

## Maclaurin Series

if you know about differentiation and integration ...

Sometimes we can write a function  $f(x)$  as a power series:

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

Put  $x=0$

$$f(0) = a$$

Differentiate

$$f'(x) = b + 2cx + 3dx^2 + 4ex^3 + \dots$$

Put  $x=0$

$$f'(0) = b$$

Differentiate

$$f''(x) = 2c + (3 \times 2)dx + (4 \times 3)ex^2 + \dots$$

Put  $x=0$

$$f''(0) = 2c$$

Differentiate

$$f'''(x) = (3 \times 2)d + (4 \times 3 \times 2)ex + \dots$$

Put  $x=0$

$$f'''(0) = (3 \times 2)d$$

etc

$$\text{So } f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

### Example 1

Lets try this out with  $f(x) = \sin x$

$$f(x) = \sin x$$

$$f(0) = 0$$

$$f'(x) = \cos x$$

$$f'(0) = 1$$

$$f''(x) = -\sin x$$

$$f''(0) = 0$$

$$f'''(x) = -\cos x$$

$$f'''(0) = -1$$

$$f^{(4)}(x) = \sin x$$

$$f^{(4)}(0) = 0$$

etc

$$\text{So } \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

Find a graph plotter online and plot the following graphs:

$$y = x$$

$$y = x - \frac{1}{3!}x^3$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

etc

And see how these graphs get more and more like  $y = \sin x$

### Example 2

Show that:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots$$

### Example 3

Show that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

### Example 4

There is a sneaky way to find the Maclaurin series for  $\ln(1+x)$

We start with:

$$1 - x + x^2 - x^3 + \dots = \frac{1}{1+x} \quad \text{it is a geometric series}$$

So:

$$\int \frac{1}{1+x} dx = \int (1 - x + x^2 - x^3 + \dots) dx$$

So:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad \text{this is only valid for } -1 < x \leq 1$$

### Example 5

There is also a sneaky way to find the Maclaurin series for  $\tan^{-1}x$

We start with:

$$1 - x^2 + x^4 - x^6 + \dots = \frac{1}{1+x^2} \quad \text{it is a geometric series}$$

So:

$$\int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + \dots) dx$$

So:

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad \text{this is only valid for } -1 \leq x \leq 1$$

As a bonus:

show that:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$