Propositions

A proposition is a statement that is either true or false.

Newton was born in England this is a true proposition

Fermat was born in Poland this is a false proposition

Please shut the door this is not a proposition

Example

a : Today is Monday b : I go to work today

We can combine two propositions using AND

$a \wedge b$: Today is Monday and I go to work today

Today is Monday and I go to work today is true if:

Today is Monday is true

and I go to work today is true

Otherwise it is false.

So $a \wedge b$ is true if a is true and b is true, otherwise it is false.

We can set this out in a truth table where 0 means false and 1 means true:

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

We can combine two propositions using OR

$a \vee b$: Today is Monday or I go to work today (or both)

Today is Monday or I go to work today is true if:

Today is Monday is true

or I go to work today is true

or both are true

Otherwise it is false.

So $a \vee b$ is true if a is true or b is true (or both), otherwise it is false.

Truth table:

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

Notes:

In ordinary English we use OR in two different ways.

I will buy you a beer or a lemonade (but not both)

You will get the job if you can sing or dance (or both)

$a \vee b$ always means a or b or both

Negation

The negation of a is a'

a : Today is Monday a' : Today is not Monday

Truth table:

a	a'
0	1
1	0

Examples

Fill in the truth table for $(a \vee b)'$

a	b	$a \vee b$	$(a \vee b)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

Fill in the truth table for $a \vee (b \wedge c)$

a	b	c	$b \wedge c$	$a \vee (b \wedge c)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

see EXERCISE 1

Look at Exercise 1, questions (3) and (4)

You should have found that the columns for $(a \wedge b) \vee (a \wedge c)$ and $a \wedge (b \vee c)$ are the same.

We say $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$

See EXERCISE 2

Note:

$a \wedge 1 = 1$ if $a = 1$ and $a \wedge 1 = 0$ if $a = 0$ so $a \wedge 1 = a$

$a \vee 1 = 1$ if $a = 1$ and $a \vee 1 = 1$ if $a = 0$ so $a \vee 1 = 1$

Similarly:

$a \wedge 0 = 0$ and $a \vee 0 = a$

Use truth tables to prove the following rules: (no need to do them all)

$(a')' = a$

$a \wedge a = a$	$a \vee a = a$
$a \wedge a' = 0$	$a \vee a' = 1$
$a \wedge b = b \wedge a$	$a \vee b = b \vee a$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \vee b) \vee c = a \vee (b \vee c)$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$(a \wedge b)' = a' \vee b'$	$(a \vee b)' = a' \wedge b'$
$a \wedge (a \vee b) = a$	$a \vee (a \wedge b) = a$

It's like a new type of algebra!

EXERCISE 1

1) fill in the truth table:

a	b	b'	$a \wedge b'$
0	0		
0	1		
1	0		
1	1		

2) fill in the truth table:

a	b	a'	$a' \vee b$
0	0		
0	1		
1	0		

1	1		

3) fill in the truth table:

a	b	c	$a \wedge b$	$a \wedge c$	$(a \wedge b) \vee (a \wedge c)$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

4) fill in the truth table:

a	b	c	$b \vee c$	$a \wedge (b \vee c)$
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

EXERCISE 2

Use truth tables to show that:

1) $(a \wedge b)' = a' \vee b'$

2) $(a \vee b)' = a' \wedge b'$

SOLUTIONS 1

1)

a	b	b'	$a \wedge b'$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

2)

a	b	a'	$a' \vee b$
0	0	1	1

0	1	1	1
1	0	0	0
1	1	0	1

3)

a	b	c	$a \wedge b$	$a \wedge c$	$(a \wedge b) \vee (a \wedge c)$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

4)

a	b	c	$b \vee c$	$a \wedge (b \vee c)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

SOLUTIONS 2

1)

a	b	$a \wedge b$	$(a \wedge b)'$	a'	b'	$a' \vee b'$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0-	1	1
1	1	1	0	0	0	0

2)

a	b	$a \vee b$	$(a \vee b)'$	a'	b'	$a' \wedge b'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

If … Then

Example

p : Today is Friday q : I go to yoga today

We can combine two propositions using IF … THEN

$p \Rightarrow q$:If today is Friday then I go to yoga today

What does this tell you?

If today is Friday then it tells you that I go to yoga today.

If today is not Friday then it tells you nothing.

If I go to yoga today then it tells you nothing.

If I do not go to yoga today then it tells you that today is not Friday.

The only way

If today is Friday then I go to yoga today

can be false, is if:

today is Friday, is true

and I go to yoga today, is false.

The only way $p \Rightarrow q$ can be false is if p is true and q is false.

So we have this truth table:

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

$p \Rightarrow q$ can be read as:

if p then q

p implies q

q if p

p only if q

q is necessary for p

p is sufficient for q

$p \Rightarrow q$ is not the same as $p' \Rightarrow q'$

Compare:

If today is Friday then I go to yoga today

If today is not Friday then I do not go to yoga today

$p \Rightarrow q$ is not the same as $q \Rightarrow p$

Compare:

If today is Friday then I go to yoga today

If I go to yoga today then today is Friday

Note: $q \Rightarrow p$ is called the converse of $p \Rightarrow q$

$p \Rightarrow q$ is the same as $q' \Rightarrow p'$

Compare:

If today is Friday then I go to yoga today

If I do not go to yoga today then today is not Friday

Note: $q' \Rightarrow p'$ is called the contrapositive of $p \Rightarrow q$

Note: A common error or fudge is to prove $p \Rightarrow q$ and then pretend you have proved $q \Rightarrow p$

For example, in the chapter Euler Tours, we proved:

If a closed Euler tour exits then every vertex is even.

We then pretended to have proved:

If every vertex is even then a closed Euler tour exists.

This is very naughty.

see Exercise 1

We can combine two propositions using IF AND ONLY IF

$p \Leftrightarrow q$:Today is Friday if and only if I go to yoga today

What does this tell you?

If today is Friday then it tells you that I go to yoga today.

If today is not Friday then it tells you that I do not go to yoga today.

If I go to yoga today then it tells you that today is Friday.

If I do not go to yoga today then it tells you that today is not Friday.

So $p \Leftrightarrow q$ is true if p and q are both true or both false, otherwise it is false.

Truth table:

p	q	$p \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

$p \Leftrightarrow q$ can be read as:

p if and only if q

p is necessary and sufficient for q

To prove $p \Leftrightarrow q$ we have to prove $p \Rightarrow q$ and we have to prove $q \Rightarrow p$

For example, in the chapter Rationals and Irrationals, we proved:

x is rational $\Leftrightarrow$ x is a terminating or recurring decimal

See Exercise 2

EXERCISE 1

1.

Use a truth table to show that:

$p \Rightarrow q$ is not the same as $p' \Rightarrow q'$

$p \Rightarrow q$ is not the same as $q \Rightarrow p$

$p \Rightarrow q$ is the same as $q' \Rightarrow p'$

2.Use a truth table to show that:

$(p \Rightarrow q) = (p' \vee q)$ and $(p \Rightarrow q) = (p \wedge q')'$

3.

Fill in the truth table

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					

1	1	0					
1	1	1					

SOLUTIONS 1

1.

p	q	$p\Rightarrow q$	p'	q'	$p'\Rightarrow q'$	$q\Rightarrow p$	$q'\Rightarrow p'$
0	0	1	1	1	1	1	1
0	1	1	1	0	0	0	1
1	0	0	0	1	1	1	0
1	1	1	0	0	1	1	1

2.

p	q	$p\Rightarrow q$	p'	$p'\vee q$	q'	$p\wedge q'$	$(p\wedge q')'$
0	0	1	1	1	1	0	1
0	1	1	1	1	0	0	1
1	0	0	0	0	1	1	0
1	1	1	0	1	0	0	1

3.

p	q	r	$p\Rightarrow q$	$q\Rightarrow r$	$(p\Rightarrow q)\wedge(q\Rightarrow r)$	$p\Rightarrow r$	$((p\Rightarrow q)\wedge(q\Rightarrow r))\Rightarrow(p\Rightarrow r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

So $((p\Rightarrow q)\wedge(q\Rightarrow r))\Rightarrow(p\Rightarrow r)$ is always true, regardless of the truth of p , q , r We call this a tautology.

EXERCISE 2

For each of the following, does $p \Rightarrow q$ or $q \Rightarrow p$ or $p \Leftrightarrow q$?

	p	q
a)	$x=4$	$2x=8$
b)	n is a multiple of 5	n is a multiple of 15
c)	n is not a multiple of 10	n is a prime
d)	$ABCD$ is a parallelogram	$ABCD$ is a square
e)	$x^2-6x+8=0$	$x=2$
f)	$x^2-4x+4=0$	$x=2$
g)	$x=4$	$x^2=16$
h)	$x>7$	$x>4$
i)	x is an integer	x is rational
j)	$x>2$	$x^2>4$
k)	$x<4$	$x^2<16$

SOLUTIONS 2

a) $p \Leftrightarrow q$ b) $q \Rightarrow p$ c) $q \Rightarrow p$ d) $q \Rightarrow p$ e) $q \Rightarrow p$ f) $p \Leftrightarrow q$

g) $p \Rightarrow q$ h) $p \Rightarrow q$ i) $p \Rightarrow q$ j) $p \Rightarrow q$ k) $q \Rightarrow p$

Arguments with If … Then

Example 1

If today is Monday then I go to work today

Today is Monday

So I go to work today

This argument consists of three propositions:

Premise: If today is Monday then I go to work today

Premise: Today is Monday

Conclusion: I go to work today

We are not interested in whether these propositions are true or false. We are interested in whether this argument is valid or invalid. An argument is only valid if the conclusion must be true whenever both the premises are true.

p : Today is Monday q : I go to work today

p' : Today is not Monday q' : I do not go to work today

The above argument is valid and has the form:

$p \Rightarrow q$

p

So q

Example 2

If today is Monday then I go to work today

I do not go to work today

So today is not Monday

This argument is valid and has the form:

$p \Rightarrow q$

q'

So p'

Example 3

If today is Monday then I go to work today

Today is not Monday

So I do not go to work today

This argument is invalid and has the form:

$p \Rightarrow q$

p'

So q'

Example 4

If today is Monday then I go to work today

I go to work today

So today is Monday

This argument is invalid and has the form:

$p \Rightarrow q$

q

So p

See Exercise 1

I have stated which of the above arguments are valid and which are invalid. It is just obvious.
Well … we can do better than this. We can test arguments using truth tables.

Example

Test the argument:

$p \Rightarrow q$

q'

So p'

p	q	$p \Rightarrow q$	q'	p'
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	1	0	0

Remember, an argument is only valid if the conclusion must be true whenever both the premises are true.
Both the premises are true in line 1 and the conclusion is true.
So this argument is valid.

Example

$p \Rightarrow q$

p'

So q'

p	q	$p \Rightarrow q$	p'	q'
0	0	1	1	1
0	1	1	1	0
1	0	0	0	1
1	1	1	0	0

Both the premises are true in line 2 but the conclusion is false.
So this argument is invalid.

See Exercise 2

If an argument is invalid then we might be able to show this by finding a counter-example.

Consider the argument

$p \Rightarrow q$

p'

So q'

If we put:

p dogs are birds q dogs are animals

then we get:

If dogs are birds then dogs are animals

Dogs are not birds

So dogs are not animals

An argument is only valid if the conclusion must be true whenever both the premises are true. But here, both premises are true and the conclusion is false. So the argument must be invalid.

See Exercise 3

EXERCISE 1

p : It is raining q : I take my umbrella

p' : It is not raining q' : I do not take my umbrella

Which of the following arguments are valid?

1) If it is raining then I take my umbrella
I do not take my umbrella
So it is not raining

2) If it is raining then I take my umbrella
I take my umbrella
So it is raining

3) If it is raining then I take my umbrella
It is not raining
So I do not take my umbrella

4) If it is raining then I take my umbrella
It is raining
So I take my umbrella

EXERCISE 2

Test the following arguments using truth tables:

1.
$p \Rightarrow q$
p
So q

2.
$p \Rightarrow q$
q
So p

3.
$(p \wedge q)'$
p'
So q

4.
$p \vee q$
p'
So q

5.
$p \vee q$

$p \Rightarrow r$

$q \Rightarrow r$

So r

6.

$p \Rightarrow q$

$q \Rightarrow r$

So $p \Rightarrow r$

EXERCISE 3

Show that the negation of $p \Rightarrow q$ is $p \wedge q'$

SOLUTIONS 1

1) This argument has the form:

$p \Rightarrow q$

q'

So p'

Like example 2, this is valid.

2) This argument has the form:

$p \Rightarrow q$

q

So p

Like example 4, this is invalid.

3) This argument has the form:

$p \Rightarrow q$

p'

So q'

Like example 3, this is invalid.

4) This argument has the form:

$p \Rightarrow q$

p

So q

Like example 1, this is valid.

SOLUTIONS 2

1.

p	q	$p \Rightarrow q$	p	q
0	0	1	0	0
0	1	1	0	1
1	0	0	1	0
1	1	1	1	1

Look at line 4. This argument is valid.

2.

p	q	$p \Rightarrow q$	q	p
0	0	1	0	0
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

Look at line 2. This argument is invalid.

3.

p	q	$p \wedge q$	$(p \wedge q)'$	p'	q
0	0	0	1	1	0
0	1	0	1	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Look at line 1. This argument is invalid.

4.

p	q	$p \vee q$	p'	q
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	1	0	1

Look at line 2. This argument is valid.

5.

p	q	r	$p \vee q$	$p \Rightarrow r$	$q \Rightarrow r$	r
0	0	0	0	1	1	0
0	0	1	0	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	1	0	1	0
1	0	1	1	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Look at lines 4, 6 and 8. This argument is valid.

6.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	1	1

Look at lines 1, 2, 4, 8. This argument is valid

SOLUTIONS 3

p	q	$p \Rightarrow q$	$(p \Rightarrow q)'$	q'	$p \wedge q'$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

The columns for $(p \Rightarrow q)'$ and $p \wedge q'$ are the same.

Arguments with All, None, Some

Example 1

All teachers are honest

All honest people like sprouts

So all teachers like sprouts

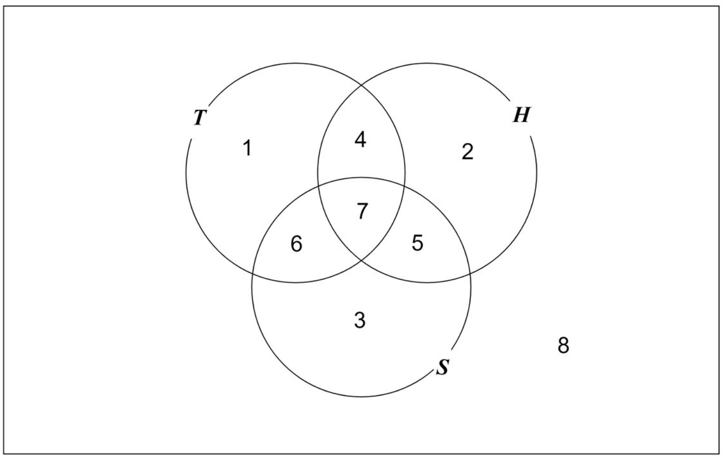
This argument consists of two premises and a conclusion:

Premise: All teachers are honest

Premise: All honest people like sprouts

Conclusion: All teachers like sprouts

We are not interested in whether the premises are true or false. We are interested in whether the argument is valid or invalid. An argument is only valid if the conclusion must be true whenever both the premises are true. We can use a Venn diagram to decide if an argument is valid or invalid. We have a rectangular room and there are three loops drawn on the floor.



Everyone is standing somewhere in the room. Teachers must stand inside the T loop. Honest people must stand inside the H loop. People who like sprouts must stand inside the S loop.

People standing inside region 1 are teachers, not honest, do not like sprouts

People standing inside region 5 are not teachers, honest, do like sprouts

People standing inside region 7 are teachers, honest, do like sprouts

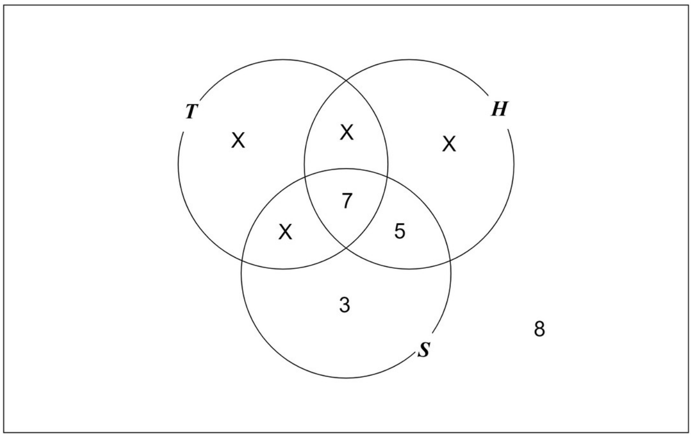
etc

Premise: All teachers are honest

This means there is no-one standing in regions 1 or 6. So we put a cross in each of these regions.

Premise: All honest people like sprouts

This means there is no-one standing in regions 2 or 4. So we put a cross in each of these regions.



Conclusion: All teachers like sprouts

This means there should be crosses in regions 1 and 4. Yes!

So if both premises are true then the conclusion must be true. So the argument is valid.

We are not really interested in teachers, honest people and people who like sprouts. We are interested in the form of the argument. We could say:

All A are B

All B are C

So all A are C

This argument form is valid. So any argument of this form is valid, such as:

All lawyers are happy

All happy people like Beethoven

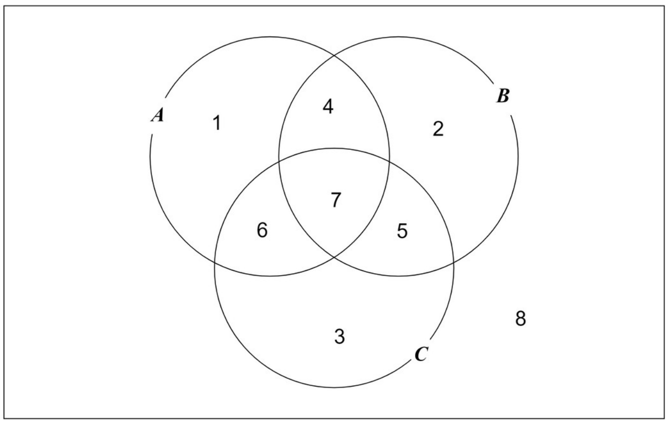
So all lawyers like Beethoven

Example 2

All A are B

No C are A

So no C are B

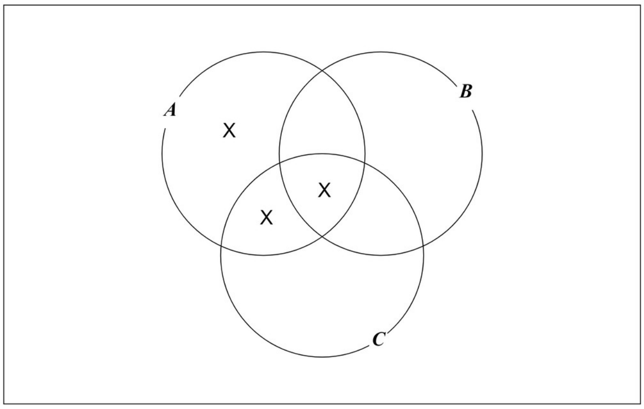


Premise: All A are B

This means there is no-one standing in regions 1 or 6

Premise: No C are A

This means there is no-one standing in regions 6 or 7



Conclusion: No C are B

This means there should be crosses in regions 5 and 7. No!

So the argument is invalid. So any argument of this form is invalid, such as:

All dentists are polite.

No hoodlums are dentists.

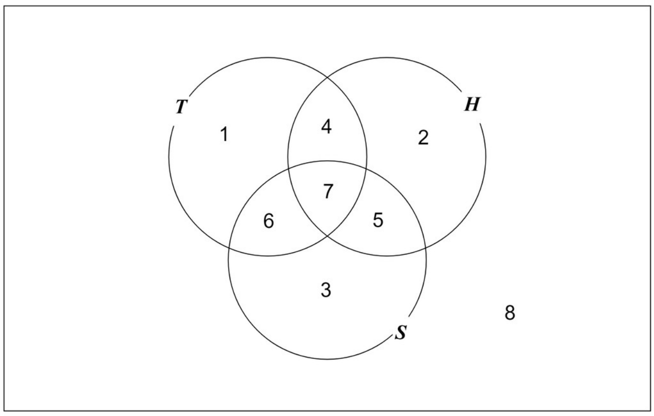
So no hoodlums are polite.

Example 3

All teachers are honest

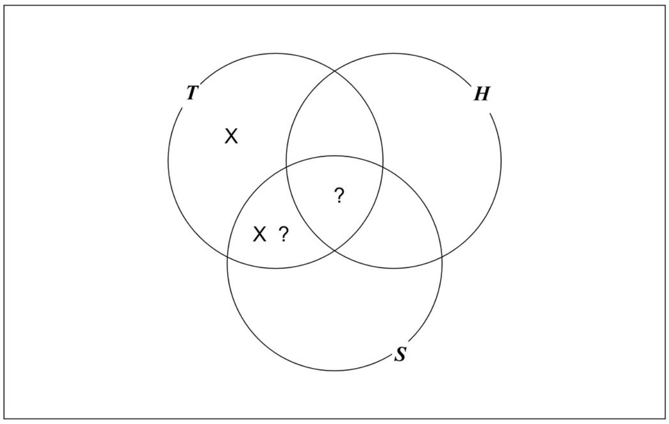
Some teachers like sprouts

So some honest people like sprouts



All teachers are honest we put a cross in regions 1 and 6

Some teachers like sprouts this means that there is at least one teacher who likes sprouts, so there is at least one person in region 6 or 7, so we put question-marks in regions 6 and 7



Conclusion: Some honest people like sprouts

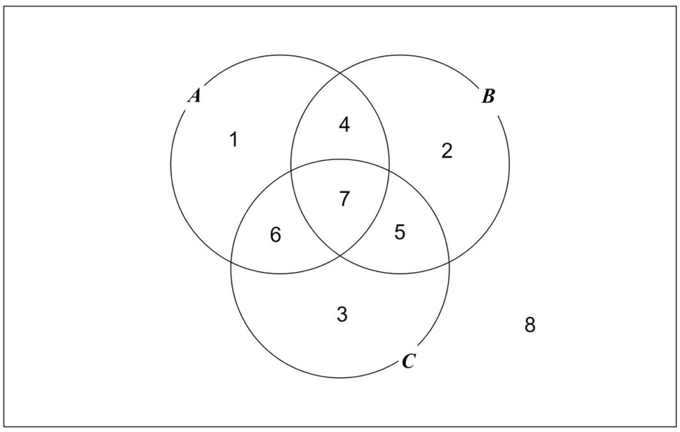
There is at least one person in region 6 or 7, but there is no-one in region 6 (because region 6 has a cross), so there must be at least one person in region 7, so there must be at least one person who is honest and likes sprouts, so some honest people like sprouts, so the argument is valid.

Example 4

All A are B

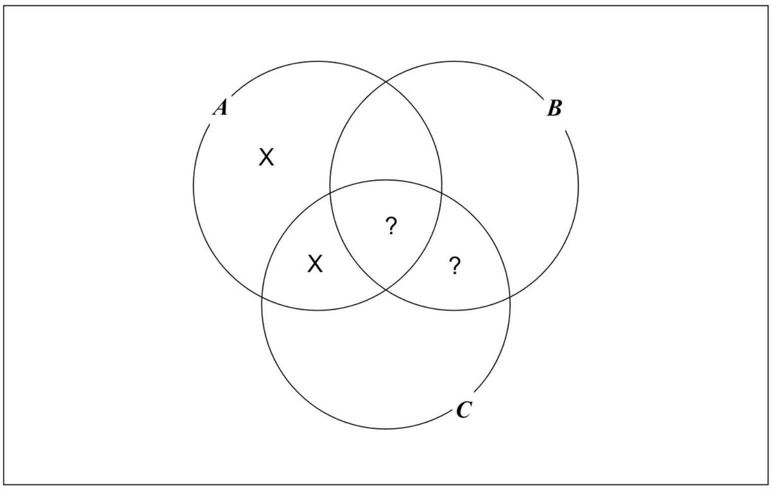
Some B are C

So some A are C



All A are B put crosses in regions 1 and 6

Some B are C put question-marks in regions 5 and 7



Conclusion: Some A are C

There is at least one person in region 5 or 7, but we cannot be certain that there is anybody in region 7

So the argument is invalid. So any argument of this form is invalid, such as:

All cats are mammals.

Some mammals are ferocious.

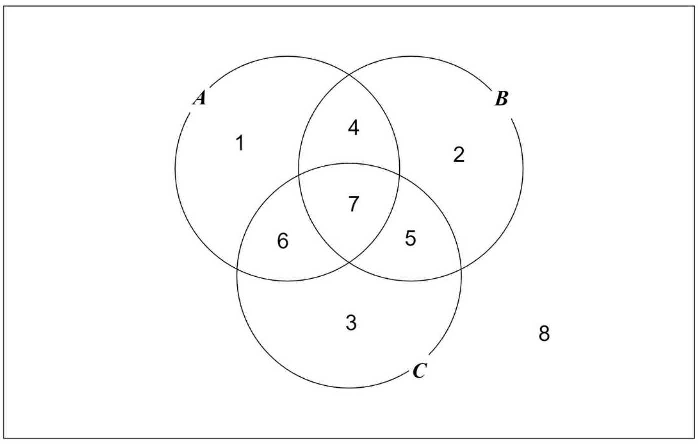
So some cats are ferocious.

Example 5

Some A are B

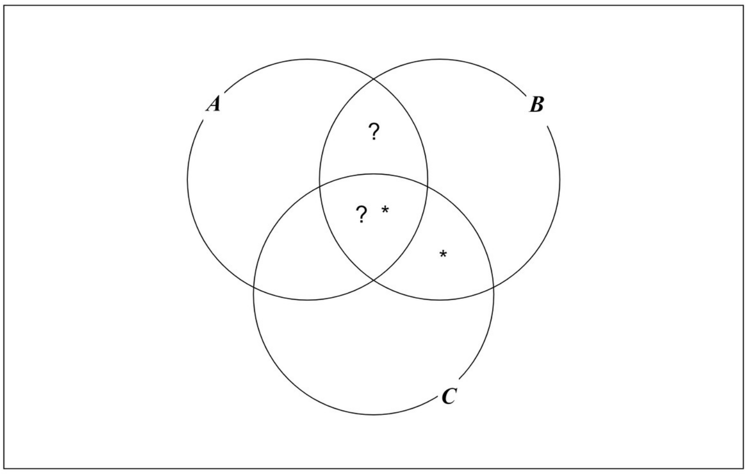
Some B are C

So some A are C



Some A are B put question-marks in regions 4 and 7

Some B are C we cannot just put question-marks in regions 5 and 7 because we will get these mixed up with the question-marks from Some A are B, so we will use asterisks instead



Conclusion Some A are C

There is at least one person in regions 4 or 7 and there is at least one person in regions 5 or 7, but we cannot be certain that there is anybody in region 7

So the argument is invalid.

In fact, if our two premisses both begin with Some … then the argument will always be invalid. Think about it!

EXERCISE

Show that these two arguments are invalid:

1. All A are B
   All C are B
   So all C are A

2. All A are B
   All A are C
   So all C are B

Now try these:

3. No bankers are poor.
   No poor people eat carrots.
   So all bankers eat carrots.

4. All plumbers are rich.

No rich people are happy.

So no plumbers are happy.

5. No dentists eat sugar.

All communists eat sugar.

So no dentists are communists.

6. All astronauts are handsome.

All handsome people like jazz.

So all astronauts like jazz.

7. Some teachers eat bananas.

All teachers like spaghetti.

So some people who like spaghetti eat bananas.

8. No A are B

Some B are C

So no A are C

9. All A are B

Some C are B

So some A are C

10. The negation of:

it is raining

is:

it is not raining

Write down the negations of the following:

a) All astronauts like jazz

b) No astronauts like jazz

c) Some astronauts like jazz

d) All A are B

e) No A are B

f) Some A are B

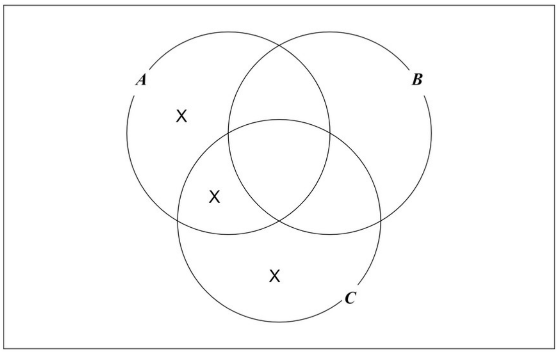
g) Some A are not B

11. Only A are B   All B are A

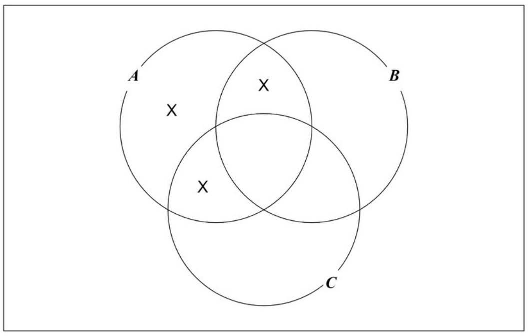
Do these mean the same thing?

SOLUTIONS

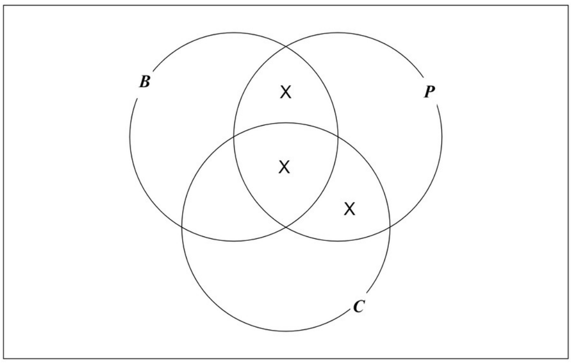
1. invalid



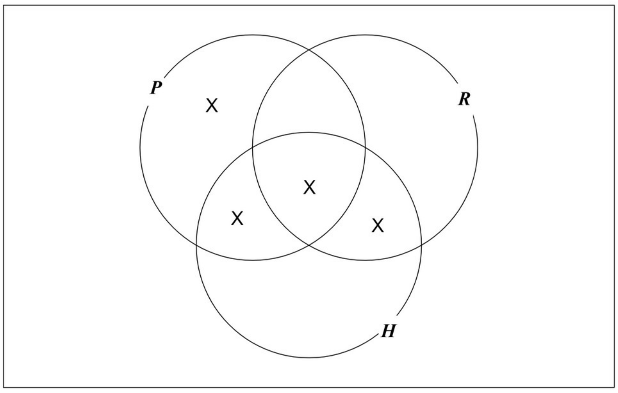
2. invalid



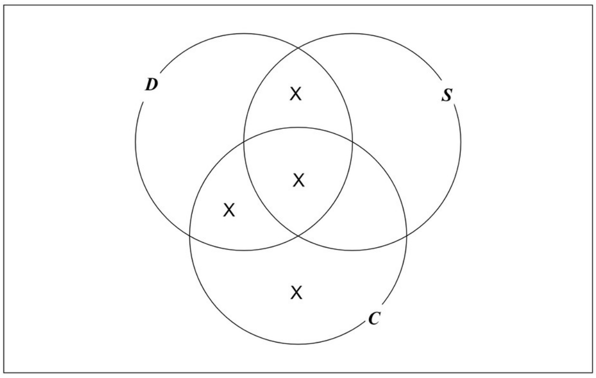
3. invalid



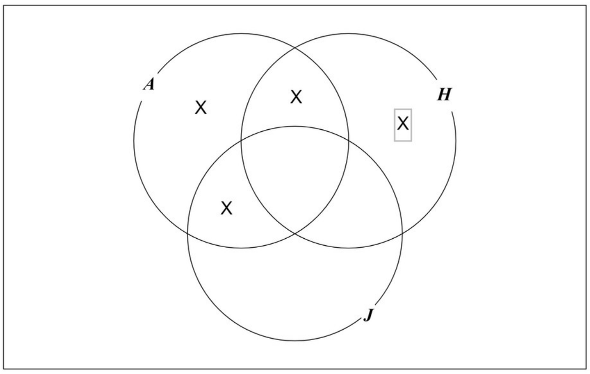
4. valid



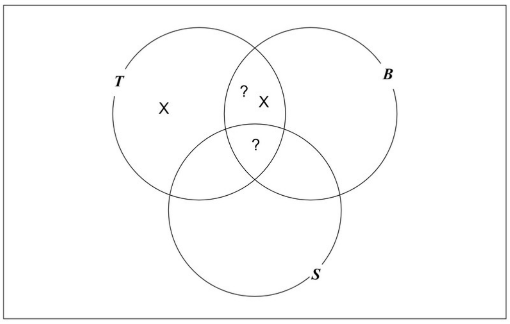
5. valid



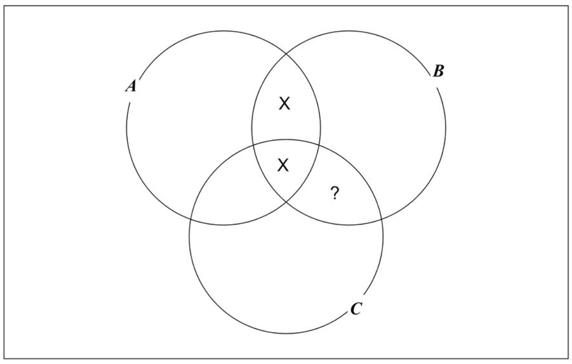
6. valid



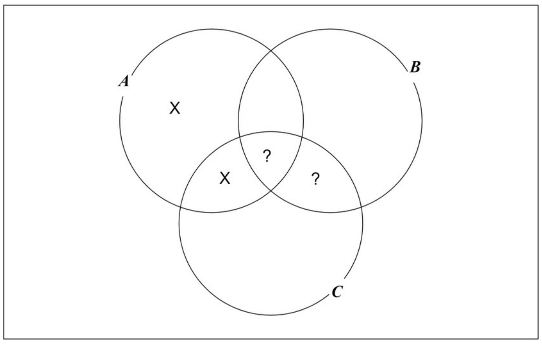
7. valid



8. invalid



9. invalid



10.

a) Some astronauts do not like jazz

b) Some astronauts like jazz

c) No astronauts like jazz

d) Some A are not B

e) Some A are B

f) Some A are B

g) All A are B

11. Yes

Hat Puzzles

There are three players called A, B and C. Each player is capable of making logical deductions.
I place a hat on the head of each player. I have three silver coloured hats and two gold coloured hats to choose from. No player can see their own hat. Each player tries to deduce the colour of their own hat. If a player can deduce the colour of their own hat then they shout.
The players stand in a queue.
A (at the back of the queue) can see B's hat and C's hat
B (in the middle of the queue) can see C's hat
C (at the front of the queue) can see no hats
Who can deduce the colour of their own hat in these examples?

a) A has a silver hat	B has a gold hat	C has a gold hat
b) A has a silver hat	B has a silver hat	C has a gold hat
c) A has a silver hat	B has a silver hat	C has a silver hat

SOLUTIONS
a) A thinks:
I can see two gold hats. So my hat must be silver.
b) B thinks:
If my hat is gold then A can deduce her hat is silver (see previous example). So if A does not shout then my hat is silver.
c) C thinks:
If my hat is gold then B can deduce his hat is silver (see previous example). So if B does not shout then my hat is silver.