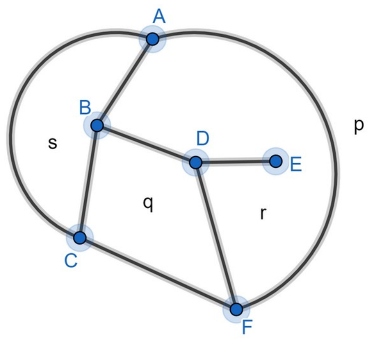
Euler's Formula

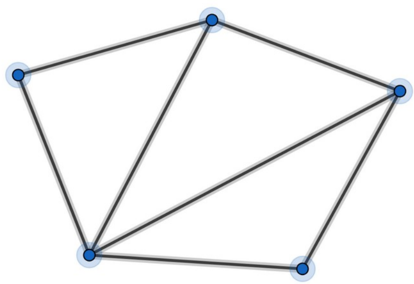
Here is a planar graph:



It is planar because no two edges cross-over each other.

The graph divides the plane into regions p, q, r, s (called faces).

Euler's formula for planar graphs



For this planar graph:

the number of vertices is: $V=5$

the number of edges is: $E=7$

the number of faces is: $F=4$ (remember to include the outer face)

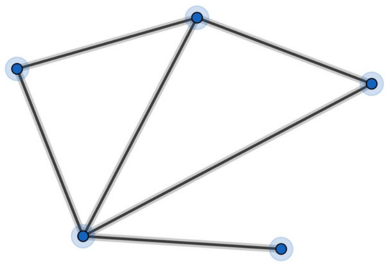
Euler's formula:

For any planar graph $F+V-E=2$

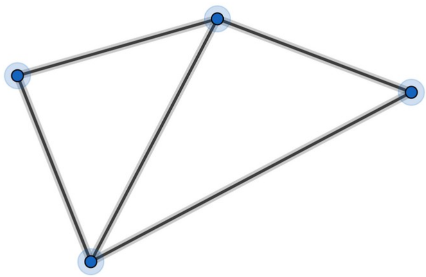
Proof

Start with the graph above.

You can rub-out an edge so $E \rightarrow E-1$ and $F \rightarrow F-1$ and $F+V-E$ stays unchanged.



You can rub-out an edge so $E \rightarrow E-1$ and $V \rightarrow V-1$ and $F+V-E$ stays unchanged.



Once all the rubbing-out has been done, you will be left with:

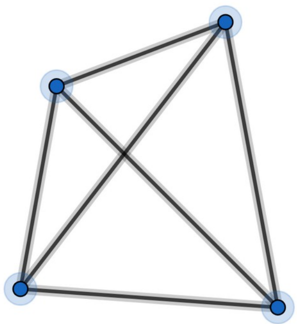
one face $F=1$ one vertex $V=1$ no edges $E=0$ and $F+V-E=2$

But all the rubbing-out leaves $F+V-E$ unchanged.

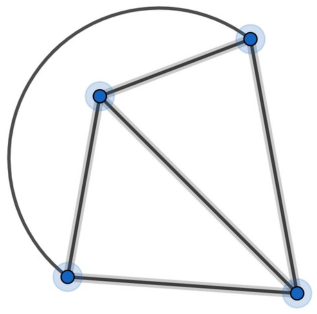
So $F+V-E=2$ for the original graph.

EXAMPLE

We can redraw this graph:

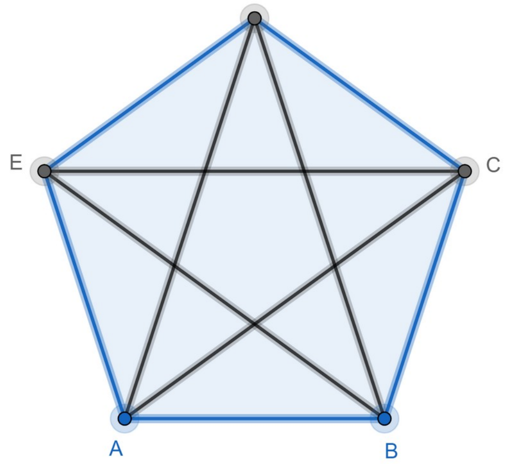


so that it is planar:



Theorem

We cannot redraw this graph so that it is planar:



Proof (by contradiction)

Assume we can redraw this graph so that it is planar.

$V = 5$

Each vertex is joined to 4 edges.

So:

$E = 4 \times 5$ No!

Each edge is shared with 2 vertices.

So:

$$E = \frac{4 \times 5}{2} = 10$$

So by Euler's formula $F = 7$

Each face has at least 3 edges.

So:

$E \geq 3F$ No!

Each edge is shared by 2 faces.

So:

$$E \geq \frac{3F}{2}$$

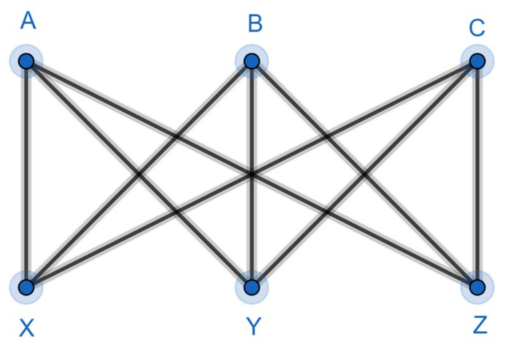
So:

$$10 \geq \frac{3 \times 7}{2}$$

Contradiction.

Theorem

We cannot redraw this graph so that it is planar:



Proof (by contradiction)

Assume we can redraw this graph so that it is planar.

$V=6$ and $E=9$ so by Euler's formula $F=5$

A face cannot have just 3 edges – try drawing one!

Each face has at least 4 edges.

So:

$E \geq 4F$ No!

Each edge is shared by 2 faces.

So:

$$E \geq \frac{4F}{2}$$

So:

$$9 \geq \frac{4 \times 5}{2}$$ Contradiction.

This is known as the utilities problem. Imagine A, B, C are houses and X, Y, Z are gas, water, electricity supply points. Each house needs to be connected, by pipe, to each utility. Can we do this without any pipes crossing over each-other? No!