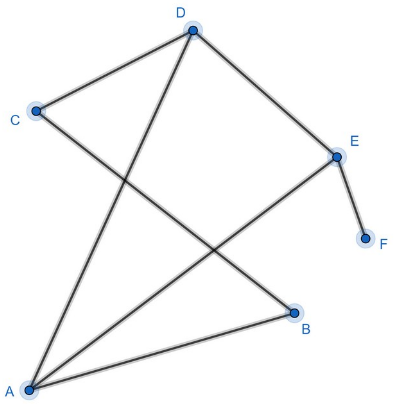
Graphs

Here is a graph:



Note: this has nothing to do with graphs that have x and y axes.

We have points A, B, C, D, E, F (called vertices) connected by lines (called edges).

The degree of a vertex is the number of edges joined to that vertex.

vertex	A	B	C	D	E	F
degree	3	2	2	3	3	1

The sum of the degrees of the vertices is $3+2+2+3+3+1=14$

The number of edges is 7

Look at the edge AD. It is counted once when we find the degree of A and counted once again when we find the degree of D. This is true of all the edges. So ...

The handshaking rule:

For any graph, the sum of the degrees of the vertices is twice the number of edges.

We say a vertex is even if its degree is even and a vertex is odd if its degree is odd.

When you add up some positive integers the total will be even if and only if there are an even number of odd integers. The sum of the degrees of the vertices is even. So from the handshaking rule we can deduce:

For any graph, there are an even number of odd vertices.

Why is it called the handshaking rule?

Imagine A, B, C, D, E, F are people who went to a party. During the party, some handshaking took place. We recorded this on the graph.

A and D shook hands so our graph has an edge joining A and D.

B and F did not shake hands so our graph does not have an edge joining B and F.

Vertex D has degree 3, so D shook hands with 3 people.

At the end of the party we ask everyone how many hands they shook. The replies $(3,2,2,3,3,1)$ add up to 14. Each act of handshaking has been counted twice. When A and D shake hands, this contributes to A's total and it contributes to D's total. So the sum of the replies must equal twice the number of handshakes.

EXERCISE

Both these problems are about people at a party where handshaking took place.

1) Everyone shakes hands with 3 people. Why must there be an even number of people at the party?

2) Alice and Bill are at a party with 5 other people. Everyone shook hands with at least one other person. Only Alice and Bill shook hands with the same number of people. Why must Alice have shaken hands with an odd number of people?

SOLUTIONS

1) The sum of the degrees of the vertices is even. If each vertex has degree is 3 then there must be an even number of vertices.

2) There are 7 people at the party so everyone must have shaken hands with 1, 2, 3, 4, 5 or 6 people.

Alice shakes hands with A people.

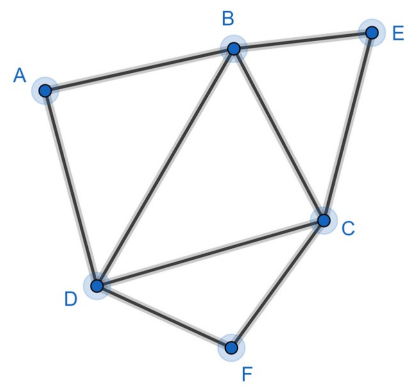
The other 6 people at the party each shook hands with a different number of people.

So the number of people they shook hands with must be 1, 2, 3, 4, 5 and 6

Now $1+2+3+4+5+6+A$ must be even so A must be odd.

Euler Tours

Example 1

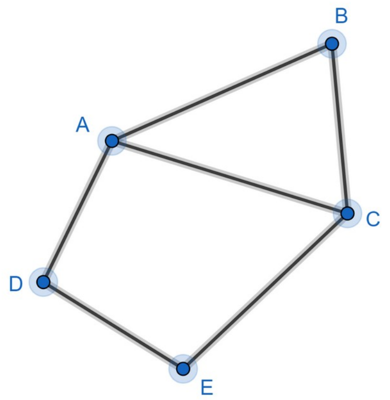


If you walk along the edges of this graph and visit the vertices in the order ABDCBECFDA then you have completed an Euler tour because you have walked along each edge once (and only once). This is a closed tour because the end vertex (A) is the same as the start vertex (A).

Any tour can be reversed. You could walk ADFCEBCDBA

A closed tour can start on any vertex. We can think of the above tour as starting on vertex F. You could walk: FDABDCBECF

Example 2

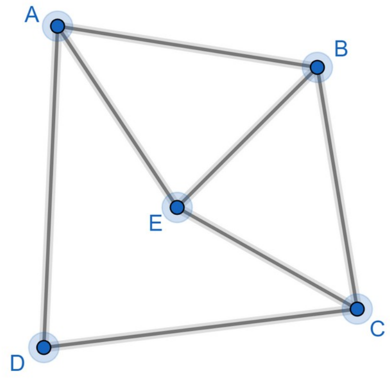


ABCEDAC is an open Euler tour because the end vertex (C) is not the same as the start vertex (A).

Any tour could be reversed. You could walk CADECBA

An open tour cannot start on any vertex.

Example 3



Does this graph have a closed Euler tour?

Let's think about the start/end vertex of a tour:

If you start on vertex C, you can leave via one edge, return via another edge, leave again via the third edge but there is no edge left for your final return.

So you cannot start/end on vertex C (or vertex A, B or E)

The start/end vertex must be connected to an even number of edges.

Let's think about the vertices that are not at the start/end of a tour:

Each time you go to a vertex via an edge, you have to leave this vertex via a different edge.

So you need an even number of edges connected to this vertex.

For a closed Euler tour, you need an even number of edges connected to every vertex.

So in our example, there is no closed Euler tour.

Does this graph have an open Euler tour?

Let's think about the start/end vertex of a tour:

If you start on vertex C, you can leave via one edge, return via another edge, leave again via the third edge and you don't need to return to vertex C.

The start/end vertex must be connected to an odd number of edges.

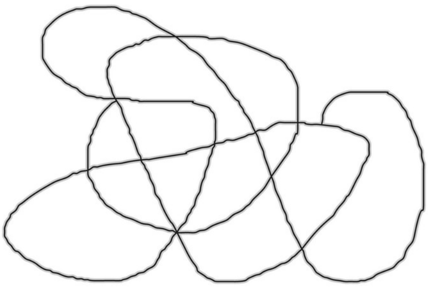
For an open Euler tour, you need an odd number of edges connected to the start and end vertices and an even number of edges connected to all the other vertices.

So in our example, there is no open Euler tour.

When you visit an art gallery, you want to walk along all the corridors, so you don't miss any of the paintings. It would be good if you could return to the entrance without having to walk along any corridor more than once. Think of the corridors as edges and where corridors meet as vertices. A good art gallery lay-out would have no odd vertices.

Doodle Problem

Here is a doodle:



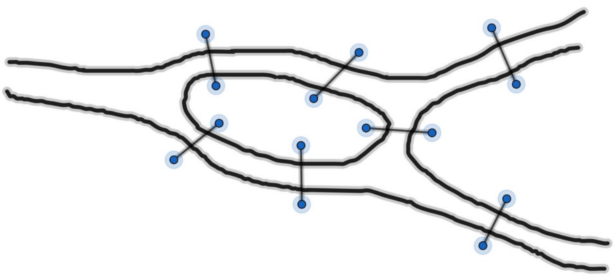
Can you draw this doodle without taking your pencil off the paper and without going over any part of the doodle more than once?

Put a vertex where lines meet and we have a graph. Drawing this doodle without taking your pencil off the paper and without going over any part of the doodle more than once is the same as finding an Euler tour. So we just need to count how many vertices are odd.

In this example there are 2 odd vertices, so you can draw this doodle by starting at one of the odd vertex and finishing at the other odd vertex.

Konigsberg Bridges Problem

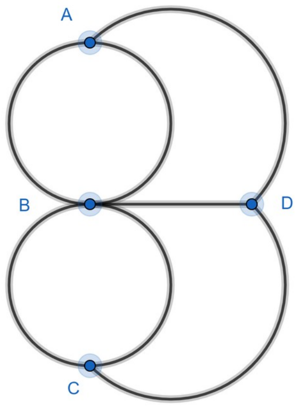
Here is a map of the city of Konigsberg (now known as Kaliningrad):



There is an island in the middle of a river. There are seven bridges connecting the island, the north bank, the south bank and the east area.

The story goes that citizens of Konigsberg wanted to go for a walk and cross each bridge once (and only once). Is this possible?

We can represent this map as a graph showing how the areas are connected by the bridges.



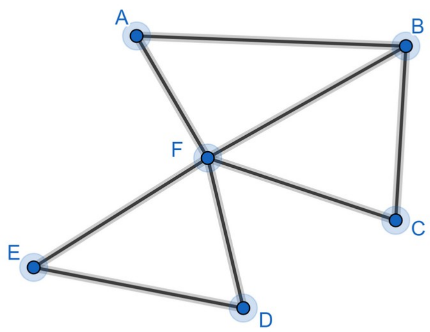
B is the island, A is the north bank, C is the south bank and D is the east area.

The citizens are trying to find an Euler tour. But this is not possible because there are 4 odd vertices.

It was Euler who first solved this problem, thereby starting a whole new branch of mathematics.

Chinese Postman Problem

Here is a street map of a town (drawn to scale):



The postman starts at A, walks along every street delivering the mail, and then returns to A. The problem is to find the shortest route.

Think of the streets as edges and the street junctions as vertices and we have a graph.

If every vertex was even then the postman could take a closed Euler tour and walk along every edge once (and only once). But as some of the vertices are odd, the postman will have to walk along some edges twice.

B and F are the only odd vertices. The postman could start at B, walk along every edge once (and only once) and end at F. A possible route is BFCBAFEDF. The postman could then find the shortest route from F back to B. This is along edge FB.

We now have the closed non-Euler tour BFCBAFEDFB.

But the postman has to start and finish at A not at B. No problem. We can start a closed tour at any vertex. The tour AFEDFBFCBA will start and finish at A.

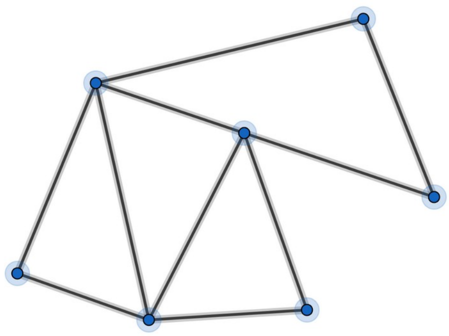
If there are more than 2 odd vertices then this problem is more difficult.

EXERCISE

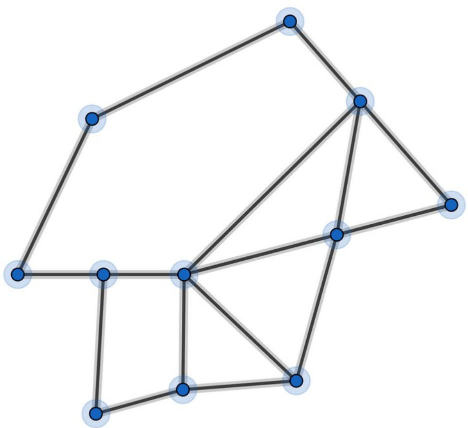
1)

Do these graphs have Euler tours? If so, are they open or closed?

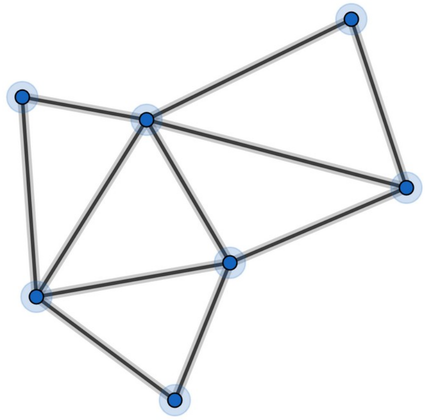
a)



b)



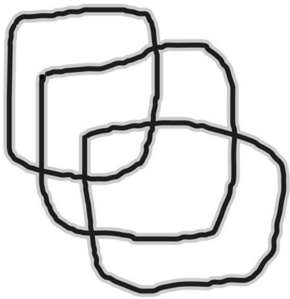
c)



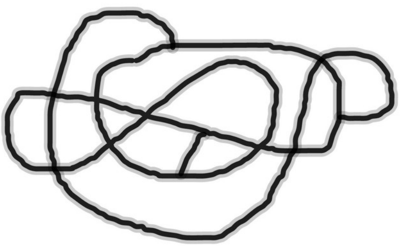
2)

Can you draw these doodles without taking your pencil off the paper and without going over any part of the doodle more than once?

a)



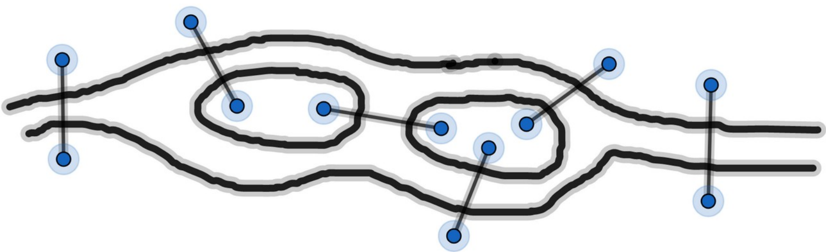
b)



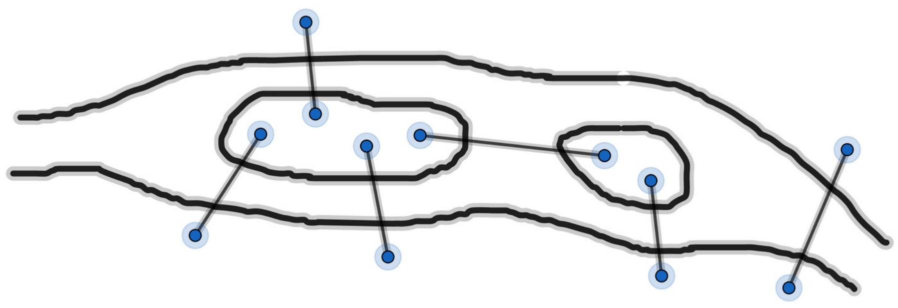
3)

Can you go for a walk and cross each bridge exactly once?

a)



b)

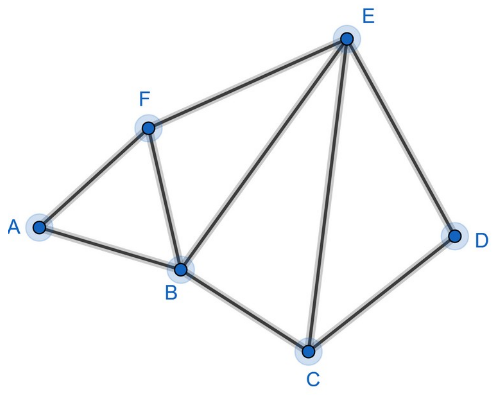


c) The mayor of Konigsberg wants to build another bridge so that she can go for a walk, starting from the north bank, crossing every bridge once (and only once) and ending on the island. Where should she build this new bridge?

4)

Here is a street map of a town (drawn to scale):

Find the shortest postman route starting and finishing at A.



SOLUTIONS

1)

a) No odd vertices. So there is a closed Euler tour.

b) Four odd vertices. So there is no Euler tour.

c) Two odd vertices. So there is an open Euler tour.

2)

a) No odd vertices. Answer: Yes

b) Four odd vertices. Answer: No

3)

a) Two odd vertices. Answer: Yes

b) No odd vertices: Answer: Yes

c) The bridge must go between the south bank and the east area.

4)

F and C are odd vertices so let's find an open tour starting at F and ending at C

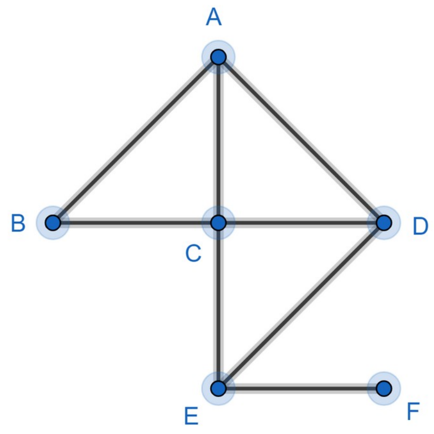
for example: FEDCEBAFBC

Now let's add on the shortest route from C back to F to get FEDCEBAFBCBF

Now let's start/end this tour at A to get AFBCBFEDCEBA

Hamilton Tours

Example 1



If you walk along the edges of this graph and visit the vertices in the order BADCEF then you have completed a Hamilton tour because you have visited every vertex once (and only once).

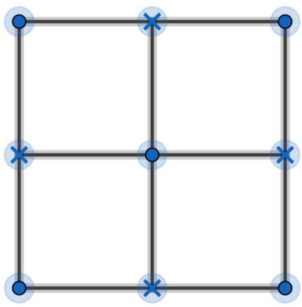
This is an open Hamilton tour because the end vertex (F) is not the same as the start vertex (B).

Unlike Euler tours, there are no simple rules to decide if Hamilton tours exists.

Sometimes you can find a Hamilton tour by trial and error. Sometimes you can prove a Hamilton tour does not exist. Sometimes you just don’t know.

Example 2

Look at the graph below where each vertex is marked with a cross or a dot:



I can find an open Hamilton tour by trial and error. Can you?

I can prove a closed Hamilton tour does not exist:

> any Hamilton tour must visit a dot vertex then a cross vertex then a dot vertex then ...
>
> so a closed Hamilton tour must have the same number of dot and cross vertices
>
> but there are 5 dot vertices and 4 cross vertices

Travelling Salesman Problem

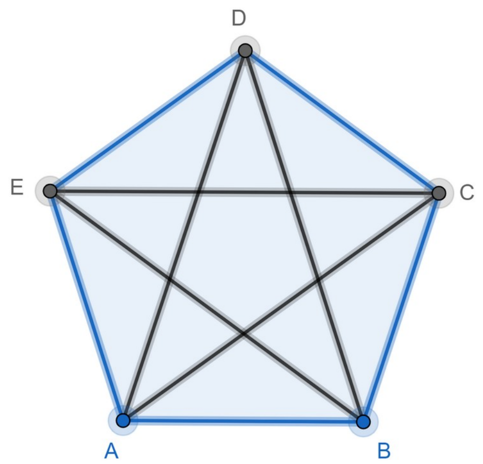
Example 2

Here are the distances between five airports A, B, C, D, E:

	A	B	C	D	E
A	-	75	132	125	73
B	75	-	65	96	109
C	132	65	-	72	137
D	125	96	72	-	91
E	73	109	137	91	-

There are direct flights between all these airports.

Think of the flights as edges and the airports as vertices and we have a graph. (not drawn to scale)



The salesman starts at A, visits every airport once (and only once) and then returns to A. The problem is to find the shortest route. The salesman is looking for the shortest closed Hamilton tour.

The salesman could try the nearest neighbour algorithm:

start at A then fly to the nearest airport not already visited then fly to the nearest airport not already visited then … then return to A

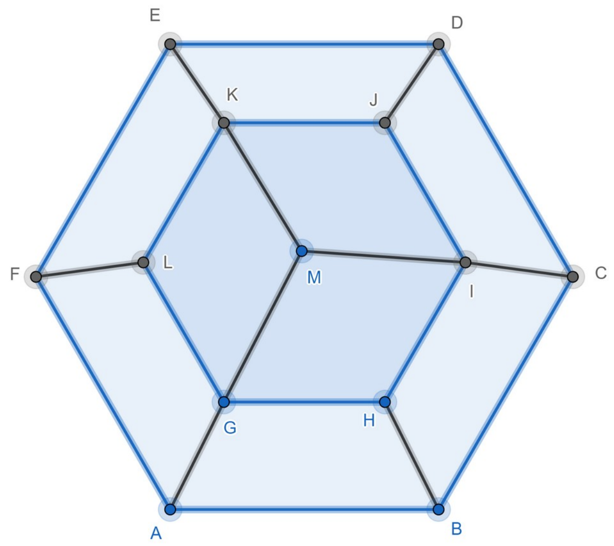
This gives the route: AEDCBA which has length 376

Unfortunately, this algorithm does not always find the shortest route.

So why don't I give you the algorithm that does always find the shortest route? Because no-one has found such an algorithm!

EXERCISE

1) Does this graph have a Hamilton tour?



2) There are 4 flowers (A, B, C, D) in a field. A bee starts on flower A, visits each of the other flowers once (and only once) to collect pollen and returns to flower A. Use the nearest neighbour algorithm to find a route.

Here are the distances between the flowers:

	A	B	C	D
A	-	85	105	92
B	85	-	73	115
C	105	73	-	65
D	92	115	65	-

SOLUTIONS

1) Here is an open tour: ABCDEFLGHIJKM. I found it by trial and error.

We can prove there is no closed tour

Colour vertices A, C, E, L, H, J, M green. Colour vertices B, D, F, G, I, K pink.

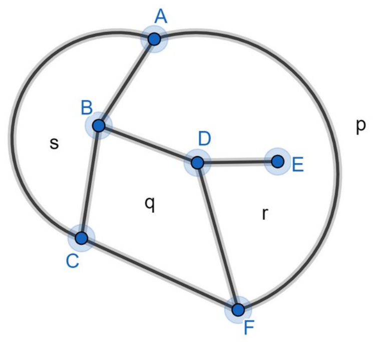
Any tour must alternate green, pink, green, pink, ...

A closed tour must have the same number of green and pink vertices but there are 7 green vertices and 6 pink vertices.

2) ABCDA

Euler’s Formula

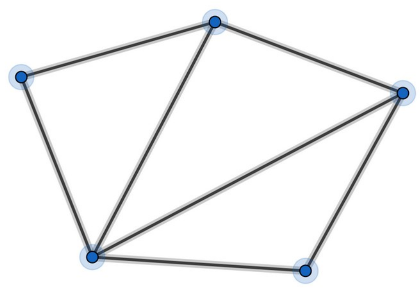
Here is a planar graph:



It is planar because no two edges cross-over each other.

The graph divides the plane into regions p, q, r, s (called faces).

Euler’s formula for planar graphs



For this planar graph:

the number of vertices is: $V=5$

the number of edges is: $E=7$

the number of faces is: $F=4$ (remember to include the outer face)

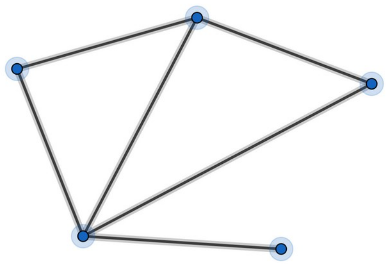
Euler’s formula:

For any planar graph $F+V-E=2$

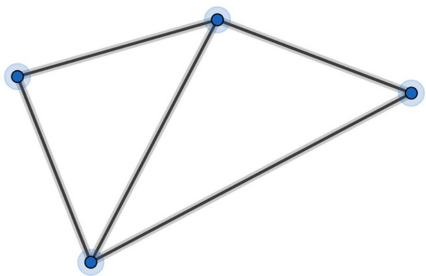
Proof

Start with the graph above.

You can rub-out an edge so $E \rightarrow E-1$ and $F \rightarrow F-1$ and $F+V-E$ stays unchanged.



You can rub-out an edge so $E \to E-1$ and $V \to V-1$ and $F+V-E$ stays unchanged.



Once all the rubbing-out has been done, you will be left with:

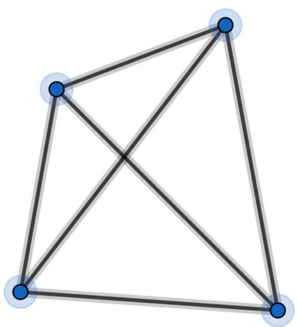
one face $F=1$ one vertex $V=1$ no edges $E=0$ and $F+V-E=2$

But all the rubbing-out leaves $F+V-E$ unchanged.

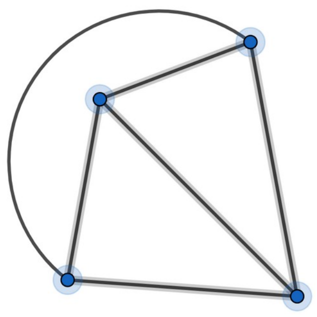
So $F+V-E=2$ for the original graph.

EXAMPLE

We can redraw this graph:

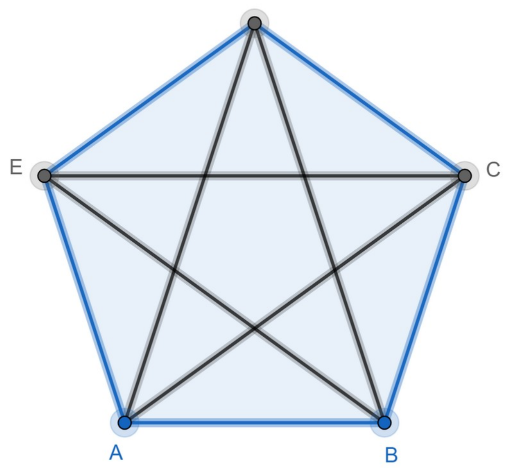


so that it is planar:



Theorem

We cannot redraw this graph so that it is planar:



Proof (by contradiction)

Assume we can redraw this graph so that it is planar.

$V=5$

Each vertex is joined to 4 edges.

So:

$E=4\times 5$ No!

Each edge is shared with 2 vertices.

So:

$$E=\frac{4\times 5}{2}=10$$

So by Euler's formula $F=7$

Each face has at least 3 edges.

So:

$E \geq 3F$ No!

Each edge is shared by 2 faces.

So:

$$E \geq \frac{3F}{2}$$

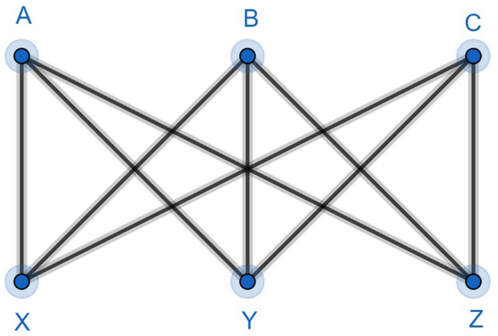
So:

$$10 \geq \frac{3 \times 7}{2}$$

Contradiction.

Theorem

We cannot redraw this graph so that it is planar:



Proof (by contradiction)

Assume we can redraw this graph so that it is planar.

$V=6$ and $E=9$ so by Euler's formula $F=5$

A face cannot have just 3 edges – try drawing one!

Each face has at least 4 edges.

So:

$E \geq 4F$ No!

Each edge is shared by 2 faces.

So:

$$E \geq \frac{4F}{2}$$

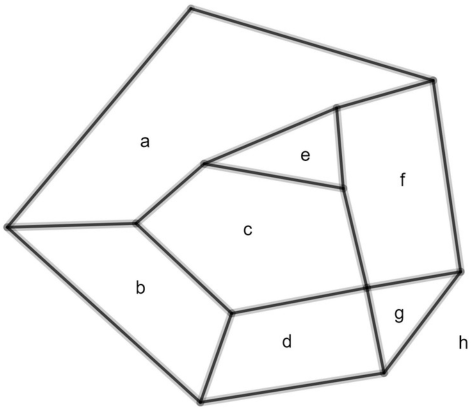
So:

$$9 \geq \frac{4 \times 5}{2}$$ Contradiction.

This is known as the utilities problem. Imagine A, B, C are houses and X, Y, Z are gas, water, electricity supply points. Each house needs to be connected, by pipe, to each utility. Can we do this without any pipes crossing over each-other? No!

Map Colouring

Here is a map:



This map has 8 regions a, b, c, d, e, f, g, h We want to colour the regions. Two regions that share a border like c and f must have different colours. Two regions that meet at a point like c and g can have the same colour. What is the minimum number of colours required?

Think of the regions as faces. Think of the borders as edges. Put a vertex where borders meet. We have a planar graph and we can use Euler's formula.

Theorem

Every planar graph has a face with five (or fewer) edges.

Proof (by contradiction)

Assume there is a planar graph where every face has at least six edges.

Every face has at least six edges.

So:

$E \geq 6F$ No!

Each edge is shared by 2 faces.

So:

$E \geq \frac{6F}{2}$ so $F \leq \frac{E}{3}$

Every vertex has at least three edges.

So:

$E \geq 3V$ No!

Each edge is shared by two vertices.

So:

$E \geq \frac{3V}{2}$ so $V \leq \frac{2E}{3}$

So:

$$F+V-E\leq\frac{E}{3}+\frac{2E}{3}-E$$

So:

$F+V-E\leq 0$ But, by Euler's formula, $F+V-E=2$

Contradiction

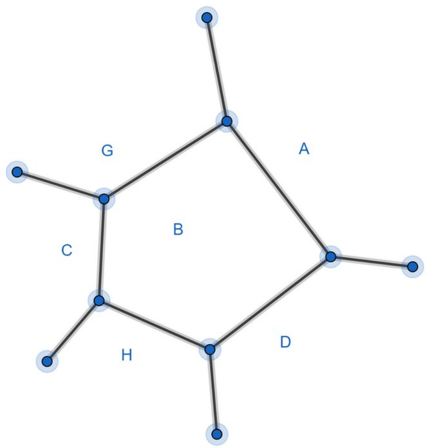
Six colour theorem

Every map can be coloured with at most six colours.

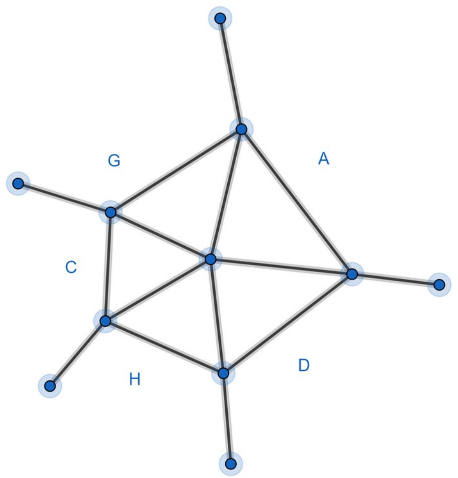
Proof

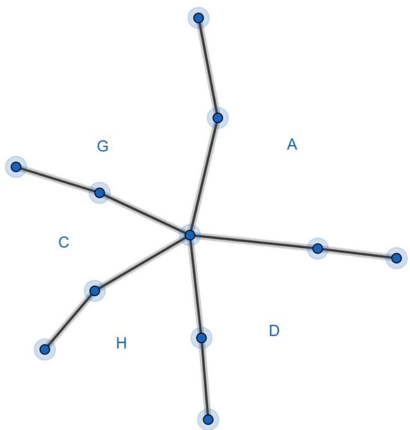
Say a map has 17 faces. We find a face with five (or fewer) edges.

The diagram below is just part of this map. Face B has five (or fewer) edges.



Remove this face in the following way:





We now have a map with 16 faces. If we can colour the 16 face map with just six colours then we can colour the 17 face map with just six colours because we will only use five colours for the faces A, D, H, C, G and this leaves a sixth colour for when we reinstate face B.

We can now repeat the process.

We start with the 16 face map. We find a face with five (or fewer) edges.

We remove this face …

We start with the 15 face map ...

Eventually

We start with the 6 face map. We can colour this with six colours.

Then we go back and replace all the faces we have removed. Job done.

Four Colour Theorem

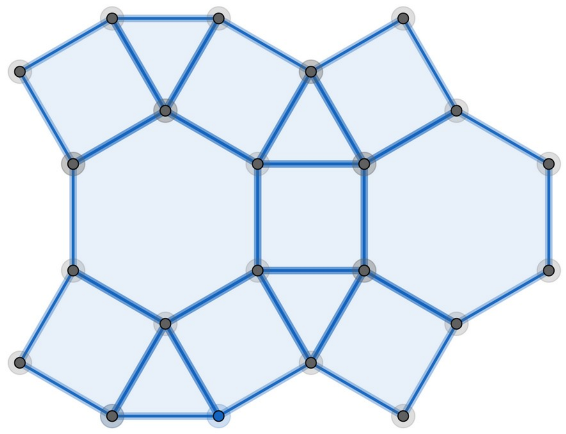
Every map can be coloured with at most four colours.

Proof

This was proved in 1976 by Appel and Haken. The proof is very difficult.

Tessellations

Here is part of a tessellation:



It is a tiling of the plane, with no gaps. You have to imagine the tiling extends in all directions.

If you walk (clockwise) around any vertex you will pass through a triangle then a square then a hexagon and then a square. We call this the $3,4,6,4$ tessellation.

The angles at a vertex must add-up to 360°

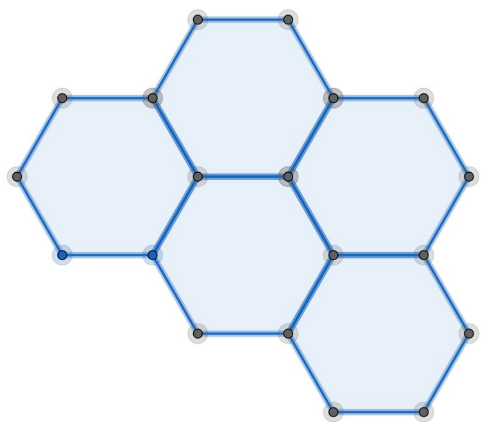
Here $60°+90°+120°+90°=360°$

In case you had forgotten, for regular polygons:

Number of sides	3	4	5	6	n
Internal angle	60°	90°	108°	120°	$180°-\frac{360°}{n}$

The regular tessellations:

Here is the $6,6,6$ tessellation:



This is a regular tessellation because:

All the faces are identical.

All the faces are regular polygons.

All the vertices are surrounded by the same number of faces.

Theorem

There are only 3 regular tessellations

See Exercise 1

The semi-regular tessellations:

Look at the above $3,4,6,4$ tessellation:

This is a semi-regular tessellation because:

All the triangles are identical.

All the squares are identical.

All the hexagons are identical.

All the faces are regular polygons.

All the vertices are surrounded by the same set of faces in the same order

Theorem

There are only 8 semi-regular tessellations

There are no semi-regular tessellations involving pentagons. Why not?

Remember, the angles at a vertex must add up to 360°

see Exercise 2

EXERCISE 1

Find the other 2 regular tessellations.

EXERCISE 2

Find the other 7 semi-regular tessellations. This is not easy!

SOLUTIONS 1

Hint: The angles at a vertex must add-up to 360°

This gives us the following regular tessellations:

4,4,4,4 and 3,3,3,3,3,3

SOLUTIONS 2

Hint: The angles at a vertex must add-up to 360°

This gives us the following semi-regular tessellations:

6,3,6,3   4,8,8   3,3,4,3,4   3,3,3,4,4   3,3,3,3,6   3,12,12 and 6,4,12

Polyhedrons

Here is a football:

WE NEED A PICTURE OF A FOOTBALL

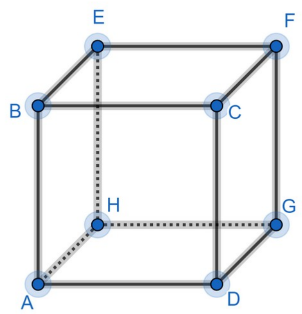
Flatten each face and we have a polyhedron.

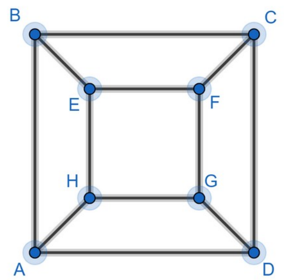
WE NEED A DIAGRAM

Euler's Formula:

$F+V=E+2$ applies to any planar graph. It also applies to any polyhedron.

Here is a cube and we can represent it by a planar graph:





The cube has 8 vertices, the graph has 8 vertices. The cube has 12 edges, the graph has 12 edges.

The cube has 6 faces, the graph has 6 faces.

The face $EFGH$ on the cube corresponds to the face $EFGH$ on the planar graph. etc

The face $ABCD$ on the cube corresponds to the outside face on the planar graph.

The cube and the planar graph have their vertices connected in the same way.

So the Euler formula must apply to the cube just as it applies to the planar graph.

Interior angles:

A cube has 6 faces. Each face has 4 interior angles. Each interior angle is $90°$

So the sum of all the interior angles of a cube is $6\times4\times90°=2160°$

Theorem

For any polyhedron

$$\sum(interior\,angles)=(V-2)360°$$

Proof

See footnote 1

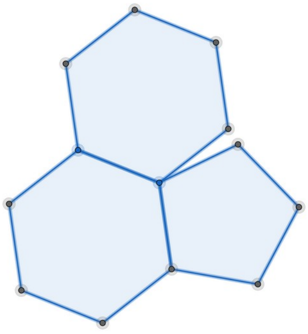
Let's check this out for the cube:

$V=8$ so $(V-2)360°=2160°$ as expected

Gaps:

Look at the football polyhedron. At each vertex, a regular pentagon and two regular hexagons meet.

If you put a regular pentagon and two regular hexagons together on a flat table then there is a gap:



Each interior angle of a regular pentagon is $108°$

Each interior angle of a regular hexagon is $120°$

So:

$108°+120°+120°+gap=360°$

So:

$gap=12°$

Descartes' theorem

Take any polyhedron. Find the gap at each vertex. The sum of all these gaps will be 720°

Proof

see footnote 2

Example 1

A polyhedron has 3 regular pentagons meeting at each vertex. How many vertices are there?

$108°+108°+108°+gap=360°$ so $gap=36°$

Now:

$\frac{720}{36}=20$ so this polyhedron has 20 vertices.

Example 2

A polyhedron has 2 regular pentagons and a square meeting at each vertex. How many vertices are there?

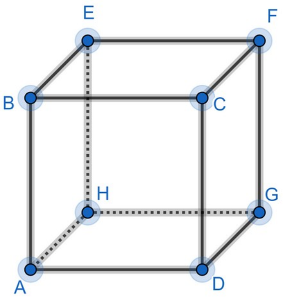
$108°+108°+90°+gap=360°$ so $gap=54°$

Now:

$\frac{720}{54}=13.3$ so this polyhedron does not exist.

The regular polyhedrons.

Look at this cube:



This is a regular polyhedron because:

All the faces are identical.

All the faces are regular polygons.

All the vertices are surrounded by the same number of faces.

Theorem

There are only 5 regular polyhedrons

Proof

Consider a regular polyhedron where each face has n sides and r faces meet at each vertex.

Note:

$n \geq 3$ and $r \geq 3$ Can you see why?

If $n=3$ and $r=3$ then three triangles meet at each vertex.

$60° + 60° + 60° + gap = 360°$ so $gap = 180°$ and $\frac{720}{180} = 4$ so we have 4 vertices

If $n=3$ and $r=4$ then four triangles meet at each vertex.

$60° + 60° + 60° + 60° + gap = 360°$ so $gap = 120°$ and $\frac{720}{120} = 6$ so we have 6 vertices

If $n=3$ and $r=5$ then five triangles meet at each vertex.

$60° + 60° + 60° + 60° + 60° + gap = 360°$ so $gap = 60°$ and $\frac{720}{60} = 12$ so we have 12 vertices

If $n=4$ and $r=3$ then three squares meet at each vertex.

$90° + 90° + 90° + gap = 360°$ so $gap = 90°$ and $\frac{720}{90} = 8$ so we have 8 vertices

If $n=5$ and $r=3$ then three pentagons meet at each vertex.

$108° + 108° + 108° + gap = 360°$ so $gap = 36°$ and $\frac{720}{36} = 20$ so we have 20 vertices

If $n = 3$ and $r \geq 6$ or $n = 4$ and $r \geq 4$ or $n = 5$ and $r \geq 4$ or $n \geq 6$ and $r \geq 3$

then you can easily check that the gaps are zero or negative and this is no good.

So there are only these 5 possibilities.

Example 3

Let's check out the $n = 5$ and $r = 3$ polygon.

We know $V = 20$ What about E and F ?

Three faces meet at each vertex, so 3 edges meet at each vertex.

So $E = 3V$ No!

Each edge is shared with 2 vertices.

So $E = \frac{3V}{2} = 30$

But $F + V = E + 2$ so $F = 12$

This polyhedron is called a dodecahedron.

see Exercise 1

The semi-regular polyhedrons.

Look at the football polyhedron:

This is semi-regular polyhedron because:

    All the pentagons are identical

    All the hexagons are identical.

    All the faces are regular polygons.

    All the vertices are surrounded by the same set of faces in the same order.

Theorem

There are only 13 semi-regular polyhedrons.

We will not prove this and we will not try to find them all.

But let's see if we can find some.

Example 4

Let's check out the football polyhedron.

One pentagon and two hexagons meet at each vertex.

$gap = 12°$ and $\frac{720}{12} = 60$ So $V = 60$ What about E and F ?

Three faces meet at each vertex. So 3 edges meet at each vertex.

So $E = 3V$ No!

Each edge is shared with 2 vertices.

So $E = \frac{3V}{2} = 90$

But $F + V = E + 2$ So $F = 32$

We have 32 faces. P pentagons and H hexagons.

One pentagon meets at each vertex.

So $P = V$ No!

Each pentagon joins five vertices.

So $P = \frac{V}{5} = 12$

Two hexagons meet at each vertex.

So $H = 2V$ No!

Each hexagon joins six vertices.

So $H = \frac{2V}{6} = 20$

Check: $T + H = F$ Good!

WARNING

We have been a bit sloppy.

SEE NOTES IN LEVER ARCH FILE

There are some polyhedrons, called prisms and anti-prisms, that seem to fit the description of a semi-regular polyhedron but are not included in the 13 semi-regular polyhedrons – check them out

see Exercise 2

see Exercise 3

EXERCISE 1

Check out the other 4 regular polyhedrons

EXERCISE 2

1) Can you find a semi-regular polyhedron where one triangle and two hexagons meet at each vertex?

2) Can you find a semi-regular polyhedron where one square and two pentagons meet at each vertex? CHANGE THIS – SAME AS EXAMPLE 2

EXERCISE 3

Use the pigeon-hole principle to show that you cannot have a polyhedron where every face has a different number of edges.

SOLUTIONS 1

tetrahedron $n=3$ $r=3$ $V=4$ $E=\frac{3V}{2}=6$ $F=4$

octahedron $n=3$ $r=4$ $V=6$ $E=\frac{4V}{2}=12$ $F=8$

icosahedron $n=3$ $r=5$ $V=12$ $E=\frac{5V}{2}=30$ $F=20$

cube $n=4$ $r=3$ $V=8$ $E=\frac{3V}{2}=12$ $F=6$

SOLUTIONS 2

1) $gap=60°$ and $\frac{720}{60}=12$ So $V=12$

Three faces meet at each vertex. So 3 edges meet at each vertex. So $E=3V$ No!

Each edge is shared with two vertices. So $E=\frac{3V}{2}=18$

But $F+V=E+2$ So $F=8$

We have 8 faces. T triangles and H hexagons.

One triangle meets at each vertex. So $T=V$ No!

Each triangle joins three vertices. So $T=\frac{V}{3}=4$

Two hexagons meet at each vertex. So $H=2V$ No!

Each hexagon joins six vertices. So $H=\frac{2V}{6}=4$

Check: $T+H=F$ Good!

2) $gap=54°$ and $\frac{720}{54}=13.33$ So this is no good! CHANGE THIS – SAME AS EXAMPLE 2

SOLUTIONS 3

A polyhedron has 10 faces.

I have 7 boxes, labelled 3, 4, 5, 6, 7, 8, 9. I put each face in a box.

If a face has 7 edges then I put it in the box with 7 on the label. etc

There are 7 boxes and 10 faces. One (or more) box must contain two (or more) faces.

So two (or more) faces have the same number of edges.

note: each face has at least 3 edges

note: there are only 10 faces so a face cannot have 10 or more edges because each edge is connected to another face.

This proof will work however many faces the polyhedron has.

Footnote 1

Interior angles rule

Let's find the sum of all the interior angles of any polyhedron.

The first face of our polyhedron has n_1 sides.

The interior angles of this face add up to $n_1(180°)-360°$ (remember?)

The second face of our polyhedron has n_2 sides.

The interior angles of this face add up to $n_2(180°)-360°$

etc

There are F faces

So:

$$\sum(\mathit{interior\,angles})=(n_1+n_2+n_3+...+n_F)180°-F(360°)$$

Now:

$$E=n_1+n_2+n_3+...+n_F \quad \text{No!}$$

Each edge is shared with two faces.

So:

$$E=\frac{1}{2}(n_1+n_2+n_3+...+n_F) \quad \text{So} \quad n_1+n_2+n_3+...+n_F=2E$$

So:

$$\sum(\mathit{interior\,angles})=2E(180°)-F(360°)=(E-F)360°$$

Now:

$$F+V=E+2 \quad \text{so} \quad E-F=V-2$$

So:

$$\sum(\mathit{interior\,angles})=(V-2)360°$$

Footnote 2

To find $\sum(\mathit{interior\,angles})$ we looked at the faces of our polyhedron.

We looked at the faces of a cube and said:

A cube has 6 faces. Each face has 4 interior angles. Each interior angle is $90°$

So the sum of all the interior angles of a cube is $6\times4\times90°=2160°$

In general, we proved:

$$\sum(\mathit{interior\,angles})=(V-2)360°$$

by finding the sum of the interior angles of each face and then adding these up.

An alternative approach

We will look at the vertices of our polyhedron.

We look at the vertices of a cube and say:

A cube has 8 vertices. Each vertex is surrounded by 3 interior angles. Each interior angle is $90°$

So the sum of all the interior angles of a cube is $8\times3\times90°=2160°$

We will use this approach to prove Descartes' theorem.

At each vertex:

$$interior\,angles+gap=360°$$

So if we visit each vertex and add up all these angles:

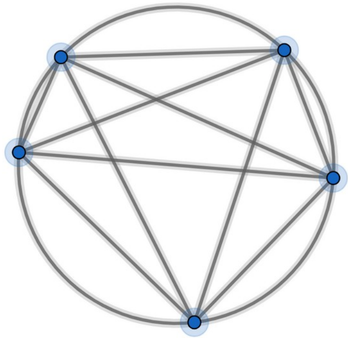
$$\sum(interior\,angles)+\sum(gap)=\sum 360°$$

Now:

$\sum(interior\,angles)=(V-2)360°$ and $\sum 360°=(V)360°$ so $\sum(gap)=720°$

Points and Regions

We put 5 points on a circle and join these points with straight lines:



This divides the circle into a maximum of 16 regions (count them)

Note: to get the maximum number of regions, we must not allow three or more lines to cross at the same point.

If you draw diagrams and count the regions then you will find:

number of points on a circle	1	2	3	4	5
maximum number of regions	1	2	4	8	16

Instead of drawing diagrams and counting the regions, let's calculate.

What is the maximum number of regions with 5 points on a circle?

Think of the lines as edges and the regions as faces and put a vertex wherever two lines or a line and the circle intersect and we have a planar graph.

As we are not counting the region on the outside, Euler's formula becomes $F+V=E+1$

Let's calculate the number of vertices:

(a) There are 5 points on the circle. That's 5 vertices.

(b) For every choice of 4 points on the circle, you can draw two lines that intersect.

There are $(5C4)=5$ ways to choose 4 points on the circle so there are 5 lines that intersect giving us another 5 vertices.

So $V=5+5=10$

Let's calculate the number of edges:

(a) There are 5 points on the circle. Each of these points is attached to 6 edges.

That's $(5\times6)=30$ edges. No!

Each edge is shared with two points.

So that's $\frac{30}{2}=15$ edges.

(b) There are $(5C4)$ points where two lines intersect. Each of these points is attached to 4 edges.

That's $(5C4)\times4=20$ edges. No!

Each edge is shared with two points.

So that's $\frac{20}{2}=10$ edges.

So $E=15+10=25$

Let's calculate the number of faces:

$F+V-E=1$ so $F=16$ as expected.

Repeat this calculation for 6 points on the perimeter. You should find there are 31 regions.

(surprised?)

Repeat this calculation for n points on the perimeter.

Let's calculate the number of vertices:

$$V=n+(nC4)$$

Let's calculate the number of edges:

$$E=\frac{n(n+1)+4(nC4)}{2}$$

Let's calculate the number of faces:

$$F=\frac{n(n+1)+4(nC4)}{2}+1-(n+(nC4))$$

You can simplify this to: $F=1+(nC2)+(nC4)$

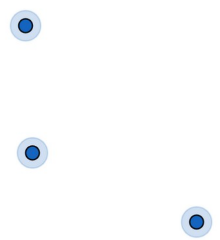
Or if you prefer: $F=\frac{1}{24}(n^4-6n^3+23n^2-18n+24)$

Sprouts

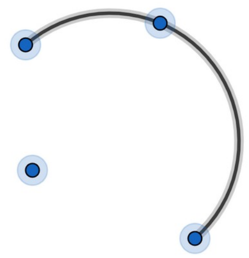
A game for two players

Start with three dots on a piece of paper. Players take turns to draw a line starting on a dot and ending on the same or a different dot and then putting another dot on this line. A line must not cross another line. A dot cannot be attached to more than three lines. The first player who cannot go is the loser.

position at start



possible position after one turn



Can you think of a good strategy to play this game? (I can't)

Note: each dot can attach to three lines and each turn uses up two attachments and creates one new attachment so the game must eventually end.

Note: you can vary the number of dots at the start.

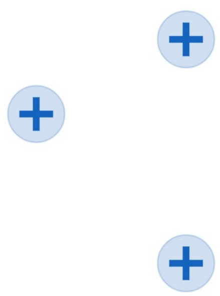
Note: think of the dots as vertices and the lines as edges and then every position is a planar graph.

Brussel sprouts

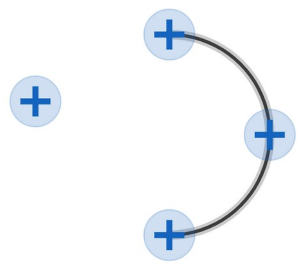
A game for two players

Start with three crosses on a piece of paper. Players take turns to draw a line starting on an arm of a cross and ending on an arm of the same or a different cross and then putting another cross on this line. A line must not cross another line. The first player who cannot go is the loser.

position at start



possible position after one turn



Note: each cross can attach to four lines and each turn uses up two attachments and creates two new attachments so it is not obvious if the game will ever end.
Note: you can vary the number of crosses at the start.
Note: think of the crosses as vertices and the lines as edges and then every position is a planar graph.

Brussel Sprouts is a con. If you start with three crosses then the game will always end after 13 turns. So the first player will always win.
Play a game and look at the final diagram. We start with 3 crosses, that's 12 attachments. At the end of the game each attachment is in a separate face, so $F=12$
If the game ends after n turns:
Each turn adds one vertex.
So:
$V=3+n$
Each turn adds 2 edges.
So:
$E=2n$
Euler's formula says:
$F+V=E+2$
So:
$12+(3+n)=2n+2$
So:
$n=13$
In general:
if we start with c crosses, that's $4c$ attachments.
If the game ends after n turns:
Then:
$F=4c \quad V=c+n \quad E=2n$
Euler's formula says:
$F+V=E+2$
So:
$4c+(3c+n)=2n+2$
So:
$n=5c-2$

Topology

In geometry we study rigid objects and we are concerned with lengths, areas, volumes, angles, etc

This is not the case in topology.

Example 1

When you are planning how to travel between two stations on the London underground, you need to know the order of the stations along the lines and where the lines connect. You don't need to know anything about lengths or angles. The London underground map is topological.

Example 2

In the chapter: Polyhedrons, we proved there are only 5 regular polyhedrons.

The proof relied on all the faces being regular polygons. This means that every edge has the same length and every interior angle is the same. We were doing geometry.

Here is another proof:

Consider a polyhedron where each face has n sides and r faces meet at each vertex.

Note:

If r faces meet at each vertex then r edges meet at each vertex.

Note:

$n \geq 3$ and $r \geq 3$

There are F faces and each face has n edges.

So: $E = Fn$ No!

Each edge is shared by 2 faces.

So $E = \frac{Fn}{2}$

There are V vertices and each vertex is joined to r edges.

So $E = Vr$ No!

Each edge is shared by 2 vertices.

So $E = \frac{Vr}{2}$

Now $E = \frac{Fn}{2}$ and $E = \frac{Vr}{2}$ so $\frac{Fn}{2} = \frac{Vr}{2}$ so $V = \frac{Fn}{r}$

Now $F + V = E + 2$

So $F + \frac{Fn}{r} = \frac{Fn}{2} + 2$

So $2rF + 2Fn = Fnr + 4r$

So $F(2n - nr + 2r) = 4r$

So $2n - nr + 2r > 0$

So $2n+2r>nr$

Together with $n\geq 3$ and $r\geq 3$ we get just five possible n and r values:

n	r	$2n+2r$	nr
3	3	12	9
3	4	14	12
3	5	16	15
4	3	14	12
5	3	16	15

In this proof we have not talked about regular polygons. The proof works even if the edges have different lengths and the interior angles are not the same. The edges don't even have to be straight. We have only assumed that all the faces have the same number of edges and all the vertices are connected to the same number of edges. So our result is more general. It is really a topological not a geometrical result.

Example 3

Can we find any polyhedrons whose faces are pentagons and hexagons with three faces meeting at each vertex?

If there are P pentagons and H hexagons then:

$F=P+H$

Each pentagon has 5 edges and each hexagon has 6 edges.

So $E=5P+6H$ No!

Each edge is shared with 2 faces.

So $E=\frac{1}{2}(5P+6H)$

Three faces meet at each vertex so 3 edges meet at each vertex.

So $E=3V$ No!

Each edge is shared with 2 vertices.

So $E=\frac{3V}{2}$

So $V=\frac{1}{3}(5P+6H)$

Now $F+V=E+2$

So $P=12$