

Polygonal numbers

The triangle numbers:

1, 3, 6, 10, 15, ... are given by the recurrence relation:

$$T_1 = 1 \quad \text{and} \quad T_{n+1} = T_n + n + 1$$

So:

$$T_2 = T_1 + 2 \quad \text{and} \quad T_3 = T_2 + 3 \quad \text{and} \quad T_4 = T_3 + 4 \quad \text{etc}$$

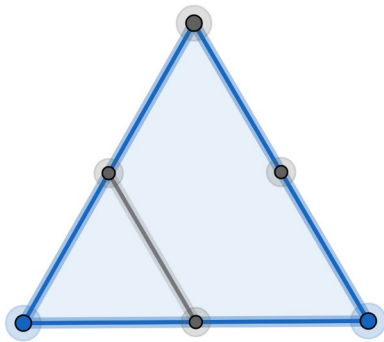
So:

$$T_4 = T_3 + 4 = (T_2 + 3) + 4 = ((T_1 + 2) + 3) + 4 = 1 + 2 + 3 + 4$$

In general:

$$T_n = 1 + 2 + 3 + \dots + n$$

NEED BETTER DIAGRAM



Theorem:

$$T_n = \frac{1}{2}n(n+1)$$

Proof (by induction)

part 1:

If $n=1$ then:

$$LHS = T_1 = 1 \quad \text{and} \quad RHS = \frac{1}{2}(1)(2) = 1 \quad \text{So the formula is true when } n=1$$

part 2:

If $T_n = \frac{1}{2}n(n+1)$ is true when $n=k$ then:

$$T_k = \frac{1}{2}k(k+1) \quad \text{but} \quad T_{k+1} = T_k + k + 1$$

So:

$$T_{k+1} = \frac{1}{2}k(k+1) + k + 1 = \dots = \frac{1}{2}(k+1)(k+2)$$

So:

$$T_n = \frac{1}{2}n(n+1) \text{ is true when } n = k+1$$

The square numbers:

1, 4, 9, 16, 25, ... are given by the recurrence relation:

$$S_1 = 1 \text{ and } S_{n+1} = S_n + 2n + 1$$

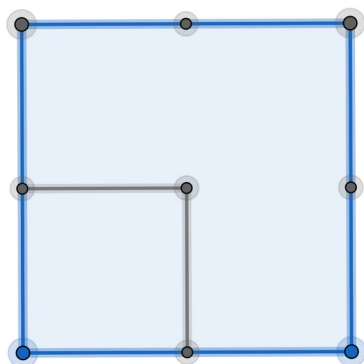
So:

$$S_2 = S_1 + 3 \text{ and } S_3 = S_2 + 5 \text{ and } S_4 = S_3 + 7 \text{ etc}$$

Show that:

$$S_n = 1 + 3 + 5 + \dots + (2n - 1)$$

NEED BETTER DIAGRAM



Theorem:

$$S_n = \frac{1}{2}n(2n) \text{ or if you prefer } S_n = n^2$$

see Exercise 1

The pentagonal numbers:

1, 5, 12, 22, 35, ... are given by the recurrence relation:

$$P_1 = 1 \text{ and } P_{n+1} = P_n + 3n + 1$$

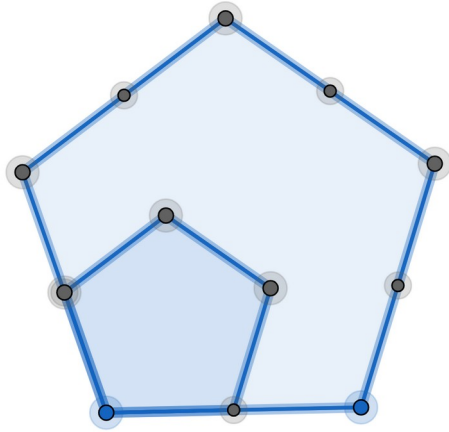
So:

$$P_2 = P_1 + 4 \text{ and } P_3 = P_2 + 7 \text{ and } P_4 = P_3 + 10 \text{ etc}$$

Show that:

$$P_n = 1 + 4 + 7 + \dots + (3n - 2)$$

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Theorem:

$$P_n = \frac{1}{2}n(3n - 1)$$

The hexagonal numbers:

1, 6, 15, 28, 45, ... are given by the recurrence relation:

$$H_1 = 1 \quad \text{and} \quad H_{n+1} = H_n + 4n + 1$$

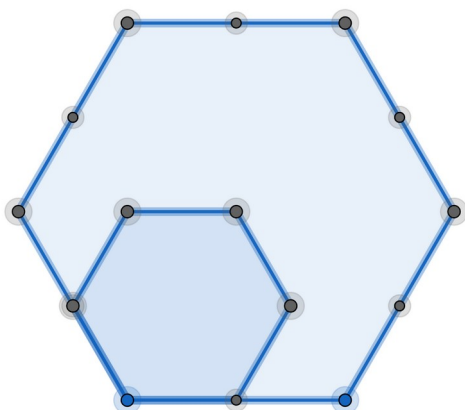
So:

$$H_2 = H_1 + 5 \quad \text{and} \quad H_3 = H_2 + 9 \quad \text{and} \quad H_4 = H_3 + 13 \quad \text{etc}$$

Show that:

$$H_n = 1 + 5 + 9 + \dots + (4n - 3)$$

NEED BETTER DIAGRAM



Theorem:

$$H_n = \frac{1}{2}n(4n-2)$$

The polygonal number theorem (difficult to prove)

Every positive integer can be written as the sum of:

3 (or fewer) triangle numbers

4 (or fewer) square numbers

5 (or fewer) pentagonal numbers

etc

see Exercise 2

EXERCISE 1

Square numbers:

Show that $S_n = \frac{1}{2}n(2n)$ Use proof by induction

EXERCISE 2

Prove the following:

1. The sum of two consecutive triangle numbers is a square number.
2. If T is a triangle number then $8T+1$ is a square number.
3. If T is a triangle number then $9T+1$ is a triangle number.
4. The difference of the squares of two consecutive triangle numbers is a cube.
5. $P_n = T_n + 2T_{n-1}$
6. $H_n = T_n + 3T_{n-1}$
7. The last digit of a triangle number can't be 2, 4, 7 or 9

SOLUTIONS 1

Proof (by induction)

part 1:

If $n=1$ then $LHS = S_1 = 1$ and $RHS = \frac{1}{2}(1)(2) = 1$ So the formula is true when $n=1$

part 2:

If $S_n = \frac{1}{2}n(2n)$ is true when $n=k$ then:

$$S_k = \frac{1}{2}k(2k) \text{ but } S_{k+1} = S_k + 2k + 1$$

$$\text{So } S_{k+1} = \frac{1}{2}k(2k) + 2k + 1 = \dots = \frac{1}{2}(k+1)2(k+1)$$

$$\text{So } S_n = \frac{1}{2}n(2n) \text{ is true when } n=k+1$$

SOLUTIONS 2

$$1) T_k + T_{k+1} = \frac{1}{2}k(k+1) + \frac{1}{2}(k+1)(k+2) = \dots = (k+1)^2$$

Or:

In this diagram we have got: T_5 Ps and T_6 Qs

How many letters have we got?

Q	Q	Q	Q	Q	Q
P	Q	Q	Q	Q	Q
P	P	Q	Q	Q	Q
P	P	P	Q	Q	Q
P	P	P	P	Q	Q
P	P	P	P	P	Q

$$T_5 + T_6 = 6^2$$

$$\text{In general } T_k + T_{k+1} = (k+1)^2$$

$$2) 8T_k + 1 = 8 \cdot \frac{1}{2}k(k+1) + 1 = \dots = (2k+1)^2$$

$$3) 9T_k + 1 = 9 \cdot \frac{1}{2}k(k+1) + 1 = \dots = \frac{1}{2}(3k+1)(3k+2)$$

$$4) (T_{k+1})^2 - (T_k)^2 = \left(\frac{1}{2}(k+1)(k+2)\right)^2 - \left(\frac{1}{2}k(k+1)\right)^2 = \dots = (k+1)^3$$

$$5) P_n = \frac{1}{2}n(3n-1)$$

$$T_n + 2T_{n-1} = \frac{1}{2}n(n+1) + 2 \cdot \frac{1}{2}(n-1)n = \dots = \frac{1}{2}n(3n-1)$$

6) $H_n = \frac{1}{2}n(4n-2)$

$$T_n+3T_{n-1}=\frac{1}{2}n(n+1)+3\frac{1}{2}(n-1)n=...=\frac{1}{2}n(4n-2)$$

7) mod 10:

n	0	1	2	3	4	5	6	7	8	9
$n+1$	1	2	3	4	5	6	7	8	9	0
$n(n+1)$	0	2	6	2	0	0	2	6	2	0

now $T=\frac{1}{2}n(n+1)$ so $2T=n(n+1)$ so the last digit of $2T$ is 0, 2 or 6

mod 10:

T	0	1		3		5	6		8	
$2T$	0	2		6		0	2		6	

If the last digit of $2T$ is 0, 2 or 6 then the last digit of T must be 0, 1, 3, 5, 6 or 8