Group Theory

a) Groups

We can combine two numbers, using addition, to get another number:

$5+7=12$

We can combine two sets, using union, to get another set:

$(a,b,e,g)\cup(a,c,e,h,k)=(a,b,c,e,g,h,k)$

etc

A binary operation * combines two “things” to get another “thing”.

A binary operation * is commutative if $p*q$ is always the same as $q*p$

For example, if we are combining numbers:

addition is commutative	$4+8=8+4$
subtraction is not commutative	$10-3\neq 3-10$
multiplication is commutative	$3\times 5=5\times 3$
division is not commutative	$24\div 6\neq 6\div 24$

A binary operation * is associative if $p*(q*r)$ is always the same as $(p*q)*r$

For example, if we are combining numbers:

addition is associative	$4+(3+8)=(4+3)+8$
subtraction is not associative	$20-(12-8)\neq(20-12)-8$
multiplication is associative	$3\times(4\times 5)=(3\times 4)\times 5$
division is not associative	$24\div(6\div 2)\neq(24\div 6)\div 2$

Example 1

Set $\{1,2,3,4,5,6\}$

Binary operation * where $p*q=pq, mod\, 7$

For example:

$5*6=5\times 6=30=2, mod\, 7$

Here is the combination table:

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3

5	5	3	1	6	4	2
6	6	5	4	3	2	1

Note: $5*6$ appears in the 5 row and the 6 column etc

Note: the binary operation * is commutative because $5*6=6*5$ etc

(a) The set is closed under the binary operation. This means:

For all p and q in the set $p*q$ is in the set.

So all the numbers in the combination table are in the set.

(b) The set contains an identity element e This means:

For all p in the set $p*e=p$ and $e*p=p$

Here the identity element is 1

$1*1=1 \quad 2*1=2 \quad 3*1=3 \quad 4*1=4 \quad 5*1=5 \quad 6*1=6$

$1*1=1 \quad 1*2=2 \quad 1*3=3 \quad 1*4=4 \quad 1*5=5 \quad 1*6=6$

(c) Every element in the set has an inverse element in the set. This means:

For all p in the set there is an element p' in the set where $p*p'=e$ and $p'*p=e$

1 is it's own inverse $1*1=1$

2 and 4 are inverses $2*4=1$ and $4*2=1$

3 and 5 are inverses $3*5=1$ and $5*3=1$

6 is it's own inverse $6*6=1$

(d)The binary operation is associative.

You can check this for the above combination table.

Rules for a group:

A set of elements and a binary operation * is a group if:

The set is closed under *

The set contains an identity element

Every element in the set has an inverse element in the set

* is associative

So the set $\{1,2,3,4,5,6\}$ with the binary operation * where $p*q=pq, mod\,7$ is a group.

See Exercise

EXERCISE

1) Set $\{0,1,2,3\}$

Binary operation * where $p*q=p+q, mod\, 4$

Complete the combination table and show we have a group.

2) We have these functions: $e(x)=x$ $f(x)=\frac{1}{x}$ $g(x)=-x$ $h(x)=-\frac{1}{x}$

Set $\{e,f,g,h\}$

Binary operation * where $f(x)*g(x)=f(g(x))$

Complete the combination table and show we have a group.

SOLUTIONS

1)

*	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Closed: all the numbers in the combination table are in the set.

Identity: 0

Inverses: 0 is its own inverse 1 and 3 are inverses 2 is its own inverse

Associative: you can check this for the above combination table.

2)

*	e	f	g	h
e	e	f	g	h
f	f	e	h	g
g	g	h	e	f
h	h	g	f	e

Closed: all the functions in the combination table are in the set.

Identity: e

Inverses: every function is its own inverse

Associative: you can check this for the above combination table.