Using Complex Numbers

Some real problems can be solved using complex numbers. Here are some examples.

1. Deriving trig identities

a) Pythagoras identity:

We know:

$$cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \quad \text{and} \quad sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

Show that:

$$\cos^2\theta+\sin^2\theta=1$$

b) Addition identity:

$$(cos\theta+isin\theta)(cos\phi+isin\phi)=e^{i\theta}e^{i\phi}=e^{i(\theta+\phi)}=\cos(\theta+\phi)+isin(\theta+\phi)$$

Equating real parts:

$$\cos(\theta+\phi)=cos\theta cos\phi-sin\theta sin\phi$$

Equating imaginary parts:

$$\sin(\theta+\phi)=cos\theta sin\phi+sin\theta cos\phi$$

c) Double angle identity:

$$(cos\theta+isin\theta)^2=(e^{i\theta})^2=e^{i2\theta}=\cos(2\theta)+isin(2\theta)$$

Equating real parts:

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

Equating imaginary parts:

$$\sin 2\theta=2\,cos\theta sin\theta$$

Note:

de Moivre’s theorem:

$$(cos\theta+isin\theta)^n=\cos(n\theta)+isin(n\theta)$$

d) Half angle identity:

We know:

$$2\,cos\theta=e^{i\theta}+e^{-i\theta}$$

So:

$$(2\,cos\theta)^2=(e^{i\theta}+e^{-i\theta})^2$$

Multiply out the brackets:

$$2^2\cos^2\theta = e^{i2\theta} + 2 + e^{-i2\theta}$$

Collect up terms:

$$2^2\cos^2\theta = \left(e^{i2\theta} + e^{-i2\theta}\right) + 2$$

Write with cosines:

$$2^2\cos^2\theta = 2\cos 2\theta + 2$$

So:

$$\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$$

e) Factor identity:

We know:

$$2\,cos\theta = e^{i\theta} + e^{-i\theta} \quad \text{and} \quad 2\,isin\theta = e^{i\theta} - e^{-i\theta}$$

So:

$$sin\theta cos\phi = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$$

Show that:

$$sin\theta cos\phi = \frac{1}{2}\sin(\theta+\phi) + \frac{1}{2}\sin(\theta-\phi)$$

We could write:

$$sin\alpha + sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \quad \text{can you see how?}$$

I could go on …

2. Integration

Example

We want to work out:

$$\int e^{-x} cosx\,dx \quad \text{and} \quad \int e^{-x} cosx\,dx$$

Here we go:

$$\int e^{-x}(cosx + isinx)\,dx = \int e^{-x}e^{ix}\,dx = \int e^{(-1+i)x}\,dx = \frac{1}{(-1+i)}e^{(-1+i)x} + c = \frac{1}{(-1+i)}e^{-x}e^{ix} + c$$

But

$$\frac{1}{(-1+i)} = \frac{-1-i}{(-1+i)(-1-i)} = -\frac{1}{2}(1+i)$$

So

$$\int e^{-x}(cosx+isinx)\,dx=-\frac{1}{2}(1+i)\,e^{-x}(cosx+isinx)+c$$

Equating real parts:

$$\int e^{-x}cosx\,dx=-\frac{1}{2}e^{-x}(cosx-sinx)+c' \quad \text{where} \quad c' \quad \text{is the real part of} \quad c$$

Equating imaginary parts:

$$\int e^{-x}sinx\,dx=-\frac{1}{2}e^{x}(cosx+sinx)+c'' \quad \text{where} \quad c'' \quad \text{is the imaginary part of} \quad c$$

3. A formula for $\ln\sqrt{2}$ and π

a) $1+i=\sqrt{2}e^{i\pi/4}$

So:

$$\ln(1+i)=\ln(\sqrt{2})+\ln(e^{i\pi/4})=\ln(\sqrt{2})+\frac{i\pi}{4}$$

b) $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots$ this is the Maclaurin series

So:

$$\ln(1+i)=i-\frac{1}{2}i^2+\frac{1}{3}i^3-\frac{1}{4}i^4+\ldots=i+\frac{1}{2}-\frac{1}{3}i-\frac{1}{4}+\ldots$$

c) From (a) and (b) we have:

$$\ln(\sqrt{2})+\frac{i\pi}{4}=i+\frac{1}{2}-\frac{1}{3}i-\frac{1}{4}+\ldots$$

Equating real parts:

$$\ln(\sqrt{2})=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\ldots$$

Equating imaginary parts:

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

4. Van Aubel's Theorem if you know about vectors ...

What do you make of the following proof, where we have recklessly mixed up complex numbers and vectors?

If $\boldsymbol{v}$ is a vector then $i\boldsymbol{v}$ is the vector you get by rotating $\boldsymbol{v}$ anti-clockwise by $90°$

Draw some diagrams and convince yourself that:

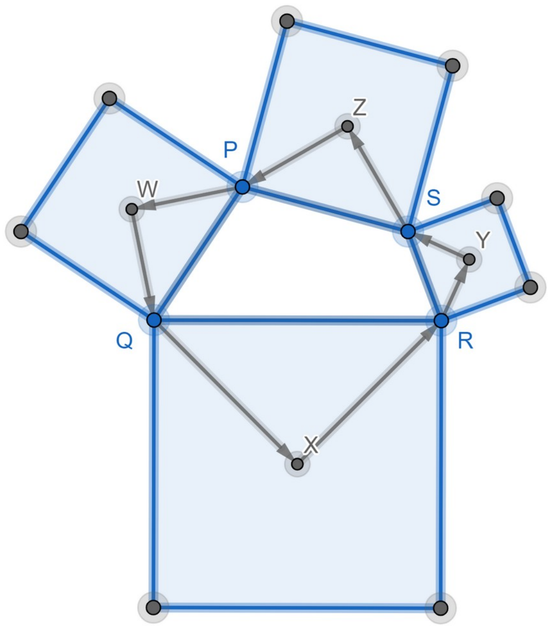
$i\boldsymbol{w}+i\boldsymbol{v}=i(\boldsymbol{w}+\boldsymbol{v})$ and $i^2\boldsymbol{w}=-\boldsymbol{w}$ and if $\boldsymbol{w}+i\boldsymbol{w}=\boldsymbol{0}$ then $\boldsymbol{w}=\boldsymbol{0}$

Given any quadrilateral PQRS, draw a square on each side.

W, X, Y and Z are the centres of these squares.

Theorem

ZX and YW will have the same length and are at right-angles.



Proof

Let $\boldsymbol{a}=\overrightarrow{PW}$ so $i\boldsymbol{a}=\overrightarrow{WQ}$ Let $\boldsymbol{b}=\overrightarrow{QX}$ so $i\boldsymbol{b}=\overrightarrow{XR}$

Let $\boldsymbol{c}=\overrightarrow{RY}$ so $i\boldsymbol{c}=\overrightarrow{YS}$ Let $\boldsymbol{d}=\overrightarrow{SZ}$ so $i\boldsymbol{d}=\overrightarrow{ZP}$

From the diagram:

$$\boldsymbol{a}+i\boldsymbol{a}+\boldsymbol{b}+i\boldsymbol{b}+\boldsymbol{c}+i\boldsymbol{c}+\boldsymbol{d}+i\boldsymbol{d}=\boldsymbol{0}$$

So:

$$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})+i(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})=\boldsymbol{0}$$

So:

$$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})=\boldsymbol{0}$$

Now:

$$\overrightarrow{YW}=i\boldsymbol{c}+\boldsymbol{d}+i\boldsymbol{d}+\boldsymbol{a}$$

So:

$$i\overrightarrow{YW}=i^2\boldsymbol{c}+i\boldsymbol{d}+i^2\boldsymbol{d}+i\boldsymbol{a}=-\boldsymbol{c}+i\boldsymbol{d}-\boldsymbol{d}+i\boldsymbol{a}$$

Now:

$$\overrightarrow{ZX}=i\boldsymbol{d}+\boldsymbol{a}+i\boldsymbol{a}+\boldsymbol{b} \qquad \text{But} \quad \boldsymbol{b}=-(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})$$

So:

$$\overrightarrow{ZX}=i\boldsymbol{d}+\boldsymbol{a}+i\boldsymbol{a}-(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})=-\boldsymbol{c}+i\boldsymbol{d}-\boldsymbol{d}+i\boldsymbol{a}$$

So:

$\vec{ZX} = i\,\vec{YW}$ as required.

EXERCISE

1)

Derive trig identities for:

$\cos(\theta-\phi)$ and $\sin(\theta-\phi)$

2)

Derive trig identities for:

$\cos 3\theta$ and $\sin 3\theta$

3)

Derive the half angle formula for $\sin^2\theta$

4)

Use the method of question (3) to write $\cos^5\theta$ in terms of $\cos 5\theta$ and $\cos 3\theta$ and $cos\theta$

5)

Show that:

$$cos\theta cos\phi = \frac{1}{2}\cos(\theta+\phi)+\frac{1}{2}\cos(\theta-\phi)$$

or if you prefer:

$$cos\alpha + cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

6)

We want to evaluate:

$$S=\frac{1}{2}cos\theta+\frac{1}{4}\cos 2\theta+\frac{1}{8}\cos 3\theta+\ldots$$

Now:

$$cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta}) \quad \text{so} \quad \cos 2\theta=\frac{1}{2}(e^{i2\theta}+e^{-i2\theta}) \quad \text{and} \quad \cos 3\theta=\frac{1}{2}(e^{i3\theta}+e^{-i3\theta}) \quad \text{etc}$$

Show that:

$$S=\left(\frac{1}{4}(e^{i\theta})+\frac{1}{8}(e^{i2\theta})+\frac{1}{16}(e^{i3\theta})+\ldots\right)+\left(\frac{1}{4}(e^{-i\theta})+\frac{1}{8}(e^{-i2\theta})+\frac{1}{16}(e^{-i3\theta})+\ldots\right)$$

Show that:

$$\frac{1}{4}(e^{i\theta})+\frac{1}{8}(e^{i2\theta})+\frac{1}{16}(e^{i3\theta})+\ldots=\frac{\frac{1}{4}(e^{i\theta})}{1-\frac{1}{2}(e^{i\theta})} \qquad \text{hint, geometric series}$$

Show that:

$$\frac{1}{4}\left(e^{-i\theta}\right)+\frac{1}{8}\left(e^{-i2\theta}\right)+\frac{1}{16}\left(e^{-i3\theta}\right)+...=\frac{\frac{1}{4}\left(e^{-i\theta}\right)}{1-\frac{1}{2}\left(e^{-i\theta}\right)}$$ hint, geometric series

Show that:

$$S=\frac{\frac{1}{4}\left(e^{i\theta}\right)}{1-\frac{1}{2}\left(e^{i\theta}\right)}+\frac{\frac{1}{4}\left(e^{-i\theta}\right)}{1-\frac{1}{2}\left(e^{-i\theta}\right)}$$

Show that:

$$S=\frac{2\cos\theta-1}{5-4\cos\theta}$$

7)

Another formula for π

a) $\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+...$ this is the Maclaurin series

b)Show that:

$$(2+i)(3+i)=5+5i$$

So:

$$arg(2+i)+arg(3+i)=arg(5+5i)$$

So:

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left(\frac{5}{5}\right) \quad \text{but} \quad \tan^{-1}\left(\frac{5}{5}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

So:

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

c) Write down the Maclaurin series for:

$$\tan^{-1}\left(\frac{1}{2}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{1}{3}\right)$$

Show that:

$$\frac{\pi}{4}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)+...$$

8)

We can make up more formulas for π using the method of question (7)

We need to find a,b,c where $(a+i)(b+i)=c+ci$

Show that:

$$ab-1=a+b$$

Show that:

$$b=\frac{a+1}{a-1}$$

Now let:

$a=\frac{p}{q}$ where p and q are integers and $p>q>0$

Show that:

$$b=\frac{p+q}{p-q}$$

So:

$$\left(\frac{p}{q}+i\right)\left(\frac{p+q}{p-q}+i\right)=c+ci$$

Show that:

$$(p+iq)((p+q)+i(p-q))=(p^2+q^2)+i(p^2+q^2)$$

So:

$$arg(p+iq)+arg((p+q)+i(p-q))=(p^2+q^2)+i(p^2+q^2)$$

So:

$$\tan^{-1}\left(\frac{q}{p}\right)+\tan^{-1}\left(\frac{p-q}{p+q}\right)=\frac{\pi}{4}$$

Write down the Maclaurin series for:

$\tan^{-1}\left(\frac{q}{p}\right)$ and $\tan^{-1}\left(\frac{p-q}{p+q}\right)$ to get a formula for π

I chose $p=17$ and $q=4$

So:

$$\tan^{-1}\left(\frac{4}{17}\right)+\tan^{-1}\left(\frac{13}{21}\right)=\frac{\pi}{4}$$

and then I got:

$$\frac{\pi}{4}=\left(\frac{4}{17}+\frac{13}{21}\right)-\frac{1}{3}\left(\frac{4^3}{17^3}+\frac{13^3}{21^3}\right)+\frac{1}{5}\left(\frac{4^5}{17^5}+\frac{13^5}{21^5}\right)+\ldots$$

SOLUTIONS

1)

$$(cos\theta+isin\theta)(cos\phi-isin\phi)=e^{i\theta}e^{-i\phi}=e^{i(\theta-\phi)}=\cos(\theta-\phi)+isin(\theta-\phi)$$

Equating real parts:

$$cos\theta cos\phi+sin\theta sin\phi=\cos(\theta-\phi)$$

Equating imaginary parts:

$$-cos\theta sin\phi+sin\theta cos\phi=\sin(\theta-\phi)$$

2)

$$(cos\theta + isin\theta)^3 = (e^{i\theta})^3 = e^{i3\theta} = \cos(3\theta) + isin(3\theta)$$

Equating real parts:

$$\cos^3\theta - 3\, cos\theta \sin^2\theta = \cos 3\theta$$

We could replace $\sin^2\theta$ by $1-\cos^2\theta$ and write $\cos 3\theta = 4\cos^3\theta - 3\, cos\theta$

Equating imaginary parts:

$$3\cos^2\theta sin\theta - \sin^3\theta = \sin 3\theta$$

We could replace $\cos^2\theta$ by $1-\sin^2\theta$ and write $\sin 3\theta = 3\, sin\theta - 4\sin^3\theta$

Dividing:

$$\frac{3\cos^2\theta sin\theta - \sin^3\theta}{\cos^3\theta - 3\, cos\theta \sin^2\theta} = \frac{\sin 3\theta}{\cos 3\theta}$$

Divide top and bottom of the left-hand-side by $\cos^3\theta$

$$\frac{3\, tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta$$

3)

$$(2\, isin\theta)^2 = (e^{i\theta} - e^{-i\theta})^2$$

Multiply out the brackets:

$$2^2 i^2 \sin^4\theta = e^{i2\theta} - 2 + e^{-i2\theta}$$

Collect up terms:

$$-2^2 \sin^2\theta = (e^{i2\theta} + e^{-i2\theta}) - 2$$

Write with cosines:

$$-2^2 \sin^2\theta = 2\cos 2\theta - 2$$

So:

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

4)

$$(2\, cos\theta)^5 = (e^{i\theta} + e^{-i\theta})^5$$

Multiply out the brackets:

$$2^5 \cos^5\theta = e^{i5\theta} + 5\, e^{i3\theta} + 10\, e^{i\theta} + 10\, e^{-i\theta} + 5\, e^{-i3\theta} + e^{-i5\theta}$$

Collect up terms:

$$2^5 \cos^5\theta = (e^{i5\theta} + e^{-i5\theta}) + 5\,(e^{i3\theta} + e^{-i3\theta}) + 10\,(e^{i\theta} + e^{-i\theta})$$

Write with cosines:

$$2^5 \cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\, cos\theta$$

So:

$$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}cos\theta$$

5)

$$cos\theta cos\phi = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$$

So:

$$cos\theta cos\phi = \frac{1}{4}\left(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{i(-\theta-\phi)}\right)$$

So:

$$cos\theta cos\phi = \frac{1}{4}\left(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)}\right) + \frac{1}{4}\left(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}\right)$$

So:

$$cos\theta cos\phi = \frac{1}{2}\cos(\theta+\phi) + \frac{1}{2}\cos(\theta-\phi)$$