Collecting Cards if you know about probability distributions ...

Every packet of cornflakes contains a card with a picture of a mathematician. There are 20 different mathematicians to collect. On average, how many cornflakes packets will I have to buy to get the complete set?

Let X_1 be the number of cereal packets I will have to buy to get my first mathematician.
Let X_2 be the number of cereal packets I will have to buy, after I have got my first mathematician, to get my second mathematician.
Let X_3 be the number of cereal packets I will have to buy, after I have got my second mathematician, to get my third mathematician.
etc
Let X be the total number of cereal packets I will have to buy to get the complete set of 20 mathematicians.
So:
$X = X_1 + X_2 + X_3 + ... + X_{20}$ and we want to find $E(X)$
We know that:
$E(X) = E(X_1) + E(X_2) + E(X_3) + ... + E(X_{20})$

When I buy my first packet, I will get my first mathematician.
So:
$X_1 = 1$ and so $E(X_1) = 1$

Now I have got my first mathematician. How many more packets will I have to buy to get my second mathematician?
Look at this table.

x_2	$p(x_2)$	$x_2 p(x_2)$
1	$\frac{19}{20}$	$1 \times \frac{19}{20}$
2	$\frac{1}{20} \times \frac{19}{20}$	$2 \times \frac{1}{20} \times \frac{19}{20}$
3	$\frac{1}{20} \times \frac{1}{20} \times \frac{19}{20}$	$3 \times \frac{1}{20} \times \frac{1}{20} \times \frac{19}{20}$
...	...	...

Now:

$$E(X_2)=\sum x_2\, p(x_2)$$

So:

$$E(X_2)=\frac{19}{20}\left(1+2\left(\frac{1}{20}\right)+3\left(\frac{1}{20}\right)^2+4\left(\frac{1}{20}\right)^3+\ldots\right)$$

So:

$$E(X_2)=\frac{19}{20}\frac{1}{\left(1-\frac{1}{20}\right)^2}=\frac{20}{19}\quad \text{see Footnote with}\quad x=\frac{1}{20}$$

Now I have got my second mathematician. How many more packets will I have to buy to get my third mathematician?

Look at this table.

x_3	$p(x_3)$	$x_3\, p(x_3)$
1	$\frac{18}{20}$	$1\times\frac{19}{20}$
2	$\frac{2}{20}\times\frac{18}{20}$	$2\times\frac{1}{20}\times\frac{19}{20}$
3	$\frac{2}{20}\times\frac{2}{20}\times\frac{18}{20}$	$3\times\frac{1}{20}\times\frac{1}{20}\times\frac{19}{20}$
...	...	...

Repeat the above calculation and show that:

$$E(X_3)=\frac{20}{18}$$

Find:

$$E(X_4)$$

etc

Show that:

$$E(X)=20\left(\frac{1}{20}+\frac{1}{19}+\frac{1}{18}+\ldots+\frac{1}{1}\right)$$

Footnote

We know that:

$$1+x+x^2+x^3+x^4+x^5+\ldots=\frac{1}{1-x}\qquad \text{it is a geometric series}$$

Differentiate both sides and show that:

$$1+2x+3x^2+4x^3+5x^4+\ldots=\frac{1}{(1-x)^2}$$