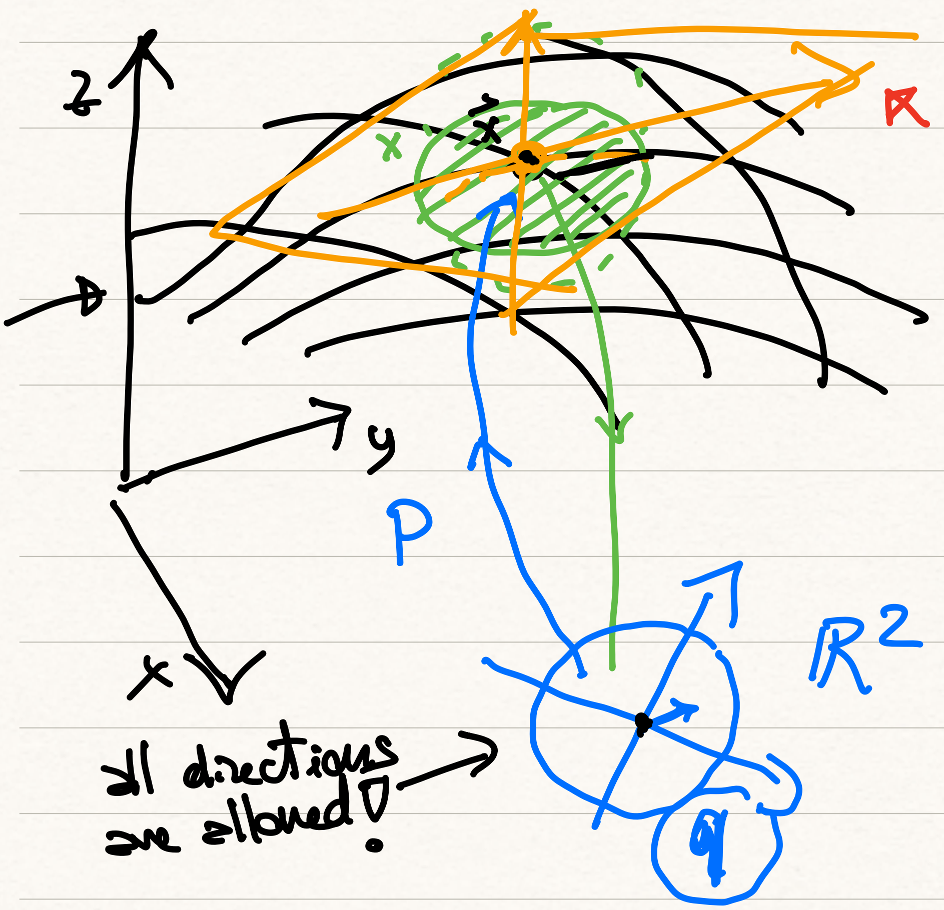
LAGRANGIAN MODELING

$\hookrightarrow$ Newton's law $\quad m\vec{a} = \vec{F}$

has some drawbacks ... (last time)



$x \in \mathbb{R}^3$

constrained to a set

$$\Psi(x) = 0$$

if Ψ is well-behaved (constant rank of $\frac{\partial \Psi}{\partial x}$)

$$P(\vec{q}) = \underline{\vec{x}} \quad , \qquad P \text{ s.t. } \left[\frac{\partial P}{\partial q}\right](q)$$

is **full rank**

q : generalized coordinates remaining after we consider the constraints.

TASK : PRESENT A SYSTEMATIC METHOD TO DERIVE ODE'S DESCRIBING THE SYSTEMS DYNAMICS (in q)

1^{ST} STEP) WRITE THE KINETIC ENERGIES IN THE NEW COORDINATES.

- Since $\vec{x} = P(\vec{q})$

$$\vec{v} = \frac{d}{dt}\vec{x} = \left[\frac{\partial P}{\partial q}\right](\vec{q})\,\underbrace{\frac{d}{dt}\vec{q}}_{\dot{\vec{q}}}$$

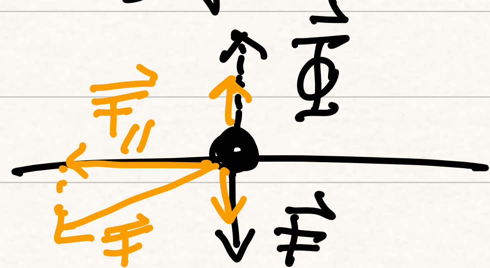
KINETIC ENERGY:

Define $M = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_e \end{bmatrix}$

$$\begin{aligned} T(\vec{v}) &= \frac{1}{2}\sum_e m_e v_e^2 \\ &= \frac{1}{2}\vec{v}^T M \vec{v} \\ &= \frac{1}{2}\dot{\vec{q}}^T\left[\frac{\partial P}{\partial q}\right]^T M \left[\frac{\partial P}{\partial q}\right]\dot{\vec{q}} \\ &= T(\dot{\vec{q}}) \end{aligned}$$

2^{nd} STEP) REWRITE FORCES in this framework

IN NEWTON'S APPROACH



Let's project forces onto "allowed directions": Tangent space

elements of the tangent space : δP

$\vec{x} = P(\vec{q})$

$$\Rightarrow \quad \delta \vec{P} = \left[\frac{\partial P}{\partial q}(q)\right] \delta \vec{q}$$

$\frac{\partial P}{\partial q}$: best lin approx of P

$\Rightarrow$ PROJECT FORCES along allowed directions

$$\langle F, \delta P \rangle = \langle F, \left[\frac{\partial P}{\partial q}\right] \delta q \rangle$$

$\delta q = \sum_h c_h \, \delta q_h$

orth. basis for the new coord

$$= \sum_h c_h \langle F, \frac{\partial P}{\partial q_h} \rangle \delta q_h$$

h-th column of $\frac{\partial P}{\partial q}$

$$= \sum_h Q_h \, c_h \, \delta q_h$$

$$Q_h := \langle F, \frac{\partial P}{\partial q_h} \rangle$$

FOR CONSERVATIVE FORCES $F = -\nabla V(x)$

$$\Rightarrow Q_h = -\frac{\partial V(P(q))}{\partial q_h} = -\frac{\partial \tilde{V}}{\partial q_h}(q) \; : \; \tilde{V}(q) = V(P(q))$$

3rd STEP) DERIVE Eq. of Motions starting from $m\vec{a} = \vec{F}$

$$m\vec{a} - \vec{F} = \vec{\Phi}$$

all forces but the reactive

only forces relevant for constraints.

$\Rightarrow$ PROJECT onto $\vec{\delta P}$:

$$\partial P \quad \langle m\vec{a} - \vec{F}, \vec{\delta P} \rangle = \cancel{\langle \vec{\Phi}, \vec{\delta P} \rangle = 0}$$

Given the decomposition of $\delta P = \sum_k \frac{\partial P}{\partial q_k} \delta q_k$

$$\forall k \quad \langle m\vec{a} - \vec{F}, \frac{\partial P}{\partial q_k} \rangle = 0$$

$$\Rightarrow (1) \quad \langle m\vec{a}, \frac{\partial P}{\partial q_k} \rangle = Q_k$$

$\Rightarrow$ It can be proved that [calculation]

$$(2) \quad \langle m\vec{a}, \frac{\partial P}{\partial q_k} \rangle = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \dot q_h} - \frac{\partial T}{\partial q_h} = Q_h = \tilde{Q}_h - \frac{\partial \tilde{V}(q)}{\partial q_h}$$

Non conserv. forces

cons. forces

Define the LAGRANGIAN

$$L(q,\dot q) := T(\dot q) - \tilde{V}(q)$$

the previous equation is equivalent to:

$$\Rightarrow \quad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_h} - \frac{\partial L}{\partial q_h} = \tilde{Q}_h} \quad \forall h$$

EULER-LAGRANGE EQUATIONS.

(equiv. to Newton's law)

Algorithm:

1) choose q (free parameters), $P(q)=x$

2) rewrite $T(v) \rightarrow T(\dot q)$

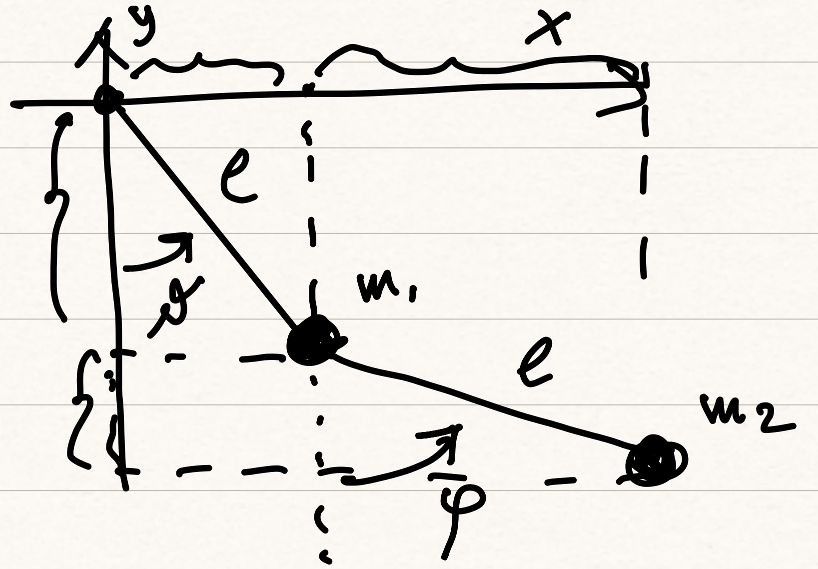
3) " $V(x) \rightarrow V(q)$

4) Define $L \doteq T - V$

5) compute $\tilde{Q}_h = \langle \underset{\text{non.cons.}}{\underset{\not{}}{F}} , \frac{\partial P}{\partial q_h} \rangle$

6) compute $\forall h$ $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q} = \tilde{Q}_h$

EX (Double Pendulum)



$m_1 = m_2 = m$

1) choose $q_1 = \vartheta$, $q_2 = \varphi$

2) $T_1 = \frac{1}{2} m v_1^2 = \frac{1}{2} m l^2 \dot{q}_1^2$

$(v_1 = l \dot{q}_1)$

T_2?

$$\begin{cases} x_2 = l \sin\vartheta + l \sin\varphi \\ y_2 = -l\cos\vartheta - l\cos\varphi \end{cases}$$

$$\begin{cases} \dot{x}_2 = \ell\cos\vartheta\,\dot{\vartheta} + \ell\cos\varphi\,\dot{\varphi} = v_{2,x} \\ \dot{y}_2 = \ell\sin\vartheta\,\dot{\vartheta} + \ell\sin\varphi\,\dot{\varphi} = v_{2,y} \end{cases}$$

$$T_2 = \tfrac{1}{2} m_2 v_2^2 \quad \leftarrow$$

$$\vdots$$

$$= \frac{1}{2} m\ell^2\left(\dot{\vartheta}^2 + \dot{\varphi}^2 + 2\cos(\vartheta-\varphi)\dot{\vartheta}\dot{\varphi}\right)$$

3) Potential

$$V = -mg\ell\,(2\cos\vartheta - \cos\varphi)$$

4) Lagrangian : $L(\vartheta,\varphi,\dot{\vartheta},\dot{\varphi}) = T(\dot{\vartheta},\dot{\varphi}) - V(\vartheta,\varphi)$

5) E-L equations

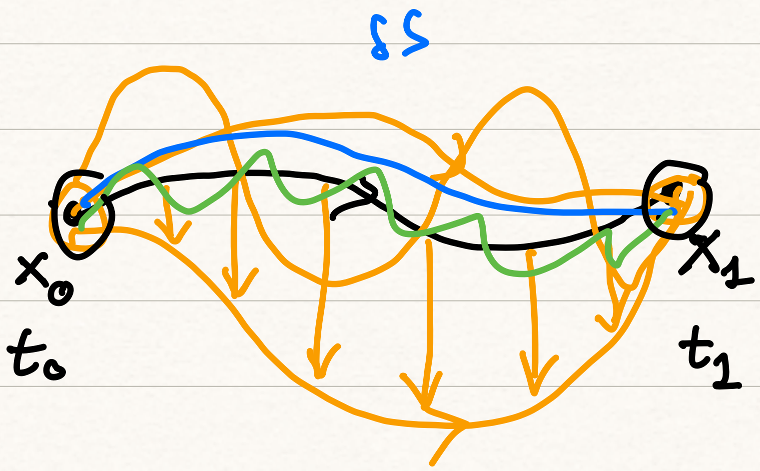
start computing $\dfrac{\partial L}{\partial\vartheta}$, $\dfrac{\partial L}{\partial\dot{\vartheta}}$, $\dfrac{\partial L}{\partial\varphi}$, $\dfrac{\partial L}{\partial\dot{\varphi}}$

$$\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{\vartheta}}\,,\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}}\right)$$

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\vartheta}} - \dfrac{\partial L}{\partial\vartheta} = 2\ddot{\vartheta} + \cos(\vartheta-\varphi)\ddot{\varphi} + \sin(\vartheta-\varphi)\dot{\varphi}^2 + 2\frac{g}{\ell}\sin\vartheta \overset{!}{=} 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\varphi}} - \dfrac{\partial L}{\partial\varphi} \overset{!}{=} \ddot{\varphi} + \cos(\vartheta-\varphi)\ddot{\vartheta} - \sin(\vartheta-\varphi)\dot{\vartheta}^2 + \frac{g}{\ell}\sin\vartheta \end{cases}$$

$= 0$

ANOTHER LOOK at E-L Equation



$\forall t \in [t_0, t_1]$

All possible ways of going from x_0 @ t_0 to x_1 @ t_1

Sol. of E-L

$$\tilde{x}(t) = \underline{x(t)} + \underline{y(t)}$$

s.t. $\begin{cases} y(t_0) = 0 \\ y(t_1) = 0 \end{cases}$

Recall : $L = T - V$

Define ACTION : $S(x, \dot{x}) = \int_{t_0}^{t_1} L(x, \dot{x})\,dt$

Define

$$\delta S(x; y) := \lim_{\varepsilon \to 0} \frac{S(x + \varepsilon y) - S(x)}{\varepsilon}$$

x satisfies E-L equations iff $\delta S(x;y)=0$ $\forall y$

$L(x, \dot{x})$

$$0 = \delta S(\bar{x};y) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x}(y) - \frac{\partial L}{\partial \dot{x}}(\dot{y})\right] dt$$

$\forall y$

$$= \int_{t_0}^{t_1} \frac{\partial L}{\partial x} y \, dt + \cancel{\left[\frac{\partial L}{\partial \dot{x}}(y)\right]_{t_0}^{t_1}} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} y \, dt$$

$$= -\int_{t_0}^{t_1} \underbrace{\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right]}_{} y \, dt$$

$\Updownarrow$

$$0 = \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right] \sim \text{E.L.}$$