Probability

Here are 8 ways to calculate probabilities.

1. Counting

Example 1

I pick a card from a pack of cards. What is the probability I get a red picture card?

There are 52 possible outcomes – the 52 cards in the pack.

Each possible outcome is equally.

There are 6 desired outcomes – the 6 red picture cards.

So answer is $6/52$

Example 2

I roll two dice. What is the probability the sum of the scores is 8?

6		*				
5			*			
4				*		
3					*	
2						*
1						
	1	2	3	4	5	6

The numbers along the bottom of the grid are the possible scores on one dice and the numbers up the side of the grid are the possible scores on the other dice.

There are 36 possible outcomes – the 36 cells in the grid

Each possible outcome is equally likely.

There are 5 desired outcomes – the 5 cells marked with a *

So answer is $5/36$

see Exercise 1

2. Using a table

Example 3

We have a group of 50 students.

38 study Art. 32 study Biology. 24 study Art and Biology.

We put the information in a table:

	A	A'	
B	24		32
B'			
	38		50

A studies Art A' does not study Art

B studies Biology B' does not study Biology

50 goes in the bottom right-hand corner cell.

38 goes at the end of the A column. This is the total number of students who study Art.

32 goes at the end of the B row. This is the total number of students who study Biology.

24 goes in the A column and the B row because 24 students study both Art and Biology.

When I say 38 students study Art, this is the total number of students who study Art.

24 of these 38 students also study Biology. The other 14 of these 38 students do not study Biology.

So 14 goes in the A column and the B' row.

We can now complete the table

	A	A'	
B	24	8	32
B'	14	4	18
	38	12	50

The table shows the number of students in each category. We can use this table to read off probabilities:

The probability that the student studies Art:

$$p(A)=\frac{38}{50}$$

The probability that the student studies both Art and Biology:

$$p(A\cap B)=\frac{24}{50}$$

The probability that the student studies Art or Biology or both:

$$p(A \cup B) = \frac{24+14+8}{50} \quad \text{or if you prefer} \quad p(A \cup B) = \frac{50-4}{50}$$

The probability that the student studies Art given that they study Biology:

$$p(A \mid B) = \frac{24}{32}$$

etc

We can use this table to explain the laws of probability:

1. $p(A) + p(A') = \frac{38}{50} + \frac{12}{50} = \frac{50}{50} = 1$

In general:

$$p(A) + p(A') = 1$$

2. $p(A \cap B) + p(A \cap B') = \frac{24}{50} + \frac{14}{50} = \frac{38}{50} = p(A)$

In general:

$$p(A \cap B) + p(A \cap B') = p(A)$$

3. $p(A \cup B) = \frac{24}{50} + \frac{14}{50} + \frac{8}{50} = \frac{46}{50}$

And:

$$p(A) + p(B) - p(A \cap B) = \frac{38}{50} + \frac{32}{50} - \frac{24}{50} = \frac{46}{50}$$

In general:

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

4. $p(A \mid B) = \frac{24}{32} = \frac{(24/50)}{(32/50)} = \frac{p(A \cap B)}{p(B)}$

In general:

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} \quad \text{so} \quad p(A \cap B) = p(A \mid B) \times p(B)$$

5. $p(A \cup B) = \frac{24}{50} + \frac{14}{50} + \frac{8}{50} = \frac{46}{50} = 1 - \frac{4}{50}$

In general:

$$p(A\cup B)=1-p(A'\cap B')$$

6. $p(A\cap B)=\frac{24}{50}=1-\left(\frac{8}{50}+\frac{4}{50}+\frac{14}{50}\right)$

In general:

$$p(A\cap B)=1-p(A'\cup B')$$

We can divide all the numbers in the table by 50

	A	A'	
B	0.48	0.16	0.64
B'	0.28	0.08	0.36
	0.76	0.24	1

The table shows the probability of students in each category. We can use this table to read off probabilities:

The probability that the student studies Art:

$$p(A)=0.76$$

The probability that the student studies both Art and Biology:

$$p(A\cap B)=0.48$$

The probability that the student studies Art or Biology or both:

$p(A\cup B)=0.48+0.28+0.16$ or if you prefer $p(A\cup B)=1-0.08$

The probability that the student studies Art given that they study Biology:

$$p(A \mid B)=\frac{0.48}{0.64}$$

etc

Example 4

$p(A'\cap B)=0.3$ $p(A\cap B')=0.2$ $p(B')=0.6$

We can fill in these probabilities

	A	A'	
B		0.3	
B'	0.2		0.6
			1

then we can fill in the other probabilities:

	A	A'	
B	0.1	0.3	0.4
B'	0.2	0.4	0.6
	0.3	0.7	1

then we can read off any probability we want:

$$p(A')=0.7 \quad p(A\cap B')=0.2 \quad p(A'\cup B)=0.8 \quad p(A' \mid B)=\frac{0.3}{0.4} \text{ etc}$$

Example 5

$$p(A\cup B)=0.8 \quad p(A)=0.4 \quad p(A\cap B)=0.3$$

We can fill in these probabilities:

note: $p(A'\cap B')=1-p(A\cup B)=1-0.8=0.2$

	A	A'	
B	0.3		
B'		0.2	
	0.4		1

then we can fill in the other probabilities:

	A	A'	
B	0.3	0.4	0.7
B'	0.1	0.2	0.3
	0.4	0.6	1

then we can read off any probability we want.

Note:

Some people like to use Venn diagrams instead of tables but I prefer tables.

See Exercise 2

3. Using the laws of probability

Example 6

Every day I walk to work or I cycle to work.

The probability I walk is 0.8 and the probability I cycle is 0.2

If I walk, the probability I am late is 0.4 and if I cycle, the probability I am late is 0.3

(a) What is the probability I will be late?

(b) What is the probability I walked given that I was late?

a) Now:

$$p(walk \cap late) = p(late \mid walk) \times p(walk) = 0.8 \times 0.4 = 0.32$$

And:

$$p(cycle \cap late) = p(late \mid cycle)\, p(cycle) = 0.2 \times 0.3 = 0.06$$

So:

$$p(late) = p(walk \cap late) + p(cycle \cap late) = 0.32 + 0.06 = 0.38$$

b)

$$p(walk \mid late) = p\frac{(walk \cap late)}{p(late)} = \frac{0.32}{0.38}$$

see Exercise 3

4. Using a tree diagram

Example 7

Every day I walk to work or I cycle to work.

The probability I walk is 0.8 and the probability I cycle is 0.2

If I walk, the probability I am late is 0.4 and if I cycle, the probability I am late is 0.3

(a) What is the probability I will be late?

(b) What is the probability I walked given that I was late?

WE NEED A TREE DIAGRAM

Note:

$$p(walk \cap late) = p(walk) \times p(late \mid walk) = 0.8 \times 0.4 = 0.32$$

And:

$$p(cycle \cap late) = p(cycle) \times p(late \mid cycle) = 0.2 \times 0.3 = 0.06$$

So to find probabilities we just multiply the probabilities along the branches.

WE NEED A TREE DIAGRAM

a) $p(late) = 0.32 + 0.06 = 0.38$

b) $p(walk \mid late) = \dfrac{p(walk \cap late)}{p(late)} = \dfrac{0.32}{0.38}$

see Exercise 4

5. Considering a number of cases.

Example 8

Every day I walk to work or I cycle to work.

The probability I walk is 0.8 and the probability I cycle is 0.2

If I walk, the probability I am late is 0.4 and if I cycle, the probability I am late is 0.3

(a) What is the probability I will be late?

(b) What is the probability I walked given that I was late?

Consider 100 days. On average:

I will walk on 80 days and be late on $0.4 \times 80 = 32$ of these walking days.

I will cycle on 20 days and be late on $0.3 \times 20 = 6$ of these cycling days.

(a) I will be late on 38 days out of 100 days.

Answer: $38/100$

(b) I will walk and be late on 32 days. I will be late on 38 days.

Answer: $32/38$

EXPECTED FREQ TREE DIAGRAM? SEE MATHS CAFE NOTES

see Exercise 5

6. Using Arrangements and Selections.

See chapter: Arrangements and Selections

See Exercise 6

7. Using computer simulation

Example 9

In a game of chuck-a-luck, I roll 3 dice.

If I get 3 sixes, then I win £3.

If I get 2 sixes, then I win £2.

If I get 1 six, then I win £1.

If I get 0 sixes, then I lose £1.

What will be my average winnings, per game, in the long run?

If you can’t work this out, you can do a simulation.

Write a computer program to play this game 1,000,000 times and record my total winnings. If my total winnings is, say £58743 then $£58743/1{,}000{,}000$ is an estimate of my average winnings per game.

See Appendix 1

8. Doing an experiment

I have a wooden cone. When I throw it up in the air it can land on its side or it can land on its base, point up. What is the probability that it lands on its side?

There are 2 possible outcomes but there is no reason to think that these outcomes are equally likely.

Perhaps, if I was good at Mechanics I could work it out. But I’m not.

I can do an experiment. I can throw it up in the air a very large number of times and record how many times it lands on its side.

See Exercise 7

EXERCISE 1

1.

I pick a card from a pack of cards. What is the probability I get:

a) a spade

b) a picture card

c) a spade and a picture card

d) a spade or a picture card

e) a spade or a picture card but not both

2.

I roll two dice. What is the probability:

a) the sum of the scores is 7

b) the product of the scores is 12

c) the difference of the scores is less than 4

d) I get at least one 6

EXERCISE 2

1.

There are 42 people at a party. 24 drink wine, 22 drink beer and 6 drink neither. Find the probability that a person at the party drinks wine and beer.

2.

There are 40 students in a class.

23 are girls, 17 are boys, 32 are right-handed and 3 are left-handed boys.

If I pick a left-hander, what is the probability they are a girl?

3.

$p(A\cap B)=0.2$ $p(A')=0.7$ $p(B')=0.4$ Find $p(B\rfloor A)$

4.

$p(A\cup B)=0.8$ $p(A\cap B)=0.6$ $p(B)=0.7$ Find $p(B\rfloor A')$

EXERCISE 3

1) The probability I revise for a test is 0.3

If I revise, the probability I pass is 0.9 and if I don’t revise, the probability I pass is 0.4

What is the probability I pass my next test?

2) 60% of students love jazz. Only $1/3$ of jazz lovers like carrots.

40% of students hate jazz. Only $1/4$ of jazz haters like carrots.

I meet a student who likes carrots. What is the probability they like jazz?

EXERCISE 4

1) The probability I revise for a test is 0.3

If I revise, the probability I pass is 0.9 and if I don’t revise, the probability I pass is 0.4

What is the probability I pass my next test?

2) 60% of students love jazz. Only $1/3$ of jazz lovers like carrots.

40% of students hate jazz. Only $1/4$ of jazz haters like carrots.

I meet a student who likes carrots. What is the probability they like jazz?

EXERCISE 5

1) The probability I revise for a test is 0.3

If I revise, the probability I pass is 0.9 and if I don't revise, the probability I pass is 0.4

What is the probability I pass my next test?

2) 60% of students love jazz. Only $1/3$ of jazz lovers like carrots.

40% of students hate jazz. Only $1/4$ of jazz haters like carrots.

I meet a student who likes carrots. What is the probability they like jazz?

EXERCISE 6

1) A bag contains 8 red counters and 15 blue counters. I take 5 counters out of the bag (without replacement) What is the probability I get 2 reds and 3 blues?

2) A bag contains 6 red counters and 43 blue counters. I take 6 counters out of the bag (without replacement) What is the probability I get:

(a) 6 reds (b) 4 reds, 2 blues (c) 2 reds and 4 blues

Why is this set-up the same as the lottery?

3) A bag contains 5 red, 7 green and 12 yellow counters. I take 5 counters out of the bag (without replacement) What is the probability I get:

(a) no reds (b) at least one red(c) all the same colour(d) 2 reds, 1 green, 2 yellows

4) In a game of bridge, there are 4 players and each player gets 13 cards. What is the probability that each player gets exactly one ace?

EXERCISE 7

Look at the questions in Dodgy Probability

SOLUTIONS 1

1.

a) $13/52$ b) $12/52$ c) $3/52$ d) $22/52$ e) $19/52$

2.

a) $6/36$ b) $4/36$ c) $30/36$ d) $11/36$

SOLUTIONS 2

1.

We can fill in these numbers:

	wine	not wine	
beer			22
not beer		6	
	24		42

then we can fill in the other numbers:

	wine	not wine	
beer	10	12	22
not beer	14	6	20
	24	18	42

Answer: $10/42$

2.

We can fill in these numbers:

	girl	boy	
right-handed			32
left-handed		3	
	23	17	40

then we can fill in the other numbers:

	girl	boy	
right-handed	18	14	32
left-handed	5	3	8
	23	17	40

Answer: $5/8$

3.

We can fill in these probabilities:

	A	A'	
B	0.2		
B'			0.4
		0.7	1

Then we can fill in the other probabilities

	A	A'	
B	0.2	0.4	0.6
B'	0.1	0.3	0.4
	0.3	0.7	1

$$p(B \rfloor A) = \frac{p(B \cap A)}{p(A)}$$

Answer: $0.2/0.3$

4.

We can fill in these probabilities:

	A	A'	
B	0.6		0.7
B'		0.2	
			1

Then we can fill in the other probabilities

	A	A'	
B	0.6	0.1	0.7
B'	0.1	0.2	0.3
	0.7	0.3	1

$$p(B \rfloor A') = \frac{p(B \cap A')}{p(A')}$$

Answer: $0.1/0.3$

SOLUTIONS 3

1.

R means I revise P means I pass

$p(R \cap P) = p(R) \times p(P \rfloor R) = 0.3 \times 0.9 = 0.27$

$p(R' \cap P) = p(R') \times p(P \rfloor R') = 0.7 \times 0.4 = 0.28$

$p(P) = p(R \cap P) + p(R' \cap P) = 0.27 + 0.28 = 0.55$

2.

J means likes jazz C means likes carrots

$p(J \cap C) = p(J) \times p(C \rfloor J) = 0.6 \times 1/3 = 0.2$

$p(J' \cap C) = p(J') \times p(C \rfloor J') = 0.4 \times 1/4 = 0.1$

$p(C) = p(J \cap C) + p(J' \cap C) = 0.2 + 0.1 = 0.3$

$p(J \rfloor C) = p\frac{(J \cap C)}{p(C)} = \frac{0.2}{0.3}$

SOLUTIONS 4

1.

TREE DIAGRAM

$p(pass) = 0.27 + 0.28 = 0.55$

2.

TREE DIAGRAM

$p(likes\ jazz \rfloor likes\, carrots) = \frac{p(likes\ jazz \cap likes\ carrots)}{p(likes\, carrots)} = \frac{0.2}{0.3}$

SOLUTIONS 5

1) Consider 100 tests. On average:

I revise for 30 tests and I pass 27 of these tests

I don’t revise for 70 tests and I pass 28 of these tests.

I pass 55 tests.

Answer: $55/100$

2) Consider 100 students. On average:

60 students love jazz and 20 of these students will like carrots.

40 students hate jazz and 10 of these students will like carrots.

30 students like carrots. Of these 20 like jazz.

Answer: $20/30$

SOLUTIONS 6

1)

There are (23C5) ways to select 5 counters.

There are $(8C2)(15C3)$ ways to select 2 reds and 3 blues.

Answer is: $\frac{(8C2)(15C3)}{(23C5)}$

2)

(a) $\frac{(6C6)}{(49C6)}$ (b) $\frac{(6C4)(43C2)}{(49C6)}$ (c) $\frac{(6C2)(43C4)}{(49C6)}$

In the lottery there are 6 winning balls and 43 non-winning balls.

3)

(a) $\frac{(19C5)}{(24C5)}$ (b) $1-\frac{(19C5)}{(24C5)}$ (c) $\frac{(5C5)+(7C5)+(12C5)}{(24C5)}$

(d) $\frac{(5C2)(7C1)(12C2)}{(24C5)}$

4)

We start by giving player A, 13 cards from the pack.

There are (52C13) ways to give player A, 13 cards.

There are (4C1)(48C12) ways to give player A, 1 ace and 12 non-aces.

We now give player B, 13 cards from the remaining cards in the pack.

There are (39C13) ways to give player B, 13 cards.

There are (3C1)(36C12) ways to give player B, 1 ace and 12 non-aces.

We now give player C, 13 cards from the remaining cards in the pack.

There are (26C13) ways to give player C, 13 cards.

There are (2C1)(24C12) ways to give player C, 1 ace and 12 non-aces.

We now give player D the remaining 13 cards.

Answer: $\frac{(4C1)(48C12)}{(52C13)}\times\frac{(3C1)(36C12)}{(39C13)}\times\frac{(2C1)(24C12)}{(26C13)}$

SOLUTIONS 7

1) These three possible outcomes are not equally likely.

head	*	
tail		
	head	tail

Answer: 1/4

2) When you spin three coins there are 8 possible outcomes:

HHH, HHT, HTH, THH, TTH, THT, HTT, TTT

Of these 8 possible outcomes, 2 outcomes are three heads or three tails.

Answer: 2/8

3) When you spin 2 coins, there are 4 possible, equally likely, outcomes:

(head, head) (head, tail) (tail, head) (tail, tail)

Given that one of the coins lands heads means that we can eliminate (tail, tail) leaving 3 possible, equally likely, outcomes.

Answer:1/3

4) The only way you get HH before you get TH is if the first two spins are HH (think about it)

The probability of this is 1/4

Amusingly, if you spin a coin repeatedly:

the sequence TTH is likely to appear before the sequence THH

the sequence THH is likely to appear before the sequence HHT

the sequence HHT is likely to appear before the sequence HTT

the sequence HTT is likely to appear before the sequence TTH

Look up the game Penney-Ante

5) We must count correctly! Look back to example 2.

6) Consider 100 cases

a) Start with a red ball in the bag (50 cases)

In 50 cases we add a red then remove a red, leaving a red.

b) Start with a green ball in the bag (50 cases)

In 25 cases we add a red then remove a red, leaving a green.

In 25 case we add a red then remove a green, leaving a red.

In 75 cases we remove a red and of these, 50 cases we leave a red.

Answer: $50/75$

7) Consider 300 cases

a) Card is red/red (100 cases)

In 100 cases we see a red side and the other side is red.

b) Card is green/green (100 cases)

In 100 cases we see a green side and the other side is green.

c) Card is red/green (100 cases)

In 50 cases we see a red card and the other side is green.

In 50 cases we see a green side and the other side is red.

In 150 cases we see a red card and of these, 100 cases the other side is red.

Answer: $100/150$

8) You point to one of three closed doors. The probability there is a car behind that door is $1/3$

So if you decide to stick, the probability you win the car is $1/3$

So if you decide to switch, the probability you win the car is $2/3$

9) There is 1 person in the room. Another person walks in. The probability these 2 people have different birthdays is $364/365$ Another person walks in. The probability these 3 people have different birthdays is $\frac{364}{365}\times\frac{364}{365}$ Another person walks in … etc

Answer: $\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times...\times\frac{343}{365}$

Surprisingly, this works out to be approximately 0.5

So with 23 people in a room, there is a probability of 0.5 that at least 2 of them have the same birthday!