

## Proof by Contradiction

To prove a theorem is true, we assume it is false and then show that this cannot be the case as it leads to a contradiction.

### Theorem 1

$4x - 2y = 1$  has no solution where  $x$  and  $y$  are integers.

Proof

Assume we have found a solution where  $x$  and  $y$  are integers:

LHS is even. RHS is odd.

Contradiction.

### Theorem 2

No prime (except 3) is one less than a square.

Proof

Assume the prime  $p$  is one less than the square  $n^2$

So:

$$p = n^2 - 1 = (n - 1)(n + 1)$$

So:

the prime  $p$  can be written as the product of two integers.

Contradiction. Unless  $(n - 1) = 1$  So  $n = 2$  so  $p = 3$

### Theorem 3

$\log 5$  is irrational

Proof

Assume  $\log 5$  is rational

So:

$$\log 5 = \frac{p}{q} \text{ where } p \text{ and } q \text{ are positive integers.}$$

So:

$$5 = 10^{p/q} \text{ So } 5^q = 10^p$$

Now:

LHS is odd. RHS is even.

Contradiction

#### Theorem 4

$\sqrt{2}$  is irrational

#### Proof

Assume  $\sqrt{2}$  is rational

So:

$$\sqrt{2} = \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers}$$

We can say  $p$  and  $q$  are not both multiples of 2 because if they had both been multiples of 2 then we would have cancelled them down before we started.

Now:

$$2q^2 = p^2$$

So:

$p^2$  is a multiple of 2. So  $p$  is a multiple of 2. Let  $p = 2r$

So:

$$2q^2 = 4r^2 \text{ So } q^2 = 2r^2 \text{ So } q^2 \text{ is a multiple of 2. So } q \text{ is a multiple of 2.}$$

So:

$p$  and  $q$  are not both multiples of 2 but  $p$  is a multiple of 2 and  $q$  is a multiple of 2.

Contradiction.