

Every packet of cornflakes contains a card with a picture of a mathematician. There are 20 different mathematicians to collect. On average, how many cornflakes packets will I have to buy to get the complete set?

Let X_1 be the number of cereal packets I will have to buy to get my first mathematician.

Let X_2 be the number of cereal packets I will have to buy, after I have got my first mathematician, to get my second mathematician.

Let X_3 be the number of cereal packets I will have to buy, after I have got my second mathematician, to get my third mathematician.

etc

Let X be the total number of cereal packets I will have to buy to get the complete set of 20 mathematicians.

So:

$$X = X_1 + X_2 + X_3 + \dots + X_{20} \quad \text{and we want to find } E(X)$$

We know that:

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \dots + E(X_{20})$$

When I buy my first packet, I will get my first mathematician.

So:

$$X_1 = 1 \quad \text{and so } E(X_1) = 1$$

Now I have got my first mathematician. How many more packets will I have to buy to get my second mathematician?

Look at this table.

x_2	$p(x_2)$	$x_2 p(x_2)$
1	$\frac{19}{20}$	$1 \times \frac{19}{20}$
2	$\frac{1}{20} \times \frac{19}{20}$	$2 \times \frac{1}{20} \times \frac{19}{20}$
3	$\frac{1}{20} \times \frac{1}{20} \times \frac{19}{20}$	$3 \times \frac{1}{20} \times \frac{1}{20} \times \frac{19}{20}$
...

Now:

$$E(X_2) = \sum x_2 p(x_2)$$

So:

$$E(X_2) = \frac{19}{20} \left(1 + 2 \left(\frac{1}{20} \right) + 3 \left(\frac{1}{20} \right)^2 + 4 \left(\frac{1}{20} \right)^3 + \dots \right)$$

So:

$$E(X_2) = \frac{19}{20} \frac{1}{\left(1 - \frac{1}{20} \right)^2} = \frac{20}{19} \quad \text{see Footnote with } x = \frac{1}{20}$$

Now I have got my second mathematician. How many more packets will I have to buy to get my third mathematician?

Look at this table.

x_3	$p(x_3)$	$x_3 p(x_3)$
1	$\frac{18}{20}$	$1 \times \frac{19}{20}$
2	$\frac{2}{20} \times \frac{18}{20}$	$2 \times \frac{1}{20} \times \frac{19}{20}$
3	$\frac{2}{20} \times \frac{2}{20} \times \frac{18}{20}$	$3 \times \frac{1}{20} \times \frac{1}{20} \times \frac{19}{20}$
...

Repeat the above calculation and show that:

$$E(X_3) = \frac{20}{18}$$

Find:

$$E(X_4)$$

etc

Show that:

$$E(X) = 20 \left(\frac{1}{20} + \frac{1}{19} + \frac{1}{18} + \dots + \frac{1}{1} \right)$$

Footnote

We know that:

$$1 + x + x^2 + x^3 + x^4 + x^5 + \dots = \frac{1}{1-x} \quad \text{it is a geometric series}$$

Differentiate both sides and show that:

$$1+2x+3x^2+4x^3+5x^4+\ldots=\frac{1}{(1-x)^2}$$