FEEDBACK CONTROL & STATE OBSERVERS IN DISCRETE TIME (STATE SPACE APPROACH)

→ REDUCED – ORDER OBSERVER

STATE-SPACE MODEL

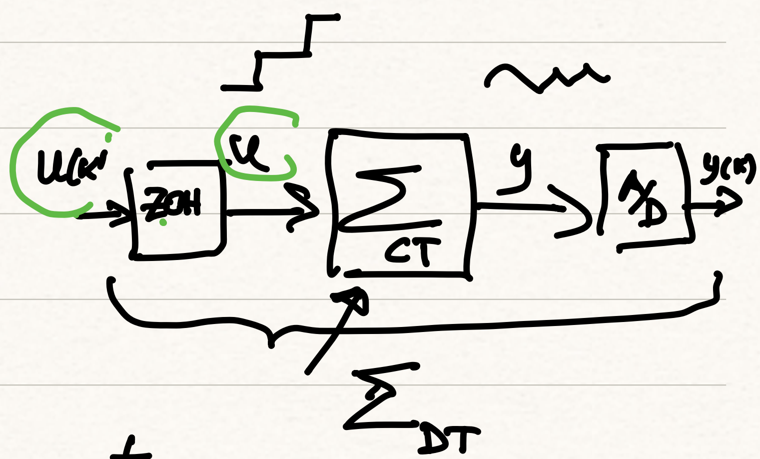
$$\begin{cases} x(k+1) = A\,x(k) + B\,u(k) \\ y(k) = C\,x(k) \end{cases}$$

Assume $D=0$ (1-STEP DELAY)

▷ How to obtain ↑ from continuous model ?

$$\rightarrow \quad \dot{x} = A\,x + B\,u$$

$$y = C\,x$$



$$P(s) = \frac{N_P(s)}{D_P(s)}$$

$$x(t) = e^{A(t-t_0)}\,x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\,B\,u(\tau)\,d\tau$$

fix $t_0 = kT$ $\quad t = (k+1)T$

T sampling period

$(k+1)T$

$$x((k+1)T) = \underbrace{e^{AT}}_{\Phi}\, x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\, u(\tau)\, d\tau$$

$u(\tau) = u(kT)$

$$x(k+1) = \Phi\, x(k) + \underbrace{\left(\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\, d\tau\right)}_{=:\Gamma} u(kT)$$

$$\Rightarrow \boxed{\begin{aligned} x(k+1) &= \Phi\, x(k) + \Gamma\, u(k) \\ y(k) &= C\, x(k) \end{aligned}} \quad \Sigma_{D.T.}$$

$$\Rightarrow \quad P(z) = C\,(zI - \Phi)^{-1}\,\Gamma$$

(c2d in MATLAB)

$P_D = e^{P_c T}$

To choose where we want the eigs$(\Phi - \Gamma K_D)$

we consider: formulas $\nabla \Sigma_c$ $P_D = e^{P_c T}$

specs$(m_p, t_r, t_s) \longrightarrow$ Acc. area of $\mathbb{C}$ plane $\longrightarrow$ Acc. area for D.T. Σ

Consider $u(k) = -K_D \, x(k)$

$\Rightarrow$ closed loop Dynamics

$$x(k+1) = \underbrace{\left(\Phi - \Gamma K_D\right)}_{\Phi_{F.B.}} x(k)$$

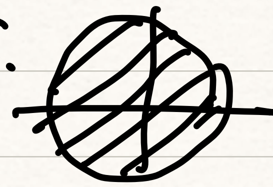
$$\Phi_{F.B.} \sim A - BK$$

$\Rightarrow$ use place, acker matlab funct.

to compute K_D s.t.

$$\text{eig}(\Phi_{F.B}) = \{\lambda_1, \dots, \lambda_n\}$$

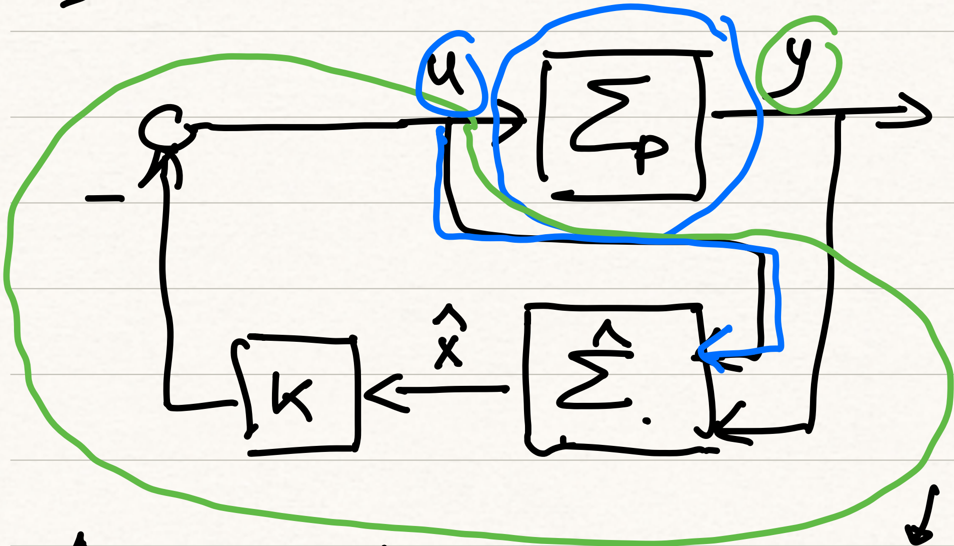
TARGET.



possible if (Φ, Γ) reachable.

What if we do not have $x(k)$?
only $y(k)$, $u(k)$

$\Rightarrow$ OBSERVER



$$\hat{\Sigma} : \quad \hat{x}(k+1) = \underbrace{(A - LC)}_{\Phi}\hat{x}(k) + B\overset{=-K\hat{x}}{\hat{u}(k)} + Ly(k)$$

DISCRETE TIME

$$\begin{cases} \hat{x}(k+1) = (\underbrace{A - LC - BK}_{F})\hat{x}(k) + L\,y(k) \\ u(k) = -K\hat{x}(k) \end{cases}$$

$$C(z) = -K(zI - F)^{-1}L$$

Can we simplify and render more efficient the estimator ?

NOTE : Part of the state is already available from y with a static, linear transformation

$$y = C x \qquad T: \text{inv. matrix} \qquad (y \in \mathbb{R}^p)$$

$$y = \underbrace{C T}_{\tilde{C}} \underbrace{T^{-1} x}_{\tilde{x}}$$

$$C = \begin{array}{c} \\ \underbrace{\begin{bmatrix} - - & c_1 & - - \\ - - & c_2 & - - \\ - - & \vdots & - - \end{bmatrix}}_{n} \end{array} \Big\} p \qquad T$$

$\Rightarrow$ find a T s.t. $\{c_i T\}$ are orthonormal row vectors

$$\tilde{C} = C T = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \cdots & 1 \end{bmatrix} \cdots \; 0 \right]$$

$$\tilde{C} = \left[\; I \;\vdots\; \emptyset \; \right]$$ $\leftarrow$ this extracts the first p-components of $\tilde{x}$

$$y = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{bmatrix} \Longleftrightarrow \tilde{x} = \begin{bmatrix} y \\ \hline v \end{bmatrix} \begin{matrix} \}p \\ \}n-p \end{matrix}$$

DYNAMICS of $\tilde{x}$ $\rightarrow$ dynamics of v

$$\begin{bmatrix} y(k+1) \\ \cdots \\ v(k+1) \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right]}_{\tilde{A} = T^{-1}AT} \begin{bmatrix} y(k) \\ \cdots \\ v(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \tilde{B}_1 \\ \hline \tilde{B}_2 \end{bmatrix}}_{\tilde{B} = T^{-1}B} u(k)$$

$$\begin{cases} \bar{u}(k) := \tilde{B}_2 u(k) + \tilde{A}_{21} y(k) \\ \bar{y}(k) = y(k+1) - \tilde{A}_{11} y(k) - \tilde{B}_1 u(k) \end{cases}$$

$$\Rightarrow \begin{cases} v(k+1) = \tilde{A}_{22} v(k) + \bar{u}(k) \\ \bar{y}(k) = \underbrace{\tilde{A}_{12}}_{\tilde{C}} v(k) \end{cases}$$

$$\Rightarrow \hat{v}(k+1) = \tilde{A}_{22} \hat{v}(k) + \bar{u}(k) + L\left(\bar{y}(k) - \tilde{A}_{12} \hat{v}(k)\right)$$

$$= \tilde{A}_{22} \hat{v}(k) + \tilde{A}_{21} y(k) + \tilde{B}_2 u(k)$$

$$+ L\left(y(k+1) - \tilde{A}_{11} y(k) - \tilde{B}_1 u(k) - \tilde{A}_{12} \hat{v}(k)\right)$$

$$= \Big| (\tilde{A}_{22} - L\tilde{A}_{12})\hat{v}(k) + \tilde{A}_{21}\, y(k) + B_2\, u(k)$$
$$+ L\big(y(k+1) - \tilde{A}_{11}\, y(k) - \tilde{B}_1\, u(k) \big)$$

$\Rightarrow$ Define $z(k) = \hat{v}(k) - L\, y(k)$

$$z(k+1) = \hat{v}(k+1) - L y(k+1)$$

$$\Rightarrow z(k+1) = (\tilde{A}_{22} - L\tilde{A}_{12})\big(z(k) + L y(k)\big) + \tilde{A}_{21}\, y(k)$$
$$\Big| + \tilde{B}_2\, u(k) + L\big(- \tilde{A}_{11}\, y(k) - \tilde{B}_1\, u(k) \big)$$

$$= A_o\, z(k) + B_o \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\begin{cases} A_o = \tilde{A}_{22} - L\tilde{A}_{12} \quad (\text{same of } \hat{v}) \\ B_o = \big[\, \tilde{B}_2 - L\tilde{B}_1 \;\vdots\; (\tilde{A}_{22} - L\tilde{A}_{12})L + \tilde{A}_{21} - L\tilde{A}_{11} \,\big] \end{cases}$$

Recalling : $\hat{v}(k) = z(k) + L y(k)$

$$\Rightarrow \hat{x}(k) = \begin{array}{l} \begin{bmatrix} y(k) \\ \cdots \\ \hat{v}(k) \end{bmatrix} \begin{array}{l} \}\, p \\ \\ \}\, n-p \end{array} \end{array} = \begin{bmatrix} y(k) \\ \cdots \\ z(k) + L y(k) \end{bmatrix}$$

Reduced order estimator, discrete time

$$\begin{cases} z(k+1) = A_o z(k) + B_o \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \\ \hat{x}(k) = C_o z(k) + D_o \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \end{cases}$$

A_o, B_o : as before

$$C_o = \begin{bmatrix} 0 \\ \cdots \\ I \end{bmatrix} \qquad D_o = \left[\begin{array}{c|c} 0 & I \\ \hline 0 & L \end{array}\right]$$

$A_o = \tilde{A}_{22} - L\tilde{A}_{12}$ ← eigs. can be allocated if $(\tilde{A}_{22}, \tilde{A}_{12})$ is OBSERVABLE

OBSERVABILITY of $(\tilde{A}, \tilde{C})$

$\Updownarrow$

$$\text{rank}\begin{pmatrix} \tilde{C} \\ \cdots \\ sI - \tilde{A} \end{pmatrix} = n \quad \forall s \in \mathbb{C}$$

in our case $\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$ $\tilde{C} = [I \, \vdots \, 0]$

p x p

$$\operatorname{rank}\begin{bmatrix} I & 0 \\ sI-\tilde{A}_{11} & -\tilde{A}_{12} \\ -\tilde{A}_{21} & \underbrace{sI-\tilde{A}_{22}}_{n-p} \end{bmatrix} = n \qquad \forall s$$

$$\operatorname{rank}\begin{bmatrix} -\tilde{A}_{12} \\ sI-\tilde{A}_{22} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \tilde{A}_{12} \\ sI-\tilde{A}_{22} \end{bmatrix} = n-p$$

MOTOR

Real. in state space

$$\begin{bmatrix} \dot{\vartheta} \\ \ddot{\vartheta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}\begin{bmatrix} \vartheta \\ \dot{\vartheta} \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u$$

$$\vartheta = y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \vartheta \\ \dot{\vartheta} \end{bmatrix}$$