Complex Numbers

Example

We can try to solve the quadratic equation

$$x^2-4x+13=0$$

using the quadratic formula

$$x=\frac{4\pm\sqrt{16-52}}{2}=\frac{4\pm\sqrt{-36}}{2}$$

At this point we give up because $\sqrt{-36}$ does not exist.

Let's introduce a new number i where $i^2=-1$

Now

$$(6i)^2=6i\times 6i=36i^2=-36$$

So we can now solve our equation

$$x=\frac{4\pm 6i}{2}=2\pm 3i$$

Check:

If

$$x=2+3i$$

then

$$x^2=(2+3i)(2+3i)=4+6i+6i+9i^2=4+12i-9=-5+12i$$

So

$$x^2-4x+13=(-5+12i)-4(2+3i)+13=-5+12i-8-12i+13=0$$ Good.

And if

$x=2-3i$ etc

A number like $2+3i$ is called a complex number. We can add, subtract, multiply and divide complex numbers:

Addition

$$(3+7i)+(2-5i)=5+2i$$

Subtraction

$$(3+7i)-(2-5i)=1+12i$$

Multiplication

$$(3+7i)(2-5i)=6-15i+14i-35i^2=6-15i+14i+35=41-i$$

Division

Here we need a trick

$$\frac{(3+7i)}{(2-5i)}=\frac{(3+7i)(2+5i)}{(2-5i)(2+5i)}=...=\frac{-29+29i}{29}=-1+i$$

Squaring

$$(3+7i)^2=9+42i+49i^2=9+42i-49=-40+42i$$

Powers of i

$$i^3=(i^2)i=(-1)i=-i$$

$$i^4=(i^2)(i^2)=(-1)(-1)=1$$

$$i^5=(i^2)(i^2)i=(-1)(-1)i=i$$

$$i^{379}=...=i^3=-i$$

Quadratic equations

$$x^2-4x+29=0$$

$$x=\frac{4\pm\sqrt{-100}}{2}=2\pm5i$$

Real and imaginary parts.

We say 2 is the real part of $2+3i$ and we say 3 is the imaginary part of $2+3i$

Example

$2x+y+3i-4iy=10-5i$ where x and y are real numbers

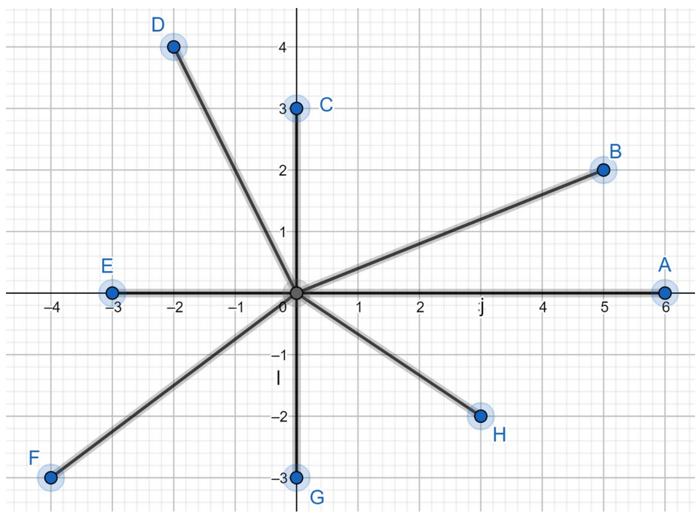
We can rewrite this as:

$$(2x+y)+i(3-4y)=(10)+i(-5)$$

If two complex numbers are equal then their real parts must be equal and their imaginary parts must be equal.

So $2x+y=10$ and $3-4y=-5$ so $x=4$ and $y=2$

We can think of a complex number as a point on a number plane:



A is the complex number: 6 B is the complex number: $5+2i$

C is the complex number: $3i$ D is the complex number: $-2+4i$

E is the complex number: -3 F is the complex number: $-4-3i$

G is the complex number: $-3i$ H is the complex number: $3-2i$

If $z=4+2i$ then:

a) $iz=i(4+2i)=-2+4i$

Draw a line from the origin O to z Draw a line from the origin O to iz

Multiplying z by i is the same as rotating Oz by $\frac{\pi}{2}$

b) $-z=-(4+2i)=-4-2i$

Multiplying z by -1 is the same as rotating Oz by π

c) $-iz=-i(4+2i)=2-4i$

Multiplying z by $-i$ is the same as rotating Oz by $-\frac{\pi}{2}$

Fundamental Theorem of Algebra

Without complex numbers, some polynomials can be factorised:

$x^2-5x+6=(x-2)(x-3)$

but other polynomials cannot be factorised:

$x^2-4x+13$

However, with complex numbers we have a nice result:

Every polynomial of degree n can be factorised into n brackets.

For example $x^4-4x^3+3x^2+2x-6=(x+1)(x-3)(x-1+i)(x-1-i)$

Footnote:

Did mathematicians invent complex numbers or did they invent them?

EXERCISE

1)

Evaluate

a) $(3+5i)+(-2+7i)$

b) $(5-3i)-(8+4i)$

c) $(1+3i)(5-2i)$

d) $\frac{(8+5i)}{(7+2i)}$ hint multiply top and bottom by $(7-2i)$

e) $(2-5i)^2$

2)

Solve

a) $x^2-6x+13=0$

b) $x^2-14x+58=0$

3)

Solve $3x+iy-6+2i=2ix+3y+8i$ where x and y are real numbers

SOLUTIONS

1)

a) $1+12i$

b) $-3-7i$

c) $11+13i$

d) $\frac{66}{53}+\frac{19}{53}i$

e) $-21-20i$

2)

a) $x=\frac{2\pm\sqrt{4-52}}{2}=\frac{2\pm\sqrt{-48}}{2}=1\pm12i$

b) $x=\frac{14\pm\sqrt{196-232}}{2}=\frac{14\pm\sqrt{-36}}{2}=7\pm3i$

3)

equating real parts:

$3x-6=3y$

equating imaginary parts:

$y+2=2x+8$

$x=-8 \quad y=-10$

Euler’s Identity

In the chapter, Maclaurin Series we saw that:

$$cosx=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\ldots$$

$$sinx=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5$$

$$e^x=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots$$

So

$$e^{(i\theta)}=1+(i\theta)+\frac{1}{2!}(i\theta)^2+\frac{1}{3!}(i\theta)^3+\frac{1}{4!}(i\theta)^4+\frac{1}{5!}(i\theta)^5+\ldots$$

$$e^{i\theta}=1+i\theta-\frac{1}{2!}\theta^2-\frac{1}{3!}i\theta^3+\frac{1}{4!}\theta^4+\frac{1}{5!}i\theta^5+\ldots$$

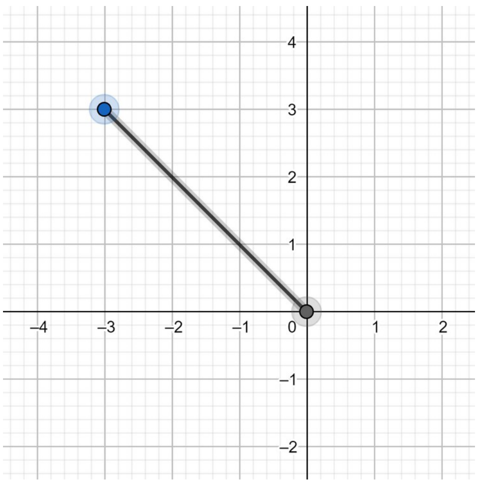
$$e^{(i\theta)}=\left(1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4+\ldots\right)+i\left(\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5+\ldots\right)$$

$$e^{(i\theta)}=cos\theta+isin\theta$$

This is Euler’s identity. It is simply astounding.

Mod-arg form

The point $(-3,3)$ represents the complex number $-3+3i$



The length of the line from the origin to $-3+3i$ is:

$r=\sqrt{(-3)^2+(3)^2}=\sqrt{18}$ We say $mod(-3+3i)=\sqrt{18}$

The angle between the positive x axis and the line from the origin to $-3+3i$ is:

$\theta=\frac{3\pi}{4}$ We say $arg(-3+3i)=\frac{3\pi}{4}$

We know from trigonometry that:

$$cos\theta=\frac{-3}{\sqrt{18}} \quad \text{and} \quad sin\theta=\frac{3}{\sqrt{18}} \quad \text{so} \quad -3=\sqrt{18}\,cos\theta \quad \text{and} \quad 3=\sqrt{18}\,sin\theta$$

So:

$$-3+3i=\sqrt{18}\cos\frac{3\pi}{4}+i\sqrt{18}\sin\frac{3\pi}{4}=\sqrt{18}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{18}\,e^{i3\pi/4}$$

In general

We can write $x+iy$ in the form $r\,e^{i\theta}$ where $r=mod(x+iy)$ and $\theta=arg(x+iy)$

Where

$$r=\sqrt{x^2+y^2}$$

and

$$cos\theta=\frac{x}{r} \quad \text{and} \quad sin\theta=\frac{y}{r} \quad \text{so} \quad tan\theta=\frac{y}{x} \quad \text{so} \quad \theta=\tan^{-1}\left(\frac{y}{x}\right) \quad \text{We usually choose} \quad -\pi<\theta\leq\pi$$

If we want to convert a complex number to mod-arg form then it is a good idea to mark the number on the number plane. Do this and show that:

$$mod(1+i\sqrt{3})=2 \qquad arg(1+i\sqrt{3})=\frac{\pi}{3} \qquad \text{so} \quad 1+i\sqrt{3}=2\,e^{i\pi/3}$$

$$mod(4-4i)=\sqrt{32} \qquad arg(4-4i)=-\frac{\pi}{4} \qquad \text{so} \quad 4-4i=\sqrt{32}e^{-i\pi/4}$$

$$mod(-1)=1 \qquad arg(-1)=\pi \qquad \text{so} \quad -1=e^{i\pi}$$

$$mod(i)=1 \qquad arg(i)=\frac{\pi}{2} \qquad \text{so} \quad i=e^{i\pi/2}$$

$$mod(-i)=1 \qquad arg(-i)=-\frac{\pi}{2} \qquad \text{so} \quad -i=e^{-i\pi/2}$$

There are advantages in writing complex numbers in mod-arg form, for example:

multiplication

$$(3\,e^{i\pi/4})(5\,e^{i\pi/2})=15\,e^{i3\pi/4}$$

division

$$\frac{12\,e^{i\pi}}{4\,e^{i\pi/3}}=3\,e^{i2\pi/3}$$

powers

$$(2\,e^{i\pi/7})^5=32\,e^{i5\pi/7}$$

Note:

If $w=re^{i\theta}$ and $z=te^{i\phi}$ then $wz=rte^{i(\theta+\phi)}$

So

$$mod(wz)=mod(w)\times mod(z)$$

And

$$arg(wz)=arg(w)+arg(z)$$

Note:

If $z=re^{i\theta}$ then:

a) $iz=(e^{i\pi/2})re^{i\theta}=re^{i(\theta+\pi/2)}$

Multiplying z by i is the same as rotating Oz by $\frac{\pi}{2}$

b) $-z=(e^{i\pi})re^{i\theta}=re^{i(\theta+\pi)}$

Multiplying z by -1 is the same as rotating Oz by π

c) $-iz=(e^{-i\pi/2})re^{i\theta}=re^{i(\theta-\pi/2)}$

Multiplying z by $-i$ is the same as rotating Oz by $-\frac{\pi}{2}$

Let's fool around with Euler's identity $e^{i\theta}=cos\theta+isin\theta$

a) Put $\theta=\pi$ in Euler's identity

$e^{i\pi}=cos\pi+isin\pi=-1$ so $-1=e^{i\pi}$

So:

$(-1)^i=(e^{i\pi})^i=e^{-\pi}$ which is real

And:

$$\ln(-1)=\ln(e^{i\pi})=i\pi$$

And:

$e^{i\pi}+1=0$ My five favourite numbers all in one neat formula.

b) Put $\theta=\pi/2$ in Euler's identity

$e^{i\pi/2}=\cos\pi/2+isin\pi/2=i$ so $i=e^{i\pi/2}$

So:

$(i)^i=e^{-\pi/2}$ which is real

And:

$\ln(i)=i\pi/2$

c) We have shown that $e^{i\theta}=cos\theta+isin\theta$ We can also show that $e^{-i\theta}=cos\theta-isin\theta$

Adding gives:

$2cos\theta=e^{i\theta}+e^{-i\theta}$ So $cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$

Subtracting gives:

$2isin\theta=e^{i\theta}-e^{-i\theta}$ So $sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})$

So:

$\cos(i)=\frac{1}{2}\left(\frac{1}{e}+e\right)$ and $\sin(i)=\frac{1}{2i}\left(\frac{1}{e}-e\right)$

d) Solve $cos\theta=2$

$\frac{1}{2}(e^{i\theta}+e^{-i\theta})=2$ so $e^{i\theta}+e^{-i\theta}=4$ so $e^{i2\theta}-4e^{i\theta}+1=0$ so $e^{i\theta}=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$

Now:

$e^{i\theta}=2+\sqrt{3}$ so $\theta=\frac{1}{i}\ln(2+\sqrt{3})$

And:

$e^{i\theta}=2-\sqrt{3}$ so $\theta=\frac{1}{i}\ln(2-\sqrt{3})$

Footnote:

We have been rather casual in our approach to complex numbers. For example, we know that $(e^a)(e^b)=e^{a+b}$ is true if a and b are real numbers and we have just assumed it is also true if a and b are complex numbers. Which of the rules that apply to real numbers still apply to complex numbers? We need to be careful about this or we can run into problems ...

a) $1=\sqrt{1}=\sqrt{-1\times-1}=\sqrt{-1}\times\sqrt{-1}=i^2$ so $1=-1$

b) $e^{i2\pi}=1$ and $e^{i4\pi}=1$ so $e^{i2\pi}=e^{i4\pi}$ so $\ln\left(e^{i2\pi}\right)=\ln\left(e^{i4\pi}\right)$ so $i2\pi=i4\pi$ so $2\pi=4\pi$

c) $(-1)^2=1$ so $\ln((-1)^2)=\ln(1)$ so $2\ln(-1)=0$ so $\ln(-1)=0$ so $-1=e^0$

EXERCISE

1)

Write in mod-arg form:

a) $\sqrt{3}+i$ b) $-1+i$ c) $-2-2i$ d) $1-\sqrt{3}i$

2)

If $w=2e^{i\pi/4}$ and $z=3e^{i2\pi/3}$ write down:

Write in mod-arg form:

a) wz b) $\frac{w}{z}$ c) w^7 d) iw e) $-z$ f) $-iw$

3)

Convert $(1+i)$ to mod-arg form and hence find $(1+i)^{16}$

SOLUTIONS

1)

Mark these numbers on a number plane

a) $\sqrt{3}+i=2e^{i\pi/6}$ b) $-1+i=\sqrt{2}e^{i3\pi/4}$ c) $-2-2i=\sqrt{8}e^{(-3\pi/4)}$ d) $1-\sqrt{3}i=2e^{-i\pi/3}$

2)

a) $6e^{i11\pi/12}$ b) $\frac{2}{3}e^{-i5\pi/12}$ c) $128e^{i7\pi/4}$ which we can write as $128e^{-i\pi/4}$

d) $2e^{i3\pi/4}$ e) $3e^{i5\pi/3}$ which we can write as $3e^{-i\pi/3}$ f) $2e^{-i\pi/4}$

3)

$1+i=\sqrt{2}e^{i\pi/4}$ so $(1+i)^{16}=(\sqrt{2})^{16}(e^{i\pi/4})^{16}=256e^{i4\pi}=256$

Using Complex Numbers

Some real problems can be solved using complex numbers. Here are some examples.

1. Deriving trig identities

a) Pythagoras identity:

We know:

$$\cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \quad \text{and} \quad \sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

Show that:

$$\cos^2\theta+\sin^2\theta=1$$

b) Addition identity:

$$(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)=e^{i\theta}e^{i\phi}=e^{i(\theta+\phi)}=\cos(\theta+\phi)+i\sin(\theta+\phi)$$

Equating real parts:

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$

Equating imaginary parts:

$$\sin(\theta+\phi)=\cos\theta\sin\phi+\sin\theta\cos\phi$$

c) Double angle identity:

$$(\cos\theta+i\sin\theta)^2=\left(e^{i\theta}\right)^2=e^{i2\theta}=\cos(2\theta)+i\sin(2\theta)$$

Equating real parts:

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

Equating imaginary parts:

$$\sin 2\theta=2\cos\theta\sin\theta$$

Note:

de Moivre’s theorem:

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

d) Half angle identity:

We know:

$$2\cos\theta=e^{i\theta}+e^{-i\theta}$$

So:

$$(2\cos\theta)^2=\left(e^{i\theta}+e^{-i\theta}\right)^2$$

Multiply out the brackets:

$$2^2\cos^2\theta = e^{i2\theta}+2+e^{-i2\theta}$$

Collect up terms:

$$2^2\cos^2\theta = \left(e^{i2\theta}+e^{-i2\theta}\right)+2$$

Write with cosines:

$$2^2\cos^2\theta = 2\cos 2\theta + 2$$

So:

$$\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$$

e) Factor identity:

We know:

$2\,cos\theta = e^{i\theta}+e^{-i\theta}$ and $2\,isin\theta = e^{i\theta}-e^{-i\theta}$

So:

$$sin\theta cos\phi = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\frac{1}{2}\left(e^{i\phi}+e^{-i\phi}\right)$$

Show that:

$$sin\theta cos\phi = \frac{1}{2}\sin(\theta+\phi)+\frac{1}{2}\sin(\theta-\phi)$$

We could write:

$sin\alpha + sin\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$ can you see how?

I could go on …

2. Integration

Example

We want to work out:

$\int e^{-x} cosx\,dx$ and $\int e^{-x} cosx\,dx$

Here we go:

$$\int e^{-x}(cosx + isinx)\,dx = \int e^{-x}e^{ix}dx = \int e^{(-1+i)x}dx = \frac{1}{(-1+i)}e^{(-1+i)x}+c = \frac{1}{(-1+i)}e^{-x}e^{ix}+c$$

But

$$\frac{1}{(-1+i)} = \frac{-1-i}{(-1+i)(-1-i)} = -\frac{1}{2}(1+i)$$

So

$$\int e^{-x}(cosx+isinx)\,dx=-\frac{1}{2}(1+i)\,e^{-x}(cosx+isinx)+c$$

Equating real parts:

$$\int e^{-x}cosx\,dx=-\frac{1}{2}e^{-x}(cosx-sinx)+c' \quad \text{where} \quad c' \quad \text{is the real part of} \quad c$$

Equating imaginary parts:

$$\int e^{-x}sinx\,dx=-\frac{1}{2}e^{x}(cosx+sinx)+c'' \quad \text{where} \quad c'' \quad \text{is the imaginary part of} \quad c$$

3. A formula for $\ln\sqrt{2}$ and π

a) $1+i=\sqrt{2}e^{i\pi/4}$

So:

$$\ln(1+i)=\ln(\sqrt{2})+\ln(e^{i\pi/4})=\ln(\sqrt{2})+\frac{i\pi}{4}$$

b) $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots$ this is the Maclaurin series

So:

$$\ln(1+i)=i-\frac{1}{2}i^2+\frac{1}{3}i^3-\frac{1}{4}i^4+\ldots=i+\frac{1}{2}-\frac{1}{3}i-\frac{1}{4}+\ldots$$

c) From (a) and (b) we have:

$$\ln(\sqrt{2})+\frac{i\pi}{4}=i+\frac{1}{2}-\frac{1}{3}i-\frac{1}{4}+\ldots$$

Equating real parts:

$$\ln(\sqrt{2})=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\ldots$$

Equating imaginary parts:

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

4. Van Aubel’s Theorem if you know about vectors ...

What do you make of the following proof, where we have recklessly mixed up complex numbers and vectors?

If $\boldsymbol{v}$ is a vector then $i\boldsymbol{v}$ is the vector you get by rotating $\boldsymbol{v}$ anti-clockwise by $90°$

Draw some diagrams and convince yourself that:

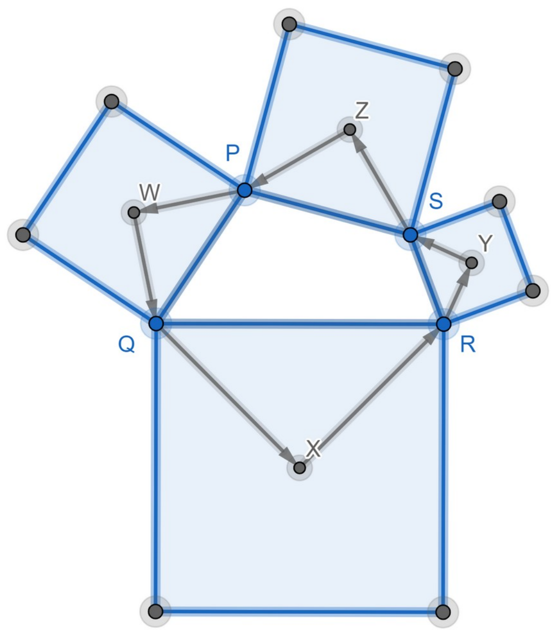
$i\boldsymbol{w}+i\boldsymbol{v}=i(\boldsymbol{w}+\boldsymbol{v})$ and $i^2\boldsymbol{w}=-\boldsymbol{w}$ and if $\boldsymbol{w}+i\boldsymbol{w}=\boldsymbol{0}$ then $\boldsymbol{w}=\boldsymbol{0}$

Given any quadrilateral PQRS, draw a square on each side.

W, X, Y and Z are the centres of these squares.

Theorem

ZX and YW will have the same length and are at right-angles.



Proof

Let $\boldsymbol{a}=\overrightarrow{PW}$ so $i\boldsymbol{a}=\overrightarrow{WQ}$ Let $\boldsymbol{b}=\overrightarrow{QX}$ so $i\boldsymbol{b}=\overrightarrow{XR}$

Let $\boldsymbol{c}=\overrightarrow{RY}$ so $i\boldsymbol{c}=\overrightarrow{YS}$ Let $\boldsymbol{d}=\overrightarrow{SZ}$ so $i\boldsymbol{d}=\overrightarrow{ZP}$

From the diagram:

$$\boldsymbol{a}+i\boldsymbol{a}+\boldsymbol{b}+i\boldsymbol{b}+\boldsymbol{c}+i\boldsymbol{c}+\boldsymbol{d}+i\boldsymbol{d}=\boldsymbol{0}$$

So:

$$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})+i(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})=\boldsymbol{0}$$

So:

$$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})=\boldsymbol{0}$$

Now:

$$\overrightarrow{YW}=i\boldsymbol{c}+\boldsymbol{d}+i\boldsymbol{d}+\boldsymbol{a}$$

So:

$$i\overrightarrow{YW}=i^2\boldsymbol{c}+i\boldsymbol{d}+i^2\boldsymbol{d}+i\boldsymbol{a}=-\boldsymbol{c}+i\boldsymbol{d}-\boldsymbol{d}+i\boldsymbol{a}$$

Now:

$$\overrightarrow{ZX}=i\boldsymbol{d}+\boldsymbol{a}+i\boldsymbol{a}+\boldsymbol{b} \qquad \text{But} \quad \boldsymbol{b}=-(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})$$

So:

$$\overrightarrow{ZX}=i\boldsymbol{d}+\boldsymbol{a}+i\boldsymbol{a}-(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})=-\boldsymbol{c}+i\boldsymbol{d}-\boldsymbol{d}+i\boldsymbol{a}$$

So:

$\overrightarrow{ZX}=i\,\overrightarrow{YW}$ as required.

EXERCISE

1)

Derive trig identities for:

$\cos(\theta-\phi)$ and $\sin(\theta-\phi)$

2)

Derive trig identities for:

$\cos 3\theta$ and $\sin 3\theta$

3)

Derive the half angle formula for $\sin^2\theta$

4)

Use the method of question (3) to write $\cos^5\theta$ in terms of $\cos 5\theta$ and $\cos 3\theta$ and $cos\theta$

5)

Show that:

$$cos\theta cos\phi=\frac{1}{2}\cos(\theta+\phi)+\frac{1}{2}\cos(\theta-\phi)$$

or if you prefer:

$$cos\alpha+cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

6)

We want to evaluate:

$$S=\frac{1}{2}cos\theta+\frac{1}{4}\cos 2\theta+\frac{1}{8}\cos 3\theta+\ldots$$

Now:

$cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ so $\cos 2\theta=\frac{1}{2}(e^{i2\theta}+e^{-i2\theta})$ and $\cos 3\theta=\frac{1}{2}(e^{i3\theta}+e^{-i3\theta})$ etc

Show that:

$$S=\left(\frac{1}{4}(e^{i\theta})+\frac{1}{8}(e^{i2\theta})+\frac{1}{16}(e^{i3\theta})+\ldots\right)+\left(\frac{1}{4}(e^{-i\theta})+\frac{1}{8}(e^{-i2\theta})+\frac{1}{16}(e^{-i3\theta})+\ldots\right)$$

Show that:

$$\frac{1}{4}(e^{i\theta})+\frac{1}{8}(e^{i2\theta})+\frac{1}{16}(e^{i3\theta})+\ldots=\frac{\frac{1}{4}(e^{i\theta})}{1-\frac{1}{2}(e^{i\theta})}$$ hint, geometric series

Show that:

$$\frac{1}{4}(e^{-i\theta})+\frac{1}{8}(e^{-i2\theta})+\frac{1}{16}(e^{-i3\theta})+...=\frac{\frac{1}{4}(e^{-i\theta})}{1-\frac{1}{2}(e^{-i\theta})}$$ hint, geometric series

Show that:

$$S=\frac{\frac{1}{4}(e^{i\theta})}{1-\frac{1}{2}(e^{i\theta})}+\frac{\frac{1}{4}(e^{-i\theta})}{1-\frac{1}{2}(e^{-i\theta})}$$

Show that:

$$S=\frac{2\cos\theta-1}{5-4\cos\theta}$$

7)

Another formula for π

a) $\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+...$ this is the Maclaurin series

b)Show that:

$$(2+i)(3+i)=5+5i$$

So:

$$arg(2+i)+arg(3+i)=arg(5+5i)$$

So:

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left(\frac{5}{5}\right)$$ but $\tan^{-1}\left(\frac{5}{5}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

So:

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

c) Write down the Maclaurin series for:

$\tan^{-1}\left(\frac{1}{2}\right)$ and $\tan^{-1}\left(\frac{1}{3}\right)$

Show that:

$$\frac{\pi}{4}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)+...$$

8)

We can make up more formulas for π using the method of question (7)

We need to find a,b,c where $(a+i)(b+i)=c+ci$

Show that:

$$ab-1=a+b$$

Show that:

$$b=\frac{a+1}{a-1}$$

Now let:

$a=\frac{p}{q}$ where p and q are integers and $p>q>0$

Show that:

$$b=\frac{p+q}{p-q}$$

So:

$$\left(\frac{p}{q}+i\right)\left(\frac{p+q}{p-q}+i\right)=c+ci$$

Show that:

$$(p+iq)((p+q)+i(p-q))=(p^2+q^2)+i(p^2+q^2)$$

So:

$$arg(p+iq)+arg((p+q)+i(p-q))=(p^2+q^2)+i(p^2+q^2)$$

So:

$$\tan^{-1}\left(\frac{q}{p}\right)+\tan^{-1}\left(\frac{p-q}{p+q}\right)=\frac{\pi}{4}$$

Write down the Maclaurin series for:

$\tan^{-1}\left(\frac{q}{p}\right)$ and $\tan^{-1}\left(\frac{p-q}{p+q}\right)$ to get a formula for π

I chose $p=17$ and $q=4$

So:

$$\tan^{-1}\left(\frac{4}{17}\right)+\tan^{-1}\left(\frac{13}{21}\right)=\frac{\pi}{4}$$

and then I got:

$$\frac{\pi}{4}=\left(\frac{4}{17}+\frac{13}{21}\right)-\frac{1}{3}\left(\frac{4^3}{17^3}+\frac{13^3}{21^3}\right)+\frac{1}{5}\left(\frac{4^5}{17^5}+\frac{13^5}{21^5}\right)+\ldots$$

SOLUTIONS

1)

$$(cos\theta+isin\theta)(cos\phi-isin\phi)=e^{i\theta}e^{-i\phi}=e^{i(\theta-\phi)}=\cos(\theta-\phi)+isin(\theta-\phi)$$

Equating real parts:

$$cos\theta cos\phi+sin\theta sin\phi=\cos(\theta-\phi)$$

Equating imaginary parts:

$$-cos\theta sin\phi+sin\theta cos\phi=\sin(\theta-\phi)$$

2)

$$(cos\theta+isin\theta)^3=(e^{i\theta})^3=e^{i3\theta}=\cos(3\theta)+isin(3\theta)$$

Equating real parts:

$$\cos^3\theta-3cos\theta\sin^2\theta=\cos 3\theta$$

We could replace $\sin^2\theta$ by $1-\cos^2\theta$ and write $\cos 3\theta=4\cos^3\theta-3cos\theta$

Equating imaginary parts:

$$3\cos^2\theta sin\theta-\sin^3\theta=\sin 3\theta$$

We could replace $\cos^2\theta$ by $1-\sin^2\theta$ and write $\sin 3\theta=3sin\theta-4\sin^3\theta$

Dividing:

$$\frac{3\cos^2\theta sin\theta-\sin^3\theta}{\cos^3\theta-3cos\theta\sin^2\theta}=\frac{\sin 3\theta}{\cos 3\theta}$$

Divide top and bottom of the left-hand-side by $\cos^3\theta$

$$\frac{3tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\tan 3\theta$$

3)

$$(2isin\theta)^2=(e^{i\theta}-e^{-i\theta})^2$$

Multiply out the brackets:

$$2^2i^2\sin^4\theta=e^{i2\theta}-2+e^{-i2\theta}$$

Collect up terms:

$$-2^2\sin^2\theta=(e^{i2\theta}+e^{-i2\theta})-2$$

Write with cosines:

$$-2^2\sin^2\theta=2\cos 2\theta-2$$

So:

$$\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$$

4)

$$(2cos\theta)^5=(e^{i\theta}+e^{-i\theta})^5$$

Multiply out the brackets:

$$2^5\cos^5\theta=e^{i5\theta}+5e^{i3\theta}+10e^{i\theta}+10e^{-i\theta}+5e^{-i3\theta}+e^{-i5\theta}$$

Collect up terms:

$$2^5\cos^5\theta=(e^{i5\theta}+e^{-i5\theta})+5(e^{i3\theta}+e^{-i3\theta})+10(e^{i\theta}+e^{-i\theta})$$

Write with cosines:

$$2^5\cos^5\theta=2\cos 5\theta+10\cos 3\theta+20cos\theta$$

So:

$$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}cos\theta$$

5)

$$cos\theta cos\phi = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$$

So:

$$cos\theta cos\phi = \frac{1}{4}\left(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{i(-\theta-\phi)}\right)$$

So:

$$cos\theta cos\phi = \frac{1}{4}\left(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)}\right) + \frac{1}{4}\left(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}\right)$$

So:

$$cos\theta cos\phi = \frac{1}{2}\cos(\theta+\phi) + \frac{1}{2}\cos(\theta-\phi)$$

Julia Sets WE NEED DIAGRAMS

Example 1

Choose a complex number z_0 and look at the sequence:

$z_0 \quad z_1 \quad z_2 \quad z_3$ … where $z_{n+1}=z_n^2$

If we choose:

$z_0=4\,e^{i\pi/5}$

We get:

$4\,e^{i\pi/5} \qquad 16\,e^{i2\pi/5} \qquad 256\,e^{i4\pi/5} \qquad 65536\,e^{i8\pi/5} \quad \ldots$

If we choose:

$z_0=\frac{1}{3}e^{i\pi/5}$

We get:

$\frac{1}{3}e^{i\pi/5} \qquad \frac{1}{9}e^{i2\pi/5} \qquad \frac{1}{81}e^{i4\pi/5} \qquad \frac{1}{6561}e^{i8\pi/5} \quad \ldots$

If $mod(z_0)>1$ then $mod(z_n)$ tends to infinity as n tends to infinity.

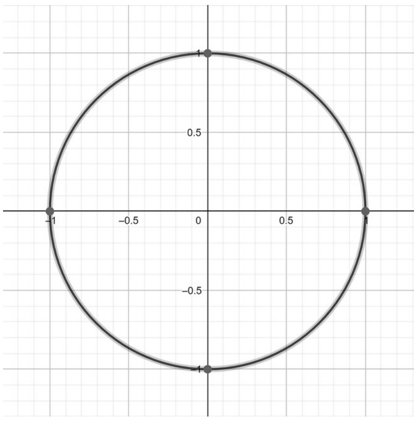
If $mod(z_0)<1$ then $mod(z_n)$ does not tend to infinity as n tends to infinity.

We could colour the number plane.

z_0 is in the red region if $mod(z_0)>1$ and z_0 is in the blue region if $mod(z_0)<1$

The Julia set is the boundary between the red region and the blue region.

A circle.



So far, so boring ...

Example 2

Choose a complex number z_0 and look at the sequence:

$z_0 \quad z_1 \quad z_2 \quad z_3$ … where $z_{n+1} \to z_n^2 - 0.5 + 0.3i$

For some values of z_0 we find $mod(z_n)$ tends to infinity as n tends to infinity.

For other vales of z_0 we find $mod(z_n)$ does not tend to infinity as n tends to infinity.

We could colour the number plane.

z_0 is in the red region if $mod(z_n)$ tends to infinity as n tends to infinity.

z_0 is in the blue region if $mod(z_n)$ does not tend to infinity as n tends to infinity.

The Julia set is the boundary between the red region and the blue region.

A slightly deformed circle.

WE NEED A DIAGRAM

So far, so slightly interesting ...

In general:

Choose a complex number z_0 and look at the sequence:

$z_0 \quad z_1 \quad z_2 \quad z_3$ … where $z_{n+1} \to z_n^2 + c$

Look at the Julia set for different values of c The results are truly amazing.

WE NEED DIAGRAMS