e

Here are some nice formulas for e – there are lots more:

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

$$\frac{1}{e}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots$$

If $n\to\infty$ then $\left(1+\frac{1}{n}\right)^n\to e$

If $n\to\infty$ then $\left(1-\frac{1}{n}\right)^n\to\frac{1}{e}$

If $n\to\infty$ then $\frac{n}{(n!)^{1/n}}\to e$

Example 1

e and compound interest

I invest £1 for one year. How much is my investment worth if ...

a) the interest rate is 100% per year

answer $£(1+1)^1$

b) the interest rate is 50% per $1/2$ year

answer $£\left(1+\frac{1}{2}\right)^2$

c) the interest rate is 10% per $1/10$ year

answer $£\left(1+\frac{1}{10}\right)^{10}$

d) the interest rate is 5% per $1/20$ year

answer $£\left(1+\frac{1}{20}\right)^{20}$

e) the interest rate is $(100/n)\%$ per $1/n$ year

answer $£\left(1+\frac{1}{n}\right)^n$

Note:

as $n\to\infty$ so $answer\to£e$

Example 2

e and arranging ornaments

I have 3 ornaments. In how many ways can I arrange some (or none) ornaments in a line on my mantelpiece? see chapter: Arrangements and Selections

There are $(3P0)$ arrangements with no ornaments.

There are $(3P1)$ arrangements with one ornament.

There are $(3P2)$ arrangements with two ornaments.

There are $(3P3)$ arrangements with three ornaments.

The total number of arrangements is:

$$(3P0)+(3P1)+(3P2)+(3P3)=\frac{3!}{3!}+\frac{3!}{2!}+\frac{3!}{1!}+3!=3!\left(\frac{1}{3!}+\frac{1}{2!}+\frac{1}{1!}+1\right)=3!\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\right)$$

In general:

If I have n ornaments then the number of arrangements is:

$$n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)$$

If n is large then the number of arrangements is about $n!e$

Example 3

Theorem

$e<3$

Proof:

$$e=1+\left(1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right)<1+\left(1+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\right)=1+\frac{1}{1-1/2}=3$$

Example 4

Theorem

e is irrational

Note:

$$\frac{7!}{8!}+\frac{7!}{9!}+\frac{7!}{10!}+\ldots=\left(\frac{1}{8}\right)+\left(\frac{1}{8\times 9}\right)+\left(\frac{1}{8\times 9\times 10}\right)+\ldots<\frac{1}{8^1}+\frac{1}{8^2}+\frac{1}{8^3}+\ldots=\frac{\frac{1}{8}}{\left(1+\frac{1}{8}\right)}=\frac{1}{7}$$

So:

$$\frac{7!}{8!}+\frac{7!}{9!}+\frac{7!}{10!}+\ldots<\frac{1}{7}$$

Proof (by contradiction)

Assume e is rational, say $e=\frac{19}{7}$

Now:

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

So:

$$\frac{19}{7}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

So:

$$\frac{19}{7}\times 7!=\left(7!+\frac{7!}{1!}+\frac{7!}{2!}+\frac{7!}{3!}+\ldots+\frac{7!}{7!}\right)+\left(\frac{7!}{8!}+\frac{7!}{9!}+\frac{7!}{10!}+\ldots\right)$$

The LHS is an integer and the first bracket on the RHS is an integer but the second bracket isn't – see above.

Contradiction.

If we assume:

$e=\frac{p}{q}$ where p and q are any positive integers, then we can repeat the above argument and again get a contradiction. So e must be irrational.

It has been proved that e π and e^{π} are irrational

We do not know about $\pi+e$ πe π^{π} e^{e} π^{e}

Example 5 if you know about differentiation ...

Investigation

$2^4=4^2$ Can you think of another pair of positive integers with this property?

Hint:

If:

$$p^q=q^p$$

show that:

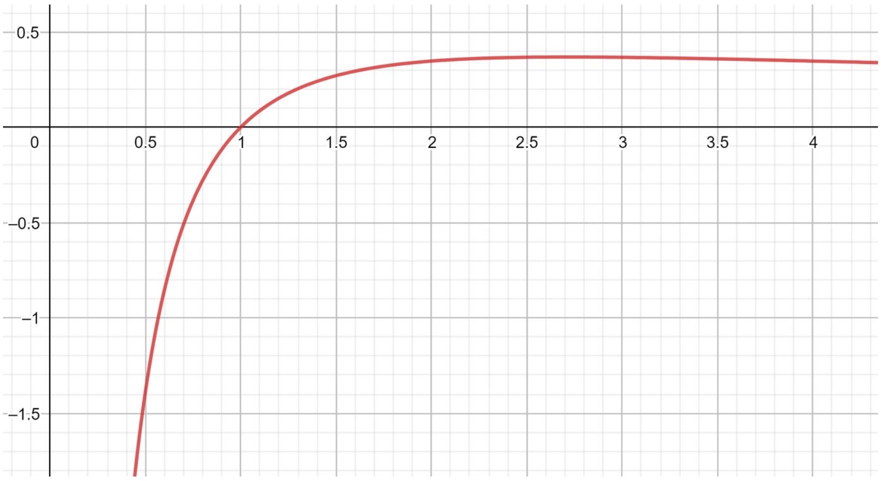
$$q\ln p=p\ln q$$

so:

$$\frac{\ln p}{p}=\frac{\ln q}{q}$$

Here is the graph:

$$y=\frac{\ln x}{x}$$



We want two different x values, call them p and q with the same y value.

So we want to be able to draw a horizontal line that cuts the graph twice.

Use differentiation to show that the maximum point on the graph occurs at $x=e$

Hence show that $2^4=4^2$ is the only solution of $p^q=q^p$ if p and q are positive integers.

Also

The maximum on the graph occurs at $x=e$

So:

$\frac{\ln e}{e}>\frac{\ln x}{x}$ for any x value, in particular $\frac{\ln e}{e}>\frac{\ln \pi}{\pi}$

Show:

$\pi \ln e > e \ln \pi$

Show:

$e^{\pi}>\pi^{e}$