Fibonacci Numbers

The Fibonacci numbers:

$1,1,2,3,5,8,13,\ldots$ are given by the recurrence relation:

$F_1=1 \quad F_2=1 \quad F_{n+2}=F_{n+1}+F_n$

So:

$F_3=F_2+F_1$ and $F_4=F_3+F_2$ and $F_5=F_4+F_3$ etc

We want a formula for F_n We will use the guess and prove method.

Guess:

$F_n=\frac{a^n-b^n}{\sqrt{5}}$ where $a=\frac{1+\sqrt{5}}{2}$ and $b=\frac{1-\sqrt{5}}{2}$ (where did that come from?)

Proof:

Consider the equation $x^2=x+1$ (where did that come from?)

Solving with the quadratic equation formula gives:

$x=\frac{1\pm\sqrt{5}}{2}$ so $x=a$ or $x=b$

Now:

a and b satisfy $x^2=x+1$

So:

$a^2=a^1+1 \quad a^3=a^2+a^1 \quad \ldots \quad a^{n+2}=a^{n+1}+a^n$

And:

$b^2=b^1+1 \quad b^3=b^2+b^1 \quad \ldots \quad b^{n+2}=b^{n+1}+b^n$

According to our guess:

$$F_1=\frac{a^1-b^1}{\sqrt{5}}=\frac{\sqrt{5}}{\sqrt{5}}=1$$

Correct

$$F_2=\frac{a^2-b^2}{\sqrt{5}}=\frac{(a^1+1)-(b^1+1)}{\sqrt{5}}=\frac{a^1-b^1}{\sqrt{5}}=1$$

Correct

$$F_{n+2}=\frac{a^{n+2}-b^{n+2}}{\sqrt{5}}=\frac{(a^{n+1}+a^n)-(b^{n+1}+b^n)}{\sqrt{5}}=\frac{(a^{n+1}-b^{n+1})+(a^n-b^n)}{\sqrt{5}}=F_{n+1}+F_n$$

Correct

Let’s look at the ratio of consecutive Fibonacci numbers:

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$

As $n \to \infty$ these ratios tend to a limit called ϕ

Now:

$$F_{n+2} = F_{n+1} + F_n \quad \text{so} \quad \frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1}}{F_{n+1}} + \frac{F_n}{F_{n+1}} \quad \text{so} \quad \frac{F_{n+2}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}$$

Letting $n \to \infty$ we get $\phi = 1 + \frac{1}{\phi}$ so $\phi^2 = \phi + 1$ so $\phi = \frac{1+\sqrt{5}}{2}$

Note:

ϕ is called the golden ratio. Look it up!

Theorem:

If F_n is prime then n is prime.

The converse of this theorem is not true – for example F_{19} is not prime.

Conjecture:

There are an infinite number of Fibonacci numbers that are prime.

EXERCISE

1) Write the Fibonacci sequence in mod 2.

Show that the 3rd, 6th, 9th, 12th … Fibonacci numbers are all multiples of 2

2) Write the Fibonacci sequence in mod 3.

Show that the 4th, 8th, 12th, 16th … Fibonacci numbers are all multiples of 3

3) Write the Fibonacci sequence in mod 5.

Show that the 5th, 10th, 15th, 20th ... Fibonacci numbers are all multiples of 5

4)

If d is a factor of F_{17} and F_{18} show that d is a factor of F_{16}

If d is a factor of F_{16} and F_{17} show that d is a factor of F_{15}

Show that consecutive Fibonacci numbers have no common factor.

5)

$F_1 = F_3 - F_2$

$F_2 = F_4 - F_3$

$F_3 = F_5 - F_4$

…

$F_{n-1} = F_{n+1} - F_n$

$F_n = F_{n+2} - F_{n+1}$

Show that:

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$

SOLUTIONS

1) mod 2:

1, 1, 0, 1, 1, 0, 1, 1, …

The sequence must now repeat because we are back to 1, 1

So the 3rd, 6th, 9th, 12th … terms are all multiples of 2

2) mod 3:

1, 1, 2, 0, 2, 2, 1, 0, 1, 1, …

The sequence must now repeat because we are back to 1, 1

So the 4th, 8th, 12th, 16th … terms are all multiples of 3

3) mod 5:

1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, …

The sequence must now repeat because we are back to 1, 1

So the 5th, 10th, 15th, 20th ... terms are all multiples of 5

4)

d is a factor of F_{16} because $F_{16} = F_{18} - F_{17}$

d is a factor of F_{15} because $F_{15} = F_{17} - F_{16}$

etc

d is a factor of F_1

So consecutive Fibonacci numbers have no common factor

5)

add up the left-hand-sides: $F_1 + F_2 + F_3 + \ldots + F_n$

add up the right-hand -sides: $F_{n+2} - F_2$

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - F_2$

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$