Euler’s Zeta Function

Euler introduced the zeta function:

$$\zeta(x)=\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\frac{1}{5^x}+\frac{1}{6^x}+\frac{1}{7^x}+\frac{1}{8^x}+\ldots$$

where x is a real number and $x>1$ which guarantees the series converges.

Look at this infinite product of infinite series:

$$\left(1+\frac{1}{2^1}+\frac{1}{2^2}+\ldots\right)\left(1+\frac{1}{3^1}+\frac{1}{3^2}+\ldots\right)\left(1+\frac{1}{5^1}+\frac{1}{5^2}+\ldots\right)\left(1+\frac{1}{7^1}+\frac{1}{7^2}+\ldots\right)\left(1+\frac{1}{11^1}+\frac{1}{11^2}+\ldots\right)\ldots$$

where the denominators of the fractions are powers of the prime numbers.

First attempt:

Each bracket is a geometric series. So this infinite product is equal to:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)\ldots$$

Second attempt:

If we multiply out the brackets, we get a lot of fractions. All these fractions will have 1 as the numerator. No two fractions will have the same denominator. The denominator of each fraction will be a product of powers of primes. Every possible product of powers of primes will appear as a denominator. So this infinite product is equal to:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$$

Equating:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)\ldots=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$$

Now look at this infinite product of infinite series:

$$\left(1+\frac{1}{2^x}+\frac{1}{2^{2x}}+\ldots\right)\left(1+\frac{1}{3^x}+\frac{1}{3^{2x}}+\ldots\right)\left(1+\frac{1}{5^x}+\frac{1}{5^{2x}}+\ldots\right)\left(1+\frac{1}{7^x}+\frac{1}{7^{2x}}+\ldots\right)\left(1+\frac{1}{11^x}+\frac{1}{11^{2x}}+\ldots\right)\ldots$$

By repeating what we did above we (eventually) get:

$$\left(\frac{1}{1-\frac{1}{2^x}}\right)\left(\frac{1}{1-\frac{1}{3^x}}\right)\left(\frac{1}{1-\frac{1}{5}^x}\right)\left(\frac{1}{1-\frac{1}{7^x}}\right)\left(\frac{1}{1-\frac{1}{11^x}}\right)\ldots=\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\frac{1}{5^x}+\frac{1}{6^x}+\frac{1}{7^x}+\frac{1}{8^x}+\ldots \quad ***$$

Notice, the right-hand side is $\zeta(x)$

So we can write the zeta function in terms of primes:

$$\zeta(x)=\left(\frac{1}{1-\frac{1}{2^x}}\right)\left(\frac{1}{1-\frac{1}{3^x}}\right)\left(\frac{1}{1-\frac{1}{5^x}}\right)\left(\frac{1}{1-\frac{1}{7^x}}\right)\left(\frac{1}{1-\frac{1}{11^x}}\right)\dots$$

Note:

In the chapter, Euler's Sine Formula, we got the result:

$$\zeta(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots=\frac{\pi^2}{6}$$

So:

$$\frac{\pi^2}{6}=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{1}{1-\frac{1}{3^2}}\right)\left(\frac{1}{1-\frac{1}{5^2}}\right)\left(\frac{1}{1-\frac{1}{7^2}}\right)\left(\frac{1}{1-\frac{1}{11^2}}\right)\dots$$

And we have a formula for π in terms of primes.

Note:

If we sub $x=1$ into ***

We know the RHS diverges, so the LHS diverges, so there must be an infinite number of primes!