

## Rearrangements

I have three ornaments in a line on my mantelpiece.

Let's call the left hand end of the mantelpiece, position 1. The middle, position 2 and the right hand end of the mantelpiece, position 3

Occasionally I decide to rearrange these ornaments. This means that I put them in a different order on the mantelpiece. The possible rearrangements are:

$P1$  Don't do anything

$P2$  Swap over the ornaments in positions 2 and 3

$P3$  Swap over the ornaments in positions 1 and 3

$P4$  Swap over the ornaments in positions 1 and 2

$P5$  Move each ornament one position to the left. The ornament that started in position 1 falls off the mantelpiece and is then put in position 3

$P6$  Move each ornament one position to the right. The ornament that started in position 3 falls off the mantelpiece and is then put in position 1

Let's call the ornaments  $A$  and  $B$  and  $C$

If the ornaments start in the order  $A, B, C$  and I do  $P5$  they will end up in the order  $B, C, A$

If the ornaments start in the order  $C, B, A$  and I do  $P5$  they will end up in the order  $B, A, C$   
etc

We can combine rearrangements.

$P4 * P2$  means you do  $P2$  and then you do  $P4$  This means you do  $P2$  first.

Put the ornaments on the mantelpiece in any order.

If you do  $P2$  and then do  $P4$  they will end up in the same order as if you had just done  $P6$   
Try it.

So  $P4 * P2 = P6$

Show that  $P2 * P4 = P5$  So  $P4 * P2$  and  $P2 * P4$  are not the same.

$*$  is not commutative.

Here is the combination table. You should check some of these.

*	P1	P2	P3	P4	P5	P6
P1	P1	P2	P3	P4	P5	P6
P2	P2	P1	P6	P5	P4	P3
P3	P3	P5	P1	P6	P2	P4
P4	P4	P6	P5	P1	P3	P2
P5	P5	P3	P4	P2	P6	P1
P6	P6	P4	P2	P3	P1	P5

Note  $P2 * P4$  goes in the P2 row and the P4 column.

And  $P4 * P2$  goes in the P4 row and the P2 column.

The set  $\{P1, P2, P3, P4, P5, P6\}$  with the binary operation  $*$  forms a group.

A final thought ...

Look at the chapter: Symmetry of a Triangle. We can pair-up these rearrangements with the symmetries of the triangle:

$$P1 \rightarrow e \quad P2 \rightarrow p \quad P3 \rightarrow q \quad P4 \rightarrow r \quad P5 \rightarrow b \quad P6 \rightarrow a$$

We find that these two groups are basically the same.

For example:

$$P3 * P5 = P2 \quad \text{and} \quad q * b = p$$

$$P2 * P4 = P5 \quad \text{and} \quad p * r = b$$

$$P3 * P2 = P5 \quad \text{and} \quad q * p = b$$

etc

We say these two groups are isomorphic which is a fancy way of saying they are basically the same.