How To Tune A Piano

When you press a key on a piano, a hammer hits a string and the string vibrates. The frequency of this vibration determines the pitch of the note. For example, a string vibrating 261.6 times per second will produce the note middle C. We write 261.6 times per second as 261.6 Hz

The interval between two notes is the ratio of their frequencies.
The interval between 800 Hz and 600 Hz is $800/600=4/3$

Look at this sequence of notes:

200 Hz, 300 Hz, 450 Hz, 675 Hz, …

Now

$$\frac{300}{200}=\frac{450}{300}=\frac{675}{450}=...=\frac{3}{2}$$

So the intervals between consecutive notes are the same. The frequencies of the notes form a geometric sequence with common ratio $3/2$ (see Appendix 2: Geometric Sequence)

If one note has twice the frequency of another note then the interval between these notes is called an octave. In western European music, the octave is divided into 12 intervals. So if one note on the piano has frequency 440 Hz then the note, one octave above, has frequency 880 Hz and we need to put another eleven notes in-between. What are the frequencies of these other notes?
If we want equal intervals between consecutive notes then the frequencies must form a geometric sequence:

$440 \quad 440r^1 \quad 440r^2 \quad 440r^3 \quad \ldots \quad 440r^{11} \quad 440r^{12}$

We want:

$440r^{12}=880$ so $r^{12}=2$ so $r=2^{1/12}$

This gives the following frequencies:

440 A	466.2 A#	493.9 B	523.3 C	554.4 C#	587.3 D	622.3 D#	659.3 E	698.5 F	740.0 F#	784.0 G	830.6 G#	880 A

The names of the notes are given below the frequencies.
Notes one octave apart are given the same name.

Once we know the frequencies of all the notes in one octave we can calculate the frequencies of all the notes on the piano.
To find the frequencies of the notes in the octave above we just double these frequencies.
To find the frequencies of the notes in the octave below we just halve these frequencies.
etc
This method for tuning a piano is called equal-tempered tuning because there are equal intervals between the consecutive notes.

Why do we want equal intervals between consecutive notes on the piano?
Here are the first seven notes of “Twinkle Twinkle Little Star”:
523.3 Hz   523.3 Hz   784.0 Hz   784.0 Hz   880 Hz   880 Hz   784.0 Hz
We can play this on the piano. But what if someone says they can’t sing notes that low. Can we play the same tune only higher?
If we multiply the frequencies of all these notes by say r^3 then we get:
622.3 Hz   622.3 Hz   932.3 Hz   932.3 Hz   1046.5 Hz   1046.5 Hz   932.3 Hz
All these notes are on the piano. This version of the song has all the same intervals as the first version. It will sound just the same. Only higher.

Musicologists (including Pythagoras) have claimed that two notes sound nice when played together if the interval between them is a simple ratio. So 600 Hz and 400 Hz sound nice when played together because $600/400=3/2$ is a simple ratio.
Unfortunately, if we use equal-tempered tuning then the interval between any two notes (apart from octaves) is not a simple ratio. It is irrational.
Fortunately, some interval are nearly simple ratios.
The interval between say C and G is $2^{7/12}$ which is nearly $3/2$
The interval between say C and F is $2^{5/12}$ which is nearly $4/3$
The interval between say C and E is $2^{4/12}$ which is nearly $5/4$
Musicologists (including Pythagoras) have devised tuning schemes with lots of simple ratios. But these do not have equal intervals between consecutive notes. So we have a problem.
We want equal intervals. We want simple ratios. We can’t have both.

A piano creates a sound with a vibrating string. A bugle creates a sound with a vibrating column of air. A skilled player can produce different notes on a bugle by altering the way their lips vibrate.
Physics tells us that if the lowest note you can get on a bugle is 110Hz then the other notes you can get are: 220 Hz, 330 Hz, 440 Hz, 550 Hz, 660 Hz, ...

110 Hz is an A

220 Hz is an A

330 Hz is nearly an E (329.6Hz)

440 Hz is an A

550 Hz is nearly a C# (554.4Hz)

660 Hz is nearly an E (659.3Hz)

So the bugle notes don’t quite match up with the piano notes.

“The Last Post” is played on a bugle with the notes:

220 Hz, 330 Hz, 440 Hz, 550 Hz, 660 Hz

It will sound slightly different if you play it on a piano.

J.S.Bach wrote a piece of music called The Well-Tempered Clavier to demonstrate the advantages of equal-tempered tuning. Check it out.