Euler’s Identity

In the chapter, Maclaurin Series we saw that:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

So

$$e^{(i\theta)} = 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots$$

$$e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 + \ldots$$

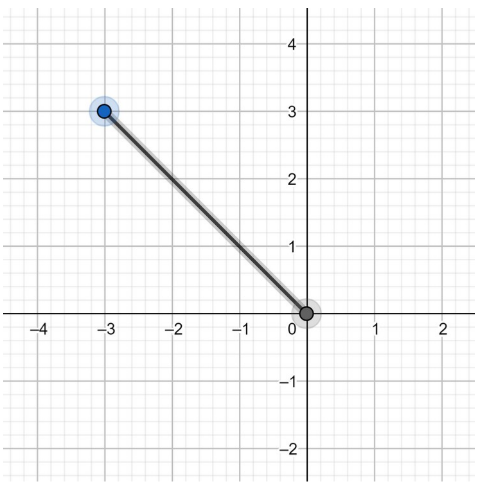
$$e^{(i\theta)} = \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \ldots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots\right)$$

$$e^{(i\theta)} = \cos\theta + i\sin\theta$$

This is Euler’s identity. It is simply astounding.

Mod-arg form

The point $(-3,3)$ represents the complex number $-3+3i$



The length of the line from the origin to $-3+3i$ is:

$r = \sqrt{(-3)^2 + (3)^2} = \sqrt{18}$ We say $mod(-3+3i) = \sqrt{18}$

The angle between the positive x axis and the line from the origin to $-3+3i$ is:

$\theta = \frac{3\pi}{4}$ We say $arg(-3+3i) = \frac{3\pi}{4}$

We know from trigonometry that:

$$cos\theta=\frac{-3}{\sqrt{18}} \quad \text{and} \quad sin\theta=\frac{3}{\sqrt{18}} \quad \text{so} \quad -3=\sqrt{18}\,cos\theta \quad \text{and} \quad 3=\sqrt{18}\,sin\theta$$

So:

$$-3+3i=\sqrt{18}\cos\frac{3\pi}{4}+i\sqrt{18}\sin\frac{3\pi}{4}=\sqrt{18}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{18}\,e^{i3\pi/4}$$

In general

We can write $x+iy$ in the form $r\,e^{i\theta}$ where $r=mod(x+iy)$ and $\theta=arg(x+iy)$

Where

$$r=\sqrt{x^2+y^2}$$

and

$$cos\theta=\frac{x}{r} \quad \text{and} \quad sin\theta=\frac{y}{r} \quad \text{so} \quad tan\theta=\frac{y}{x} \quad \text{so} \quad \theta=\tan^{-1}\left(\frac{y}{x}\right) \quad \text{We usually choose} \quad -\pi<\theta\le\pi$$

If we want to convert a complex number to mod-arg form then it is a good idea to mark the number on the number plane. Do this and show that:

$mod(1+i\sqrt{3})=2$ $\qquad arg(1+i\sqrt{3})=\frac{\pi}{3}$ $\qquad$ so $1+i\sqrt{3}=2\,e^{i\pi/3}$

$mod(4-4i)=\sqrt{32}$ $\qquad arg(4-4i)=-\frac{\pi}{4}$ $\qquad$ so $4-4i=\sqrt{32}e^{-i\pi/4}$

$mod(-1)=1$ $\qquad arg(-1)=\pi$ $\qquad$ so $-1=e^{i\pi}$

$mod(i)=1$ $\qquad arg(i)=\frac{\pi}{2}$ $\qquad$ so $i=e^{i\pi/2}$

$mod(-i)=1$ $\qquad arg(-i)=-\frac{\pi}{2}$ $\qquad$ so $-i=e^{-i\pi/2}$

There are advantages in writing complex numbers in mod-arg form, for example:

multiplication

$$(3\,e^{i\pi/4})(5\,e^{i\pi/2})=15\,e^{i3\pi/4}$$

division

$$\frac{12\,e^{i\pi}}{4\,e^{i\pi/3}}=3\,e^{i2\pi/3}$$

powers

$$(2\,e^{i\pi/7})^5=32\,e^{i5\pi/7}$$

Note:

If $w = re^{i\theta}$ and $z = te^{i\phi}$ then $wz = rte^{i(\theta+\phi)}$

So

$mod(wz) = mod(w) \times mod(z)$

And

$arg(wz) = arg(w) + arg(z)$

Note:

If $z = re^{i\theta}$ then:

a) $iz = (e^{i\pi/2}) re^{i\theta} = re^{i(\theta+\pi/2)}$

Multiplying z by i is the same as rotating Oz by $\frac{\pi}{2}$

b) $-z = (e^{i\pi}) re^{i\theta} = re^{i(\theta+\pi)}$

Multiplying z by -1 is the same as rotating Oz by π

c) $-iz = (e^{-i\pi/2}) re^{i\theta} = re^{i(\theta-\pi/2)}$

Multiplying z by $-i$ is the same as rotating Oz by $-\frac{\pi}{2}$

Let's fool around with Euler's identity $e^{i\theta} = cos\theta + isin\theta$

a) Put $\theta = \pi$ in Euler's identity

$e^{i\pi} = cos\pi + isin\pi = -1$ so $-1 = e^{i\pi}$

So:

$(-1)^i = (e^{i\pi})^i = e^{-\pi}$ which is real

And:

$\ln(-1) = \ln(e^{i\pi}) = i\pi$

And:

$e^{i\pi} + 1 = 0$ My five favourite numbers all in one neat formula.

b) Put $\theta = \pi/2$ in Euler's identity

$e^{i\pi/2} = \cos\pi/2 + isin\,\pi/2 = i$ so $i = e^{i\pi/2}$

So:

$(i)^i = e^{-\pi/2}$ which is real

And:

$\ln(i) = i\pi/2$

c) We have shown that $e^{i\theta} = cos\theta + isin\theta$ We can also show that $e^{-i\theta} = cos\theta - isin\theta$
Adding gives:

$2cos\theta = e^{i\theta} + e^{-i\theta}$ So $cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

Subtracting gives:

$2isin\theta = e^{i\theta} - e^{-i\theta}$ So $sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

So:

$\cos(i) = \frac{1}{2}\left(\frac{1}{e} + e\right)$ and $\sin(i) = \frac{1}{2i}\left(\frac{1}{e} - e\right)$

d) Solve $cos\theta = 2$

$\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = 2$ so $e^{i\theta} + e^{-i\theta} = 4$ so $e^{i2\theta} - 4e^{i\theta} + 1 = 0$ so $e^{i\theta} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

Now:

$e^{i\theta} = 2 + \sqrt{3}$ so $\theta = \frac{1}{i}\ln(2 + \sqrt{3})$

And:

$e^{i\theta} = 2 - \sqrt{3}$ so $\theta = \frac{1}{i}\ln(2 - \sqrt{3})$

Footnote:
We have been rather casual in our approach to complex numbers. For example, we know that $(e^a)(e^b) = e^{a+b}$ is true if a and b are real numbers and we have just assumed it is also true if a and b are complex numbers. Which of the rules that apply to real numbers still apply to complex numbers? We need to be careful about this or we can run into problems ...

a) $1 = \sqrt{1} = \sqrt{-1 \times -1} = \sqrt{-1} \times \sqrt{-1} = i^2$ so $1 = -1$

b) $e^{i2\pi} = 1$ and $e^{i4\pi} = 1$ so $e^{i2\pi} = e^{i4\pi}$ so $\ln\left(e^{i2\pi}\right) = \ln\left(e^{i4\pi}\right)$ so $i2\pi = i4\pi$ so $2\pi = 4\pi$

c) $(-1)^2 = 1$ so $\ln((-1)^2) = \ln(1)$ so $2\ln(-1) = 0$ so $\ln(-1) = 0$ so $-1 = e^0$

EXERCISE

1)

Write in mod-arg form:

a) $\sqrt{3}+i$ b) $-1+i$ c) $-2-2i$ d) $1-\sqrt{3}i$

2)

If $w=2e^{i\pi/4}$ and $z=3e^{i2\pi/3}$ write down:

Write in mod-arg form:

a) wz b) $\frac{w}{z}$ c) w^7 d) iw e) $-z$ f) $-iw$

3)

Convert $(1+i)$ to mod-arg form and hence find $(1+i)^{16}$

SOLUTIONS

1)

Mark these numbers on a number plane

a) $\sqrt{3}+i=2e^{i\pi/6}$ b) $-1+i=\sqrt{2}e^{i3\pi/4}$ c) $-2-2i=\sqrt{8}e^{(-3\pi/4)}$ d) $1-\sqrt{3}i=2e^{-i\pi/3}$

2)

a) $6e^{i11\pi/12}$ b) $\frac{2}{3}e^{-i5\pi/12}$ c) $128e^{i7\pi/4}$ which we can write as $128e^{-i\pi/4}$

d) $2e^{i3\pi/4}$ e) $3e^{i5\pi/3}$ which we can write as $3e^{-i\pi/3}$ f) $2e^{-i\pi/4}$

3)

$1+i=\sqrt{2}e^{i\pi/4}$ so $(1+i)^{16}=(\sqrt{2})^{16}(e^{i\pi/4})^{16}=256e^{i4\pi}=256$