Tiling

Example 1

I want to cover my 8×8 chess-board with thirty-two 2×1 tiles. Try it. It's easy.

Example 2

Someone has removed two squares from my chess-board. The bottom left-hand corner square and the top right-hand corner square. Can I cover my mutilated chess-board with thirty-one 2×1 tiles?

Each 2×1 tile covers one black square and one white square. So however I arrange the tiles, I will always cover the same number of black squares and white squares. The bottom left-hand corner square and the top right-hand corner are both black. So the mutilated chess-board has 30 black squares and 32 white squares. So I cannot cover my mutilated chess-board with thirty-one 2×1 tiles.

Example 3

Someone has removed two squares from my chess-board. Can I cover my mutilated chess-board with thirty-one 2×1 tiles?

The answer will depend on which squares have been removed. We know, from the previous example that if two black squares have been removed or if two white squares have been removed then the answer is: No!

Gomory's Theorem

I can cover my mutilated chess-board with thirty-one 2×1 tiles if any black square and any white square have been removed.

Proof

Here is a chess-board:

1	64	63	62	61	60	59	58
2	51	52	53	54	55	56	57
3	50	49	48	47	46	45	44
4	37	38	39	40	41	42	43
5	36	35	34	33	32	31	30
6	23	24	25	26	27	28	29
7	22	21	20	19	18	17	16
8	9	10	11	12	13	14	15

Look at the way I have numbered the squares. I can go for a walk around the board, visiting every square in the order 1, 2, 3, … 64

Say, the black square 24 and the white square 41 have been removed on my mutilated chess-board. I can cover the board as follows:

Put the first tile on squares 25 and 26, the next tile on squares 27 and 28, … on squares 39 and 40, the next tile on squares 42 and 43, the next tile on squares 44 and 45, … on squares 22 and 23.

This method will work whichever black square and whichever white square have been removed.

Investigation

Someone has removed two black squares and two white squares from my chess-board. Can I cover my mutilated chess-board with thirty 2×1 tiles?

Example 4

Can I cover a chess-board with 3×1 tiles?

Each tile covers 3 squares. There are 64 squares on the board. So the answer is: No!

Example 5

Someone has removed one square from my chess-board. Can I cover my mutilated chess-board with twenty-one 3×1 tiles?

The answer will depend on which square has been removed. Let's colour the squares white, red and black:

W	R	B	W	R	B	W	R
B	W	R	B	W	R	B	W
R	B	W	R	B	W	R	B
W	R	B	W	R	B	W	R
B	W	R	B	W	R	B	W
R	B	W	R	B	W	R	B
W	R	B	W	R	B	W	R
B	W	R	B	W	R	B	W

Each 3×1 tile covers one white square, one red square and one black square. So however I arrange the tiles, I will always cover the same number of white squares, red squares and black squares. The chess-board (before the square has been removed) has 22 white squares, 21 red squares and 21 black squares. So the removed square must be white. But can it be any white square?

If I could cover the mutilated chess-board with the bottom right-hand corner square removed then I could rotate the board $90°$ clockwise and I would have covered the mutilated chess-board with the bottom left-hand corner removed. But we know that this is not possible.

To have any hope of covering my mutilated chess-board, the removed square must be one of the * squares:

		*			*		
		*			*		

Can I can cover my mutilated chess-board if the removed square is one of the * squares?

I don’t know. Try it.

Example 6

This final example is a bit different and involves no mutilation.

I have a 100×101 board.

I have lots of 2×2 tiles 4×4 tiles 6×6 tiles and 13×13 tiles available.

Can I cover my board with tiles?

We are going to colour the squares black and white, but in an unusual way:

B	B	B	B	B	B	...
W	W	W	W	W	W	...
B	B	B	B	B	B	...
W	W	W	W	W	W	...
B	B	B	B	B	B	...
W	W	W	W	W	W	...
...	...	...	...	...	...	...

Each 2×2 tile covers 2 black squares and 2 white squares.

Each 4×4 tile covers 8 black squares and 8 white squares.

Each 6×6 tile covers 18 black squares and 18 white squares.

Each 13×13 tile covers 91 black squares and 78 white squares or 78 black squares and 91 white squares. So the difference between the number of black squares I can cover and the number of white squares I can cover must be a multiple of 13. But the board has 5100 black squares and 5000 white squares. So the answer is: No!