CONTROL DESIGN IN STATE-SPACE

Basic Steps : same as in freq. domain (s, T.F.)

STEP I : obtain a model (A, B, C, D)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

ODE's

TF

STEP II : check $\underbrace{\text{Reachability}}$ (A, B)

↳ ok → allocation of eigs $A_{FB} = A - BK$

↳ NO → AT LEAST STABILIZABILITY

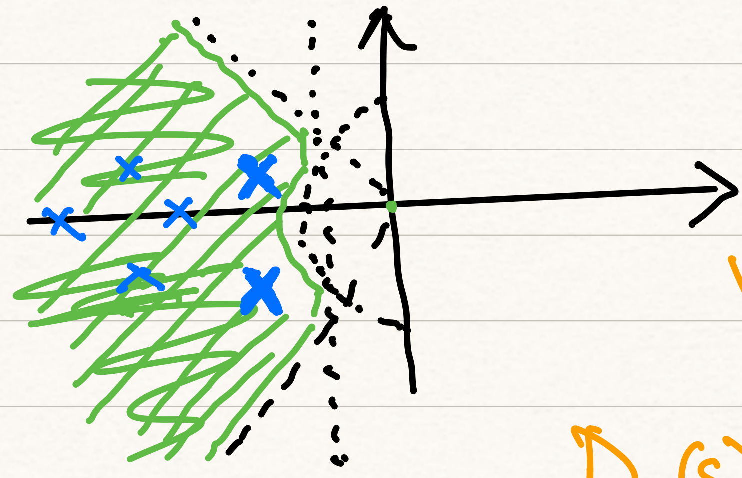
STEP III : obtain and translate PERFORMANCE SPECIFICATIONS

↳ (0) STABILITY

↳ (1) TRANSIENT

$$\underbrace{t_r}_{\text{"speed"}} , \underbrace{t_s}_{\text{"Transient lenght"}} , \underbrace{m_p} \leftarrow \text{"PRECISION"}$$

→ can be translated to acc. area in $\mathbb{C}$ for the poles (in the freq. dom)



Recall

$$W(s) = \frac{N_W(s)}{D_W(s)}$$

Dominant eigs ~ close to spec. limits

$D_W(s)$ has poles

roots $D_W(s) \subseteq$ eigs$\left(\underbrace{A-BK}_{A_{FB}}\right)$

others?

↳ (2) ASYMPT. BEHAVIOR

↳ TRACKING OF "KNOWN" CLASSES OF REFERENCE

TRACKING STEPS : REGULATION

↳ DISTURBANCE REJECT. for "classes" of external inputs

INTERNAL MODEL PRINCIPLE

① POLE / EIGENVALUE POSITIONING

↳ How ? ↳ Where ?

↳ STATE FEEDBACK, LINEAR & STATIC

$$u(t) = -K\,x(t)$$

if (A,B) is reachable → it can be put in CONTROLLABILITY FORM. (T)

$$A_c = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ -a_0 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

PLANT POLES (if observable) : roots $A(s) = \det(sI - A_c) \stackrel{!}{=} \sum_{i=0}^{n} a_i s^i$

▷ Assume we want poles for the closed-loop syst. in the roots of $D(s) = s^n + \gamma_{n-1}s^{n-1} + \ldots + \gamma_0$

$$\left(\stackrel{!}{=} (s-p_1)(s-p_2)\,D_a(s) \right)$$

"faster poles"

▷ FIND K TO MAKE THIS HAPPEN?

$[k_0 \ldots k_{n-1}]$

$$A_{FB} = A_c - B_c K = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0-k_0 & \cdots & \cdots & -a_{n-1}-k_{n-1} \end{bmatrix}$$

$\begin{bmatrix} 0 \\ -K \end{bmatrix}$ ↳ $\begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$

Poles of closed-loop system: roots$\left(\det(sI - A_{FB})\right)$

$$= \sum_i (a_i + K_i) s^i$$

$$\Rightarrow \gamma_i = a_i + K_i \quad \forall i \qquad \sum_i \gamma_i s^i = D(s)$$

$$\boxed{\Rightarrow K_i = \gamma_i - a_i \quad \forall i}$$

MATLAB : → acker $(A, B, [p_1, p_2 \dots])$

→ place

WHERE ? (we would like to keep K_i small)

↙ control has a cost ↘ saturation

↳ MOVE THE EIGS AS LITTLE AS POSSIBLE

$\lambda(A_{FB})$ continuous functions of K

EX $P(s) = \dfrac{s - z_0}{(s-p_1)(s-p_2)} = \dfrac{s - z_0}{s^2 - (p_1+p_2)s + p_1 p_2}$

PLANT

$$W(s) = \frac{N_W(s)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

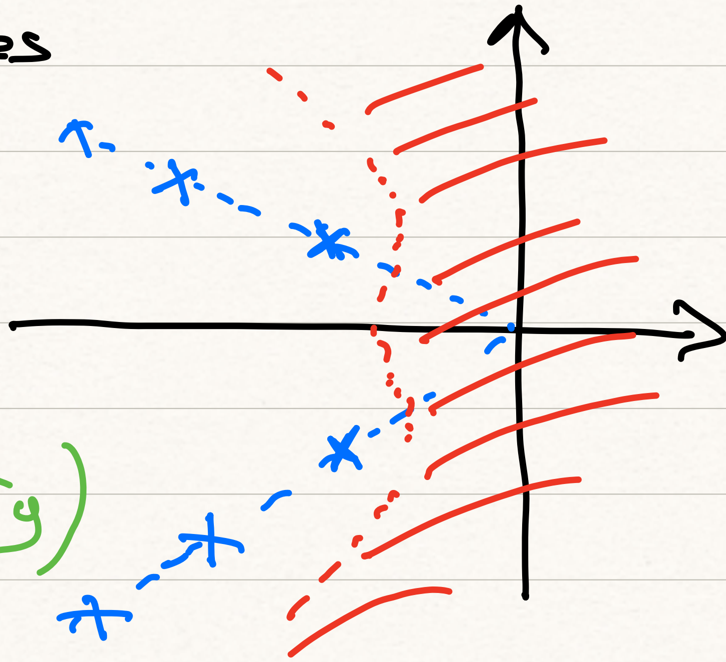
← TARGET CLOSED LOOP EV.

$P(s) \rightarrow$ ST. SP.

$$A_c = \begin{bmatrix} 0 & 1 \\ -p_1 p_2 & p_1 + p_2 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \omega_n^2 - p_1 p_2 & 2\xi\omega_n + (p_1 + p_2) \end{bmatrix}$$

ω_n : modulus of poles



$\omega_n \nearrow \qquad K_1 \nearrow \omega_n^2$

BE CAREFUL !

(saturation, non ideality)

▷ AVOID DOUBLE/MULTIPLE EIGS.

$$(s-\lambda_1)\, D_0(s) \quad \text{vs} \quad (s-\lambda_1)^2\, \tilde{D}_0(s)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$e^{\lambda_1 t} \qquad\qquad\qquad e^{\lambda_1 t},\ t e^{\lambda_1 t}$$

→ slower response

→ FRAGILE   small errors ⇓ potentially big errors.

▷ CARE ABOUT WHERE THE ZEROES ARE!

$$N_w(s) = N_p(s)$$

zeroes : the "s" s.t. rank $\underbrace{\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}}_{n+1} < n+1$ (SISO) $M(s)$

(~P.B.H criterion)

$$M_{FB}(s) = M(s) \underbrace{\begin{bmatrix} I & 0 \\ K & 1 \end{bmatrix}}_{\text{INVERTIBLE}}$$

▷ AVOID PLACING POLES CLOSE TO ZEROS (of $P(s)$)

↳ THESE ARE COMPETING ACTIONS :

" $\lim_{s \to p} W(s) = \infty$ "

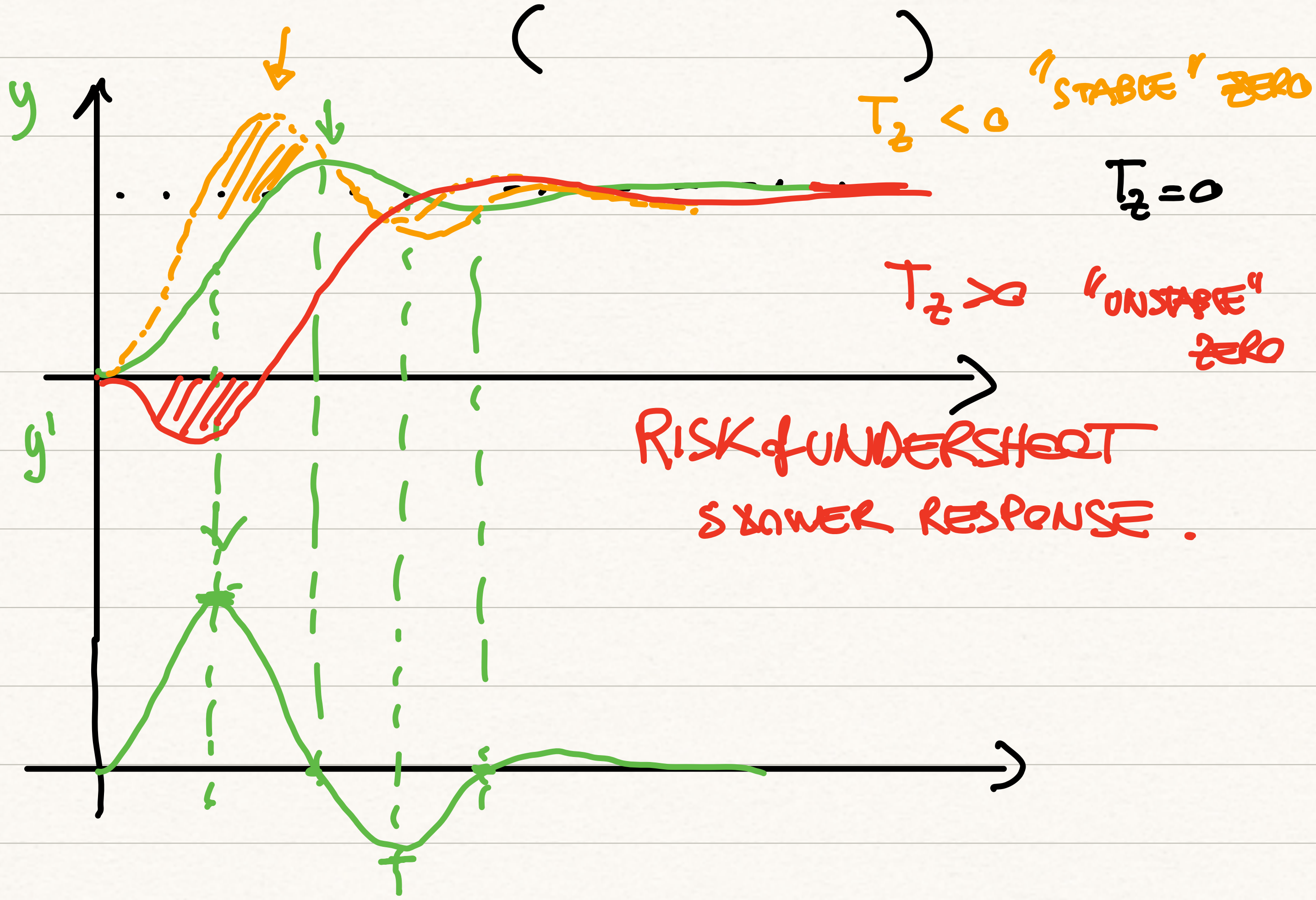
$\lim_{s \to z} W(s) = 0$

↳ RESULT IN "HIGH" CONTROL EFFORT...

▷ POSITION OF ZEROS INFLUENCES TRANSIENT

Ex $$W(s) \simeq \frac{\omega_n^2 - \omega_n^2 T_z s}{s^2 + \xi \omega_n s + \omega_n^2}$$

$$\simeq \frac{\omega_n^2 (1 - T_z s)}{(\qquad)}$$



RISK of UNDERSHOOT
SLOWER RESPONSE.

▷ REACHABILITY : yes/No ?

EX $$A = \begin{bmatrix} 0 & \varepsilon \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & \vdots & \varepsilon \\ 1 & \vdots & 1 \end{bmatrix}$$ if $\varepsilon \neq 0$ REACHABLE !

$\hookrightarrow \quad \underbrace{D(s) = s^2 + 2\xi\omega_n s + \omega_n^2}_{\text{Target deter}} = \underbrace{\det\left(sI - (A - BK)\right)}_{\text{poles of FB. syst.}}$

$$\det \begin{bmatrix} s & -\varepsilon \\ K_1 & s - 1 + K_2 \end{bmatrix} = s^2 + s(K_2 - 1) + K_1 \varepsilon$$

IMPOSE : $\quad K_2 = 2\xi\omega_n + 1$

$$\boxed{K_1 = \frac{\omega_n^2}{\varepsilon}} \nearrow +\infty \quad \varepsilon \searrow 0$$

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② ASYMPT. SPECS

REGULATION : $\quad y(t) \xrightarrow[t \to \infty]{} r(t) = r_s \mathbb{1}(t)$

Unknown

Recall if A_{fb} STABLE $\Rightarrow$ $x(t) \xrightarrow{t \to +\infty} 0$

if $r(t) = 0$ $\quad y(t) \longrightarrow 0$

Here $\quad r(t) \xrightarrow[t \to +\infty]{} r_s \quad \Rightarrow \quad x(t) \to ?$

$y(t) \to ?$

WRITE DYNAMICS : (for $t \to +\infty$)

$$\dot{x} = A_{fb}\, x + B\, r_s$$

New Equilibrium : check

$$0 = A_{fb}\, \bar{x} + B r_s$$

$$\boxed{\bar{x} = - A_{fb}^{-1} B\, r_s}$$

a new Eq for each value of r_s

Let's define : $\Delta x = x - \bar{x}$

New reference $\bar{x}$ is now 0

$$\dot{\Delta x} = A_{fb}\, x + \cancel{B r_s} - \underbrace{A_{fb}\, \bar{x} - \cancel{B r_s}}$$

$$= A_{fb} (\underbrace{x - \bar{x}}_{\Delta x})$$

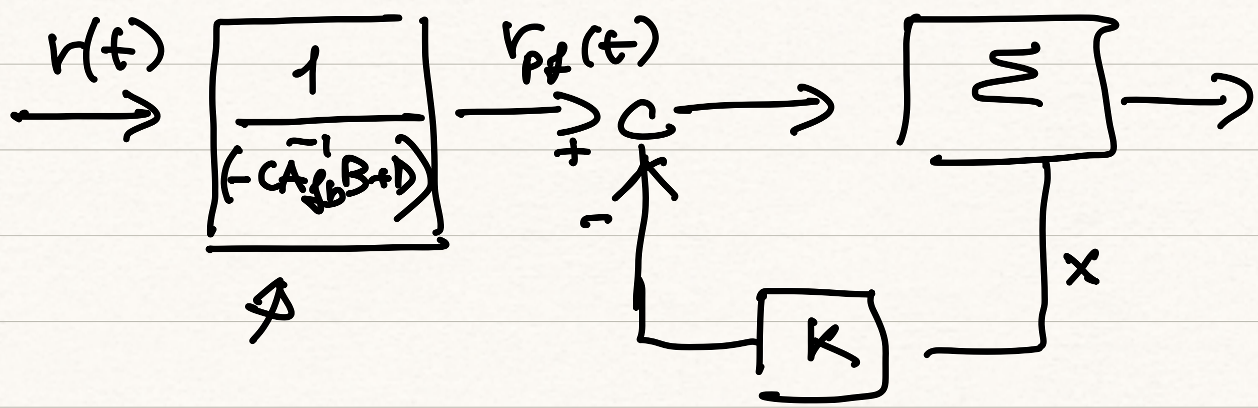
$$\Rightarrow \Delta x(t) \xrightarrow[t \to +\infty]{} 0$$

is this $= 1$?

$$y(t) \xrightarrow[t \to +\infty]{} C \underbrace{\bar{x}}_{= -A_{fb}^{-1} B\, r_s} + D\, r_s = \left(-C A_{fb}^{-1} B + D\right) r_s$$

New $y_\infty \neq 0$

OPTION 1 — Add a prefilter gain



OPTION 2 — add two gains N_x (rel. to Δx)

N_u (rel to u)

st. eq

ASYMT

out. eq

$$\left\{\begin{aligned} \dot{x} = 0 &= A_{fb}\,\bar{x} + B N_u r_s \\ r_s &= C\,\bar{x} + D N_u r_s \end{aligned}\right.$$

$r \xrightarrow[t\to+\infty]{} r_s$

$$\left\{\begin{aligned} 0 &= A_{fb}\,\frac{\bar{x}}{r_s}\,r_s + B N_u r_s \\ r_s &= C\,\frac{\bar{x}}{r_s}\,r_s + D N_u r_s \end{aligned}\right.$$

$$\frac{\bar{x}}{r_s} = -\frac{A_{fb}^{-1} B N_u \cancel{r_s}}{\cancel{r_s}} = N_x$$

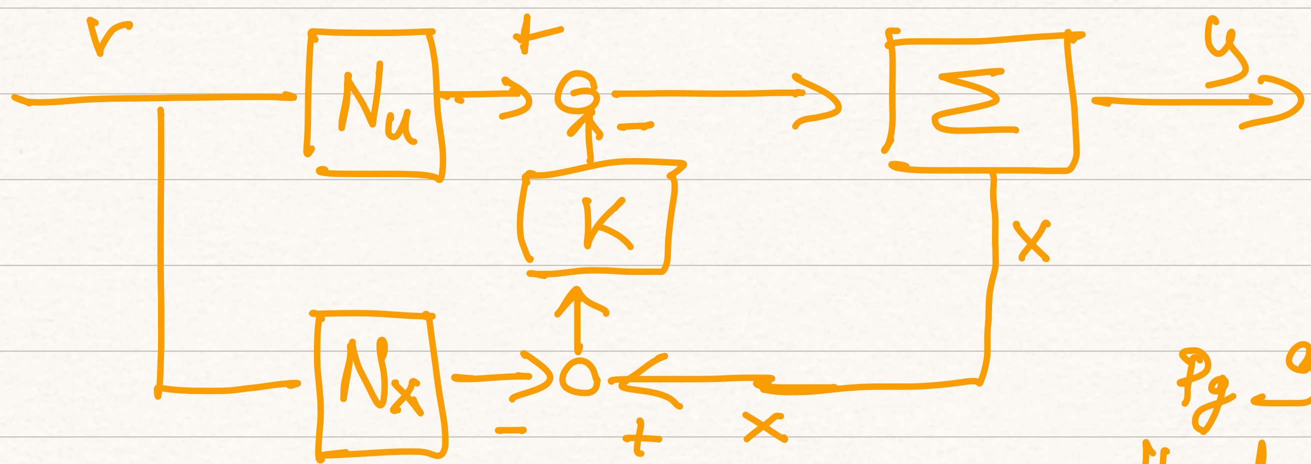
Equiv. Matrix eq.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cancel{r_s} = \underbrace{\begin{bmatrix} A_{fb} & B \\ C & D \end{bmatrix}} \begin{bmatrix} N_x \\ N_u \end{bmatrix} \cancel{r_s}$$

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \left(\quad \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u(t) = v_s(t) - K\,\overset{x - \bar{x}}{\Delta x}(t)$$

$$= N_u r_s - K\left(x(t) - N_x r_s\right)$$



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Handout