DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Control Laboratory

Lab Activity 4: Longitudinal state-space control of the balancing robot

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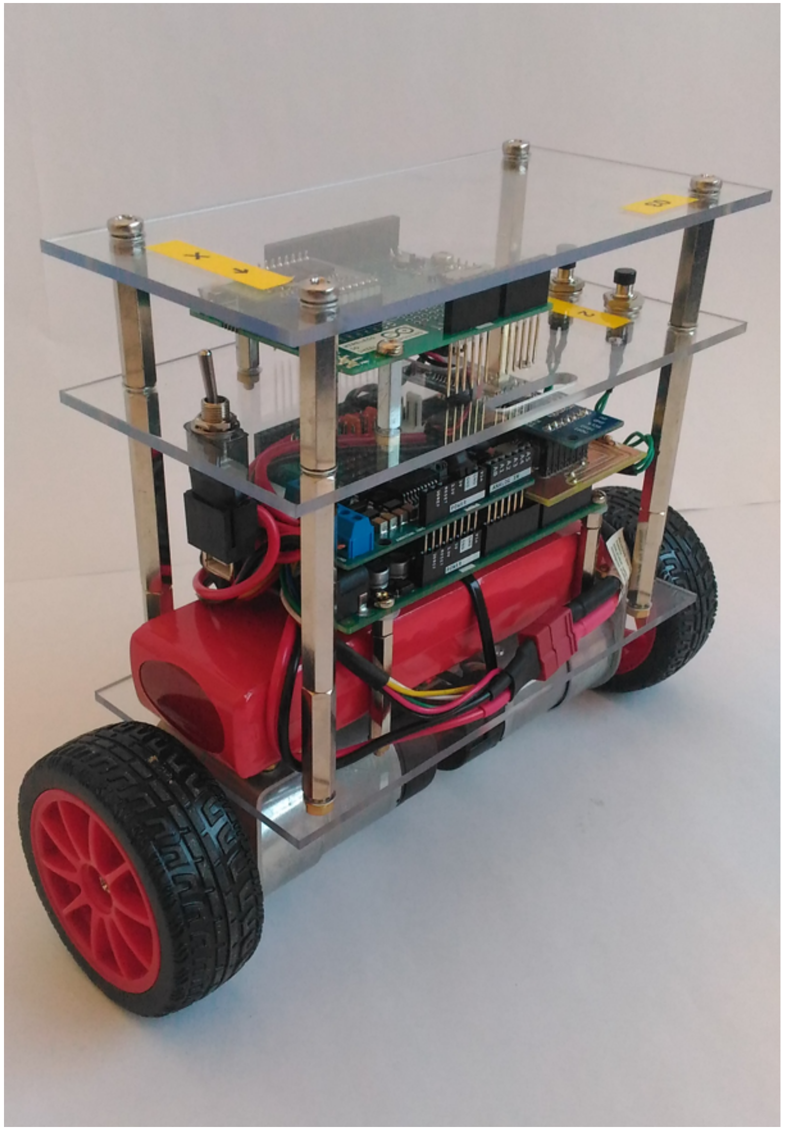
Outline

- Activity goal
- Balancing robot structure
- Derivation of the analytical model with the *Lagrangian approach*
- Tilt angle estimation by *complementary filtering* of inertial measurements
- Balance and position control
- Simplified yaw angle control

Balancing robot

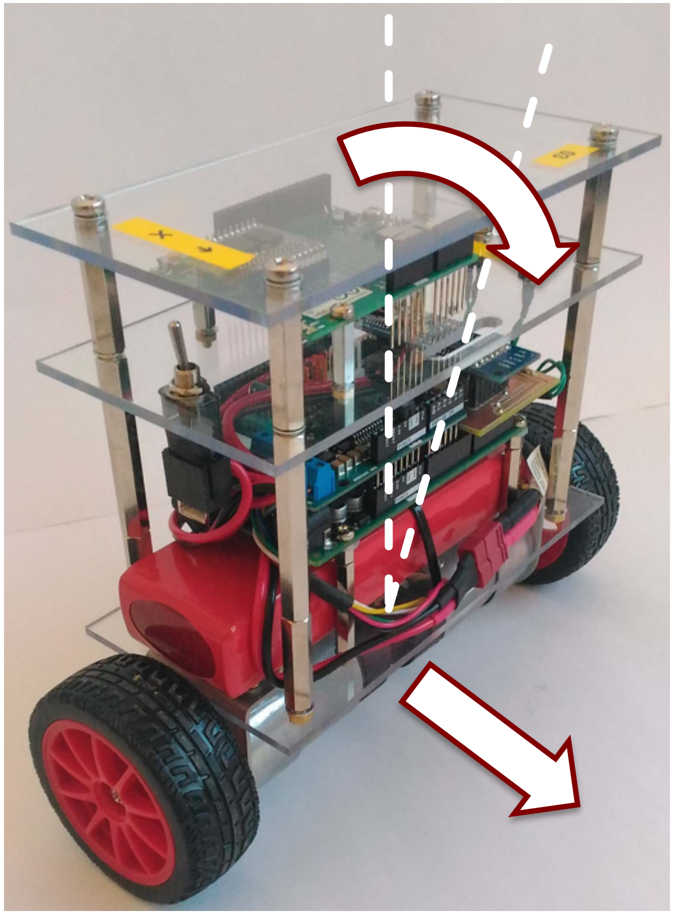


Balancing robot [(1)] :
a mobile robot that moves while keeping its body balanced on two coaxial wheels.

(1) Also known as *"two-wheeled inverted pendulum robot"* or *"Segway-like"* robot.

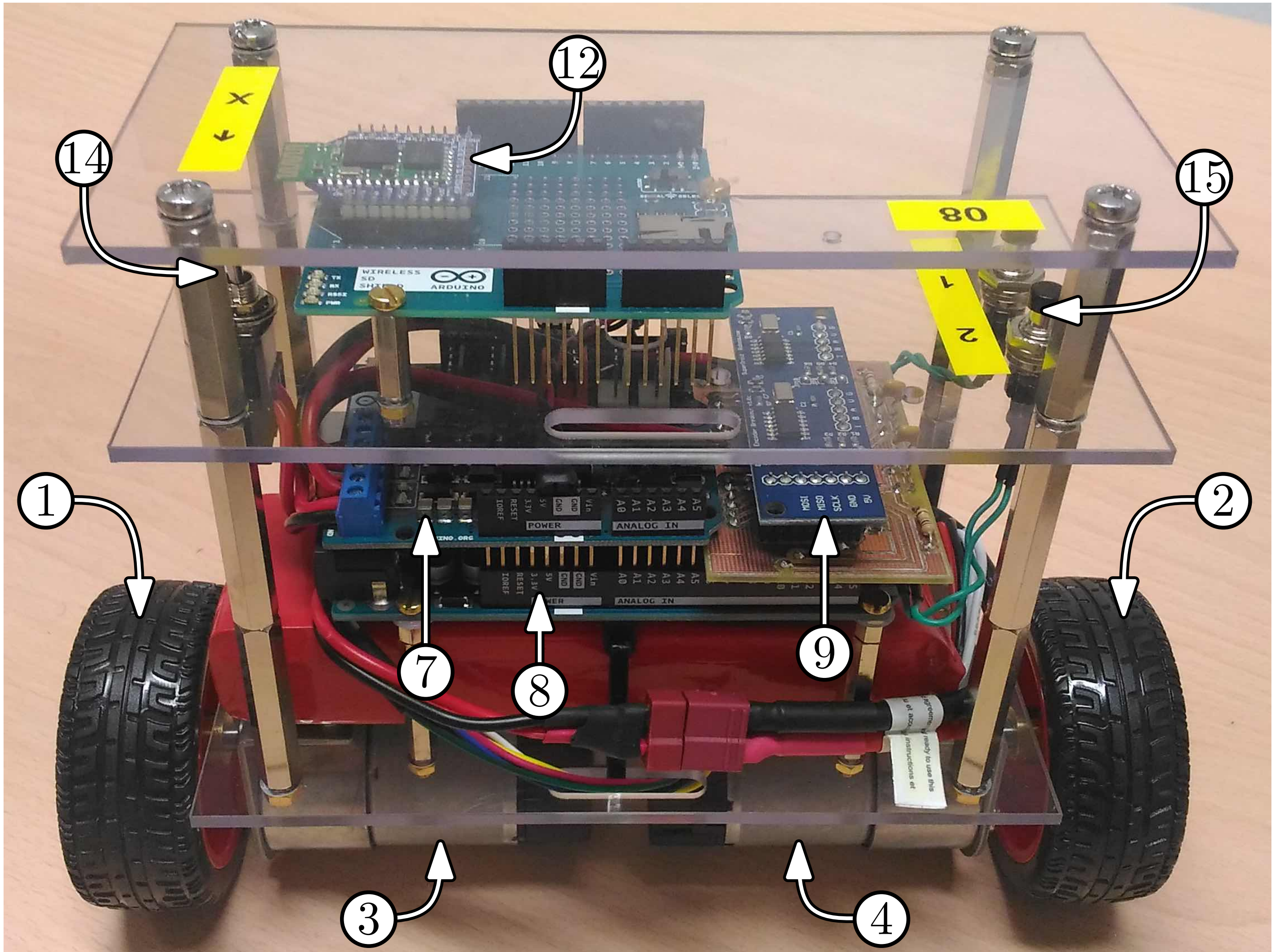
Activity goal

<u>Activity goal</u>: design a state-space controller for:

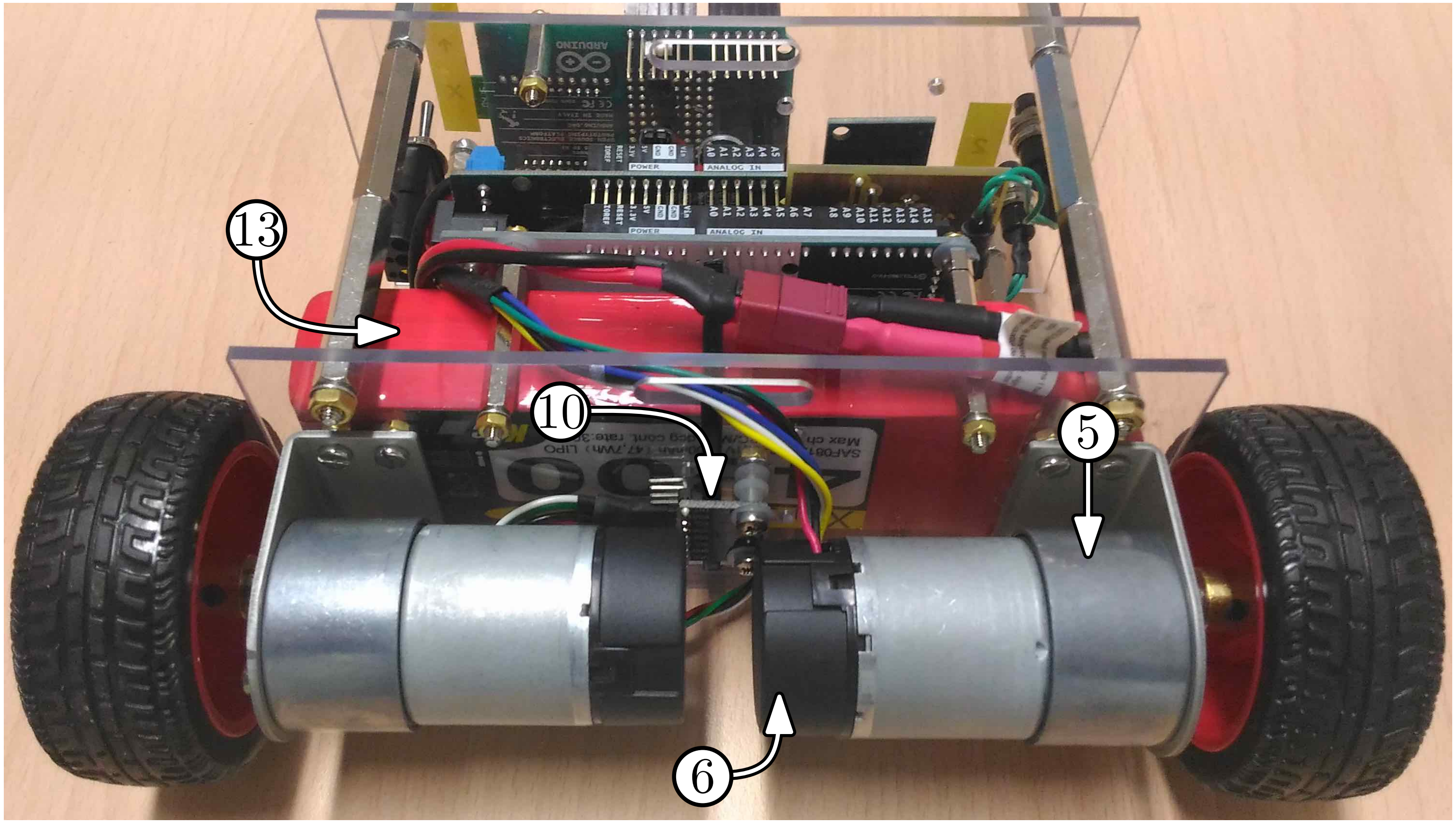


- stabilizing the robot body to its upward vertical position;
- regulating the robot base to a desired longitudinal position set-point.

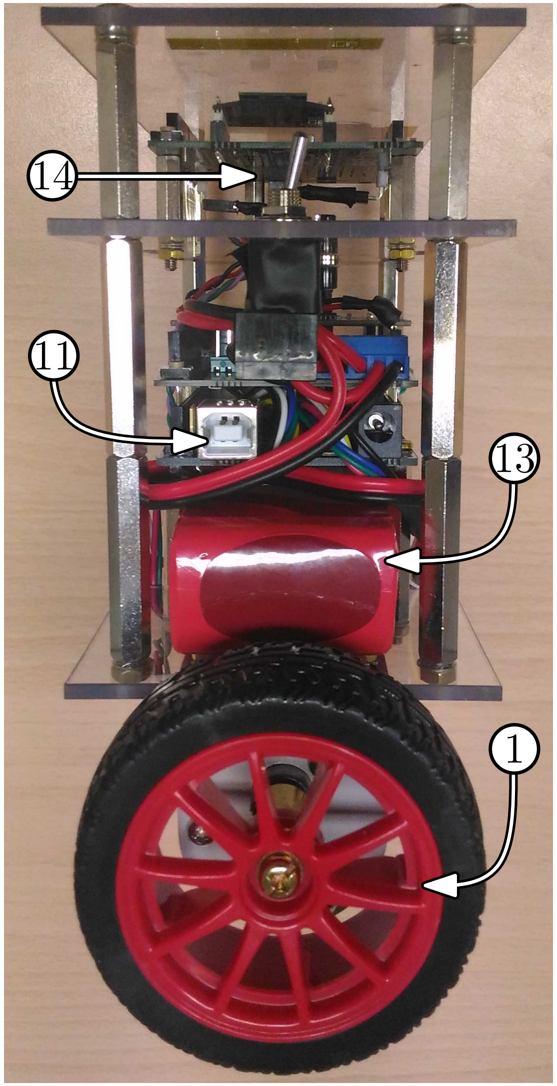
Balancing robot structure



Balancing robot structure



Balancing robot structure

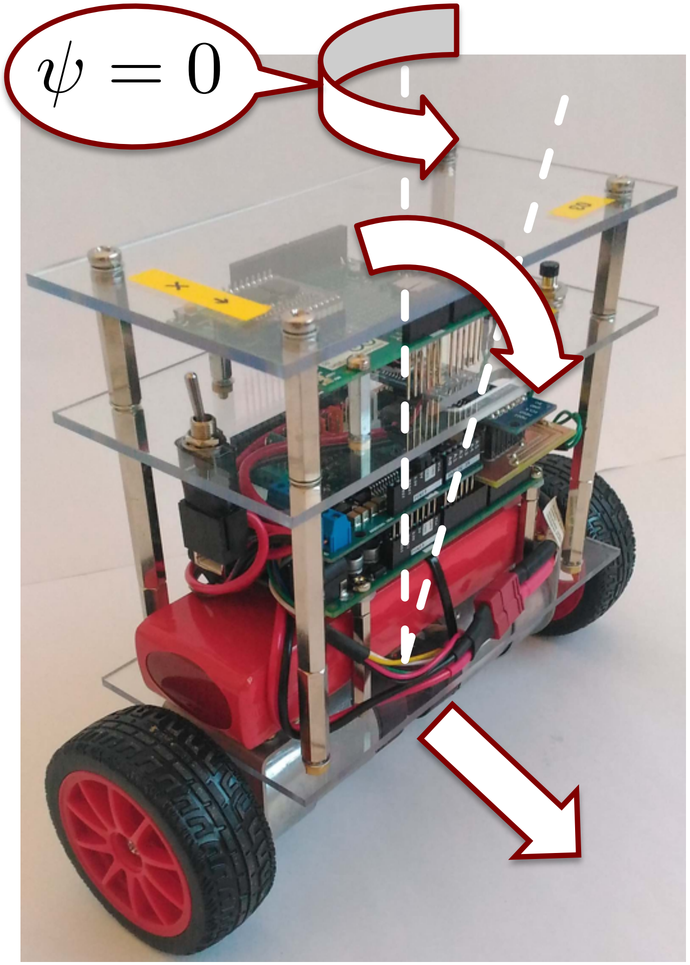


System modeling approach

Modeling approach:

- Represent the robot as a *multi-body system.*
- Identify the *generalized coordinates* required to describe the robot configuration.
- Derive the equations of motion (EoM) with the *Lagrangian approach*.
- Obtain the state-space model by *linearizing* the EoM around the upward vertical equilibrium.

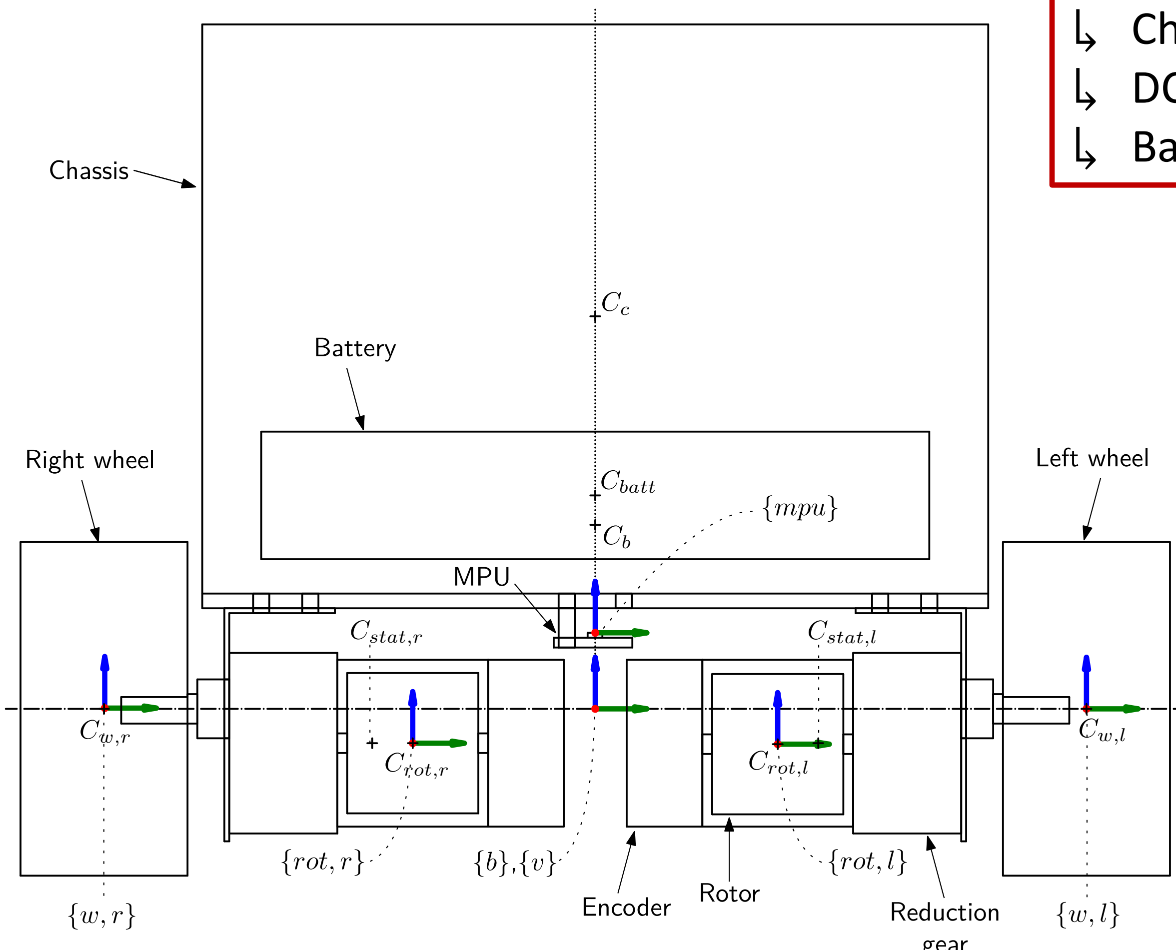
System modeling approach



<u>Modeling simplification:</u>

a simplified *planar model* is used, by assuming that the motion is constrained along a straight line.

Multi-body representation



1) Robot body:
- ↳ Chassis
- ↳ DC motor stators
- ↳ Battery

2) Wheels

3) DC motor rotors

Multi-body representation

Rigid bodies have their own *geometrical* and *inertial properties*:

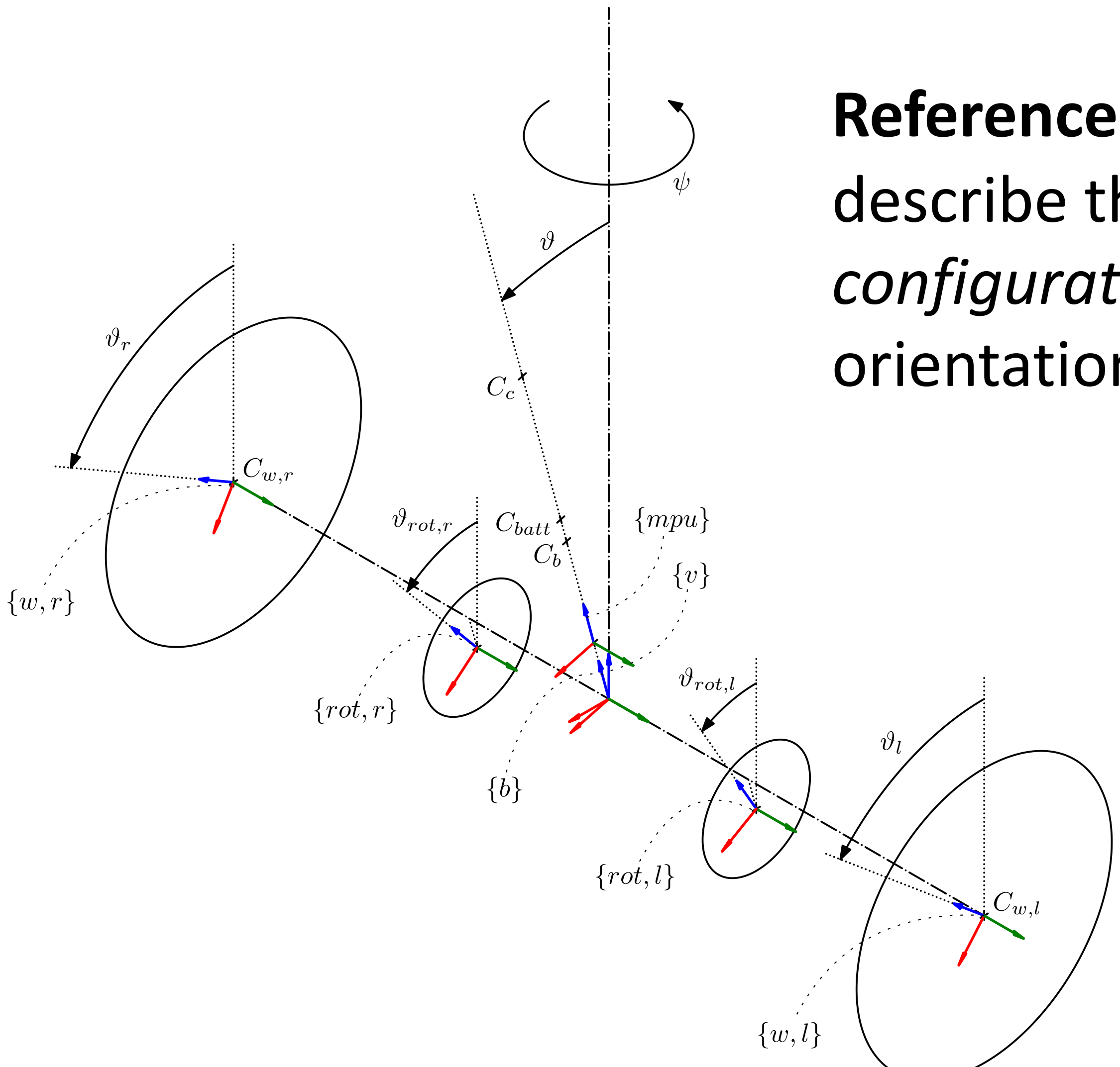
• Robot body			
Center–of–Mass coords wrt body frame $\{b\}$	$x_b^b,\ y_b^b,\ z_b^b$	$0,\ 0,\ 46.05$	[mm]
Mass	m_b	1.06	[kg]
Principal Moments–of–Inertia	$I_{b,xx},\ I_{b,yy},\ I_{b,zz}$	$4.22,\ 2.20,\ 2.65$	[gm^2]
↳ Robot chassis			
Dimensions (width, height, depth)	$w_c,\ h_c,\ d_c$	$160,\ 119,\ 80$	[mm]
Center–of–Mass coords wrt body frame $\{b\}$	$x_c^b,\ y_c^b,\ z_c^b$	$0,\ 0,\ 80$	[mm]
Mass	m_c	456	[g]
Principal Moments–of–Inertia	$I_{c,xx},\ I_{c,yy},\ I_{c,zz}$	$1.5,\ 0.78,\ 1.2$	[gm^2]
↳ Battery			
Dimensions (width, height, depth)	$w_{batt},\ h_{batt},\ d_{batt}$	$136,\ 26,\ 44$	[mm]
Center–of–Mass coords wrt body frame $\{b\}$	$x_{batt}^b,\ y_{batt}^b,\ z_{batt}^b$	$0,\ 0,\ 44$	[mm]
Mass	m_{batt}	320	[g]
Principal Moments–of–Inertia	$I_{batt,xx},\ I_{batt,yy},\ I_{batt,zz}$	$0.51,\ 0.07,\ 0.06$	[gm^2]
↳ DC gearmotor stator			
Dimensions (height, radius)	$h_{stat},\ r_{stat}$	$68.1,\ 17$	[mm]
Center–of–Mass coords wrt body frame $\{b\}$	$x_{stat}^b,\ y_{stat}^b,\ z_{stat}^b$	$0,\ \pm 52.1,\ -7$	[mm]
Mass	m_{stat}	139.75	[g]
Principal Moments–of–Inertia	$I_{stat,xx} = I_{stat,zz},\ I_{stat,yy}$	$0.064,\ 0.02$	[gm^2]

Multi-body representation

• **DC gearmotor rotor**			
Dimensions (height, radius)	$h_{rot},\ r_{rot}$	$30.7,\ 15.3$	[mm]
Center–of–Mass coords wrt body frame $\{b\}$	$x^b_{rot},\ y^b_{rot},\ z^b_{rot}$	$0,\ \pm 42.7,\ -7$	[mm]
Mass	m_{rot}	75.25	[g]
Principal Moments–of–Inertia	$I_{rot,xx} = I_{rot,zz},\ I_{rot,yy}$	$0.01,\ 0.009$	[gm^2]

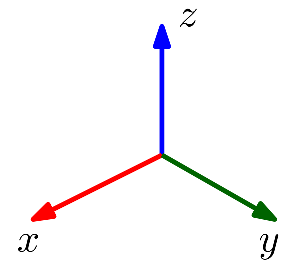
• **Wheels**			
Dimensions (height, radius)	$h_w,\ r_w$	$26,\ 34$	[mm]
Center–of–Mass coords wrt body frame $\{b\}$	$x^b_w,\ y^b_w,\ z^b_w$	$0,\ \pm 100,\ 0$	[mm]
Mass	m_w	50	[g]
Principal Moments–of–Inertia	$I_{w,xx} = I_{w,zz},\ I_{w,yy}$	$0.017,\ 0.029$	[gm^2]

Multi-body representation

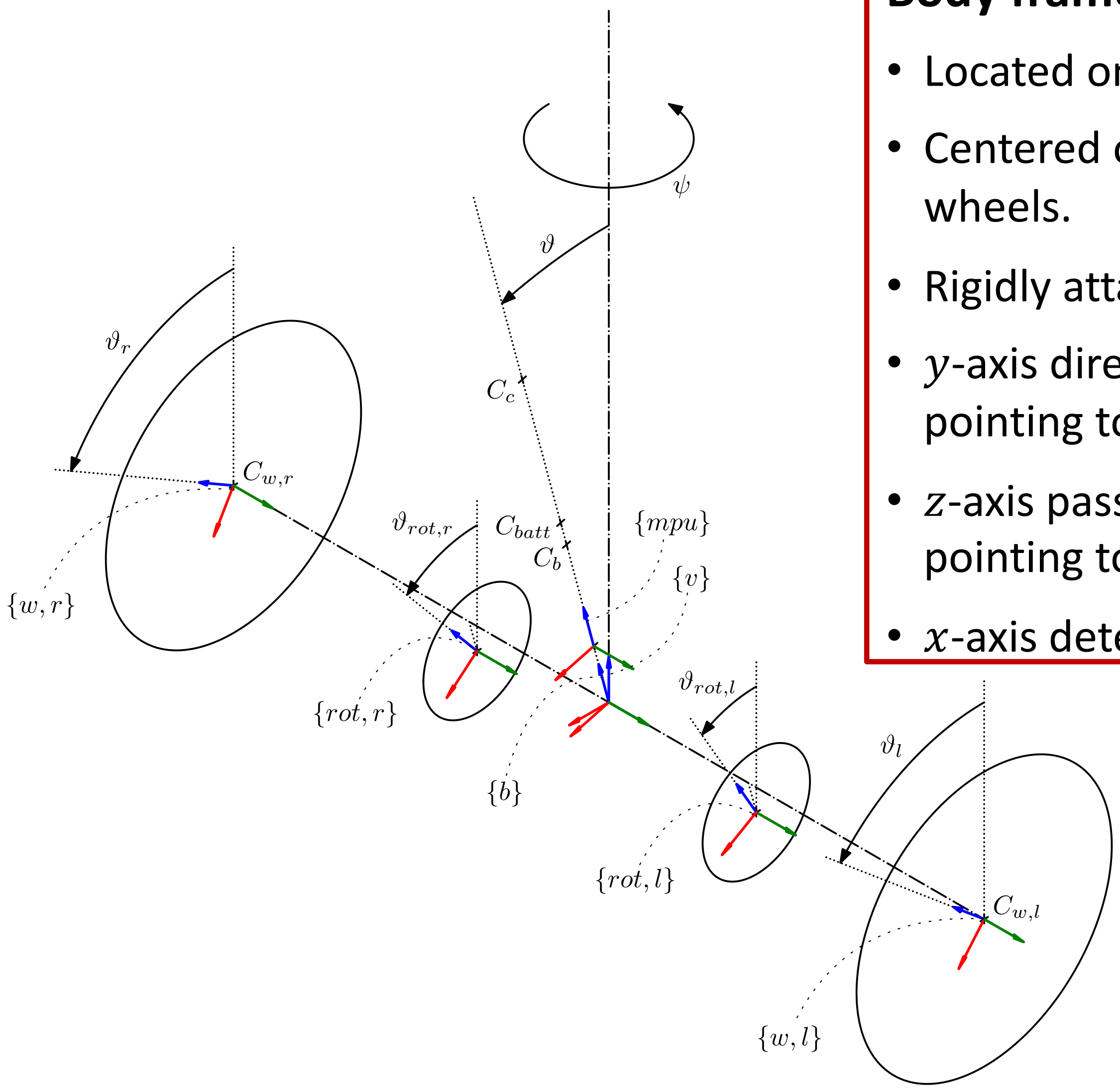


Reference frames are used to describe the robot *structure* and *configuration (i*.e. position and orientation of *each body* in space*).*

Reference frame color convention



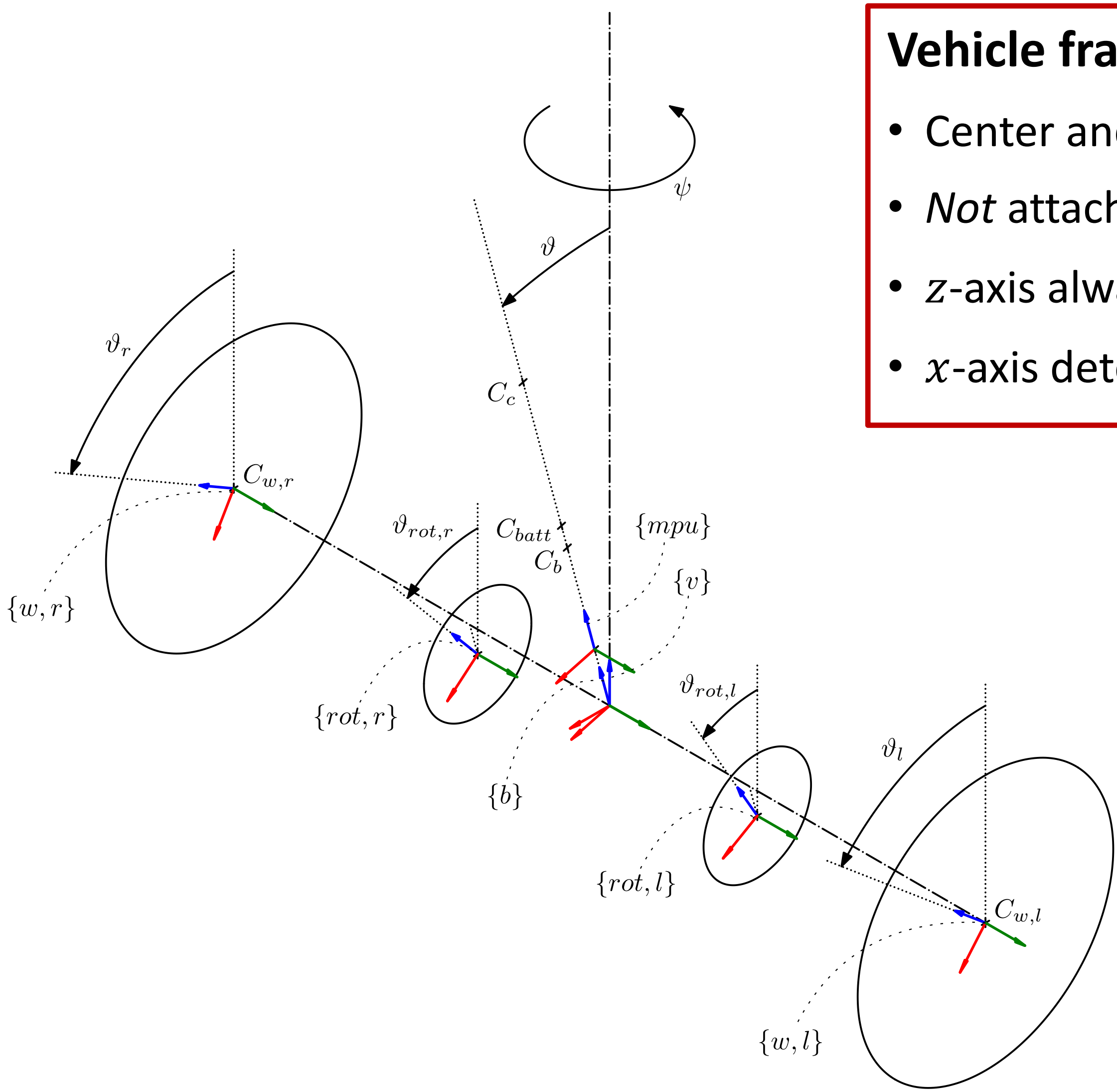
Multi-body representation



Body frame $\{b\}$

- Located on the wheels rotation axis.
- Centered on the midpoint between the two wheels.
- Rigidly attached to the robot chassis.
- y-axis directed along the wheel axis, pointing toward the left wheel.
- z-axis passing through the body CoM, pointing toward it.
- x-axis determined by the right-hand rule.

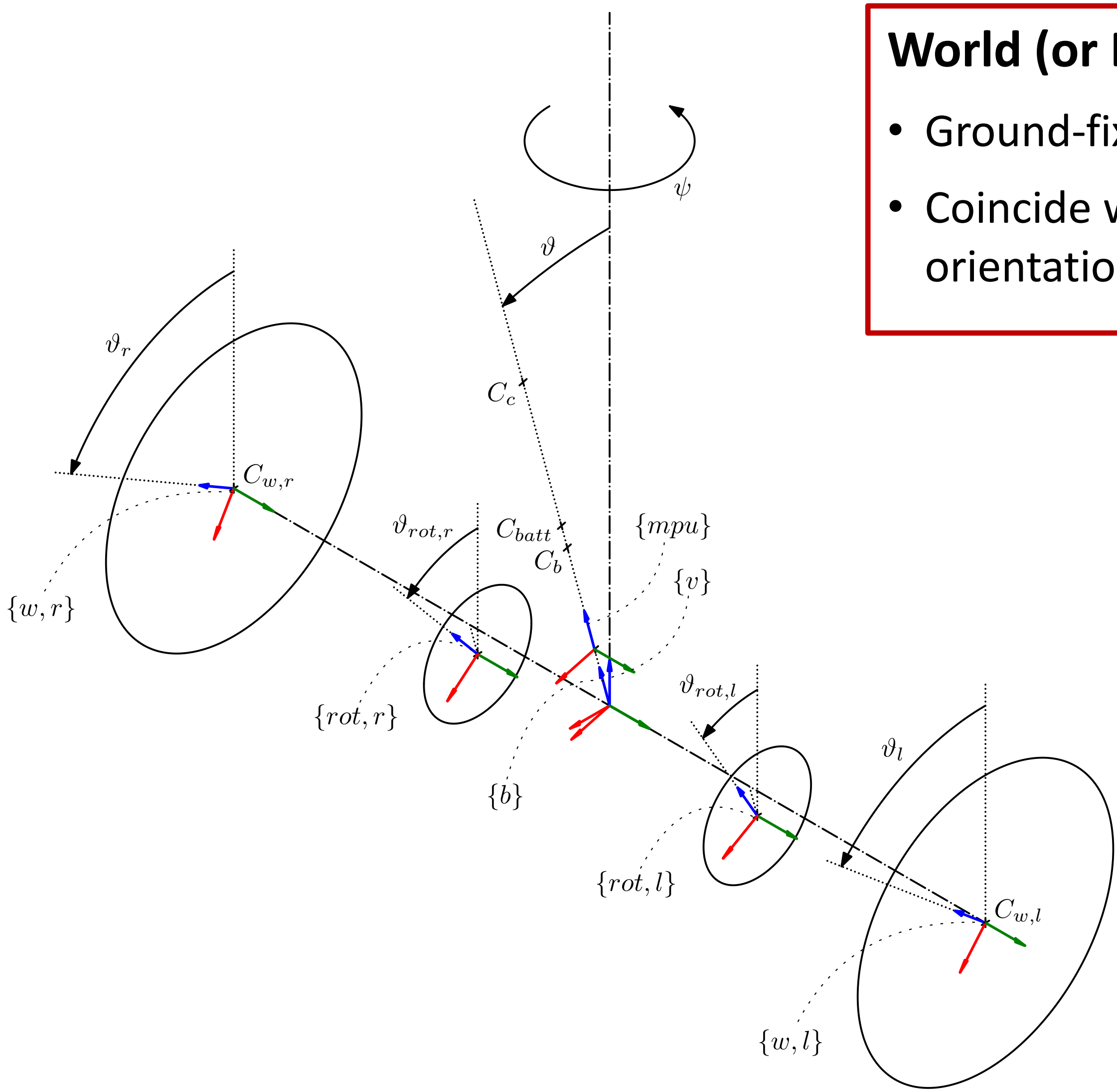
Multi-body representation



Vehicle frame $\{v\}$

- Center and y-axis as the body frame.
- *Not* attached to the robot chassis.
- z-axis always opposed to the gravity vector.
- x-axis determined by the right-hand rule.

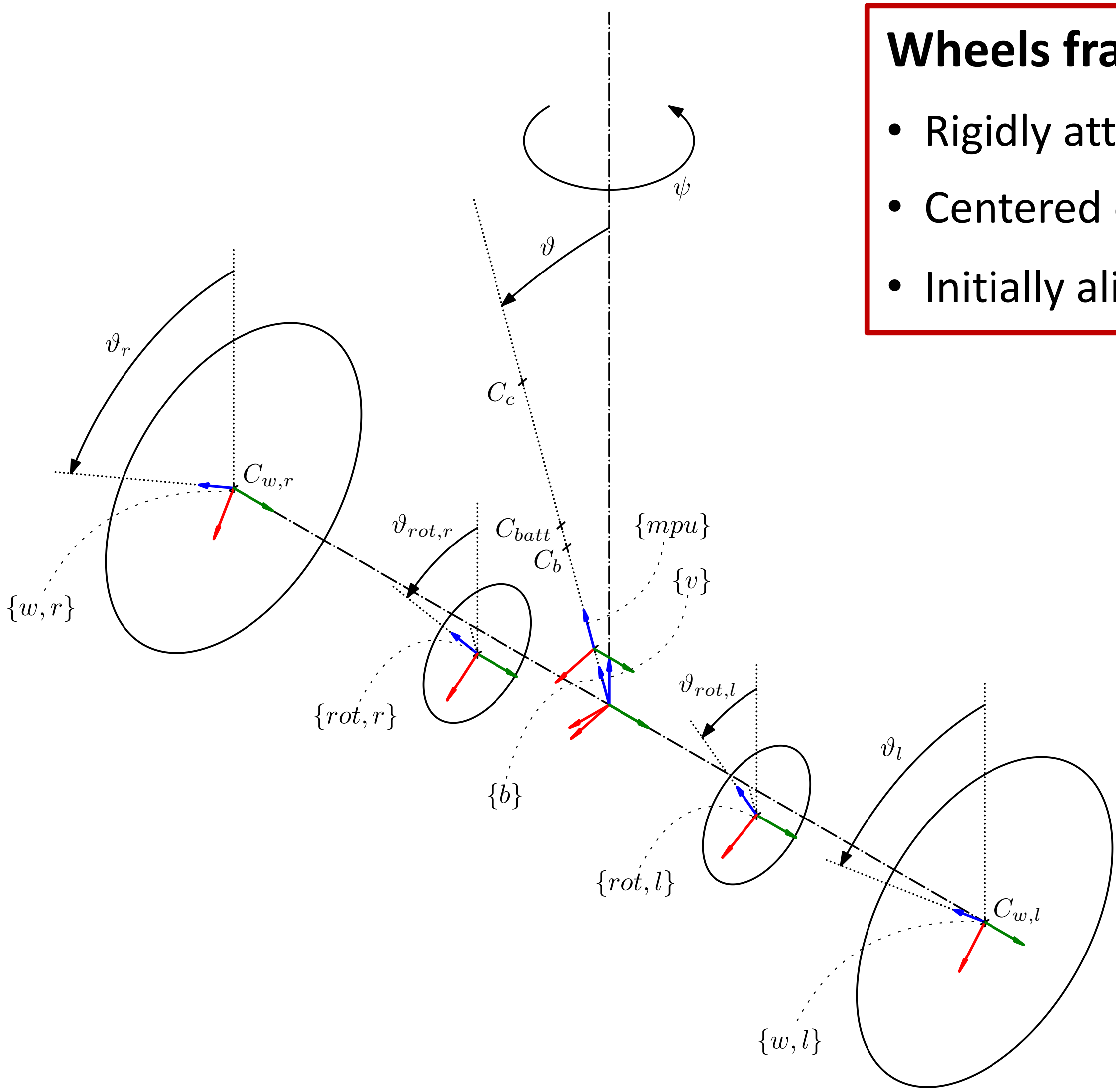
Multi-body representation



World (or Earth) frame $\{o\}$

- Ground-fixed position and orientation.
- Coincide with the initial position and orientation of the vehicle frame.

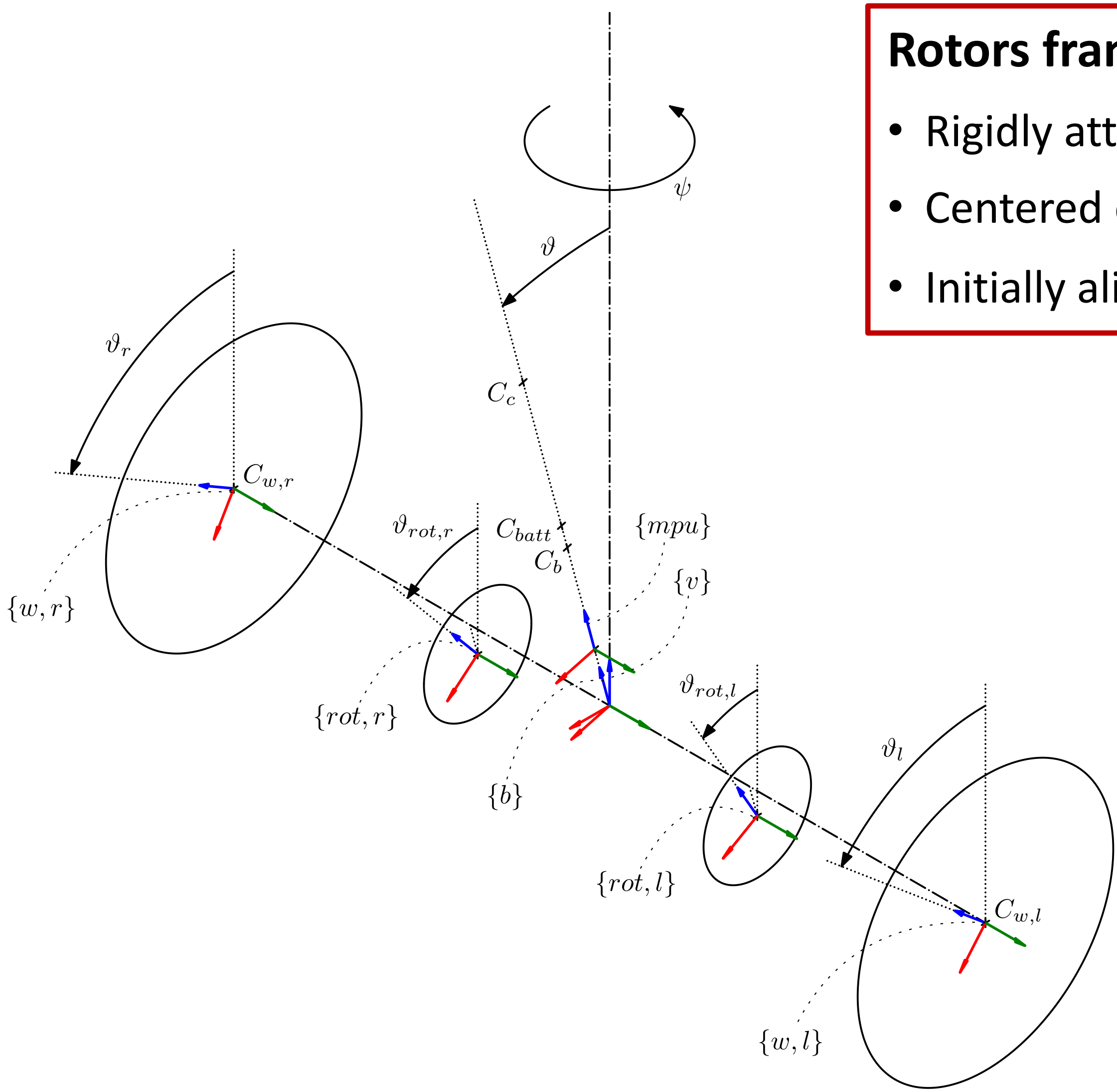
Multi-body representation



Wheels frames $\{w,l\}$, $\{w,r\}$

- Rigidly attached to the wheels bodies.
- Centered on the wheels CoM.
- Initially aligned with the vehicle frame.

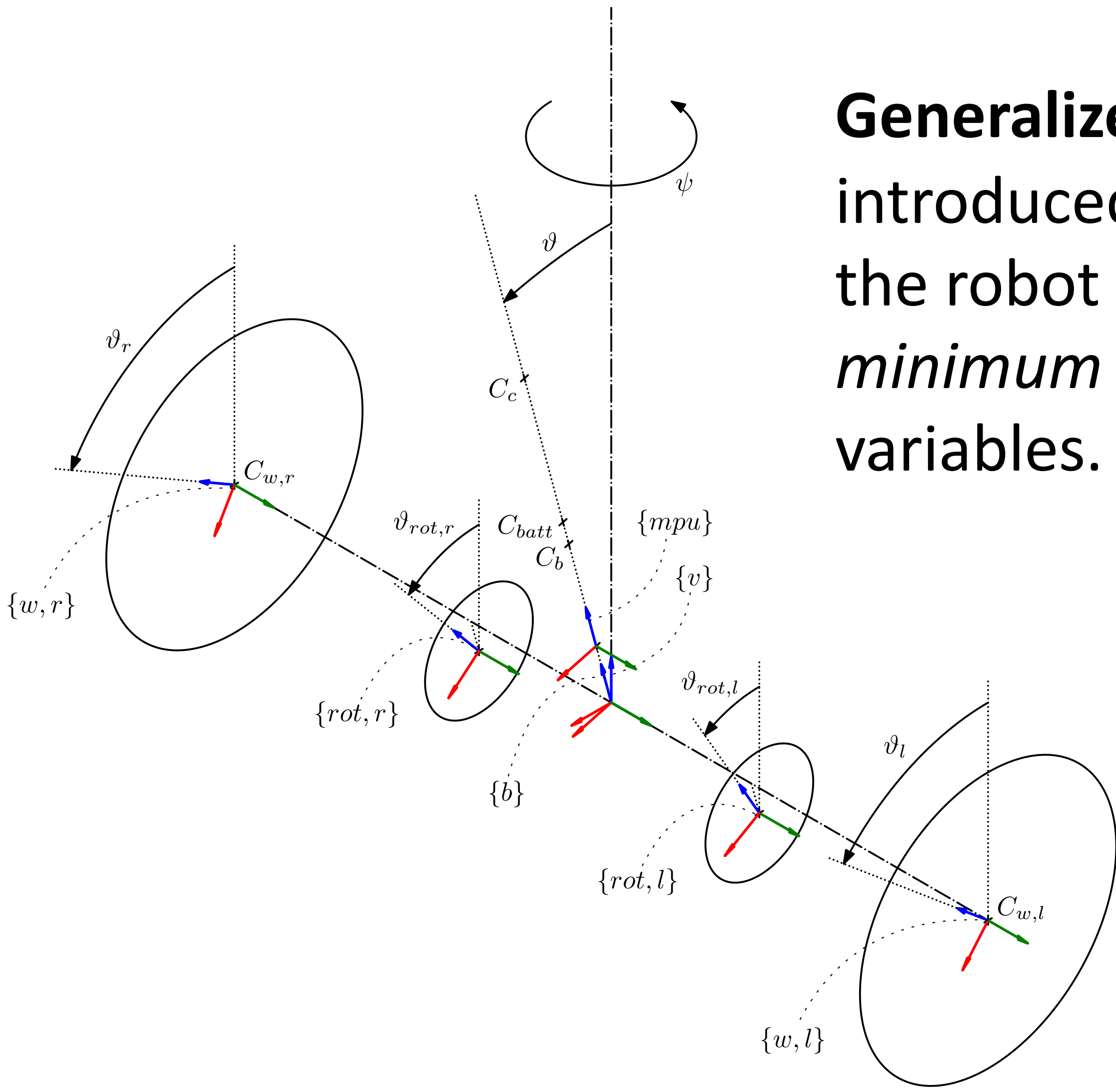
Multi-body representation



Rotors frames $\{rot,l\}$, $\{rot,r\}$

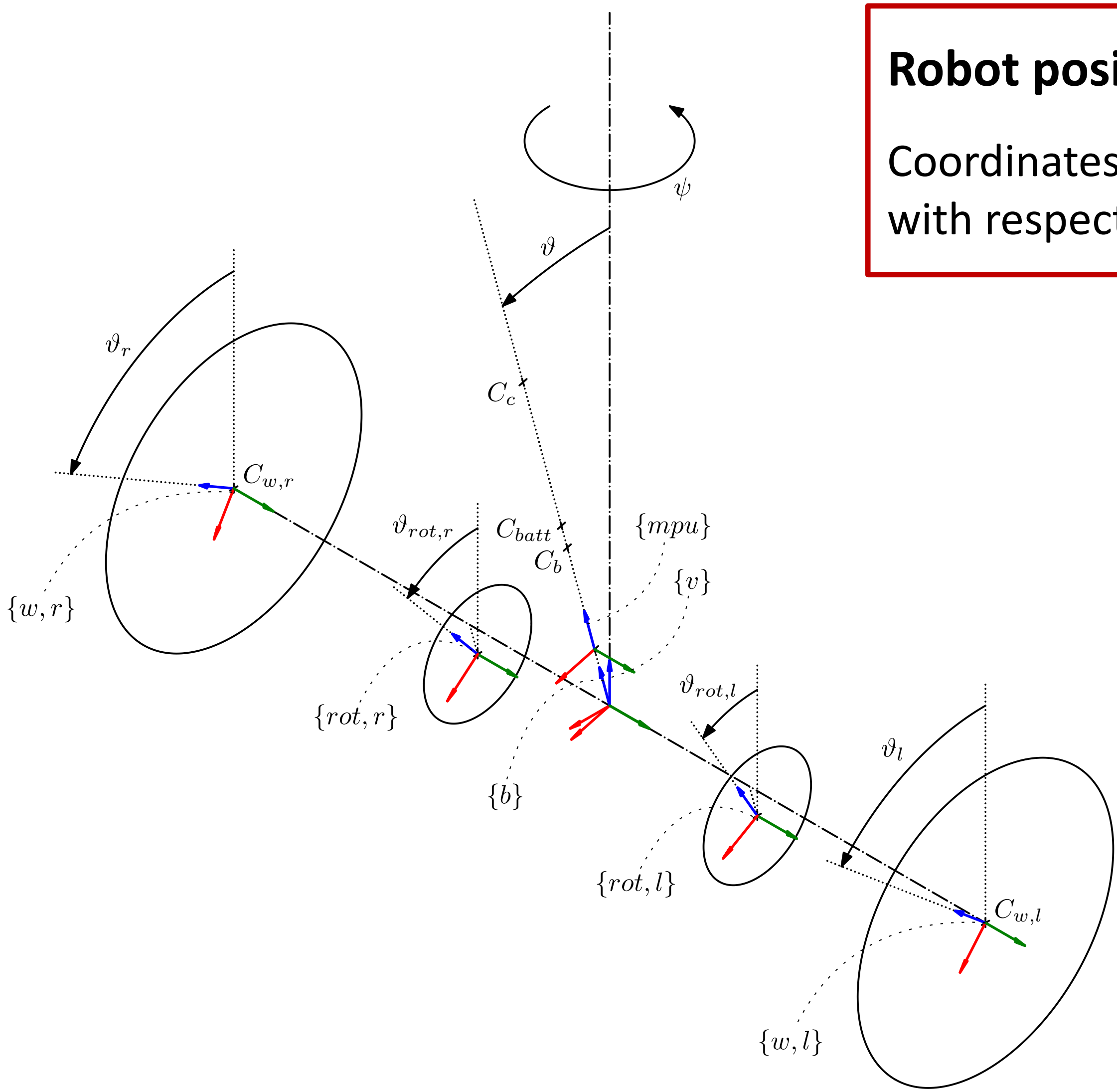
- Rigidly attached to the motors rotors.
- Centered on the rotors CoM.
- Initially aligned with the vehicle frame.

Generalized coordinates



Generalized coordinates are introduced to *uniquely* describe the robot configuration with a *minimum* set of independent variables.

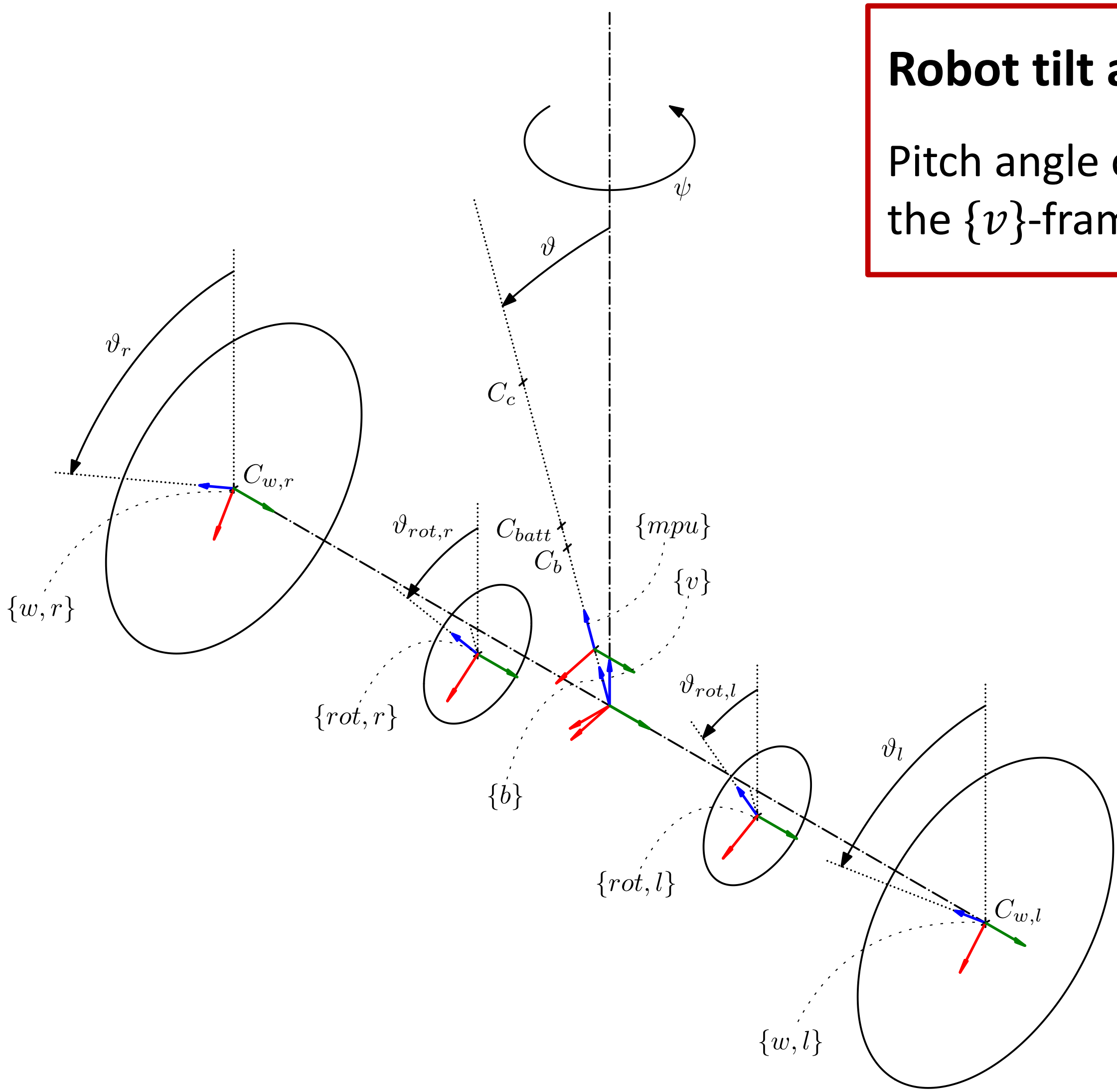
Generalized coordinates



Robot position $\boldsymbol{p}_v^o = [\, x_v, y_v, \ z_v \,]^T$

Coordinates of the origin of the $\{v\}$-frame with respect to the $\{o\}$-frame.

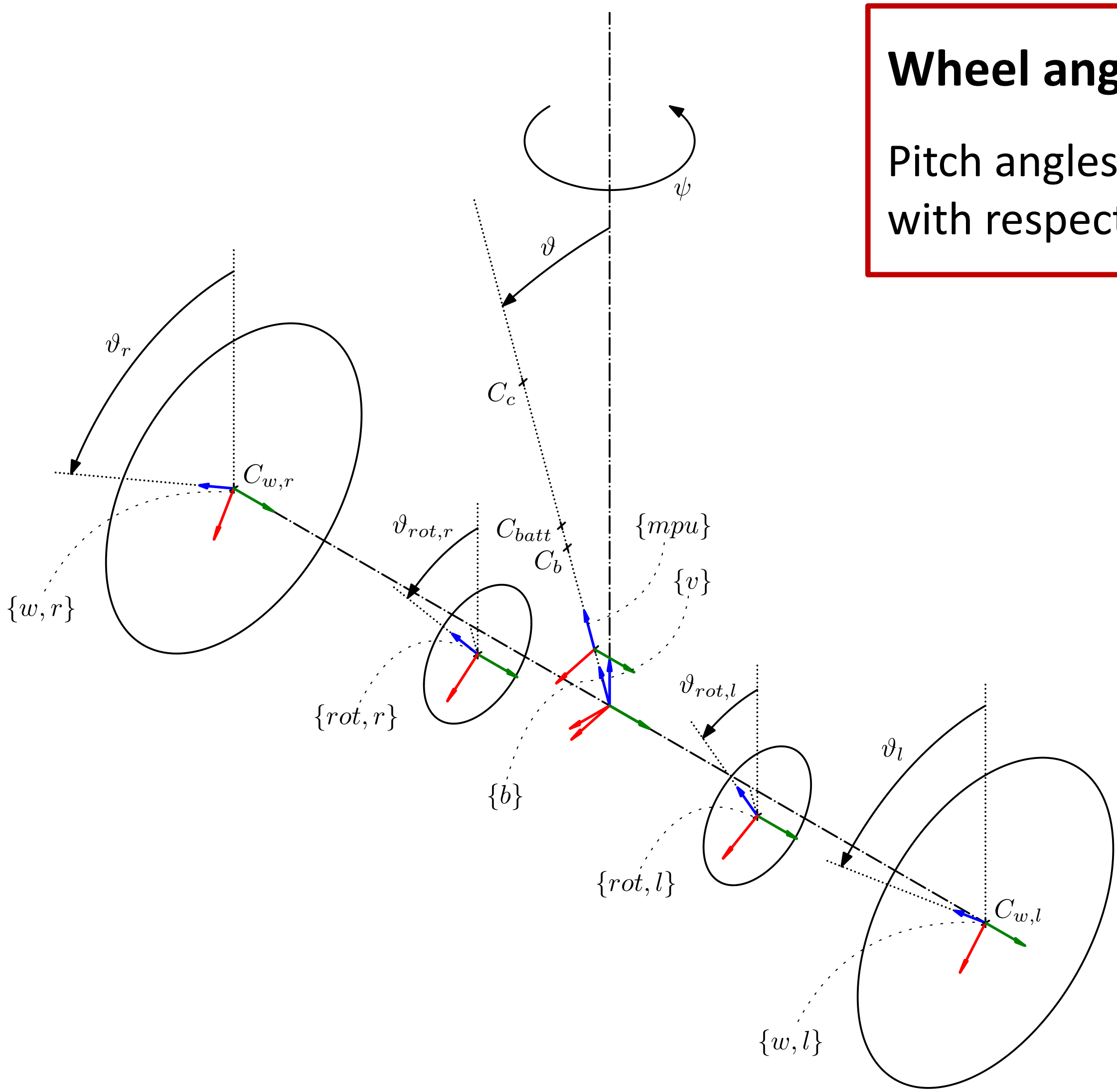
Generalized coordinates



Robot tilt angle ϑ

Pitch angle of the $\{b\}$-frame with respect to the $\{v\}$-frame.

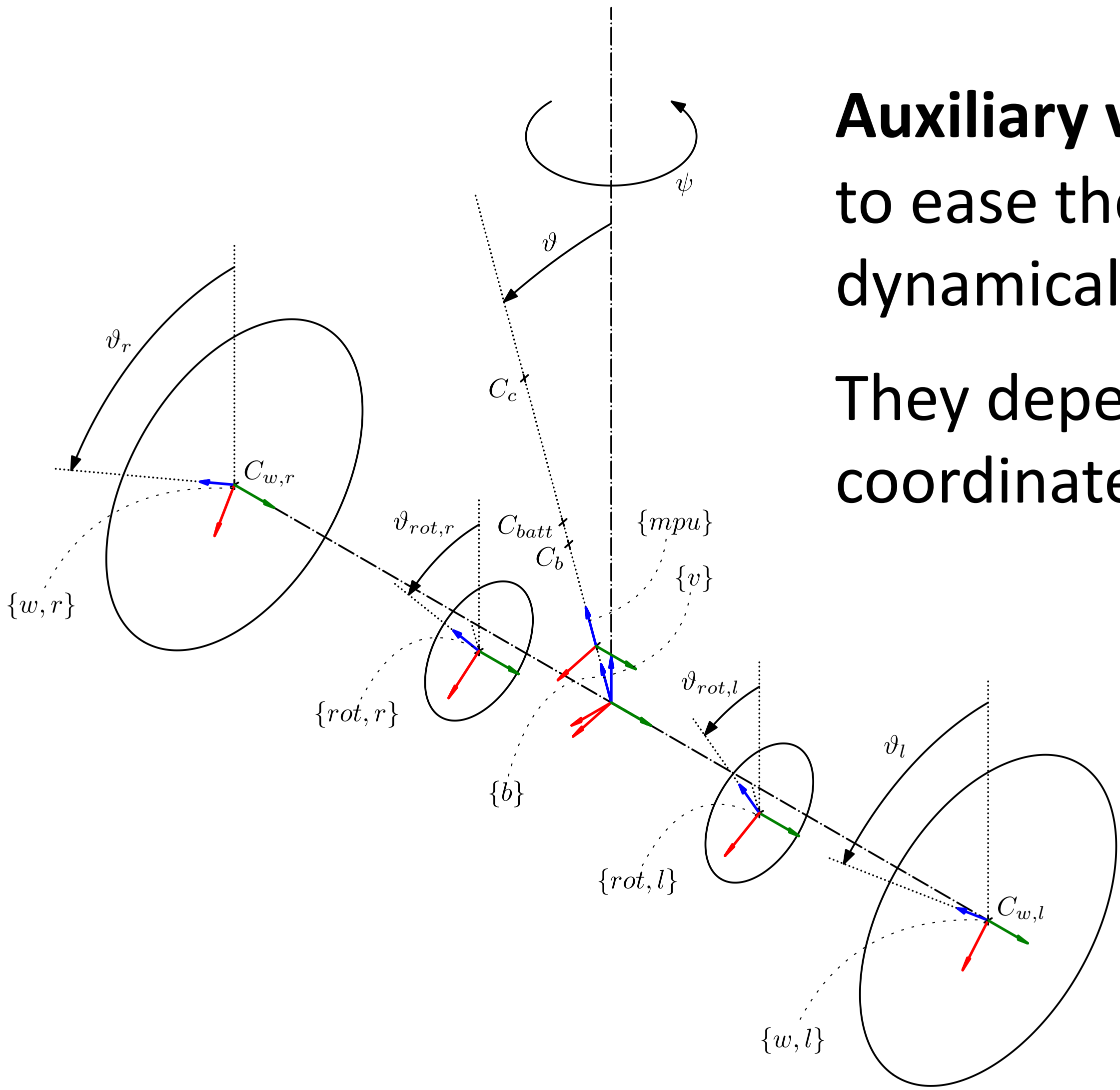
Generalized coordinates



Wheel angles ϑ_l and ϑ_r

Pitch angles of the $\{w,l\}$ and $\{w,l\}$ frames with respect to the $\{v\}$-frame.

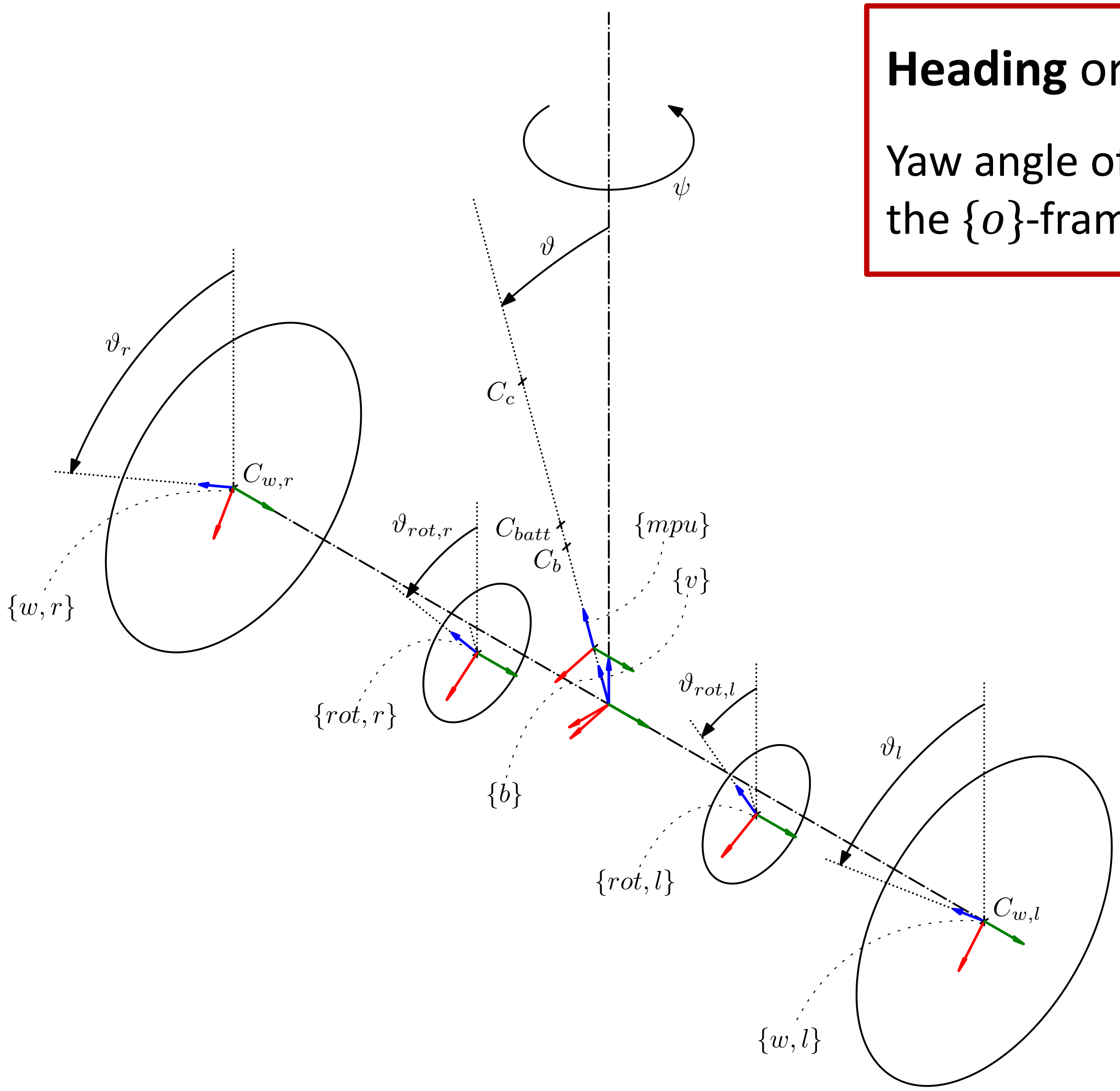
Generalized coordinates



Auxiliary variables are introduced to ease the derivation of the dynamical model.

They depend on the generalized coordinates.

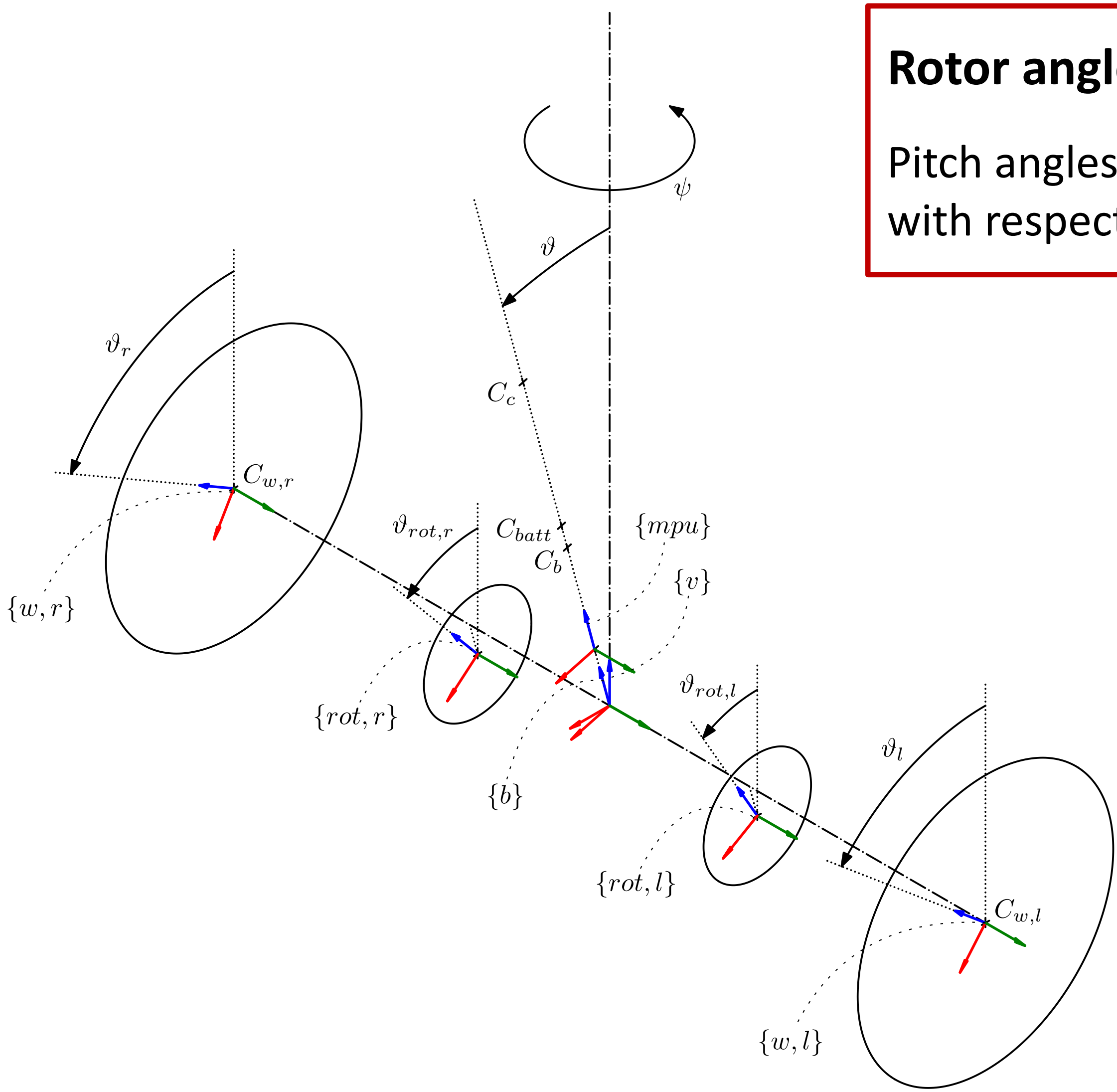
Generalized coordinates



Heading or **yaw angle** ψ

Yaw angle of the $\{v\}$-frame with respect to the $\{o\}$-frame.

Generalized coordinates



Rotor angles $\vartheta_{rot,l}$ and $\vartheta_{rot,r}$

Pitch angles of the $\{rot, l\}$ and $\{rot, l\}$ frames with respect to the $\{v\}$-frame.

Generalized coordinates

Auxiliary variables depend on the generalized coordinates because of the presence of **kinematics constraints**:

- *Gearbox mechanical coupling*: relate the rotors angles $\vartheta_{rot,l/r}$ to wheels angles $\vartheta_{l/r}$ and robot tilt angle ϑ.

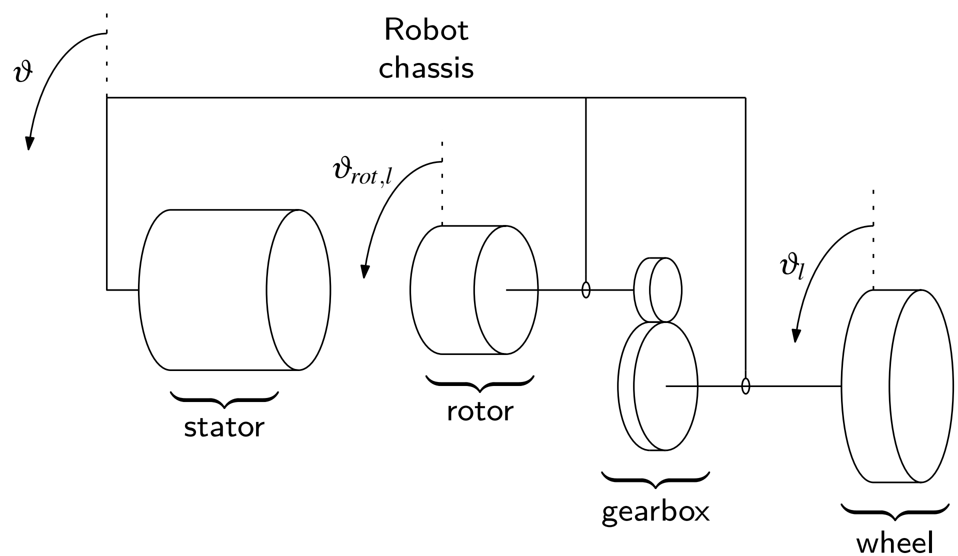
- *No side-slip* and *pure rolling conditions*: relate the robot heading angle ψ to the wheels angles $\vartheta_{l/r}$.

Generalized coordinates

<u>Gearbox mechanical coupling:</u> let

$$\Delta\vartheta_l = \vartheta_l - \vartheta\,, \qquad \Delta\vartheta_{rot,l} = \vartheta_{rot,l} - \vartheta$$

be the angles by which the rotor and wheel rotates with respect to the robot body.

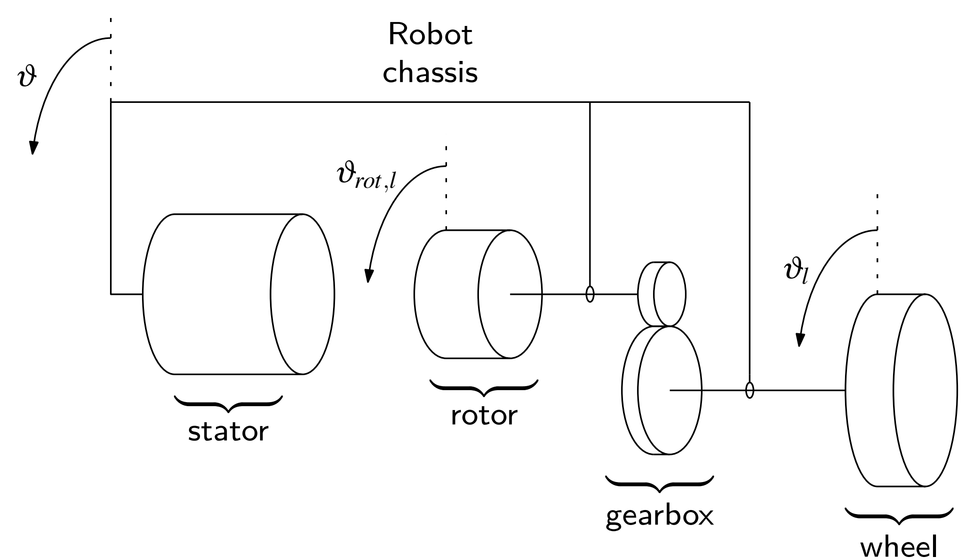


Generalized coordinates

The gearbox imposes that:

$$\Delta\vartheta_{rot,l} = N\,\Delta\vartheta_l \quad \Rightarrow \quad \vartheta_{rot,l} = \vartheta + N\,(\vartheta_l - \vartheta)$$

A similar result holds for the right rotor angle $\vartheta_{rot,r}$.



Generalized coordinates

<u>No side-slip and pure rolling conditions</u>: the velocity of the wheel center is parallel to the wheel sagittal plane, and is proportional to the wheel rotational velocity.

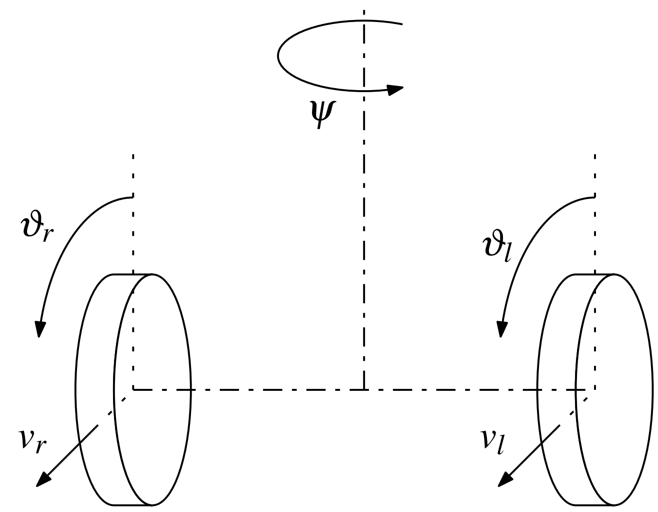
$$w\,\dot{\psi} = v_r - v_l\,, \qquad \begin{aligned} v_l &= r\,\dot{\vartheta}_l \\ v_r &= r\,\dot{\vartheta}_r \end{aligned}$$

$$\Downarrow$$

$$\dot{\psi} = \frac{r}{w}\left(\dot{\vartheta}_r - \dot{\vartheta}_l\right)$$

$$\Downarrow$$

$$\psi = \psi(0) + \frac{r}{w}\left(\vartheta_r - \vartheta_l\right)$$



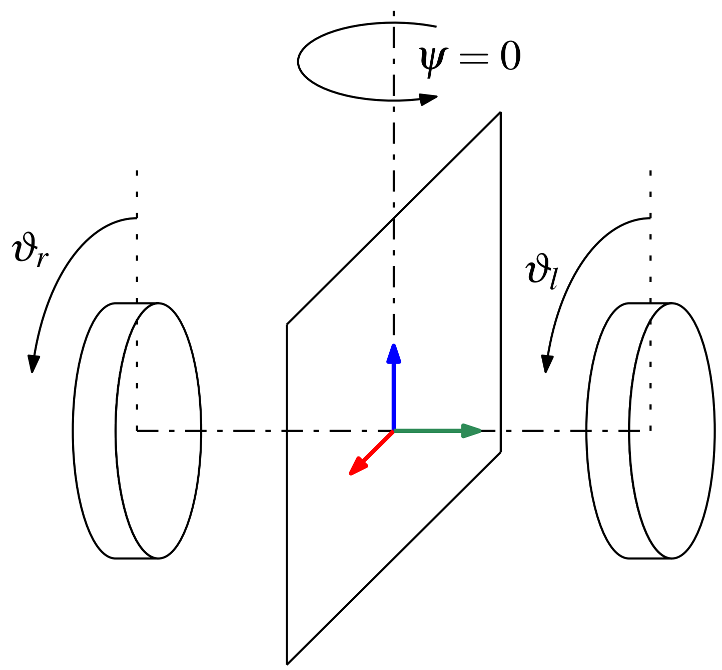
r : wheel radius

w : distance between wheels ($w = 2|y_w^b|$)

Planar model simplification

<u>Planar model simplification</u>: the robot moves along a *straight line*, i.e. the heading angle is *constant*.

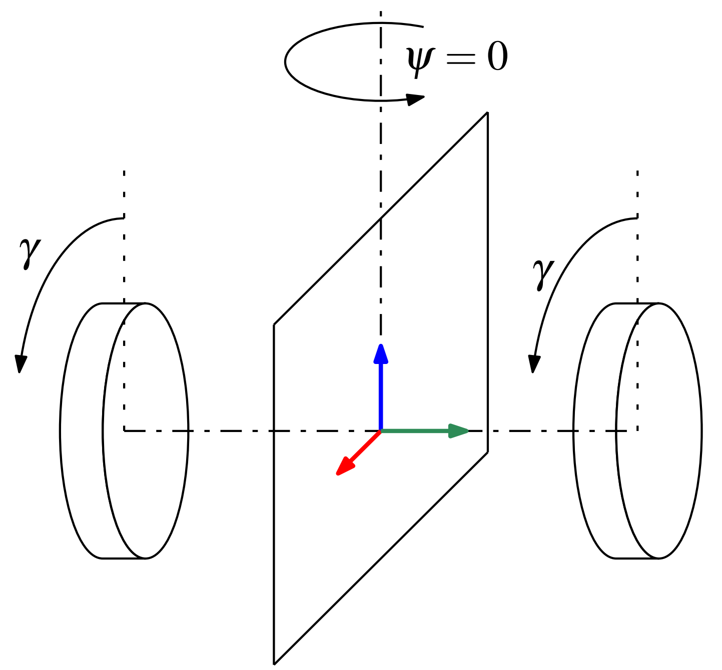
W.l.o.g. assume $\psi = 0$ so that $y_v = 0$ and the motion is confined on the xz-plane of the world frame $\{o\}$.



Planar model simplification

If the robot moves along a straight line, the wheels angles $\vartheta_{l/r}$ are identical. Let

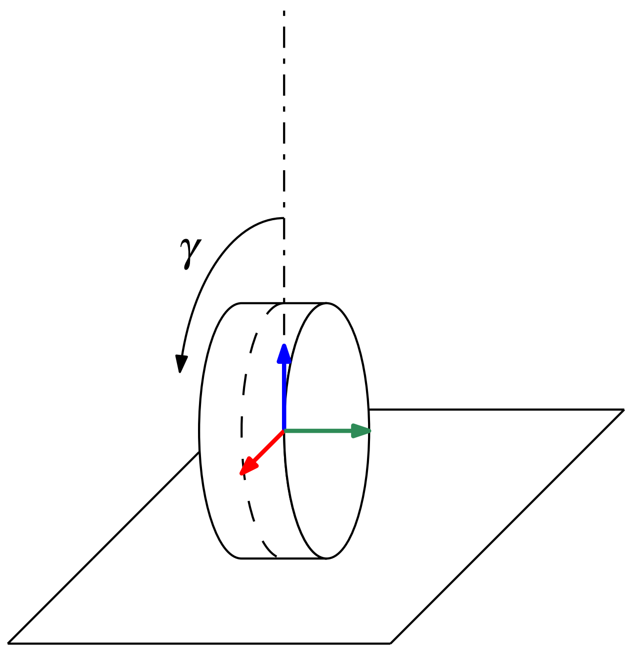
$$\gamma \triangleq \text{common value of the two wheels angles}$$



Planar model simplification

W.l.o.g. assume also that the robot moves on a horizontal flat surface, so that:

$$z_v = 0 \quad \text{and} \quad x_v = r\gamma$$



Derivation of the analytical model

The Equations-of-Motion (EoM) are derived by using the **Lagrangian approach**:

1) obtain the *kinematic equations* of each body;

2) compute the *kinetic* and *potential energies* of each body;

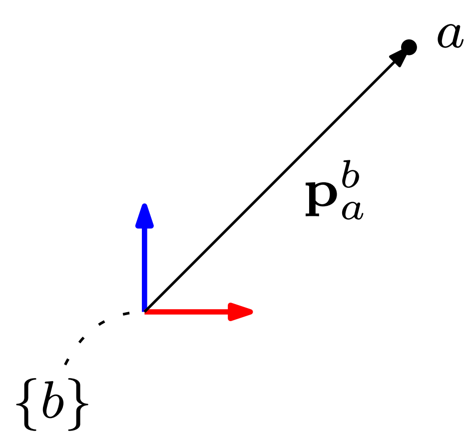
3) compute the *Lagrangian function*;

4) obtain the EoM by evaluating the *Lagrange equations*.

Derivation of the analytical model

Notation:

$\boldsymbol{p}_a^b = [x_a^b,\ z_a^b]^T$: **position vector** of point a, expressed with respect to frame $\{b\}$.

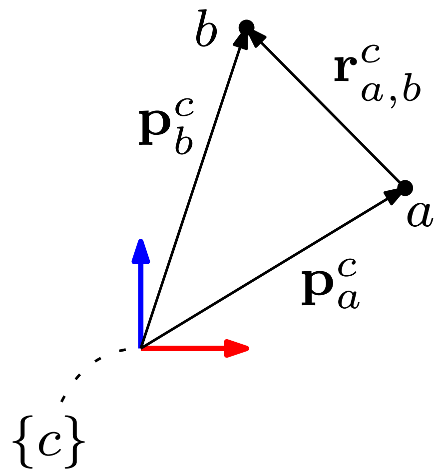


If $\{b\}$ coincides with the world frame $\{o\}$, then the superscripts in the position coordinates are omitted.

Derivation of the analytical model

Notation:

$\boldsymbol{r}^{c}_{a,b} = \left[x^{c}_{a,b},\ z^{c}_{a,b}\right]^{T}$: **displacement vector** from point a to point b, expressed with respect to frame $\{c\}$.

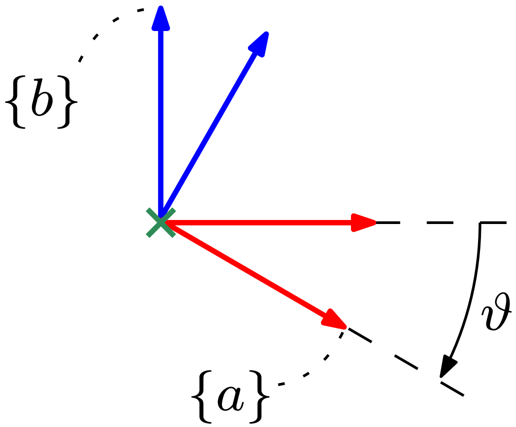


$$\boldsymbol{r}^{c}_{a,b} = \boldsymbol{p}^{c}_{b} - \boldsymbol{p}^{c}_{a}$$

Derivation of the analytical model

<u>Notation</u>:

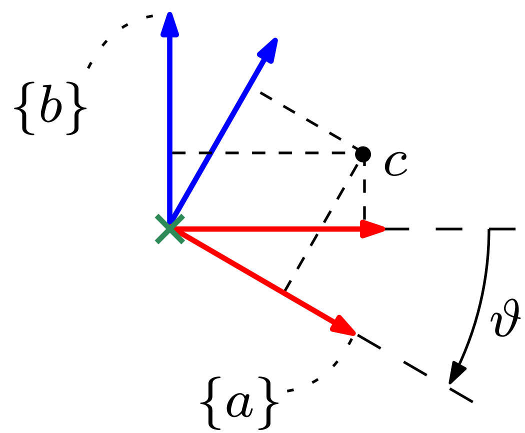
$\boldsymbol{R}_a^b$: **rotation matrix** of frame $\{a\}$ w.r.t. $\{b\}$.



$$\boldsymbol{R}_a^b = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}$$

Derivation of the analytical model

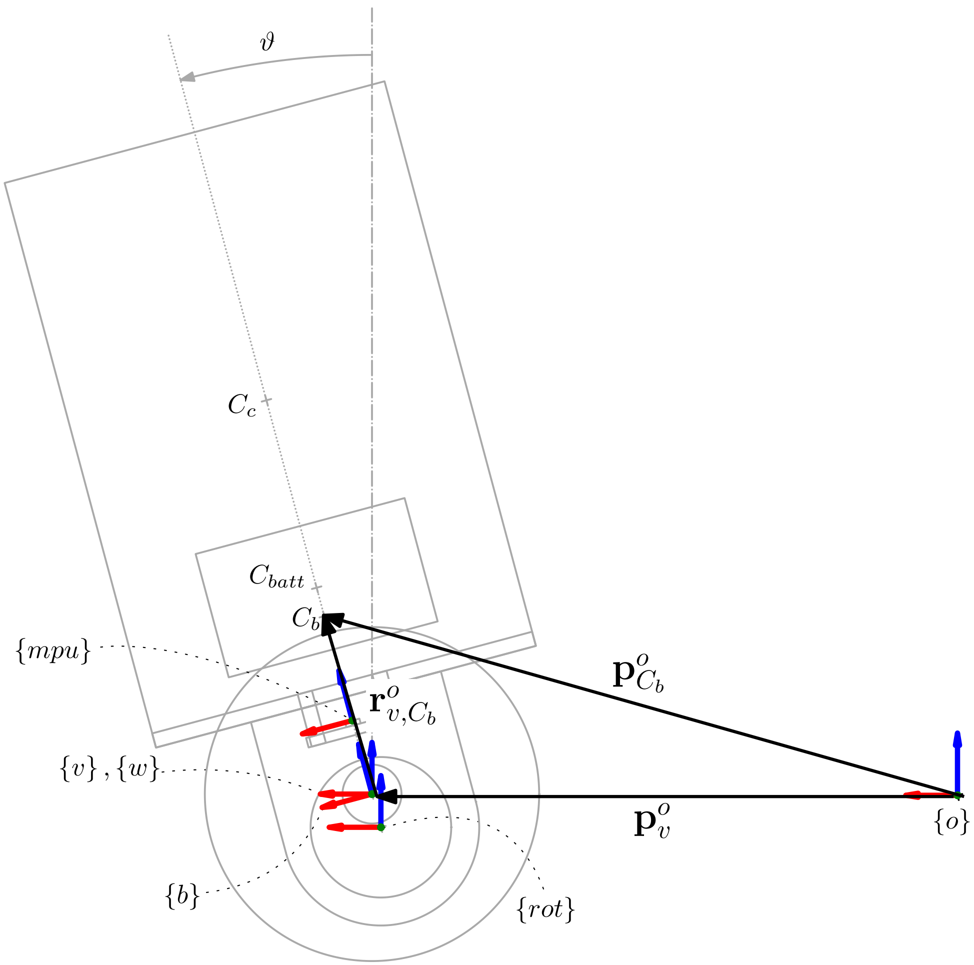
<u>Note</u>: the rotation matrix can be used to express a given position vector w.r.t. a different reference frame.



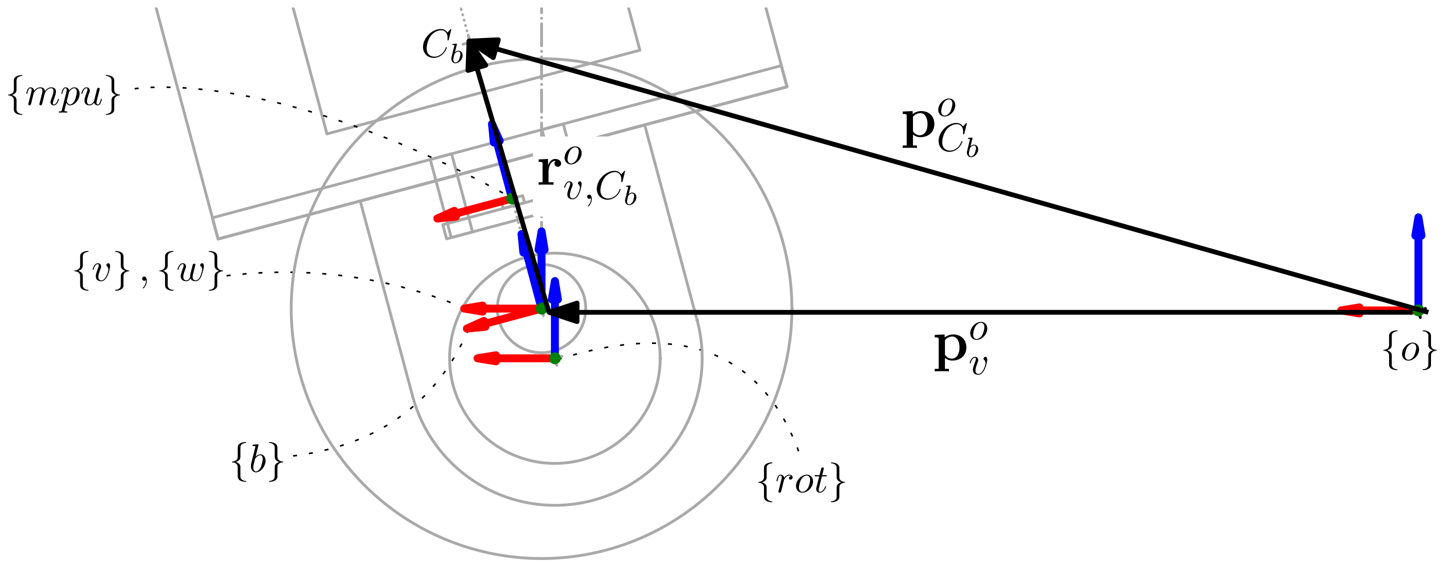
$$\boldsymbol{p}_c^b = \boldsymbol{R}_a^b \, \boldsymbol{p}_c^a$$

Derivation of the analytical model

Robot body kinematics:



Derivation of the analytical model



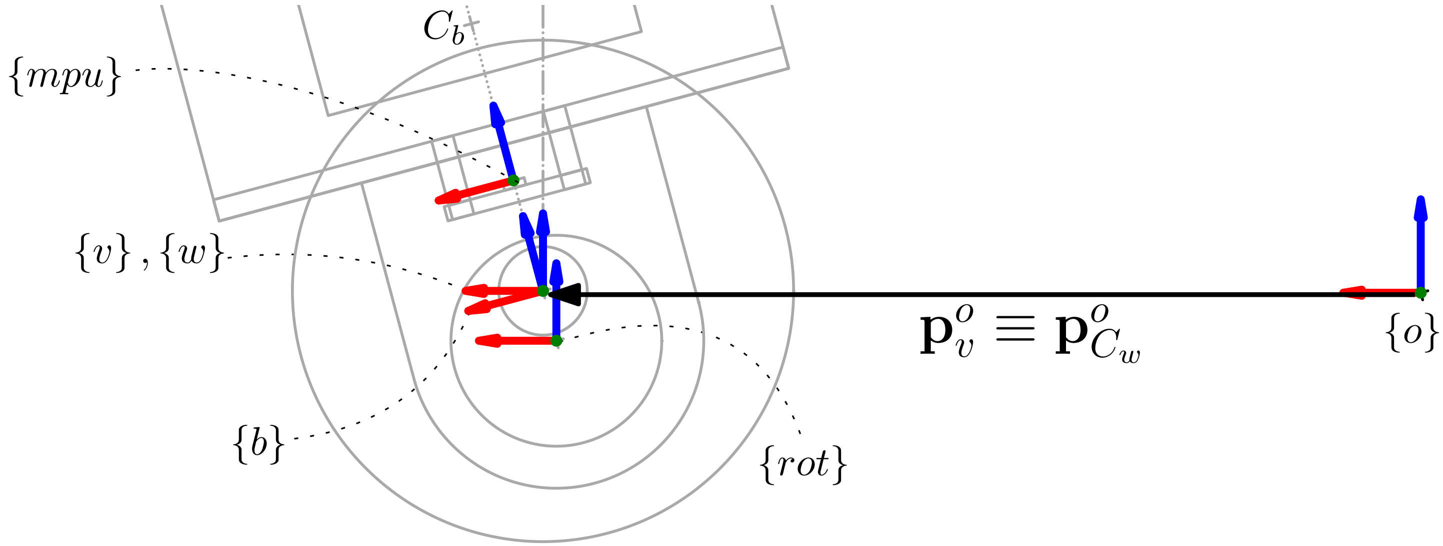
$$\boldsymbol{p}^o_{C_b} = \boldsymbol{p}^o_v + \boldsymbol{r}^o_{v,C_b} = \boldsymbol{p}^o_v + \boldsymbol{R}^o_b\,\boldsymbol{p}^b_{C_b} = \cdots$$

$$\cdots = \begin{bmatrix} r\,\gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} 0 \\ l \end{bmatrix} = \begin{bmatrix} r\,\gamma + l\,\sin\vartheta \\ l\,\cos\vartheta \end{bmatrix}$$

$$\dot{\boldsymbol{p}}^o_{C_b} = \dot{\boldsymbol{p}}^o_v + \frac{d\boldsymbol{R}^o_b}{dt}\,\boldsymbol{p}^b_{C_b} + \boldsymbol{R}^o_b\,\dot{\boldsymbol{p}}^b_{C_b} = \begin{bmatrix} r\,\dot{\gamma} + l\,\cos\vartheta\,\dot{\vartheta} \\ -l\,\sin\vartheta\,\dot{\vartheta} \end{bmatrix}$$

Derivation of the analytical model

Wheel kinematics:

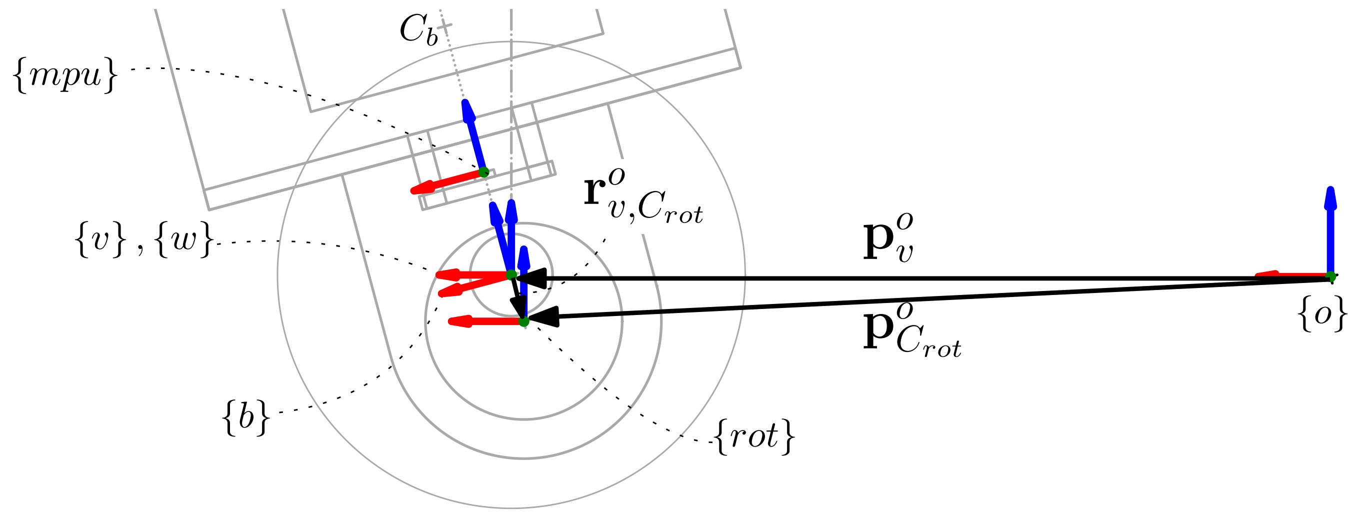


$$\boldsymbol{p}_{C_w}^o = \boldsymbol{p}_v^o + \boldsymbol{r}_{v,C_w}^o = \boldsymbol{p}_v^o = \begin{bmatrix} r\,\gamma \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{p}}_{C_w}^o = \dot{\boldsymbol{p}}_v^o = \begin{bmatrix} r\,\dot{\gamma} \\ 0 \end{bmatrix}$$

Derivation of the analytical model

Rotor kinematics:



$$\boldsymbol{p}^o_{C_{rot}} = \boldsymbol{p}^o_v + \boldsymbol{r}^o_{v,C_{rot}} = \boldsymbol{p}^o_v + \boldsymbol{R}^o_b\,\boldsymbol{p}^b_{C_{rot}} = \cdots$$

$$\cdots = \begin{bmatrix} r\,\gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} 0 \\ z^b_{rot} \end{bmatrix} = \begin{bmatrix} r\,\gamma + z^b_{rot}\,\sin\vartheta \\ z^b_{rot}\,\cos\vartheta \end{bmatrix}$$

$$\dot{\boldsymbol{p}}^o_{C_{rot}} = \dot{\boldsymbol{p}}^o_v + \frac{d\boldsymbol{R}^o_b}{dt}\,\boldsymbol{p}^b_{C_{rot}} + \boldsymbol{R}^o_b\,\dot{\boldsymbol{p}}^b_{C_{rot},l} = \begin{bmatrix} r\,\dot{\gamma} + z^b_{rot}\,\cos\vartheta\,\dot{\vartheta} \\ -\,z^b_{rot}\,\sin\vartheta\,\dot{\vartheta} \end{bmatrix}$$

Derivation of the analytical model

Robot body kinetic and potential energies:

$$T_b = \underbrace{\frac{1}{2} m_b \left(\dot{\boldsymbol{p}}^o_{C_b}\right)^T \left(\dot{\boldsymbol{p}}^o_{C_b}\right)}_{\text{Translational kinetic energy}} + \underbrace{\frac{1}{2} I_{b,yy}\, \dot{\vartheta}^2}_{\text{Rotational kinetic energy}} = \cdots$$

$$\cdots = \frac{1}{2} m_b\, r^2\, \dot{\gamma}^2 + \frac{1}{2} \left(I_{b,yy} + m_b\, l^2\right) \dot{\vartheta}^2 + m_b\, r\, l \cos\vartheta\, \dot{\gamma}\, \dot{\vartheta}$$

$$U_b = m_b\, g\, z_{C_b} = m_b\, g\, l \cos\vartheta$$

Derivation of the analytical model

Wheel kinetic and potential energies:

$$T_w = \underbrace{\frac{1}{2}(2m_w)\left(\dot{\boldsymbol{p}}^o_{C_w}\right)^T\left(\dot{\boldsymbol{p}}^o_{C_w}\right)}_{\text{Translational kinetic energy}} + \underbrace{\frac{1}{2}(2I_{w,yy})\,\dot{\gamma}^2}_{\text{Rotational kinetic energy}} = \cdots$$

$$\cdots = \left(I_{w,yy} + m_w r^2\right)\dot{\gamma}^2$$

$$U_w = (2m_w)\,g\,z_{C_w} = 0$$

Derivation of the analytical model

Rotor kinetic and potential energies:

$$
\begin{aligned}
T_{rot} &= \underbrace{\frac{1}{2}\,(2m_{rot})\left(\dot{\boldsymbol{p}}^{o}_{C_{rot}}\right)^{T}\left(\dot{\boldsymbol{p}}^{o}_{C_{rot}}\right)}_{\text{Translational kinetic energy}} \quad + \quad \underbrace{\frac{1}{2}\,(2I_{rot,yy})\,\dot{\vartheta}^{2}_{rot}}_{\text{Rotational kinetic energy}} \\
&= \left(N^{2}I_{rot,yy} + m_{rot}\,r^{2}\right)\dot{\gamma}^{2} + \left[(1-N)^{2}I_{rot,yy} + m_{rot}(z^{b}_{rot})^{2}\right]\dot{\vartheta}^{2} + \cdots \\
&\quad \cdots + 2\left[N(1-N)I_{rot,yy} + m_{rot}\,r\,z^{b}_{rot}\cos\vartheta\right]\dot{\gamma}\,\dot{\vartheta}
\end{aligned}
$$

$$
U_{rot} = (2m_{rot})\,g\,z_{C_{rot}} = (2m_{rot})\,g\,z^{b}_{rot}\cos\vartheta
$$

Derivation of the analytical model

Lagrangian function:

$$\mathcal{L} = T - U \qquad \text{with} \qquad \begin{aligned} T &= T_b + T_w + T_{rot} \\ U &= U_b + U_w + U_{rot} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\mathcal{L} = & \left[I_{w,yy} + N^2 I_{rot,yy} + \left(\frac{1}{2} m_b + m_w + m_{rot} \right) r^2 \right] \dot{\gamma}^2 + \cdots \\
& \cdots + \left[\frac{1}{2} I_{b,yy} + (1-N)^2 I_{rot,yy} + \frac{1}{2} m_b l^2 + m_{rot} (z^b_{rot})^2 \right] \dot{\vartheta}^2 + \cdots \\
& \cdots + \left[2 N (1-N) I_{rot,yy} + \left(m_b l + 2 m_{rot} z^b_{rot} \right) r \cos \vartheta \right] \dot{\gamma} \dot{\vartheta} - \cdots \\
& \cdots - \left(m_b l + 2 m_{rot} z^b_{rot} \right) g \cos \vartheta
\end{aligned}$$

Derivation of the analytical model

<u>Lagrange equations:</u>

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} - \frac{\partial \mathcal{L}}{\partial \gamma} = \xi_{\gamma}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vartheta}} - \frac{\partial \mathcal{L}}{\partial \vartheta} = \xi_{\vartheta}$$

ξ_γ and ξ_ϑ are the **generalized forces/torques** related to the generalized coordinates γ and ϑ.

Derivation of the analytical model

The **generalized force** ξ_j associated with the generalized coordinate q_j is:

$$\xi_j = \sum_{i=1}^{n} \boldsymbol{F}_i^T \frac{\partial \boldsymbol{p}_i}{\partial q_j} + \sum_{i=1}^{n} \boldsymbol{\tau}_i^T \frac{\partial \boldsymbol{\vartheta}_i}{\partial q_j}$$

$\boldsymbol{F}_i$ and $\boldsymbol{\tau}_i$ are the generic force and torque acting on the system, while $\boldsymbol{p}_i$ and $\boldsymbol{\vartheta}_i$ are their point and axis of application (expressed in world frame).

Derivation of the analytical model

Torque contributions:

- Motor torque τ
- Gearbox internal viscous friction $\tau_f' = B(\dot{\gamma} - \dot{\vartheta})$
- Rolling wheel viscous friction $\tau_f'' = B_w\,\dot{\gamma}$

$$\xi_j = \sum_{i=1}^{n} \boldsymbol{F}_i^T \frac{\partial \boldsymbol{p}_i}{\partial q_j} + \sum_{i=1}^{n} \boldsymbol{\tau}_i^T \frac{\partial \boldsymbol{\vartheta}_i}{\partial q_j}$$

Generalized torques:

$$\xi_\gamma = 2\,\tau - 2\,\tau_f' - 2\,\tau_f'' = 2\tau - 2\,(B + B_w)\,\dot{\gamma} + 2B\dot{\vartheta}$$

$$\xi_\vartheta = -2\tau + 2\tau_f' = -2\tau + 2B\dot{\gamma} - 2B\dot{\vartheta}$$

Derivation of the analytical model

Equations-of-Motion (EoM):

$$\left[2\,I_{w,yy} \,+\, 2\,N^2\,I_{rot,yy} \,+\, (m_b \,+\, 2\,m_w \,+\, 2\,m_{rot})\,r^2\right]\ddot{\gamma} \,+\, 2\,(B + B_w)\,\dot{\gamma} \,+\, \cdots$$

$$\cdots + \left[2\,N\,(1 - N)\,I_{rot,yy} \,+\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right) r\cos\vartheta\right]\,\ddot{\vartheta} \,-\, 2B\dot{\vartheta} \,-\, \cdots$$

$$\cdots \,-\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right) r\,\sin\vartheta\,\dot{\vartheta}^2 - 2\tau \;=\; 0$$

$$\left[2\,N\,(1 - N)\,I_{rot,yy} \,+\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right) r\cos\vartheta\right]\ddot{\gamma} \,-\, 2B\dot{\gamma} \,+\, \cdots$$

$$\cdots + \left[I_{b,yy} \,+\, 2\,(1 - N)^2\,I_{rot,yy} \,+\, m_b\,l^2 \,+\, 2\,m_{rot}(z^b_{rot})^2\right]\ddot{\vartheta} \,+\, 2B\dot{\vartheta} \,-\, \cdots$$

$$\cdots \,-\, \left(m_b\,l \,+\, 2\,m_{rot}\,z^b_{rot}\right)\,g\,\sin\vartheta \,+\, 2\tau \;=\; 0$$

Derivation of the analytical model

EoM – compact matrix formulation:

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Vector of generalized coordinates

$$\boldsymbol{q} = \begin{bmatrix} \gamma \\ \vartheta \end{bmatrix}$$

Derivation of the analytical model

$$\underbrace{M(q)\,\ddot{q}}_{\text{Torque contribution due to robot bodies inertia}} + C(q,\dot{q})\,\dot{q} + F_v\,\dot{q} + g(q) = \tau$$

Inertia matrix

$$M(q) = \begin{bmatrix} M_{11}(q) & M_{12}(q) \\ M_{21}(q) & M_{22}(q) \end{bmatrix}$$

$$M_{11}(q) = 2\,I_{w,yy} + 2\,N^2\,I_{rot,yy} + (m_b + 2\,m_w + 2\,m_{rot})\,r^2$$

$$M_{12}(q) = M_{21}(q) = 2\,N\,(1-N)\,I_{rot,yy} + \left(m_b\,l + 2\,m_{rot}\,z^b_{rot}\right) r\cos\vartheta$$

$$M_{22}(q) = I_{b,yy} + 2\,(1-N)^2\,I_{rot,yy} + m_b\,l^2 + 2\,m_{rot}(z^b_{rot})^2$$

Derivation of the analytical model

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \overbrace{\boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})}^{\text{Torque contribution due to } \textit{centrifugal} \text{ and } \textit{Coriolis accelerations}}\,\dot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \;=\; \boldsymbol{\tau}$$

Matrix of Centrifugal and Coriolis-related coefficients

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix}$$

$$C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) \;=\; C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) \;=\; C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) \;=\; 0$$

$$C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \;=\; -\left(m_b\,l \;+\; 2\,m_{rot}\,z_{rot}^{b}\right) r\;\sin\vartheta\,\dot{\vartheta}$$

Derivation of the analytical model

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \underbrace{\boldsymbol{F}_v}^{\text{Torque contribution due to \textit{viscous friction}}}\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \;=\; \boldsymbol{\tau}$$

Matrix of viscous friction coefficients

$$\boldsymbol{F}_v = \begin{bmatrix} F_{v,11} & F_{v,12} \\ F_{v,21} & F_{v,22} \end{bmatrix}$$

$$F_{v,11} \;=\; 2(B + B_w)$$

$$F_{v,12} \;=\; F_{v,21} \;=\; -2B$$

$$F_{v,22} \;=\; 2B$$

Derivation of the analytical model

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \overbrace{\boldsymbol{g}(\boldsymbol{q})}^{\text{Torque contribution due to } \textit{gravity}} = \boldsymbol{\tau}$$

Torque contribution due to gravity

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ -\left(m_b\, l + 2\, m_{rot}\, z^b_{rot}\right) g \sin\vartheta \end{bmatrix}$$

Derivation of the analytical model

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Motor torque

$$\boldsymbol{\tau} = \begin{bmatrix} 2\,\tau \\ -2\,\tau \end{bmatrix}$$

Derivation of the analytical model

The EoM are coupled 2nd order *nonlinear* ODEs.

For control design, they are linearized w.r.t. the unstable equilibrium point $P_0 = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \ddot{\boldsymbol{q}}_0, \tau_0)$:

$$\boldsymbol{q}_0 = [\gamma_0,\, 0]^T\,, \quad \dot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}_0 = [0,\, 0]^T\,, \quad \tau_0 = 0$$

<u>Note</u>: P_0 is an equilibrium point for any $\gamma_0 \in \mathbb{R}$.

Derivation of the analytical model

Model linearization around P_0:

$$\boldsymbol{f}(P_0) + \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \boldsymbol{q}}\,\delta\boldsymbol{q} + \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \dot{\boldsymbol{q}}}\,\delta\dot{\boldsymbol{q}} + \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \ddot{\boldsymbol{q}}}\,\delta\ddot{\boldsymbol{q}} + \frac{\partial \boldsymbol{f}\,(P_0)}{\partial \tau}\delta\tau \;=\; \mathbf{0}$$

where

$$\boldsymbol{f}\,(\boldsymbol{q},\,\dot{\boldsymbol{q}},\,\ddot{\boldsymbol{q}},\,\tau) \triangleq \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{\tau}$$

and $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\ddot{\boldsymbol{q}}$, $\delta\tau$ are small deviations from the equilibrium values (e.g. $\delta\boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}_0$, …).

Derivation of the analytical model

$$\boldsymbol{f}(P_0) + \frac{\partial \boldsymbol{f}(P_0)}{\partial \boldsymbol{q}}\,\delta\boldsymbol{q} + \frac{\partial \boldsymbol{f}(P_0)}{\partial \dot{\boldsymbol{q}}}\,\delta\dot{\boldsymbol{q}} + \frac{\partial \boldsymbol{f}(P_0)}{\partial \ddot{\boldsymbol{q}}}\,\delta\ddot{\boldsymbol{q}} + \frac{\partial \boldsymbol{f}(P_0)}{\partial \tau}\delta\tau = \boldsymbol{0}$$

$$\Downarrow$$

<u>Notation abuse</u>

Use γ, ϑ, τ in place of $\delta\gamma$, $\delta\vartheta$, $\delta\tau$

$$\left[2\,I_{w,yy} + 2\,N^2\,I_{rot,yy} + (m_b + 2\,m_w + 2\,m_{rot})\,r^2\right]\ddot{\gamma} + 2\,(B + B_w)\,\dot{\gamma} + \cdots$$

$$\cdots + \left[2\,N\,(1-N)\,I_{rot,yy} + \left(m_b\,l + 2\,m_{rot}\,z^b_{rot}\right)r\right]\ddot{\vartheta} - 2B\dot{\vartheta} - 2\tau = 0$$

$$\left[2\,N\,(1-N)\,I_{rot,yy} + \left(m_b\,l + 2\,m_{rot}\,z^b_{rot}\right)r\right]\ddot{\gamma} - 2B\dot{\gamma} + \cdots$$

$$\cdots + \left[I_{b,yy} + 2\,(1-N)^2\,I_{rot,yy} + m_b\,l^2 + 2\,m_{rot}(z^b_{rot})^2\right]\ddot{\vartheta} + 2B\dot{\vartheta} - \cdots$$

$$\cdots - \left(m_b\,l + 2\,m_{rot}\,z^b_{rot}\right)g\,\vartheta + 2\tau = 0$$

Derivation of the analytical model

Linearized model – compact matrix formulation:

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{G}\,\boldsymbol{q} = \boldsymbol{\tau}$$

$$\boldsymbol{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad \boldsymbol{F}_v = \begin{bmatrix} F_{v,11} & F_{v,12} \\ F_{v,21} & F_{v,22} \end{bmatrix}, \qquad \boldsymbol{G} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$M_{11} = 2\,I_{w,yy} + 2\,N^2\,I_{rot,yy} + (m_b + 2\,m_w + 2\,m_{rot})\,r^2$$

$$M_{12} = M_{21} = 2\,N\,(1-N)\,I_{rot,yy} + \left(m_b\,l + 2\,m_{rot}\,z^b_{rot}\right)r$$

$$M_{22} = I_{b,yy} + 2\,(1-N)^2\,I_{rot,yy} + m_b\,l^2 + 2\,m_{rot}(z^b_{rot})^2$$

$$F_{v,11} = 2(B + B_w)$$

$$F_{v,12} = F_{v,21} = -2B$$

$$F_{v,22} = 2B$$

$$G_{11} = G_{12} = G_{21} = 0$$

$$G_{22} = -\left(m_b\,l + 2\,m_{rot}\,z^b_{rot}\right)g$$

Derivation of the analytical model

State-space formulation:

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{G}\,\boldsymbol{q} = \boldsymbol{\tau}$$

$$\Downarrow$$

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\,(-\boldsymbol{G}\,\boldsymbol{q} - \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{\tau})$$

$$\Bigg\Downarrow \quad \begin{aligned} \boldsymbol{x} &= [\boldsymbol{q},\,\dot{\boldsymbol{q}}]^T = [\gamma,\,\vartheta,\,\dot{\gamma},\,\dot{\vartheta}]^T \\ u &= 2\tau \end{aligned}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,u$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0}_{2\times 2} & \boldsymbol{I}_{2\times 2} \\ -\boldsymbol{M}^{-1}\boldsymbol{G} & -\boldsymbol{M}^{-1}\boldsymbol{F}_v \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0}_{2\times 2} \\ \boldsymbol{M}^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Derivation of the analytical model

<u>Actuator model</u>: standard DC gearmotor model

$$\tau = N\, k_t\, i_a\,, \qquad L_a \frac{di_a}{dt} + R_a\, i_a + k_e \frac{d\Delta\vartheta_{rot}}{dt} = u_a$$

$$\Big\Downarrow \qquad \begin{aligned} \Delta\vartheta_{rot} &= \vartheta_{rot} - \vartheta = N\,(\gamma - \vartheta) \\ L_a/R_a &\ll 1 \end{aligned}$$

$$i_a = \frac{1}{R_a}\left[u_a - k_e\, N\,(\dot\gamma - \dot\vartheta)\right]$$

$$\Downarrow$$

$$\tau = \frac{N k_t}{R_a}\, u_a - \frac{N^2 k_t k_e}{R_a}\,(\dot\gamma - \dot\vartheta)$$

Derivation of the analytical model

<u>EoM with actuator model:</u>

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \;=\; \boldsymbol{\tau}$$

$$\Big\Downarrow \qquad \tau = \frac{N k_t}{R_a}\,u_a - \frac{N^2 k_t k_e}{R_a}\,(\dot{\gamma} - \dot{\vartheta})$$

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{F}'_v\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \;=\; \boldsymbol{\tau}'$$

$$\boldsymbol{F}'_v \;=\; \boldsymbol{F}_v + \frac{2\,N^2\,k_t\,k_e}{R_a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{\tau}' \;=\; \frac{2\,N\,k_t}{R_a}\begin{bmatrix} 1 \\ -1 \end{bmatrix} u_a$$

Derivation of the analytical model

Linearized EoM with actuator model:

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} + \boldsymbol{F}_v\,\dot{\boldsymbol{q}} + \boldsymbol{G}\,\boldsymbol{q} \;=\; \boldsymbol{\tau}$$

$$\Big\Downarrow \qquad \tau = \frac{N k_t}{R_a}\,u_a - \frac{N^2 k_t k_e}{R_a}\,(\dot{\gamma} - \dot{\vartheta})$$

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} + \boldsymbol{F}'_v\,\dot{\boldsymbol{q}} + \boldsymbol{G}\,\boldsymbol{q} \;=\; \boldsymbol{\tau}'$$

$$\boldsymbol{F}'_v \;=\; \boldsymbol{F}_v + \frac{2\,N^2\,k_t\,k_e}{R_a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{\tau}' \;=\; \frac{2\,N\,k_t}{R_a}\begin{bmatrix} 1 \\ -1 \end{bmatrix} u_a$$

Derivation of the analytical model

State-space formulation with actuator model:

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} + \boldsymbol{F}'_v\,\dot{\boldsymbol{q}} + \boldsymbol{G}\,\boldsymbol{q} = \boldsymbol{\tau}'$$

$$\Downarrow$$

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\left(-\boldsymbol{G}\,\boldsymbol{q} - \boldsymbol{F}'_v\,\dot{\boldsymbol{q}} + \boldsymbol{\tau}'\right)$$

$$\Bigg\Downarrow \quad \begin{aligned} \boldsymbol{x} &= [\boldsymbol{q},\, \dot{\boldsymbol{q}}]^T = [\gamma,\, \vartheta,\, \dot{\gamma},\, \dot{\vartheta}]^T \\ u &= u_a \end{aligned}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,u$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0}_{2\times 2} & \boldsymbol{I}_{2\times 2} \\ -\boldsymbol{M}^{-1}\boldsymbol{G} & -\boldsymbol{M}^{-1}\boldsymbol{F}'_v \end{bmatrix}, \qquad \boldsymbol{B} = \frac{2\,N\,k_t}{R_a}\begin{bmatrix} \boldsymbol{0}_{2\times 2} \\ \boldsymbol{M}^{-1} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Derivation of the analytical model

<u>Accelerometer model:</u> the sensor measures

- the linear acceleration $\ddot{\boldsymbol{p}}^o_{C_{mpu}}$ of its CoM C_{mpu}
- the gravity acceleration vector $\boldsymbol{g}^o = [0, -g]^T$

both expressed w.r.t. the sensor frame $\{mpu\}$ (1).

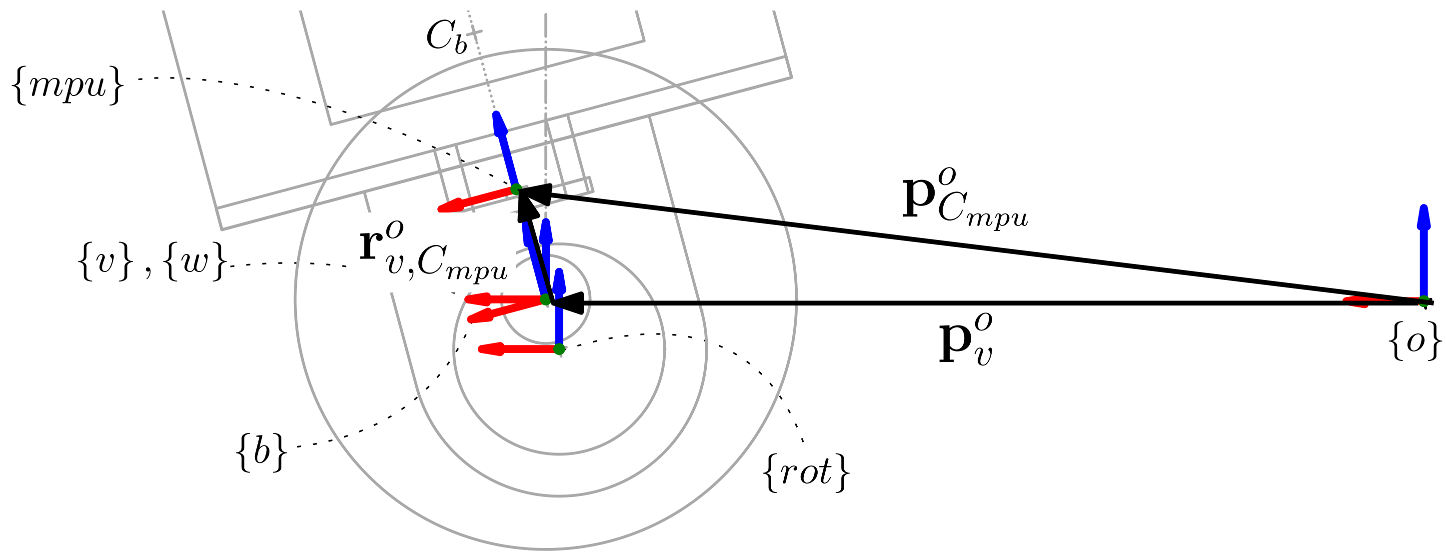
$$\boldsymbol{y}_a = \begin{bmatrix} x_a^{mpu} \\ z_a^{mpu} \end{bmatrix} = \boldsymbol{R}_o^b \left(\ddot{\boldsymbol{p}}^o_{C_{mpu}} + \boldsymbol{g}^o \right)$$

<u>Note</u>:

$$\boldsymbol{R}_o^b = (\boldsymbol{R}_b^o)^{-1} = (\boldsymbol{R}_b^o)^T = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

(1) The sensor frame $\{mpu\}$ is aligned with the robot body frame $\{b\}$.

Derivation of the analytical model



$$\boldsymbol{p}^o_{C_{mpu}} = \boldsymbol{p}^o_v + \boldsymbol{R}^o_b \, \boldsymbol{p}^b_{C_{mpu}} = \cdots$$

$$\cdots = \begin{bmatrix} r\gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} 0 \\ z^b_{mpu} \end{bmatrix} = \begin{bmatrix} r\,\gamma + z^b_{mpu}\sin\vartheta \\ z^b_{mpu}\cos\vartheta \end{bmatrix}$$

Derivation of the analytical model

$$\boldsymbol{p}^o_{C_{mpu}} = \boldsymbol{p}^o_v + \boldsymbol{R}^o_b\,\boldsymbol{p}^b_{C_{mpu}} = \begin{bmatrix} r\,\gamma + z^b_{mpu}\sin\vartheta \\ z^b_{mpu}\cos\vartheta \end{bmatrix}$$

$$\dot{\boldsymbol{p}}^o_{C_{mpu}} = \dot{\boldsymbol{p}}^o_v + \frac{d\boldsymbol{R}^o_b}{dt}\,\boldsymbol{p}^b_{C_{mpu}} + \boldsymbol{R}^o_b\,\dot{\boldsymbol{p}}^b_{C_{mpu}} = \begin{bmatrix} r\,\dot{\gamma} + z^b_{mpu}\cos\vartheta\,\dot{\vartheta} \\ -z^b_{mpu}\sin\vartheta\,\dot{\vartheta} \end{bmatrix}$$

$$\ddot{\boldsymbol{p}}^o_{C_{mpu}} = \ddot{\boldsymbol{p}}^o_v + \frac{d^2\boldsymbol{R}^o_b}{dt^2}\,\boldsymbol{p}^b_{C_{mpu}} + 2\frac{d\,\boldsymbol{R}^o_b}{dt}\,\dot{\boldsymbol{p}}_{C_{mpu}} + \boldsymbol{R}^o_b\,\ddot{\boldsymbol{p}}^b_{C_{mpu}} = \cdots$$

$$\cdots = \begin{bmatrix} r\,\ddot{\gamma} + z^b_{mpu}\,(-\sin\vartheta\,\dot{\vartheta}^2 + \cos\vartheta\,\ddot{\vartheta}) \\ -z^b_{mpu}\,(\cos\vartheta\,\dot{\vartheta}^2 + \sin\vartheta\,\ddot{\vartheta}) \end{bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{y}_a = \boldsymbol{R}^b_o\left(\ddot{\boldsymbol{p}}^o_{C_{mpu}} + \boldsymbol{g}^o\right) = \begin{bmatrix} r\,\ddot{\gamma}\cos\vartheta + z^b_{mpu}\,\ddot{\vartheta} + g\sin\vartheta \\ r\,\ddot{\gamma}\sin\vartheta - z^b_{mpu}\,\dot{\vartheta}^2 - g\cos\vartheta \end{bmatrix}$$

Derivation of the analytical model

<u>Gyroscope model</u>: the sensor measures the rate of rotation (e.g. angular speed) w.r.t its frame axes.

The y-component of the sensor output is the rate of change of the robot tilt angle:

$$y_g = \dot{\vartheta}$$

Tilt angle estimation

Tilt estimation problem: estimate the robot tilt angle ϑ from the accelerometer and gyroscope measurements.

Available options:

- use only the accelerometer data (*naïve approach 1*)
- use only the gyroscope data (*naïve approach 2*)
- use both the accelerometer and gyroscope data (*sensor fusion by complementary filtering*)

Tilt angle estimation

<u>Naïve approach 1:</u> use only the accelerometer data.

$$
\boldsymbol{y}_a = \begin{bmatrix} r\,\ddot{\gamma}\cos\vartheta + z^b_{mpu}\,\ddot{\vartheta} + g\sin\vartheta \\ r\,\ddot{\gamma}\sin\vartheta - z^b_{mpu}\,\dot{\vartheta}^2 - g\cos\vartheta \end{bmatrix}
$$

<u>Assumption:</u>

Robot motion is *slow* ...

$$
\Downarrow \quad \begin{cases} \text{linear acceleration } r\ddot{\gamma} \approx 0 \\ \text{tangential acceleration } z^b_{mpu}\ddot{\vartheta} \approx 0 \\ \text{centripetal acceleration } z^b_{mpu}\dot{\vartheta}^2 \approx 0 \end{cases}
$$

$$
\boldsymbol{y}_a = \begin{bmatrix} \cancel{r\,\ddot{\gamma}\cos\vartheta} + \cancel{z^b_{mpu}\,\ddot{\vartheta}} + g\sin\vartheta \\ \cancel{r\,\ddot{\gamma}\sin\vartheta} - \cancel{z^b_{mpu}\,\dot{\vartheta}^2} - g\cos\vartheta \end{bmatrix} \approx \begin{bmatrix} g\sin\vartheta \\ -g\cos\vartheta \end{bmatrix}
$$

Tilt angle estimation

$$\boldsymbol{y}_a = \begin{bmatrix} x_a^{mpu} \\ z_a^{mpu} \end{bmatrix} \approx \begin{bmatrix} g \sin\vartheta \\ -g\cos\vartheta \end{bmatrix}$$

$$\Downarrow$$

$$\hat{\vartheta}_a = \operatorname{atan2}\left(x_a^{mpu}, -z_a^{mpu}\right)$$

[Accelerometer–based tilt estimation]

<u>Drawbacks</u>: sensitive to

- robot accelerations
- accelerometer output noise (large at high freq)

Tilt angle estimation

Naïve approach 2: use only the gyroscope data.

$$y_g = \dot{\vartheta}$$

$$\Downarrow$$

$$\hat{\vartheta}_g = \hat{\vartheta}_g(0) + \int_0^t y_g(\tau)\, d\tau \qquad \left[\begin{array}{c}\text{Gyro–based}\\ \text{tilt estimation}\end{array}\right]$$

Drawbacks:

- Must be initialized properly, i.e. $\hat{\vartheta}_g(0) = \vartheta(0)$
- Integral may diverge because of gyro *bias* and *drift* (1)

(1) *Bias* = const error term; *drift* = linear ramp error term.

Tilt angle estimation

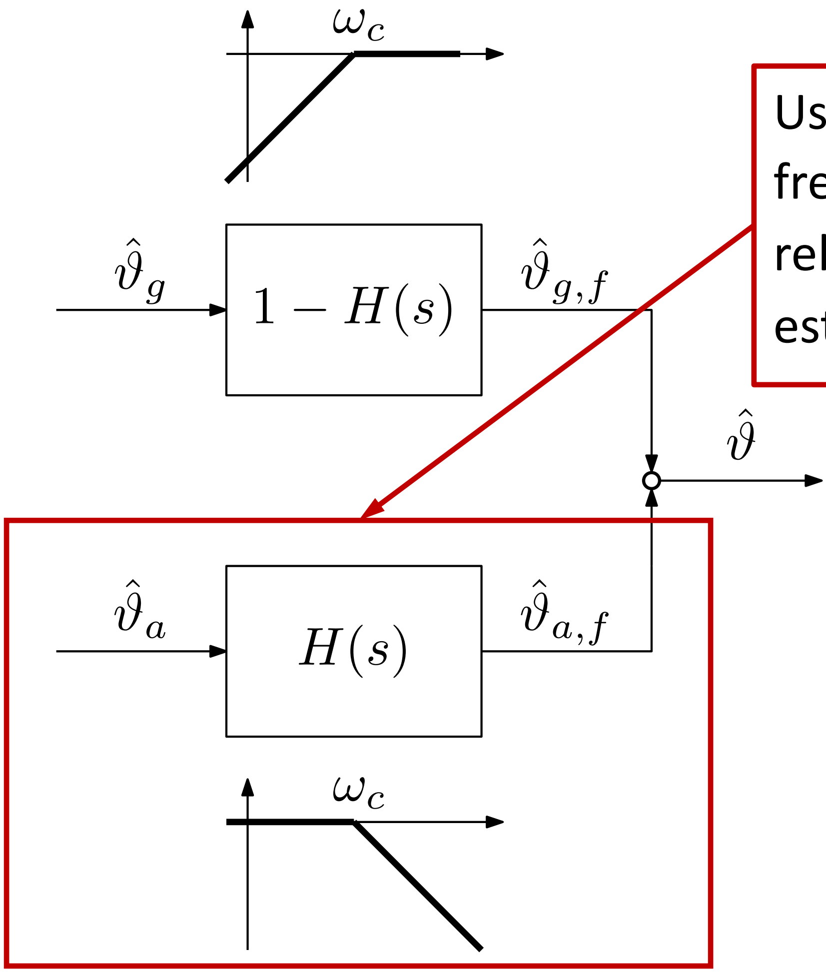
- Accelerometer-based estimate $\hat{\vartheta}_a$: more reliable at *low* frequency
- Gyroscope-based estimate $\hat{\vartheta}_g$: more reliable at *high* frequency

$$\Downarrow$$

Sensor fusion: combine the data provided by the two sensors to obtain a satisfactory tilt angle estimate at every frequency.

Tilt angle estimation

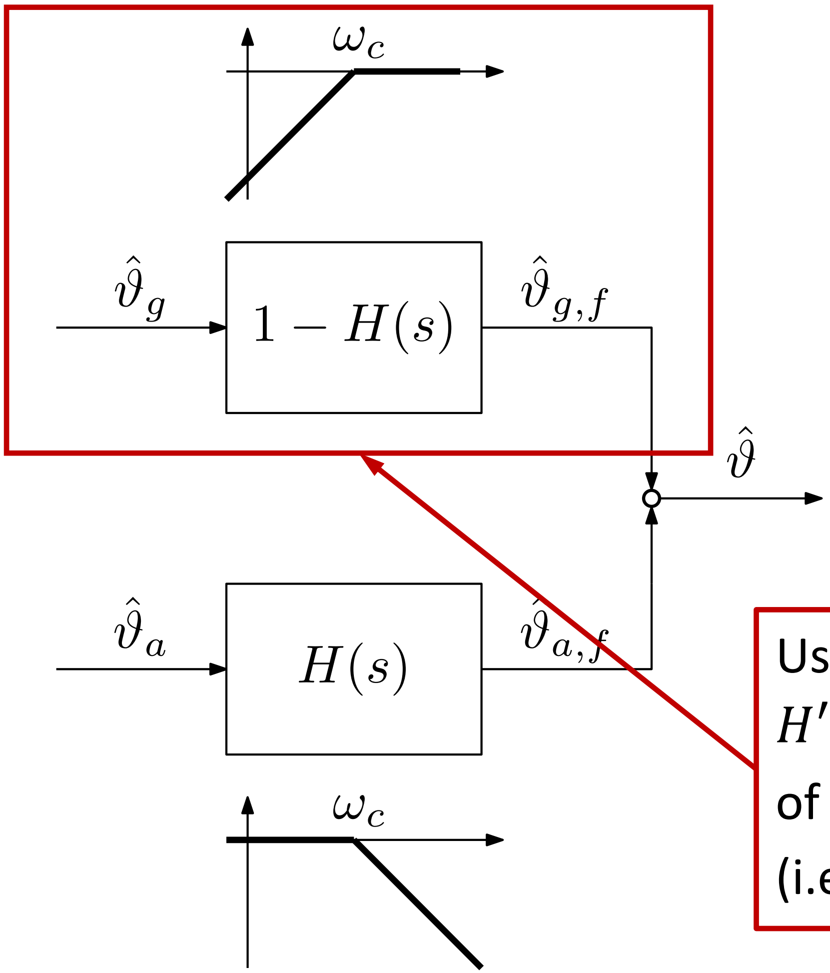
Complementary filtering approach:



Use a *low-pass* filter $H(s)$ with cut-off frequency ω_c and unit DC gain to get the relevant part of the accelerometer-based estimate $\hat{\vartheta}_a$ (i.e. low frequency spectrum).

Tilt angle estimation

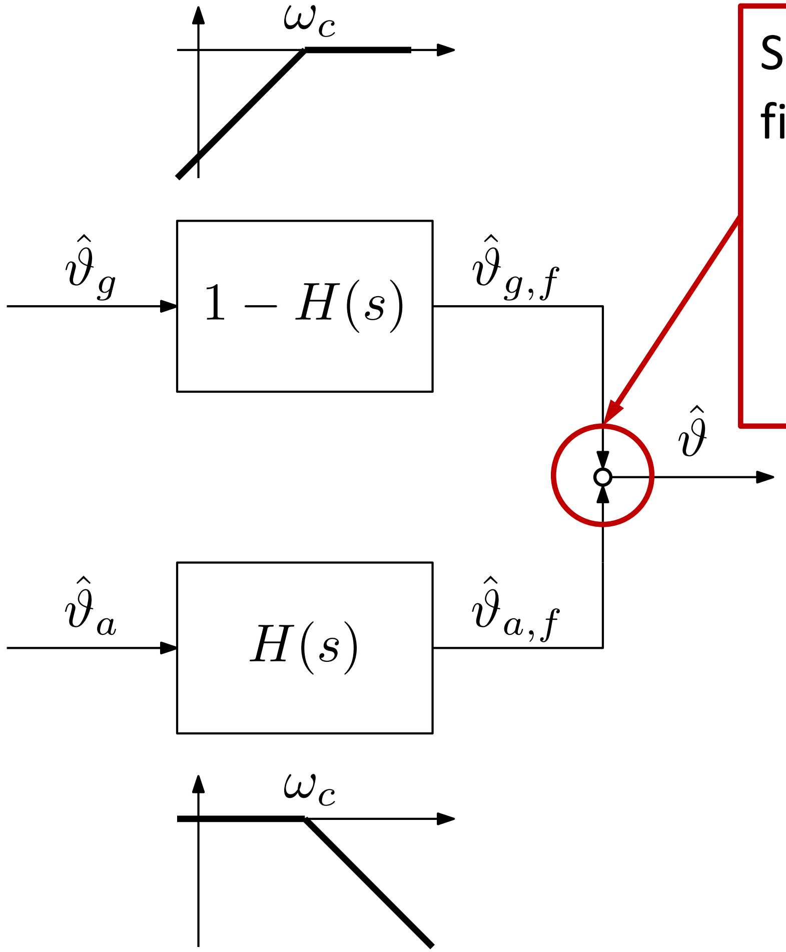
Complementary filtering approach:



Use the *complementary high-pass* filter $H'(s) = 1 - H(s)$ to get the relevant part of the gyroscope-based estimate $\hat{\vartheta}_g$ (i.e. high frequency spectrum).

Tilt angle estimation

Complementary filtering approach:



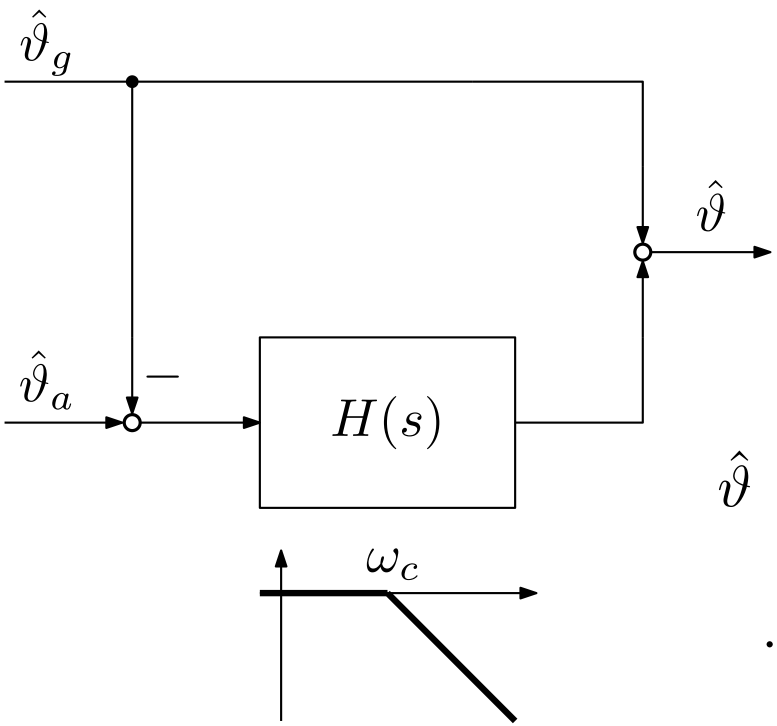
Sum the two filters outputs to obtain the final tilt angle estimate:

$$\hat{\vartheta} = \hat{\vartheta}_{a,f} + \hat{\vartheta}_{g,f} = \cdots$$

$$\cdots = H(s)\,\hat{\vartheta}_a + [1 - H(s)]\,\hat{\vartheta}_g$$

Tilt angle estimation

Alternative implementation:



$$\hat{\vartheta} = H(s)\,\hat{\vartheta}_a + [1 - H(s)]\,\hat{\vartheta}_g = \cdots$$

$$\cdots = \hat{\vartheta}_g + H(s)\left(\hat{\vartheta}_a - \hat{\vartheta}_g\right)$$

Tilt angle estimation

$$\hat{\vartheta}_{a,f} = H(s)\,\hat{\vartheta}_a\,, \qquad \hat{\vartheta}_{g,f} = [1 - H(s)]\,\hat{\vartheta}_g$$

$$\Downarrow$$

$$\hat{\vartheta} = \hat{\vartheta}_{a,f} + \hat{\vartheta}_{g,f} = H(s)\,\hat{\vartheta}_a + [1 - H(s)]\,\hat{\vartheta}_g$$

[Tilt estimation by complementary filtering]

Trade-off in the design of the complementary filters:

- attenuate the accelerometer output noise at high freq
- reject gyroscope bias and drift at low freq

Tilt angle estimation

1st order complementary filters pair:

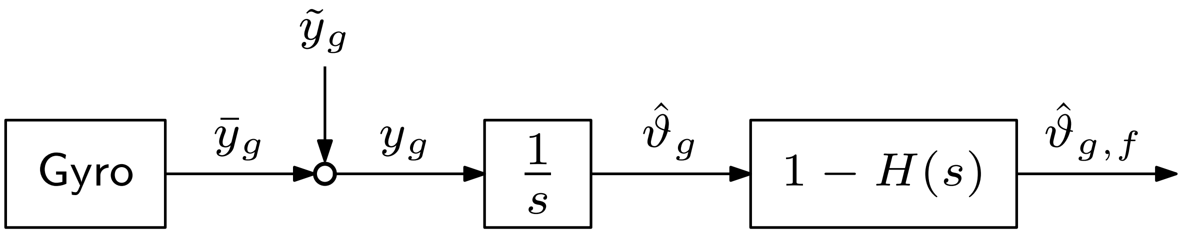
$$H(s) = \frac{1}{T_c\, s + 1}\,, \qquad 1 - H(s) = \frac{T_c\, s}{T_c\, s + 1}$$

👍 Lowest filters complexity, but ...

👎 the high-pass filter is unable to completely reject the gyroscope output bias and drift.

Tilt angle estimation

In fact:



$$\underline{\text{Gyro output error}} \qquad \underline{\text{Laplace transform}} \qquad \underline{\text{Gyro–based estimate error}}$$

$$\tilde{y}_g = d_g\, t + b_g \quad \overset{\mathcal{L}}{\Rightarrow} \quad \tilde{Y}_g(s) = \frac{d_g}{s^2} + \frac{b_g}{s} \quad \overset{\int dt}{\Rightarrow} \quad \tilde{\Theta}_g(s) = \frac{d_g}{s^3} + \frac{b_g}{s^2}$$

Tilt angle estimation

High–pass filter output

$$\tilde{\Theta}_{g,f}(s) = [1 - H(s)]\,\tilde{\Theta}_g(s) = \frac{T_c\,d_g}{(T_c\,s+1)\,s^2} + \frac{T_c\,b_g}{(T_c\,s+1)\,s} = \cdots$$

$$\cdots = -\frac{T_c^2\,d_g}{s} + \frac{T_c\,d_g}{s^2} + \frac{T_c^3 d_g}{T_c\,s+1} + \frac{T_c\,b_g}{s} - \frac{T_c^2\,b_g}{T_c\,s+1}$$

$$\Big\Downarrow\ \mathcal{L}^{-1}$$

$$\tilde{\theta}_{g,f}(t) \approx (T_c\,d_g)\,t + \left(T_c\,b_g - T_c^2\,d_g\right) \quad \text{for} \quad t \gg 0$$

The estimation error has both a drift and a bias term.

Tilt angle estimation

2nd order complementary filters pair:

$$H(s) = \frac{2\,T_c\,s + 1}{(T_c\,s + 1)^2}\,, \qquad 1 - H(s) = \frac{T_c^2\,s^2}{(T_c\,s + 1)^2}$$

👎 Increased filter complexity, but …

👍 the estimate is insensitive to the gyro output bias.

👎 Unfortunately, the estimate is still sensitive to the gyro output drift.

Tilt angle estimation

In fact:

$$\tilde{\Theta}_{g,f}(s) = [1 - H(s)]\,\tilde{\Theta}_g(s) = \frac{T_c^2\, d_g}{(T_c\, s + 1)^2\, s} + \frac{T_c^2\, b_g}{(T_c\, s + 1)^2} = \cdots$$

$$\cdots = \frac{T_c^2\, d_g}{s} - \frac{T_c^3 d_g}{T_c\, s + 1} - \frac{T_c^3 d_g}{(T_c\, s + 1)^2} + \frac{T_c^2\, b_g}{(T_c\, s + 1)^2}$$

$$\Big\Downarrow\ \mathcal{L}^{-1}$$

$$\tilde{\theta}_{g,f}(t) \approx T_c^2 d_g \quad \text{for} \quad t \gg 0$$

Depends only on the gyroscope output drift

Tilt angle estimation

3rd order complementary filters pair:

$$H(s) = \frac{3\,T_c^2\,s^2 + 3\,T_c\,s + 1}{(T_c\,s+1)^3}\,, \qquad 1 - H(s) = \frac{T_c^3\,s^3}{(T_c\,s+1)^3}$$

👎 Further increased filter complexity, but …

👍 the estimate becomes insensitive to both the gyroscope output bias and drift.

Tilt angle estimation

In fact:

$$\tilde{\Theta}_{g,f}(s) = [1 - H(s)]\,\tilde{\Theta}_g(s) = \frac{T_c^3\, d_g}{(T_c\, s + 1)^3} + \frac{T_c^3\, b_g\, s}{(T_c\, s + 1)^3}$$

$$\Big\Downarrow \; \mathcal{L}^{-1}$$

$$\tilde{\theta}_{g,f}(t) \approx 0 \quad \text{for} \quad t \gg 0$$

$\boxed{\text{Insensitive to both the gyroscope output bias and drift}}$

Balance and position control

The *balance and position control* is designed with conventional state-space methods:

- Do the design in **discrete-time domain**, by discretizing the robot model with the exact discretization method.
- Use **discrete-time LQR** methods for control design.
- Achieve robust tracking of position set-points by employing the **integral action** (1).

(1) Use a $T_s = 0.01s$ ($F_s = 100\ Hz$) sample time for the controller.

Balance and position control

State estimation ($\boldsymbol{x} = [\gamma, \vartheta, \dot{\gamma}, \dot{\vartheta}]^T$):

- For tilt angle ϑ : perform discrete-time complementary filtering of accelerometer and gyroscope measurements:

$$\hat{\vartheta} \;=\; H(z)\,\hat{\vartheta}_a \,+\, [1 - H(z)]\,\hat{\vartheta}_g \;=\; \hat{\vartheta}_g \,+\, H(z)\,(\hat{\vartheta}_a \,-\, \hat{\vartheta}_g)$$

- For wheel angle γ : use the rotor angular displacement $\Delta\vartheta_{rot}$ measured by the encoder:

$$\Delta\vartheta_{rot} = N(\gamma - \vartheta) \qquad \Rightarrow \qquad \hat{\gamma} \;=\; \Delta\vartheta_{rot}/N \,+\, \hat{\vartheta}$$

- For velocities $\dot{\vartheta}$ and $\dot{\gamma}$: derive angular positions with the “real derivative” filter:

$$H_\omega(z) \;=\; \frac{1 - z^{-N}}{N\,T} \qquad \text{with} \qquad N = 3$$

Balance and position control

For <u>nominal perfect tracking</u> of *constant* position set-points:

$$u[k] = N_u\, r[k] - \boldsymbol{K}\left(\hat{\boldsymbol{x}}[k] - \boldsymbol{N}_x\, r[k]\right)$$

- The feedforward gains $\boldsymbol{N}_x$ and N_u are obtained by solving:

$$\begin{bmatrix} \boldsymbol{\Phi} - \boldsymbol{I} & \boldsymbol{\Gamma} \\ \boldsymbol{H} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_x \\ N_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}$$

- The feedback gain $\boldsymbol{K}$ is derived with LQR methods to minimize:

$$J = \sum_{k=0}^{+\infty} \boldsymbol{x}^T[k]\, \boldsymbol{Q}\, \boldsymbol{x}[k] + \rho\, r\, u^2[k]$$

For the design, use the exact discretization $\Sigma_d = (\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{H}, 0)$ of the robot continuous-time state-space model $\Sigma = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, 0)$.

Balance and position control

For <u>robust perfect tracking</u> of *constant* position set-points:

$$\begin{cases} x_I[k+1] = x_I[k] + (\,y[k] - r[k]\,) \\ u[k] = N_u\, r[k] - \boldsymbol{K}\,(\boldsymbol{x}[k] - \boldsymbol{N}_x\, r[k]) - K_I\, x_I[k] \end{cases}$$

- The feedback gain $\boldsymbol{K_e} = [K_I, \boldsymbol{K}]^T$ is derived with LQR methods to minimize:

$$J_e = \sum_{k=0}^{+\infty} \boldsymbol{x}_e^T[k]\, \boldsymbol{Q}_e\, \boldsymbol{x}_e[k] + \rho\, r\, u^2[k]$$

subject to the *augmented model*

$$\Sigma_e \;:\; \begin{bmatrix} x_I[k+1] \\ \boldsymbol{x}[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \boldsymbol{H} \\ \boldsymbol{0} & \boldsymbol{\Phi} \end{bmatrix}}_{\triangleq\, \boldsymbol{\Phi}_e} \underbrace{\begin{bmatrix} x_I[k] \\ \boldsymbol{x}[k] \end{bmatrix}}_{\triangleq\, \boldsymbol{x}_e} + \underbrace{\begin{bmatrix} 0 \\ \boldsymbol{\Gamma} \end{bmatrix}}_{\triangleq\, \boldsymbol{\Gamma}_e} u[k] - \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} r[k]$$

Simplified yaw angle control

No perfect straight motion can be attained *in practice*, because of unavoidable unbalancing of the two motors and wheels.

$$\Downarrow$$

<u>Idea</u>: use a PI controller to regulate the yaw angle ψ to zero.

- Yaw angle has to be somehow estimated.
- Motors need to be driven "differentially":

Common-mode motor command (voltage) generated by the longitudinal state-space controller

$$u_l = u_\Sigma - u_\Delta \,, \qquad u_r = u_\Sigma + u_\Delta$$

Differential-mode motor command (voltage) generated by the yaw-angle PI controller

Simplified yaw angle control

Estimation of yaw and wheel angles:

$$\hat{\vartheta}_l = \frac{\Delta\vartheta_{rot,l}}{N} + \hat{\vartheta}\,, \qquad \hat{\vartheta}_r = \frac{\Delta\vartheta_{rot,r}}{N} + \hat{\vartheta}$$

$$\Downarrow$$

$$\hat{\psi} = \frac{r}{w}\left(\hat{\vartheta}_r - \hat{\vartheta}_l\right) \qquad \text{[Yaw angle estimate]}$$

$$\hat{\gamma} = \frac{\hat{\vartheta}_r + \hat{\vartheta}_l}{2} \qquad \text{[Wheel angle estimate]}$$

$\Delta\vartheta_{rot,l}$ and $\Delta\vartheta_{rot,r}$ are the measurement provided by the two encoders.