Chinese Remainder Theorem

I have a large bag of sweets. If I share them equally among my 10 children there are 2 sweets left over. If I share them equally among my 7 grandchildren there are 3 sweets left over. How many sweets are in my bag?

If there are $x$ sweets in the bag then $x=2, mod\,10$ and $x=3, mod\,7$

Theorem

This problem has a unique solution for $x=1,2,\ldots 70$

Proof (by contradiction)

Assume there are two solutions $x=p$ and $x=q$

$p=2, mod\,10$ $\quad p=3, mod\,7$

$q=2, mod\,10$ $\quad q=3, mod\,7$

Say $p>q$

$p=2, mod\,10$ $\quad q=2, mod\,10$

So $p=q+10m$ for some positive integer $m$

$p=3, mod\,7$ $\quad q=3, mod\,7$

So $p=q+7n$ for some positive integer $n$

So $q+10m=q+7n$ so $10m=7n$

So $10m$ is a multiple of 7

But 10 and 7 have no common factor so $m$ is a multiple of 7

So $m=7k$ for some positive integer $k$

So:

$p=q+10m$ $\quad m=7k$

So $p=q+70k$

But:

$1\le p\le 70$ $\quad 1\le q\le 70$

Contradiction!

In general:

The simultaneous equations:

$x=a, mod\,m$ $\quad x=b, mod\,n$ $\quad$ where $m$ and $n$ have no common factor have a unique solution for $x = 1, 2, \ldots mn$

A tedious method to find this solution:

$x = 2, mod\, 10$ so $x = 2, 12, 22, 32, 42, 52, 62$

$x = 3, mod\, 7$ so $x = 3, 10, 17, 24, 31, 38, 45, 52, 59, 66$

This gives the solution $x = 52$

Note:

The smallest number of sweets I could have in my bag is 52.

But I could have $52 + 70L$ sweets in my bag where $L$ is any positive integer.

Can you see why?