Complex Numbers

Example

We can try to solve the quadratic equation

$$x^2-4x+13=0$$

using the quadratic formula

$$x=\frac{4\pm\sqrt{16-52}}{2}=\frac{4\pm\sqrt{-36}}{2}$$

At this point we give up because $\sqrt{-36}$ does not exist.

Let’s introduce a new number i where $i^2=-1$

Now

$$(6i)^2=6i\times 6i=36i^2=-36$$

So we can now solve our equation

$$x=\frac{4\pm 6i}{2}=2\pm 3i$$

Check:

If

$$x=2+3i$$

then

$$x^2=(2+3i)(2+3i)=4+6i+6i+9i^2=4+12i-9=-5+12i$$

So

$$x^2-4x+13=(-5+12i)-4(2+3i)+13=-5+12i-8-12i+13=0$$ Good.

And if

$x=2-3i$ etc

A number like $2+3i$ is called a complex number. We can add, subtract, multiply and divide complex numbers:

Addition

$$(3+7i)+(2-5i)=5+2i$$

Subtraction

$$(3+7i)-(2-5i)=1+12i$$

Multiplication

$$(3+7i)(2-5i)=6-15i+14i-35i^2=6-15i+14i+35=41-i$$

Division

Here we need a trick

$$\frac{(3+7i)}{(2-5i)}=\frac{(3+7i)(2+5i)}{(2-5i)(2+5i)}=...=\frac{-29+29i}{29}=-1+i$$

Squaring

$$(3+7i)^2=9+42i+49i^2=9+42i-49=-40+42i$$

Powers of i

$$i^3=(i^2)i=(-1)i=-i$$

$$i^4=(i^2)(i^2)=(-1)(-1)=1$$

$$i^5=(i^2)(i^2)i=(-1)(-1)i=i$$

$$i^{379}=...=i^3=-i$$

Quadratic equations

$$x^2-4x+29=0$$

$$x=\frac{4\pm\sqrt{-100}}{2}=2\pm5i$$

Real and imaginary parts.

We say 2 is the real part of $2+3i$ and we say 3 is the imaginary part of $2+3i$

Example

$2x+y+3i-4iy=10-5i$ where x and y are real numbers

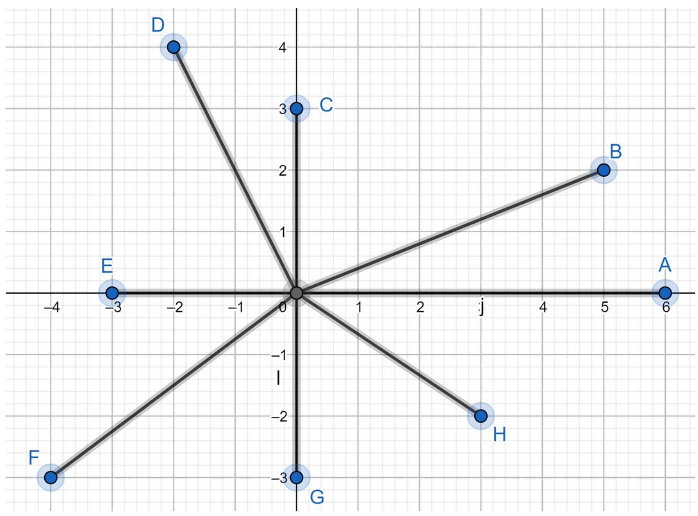
We can rewrite this as:

$$(2x+y)+i(3-4y)=(10)+i(-5)$$

If two complex numbers are equal then their real parts must be equal and their imaginary parts must be equal.

So $2x+y=10$ and $3-4y=-5$ so $x=4$ and $y=2$

We can think of a complex number as a point on a number plane:



A is the complex number: 6 B is the complex number: $5+2i$

C is the complex number: $3i$ D is the complex number: $-2+4i$

E is the complex number: -3 F is the complex number: $-4-3i$

G is the complex number: $-3i$ H is the complex number: $3-2i$

If $z=4+2i$ then:

a) $iz=i(4+2i)=-2+4i$

Draw a line from the origin O to z Draw a line from the origin O to iz

Multiplying z by i is the same as rotating Oz by $\frac{\pi}{2}$

b) $-z=-(4+2i)=-4-2i$

Multiplying z by -1 is the same as rotating Oz by π

c) $-iz=-i(4+2i)=2-4i$

Multiplying z by $-i$ is the same as rotating Oz by $-\frac{\pi}{2}$

Fundamental Theorem of Algebra

Without complex numbers, some polynomials can be factorised:

$x^2-5x+6=(x-2)(x-3)$

but other polynomials cannot be factorised:

$x^2-4x+13$

However, with complex numbers we have a nice result:

Every polynomial of degree n can be factorised into n brackets.

For example $x^4-4x^3+3x^2+2x-6=(x+1)(x-3)(x-1+i)(x-1-i)$

Footnote:

Did mathematicians invent complex numbers or did they invent them?

EXERCISE

1)

Evaluate

a) $(3+5i)+(-2+7i)$

b) $(5-3i)-(8+4i)$

c) $(1+3i)(5-2i)$

d) $\frac{(8+5i)}{(7+2i)}$ hint multiply top and bottom by $(7-2i)$

e) $(2-5i)^2$

2)

Solve

a) $x^2-6x+13=0$

b) $x^2-14x+58=0$

3)

Solve $3x+iy-6+2i=2ix+3y+8i$ where x and y are real numbers

SOLUTIONS

1)

a) $1+12i$

b) $-3-7i$

c) $11+13i$

d) $\frac{66}{53}+\frac{19}{53}i$

e) $-21-20i$

2)

a) $x=\frac{2\pm\sqrt{4-52}}{2}=\frac{2\pm\sqrt{-48}}{2}=1\pm12i$

b) $x=\frac{14\pm\sqrt{196-232}}{2}=\frac{14\pm\sqrt{-36}}{2}=7\pm3i$

3)

equating real parts:

$3x-6=3y$

equating imaginary parts:

$y+2=2x+8$

$x=-8 \quad y=-10$