Pi

Here are some nice formulas for π – there are lots more.

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

$$\frac{\pi}{2}=\frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\ldots$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

$$\frac{\pi^2}{6}=\left(\frac{1}{1-1/2^2}\right)\left(\frac{1}{1-1/3^2}\right)\left(\frac{1}{1-1/5^2}\right)\left(\frac{1}{1-1/7^2}\right)\left(\frac{1}{1-1/11^2}\right)\ldots$$

$\frac{\pi}{4}=\frac{3}{4}\times\frac{5}{4}\times\frac{7}{8}\times\frac{11}{12}\times\frac{13}{12}\ldots$ where the numerators are the primes (not including2) and each denominator is the multiple of 4 nearest to the corresponding numerator.

$$\frac{\pi}{4}=4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right)$$

$$\frac{\pi\sqrt{2}}{4}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots$$

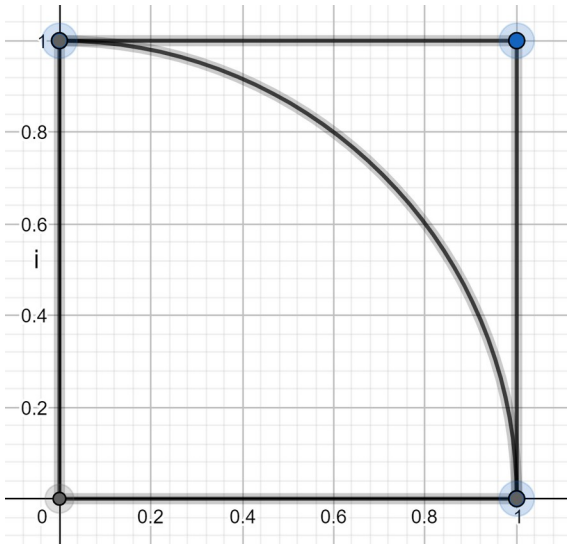
$$\frac{\pi-3}{4}=\frac{1}{2\times3\times4}-\frac{1}{4\times5\times6}+\frac{1}{6\times7\times8}-\ldots$$

$$\frac{\pi}{2}=1+\frac{1}{3}+\frac{1\times2}{3\times5}+\frac{1\times2\times3}{3\times5\times7}+\ldots$$

$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\ldots$ where $1/n$ is preceded by a $-$ sign if and only if n has an odd number of prime factors of the form $4k+1$

$$\frac{\pi}{2}-1=\frac{(2^1)(1!)^2}{3!}+\frac{(2^2)(2!)^2}{5!}+\frac{(2^3)(3!)^2}{7!}+\frac{(2^4)(4!)^2}{9!}+\ldots$$

We can estimate π using random numbers



Get two random numbers x and y where $0 \le x \le 1$ and $0 \le y \le 1$

We can think of (x, y) as the co-ordinates of a point inside the unit square. This point will be inside the quarter circle if $x^2 + y^2 < 1$ Now get lots of points.

The number of points inside the quarter circle divided by the total number of points will be approximately equal to the area of the quarter circle divided by the area of the unit square.

Show that this is $\pi/4$

So if we pick 1000 points and 763 of these points are inside the quarter circle then:

$$\frac{763}{1000} \approx \frac{\pi}{4} \quad \text{giving} \quad \pi \approx 3.05$$

To get a better approximation we need to take more points. See Appendix 1 for a computer program to estimate π

Viete's formula for π if you know about trigonometry … (all angles in radians)

a) We know:

$\cos 2\theta = 2\cos^2\theta - 1$ the double angle formula

Show that:

$cos\theta = \sqrt{\left(\frac{1+\cos 2\theta}{2}\right)}$ the half angle formula

We know:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Use the half angle formula to show that:

$$\cos\frac{\pi}{8}=\sqrt{\left(\frac{1+\cos\pi/4}{2}\right)}=\sqrt{\left(\frac{1+\sqrt{2}/2}{2}\right)}=\ldots=\frac{\sqrt{2+\sqrt{2}}}{2}$$

Use the half angle formula to show that:

$$\cos\frac{\pi}{16}=\sqrt{\left(\frac{1+\cos\pi/8}{2}\right)}=\ldots=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

etc

b) We know:

$\sin 2\theta=2\,cos\theta sin\theta$ the double angle formula

If we repeatedly use the double angle formula we get:

$$sin\theta=2\cos\frac{\theta}{2}\sin\frac{\theta}{2}=2\cos\frac{\theta}{2}\left(2\cos\frac{\theta}{4}\sin\frac{\theta}{4}\right)=2\cos\frac{\theta}{2}2\cos\frac{\theta}{4}\left(2\cos\frac{\theta}{8}\sin\frac{\theta}{8}\right)\quad\text{etc}$$

So:

$$sin\theta=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cos\frac{\theta}{16}\ldots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$$

So:

$$\frac{sin\theta}{\left(\frac{\theta}{2^n}\right)}=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots\cos\frac{\theta}{2^n}\left(\frac{\sin\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}\right)$$

So:

$$\frac{sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots\cos\frac{\theta}{2^n}\left(\frac{\sin\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}\right)$$

We know:

if $x\rightarrow 0$ then $\frac{sinx}{x}\rightarrow 1$

So:

if $n\rightarrow\infty$ then $\left(\frac{\sin\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)}\right)\rightarrow 1$

So:

$$\frac{sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ldots$$

Put:

$$\theta = \frac{\pi}{2}$$

and show that:

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\ldots$$

c) Use part (a) and part (b) to obtain Viete's formula:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\ldots$$