Perfect Numbers

Example 1

The factors of 28 are:

1, 2, 4, 7, 14 Note: we have included 1 as a factor but not 28

The sum of the factors of 28 is:

$1+2+4+7+14=28$

So 28 equals the sum of its factors. So 28 is a perfect number.

Example 2

The factors of $(2^6 p)$ where p is an odd prime, are:

1	2	2^2	2^3	2^4	2^5	2^6
p	$2p$	$2^2 p$	$2^3 p$	$2^4 p$	$2^5 p$	

The sum of the factors of $(2^6 p)$ is:

$(1+2+2^2+2^3+2^4+2^5+2^6)+p(1+2+2^2+2^3+2^4+2^5)$

but:

$(1+2+2^2+2^3+2^4+2^5+2^6)=2^7-1$ a geometric series

and:

$(1+2+2^2+2^3+2^4+2^5)=2^6-1$ a geometric series

So the sum of the factors of $(2^6 p)$ is:

$(2^7-1)+p(2^6-1)$

If:

$p=2^7-1$

then:

the sum of the factors of $(2^6 p)$ is:

$(2^7-1)+(2^7-1)(2^6-1)=(2^7-1)(1+(2^6-1))=(2^7-1)2^6=2^6 p$

So $(2^6 p)$ is a perfect number.

Euclid's theorem:

If 2^k-1 is prime then $2^{k-1}(2^k-1)$ is an even perfect number.

for example:

2^5-1 is prime so $2^4(2^5-1)$ is an even perfect number.

Euler's theorem:

All even perfect numbers are of the form $2^{k-1}(2^k-1)$ where 2^k-1 is prime

This is much more difficult to prove.

Theorem

All even perfect numbers are triangle numbers

Proof

$2^{k-1}(2^k-1)=2^k 2^{-1}(2^k-1)=\frac{1}{2}(2^k-1)(2^k)$ which is of the form $\frac{1}{2}n(n+1)$

A conjecture about even perfect numbers:

there are an infinite number of even perfect numbers

Theorem

No odd perfect number is prime

Proof

If p is prime then its only factor is: 1

If p is perfect then $1=p$ and this can't happen

Theorem

No odd perfect number is a square

Proof

Factors come in pairs (except for 1)

The factors of 24 are:

1	2 and 12	3 and 8	4 and 6

So every integer has an odd number of factors. No!

The factors of 36 are:

1	2 and 18	3 and 12	4 and 96	6

36 has an even number of factors because 36 is a square

So all squares have an even number of factors and all other integers have an odd number of factors.

225 is an odd square

225 has an even number of factors:

1	3 and 75	5 and 45	9 and 25	15

All the factors of 225 are odd. So the sum of the factors of 225 is even.

So 225 is odd but the sum of its factors is even. So 225 cannot be perfect.

All this applies to any odd square. So no odd square is perfect.

A conjecture about odd perfect numbers:

odd perfect numbers do not exist

Amicable pairs

The factors of 220 are:

$1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110$

and:

$1+2+4+5+10+11+20+22+44+55+110=284$

The factors of 284 are:

$1, 2, 4, 71, 142$

and

$1+2+4+71+142=220$

We say 220 and 284 are an amicable pair.

Pythagoras(?) discovered the amicable pair: 220 and 284

Fermat discovered the amicable pair: 17296 and 18416

Descartes discovered the amicable pair: 9363584 and 9437056

Euler then discovered another sixty amicable pairs!

They all missed the pair, 1184 and 1210 which was not discovered until 1866 (by a 16 year old)