THIS VIDEO: FROM "s" to STATE-SPACE (MP, OPT. CONT, MIMO)
- MODELS
- ANALYSIS

- Disclaimer: A BIT REDUNDANT FOR "LINEAR SYSTEM THEORY" SURVIVORS...

▷ Later: NEW IDEAS FOR ST. SP. CONTROL, TRACKING, ...

- STARTING POINT:

STATE SPACE DESCRIPTION FOR THE PLANT

$$(\Sigma_s)\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) & \rightarrow \text{STATE/DYNAMIC EQ} \\ y(t) = C\,x(t) + D\,u(t) & \rightarrow \text{OUTPUT EQ} \end{cases}$$

- Here: CONTINUOUS TIME $\boxed{t \in \mathbb{R}_{(+)}}$

$$u(t): \mathbb{R}_{(+)} \longrightarrow \mathbb{R}$$

STATE $\leftarrow$ $x(t): \mathbb{R}_{(+)} \longrightarrow \mathbb{R}^n$ $\boxed{(n = 2,\ \text{MOTOR})}$

$$y(t): \mathbb{R}_{(+)} \longrightarrow \mathbb{R}$$

SISO

HOW IS IT OBTAINED?

I) FROM ODE's OF Physical model

EX. MOTOR

$$\begin{cases} (1)\ \cancel{L_a \dot{i}_a} + R_{eq}\, i_a = K_{DRV} U - k_e \dot{\vartheta}_m & \leftarrow \text{EL} \\ \qquad\quad \underbrace{\hphantom{K_{DRV} U}}_{U_{DRV}} & \\ (2)\ J_{eq} \ddot{\vartheta}_m + B_{eq} \dot{\vartheta}_m = K_T i_a - \frac{1}{N}\tau_d & \leftarrow \text{MECH} \end{cases}$$

↳ Let's also : – Neglect $\underline{L_a}$

– Assume $U_{DRV} = \underline{K_{DRV}\, u}$

– Neglect disturbance $\underline{\tau_d}$

(1) IN (2)

⇓

$$\underline{(3)} \Rightarrow \ddot{\vartheta}_m = -\frac{B_{eq}}{J_{eq}} \dot{\vartheta}_m - \frac{K_T K_e}{R_{eq} J_{eq}} \dot{\vartheta}_m + \frac{K_T K_{DRV}}{R_{eq} J_{eq}} u$$

• NOW CHOOSE x, y, u :

GEARBOX

$$x = \begin{bmatrix} \vartheta_m \\ \dot{\vartheta}_m \end{bmatrix} \qquad y = \vartheta_L = \frac{1}{N} \vartheta_m \qquad u \equiv u$$

• Thus :

$$\dot{x} = \begin{bmatrix} \dot{\vartheta} \\ \ddot{\vartheta} \end{bmatrix} = \underbrace{\left[\begin{array}{c:c} 0 & 1 \\ \hdashline A_{21} & A_{22} \end{array}\right]}_{A} \overset{x}{\begin{bmatrix} \vartheta \\ \hline \dot{\vartheta} \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 \\ \hline B_2 \end{bmatrix}}_{B} u$$

$\dot{x}_1 = x_2$

(3)

$$\boxed{A_{21} = 0} \qquad A_{22} = -\frac{B_{eq} R_{eq} + K_T K_e}{R_{eq} J_{eq}}$$

$$B_2 = \frac{K_T K_{DRV}}{R_{eq} J_{eq}}$$

$$y = \underbrace{\begin{bmatrix} 1/N & 0 \end{bmatrix}}_{C} x$$

▷ IS IT UNIQUE ? OF COURSE NOT !

<u>Change of variables in state space</u> ⇒ same I/O ←

Change of variables in state space $\Rightarrow \begin{cases} \text{same I/O} \\ \text{same } y_e \end{cases}$

$\tilde{x} = Tx \quad , \quad TT^{-1} = I$

$\dot{\tilde{x}} = T\dot{x} = TAx + TBu = \underbrace{TAT^{-1}}_{\tilde{A}}\underbrace{Tx}_{\tilde{x}} + \underbrace{TB}_{\tilde{B}}u$

(Dyn)

$y = \underbrace{CT^{-1}}_{\tilde{C}}\underbrace{Tx}_{\tilde{x}} + Du$

HW IF NOT EXPERT:

- Verify
$P(s) = C(sI-A)^{-1}B + D \overset{!}{=} \tilde{P}(s)$
- Find and apply T
$\begin{bmatrix} \vartheta_m \\ \dot{\vartheta}_m \end{bmatrix} \longrightarrow \begin{bmatrix} \vartheta_L \\ \dot{\vartheta}_L \end{bmatrix}$, & New A, B, C.

II) FROM KNOWN $P(s) = \dfrac{b_m s^m + b_{m-1} s^{m-1} + \dots}{s^n + a_{n-1} s^{n-1} + \dots}$

Direct ST. SP. in
CANONICAL CONTROL FORM

Assume: $\boxed{m < n}$

$\hookrightarrow$ otherwise $D = b_n$ $(m=n)$

$\hookrightarrow$ Pol. division

$$A_c = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \dots & \dots & -a_{n-1} \end{bmatrix} ; \quad B_c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Den. Coeff.

$C_c = [\, b_0, b_1, \dots b_m, 0 \dots 0 \,]$

HW VERIFY $P(s)$

$$C_c = [\underbrace{b_0, b_1, \dots b_m}_{\text{Num. coeff.}}, \underbrace{0 \cdots 0}_{n-m-1}]$$

(HW) VERIFY $P(s) = C_c(sI - A_c)^{-1} B_c$ &

Eg. SIMPLIFIED $P_{MOTOR}(s) = \dfrac{K_m}{Ns(1 + T_m s)} = \dfrac{K_m/T_m N}{s^2 + s/T_m}$

$$\left\{\begin{array}{ll} A_c = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T_m} \end{bmatrix} & B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_c = \begin{bmatrix} \frac{K_m}{T_m N} & 0 \end{bmatrix} & D = 0 \end{array}\right.$$

(HW) COMPARE WITH PREVIOUS MODEL FROM ODE.

KEY PROPERTIES :

- CONTROLLABILITY / REACHABILITY
- (OBSERVABILITY)
- STABILITY / STABILIZABILITY

- REACHABLE SET : STATES I CAN REACH FROM $X = 0$ & USING CONTROL

$$X^R = \left\{ x \in \mathbb{R}^n \mid \exists u(\cdot) \text{ s.t. } x = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \right\}$$

R_t, LIN. OP.

Teo. $X^R = \text{Image}(R)$ where))

Teo $\mathcal{X}^R = \text{Image}(R)$ where

$R = [B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B]$

↳ $\mathcal{X}^R$ is a SUBSPACE

Actually just a property of $[A, B]$

- If R has full-rank, $\mathcal{X}^R \equiv \mathbb{R}^n$ and Σ_s is said REACHABLE

EX. Σ_s IN CANONICAL CONTROL FORM $\Rightarrow$ REACHABLE.

- CONTROLLABLE SET : set of states x that can be driven to $x = 0$ by some $u(\cdot)$.

$$x \in \mathcal{X}^C \text{ if: } 0 = e^{At} x + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$\Rightarrow \quad \mathcal{X}^C = -\underbrace{e^{At}}_{\text{INVERTIBLE}} \mathcal{X}^R$$

actually invariant for e^{At}
$\mathcal{X}^R = \mathcal{X}^C$

$\Rightarrow$ Σ_s CONTROLLABLE IFF REACHABLE
↳ For CONTINUOUS TIME

Alternative test

THM [PBH] $\Sigma_s([A, B])$ is REACHABLE IFF

$$\text{rank} \underbrace{[\, sI - A \mid B \,]}_{n \times n+1} = n \quad \forall s$$

L

$n \times n+1$

WHY SO INTERESTING?

▷ Consider Linear static state feedback

$$u(t) = u(x(t)) = -\underline{K}\,\underline{x(t)}$$

▷ closed-loop STATE DYN.

$$\dot{x} = Ax - BK\cancel{x} = \underbrace{(A-BK)}_{A_{FB}}x$$

THM Σ_S is REACHABLE IFF

FOR ANY POLYNOMIAL $p(s)$, $\deg(p)=n$

THERE EXISTS $\underline{K}$ SUCH THAT

$$p(s) = \det(sI - \underbrace{(A-BK)}_{A_{FB}})$$

↳ RECALL:

$$\text{ZEROS}\left(\det(sI - A_{FB})\right)$$
$$\equiv$$
$$\boxed{\text{eigs}(A_{FB})}$$
$$\cup$$
$$\left[\text{Poles of } W_{FB}(s)\right]$$

⇒ REACHABILITY OF $[A,B]$ GUARANTEES COMPLETE (IN THEORY) FREEDOM IN THE ALLOCATION OF EIGS.

FREEDOM IN THE ALLOCATION OF EIGS.

$\Rightarrow$ $\boxed{\text{CHOICE OF MODES}}$

$\Rightarrow$ IN PARTICULAR $\boxed{\text{STABILIZATION}}$

Recall : Σ_s Asymp. STABLE (for $x_e(t)$)
if $Re(eig(A)) < 0$

This also GUARANTEES $W_s(s) = C(sI-A)^{-1}B$ BIBO stable
(equivalent if also Σ_s OBSERVABLE)
(MINIMAL)

- WHAT IF $\Sigma_s \sim [A,B]$ IS NOT REACHABLE?

Recall X^R is a SUBSPACE ;

- Find T s.t. $Tx = \begin{bmatrix} x_R \\ 0 \end{bmatrix}$ if $x \in X^R$

- It can be shown that for such T

$$\tilde{A} = TAT^{-1} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

$$\tilde{B} = TB = \begin{bmatrix} B_1 \\ \hline 0 \end{bmatrix}$$

FROM $R = [B|AB|..$ & CAYLEY HAM.
X^R is an A-Invariant Subspace CONTAINING B

and $[A_{11}, B_1]$ is REACHABLE.

- SO WITH FEEDBACK $u(t) = -[K_1 | K_2] \begin{bmatrix} x_R \\ x \end{bmatrix}$

- SO, WITH FEEDBACK $u(t) = -\left[K_1 \mid K_2\right]\begin{bmatrix} x_R \\ x_N \end{bmatrix}$

$$A_{FB} = \begin{bmatrix} A_{11} - B_1 K_1 & A_{12} - B_1 K_2 \\ 0 & A_{22} \end{bmatrix}$$

No effect

- UPPER BLOCK-TRIANGULAR :

$$\text{eigs}(A_{FB}) = \underbrace{\text{eigs}(A_{11} - B_1 K_1)}_{\text{CAN BE FREELY ALLOCATED BY } K_1} \cup \underbrace{\text{eigs}(A_{22})}_{\text{FIXED!}}$$

- $[A, B]$ is said STABILIZABLE if A_{22} is s.t. $\text{Re}(\text{eigs}(A_{22})) < 0$
(or Reachable)