Euler Squares

Example 1

Here are two $3\times3$ Latin squares:

| | | |
|---|---|---|
| A | B | C |
| C | A | B |
| B | C | A |

| | | |
|---|---|---|
| a | b | c |
| b | c | a |
| c | a | b |

We can combine them to form a $3\times3$ Euler square:

| | | |
|---|---|---|
| A, a | B, b | C, c |
| C, b | A, c | B,a |
| B, c | C, a | A, b |

Each cell contains two symbols and no two cells contain the same two symbols.

Euler squares are used in agricultural research. We are growing wheat in a field and we want to compare three types of wheat, a, b, c and three types of fertilizer, A, B, C. We want to grow each type of wheat with each type of fertilizer. We divide the field into 9 plots and plant the wheat and apply the fertilizer as in the Euler square. Conditions (drainage etc) might vary across the field, so we want to try each type of wheat and each type of fertilizer in each row and column of the field.

There are no $2\times2$ Euler squares. Can you see why?

Euler tried, and failed, to find a $6\times6$ Euler square.

In 1901 a proof was discovered that $6\times6$ Euler squares do not exist.

In 1960 a proof was discovered that Euler squares exist for all sizes except $2\times2$ and $6\times6$

One way to make an Euler square is the double-diagonal method.

Example 2

| | | | | |
|---|---|---|---|---|
| A | B | C | D | E |
| E | A | B | C | D |
| D | E | A | B | C |
| C | D | E | A | B |
| B | C | D | E | A |

| | | | | |
|---|---|---|---|---|
| a | b | c | d | e |
| b | c | d | e | a |
| c | d | e | a | b |
| d | e | a | b | c |
| e | a | b | c | d |

The first Latin square is diagonal from top-left to bottom-right. The other Latin square is diagonal from top-right to bottom-left.

We can combine these two Latin squares to make an Euler square:

| | | | | |
|---|---|---|---|---|
| A, a | B, b | C, c | D, d | E, e |
| E, b | A, c | B, d | C, e | D, a |
| D, c | E, d | A, e | B, a | C, b |
| C, d | D, e | E, a | A, b | B, c |
| B, e | C, a | D, b | E, c | A, d |

Unfortunately, this method does not always work.

Investigation: When will the double-diagonal method work?

There are many other methods to find Euler squares. Look them up. (Euler squares are also called Graeco-Latin squares)

EXERCISE

Take the 16 picture cards (jacks, queens, kings, aces of spades, hearts, clubs, diamonds) from a pack of cards. Arrange them in a $4\times4$ Euler square:

a card of each suit must appear in each row and each column

a card of each rank must appear in each row and each column

SOLUTIONS

There are many solutions. Here is mine. I tried the diagonal method but it did not work.

| | | | |
|---|---|---|---|
| Ace, spade | King, heart | Queen, club | Jack, diamond |
| King, diamond | Ave club | Jack, heart | Queen, spade |
| Queen, heart | Jack, spade | Ace, diamond | King, club |
| Jack, club | Queen, diamond | King, spade | Ace, heart |