

Euler's Prime Sum

In the chapter, Euler's Zeta Function, we got the result:

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\left(\frac{1}{1-\frac{1}{11}}\right)\dots = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Taking logs of both sides:

$$\ln\left(\frac{1}{1-\frac{1}{2}}\right) + \ln\left(\frac{1}{1-\frac{1}{3}}\right) + \ln\left(\frac{1}{1-\frac{1}{5}}\right) + \ln\left(\frac{1}{1-\frac{1}{7}}\right) + \dots = \ln\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots\right)$$

So:

$$\sum_{primes} \ln\left(\frac{1}{1-\frac{1}{p}}\right) = \ln \sum_1^{\infty} \frac{1}{n} \quad *$$

Now:

$$\left(\frac{1}{1-\frac{1}{p}}\right) = \left(1-\frac{1}{p}\right)^{-1} \quad \text{So} \quad \ln\left(\frac{1}{1-\frac{1}{p}}\right) = -\ln\left(1-\frac{1}{p}\right)$$

So we can write * as:

$$\sum_{primes} -\ln\left(1-\frac{1}{p}\right) = \ln \sum_1^{\infty} \frac{1}{n} \quad **$$

Now:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad \text{this is the Maclaurin series}$$

$$\text{Put } x = -\frac{1}{p}$$

We get:

$$\ln\left(1-\frac{1}{p}\right) = -\frac{1}{p} - \frac{1}{2p^2} - \frac{1}{3p^3} - \frac{1}{4p^4} - \dots \quad \text{So} \quad -\ln\left(1-\frac{1}{p}\right) = \frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots$$

So we can write ** as:

$$\sum_{primes} \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots\right) = \ln \sum_1^{\infty} \frac{1}{n}$$

So:

$$\sum_{primes} \frac{1}{p} + \sum_{primes} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots\right) = \ln \sum_1^{\infty} \frac{1}{n} \quad ***$$

Consider:

$$\sum_{primes} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots \right)$$

Now:

$$\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots < \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \dots = \frac{\left(\frac{1}{p^2}\right)}{\left(1 - \frac{1}{p}\right)} = \dots = \frac{1}{p-1} - \frac{1}{p}$$

So:

$$\sum_{primes} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots \right) < \sum_{primes} \left(\frac{1}{p-1} - \frac{1}{p} \right) < \sum_2^{\infty} \left(\frac{1}{p-1} - \frac{1}{p} \right) = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) = \dots = 1$$

Look at *** again:

$$\sum_{primes} \frac{1}{p} + \sum_{primes} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots \right) = \ln \sum_1^{\infty} \frac{1}{n}$$

We know:

$$\ln \sum_1^{\infty} \frac{1}{n} \text{ diverges.}$$

We have just shown that:

$$\sum_{primes} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots \right) \text{ converges.}$$

So we can deduce that:

$$\sum_{primes} \frac{1}{p} \text{ diverges.}$$

So:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \dots \text{ diverges. Astonishing!}$$