Proof by Contradiction

To prove a theorem is true, we assume it is false and then show that this cannot be the case as it leads to a contradiction.

Theorem 1

$4x-2y=1$ has no solution where x and y are integers.

Proof

Assume we have found a solution where x and y are integers:

LHS is even. RHS is odd.

Contradiction.

Theorem 2

No prime (except 3) is one less than a square.

Proof

Assume the prime p is one less than the square n^2

So:

$p=n^2-1=(n-1)(n+1)$

So:

the prime p can be written as the product of two integers.

Contradiction. Unless $(n-1)=1$ So $n=2$ so $p=3$

Theorem 3

$\log 5$ is irrational

Proof

Assume $\log 5$ is rational

So:

$\log 5=\frac{p}{q}$ where p and q are positive integers.

So:

$5=10^{p/q}$ So $5^q=10^p$

Now:

LHS is odd. RHS is even.

Contradiction

Theorem 4

$\sqrt{2}$ is irrational

Proof

Assume $\sqrt{2}$ is rational

So:

$\sqrt{2}=\frac{p}{q}$ where p and q are integers

We can say p and q are not both multiples of 2 because if they had both been multiples of 2 then we would have cancelled them down before we started.

Now:

$2q^2=p^2$

So:

p^2 is a multiple of 2. So p is a multiple of 2. Let $p=2r$

So:

$2q^2=4r^2$ So $q^2=2r^2$ So q^2 is a multiple of 2. So q is a multiple of 2.

So:

p and q are not both multiples of 2 but p is a multiple of 2 and q is a multiple of 2.

Contradiction.

Proof By Induction

If a theorem is true when $n=1,2,3,\ldots$ then we might be able to prove it using proof by induction.

Theorem 1

$$1+2+3+\ldots+n=\frac{1}{2}n(n+1) \qquad \text{for } n=1,2,3,\ldots$$

Proof

part 1:

If $n=1$ then $LHS=1$ and $RHS=1$ So the formula is true when $n=1$

part 2:

If $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$ is true when $n=k$ then:

$$1+2+3+\ldots+k=\frac{1}{2}k(k+1)$$

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}(k(k+1)+2(k+1))$$

$$1+2+3+\ldots+k+(k+1)=\frac{1}{2}(k+1)(k+2)$$

So $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$ is true when $n=k+1$

End of proof

So what’s going on?

Part 1 shows that the theorem is true when $n=1$

Part 2 shows that if the theorem is true when $n=1$ then the theorem is true when $n=2$

So the theorem is true for $n=2$

Part 2 shows that if the theorem is true when $n=2$ then the theorem is true when $n=3$

So the theorem is true for $n=3$

etc

So we have shown the theorem must be true for all values of n Brilliant!

Theorem 2

$$9^n-1 \text{ is a multiple of 8} \qquad \text{for } n=1,2,3,\ldots$$

Proof

part 1:

If $n=1$ then $9^n-1=8$ So 9^n-1 is a multiple of 8 when $n=1$

part 2:

If 9^n-1 is a multiple of 8 when $n=k$ then:

9^k-1 is a multiple of 8

$9^k-1=8r$ for some integer r

Now $9^{k+1}-1=9(9^k)-1=9(9^k-1)+8=9(8r)+8=8(9r+1)$

So 9^n-1 is a multiple of 8 when $n=k+1$

End of proof

EXERCISE

Prove the following for $n=1,2,3,\ldots$

1) $1+3+5+\ldots+(2n-1)=n^2$

2) $1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$

SOLUTION

1) Proof

part 1:

If $n=1$ then $LHS=1$ and $RHS=1$ so the formula is true when $n=1$

part 2:

If $1+3+5+\ldots+(2n-1)=n^2$ is true when $n=k$ then:

$1+3+5+\ldots+(2k-1)=k^2$

$1+3+5+\ldots+(2k-1)+(2k+1)=k^2+(2k+1)$

$1+3+5+\ldots+(2k-1)+(2k+1)=(k+1)^2$

So $1+3+5+\ldots+(2n-1)=n^2$ is true when $n=k+1$

2) Proof

part 1:

If $n=1$ then $LHS=1$ and $RHS=1$ so the formula is true when $n=1$

part 2:

If $1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$ is true when $n=k$ then:

$1+2^1+2^2+2^3+...+2^k=2^{k+1}-1$

$1+2^1+2^2+2^3+...+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$

$1+2^1+2^2+2^3+...+2^k+2^{k+1}=2(2^{k+1})-1$

$1+2^1+2^2+2^3+...+2^k+2^{k+1}=2^{k+2}-1$

So $1+2^1+2^2+2^3+...+2^n=2^{n+1}-1$ is true when $n=k+1$

Incidently, and this has nothing to do with proof by induction, we can prove:

$1+3+5+\ldots+(2n-1)=n^2$ with this diagram.

E	E	E	E	E
D	D	D	D	E
C	C	C	D	E
B	B	C	D	E
A	B	C	D	E

In this diagram we have got: one A, three Bs, five Cs, seven Ds, nine Es

How many letters have we got?

$1+3+5+7+9=5^2$

In general:

$1+3+5+...+(2n-1)=n^2$

Proving The Contrapositive

In the chapter: If … Then, we showed that $p \Rightarrow q$ is the same as $q' \Rightarrow p'$

So:

to prove $p \Rightarrow q$ we can prove $q' \Rightarrow p'$ instead.

Note:

$q' \Rightarrow p'$ is called the contrapositive of $p \Rightarrow q$

Theorem

n^2 is even $\Rightarrow$ n is even

We are going to prove:

n is odd $\Rightarrow$ n^2 is odd

Proof

n is odd $\Rightarrow$ $n=(2k+1)$ for some integer k

So:

$n^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1$

So:

n^2 is odd

End of proof

Proof using Selections

Example 1

We have 10 books and we want to select 4 of them.

Method 1:

We select 4 books and keep them.

The number of ways this can be done is: $(10C4)$

Method 2:

We select 6 books and throw them away (keeping the remaining 4 books)

The number of ways this can be done is: $(10C6)$

So $(10C4)=(10C6)$

In general:

$(mCk)=(mC(m-k))$

Example 2

We have 12 books and we want to select 5 of them.

Method 1:

The number of ways this can be done is: $(12C5)$

Method 2:

One of the books is Alice's Adventures In Wonderland.

There are $(11C4)$ selections which include Alice's Adventures In Wonderland.

There are $(11C5)$ selections which exclude Alice's Adventures In Wonderland.

So there are $(11C4)+(11C5)$ selections in total.

So $(11C4)+(11C5)=(12C5)$

In general:

$(mCk)+(mC(k+1))=((m+1)C(k+1))$

Example 3

We have 4 books and we want to select some (or none) of them.

Method 1:

There are $(4C0)$ selections of 0 books.

There are $(4C1)$ selections of 1 book.

There are $(4C2)$ selections of 2 books.

There are $(4C3)$ selections of 3 books.

There are $(4C4)$ selections of 4 books.

So there are $(4C0)+(4C1)+(4C2)+(4C3)+(4C4)$ selections in total.

Method 2:

For each book, there are 2 choices. Either the book is selected or the book is not selected.

So there are 2^4 selections in total.

So $(4C0)+(4C1)+(4C2)+(4C3)+(4C4)=2^4$

In general:

$(nC0)+(nC1)+(nC2)+...+(nCn)=2^n$