A Note on the representation of zeros in state space

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1 Introduction

Consider the matrices (A,B,C,D) of a time-invariant system (we refer to the continuous-time case but the results and the arguments for the discrete-time case are identical):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{1}$$

Consider the SISO case so that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}^{1\times 1}$. Let

$$W(s) := C(sI - A)^{-1}B + D$$

be the transfer function of the system. Let $\sigma(A)$ be the *spectrum* of A, i.e. the set of the eigenvalues of A. Let $\mathscr{P}_W$ be the set of poles of $W(s)$ and $\mathscr{Z}_W$ be the set of zeros of $W(s)$. There is a well-known and important relation between $\sigma(A)$ and $\mathscr{P}_W$. Precisely,

$$\mathscr{P}_W \subseteq \sigma(A)$$

and if the realization (1) is minimal (or, equivalently, reachable and observable) then

$$\mathscr{P}_W = \sigma(A).$$

Moreover, if $p \in \sigma(A)$ and $p \notin \mathscr{P}_W$ then at least one of the following relations holds:

$$\mathrm{Rank}[pI - A \mid B] < n \qquad \mathrm{Rank}\begin{bmatrix} pI - A \\ C \end{bmatrix} < n.$$

The sets $\mathscr{P}_W$ and $\sigma(A)$ are of crucial importance as they determine the dynamics of the system.

2 Representation of zeros in state space

We know that the zeros $\mathscr{Z}_W$ also have a crucial importance for the system behaviour. For this reason it is interesting to work out a relation between $\mathscr{Z}_W$ and the matrices (A,B,C,D) associated with a state space representation of the system. To this end, it is convenient to introduce the so-called *Rosenbrock matrix*:

$$M(s) := \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \tag{2}$$

having dimensions $(n+1) \times (n+1)$, where $s \in \mathbb{C}$. The following result provides a connection between $\mathscr{Z}_W$ and (A,B,C,D) in the case in which the realization (1) is minimal (reachable and observable).

Theorem 2.1 *Consider a system (1) and its transfer function $W(s)$. Let $M(s)$ defined in (2) be the associated Rosenbrock matrix. Assume that the realization (1) is minimal.*

Then z is a zero of $W(s)$ if and only if $\mathrm{Rank}[M(z)] < n+1$.

Proof: We develop the proof for two separate cases: in the first we assume that z is not an eigenvalue of A while in the second we assume that z is an eigenvalue of A.

Case 1: $z \notin \sigma(A)$.

We need to show that $W(z) = 0 \Leftrightarrow \mathrm{Rank}[M(z)] < n+1$.

"$\Leftarrow$": assume that $\mathrm{Rank}[M(z)] < n+1$. Then there exists $w \neq 0$ such that $M(z)w = 0$. Partition w conformably with $M(z)$ as $w = \begin{bmatrix} u \\ v \end{bmatrix}$ (so that v is scalar). We have

$$M(z)w = \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

or, equivalently,

$$\begin{cases} (A - zI)u + Bv = 0 \\ Cu + Dv = 0 \end{cases} .$$

Since $z \notin \sigma(A)$ this implies

$$\begin{cases} u = -(A - zI)^{-1} B v \\ Cu + Dv = -C(A - zI)^{-1} B v + Dv = W(z) v = 0 \end{cases}$$

Assume by contradiction that $v = 0$ then $u = -(A - zI)^{-1} B v = 0$ so that $w = 0$ contradicting our assumption. Then $v \neq 0$ which, coupled with $W(z)v = 0$ implies $W(z) = 0$.
"$\Rightarrow$": Let z be such that $W(z) = 0$; we have to show that $\text{Rank}[M(z)] < n+1$.
Let $w := \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -(A - zI)^{-1} B \\ 1 \end{bmatrix} \neq 0$. Then

$$M(z)w = \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} -(A - zI)^{-1} B \\ 1 \end{bmatrix} = \begin{bmatrix} -B + B \\ W(z) \end{bmatrix} = 0$$

Hence, there is a nonzero vector in the kernel of $M(z)$ which therefore cannot have full rank. In other words, $\text{Rank}[M(z)] < n+1$.

Case 2: $z \in \sigma(A)$.
In this case, by minimality assumption z is a pole of $W(s)$ and hence it is not a zero of $W(s)$. It is therefore sufficient to show that $\text{Rank}[M(z)] = n+1$ or, equivalently, that if w is a vector such that $M(z)w = 0$ then $w = 0$. As before, partition w as $w := \begin{bmatrix} u \\ v \end{bmatrix}$ so that $M(z)w = 0$ can be written as

$$M(z)w = \begin{bmatrix} (A - zI)u + Bv \\ Cu + Dv \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, equivalently, as

$$\begin{cases} (A - zI)u + Bv = 0 \\ Cu + Dv = 0. \end{cases} \tag{3}$$

Since z is an eigenvalue of A, there exists a (column) vector q such that

$$q^\top (A - zI) = 0$$

so that, by taking into account the reachability assumption, we have

$$q^\top B \neq 0.$$

Now, by multiplying the first of (3) on the left side by $q^\top$, we get

$$q^\top (A - zI)u + q^\top Bv = \underbrace{q^\top B}_{\neq 0} \; v = 0$$

so that

$$v = 0.$$

Hence, (3) becomes

$$\begin{bmatrix} (A - zI) \\ C \end{bmatrix} u = 0$$

which, in view of the PBH observability test implies

$$u = 0.$$

□

The Rosenbrock matrix (2) has the structure of a *matrix pencil*, i.e. a polynomial of degree 1 with matricial coefficients:

$$M(s) := \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = s \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Matrices with this structure can be reduced to a canonical form which is analogous to the Jordan canonical form so that even for zeros we can define a *zero structure* i.e. algebraic and geometric multiplicity of the zeros. This becomes apparent and much easier to deal with in the case when $D \neq 0$. In fact, in this case we have the following result.

Proposition 2.1 *Let $M(s)$ defined in (2) and assume that $D \neq 0$. Let $\Gamma := A - BD^{-1}C$. Then z is such that* $\mathrm{Rank}[M(z)] < n+1$ *if and only if $z \in \sigma(\Gamma)$.*

Proof: We have $\text{Rank}[M(z)] = \text{Rank}[M(z)N]$ for any nonsingular matrix N. Let

$$N := \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}$$

(which is clearly nonsingular). We have

$$\text{Rank}[M(z)] = \text{Rank}[M(z)N] = \text{Rank}\begin{bmatrix} \Gamma - zI & B \\ 0 & D \end{bmatrix} = \text{Rank}[\Gamma - zI] + \text{Rank}[\underbrace{D}_{\neq 0}] = \text{Rank}[\Gamma - zI] + 1.$$

Therefore, $\text{Rank}[M(z)] < n+1$ if and only if $\text{Rank}[\Gamma - zI] < n$ or, equivalently, if and only if z is an eigenvalue of Γ. □

The matrix Γ is referred to as the *zero-marix* of the system and the Jordan structure of the matrix Γ is referred to as the *zero structure* system.

The definition of the zero matrix Γ and its meaning can be generalized to the MIMO case as long as D is square and nonsingular. In fact, the next result shows that in this case Γ can be regarded as the state matrix of a state-space realization of $[W(s)]^{-1}$.

Theorem 2.2 *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Assume that D is nonsingular and let*

$$W(s) := C(sI - A)^{-1}B + D.$$

Then

$$[W(s)]^{-1} = -D^{-1}C(sI-\Gamma)^{-1}BD^{-1} + D^{-1}, \quad \text{with } \Gamma := A - BD^{-1}C.$$

Proof: It is sufficient to show that $X := W(s)[-D^{-1}C(sI-\Gamma)^{-1}BD^{-1} + D^{-1}] = I$. Indeed, we have:

$$\begin{aligned} X &= [C(sI-A)^{-1}B + D][-D^{-1}C(sI-\Gamma)^{-1}BD^{-1} + D^{-1}] \\ &= -C(sI-A)^{-1}\underbrace{\underbrace{BD^{-1}C}_{A-\Gamma}}_{(sI-\Gamma)-(sI-A)}(sI-\Gamma)^{-1}BD^{-1} + C(sI-A)^{-1}BD^{-1} + C(sI-\Gamma)^{-1}BD^{-1} + I \\ &= I \end{aligned} \tag{4}$$

□

2.1 Dropping minimality assumption

If we drop the minimality assumption, we still have a relation between $\mathscr{Z}_W$ and the set of z for which $\mathrm{Rank}[M(z)] < n+1$. We discuss the SISO case but a generalization to the MIMO case is possible.

Theorem 2.3 *Consider a system (1) and its transfer function $W(s)$. Let $M(s)$ defined in (2) be the associated Rosenbrock matrix.*

If z is a zero of $W(s)$ then $\mathrm{Rank}[M(z)] < n+1$.

Proof: Observe that in the proof of the **Case 1** of Theorem 2.1, minimality assumption has not been used therefore, if $z \notin \sigma(A)$, the proof is the same as that provided for Theorem 2.1.

Now consider the case in which $z \in \sigma(A)$ and assume that $W(z) = 0$. Then $z \notin \mathscr{P}_W$ so that at least one of the following relations holds:

$$\mathrm{Rank}[zI - A \mid B] < n, \qquad \mathrm{Rank}\begin{bmatrix} zI - A \\ C \end{bmatrix} < n.$$

If $\mathrm{Rank}\begin{bmatrix} zI - A \\ C \end{bmatrix} < n$ holds then there exists a nonzero vector u such that $\begin{bmatrix} zI - A \\ C \end{bmatrix} u = 0$. Hence $M(z)\begin{bmatrix} u \\ 0 \end{bmatrix} = 0$ so that $\begin{bmatrix} u \\ 0 \end{bmatrix}$ is a nonzero vector in the kernel of $M(z)$ which therefore cannot have full rank.

If $\mathrm{Rank}[zI - A \mid B] < n$ holds then there exists a nonzero vector q such that $q^\top [zI - A \mid B] = 0$. Hence $[q^\top \mid 0]\, M(z) = 0$ so that $[q^\top \mid 0]$ is a nonzero (row) vector in the left kernel of $M(z)$ which therefore cannot have full rank. □

Notice that as much as the eigenvalues of A are associated with the *modes* of the system and to the evolution of the free response and are therefore relevant even when they are not poles of the transfer function $W(s)$, also the values z for which $\mathrm{Rank}[M(z)] < n+1$ have an important interpretation even when such values are not zeros of the transfer function $W(s)$. The values z for which $\mathrm{Rank}[M(z)] < n+1$ are called *invariant zeros* of the system.

2.2 Zeros and change of basis

Consider the system (1) and a state feedback control and a change of basis induced by the matrix T. The new system is:

$$\begin{cases} \dot{x}(t) = A_T\, x(t) + B_T\, v(t), \\ y(t) = C_T\, x(t) + D_T\, v(t) \end{cases} \tag{5}$$

with

$$A_T := T^{-1}AT, \qquad B_T := T^{-1}B, \qquad C_T := CT, \qquad D_T := D.$$

The corresponding transformed Rosenbrock matrix is

$$M_T(s) = \begin{bmatrix} A_T - sI & B_T \\ C_T & D_T \end{bmatrix}. \tag{6}$$

Notice that

$$M_T(s) = \begin{bmatrix} T^{-1}AT - sI & T^{-1}B \\ CT & D \end{bmatrix} = \underbrace{\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}}_{\text{nonsingular}} \underbrace{\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}}_{\text{nonsingular}}, \tag{7}$$

so that the Rosenbrock matrices associated to (i) the original system (1) and (ii) to the system (5), both lose rank in correspondence to exactly the same values of s.

Then we have the following result.

Theorem 2.4 *The invariant zeros of the system are invariant under any transformation induced by a change of basis in the state space.*

Notice that when D is nonsingular and the zero matrix $\Gamma = A - BD^{-1}C$ of the system (1) can be defined, also the zero matrix $\Gamma_T = A_T + B_T D_T^{-1} C_T$ of the system (5) can be defined and we have

$$\Gamma_T = T^{-1}AT + T^{-1}BD^{-1}CT = T^{-1}\Gamma T.$$

In other words the same change of basis transformation that acts on the state matrix A acts also on the zero matrix Γ.

2.3 Zeros and feedback

Consider the system (1) and a state feedback control

$$u(t) = Kx(t) + v(t).$$

The closed-loop system is therefore,

$$\begin{cases} \dot{x}(t) = (A+BK)x(t) + Bv(t), \\ y(t) = (C+DK)x(t) + Dv(t). \end{cases} \tag{8}$$

The corresponding closed-loop Rosenbrock matrix is

$$M_c(s) = \begin{bmatrix} A+BK-sI & B \\ C+DK & D \end{bmatrix}. \tag{9}$$

Notice that

$$M_c(s) = \begin{bmatrix} A+BK-sI & B \\ C+DK & D \end{bmatrix} = \underbrace{\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}}_{\text{nonsingular}}, \tag{10}$$

so that the Rosenbrock matrices associated to (i) the original system and (ii) to the closed-loop system, both lose rank in correspondence to exactly the same values of s.

Then we have the following result.

Theorem 2.5 *The invariant zeros of the system are invariant under any transformation induced by state feedback.*

Notice that when D is nonsingular and the zero matrix $\Gamma = A - BD^{-1}C$ of the system (1) can be defined, also the zero matrix $\Gamma_K = A + BK - BD^{-1}(C+DK)$ of the system (8) can be defined and we have

$$\Gamma_K = A + BK - BD^{-1}(C+DK) = A + BK - BD^{-1}C - BK = \Gamma$$

In other words, the zero matrix is invariant under state feedback transformation of the system.