Magic Squares

Example 1

Here is a 4×4 magic square:

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

The numbers in each column, each row and both diagonals add up to the same total.

Now $1+2+3+...+16=136$ Our sixteen numbers are arranged in four columns so the numbers in each column (and each row and both diagonals) must add up to $136/4=34$

Example 2

We want to arrange the numbers $1,2,3,...9$ into a 3×3 magic square.

Now $1+2+3+4+5+6+7+8+9=45$ Our nine numbers are arranged in three columns so the numbers in each column (and each row and both diagonals) must add up to $45/3=15$

Theorem

5 must go in the middle cell.

Proof

A	B	C
D	E	F
G	H	I

$A+E+I=15 \quad C+E+G=15 \quad B+E+H=15 \quad D+E+F=15$

So $A+E+I+C+E+G+B+E+H+D+E+F=60$

But $A+B+C+D+E+F+G+H+I=45$ so $E=5$

Show that we cannot put 9 in the same column/row/diagonal as 8 or 7 or 6 or 3.

Put 9 in a corner. Now 1 must go in the opposite corner. What numbers can go in the other corners?

Show that 9 cannot go in a corner. So 9 must go in the middle of a side.

Put 3 in a corner. Show that 3 cannot go in a corner. So 3 must go in the middle of a side.

Where can we put 8?

Now complete the magic square.

I got:

8	3	4
1	5	9
6	7	2

And let’s say you got:

4	3	8
9	5	1
2	7	6

We would say these are the same. Two magic squares are the same if we can change the first square into the second square by rotating the first square about its centre or by reflecting the first square about any mirror line passing through its centre.

A magic square will remain magic if:

we add k to all the numbers in the square, for any number k

we multiply all the numbers in the square by k for any number k

we swap 2 rows that are equidistant from the centre

we swap 2 columns that are equidistant from the centre

Try it!

Back to my 3×3 magic square.

If we add 9 to each number in my 3×3 magic square then we get a magic square with the numbers: 10 … 18

17	12	13
10	14	18
15	16	11

If we add another 9 to each number then we get a magic square with the numbers: 19 … 27

If we add another 9 to each number then we get a magic square with the numbers: 28 … 36

...

If we add another 9 to each number then we get a magic square with the numbers: 73 … 81

We can now assemble these nine magic squares into a 9×9 magic square:

(look carefully to see how I've arranged the nine magic squares)

71	66	67	26	21	22	35	30	31
64	68	72	19	23	27	28	32	36
69	70	65	24	25	20	33	34	29
8	3	4	44	39	40	80	75	76
1	5	9	37	41	45	73	77	81
6	7	2	42	43	38	78	79	74
53	48	49	62	57	58	17	12	13
46	50	54	55	59	63	10	14	18
51	52	47	60	61	56	15	16	11

Here is a method to find a $N \times N$ magic square if N is a multiple of 4.

Start with this square:

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Divide this square into four 4×4 squares.

Look at the numbers on the diagonals of these 4×4 squares:

1, 10, 19, 28 and 4, 11, 18, 25

5, 14, 23, 32 and 8, 15, 22, 29

33, 42, 51, 60 and 36, 43, 50, 57

37, 46, 55, 64 and 40, 47, 54, 61

Swap any two of these numbers that add up to 65.

swap: 1 and 64   swap: 10 and 55   swap: 19 and 46   etc

This gives us the magic square:

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

There are many other methods to find magic squares. Look them up.

Here is my favourite magic square:

7	53	41	27	2	52	48	30
12	58	38	24	13	63	35	17
51	1	29	47	54	8	28	42
64	14	18	36	57	11	23	37
25	43	55	5	32	46	50	4
22	40	60	10	19	33	61	15
45	31	3	49	44	26	6	56
34	20	16	62	39	21	9	59

It remains magic if all the numbers are squared!

EXERCISE

Arrange the numbers $1,2,3,...16$ into a 4×4 magic square.

SOLUTION

Start with the square:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

do the swaps to get:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Latin Squares

Example 1

Here is a 4×4 Latin square:

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

There are four symbols A, B, C, D and each symbol appears once (and only once) in each row and once (and only once) in each column.

Latin squares are used in agricultural research. We are growing oats in a field and we want to compare four types of fertilizer, A, B, C, D. We divide the field into 16 plots and apply the fertilizer as in the Latin square. Conditions (drainage etc) might vary across the field, so we want to try each type of fertilizer in each row and each column of the field.

One way to make a Latin square is the diagonal method:

A	B	C	D	E	F
F	A	B	C	D	E
E	F	A	B	C	D
D	E	F	A	B	C
C	D	E	F	A	B
B	C	D	E	F	A

Look and see how each letter is arranged along diagonals.

See EXERCISE 1

Sudoku Squares

Here is an incomplete sudoku square:

	H	I	E	G	C	D	F	B
F	G	D	I				A	E
E		C	A	F	D	H	I	
H	I				F	B	D	C
C	A	B	H	D		E		F
				E			H	
I		A			E		B	
B	E	G	D	I	A	F	C	
D				C		I	E	

There are nine symbols A, B, C, … I. The challenge is to fill in the square to make a 9×9 Latin square. But there is more. If you divide up the board into nine 3×3 squares then each 3×3 square must contain all nine symbols. So, for example, look at the top-left 3×3 square:

	H	I
F	G	D
E		C

The two empty cells must contain A and B.

The nine symbols in a sudoku square are usually 1, 2, 3, 4, 5, 6, 7, 8, 9. I've chosen to use letters instead because this puzzle is not really about numbers.

There are just 5,524,751,496,156,892,842,531,225,600 possible 9×9 Latin squares but only some of them are Sudoku squares.

See EXERCISE 2

EXERCISE 1

Arrange the letters A, B, C, D, E into a 5×5 Latin Square using the diagonal method.

EXERCISE 2

Complete the above sudoku square

SOLUTIONS 1

A	B	C	D	E
E	A	B	C	D
D	E	A	B	C
C	D	E	A	B
B	C	D	E	A

SOLUTIONS 2

A	H	I	E	G	C	D	F	B
F	G	D	I	B	H	C	A	E
E	B	C	A	F	D	H	I	G
H	I	E	G	A	F	B	D	C
C	A	B	H	D	I	E	G	F
G	D	F	C	E	B	A	H	I
I	C	A	F	H	E	G	B	D
B	E	G	D	I	A	F	C	H
D	F	H	B	C	G	I	E	A

Euler Squares

Example 1

Here are two 3×3 Latin squares:

A	B	C
C	A	B
B	C	A

a	b	c
b	c	a
c	a	b

We can combine them to form a 3×3 Euler square:

A, a	B, b	C, c
C, b	A, c	B,a
B, c	C, a	A, b

Each cell contains two symbols and no two cells contain the same two symbols.

Euler squares are used in agricultural research. We are growing wheat in a field and we want to compare three types of wheat, a, b, c and three types of fertilizer, A, B, C. We want to grow each type of wheat with each type of fertilizer. We divide the field into 9 plots and plant the wheat and apply the fertilizer as in the Euler square. Conditions (drainage etc) might vary across the field, so we want to try each type of wheat and each type of fertilizer in each row and column of the field.

There are no 2×2 Euler squares. Can you see why?

Euler tried, and failed, to find a 6×6 Euler square.

In 1901 a proof was discovered that 6×6 Euler squares do not exist.

In 1960 a proof was discovered that Euler squares exist for all sizes except 2×2 and 6×6

One way to make an Euler square is the double-diagonal method.

Example 2

A	B	C	D	E
E	A	B	C	D
D	E	A	B	C
C	D	E	A	B
B	C	D	E	A

a	b	c	d	e
b	c	d	e	a
c	d	e	a	b
d	e	a	b	c
e	a	b	c	d

The first Latin square is diagonal from top-left to bottom-right. The other Latin square is diagonal from top-right to bottom-left.

We can combine these two Latin squares to make an Euler square:

A, a	B, b	C, c	D, d	E, e
E, b	A, c	B, d	C, e	D, a
D, c	E, d	A, e	B, a	C, b
C, d	D, e	E, a	A, b	B, c
B, e	C, a	D, b	E, c	A, d

Unfortunately, this method does not always work.

Investigation: When will the double-diagonal method work?

There are many other methods to find Euler squares. Look them up. (Euler squares are also called Graeco-Latin squares)

EXERCISE

Take the 16 picture cards (jacks, queens, kings, aces of spades, hearts, clubs, diamonds) from a pack of cards. Arrange them in a 4×4 Euler square:

a card of each suit must appear in each row and each column

a card of each rank must appear in each row and each column

SOLUTIONS

There are many solutions. Here is mine. I tried the diagonal method but it did not work.

Ace, spade	King, heart	Queen, club	Jack, diamond
King, diamond	Ave club	Jack, heart	Queen, spade
Queen, heart	Jack, spade	Ace, diamond	King, club
Jack, club	Queen, diamond	King, spade	Ace, heart