Infinite series

Example 1

We say the infinite series $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\dots$ converges. What does this mean?

As we add up more and more terms ...

$$\frac{1}{2^1}=\frac{1}{2} \quad \frac{1}{2^1}+\frac{1}{2^2}=\frac{3}{4} \quad \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}=\frac{7}{8} \quad \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=\frac{15}{16} \quad \dots$$

the total gets bigger and bigger but never exceeds a certain number (in this case, 1)

Example 2

We say the infinite series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ diverges. What does this mean?

As we add up more and more terms ...

$$\frac{1}{1}=1 \quad \frac{1}{1}+\frac{1}{2}=\frac{3}{2} \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}=\frac{11}{6} \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12} \quad \dots$$

the total gets bigger and bigger and will eventually exceed any number.

Proof

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots=\left(\frac{1}{1}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots$$

The value of each bracket on the RHS is greater than $1/2$

So:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots>\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\dots$$

So the series will eventually exceed any number.

Here is another proof (by contradiction)

Assume the series converges to some finite number S

So: $S=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots$

So: $S>\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\dots$

So: $S>\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\dots$

So: $S>\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots$

So: $S>S$

Contradiction.

The series:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+...$$

is called the harmonic series. It diverges but very slowly. The sum of the first billion terms is only about 21.3

Example 3

Compare the series: $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots$ with the series: $\frac{1}{2^1}+\frac{1}{3^2}+\frac{1}{4^3}+\frac{1}{5^4}+...$

The terms in the second series are equal to or smaller than the corresponding terms in the first series. We know the first series converges so the second series must converge.

Example 4

Compare the series: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ with the series: $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+...$

The terms in the second series are equal to or larger than the corresponding terms in the first series. We know the first series diverges so the second series must diverge.

Example 5

Look at this series: $\sum_{1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...$

Show that: $\frac{1}{k^2}<\frac{2}{k(k+1)}$ for any $k>1$

Show that: $\frac{2}{k}-\frac{2}{k+1}=\frac{2}{k(k+1)}$

Show that: $\sum_{1}^{\infty}\frac{1}{k^2}<\sum_{1}^{\infty}\left(\frac{2}{k}-\frac{2}{k+1}\right)=\left(\frac{2}{1}-\frac{2}{2}\right)+\left(\frac{2}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\ldots$

Show that: $\sum_{1}^{\infty}\frac{1}{k^2}<2$

So the series converges.

If you study maths at a higher level you will learn how to manage infinite series properly.

In the meantime we can have some fun ...

Example 6

Evaluate: $S=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$

Now: $S=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots$

So: $S>0$

But: $S=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+...\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...\right)=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...\right)$

So: $S=0$

Example 7

Evaluate: $S=1-1+1-1+1-1+\ldots$

1st attempt: $S=(1-1)+(1-1)+(1-1)+\ldots$ $\quad S=0$

2nd attempt: $S=1-(1-1)-(1-1)-(1-1)-\ldots$ $\quad S=1$

3rd attempt: $S=1-(1-1+1-1+1-1+...)=1-S$ $\quad S=1/2$

Example 8

Evaluate: $S=1+2+4+8+...$

Now: $2S=2+4+8+16+\ldots$

So: $2S-S=(2+4+8+16+...)-(1+2+4+8+16+...)$

So: $S=-1$

Example 9

Evaluate: $S=1-2+4-8+\ldots$

1st attempt: $S=1+(-2+4)+(-8+16)+\ldots$ $\quad S\rightarrow\infty$

2nd attempt: $S=(1-2)+(4-8)+\ldots$ $\quad S\rightarrow-\infty$

3rd attempt: $S=1-2(1-2+4-8+...)=1-2S$ $\quad S=1/3$

Example 10

$U=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...$ and $V=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+...$

So: $U+V=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...=2V$

So: $U=V$

But: $U-V=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots$

So: $U>V$

Example 11

Now: $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots$ it’s a geometric series

So if we sub in $x=2$ we get: $1-2+4-8+16-...=\frac{1}{3}$

And if we sub in $x=-2$ we get: $1+2+4+8+16+...=-1$

Example 12

Now: $(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots$ it’s a geometric series

If we differentiate we get: $(1-x)^{-2}=1+2x+3x^2+4x^3+...$

So if we sub in $x=-1$ we get: $1-2+3-4+...=\frac{1}{4}$

Example 13

Now: $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots$ it's a geometric series

So if we sub in $x=1$ we get: $1-1+1-1+...=\frac{1}{2}$

Example 14

We know: $a=1-1+1-1+1-1+...=\frac{1}{2}$ see example 14

We know: $b=1-2+3-4+5-6+\ldots=\frac{1}{4}$ see example 13

Let: $c=1+2+3+4+5+6+\ldots$

So: $b-c=(1-2+3-4+5-6+...)-(1+2+3+4+5+6+...)$

So: $b-c=(1-1)+(-2-2)+(3-3)+(-4-4)+(5-5)+(-6-6)+...$

So: $b-c=0-4+0-8+0-12+...=-4-8-12-...=-4(1+2+3+...)=-4c$

So: $b=-3c$ but $b=\frac{1}{4}$ so $c=-\frac{1}{12}$

Hence: $1+2+3+4+5+6+\ldots=-\frac{1}{12}$

Example 15

Consider the series: $a_1+a_2+a_3+a_4+...$

$a_1=a_1$

$a_2=(a_1+a_2)-a_1$

$a_3=(a_1+a_2+a_3)-(a_1+a_2)$

$a_4=(a_1+a_2+a_3+a_4)-(a_1+a_2+a_3)$ etc

The terms on the LHS sum to:

$a_1+a_2+a_3+a_4+...$

The terms on the RHS all cancel.

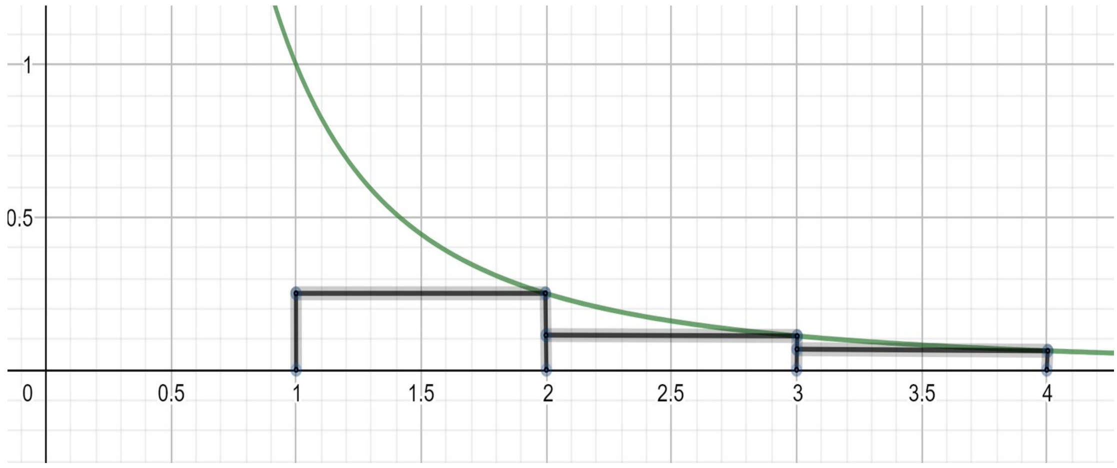
So $a_1+a_2+a_3+a_4+...=0$

Every infinite series sums to zero. We should have done this example first!

If you know about integration ...

Example

Here is the graph $y=\frac{1}{x^2}$



The diagram shows blocks between $x=1$ and $x=4$

The area of the blocks is: $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$

The area under the graph is: $\int_1^\infty \frac{1}{x^2}dx=1$

The area of the blocks is less than the area under the graph.

So: $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots<1$

So: $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$ converges.

Example

Here is the graph $y=\frac{1}{x}$

The area of the blocks is: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

The area under the graph is: $\int_1^\infty \frac{1}{x}\,dx \to \infty$

The area of the blocks is greater than the area under the graph.

So: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges.