Proof By Induction

If a theorem is true when $n=1,2,3,\ldots$ then we might be able to prove it using proof by induction.

Theorem 1

$$1+2+3+\ldots+n=\frac{1}{2}n(n+1) \qquad \text{for } n=1,2,3,\ldots$$

Proof

part 1:

If $n=1$ then $LHS=1$ and $RHS=1$ So the formula is true when $n=1$

part 2:

If $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$ is true when $n=k$ then:

$$1+2+3+\ldots+k=\frac{1}{2}k(k+1)$$

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}(k(k+1)+2(k+1))$$

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}(k+1)(k+2)$$

So $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$ is true when $n=k+1$

End of proof

So what’s going on?

Part 1 shows that the theorem is true when $n=1$

Part 2 shows that if the theorem is true when $n=1$ then the theorem is true when $n=2$

So the theorem is true for $n=2$

Part 2 shows that if the theorem is true when $n=2$ then the theorem is true when $n=3$

So the theorem is true for $n=3$

etc

So we have shown the theorem must be true for all values of n Brilliant!

Theorem 2

9^n-1 is a multiple of 8 for $n=1,2,3,\ldots$

Proof

part 1:

If $n=1$ then $9^n-1=8$ So 9^n-1 is a multiple of 8 when $n=1$

part 2:

If 9^n-1 is a multiple of 8 when $n=k$ then:

9^k-1 is a multiple of 8

$9^k-1=8r$ for some integer r

Now $9^{k+1}-1=9(9^k)-1=9(9^k-1)+8=9(8r)+8=8(9r+1)$

So 9^n-1 is a multiple of 8 when $n=k+1$

End of proof

EXERCISE

Prove the following for $n=1,2,3,\ldots$

1) $1+3+5+\ldots+(2n-1)=n^2$

2) $1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$

SOLUTION

1) Proof

part 1:

If $n=1$ then $LHS=1$ and $RHS=1$ so the formula is true when $n=1$

part 2:

If $1+3+5+\ldots+(2n-1)=n^2$ is true when $n=k$ then:

$1+3+5+\ldots+(2k-1)=k^2$

$1+3+5+\ldots+(2k-1)+(2k+1)=k^2+(2k+1)$

$1+3+5+\ldots+(2k-1)+(2k+1)=(k+1)^2$

So $1+3+5+\ldots+(2n-1)=n^2$ is true when $n=k+1$

2) Proof

part 1:

If $n=1$ then $LHS=1$ and $RHS=1$ so the formula is true when $n=1$

part 2:

If $1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$ is true when $n=k$ then:

$1+2^1+2^2+2^3+...+2^k=2^{k+1}-1$

$1+2^1+2^2+2^3+...+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$

$1+2^1+2^2+2^3+...+2^k+2^{k+1}=2(2^{k+1})-1$

$1+2^1+2^2+2^3+...+2^k+2^{k+1}=2^{k+2}-1$

So $1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$ is true when $n=k+1$

Incidently, and this has nothing to do with proof by induction, we can prove:

$1+3+5+\ldots+(2n-1)=n^2$ with this diagram.

E	E	E	E	E
D	D	D	D	E
C	C	C	D	E
B	B	C	D	E
A	B	C	D	E

In this diagram we have got: one A, three Bs, five Cs, seven Ds, nine Es

How many letters have we got?

$1+3+5+7+9=5^2$

In general:

$1+3+5+...+(2n-1)=n^2$