{ so } (x-2\pi) \text{ is a factor}$$

$$\sin(-2\pi)=0 \text{ so } (x+2\pi) \text{ is a factor}$$

$$\sin(3\pi)=0 \text{ so } (x-3\pi) \text{ is a factor}$$

$$\sin(-3\pi)=0 \text{ so } (x+3\pi) \text{ is a factor} \quad \text{etc}$$

So:

$$\sin x = c(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\dots \text{ where } c \text{ is some constant}$$

So:

$$\frac{\sin x}{x} = c(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\dots$$

Sub in  $x=0$  see Footnote

$$1 = c(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\dots$$

So:

$$c = \frac{1}{(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\dots}$$

So:

$$\sin x = \frac{x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\dots}{(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\dots}$$

Fool around with this and show that:

$$\sin x = x \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 - \frac{x}{3\pi}\right) \left(1 + \frac{x}{3\pi}\right) \dots$$

Or if you prefer:

$$\sin(x) = x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \dots \quad \text{this is Euler's sine formula}$$

We can have some fun with this.

1) We recall that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{this is the Maclaurin series}$$

Equating Maclaurin's formula and Euler's formula we have:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \dots$$

Equating coefficients of  $x^3$  we (eventually) get:

$$-\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \dots$$

Rearranging gives:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

What a surprising answer. Why is  $\pi$  doing here?

As a bonus:

Equating coefficients of  $x^5$  we (eventually) get:

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

We can also find formulas for:

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \dots \quad \text{and} \quad \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \dots \quad \text{etc}$$

What about?

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \quad \text{and} \quad \frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \dots \quad \text{etc}$$

Well Euler failed to find a formula for these series.

2) If we put  $x = \frac{\pi}{2}$  into Euler's sine formula we get:

$$1 = \frac{\pi}{2} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{2^2 2^2}\right) \left(1 - \frac{1}{2^2 3^2}\right) \dots$$

Now:

$$1 - \frac{1}{2^2 n^2} = \frac{(2n-1)(2n+1)}{2^2 n^2}$$

So:

$$1 = \frac{\pi}{2} \left(\frac{1 \times 3}{2^2}\right) \left(\frac{3 \times 5}{2^2 2^2}\right) \left(\frac{5 \times 7}{2^2 3^2}\right) \dots = \frac{\pi}{2} \left(\frac{1 \times 3}{2 \times 2}\right) \left(\frac{3 \times 5}{4 \times 4}\right) \left(\frac{5 \times 7}{6 \times 6}\right) \dots$$

Which gives us Wallis's formula:

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \dots$$

3)

$$\sin x = x \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 - \frac{x}{3\pi}\right) \left(1 + \frac{x}{3\pi}\right) \dots$$

So:

$$\ln \sin x = \ln x + \ln \left(1 - \frac{x}{\pi}\right) + \ln \left(1 + \frac{x}{\pi}\right) + \ln \left(1 - \frac{x}{2\pi}\right) + \ln \left(1 + \frac{x}{2\pi}\right) + \ln \left(1 - \frac{x}{3\pi}\right) + \ln \left(1 + \frac{x}{3\pi}\right) \dots$$

Differentiate both sides and show that:

$$\frac{\cos x}{\sin x} = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \frac{1}{3\pi - x} + \frac{1}{3\pi + x} - \dots$$

Sub in  $x = \frac{\pi}{4}$  and show that:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \dots$$

Footnote:

Am I really saying?

$$\frac{\sin 0}{0} = 1$$

Look at the Maclaurin series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

$$x \rightarrow 0 \quad \frac{\sin x}{x} \rightarrow 1$$