Recurrence Relations

Example 1

We have a sequence of numbers $u_1, u_2, u_3, \ldots$

$u_1 = 3$ and $u_{n+1} = 4u_n + 1$ This is a recurrence relation. If you know a number in this sequence then the recurrence relation will tell you how to calculate the next number in this sequence.

Now $u_1 = 3$

put $n = 1$ into $u_{n+1} = 4u_n + 1$ and we get $u_2 = 4u_1 + 1 = (4 \times 3) + 1 = 13$

put $n = 2$ into $u_{n+1} = 4u_n + 1$ and we get $u_3 = 4u_2 + 1 = (4 \times 13) + 1 = 53$

put $n = 3$ into $u_{n+1} = 4u_n + 1$ and we get $u_4 = 4u_3 + 1 = (4 \times 53) + 1 = 213$

etc

Example 2

We can define factorials using a recurrence relation:

$1! = 1$ $\qquad (n+1)! = (n+1)n!$

Now $1! = 1$

put $n = 1$ into $(n+1)! = (n+1)n!$ and we get $2! = (2)1! = 2 \times 1 = 2$

put $n = 2$ into $(n+1)! = (n+1)n!$ and we get $3! = (3)2! = 3 \times 2 = 6$

put $n = 3$ into $(n+1)! = (n+1)n!$ and we get $4! = (4)3! = 4 \times 6 = 24$

etc

Many problems give rise to recurrence relations as will will see in the next few sections.

Exercise

Write down the first 5 terms of the sequence

$u_1 = 2$ and $u_2 = 3$ $\qquad u_{n+2} = u_n \times u_{n+1}$

Solution

$u_1 = 2$

$u_2 = 3$

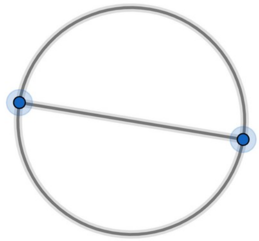
$u_3 = u_1 \times u_2 = 2 \times 3 = 6$

$u_4 = u_2 \times u_3 = 3 \times 6 = 18$

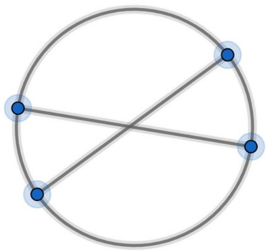
$u_5 = u_3 \times u_4 = 6 \times 18 = 108$

Cutting a Pizza

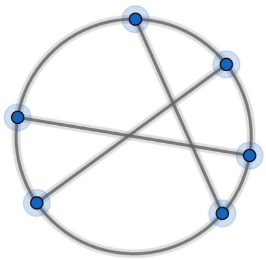
We have a pizza. We run the pizza cutter, in a straight line, across the pizza.
With 1 cut, we get 2 pieces.



With 2 cuts, we get a maximum of 4 pieces.



With 3 cuts, we get a maximum of 7 pieces.



We want the maximum number of pieces, so we want each cut to cross over all the other cuts and we don’t want three or more cuts to cross at the same point.

With 4 cuts?

The fourth cut must cross the other 3 cuts, so it must pass through 4 pieces of pizza. Each of these pieces is cut into two pieces, adding 4 more pieces.

Let P_4 be the maximum number of pieces with 4 cuts.

So $P_4 = P_3 + 4$ So $P_4 = P_3 + 4 = 7 + 4 = 11$

In general:

$P_1 = 2$ and $P_{n+1} = P_n + (n+1)$

We want a formula for P_n We will use the guess and prove method.

Guess:

$$P_n = \frac{1}{2}n(n+1)+1$$

EXERCISE

Use proof by induction to show that this guess is correct

SOLUTION

Proof part 1:

If $n=1$ then $LHS = P_1 = 2$ and $RHS = \frac{1}{2}(1)(2)+1=2$ So the formula is true when $n=1$

Proof part 2:

If $P_n = \frac{1}{2}n(n+1)+1$ is true when $n=k$ then:

$P_k = \frac{1}{2}k(k+1)+1$ but $P_{k+1} = P_k + (k+1)$

$$P_{k+1} = \frac{1}{2}k(k+1)+1+(k+1) = \ldots = \frac{1}{2}(k+1)(k+2)+1$$

So $P_n = \frac{1}{2}n(n+1)+1$ is true when $n=k+1$

So our guess is correct.

Tower of Hanoi

We have three posts. There are some discs on one of these posts. These discs are all different sizes. The discs are in order of size with the largest disc at the bottom and the smallest disc is at the top. We want to move all the discs from this post to another post. However, we can only move one disc at at time and we cannot put a disc on top of a smaller disc.

Let’s call the discs A, B, C, … with disc A being the smallest, disc B being the next smallest ...

We start with two discs, A and B on post 1. We make these moves:

A to post 2 then B to post 3 then A to post 3. Try it.

Now the discs are all on post 3. So to move two discs we need three moves.

We start with three discs, A and B and C on post 1. We make these moves:

A to post 2 then B to post 3 then A to post 3 then C to post 2 then A to post 1 then B to post 2 then A to post 2. Try it.

Now the discs are all on post 3. So to move three discs we need seven moves.

I’m not going to write out all the moves required to transfer four discs. It’s time to stop and think.

We start with four discs A and B and C and D on post 1. To move D, we first have to move A, B and C to another post. We know this takes seven moves. Then we have to move D. This takes one move. Then we have to move A, B and C back on top of D. This takes another seven moves.

So to move four discs we need $7+1+7=15$ moves.

Let $M(n)$ be the number of moves to move n discs.

So $M_4=2M_3+1$

In general:

$M_1=1$ and $M_{n+1}=2M_n+1$

We want a formula for $M(n)$ We will use the guess and prove method.

Guess:

$M_n=2^n-1$

EXERCISE

Use proof by induction to show that this guess is correct

SOLUTIONS

Proof part 1:

If $n=1$ then $LHS=M_1=1$ and $RHS=2^1-1=1$ So the formula is true when $n=1$

Proof part 2:

If $M_n=2^n-1$ is true when $n=k$ then:

$M_k=2^k-1$ but $M_{k+1}=2M_k+1$

$M_{k+1}=2(2^k-1)+1=2^{k+1}-1$

So $M_n=2^n-1$ is true when $n=k+1$

So our guess is correct.

Derangements

There are five people A, B, C, D, E and each person has a card.

A's card has a 1 printed on it, B's card has a 2 printed on it … etc

We collect in the cards, shuffle them up, and then give everyone a card.

How many possible derangements are there?

A derangement is where no-one ends up with their own card.

Let D_n be the number of ways of deranging n cards.

What if A swaps cards with another person?

For example, A gets card 3 and C gets card 1

We are now left with three people B, D, E and three cards 2, 4, 5

B could get card 4 or 5 but not card 2

D could get card 2 or 5 but not card 4

E could get card 2 or 4 but not card 5

This gives us D_3 possible derangements.

A gets card 2 and B gets card 1	D_3 derangements
A gets card 3 and C gets card 1	D_3 derangements
A gets card 4 and D gets card 1	D_3 derangements
A gets card 5 and E gets card 1	D_3 derangements

This gives a total of $4D_3$ derangements

What if A does not swap cards with another person?

For example, A gets card 3 but C does not get card 1.

We are now left with four people B, C, D, E and four cards 1, 2, 4, 5

B could get card 1 or 4 or 5 but not card 2

C could get card 2 or 4 or 5 but not card 1 (because this would be a swap)

D could get card 1 or 2 or 5 but not card 4

E could get card 1 or 2 or 4 but not card 5

This gives us D(4) possible derangements.

A gets card 2 but B does not get card 1	D_4 derangements
A gets card 3 but C does not get card 1	D_4 derangements
A gets card 4 but D does not get card 1	D_4 derangements
A gets card 5 but E does not get card 1	D_4 derangements

This gives a total of $4D_4$ derangements

So $D_5 = 4D_3 + 4D_4$

In general:

$D_1 = 0$ and $D_2 = 1$ and $D_{n+2} = (n+1)D_n + (n+1)D_{n+1}$

We want a formula for D_n We will use the guess and prove method.

Guess:

$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + ... + \frac{1}{n!}\right)$ (where did this come from?)

Proof:

$D_1 = 1!\left(1 - \frac{1}{1!}\right) = 0$ Correct

$D_2 = 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) = 1$ Correct

$(n+1)D_n + (n+1)D_{n+1} = ... = D_{n+2}$ Correct (you do this - it is very tedious!)

Now $\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$ (see chapter: e)

So if n is large $D_n \approx n!\left(\frac{1}{e}\right)$

Instead of asking about the number of derangements, we can ask about the probability of getting a derangement.

There are n people and each person owns a card. We collect in the cards, shuffle them up, and then give everyone a card. What is the probability of getting a derangement?

There are D_n ways of getting a derangement and there are $n!$ ways of giving out the cards.

So the probability of getting a derangement is $\frac{D_n}{n!}$

So if n is large, the probability of getting a derangement is approximately $1/e$

Exercise

1) I write out my thirty Christmas cards and then I write out all the envelopes. Foolishly, I put the cards into the envelopes at random.

What is the probability that all my friends get sent the wrong card?

2) You and I each have a pack of cards. Both packs are shuffled. We then play Snap. What is the probability that we get to the end of our packs with no snaps?

Answer is approximately $1/e$

3) Twenty students go to a party. When they arrive, each student drops their coat on the floor. When they leave, each student grabs a coat at random. What is the probability that no student gets their own coat?

Answer is approximately $1/e$

Solutions

1) Answer is approximately $1/e$

2) Answer is approximately $1/e$

3) Answer is approximately $1/e$

Fibonacci Numbers

The Fibonacci numbers:

$1,1,2,3,5,8,13,\ldots$ are given by the recurrence relation:

$F_1=1 \quad F_2=1 \quad F_{n+2}=F_{n+1}+F_n$

So:

$F_3=F_2+F_1$ and $F_4=F_3+F_2$ and $F_5=F_4+F_3$ etc

We want a formula for F_n We will use the guess and prove method.

Guess:

$F_n=\frac{a^n-b^n}{\sqrt{5}}$ where $a=\frac{1+\sqrt{5}}{2}$ and $b=\frac{1-\sqrt{5}}{2}$ (where did that come from?)

Proof:

Consider the equation $x^2=x+1$ (where did that come from?)

Solving with the quadratic equation formula gives:

$x=\frac{1\pm\sqrt{5}}{2}$ so $x=a$ or $x=b$

Now:

a and b satisfy $x^2=x+1$

So:

$a^2=a^1+1 \quad a^3=a^2+a^1 \quad \ldots \quad a^{n+2}=a^{n+1}+a^n$

And:

$b^2=b^1+1 \quad b^3=b^2+b^1 \quad \ldots \quad b^{n+2}=b^{n+1}+b^n$

According to our guess:

$$F_1=\frac{a^1-b^1}{\sqrt{5}}=\frac{\sqrt{5}}{\sqrt{5}}=1$$

Correct

$$F_2=\frac{a^2-b^2}{\sqrt{5}}=\frac{(a^1+1)-(b^1+1)}{\sqrt{5}}=\frac{a^1-b^1}{\sqrt{5}}=1$$

Correct

$$F_{n+2}=\frac{a^{n+2}-b^{n+2}}{\sqrt{5}}=\frac{(a^{n+1}+a^n)-(b^{n+1}+b^n)}{\sqrt{5}}=\frac{(a^{n+1}-b^{n+1})+(a^n-b^n)}{\sqrt{5}}=F_{n+1}+F_n$$

Correct

Let’s look at the ratio of consecutive Fibonacci numbers:

$$\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\ldots$$

As $n \to \infty$ these ratios tend to a limit called ϕ

Now:

$$F_{n+2}=F_{n+1}+F_n \quad \text{so} \quad \frac{F_{n+2}}{F_{n+1}}=\frac{F_{n+1}}{F_{n+1}}+\frac{F_n}{F_{n+1}} \quad \text{so} \quad \frac{F_{n+2}}{F_{n+1}}=1+\frac{F_n}{F_{n+1}}$$

Letting $n \to \infty$ we get $\phi=1+\frac{1}{\phi}$ so $\phi^2=\phi+1$ so $\phi=\frac{1+\sqrt{5}}{2}$

Note:

ϕ is called the golden ratio. Look it up!

Theorem:

If F_n is prime then n is prime.

The converse of this theorem is not true – for example F_{19} is not prime.

Conjecture:

There are an infinite number of Fibonacci numbers that are prime.

EXERCISE

1) Write the Fibonacci sequence in mod 2.

Show that the 3rd, 6th, 9th, 12th … Fibonacci numbers are all multiples of 2

2) Write the Fibonacci sequence in mod 3.

Show that the 4th, 8th, 12th, 16th … Fibonacci numbers are all multiples of 3

3) Write the Fibonacci sequence in mod 5.

Show that the 5th, 10th, 15th, 20th ... Fibonacci numbers are all multiples of 5

4)

If d is a factor of F_{17} and F_{18} show that d is a factor of F_{16}

If d is a factor of F_{16} and F_{17} show that d is a factor of F_{15}

Show that consecutive Fibonacci numbers have no common factor.

5)

$F_1=F_3-F_2$

$F_2=F_4-F_3$

$F_3=F_5-F_4$

…

$F_{n-1} = F_{n+1} - F_n$

$F_n = F_{n+2} - F_{n+1}$

Show that:

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$

SOLUTIONS

1) mod 2:

1, 1, 0, 1, 1, 0, 1, 1, …

The sequence must now repeat because we are back to 1, 1

So the 3rd, 6th, 9th, 12th … terms are all multiples of 2

2) mod 3:

1, 1, 2, 0, 2, 2, 1, 0, 1, 1, …

The sequence must now repeat because we are back to 1, 1

So the 4th, 8th, 12th, 16th … terms are all multiples of 3

3) mod 5:

1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, …

The sequence must now repeat because we are back to 1, 1

So the 5th, 10th, 15th, 20th ... terms are all multiples of 5

4)

d is a factor of F_{16} because $F_{16} = F_{18} - F_{17}$

d is a factor of F_{15} because $F_{15} = F_{17} - F_{16}$

etc

d is a factor of F_1

So consecutive Fibonacci numbers have no common factor

5)

add up the left-hand-sides: $F_1 + F_2 + F_3 + \ldots + F_n$

add up the right-hand -sides: $F_{n+2} - F_2$

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - F_2$

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$

Polygonal numbers

The triangle numbers:

$1,3,6,10,15,\ldots$ are given by the recurrence relation:

$T_1=1$ and $T_{n+1}=T_n+n+1$

So:

$T_2=T_1+2$ and $T_3=T_2+3$ and $T_4=T_3+4$ etc

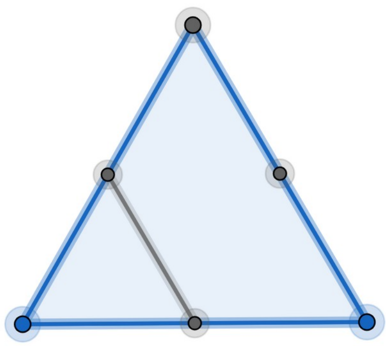
So:

$T_4=T_3+4=(T_2+3)+4=((T_1+2)+3)+4=1+2+3+4$

In general:

$T_n=1+2+3+\ldots+n$

NEED BETTER DIAGRAM



Theorem:

$$T_n=\frac{1}{2}n(n+1)$$

Proof (by induction)

part 1:

If $n=1$ then:

$LHS=T_1=1$ and $RHS=\frac{1}{2}(1)(2)=1$ So the formula is true when $n=1$

part 2:

If $T_n=\frac{1}{2}n(n+1)$ is true when $n=k$ then:

$T_k=\frac{1}{2}k(k+1)$ but $T_{k+1}=T_k+k+1$

So:

$$T_{k+1}=\frac{1}{2}k(k+1)+k+1=\ldots=\frac{1}{2}(k+1)(k+2)$$

So:

$T_n=\frac{1}{2}n(n+1)$ is true when $n=k+1$

The square numbers:

$1,4,9,16,25,\ldots$ are given by the recurrence relation:

$S_1=1$ and $S_{n+1}=S_n+2n+1$

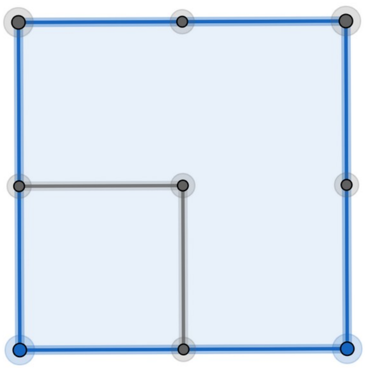
So:

$S_2=S_1+3$ and $S_3=S_2+5$ and $S_4=S_3+7$ etc

Show that:

$S_n=1+3+5+\ldots+(2n-1)$

NEED BETTER DIAGRAM



Theorem:

$S_n=\frac{1}{2}n(2n)$ or if you prefer $S_n=n^2$

see Exercise 1

The pentagonal numbers:

$1,5,12,22,35,\ldots$ are given by the recurrence relation:

$P_1=1$ and $P_{n+1}=P_n+3n+1$

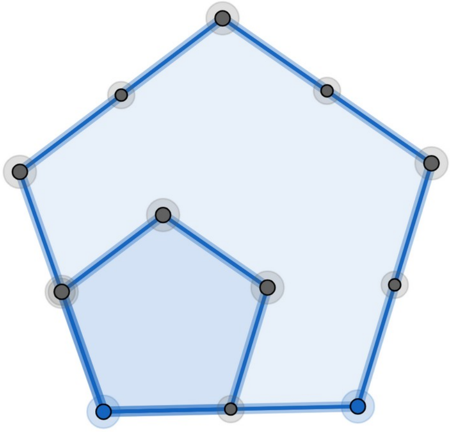
So:

$P_2=P_1+4$ and $P_3=P_2+7$ and $P_4=P_3+10$ etc

Show that:

$P_n = 1 + 4 + 7 + \ldots + (3n-2)$

NEED BETTER DIAGRAM



Theorem:

$P_n = \frac{1}{2} n(3n-1)$

The hexagonal numbers:

$1, 6, 15, 28, 45, \ldots$ are given by the recurrence relation:

$H_1 = 1$ and $H_{n+1} = H_n + 4n + 1$

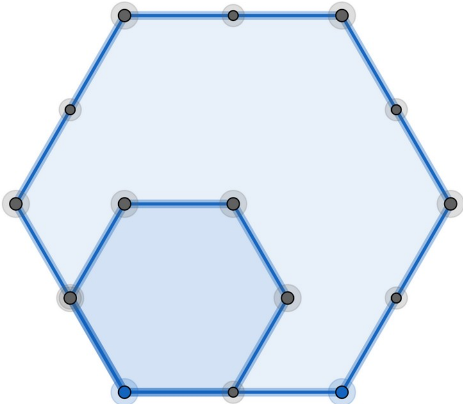
So:

$H_2 = H_1 + 5$ and $H_3 = H_2 + 9$ and $H_4 = H_3 + 13$ etc

Show that:

$H_n = 1 + 5 + 9 + \ldots + (4n-3)$

NEED BETTER DIAGRAM



Theorem:

$$H_n = \frac{1}{2} n(4n-2)$$

The polygonal number theorem (difficult to prove)

Every positive integer can be written as the sum of:

3 (or fewer) triangle numbers

4 (or fewer) square numbers

5 (or fewer) pentagonal numbers

etc

see Exercise 2

EXERCISE 1

Square numbers:

Show that $S_n = \frac{1}{2} n(2n)$ Use proof by induction

EXERCISE 2

Prove the following:

1. The sum of two consecutive triangle numbers is a square number.
2. If T is a triangle number then $8T+1$ is a square number.
3. If T is a triangle number then $9T+1$ is a triangle number.
4. The difference of the squares of two consecutive triangle numbers is a cube.
5. $P_n = T_n + 2T_{n-1}$
6. $H_n = T_n + 3T_{n-1}$
7. The last digit of a triangle number can't be 2, 4, 7 or 9

SOLUTIONS 1

Proof (by induction)

part 1:

If $n=1$ then $LHS = S_1 = 1$ and $RHS = \frac{1}{2}(1)(2) = 1$ So the formula is true when $n=1$

part 2:

If $S_n=\frac{1}{2}n(2n)$ is true when $n=k$ then:

$S_k=\frac{1}{2}k(2k)$ but $S_{k+1}=S_k+2k+1$

So $S_{k+1}=\frac{1}{2}k(2k)+2k+1=...=\frac{1}{2}(k+1)2(k+1)$

So $S_n=\frac{1}{2}n(2n)$ is true when $n=k+1$

SOLUTIONS 2

1) $T_k+T_{k+1}=\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+2)=...=(k+1)^2$

Or:

In this diagram we have got: T_5 Ps and T_6 Qs

How many letters have we got?

Q	Q	Q	Q	Q	Q
P	Q	Q	Q	Q	Q
P	P	Q	Q	Q	Q
P	P	P	Q	Q	Q
P	P	P	P	Q	Q
P	P	P	P	P	Q

$T_5+T_6=6^2$

In general $T_k+T_{k+1}=(k+1)^2$

2) $8T_k+1=8\frac{1}{2}k(k+1)+1=...=(2k+1)^2$

3) $9T_k+1=9\frac{1}{2}k(k+1)+1=...=\frac{1}{2}(3k+1)(3k+2)$

4) $(T_{k+1})^2-(T_k)^2=(\frac{1}{2}(k+1)(k+2))^2-(\frac{1}{2}k(k+1))^2=...=(k+1)^3$

5) $P_n=\frac{1}{2}n(3n-1)$

$T_n+2T_{n-1}=\frac{1}{2}n(n+1)+2\frac{1}{2}(n-1)n=...=\frac{1}{2}n(3n-1)$

6) $H_n = \frac{1}{2} n(4n-2)$

$$T_n + 3T_{n-1} = \frac{1}{2} n(n+1) + 3\frac{1}{2}(n-1)n = \ldots = \frac{1}{2} n(4n-2)$$

7) mod 10:

n	0	1	2	3	4	5	6	7	8	9
$n+1$	1	2	3	4	5	6	7	8	9	0
$n(n+1)$	0	2	6	2	0	0	2	6	2	0

now $T = \frac{1}{2} n(n+1)$ so $2T = n(n+1)$ so the last digit of $2T$ is 0, 2 or 6

mod 10:

T	0	1		3		5	6		8	
$2T$	0	2		6		0	2		6	

If the last digit of $2T$ is 0, 2 or 6 then the last digit of T must be 0, 1, 3, 5, 6 or 8

A Nice Integral if you have studied integration ...

$$I_n = \int_0^\infty x^n e^{-x} dx$$

Use integration by parts to show that:

$$I_n = n I_{n-1}$$

So:

$$I_8 = 8 \times I_7 = 8 \times 7 \times I_6 = 8 \times 7 \times 6 \times I_5 = ... = 8!$$

In general:

$$I_n = n!$$

Now:

$$n! = 1 \times 2 \times 3 \ldots \times n$$

This only makes sense if n is a positive integer.

$$I_n = \int_0^\infty x^n e^{-x} dx$$

This makes sense for any value of n

So, for example:

$$(-1)! = \int_0^\infty x^{-1} e^{-x} dx$$

I might not be able to work this out but it certainly has a value.