Euler’s Prime Sum

In the chapter, Euler’s Zeta Function, we got the result:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)\ldots=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$$

Taking logs of both sides:

$$\ln\left(\frac{1}{1-\frac{1}{2}}\right)+\ln\left(\frac{1}{1-\frac{1}{3}}\right)+\ln\left(\frac{1}{1-\frac{1}{5}}\right)+\ln\left(\frac{1}{1-\frac{1}{7}}\right)+\ldots=\ln\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots\right)$$

So:

$$\sum_{primes}\ln\left(\frac{1}{1-\frac{1}{p}}\right)=\ln\sum_{1}^{\infty}\frac{1}{n} \quad *$$

Now:

$$\left(\frac{1}{1-\frac{1}{p}}\right)=\left(1-\frac{1}{p}\right)^{-1} \quad \text{So} \quad \ln\left(\frac{1}{1-\frac{1}{p}}\right)=-\ln\left(1-\frac{1}{p}\right)$$

So we can write * as:

$$\sum_{primes}-\ln\left(1-\frac{1}{p}\right)=\ln\sum_{1}^{\infty}\frac{1}{n} \quad **$$

Now:

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots$$ this is the Maclaurin series

Put $x=-\frac{1}{p}$

We get:

$$\ln\left(1-\frac{1}{p}\right)=-\frac{1}{p}-\frac{1}{2p^2}-\frac{1}{3p^3}-\frac{1}{4p^4}-\ldots \quad \text{So} \quad -\ln\left(1-\frac{1}{p}\right)=\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots$$

So we can write ** as:

$$\sum_{primes}\left(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\right)=\ln\sum_{1}^{\infty}\frac{1}{n}$$

So:

$$\sum_{primes}\frac{1}{p}+\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\right)=\ln\sum_{1}^{\infty}\frac{1}{n} \quad ***$$

Consider:

$$\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\right)$$

Now:

$$\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots<\frac{1}{p^2}+\frac{1}{p^3}+\frac{1}{p^4}+\ldots=\frac{\left(\frac{1}{p^2}\right)}{\left(1-\frac{1}{p}\right)}=\ldots=\frac{1}{p-1}-\frac{1}{p}$$

So:

$$\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\right)<\sum_{primes}\left(\frac{1}{p-1}-\frac{1}{p}\right)<\sum_{2}^{\infty}\left(\frac{1}{p-1}-\frac{1}{p}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=\ldots=1$$

Look at *** again:

$$\sum_{primes}\frac{1}{p}+\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\right)=\ln\sum_{1}^{\infty}\frac{1}{n}$$

We know:

$\ln\sum_{1}^{\infty}\frac{1}{n}$ diverges.

We have just shown that:

$\sum_{primes}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\right)$ converges.

So we can deduce that:

$\sum_{primes}\frac{1}{p}$ diverges.

So:

$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{23}+\ldots$ diverges. Astonishing!