Derangements

There are five people A, B, C, D, E and each person has a card.

A's card has a 1 printed on it, B's card has a 2 printed on it … etc

We collect in the cards, shuffle them up, and then give everyone a card.

How many possible derangements are there?

A derangement is where no-one ends up with their own card.

Let D_n be the number of ways of deranging n cards.

What if A swaps cards with another person?

For example, A gets card 3 and C gets card 1

We are now left with three people B, D, E and three cards 2, 4, 5

B could get card 4 or 5 but not card 2

D could get card 2 or 5 but not card 4

E could get card 2 or 4 but not card 5

This gives us D_3 possible derangements.

A gets card 2 and B gets card 1	D_3 derangements
A gets card 3 and C gets card 1	D_3 derangements
A gets card 4 and D gets card 1	D_3 derangements
A gets card 5 and E gets card 1	D_3 derangements

This gives a total of $4D_3$ derangements

What if A does not swap cards with another person?

For example, A gets card 3 but C does not get card 1.

We are now left with four people B, C, D, E and four cards 1, 2, 4, 5

B could get card 1 or 4 or 5 but not card 2

C could get card 2 or 4 or 5 but not card 1 (because this would be a swap)

D could get card 1 or 2 or 5 but not card 4

E could get card 1 or 2 or 4 but not card 5

This gives us D(4) possible derangements.

A gets card 2 but B does not get card 1	D_4 derangements
A gets card 3 but C does not get card 1	D_4 derangements
A gets card 4 but D does not get card 1	D_4 derangements
A gets card 5 but E does not get card 1	D_4 derangements

This gives a total of $4D_4$ derangements

So $D_5 = 4D_3 + 4D_4$

In general:

$D_1 = 0$ and $D_2 = 1$ and $D_{n+2} = (n+1)D_n + (n+1)D_{n+1}$

We want a formula for D_n We will use the guess and prove method.

Guess:

$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + \frac{1}{n!}\right)$ (where did this come from?)

Proof:

$D_1 = 1!\left(1 - \frac{1}{1!}\right) = 0$ Correct

$D_2 = 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) = 1$ Correct

$(n+1)D_n + (n+1)D_{n+1} = \ldots = D_{n+2}$ Correct (you do this - it is very tedious!)

Now $\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$ (see chapter: e)

So if n is large $D_n \approx n!\left(\frac{1}{e}\right)$

Instead of asking about the number of derangements, we can ask about the probability of getting a derangement.

There are n people and each person owns a card. We collect in the cards, shuffle them up, and then give everyone a card. What is the probability of getting a derangement?

There are D_n ways of getting a derangement and there are $n!$ ways of giving out the cards.

So the probability of getting a derangement is $\frac{D_n}{n!}$

So if n is large, the probability of getting a derangement is approximately $1/e$

Exercise

1) I write out my thirty Christmas cards and then I write out all the envelopes. Foolishly, I put the cards into the envelopes at random.

What is the probability that all my friends get sent the wrong card?

2) You and I each have a pack of cards. Both packs are shuffled. We then play Snap. What is the probability that we get to the end of our packs with no snaps?

Answer is approximately $1/e$

3) Twenty students go to a party. When they arrive, each student drops their coat on the floor. When they leave, each student grabs a coat at random. What is the probability that no student gets their own coat?

Answer is approximately $1/e$

Solutions

1) Answer is approximately $1/e$

2) Answer is approximately $1/e$

3) Answer is approximately $1/e$