Rearrangements

I have three ornaments in a line on my mantelpiece.

Let's call the left hand end of the mantelpiece, position 1. The middle, position 2 and the right hand end of the mantelpiece, position 3

Occasionally I decide to rearrange these ornaments. This means that I put them in a different order on the mantelpiece. The possible rearrangements are:

$P1$ Don't do anything

$P2$ Swap over the ornaments in positions 2 and 3

$P3$ Swap over the ornaments in positions 1 and 3

$P4$ Swap over the ornaments in positions 1 and 2

$P5$ Move each ornament one position to the left. The ornament that started in position 1 falls off the mantelpiece and is then put in position 3

$P6$ Move each ornament one position to the right. The ornament that started in position 3 falls off the mantelpiece and is then put in position 1

Let's call the ornaments A and B and C

If the ornaments start in the order A, B, C and I do $P5$ they will end up in the order B, C, A

If the ornaments start in the order C, B, A and I do $P5$ they will end up in the order B, A, C

etc

We can combine rearrangements.

$P4*P2$ means you do $P2$ and then you do $P4$ This means you do $P2$ first.

Put the ornaments on the mantelpiece in any order.

If you do $P2$ and then do $P4$ they will end up in the same order as if you had just done $P6$

Try it.

So $P4*P2=P6$

Show that $P2*P4=P5$ So $P4*P2$ and $P2*P4$ are not the same.

* is not commutative.

Here is the combination table. You should check some of these.

*	P1	P2	P3	P4	P5	P6
P1	P1	P2	P3	P4	P5	P6
P2	P2	P1	P6	P5	P4	P3
P3	P3	P5	P1	P6	P2	P4
P4	P4	P6	P5	P1	P3	P2
P5	P5	P3	P4	P2	P6	P1
P6	P6	P4	P2	P3	P1	P5

Note $P2*P4$ goes in the P2 row and the P4 column.

And $P4*P2$ goes in the P4 row and the P2 column.

The set $\{P1, P2, P3, P4, P5, P6\}$ with the binary operation * forms a group.

A final thought …

Look at the chapter: Symmetry of a Triangle. We can pair-up these rearrangements with the symmetries of the triangle:

$P1 \to e \quad P2 \to p \quad P3 \to q \quad P4 \to r \quad P5 \to b \quad P6 \to a$

We find that these two groups are basically the same.

For example:

$P3*P5=P2$ and $q*b=p$

$P2*P4=P5$ and $p*r=b$

$P3*P2=P5$ and $q*p=b$

etc

We say these two groups are isomorphic which is a fancy way of saying they are basically the same.