Tennis

Alice and Bill play tennis. For each point they play, the probability that Alice wins the point is p (see Footnote 1) If they play just one game, what is the probability that Alice wins the game?

For Alice to win, game:love

Alice must win 4 points and Bill must win no points.

Probability: (p^4)

For Alice to win, game:15

Alice must win 4 points and Bill must win 1 point.

But Alice must win the last point – think about it!

So Alice must win 3 of the first 4 points and then win the 5th point.

The number of different orders where Alice wins 3 of the first 4 points is $(4C3)$

These are AAAB, AABA, ABAA, BAAA

(A denotes Alice wins the point and B denotes Bill wins the point)

Probability: $(4C3)(p^4)(1-p)$

For Alice to win, game:30

Alice must win 4 points and Bill must win 2 points.

But Alice must win the last point - etc

Probability: $(5C3)(p^4)(1-p)^2$

For Alice to win, game:40

This can't happen – think about it!

What is the probability the game goes to deuce?

Alice must win 3 points and Bill must win 3 points.

In any order.

Probability: $(6C3)(p^3)(1-p)^3$

What is the probability Alice wins the game, starting from deuce?

This is a bit more difficult.

note: if Alice and Bill start at deuce and play 2 points then:

either, Alice wins both points and wins the game, with probability p^2

or, Bill wins both points and wins the game, with probability $(1-p)^2$

or, Alice and Bill each win 1 point and they are back at deuce, with probability $2p(1-p)$

method 1

The score is at deuce. Alice wins the game if:

Alice wins the next 2 points

OR Alice and Bill each win 1 of the next 2 points and then Alice wins 2 points

OR Alice and Bill each win 1 of the next 2 points and then Alice and Bill each win 1 of the following 2 points and then Alice wins 2 points

OR...

So Alice wins with probability: $p^2+2p(1-p)p^2+(2p(1-p))^2p^2+(2p(1-p))^3p^2+\ldots$

We can sum this infinite series to get:

$$\frac{p^2}{1-2p(1-p)}=\frac{p^2}{2p^2-2p+1}$$

method 2

The score is at deuce. Alice wins the game if:

Alice wins the next 2 points.

OR Alice and Bill each win 1 of the next 2 points and then Alice wins the game.

So if a is the probability Alice wins the game, starting from deuce then:

$$a=p^2+2p(1-p)a$$

This is a neat trick, writing a in terms of a (see Footnote 2)

This rearranges to:

$$a=\frac{p^2}{2p^2-2p+1}$$

So the probability Alice wins the game via deuce is:

$$(6C3)(p^3)(1-p)^3\times\frac{p^2}{2p^2-2p+1}=(6C3)\frac{(p^5)(1-p)^3}{2p^2-2p+1}$$

So the total probability that Alice wins the game is simply:

$$p^4+(4C3)p^4(1-p)+(5C3)p^4(1-p)^2+(6C3)\frac{p^5(1-p)^3}{2p^2-2p+1}$$

Footnote 1:

Alice and Bill play tennis. For each point they play p is the probability that Alice wins the point. This might not be very realistic. The probability that Alice wins the point will probably depend on who is serving, etc. We have ignored such complications.

Footnote 2:

This neat trick reminds me of another problem.

Evaluate:

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$$

Let:

$$x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$$

Then:

$x=1+\frac{1}{x}$ which we can solve.