ELEMENTS OF OBSERVER DESIGN

FOR FEEDBACK !

- **KEY ISSUE** : OBTAIN $x(t)$ WHEN NOT DIRECTLY AVAILABLE

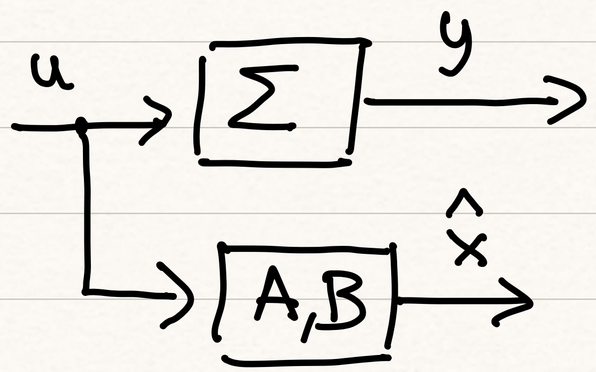
COMMON APPROACH : DESIGN **OBSERVER**

1. USES THE MODEL OF PLANT TO PROPAGATE $\hat{x}(t)$

2. USES FEEDBACK TO ENSURE
$$\hat{x}(t) \xrightarrow[t \to +\infty]{} x(t)$$

1. (OPEN-LOOP) PROP.

- assume to know $A, B, (C)$
- " " " $\hat{x}_0$
- " " " $u(t)$



$\Sigma \quad \dot{x} = Ax + Bu$
$\quad\;\; y = Cx$

$\hat{\Sigma} \quad \dot{\hat{x}} = A\hat{x} + Bu$

$\hat{x}_0 \sim x_0$

$$e_x(t) = x(t) - \hat{x}(t)$$

STATE (EST.) ERROR.

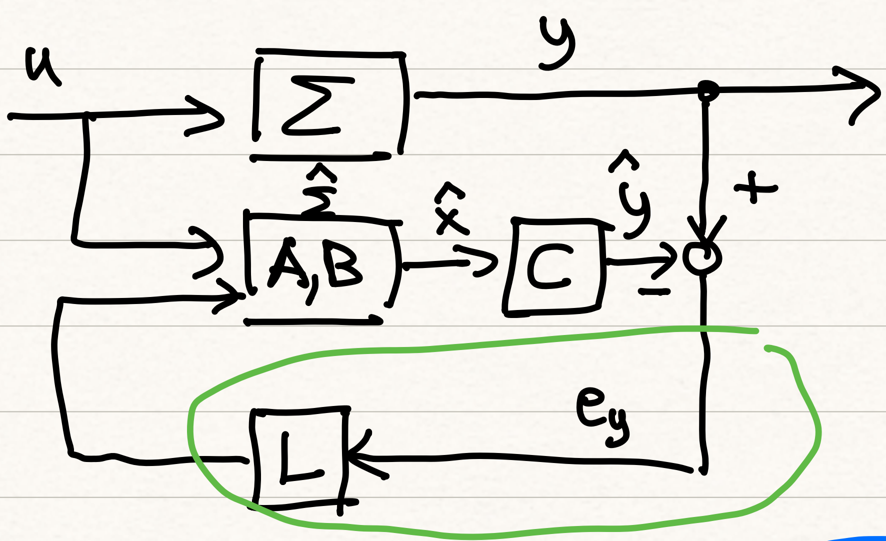
$$\dot{e}_x(t) = Ax(t) + B\cancel{u(t)} - A\hat{x}(t) - B\cancel{u(t)}$$

$$= A(x(t) - \hat{x}(t)) = A\,e_x(t)$$

↳ A.S. ⇒ convergence

$e_x \to 0$

2. FEEDBACK



$C\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y})$$

$$= A\hat{x} + Bu + Ly - LC\hat{x}$$

$$= (A-LC)\hat{x} + Bu + Ly$$

NEED FOR IMPL

NOTICE: $L(y-\hat{y}) = L(Cx - C\hat{x}) = LC(x-\hat{x})$

$$\boxed{\begin{aligned}\dot{e}_x &= A e_x - LC(x - \hat{x}) \\ &= (A - LC) e_x\end{aligned}}$$

GAIN

"FEEDBACK" MATRIX

~ WRONG ORDER

$(A - \underline{BK})$

REVIEW: DUALITY & FEEDBACK

$$\Sigma \sim (A, B, C) \longrightarrow \Sigma^* \sim (A^T, C^T, B^T)$$

STATE x_*

$$\dot{x}_* = A^T x_* + C^T u_*$$

$$y_* = B^T x_*$$

(SISO)

$u_* \sim y$

$y_* \sim u$

▷ STUDY REACHAB. OF Σ^*

if (A^T, C^T) is Reach.

$\Rightarrow$ we can allocate eigen. for

$$A^T - C^T K_*$$

GAIN VECTOR

FACT $\quad \mathrm{eig}(X) = \mathrm{eig}(X^T) \quad \forall X$
SQ. MATRICES

in particular

$$\Rightarrow \quad \mathrm{eig}(\underbrace{A^T - C^T K_*}) = \mathrm{eig}(\underbrace{A - K_*^T C}) \qquad \text{Transp.}$$

L

if we call $K_*^T = L$
$\Rightarrow$ CL.-LOOP STATE MATRIX FOR OBS.

- TO DESIGN OBSERVER DYNAMICS.
  1. check Reach. of (A^T, C^T)
  2. allocate eigs. $A^T - C^T K = A^T - C^T L^T$
     as USUAL: analytic, place, acker..
  3. $L = K^T$ as the observer gain matrix.

Reminder: What is REACH. of (A^T, C^T)?

$$\underset{\text{Reach}}{(A^T, C^T)} \Leftrightarrow \mathrm{rank} \underbrace{\left[C^T \,\vdots\, A^T C^T \,\vdots\, \dots \,\vdots\, A^{T^{n-1}} C^T \right]}_{\text{Reach. matrix } R_*} = n$$

$$\text{rank } R_* = \text{rank } R_*^T = \text{rank } \mathcal{O}$$

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ CA^{n-1} \end{bmatrix}$$

OBSERV. MATRIX FOR Σ

Σ OBSERVABLE : if we can reconstruct "x_0" from y

$\Updownarrow$

$\text{rank } \mathcal{O} = n$ (or PBH-style)

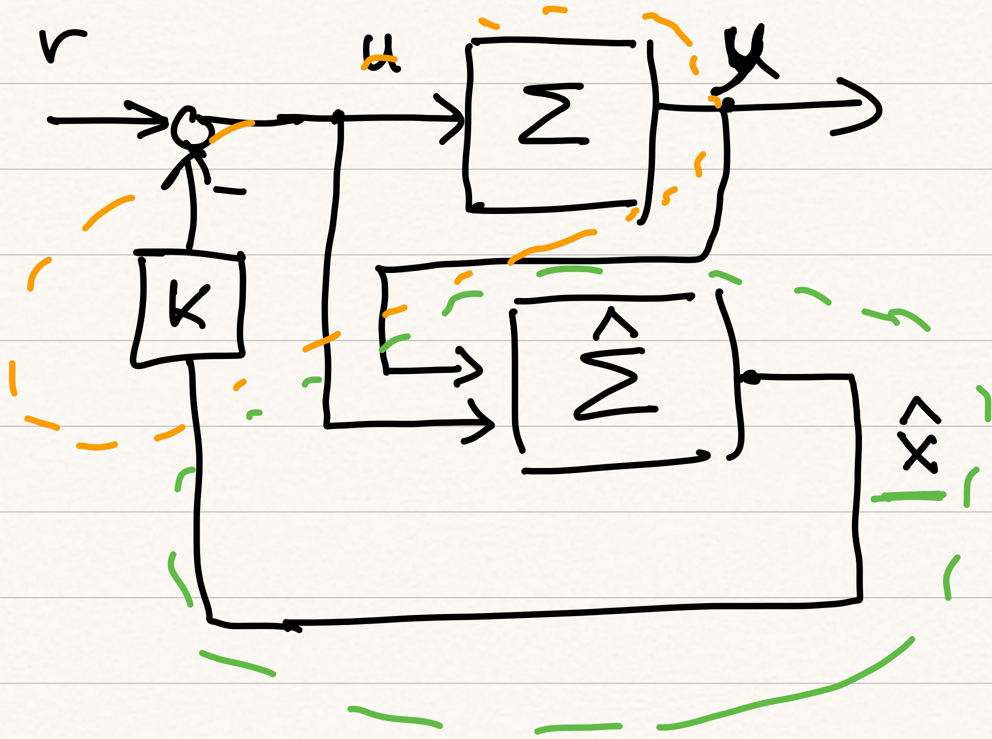
$\Updownarrow$

ability of freely assigning eigs. of OBSERVER.

- Where to allocate OBSERVER EIGS?

try 2 to 6 times faster than those of $A - BK$

FULL OBSERVER + STATE FEEDBACK



$$\hat{\Sigma} : \quad \dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$

$$= (A - LC)\hat{x} + \left[B \,\vdots\, L \right] \begin{bmatrix} u \\ y \end{bmatrix}$$

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \xrightarrow{\hat{x} \to e_x = x - \hat{x}} \begin{pmatrix} x \\ e_x \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} x \\ e_x \end{pmatrix} = \left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - LC \end{array} \right] \left[\begin{array}{c} x \\ \hline e_x \end{array} \right]$$

▷ How do we use OBSERVERS FOR TRACKING? (DIST. REJ.)

Last time : r , w solutions of ODE's

$$\Downarrow$$

$\alpha(s)$ ⟶ ERROR SPACE DESIGN

$$\Downarrow$$

Exomodels in St. sp.

$$\dot{x}_p(t) = A_p\, x_p(t)$$
$$\rho(t) = C_p\, x_p(t)$$

$$\rho = r\, w$$

INCLUDE EXOMODEL IN THE OBSERVER:

$$\begin{pmatrix} \dot{\hat{x}}_p \\ \dot{\hat{x}} \end{pmatrix} = \underbrace{\left[\begin{array}{c:c} A_p & 0 \\ \hdashline BC_p & A \end{array}\right]}_{A_{ext}} \begin{bmatrix} \hat{x}_p \\ \hat{x} \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} e_y + \begin{bmatrix} 0 \\ B \end{bmatrix} u$$

This is equiv.

$$\dot{\hat{x}} = A\hat{x} + B\,\underline{u} + B\hat{\rho}$$

• Assume (A_{ext}^T, C_{ext}^T) is Reachable

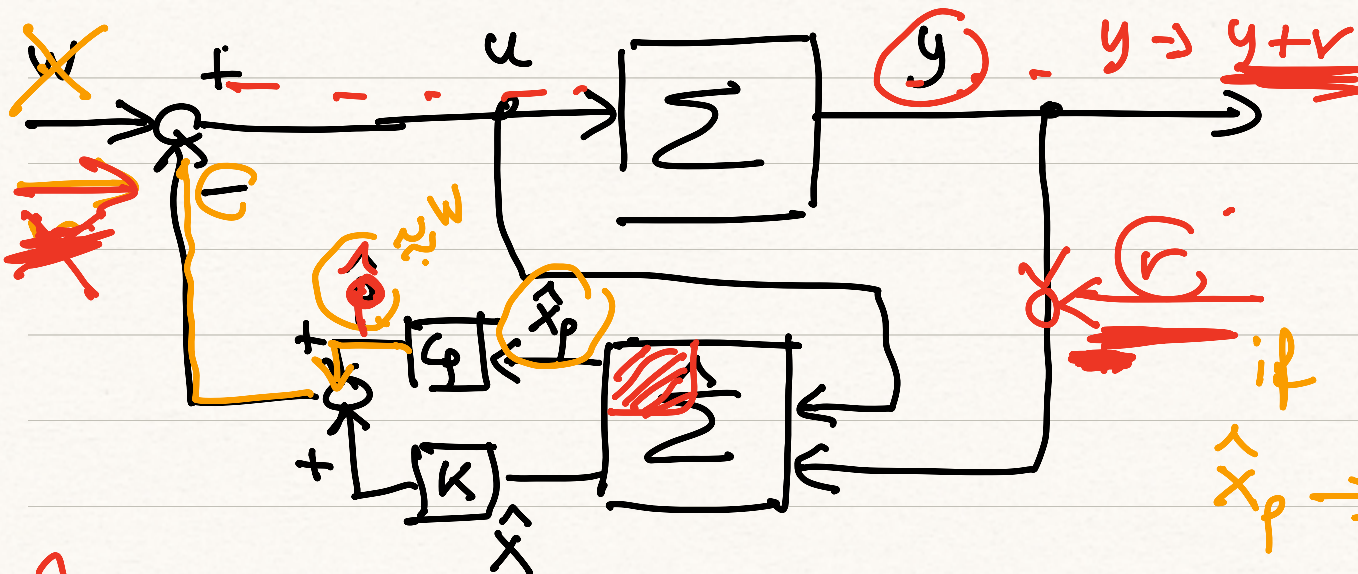
$$C_{ext} = [\,0 \;\vdots\; C\,]$$

$\Rightarrow$ choose L_1, L_2 and allocate eigs

for $A_{ext} - L_{ext} C_{ext}$, STABILIZING

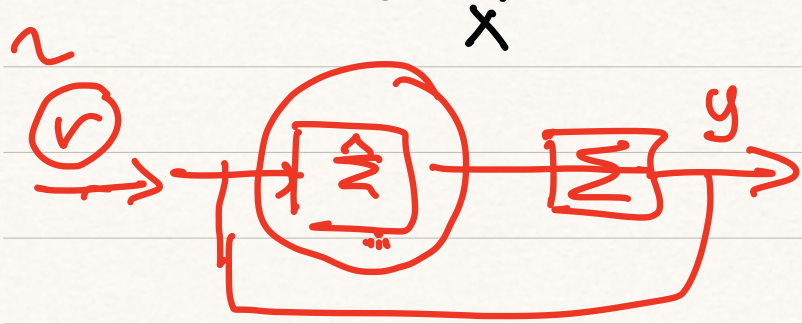
$\Rightarrow$ $\hat{x}_\rho(t), \hat{x}(t) \xrightarrow[t \to +\infty]{} x_\rho(t), x(t)$,

$\Rightarrow$ We can implement :



if $\hat{x}_\rho \to x_\rho$

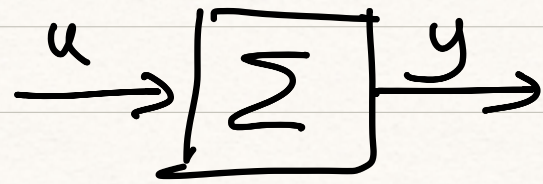
s.t



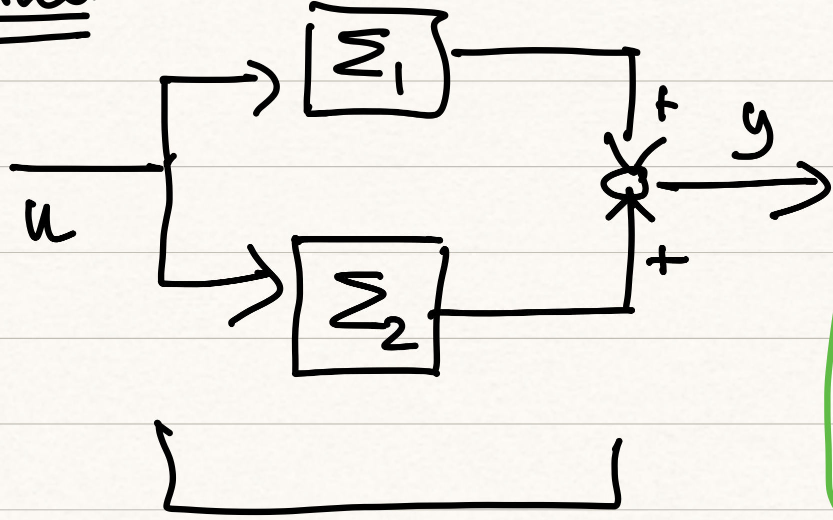
$w \simeq \rho = C_\rho x_\rho$

CONNECTING STATE-SP. MODELS

(SISO) case



PARALLEL



$\Sigma_1 : \dot{x}_1 = A_1 x_1 + B_1 u$

$y_1 = C_1 x_1$

$\Sigma_2 : \dot{x}_2 = A_2 x_2 + B_2 u$

$y_2 = C_2 x_2$

$$\Sigma_{PAR}: \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] u$$

$$y = [C_1 \mid C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

THM Σ_{PAR} is REACHABLE (OBSERV.) iff

Σ_1, Σ_2 are both reachable (OBSERV.)

and A_1, A_2 have no eigenvalues in common

IDEA PBH

$$\left[\begin{array}{cc|c} sI - A_1 & 0 & B_1 \\ 0 & sI - A_2 & B_2 \end{array}\right]$$

2 1

Same idea for OBSERVABLE

$$\xrightarrow{u} \boxed{\Sigma_1} \xrightarrow{y_1 = u_2} \boxed{\Sigma_2} \xrightarrow{} y_2 = y$$

$$\Sigma_{SER.} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{c|c} A_1 & 0 \\ \hline B_2 C_1 & A_2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] u$$

$$y = y_2 = \left[\begin{array}{c|c} 0 & C_2 \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right]$$

THM Σ_{SER} is REACHABLE iff

1) Σ_1, Σ_2 are reachable

2) the polinomials $C_1 \operatorname{adj}(sI - A_1) B_1$ and $\det(sI - A_2)$ have no roots in common