Fundamental Theorem of Arithmetic

Some positive integers are prime numbers.

2, 3, 5, 7, 11, 13, 17, 19….

All the others can be written, in just one way, as a product of prime numbers. For example:

$60 = 6\times10 = 2\times3\times2\times5 = (2^2)(3)(5)$

In general:

If N is any positive integer (except 1) then $N=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

where $a, b, c, \ldots$ are zero or positive integers.

Example 1

Highest common factor (HCF) and Lowest common multiple (LCM)

the factors of 24 are:	1, 2, 3, 4, 6, 8, 12, 24
the factors of 30 are:	1, 2, 3, 5, 6, 10, 15, 30
the common factors of 24 and 30 are:	1, 2, 3, 6

So $HCF(24,30)=6$

the multiples of 24 are:	24, 48, 72, 96, 120, 144, 168, 192, 216, 240, 264 ...
the multiples of 30 are:	30, 60, 90, 120, 150, 180, 210, 240, 270, 300, 330 ...
the common multiples of 24 and 30 are:	120, 240, ...

So $LCM(24,30)=120$

Example 2

$A=(2^5)(3^1)(5^0)(7^6)(11^2)$ and $B=(2^3)(3^7)(5^4)(7^8)(11^0)$

2^5 is a factor of A and 2^3 is a factor of B so HCF is a multiple of 2^3

3^1 is a factor of A and 3^7 is a factor of B so HCF is a multiple of 3^1

etc

So $HCF(A,B)=(2^3)(3^1)(5^0)(7^6)(11^0)$

A is a multiple of 2^5 and B is a multiple of 2^3 so LCM is a multiple of 2^5

A is a multiple of 3^1 and B is a multiple of 3^7 so LCM is a multiple of 3^7

etc

So $LCM(A,B)=(2^5)(3^7)(5^4)(7^8)(11^2)$

note

$HCF(A,B)\times LCM(A,B)=(2^8)(3^8)(5^4)(7^{14})(11^2)=AB$

Example 3

$N=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

If N is a multiple of 5 then $c\geq 1$ If N is not a multiple of 5 then $c=0$

If N is a multiple of 3 and a multiple of 7 then $b\geq 1$ and $d\geq 1$

So N is a multiple of 21

If N is a multiple of 6 then N is a multiple of 2 and a multiple of 3 so $a\geq 1$ and $b\geq 1$

If N is a multiple of 15 then N is a multiple of 3 and a multiple of 5 so $b\geq 1$ and $c\geq 1$

So if N is a multiple of 6 and a multiple of 15 then N must be a multiple of 30

In general:

If N is a multiple of a and a multiple of b then N is a multiple of $LCM(a,b)$

Example 4

$N=(2)(3^2)(13^4)$ $M=(5^3)(7^5)(13)(23)$ so $NM=(2)(3^2)(5^3)(7^5)(13^5)(23)$

NM is a multiple of 3 because N is a multiple of 3

NM is a multiple of 7 because M is a multiple of 7

NM is not a multiple of 17 because neither N nor M is a multiple of 17

but:

NM is a multiple of 14 even though neither N nor M is a multiple of 14

this is because $14=2\times 7$ and N is a multiple of 2 and M is a multiple of 7

also:

NM is a multiple of 35 but N and 35 have no common factor. So all the factors of 35 must appear in M So M must be a multiple of 35

In general: if p is prime:

NM is a multiple of p only if N or M (or both) is a multiple of p

In general: if N and r have no common factor:

NM is a multiple of r only if M is a multiple of r

see Exercise 1

Theorem

$\sqrt{2}$ is irrational

Proof (by contradiction)

Assume $\sqrt{2}$ is rational

So:

$\sqrt{2}=\frac{p}{q}$ where p and q are positive integers

So:

$2q^2=p^2$

Now:

We can write q as a product of primes:

$q=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

So:

$q^2=(2^{2a})(3^{2b})(5^{2c})(7^{2d})(11^{2e})(...)$ the powers of all the primes are even

So:

$2q^2=(2^{2a+1})(3^{2b})(5^{2c})(7^{2d})(11^{2e})(...)$ the power of 2 is odd

Now:

We can write p as a product of primes:

$p=\ldots$

So:

$p^2=\ldots$ all the powers of all the primes are even

But:

$2q^2=p^2$

LHS, power of 2 is odd. RHS, power of 2 is even.

Contradiction.

There is another proof that $\sqrt{2}$ is irrational in the chapter: Proof by Contradiction

But this proof is better, because it suggests why the result is true and it suggests further results.

See Exercise 2

EXERCISE 1

1) Write 5619250 in the form $(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

2) Find $HCF(36652, 38698)$ and $LCM(36652, 38698)$

3) $532400=(2^4)(5^2)(11^3)$ How many factors has 532400 got?

4) This question is difficult

a) If n^2 is a multiple of 7 show that n is a multiple of 7

b) If n^2 is a multiple of 6 show that n is a multiple of 6

c) If n^2 is a multiple of 12 show that n might not be a multiple of 12

d) For what values of m is the following true:

If n^2 is a multiple of m then n must be a multiple of m ?

EXERCISE 2

1) Prove $5^{1/3}$ is irrational

2) What happens when we try to prove $\sqrt{4}$ is irrational?

SOLUTIONS 1

1) $5619250=(2)(5^3)(7)(13^2)(19)$

2) $36652=(2^2)(7^2)(11^1)(17^1)$ and $38698=(2^1)(11^1)(1759^1)$

$HCF(36652, 38698)=(2^1)(11^1)=22$

$LCM(36652, 38698)=(2^2)(7^2)(11^1)(17^1)(1759^1)=64470868$

3) $532400=(2^4)(5^2)(11^3)$ so any factor can be written as $(2^p)(5^q)(11^r)$

where $p=0,1,2,3,4$ and $q=0,1,2$ and $r=0,1,2,3$

We have 5 choices for the value of p and 3 choices for the value of q and 4 choices for the value of r So there are $5\times3\times4=60$ choices for p,q,r

So 532400 has 60 factors (including 1 and 532400)

4) $n=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

$n^2=(2^{2a})(3^{2b})(5^{2c})(7^{2d})(11^{2e})(...)$

proof by contrapositive

a) If n is not a multiple of 7 then $d=0$ and n^2 is not a multiple of 7

b) If n is not a multiple of 6 then $a=0$ or $b=0$ and n^2 is not a multiple of 6

c) If n is not a multiple of 12 then we cannot say $a=0$ or $b=0$ because we could have $a=1$ and $b=1$ for example if $n=6$

6^2 is a multiple of 12 but 6 is not a multiple of 12

d) $m=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

The statement is true if $a=0,1$ $b=0,1$ $c=0,1$ etc

SOLUTIONS 2

1) Assume $5^{1/3}$ is rational

$5^{1/3}=\frac{p}{q}$ where p and q are integers

$5q^3=p^3$

We can write q as a product of powers of primes:

$q=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

$q^3=(2^{3a})(3^{3b})(5^{3c})(7^{3d})(11^{3e})(...)$ all the powers of all the primes are multiples of three.

$5q^3=(2^{3a})(3^{3b})(5^{3c+1})(7^{3d})(11^{3e})(...)$ the power of 5 is not a multiple of three.

We can write p as a product of powers of primes:

$p=\ldots$

$p^3=\ldots$ all the powers of all the primes are multiples of three

$5q^3=p^3$

LHS, power of 5 is not a multiple of three. RHS, power of 5 is a multiple of three.

Contradiction.

2) Claim

$\sqrt{4}$ is irrational

Attempted proof (by contradiction)

Assume $\sqrt{4}$ is rational

$\sqrt{4}=\frac{p}{q}$ where p and q are positive integers

$4q^2=p^2$

We can write q as a product of primes:

$q=(2^a)(3^b)(5^c)(7^d)(11^e)(...)$

$q^2=(2^{2a})(3^{2b})(5^{2c})(7^{2d})(11^{2e})(...)$ the powers of all the primes are even

$4q^2=(2^{2a+2})(3^{2b})(5^{2c})(7^{2d})(11^{2e})(...)$ the power of 2 is still even!

This is where our proof falls apart.