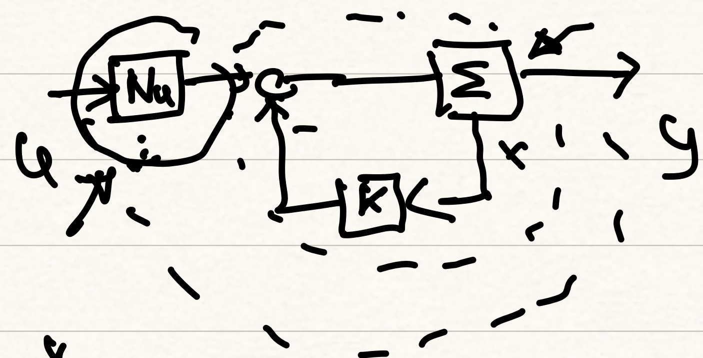
REGULATION & TRACKING

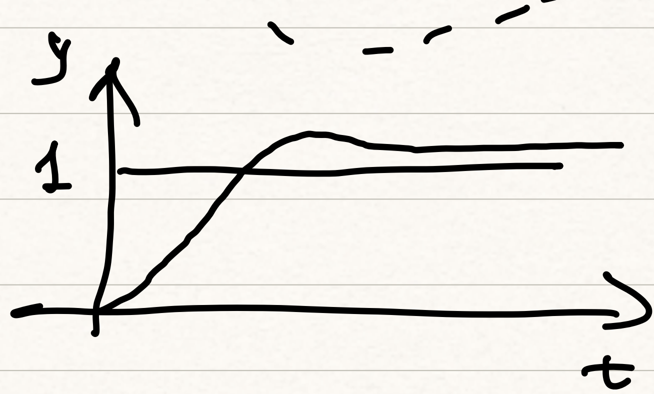
↳ $u(t) = v(t) = v_s \mathbb{1}(t)$

$y(t) \xrightarrow{t \to +\infty} v_s$

LAST TIME: we have seen "GAIN REGULATION"

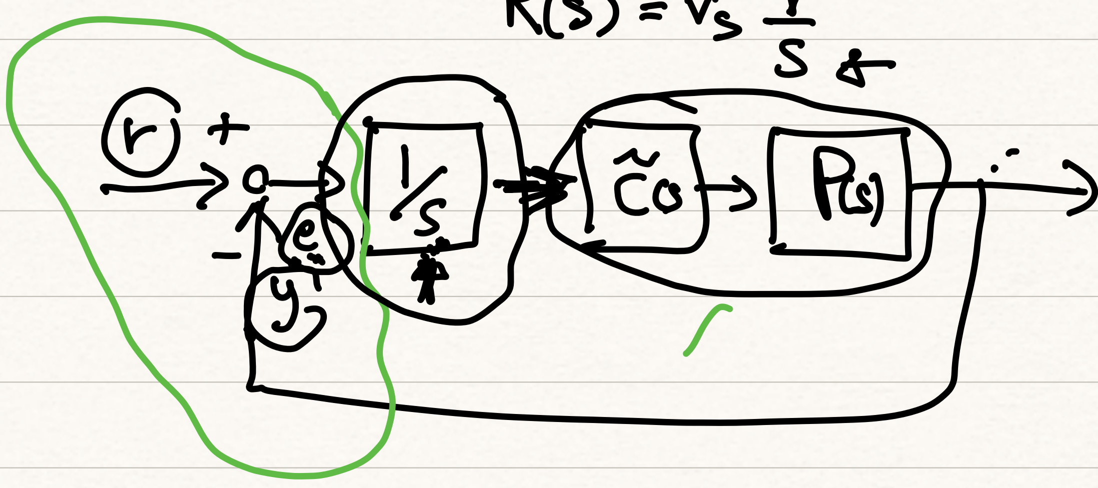
NONROBUST





▷ How to make this more robust?
(freq. domain)

$v(t) = v_s \mathbb{1}(t)$
⇓
$R(s) = v_s \frac{1}{s}$ &



▷ How to extend and use IMP (INT. MODEL PRINCIPLE) in state space?

↳ SUBPROBLEM : "ROBUST" Regulation and INTEGRAL CONTROL in ST. SP.

$$\begin{cases} \dot{x}_I = -e = -(r-y) = -r + Cx \\ \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Block MATRIX VERSION

$$\begin{pmatrix} \dot{x}_I \\ \dot{x} \end{pmatrix} = \underbrace{\overbrace{\left[\begin{array}{c|c} 0 & C \\ \hline 0 & A \end{array}\right]}^{\hat{A}} \begin{bmatrix} x_I \\ x \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\hat{B}}}_{} u + \begin{bmatrix} -1 \\ 0 \end{bmatrix} r$$

1^ST STEP Design $u = -\overbrace{\left[\begin{array}{c|c} K_I & K \end{array}\right]}^{\hat{K}} \begin{bmatrix} x_I \\ x \end{bmatrix}$

so that $\hat{A} - \hat{B}\hat{K} = \hat{A}_{FB}$ has the desired eigenvalues.

NOTICE if $\hat{A}_{FB}$ is at least as. st.

$$\begin{bmatrix} x_I \\ x \end{bmatrix} = \hat{x}(t) \xrightarrow[t \to +\infty]{} 0 \implies \left.\begin{array}{l} x_I \to 0 \\ \Downarrow \\ e \to 0 \end{array}\right\}$$

Q : when can I allocate eigs of $\hat{A}_{fb}$?

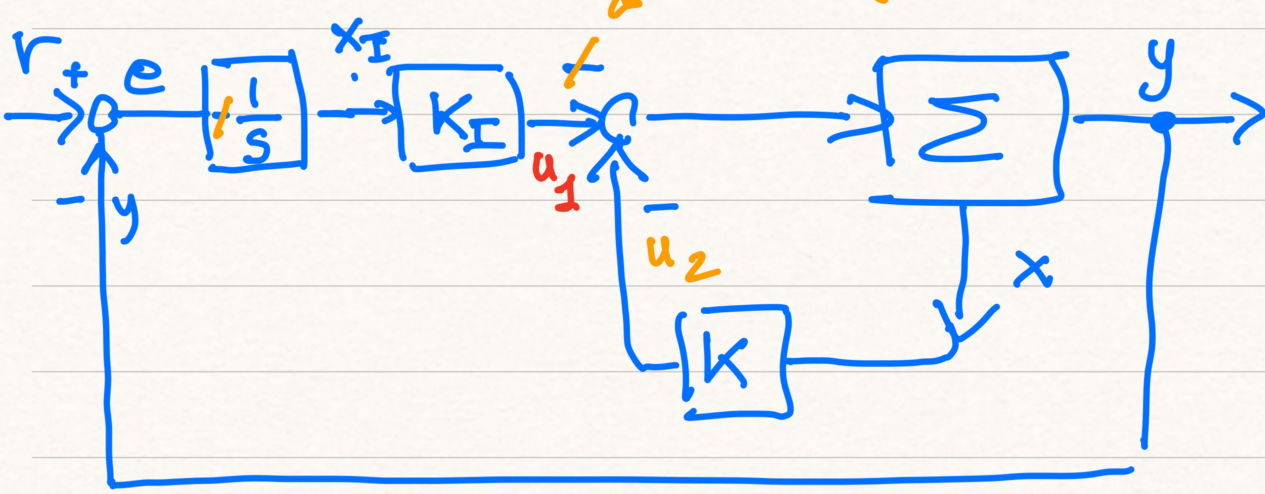
iff $(\hat{A}, \hat{B})$ is reachable

EX : USE P.B.H., WE CAN PROVE

$(\hat{A}, \hat{B})$ is reach. if (A, B) is reach and PLANT has NO ZEROS IN $S=0$

BLOCK STRUCTURE :

Reason for $\dot{x}_I = -e$



MORE GENERAL TRACKING ?

I.M.P. VIA ERROR-SPACE DESIGN

↳ "we need to include a model of the references in the CONTROLLED SYSTEM"

↳ IN STATE SPACE : we have to INCLUDE A STATE-SP. REPR.

ASSUME (as in IMP in freq. dom)

r(t) solutions of ODE's Linear T.I.

refs.

$$r^{(q)}(t) + \alpha_{q-1} r^{(q-1)}(t) + \dots + \alpha_0 r(t) = 0$$

C.I.

$$\hookrightarrow \mathcal{L}[r] = R(s) = \frac{N_r(s)}{\alpha(s)}$$

$$\alpha(s) = \sum_{i=0}^{n} \alpha_i s^i$$

↳ STATE SPACE :

$$\begin{cases} \dot{x}_r(t) = A_r x_r(t) \\ r(t) = C_r x_r(t) \end{cases}$$

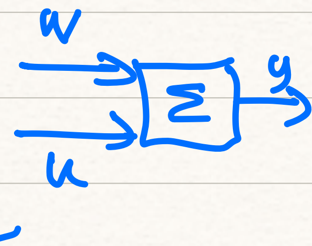
$$A_r = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ -\alpha_0 & \dots & & -\alpha_{q-1} \end{bmatrix}$$

$$C_r = [1 \; 0 \dots 0]$$

By posing $x_r(t) = [r(t) \; \dot{r}(t) \dots]^T$

Assume we also want to <u>INCLUDE DISTURB. REJ.</u>

$$\dot{x} = Ax + Bu + B_w w$$



as before : consider w solution of ODE's

$$w^{(p)}(t) + \gamma_{p-1} w^{(p-1)}(t) + \dots + \gamma_0 w(t) = 0$$

(w/o g) $\gamma_p = 1$

$$W(s) = \frac{N_w(s)}{\gamma(s)} \qquad \gamma(s) = \sum_{i=0}^{p} \gamma_i s^i$$

$\Rightarrow$ obtain

$$\dot{x}_w = A_w x_w$$
$$w = C_w x_w$$

by $x_w = \begin{bmatrix} w \\ w^{(1)} \\ \vdots \\ w^{(p-1)} \end{bmatrix}$

▷ What if I have both r, w ?

$\alpha(s)$ $\quad$ $\gamma(s)$

$\Rightarrow$ New : $\tilde{\alpha}(s) = \alpha(s)\gamma(s)$

$\Rightarrow$ CONSTRUCT ST. SP. MODEL (EXOSYSTEM) FOR BOTH.

TYPICAL SIGNALS

a) $r(t) = r_0 \mathbb{1}(t)$

ODE: $\dot{r} = 0$

$\alpha(s) = s = \alpha_1 s + \alpha_0$, $\alpha_1 = 1$, $\alpha_0 = 0$

$(-\alpha_0)$

$A_r = [0]$

$C_r = [1]$

b) STEP + RAMP: $r(t) = \underline{r_0 \mathbb{1}(t)} + \underline{r_1 t \mathbb{1}(t)}$

ODE: $\ddot{r}(t) = 0$

$\alpha(s) = s^2$

$$A_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(CHAIN OF INTEGRATORS)

$C_r = [1 \quad 0]$

c) SINUSOIDAL SIGN. $r(t) = r_0 \sin(\omega_0 t + \varphi_0) \mathbb{1}(t)$

ODE: HARMONIC OSCILLATOR

$$\ddot{r} + \omega_0^2 r = 0$$

$$\alpha(s) = s^2 + \omega_0^2$$

$\Downarrow$ $\alpha_2 = 1$ $\quad \alpha_1 = 0$ $\quad \alpha_0 = \omega_0^2$

$$\Rightarrow A_r = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \qquad C_r = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

NEXT Generalize the integral control we saw above.

↳ Need to "include $\alpha(s)$ in the model" (IMP)

↳ make $e(t) \longrightarrow 0$ using feedback (STATE-SP)

↳ construct BLOCK REP.

NOTE : abuse of notation

$$\alpha(s)\,\underbrace{y(t)}_{e(t)} \simeq \alpha\left(\frac{d}{dt}\right) y(t) = \sum_i \alpha_i \left(\frac{d}{dt}\right)^i y(t) = \sum_i \alpha_i y^{(i)}$$

$$\sum_i \alpha_i r^{(i)}$$

- in integral control : consider tracking error

$$e(t) = y(t) - r(t) = C\,x(t) - r(t)$$

STEP 1

compute

$$\alpha(s)e(t) = \alpha(s)C\,x(t) - \overbrace{\alpha(s)\,r(t)}^{=0}$$

ODE model for $r(t)$

$$= C\,\alpha(s)\,x(t)$$

$$= C\,\underbrace{\sum_i \alpha_i x^{(i)}(t)}$$

$$= C\,\xi$$

ERROR-SPACE STATE

STEP 2

$\alpha(s)X \leftrightarrow$ ER-SP. STATE : $\xi(t) = \sum_i \alpha_i x^{(i)}(t)$

$\alpha(s)u \leftrightarrow$ ER-SP. CONTROL : $\mu(t) = \sum_i \alpha_i u^{(i)}(t)$

STEP 3

FIND DYN. EQ. FOR $\xi(t)$

$$\dot{\xi}(t) = \frac{d}{dt}\underbrace{\sum_i \alpha_i \left(\frac{d}{dt}\right)^i x(t)}_{\xi(t)}$$

$$= \underbrace{\sum_i \alpha_i \left(\frac{d}{dt}\right)^i}_{\alpha(s)} \left(\frac{d}{dt}x(t)\right) \longrightarrow Ax + Bu + B_w w$$

$$\dot{\xi} \overset{!}{=} A \underbrace{\alpha(s) x}_{\xi(t)} + B \underbrace{\alpha(s) u}_{\mu(t)} + B_w \underbrace{\alpha(s) w}_{=0}$$

STEP 4 DEFINE EXTENDED STATE

$$z = \begin{bmatrix} e \\ e^{(1)} \\ \vdots \\ e^{(q-1)} \\ \hline \xi \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} e \\ e \\ \vdots \\ e \end{matrix}}\right\} q \leftarrow \text{err. \& derivatives} \\ \left.\vphantom{\xi}\right\} n \leftarrow \text{err. sp. STATE} \end{array}$$

in order to compute $\dot{z}$ (Dyn. eq. for z)
we need $\frac{d}{dt} e^{(i)}$

$$\begin{cases} \frac{d}{dt} e(t) = \dot{e}(t) = e^{(1)}(t) \\ \vdots \\ \frac{d}{dt} e^{(i)}(t) = e^{(i+1)} \\ \frac{d}{dt} e^{(q-1)} = e^{(q)} = -\alpha_0 e - \alpha_1 e^{(1)} \dots - \alpha_{q-1} e^{(q-1)} + C\xi \end{cases}$$

STARTS FROM $e^{(q)}$

$\sum_i \alpha_i e^{(i)} = C\xi$

BLOCK MATRICES

$$\dot z = \frac{d}{dt}\left[\begin{array}{c} e^{(0)} \\ \vdots \\ e^{(q-1)} \\ \hline \xi \end{array}\right] = \underbrace{\left[\begin{array}{cccccc|c} 0 & 1 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & & \vdots \\ & & & & & & 0 \\ -\alpha_0 & -\alpha_1 & \cdots & & & -\alpha_{q-1} & C \\ \hline & & & 0 & & & A \end{array}\right]}_{\bar A} z + \underbrace{\left[\begin{array}{c} \vdots \\ 0 \\ \hline B \end{array}\right]}_{\bar B} u$$

<u>FACT</u> if $[A, B]$ Reachable

$\Rightarrow$ $[\bar A, \bar B]$ reachable

because CONCATENATION OF REACH systems

$\Rightarrow$ ALLOCATE EIGEN. OF $\bar A - \bar B \bar K_2$

<u>FACT</u> $\bar A - \bar B \bar K_2$ STABLE $\qquad \bar K_2 = [K_e \mid K_\xi]$

$\Rightarrow$ $A - B K_\xi$ STABLE

(PROVE AS HW)

▷ HOW TO IMPLEMENT $u(t) = -\bar{K}_2 z$

$$[K_e \mid K_\xi] \begin{bmatrix} e^{(i)} \\ \vdots \\ \xi \end{bmatrix}$$

By def

$$\sum_i \alpha_i u^{(i)}(t) = -K_e \begin{bmatrix} e \\ \vdots \\ e^{(q-1)} \end{bmatrix} - K_\xi \xi$$

$$= -\underbrace{\sum_i K_{e,i} e^{(i)}}_{} - K_\xi \underbrace{\sum_i \alpha_i x^{(i)}(t)}_{}$$

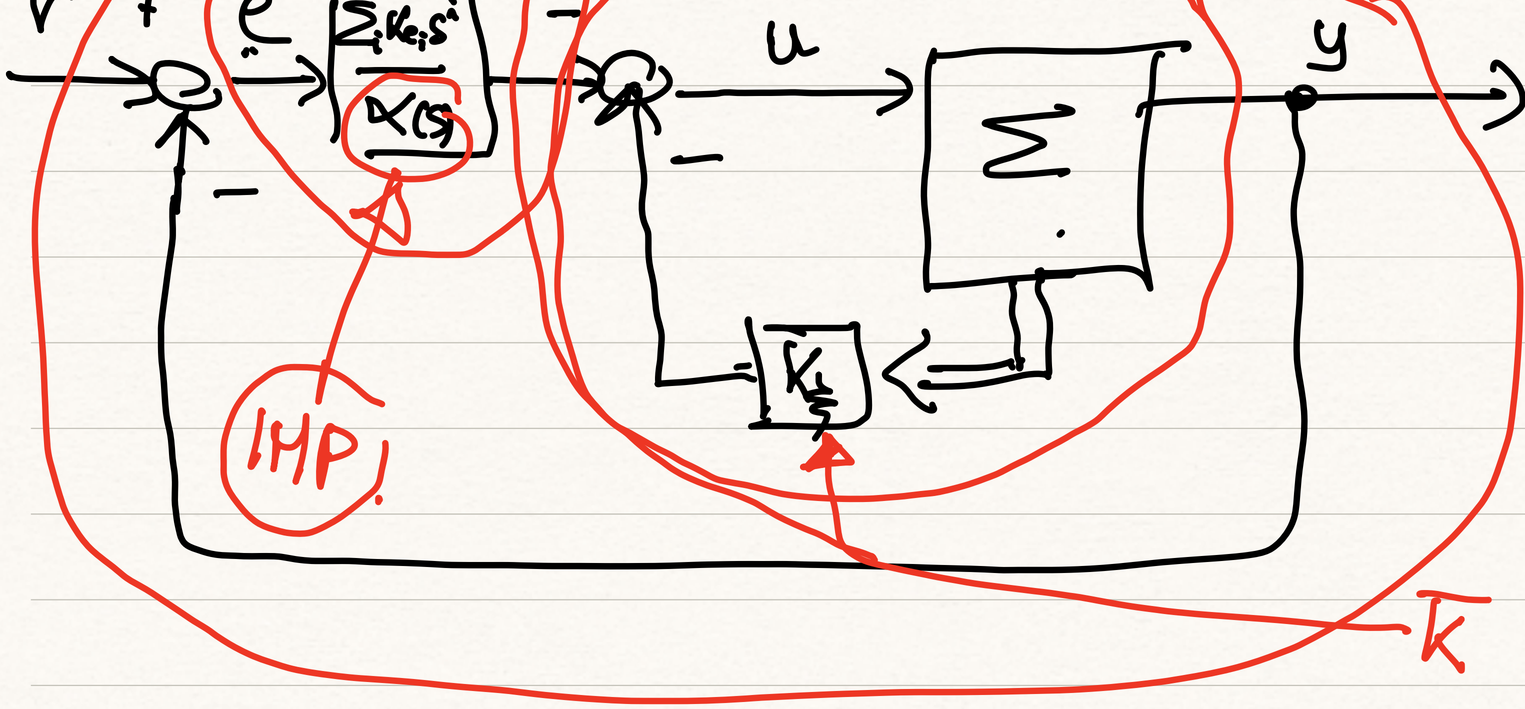
$$\Rightarrow \sum_i \alpha_i \left[u^{(i)}(t) + K_\xi x^{(i)}(t) \right] = -\sum_i K_{e,i} e^{(i)}(t)$$

$\mathcal{L}$

$$\alpha(s) \left[U(s) + K_\xi X(s) \right] = \left(-\sum_i s^i \right) E(s)$$

$$U(s) + \underbrace{K_\xi X(s)}_{} = \frac{\left(-\sum_i s^i \right)}{\alpha(s)} E(s)$$

$G_K(s)$



r
+
e
$\frac{\sum_i k_{c_i} s^i}{\alpha(s)}$
u
y
Σ
−
$\frac{K_I}{s}$
IHP!
$\bar{K}$