Euler’s Sine Formula

Reminder of the factor theorem

Example

$p(x)=x^4-17x^3+99x^2-223x+140$

Now:

$p(1)=0$ so $(x-1)$ is a factor of $p(x)$ - this is the factor theorem - see Appendix 2

$p(4)=0$ so $(x-4)$ is a factor

$p(5)=0$ so $(x-5)$ is a factor

$p(7)=0$ so $(x-7)$ is a factor

So:

$p(x)=c(x-1)(x-4)(x-5)(x-7)$ where c is some constant

Now:

$p(0)=140$ so $c=1$

Rearranging gives:

$$p(x)=(1-x)(4-x)(5-x)(7-x)=140\left(1-\frac{x}{1}\right)\left(1-\frac{x}{4}\right)\left(1-\frac{x}{5}\right)\left(1-\frac{x}{7}\right)$$

Now:

$\sin(0)=0$ so $(x-0)$ is a factor of $sinx$

$\sin(\pi)=0$ so $(x-\pi)$ is a factor

$\sin(-\pi)=0$ so $(x+\pi)$ is a factor

$\sin(2\pi)=0$ so $(x-2\pi)$ is a factor

$\sin(-2\pi)=0$ so $(x+2\pi)$ is a factor

$\sin(3\pi)=0$ so $(x-3\pi)$ is a factor

$\sin(-3\pi)=0$ so $(x+3\pi)$ is a factor etc

So:

$sinx=cx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$ where c is some constant

So:

$$\frac{sinx}{x}=c(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$$

Sub in $x=0$ see Footnote

$1=c(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\ldots$

So:

$$c=\frac{1}{(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)\ldots}$$

So:

$$sinx=\frac{x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)...}{(-\pi)(+\pi)(-2\pi)(+2\pi)(-3\pi)(+3\pi)...}$$

Fool around with this and show that:

$$sinx=x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\ldots$$

Or if you prefer:

$$\sin(x)=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\ldots$$ this is Euler's sine formula

We can have some fun with this.

1) We recall that:

$$sinx=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$ this is the Maclaurin series

Equating Maclaurin's formula and Euler's formula we have:

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\ldots$$

Equating coefficients of x^3 we (eventually) get:

$$-\frac{1}{3!}=-\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\ldots$$

Rearranging gives:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

What a surprising answer. Why is π doing here?

As a bonus:

Equating coefficients of x^5 we (eventually) get:

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...=\frac{\pi^4}{90}$$

We can also find formulas for:

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\ldots$$ and $$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\ldots$$ etc

What about?

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots$$ and $$\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\ldots$$ etc

Well Euler failed to find a formula for these series.

2) If we put $x=\frac{\pi}{2}$ into Euler's sine formula we get:

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2 2^2}\right)\left(1-\frac{1}{2^2 3^2}\right)\dots$$

Now:

$$1-\frac{1}{2^2 n^2}=\frac{(2n-1)(2n+1)}{2^2 n^2}$$

So:

$$1=\frac{\pi}{2}\left(\frac{1\times 3}{2^2}\right)\left(\frac{3\times 5}{2^2 2^2}\right)\left(\frac{5\times 7}{2^2 3^2}\right)...=\frac{\pi}{2}\left(\frac{1\times 3}{2\times 2}\right)\left(\frac{3\times 5}{4\times 4}\right)\left(\frac{5\times 7}{6\times 6}\right)..$$

Which gives us Wallis's formula:

$$\frac{\pi}{2}=\frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\dots$$

3)

$$sinx=x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\dots$$

So:

$$lnsinx=lnx+\ln\left(1-\frac{x}{\pi}\right)+\ln\left(1+\frac{x}{\pi}\right)+\ln\left(1-\frac{x}{2\pi}\right)+\ln\left(1+\frac{x}{2\pi}\right)+\ln\left(1-\frac{x}{3\pi}\right)+\ln\left(1+\frac{x}{3\pi}\right)...$$

Differentiate both sides and show that:

$$\frac{cosx}{sinx}=\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}-\frac{1}{3\pi-x}+\frac{1}{3\pi+x}-\dots$$

Sub in $x=\frac{\pi}{4}$ and show that:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\dots$$

Footnote:

Am I really saying?

$$\frac{\sin 0}{0}=1$$

Look at the Maclaurin series:

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\frac{sinx}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

$$x \to 0 \qquad \frac{sinx}{x} \to 1$$