Quantum Systems and Control: Models, Problems and Methods

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Why should I learn about

quantum information, control and all that?

A Bit of Motivation

and

Quantum Essentials

The Idea of Quantum Engineering

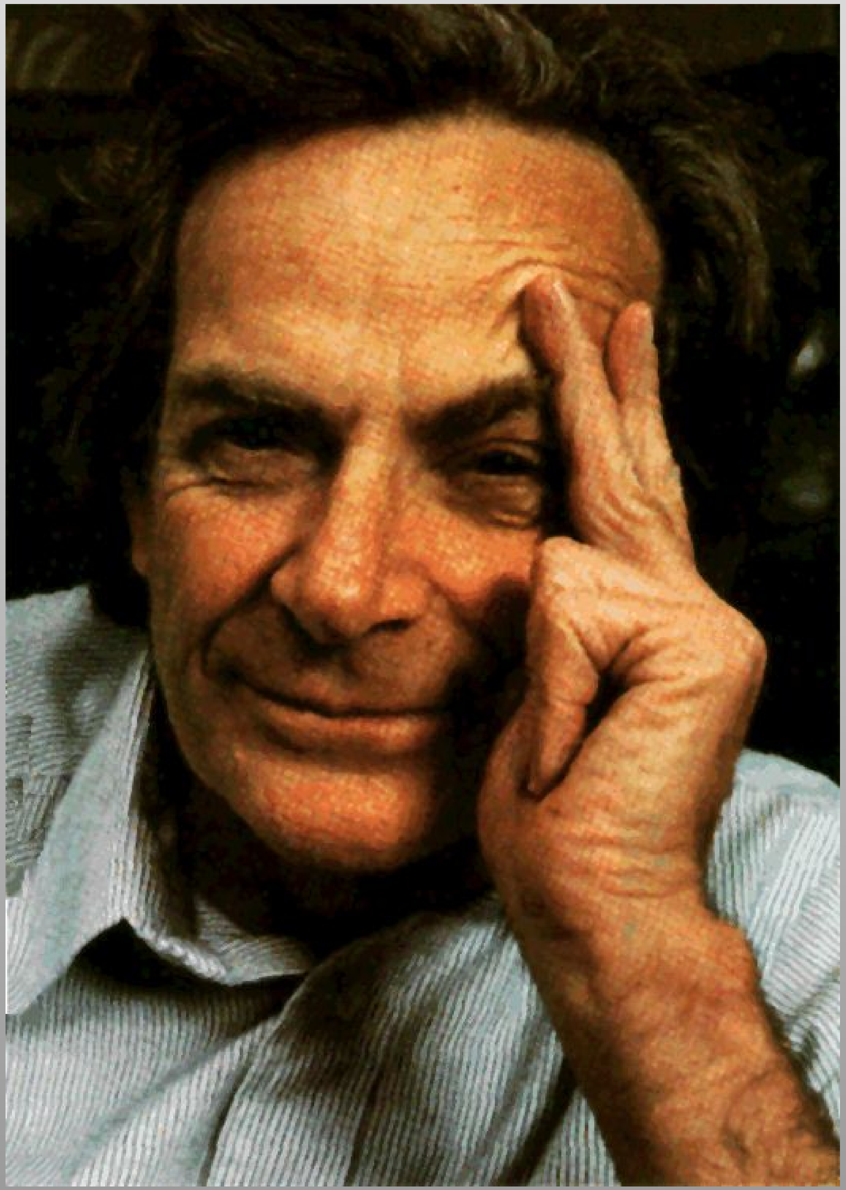
'' I would like to describe a field... which [...] would have an enormous number of technical applications.

What I want to talk about is the **problem of manipulating and controlling things on a small scale.** It is something, in principle, that can be done; but in practice, it has not because we are too big.

...In the year 2000, when they look back at this age, they will wonder why it was not until the year 1960 that anybody began seriously to move in this direction."

Richard P. Feynman,
There's Plenty of Room at the Bottom
(Caltech, APS Meeting, 29th December 1959).

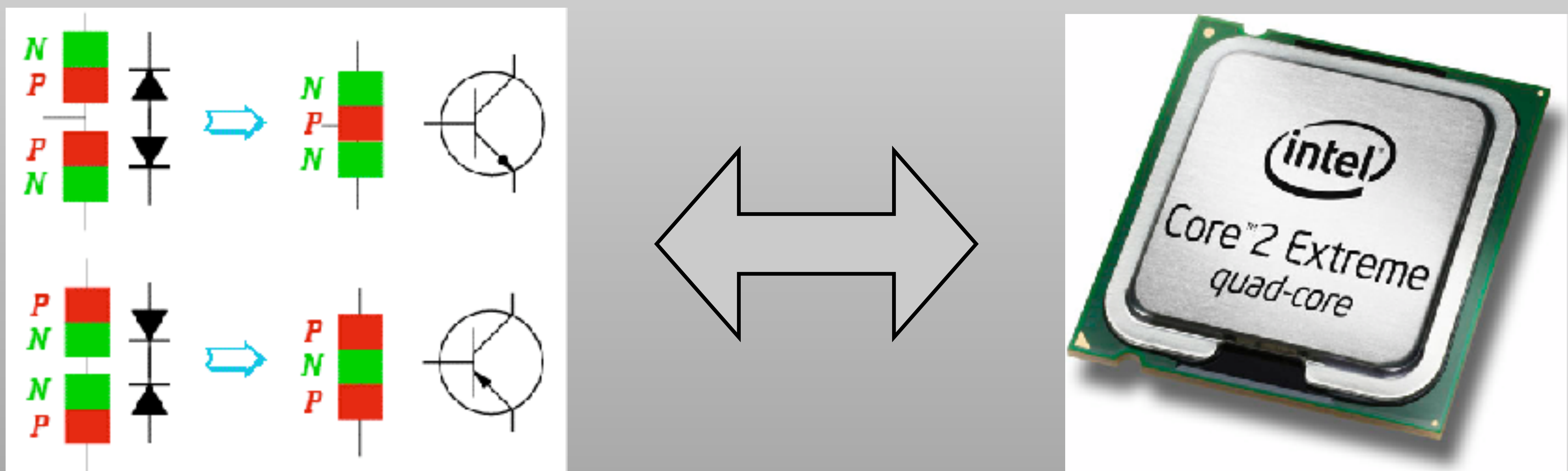
RPF Main Task:
Simulation of quantum systems;
Dimensionality curse!



Q. Technologies: We could or we should?

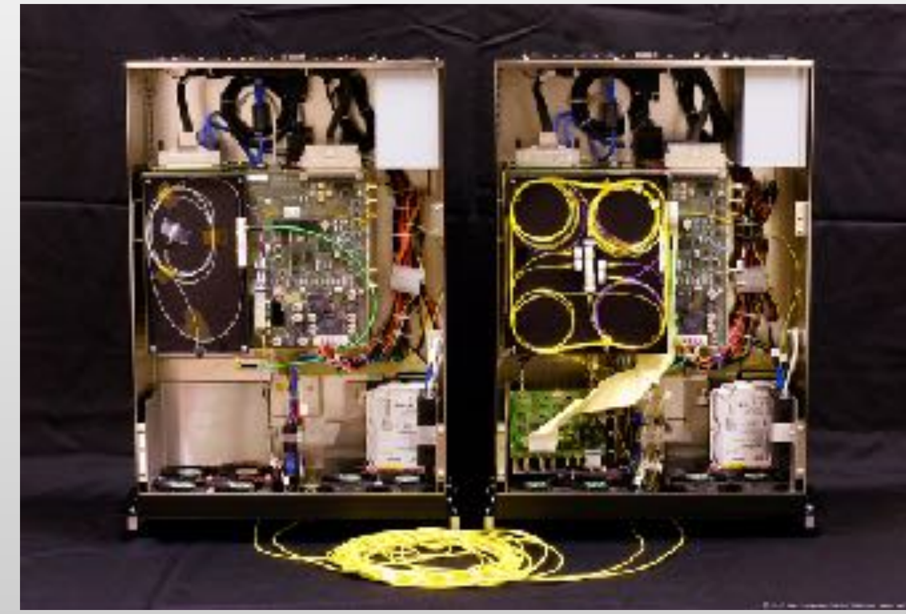
- **Information has to be encoded, stored, transmitted, processed and recovered in physical systems;**

- **Physics sets the rules of the game:**
  Physical laws define what can be done with my *source*, *code, receiver*, *controller,* etc.;



- **Quantum physics exhibits superpositions, nonlocal effects, measurements that perturb the state….**

Information and Quantum Physics today



- **Quantum cryptography:**
  Intrinsically "secure". On the market (e.g. MagiQ).

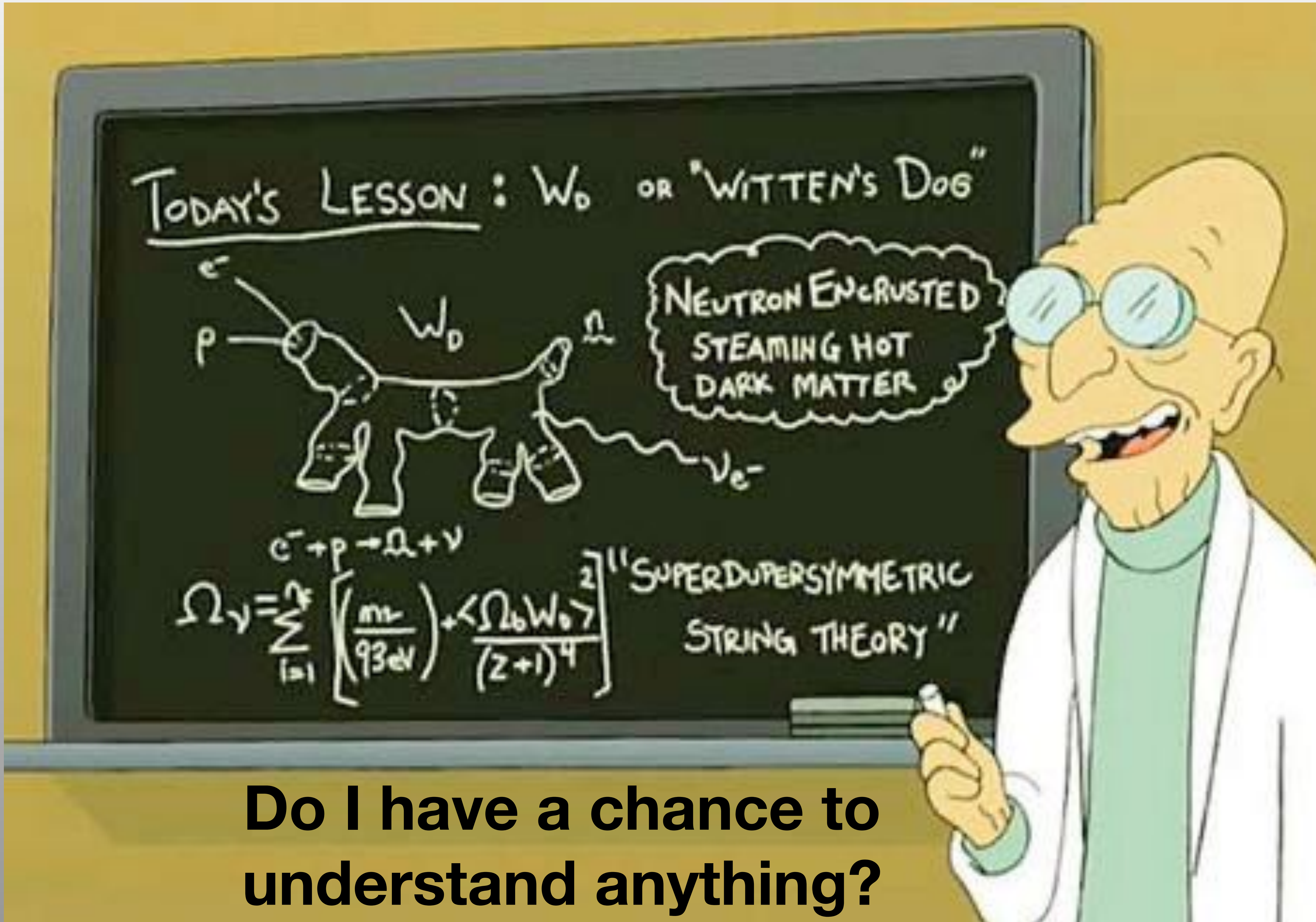
- **Full-fledged Quantum Computer (QC):**
  Theory: QC has advantage over state-of-the-art classical algorithms for factoring (exp), DFT (exp), search (quad).
  Practice: About 50 qubits (IBM)/
  72 qubits Google Bristlecone; scalability issues.

- **Dedicated quantum processors:**
  Simulated annealing/adiabatic processors are on the market (D-Wave); Google-NASA, Lockheed-Martin bought them (M$)!

- **Quantum simulation:** see QC - but needs exponentially less resources to be competitive.
  [original Feynman's task]

Also:
Metrology, Spectroscopy,
Controlled Q. chemistry,
Q. biology.

Ok, if they buy it, I am sold too. But...



Do I have a chance to understand anything?

Quantum Probability: Vectors become Matrices

✓ **CLASSICAL PROBABILITY** (finite Ω)		✓ **QUANTUM PROBABILITY** (finite dim.) $\mathcal{H}_\Omega$
• Events, σ-algebra: $\omega_i \to (0,\ldots,0,1,0,\ldots 0)^T$, $e \to (0,1,1,0,\ldots)^T$	$\Longrightarrow$	• **Orth. Projections:** $\{\Pi \mid \Pi = \Pi^2 = \Pi^\dagger\}$
• Probability Distribution: $\mathbb{P}(\omega_i) = p_i \to (p_1,\ldots,p_n)^T$	$\Longrightarrow$	• **Density matrices:** *(states)* $\rho = \sum_i p_i \Pi_i,$
• Random variables: $x(\omega_i) = x_i \to (x_1,\ldots x_n)^T$	$\Longrightarrow$	• **Hermitian matrices:** $X = \sum_i x_i \Pi_i;$
• Probability and expectation: $\mathbb{P}(e) = \sum_i p_i e_i \to \langle \vec{\mathbb{P}}, \vec{e}\rangle$, $\mathbb{E}(x) = \sum_i p_i x_i \to \langle \vec{\mathbb{P}}, \vec{x}\rangle$	$\Longrightarrow$	• **Probability and expectation:** $\mathbb{P}_\rho(\Pi) \to \langle \rho, \Pi\rangle = \mathrm{trace}(\rho\Pi)$, $\mathbb{E}_\rho(X) \to \langle \rho, X\rangle = \mathrm{trace}(\rho X)$
• Conditioning: $\mathbb{P}(\cdot\|e) \to \dfrac{\vec{e}\cdot\vec{\mathbb{P}}}{\mathbb{P}(e)}$	$\Longrightarrow$	• **Conditioning:** $\rho_{\|\Pi} = \dfrac{\Pi\rho\Pi}{\mathrm{trace}(\rho\Pi)}$

Quantum vs. Classical?

- *Matrices do not commute:* **time-ordering of measurements matters:**

$$\begin{aligned}\mathbb{P}_\rho(\Pi_a(t_1),\Pi_b(t_2),\Pi_c(t_3)) &= \mathrm{trace}(\Pi_c(t_3)(\Pi_b(t_2)\Pi_a(t_1)\rho\Pi_a(t_1)\Pi_b(t_2)) \\ &\neq \mathbb{P}_\rho(\Pi_b(t_1),\Pi_a(t_2),\Pi_c(t_3))\end{aligned}$$

- Consider a measurement of an observable

$$X=\sum_i x_i\Pi_i; \qquad \sum_i \Pi_i = I; \qquad \Pi_j\Pi_i=\delta_{ij}\Pi_i;$$

Conditioning rule implies:

$$\rho_{|\Pi_j}=\frac{1}{\mathrm{trace}(\Pi_j\rho\Pi_j)}\Pi_j\rho\Pi_j \quad\Longrightarrow\quad \sum_j \mathbb{P}_\rho(\Pi_j)\rho_{|\Pi_j}\neq\rho$$

Averaging the conditional probabilities over the outcomes does not returns me the previous probability distribution!

Measurements change the state even if the outcome is ignored.

The Control Theorist's Recipe

(aka this talk's outline)

- **Main Ingredients:**

  1. **A control task and/or performance index; Possibly useful/interesting!**

  2. **A class of dynamical systems, with inputs or tunable parameters;**

  3. **Constraints on the admissible controls;**

  4. **A grain of salt and a bit of luck;**

- **Cooking Directions:**
  Put everything together, do some math with it, and shake well!

Quantum Essentials

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Closed System Evolution

Let *H(t)* be the Hamiltonian (Energy Obs.):

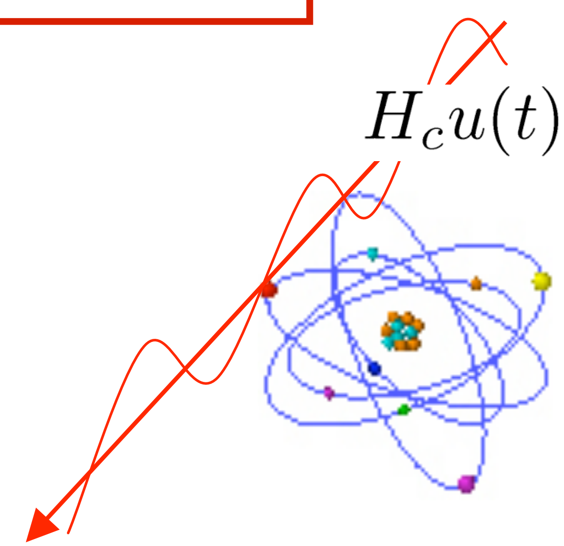
$$\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)] \text{ (Liouville Eqt.)};$$

$$\frac{d}{dt}Y(t) = i[H(t), Y(t)] \text{ (Heisenberg Eqt.)};$$

$$Y(t) = U^\dagger(t)Y(0)U(t), \quad \rho(t) = U(t)\rho(0)U^\dagger(t);$$

$$\frac{d}{dt}U(t) = -iH(t)U(t) \text{ (Schrödinger Eqt.)}.$$

$$\hbar = 1 \quad [X, Y] := XY - YX$$



Controlled Evolution: [semiclassical model]

Let *H(t)* be of the form: $H(t) = H_s + H_c u(t);$
With *u(t)* control function.

Coherent Control Issues

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Consider Problem 1. We have a $\boxed{\text{bilinear control system}}$ on *U(n)*

[Geometric Control Methods]

(e.g.) Reachability Analysis ⇨ Lie Algebraic Rank Condition

The system: $$\frac{d}{dt}U(t) = -i\left(H_0 + \sum_j H_j u_j(t)\right)U(t)$$

is controllable *iff* : $$\mathrm{Lie}(iH_0, iH_1, \ldots, iH_m) = u(n)$$

Key properties: Compactness of *U(n)* & Transitivity

[Jurdjevic-Quinn theorem; Altafini, Schirmer et al.]

(or) Optimal Control Problems:

Minimum energy control: [D'Alessandro et al.]

Time optimal control: [Khaneja et al.]

Some Existing Control Design Methods

- **Model-based feedback and Lyapunov-based design.**
  Design open-loop controls using a “fake” feedback loop based on simulations; Deeply studied, not robust. [Pavon;Altafini;Schirmer]

- **Iterative learning methods based on experiments.**
  Now also Machine Learning
  Starting from a guess, try to optimize the controls by running the experiment and adapting their action; Similar to optimal control [Rabitz]

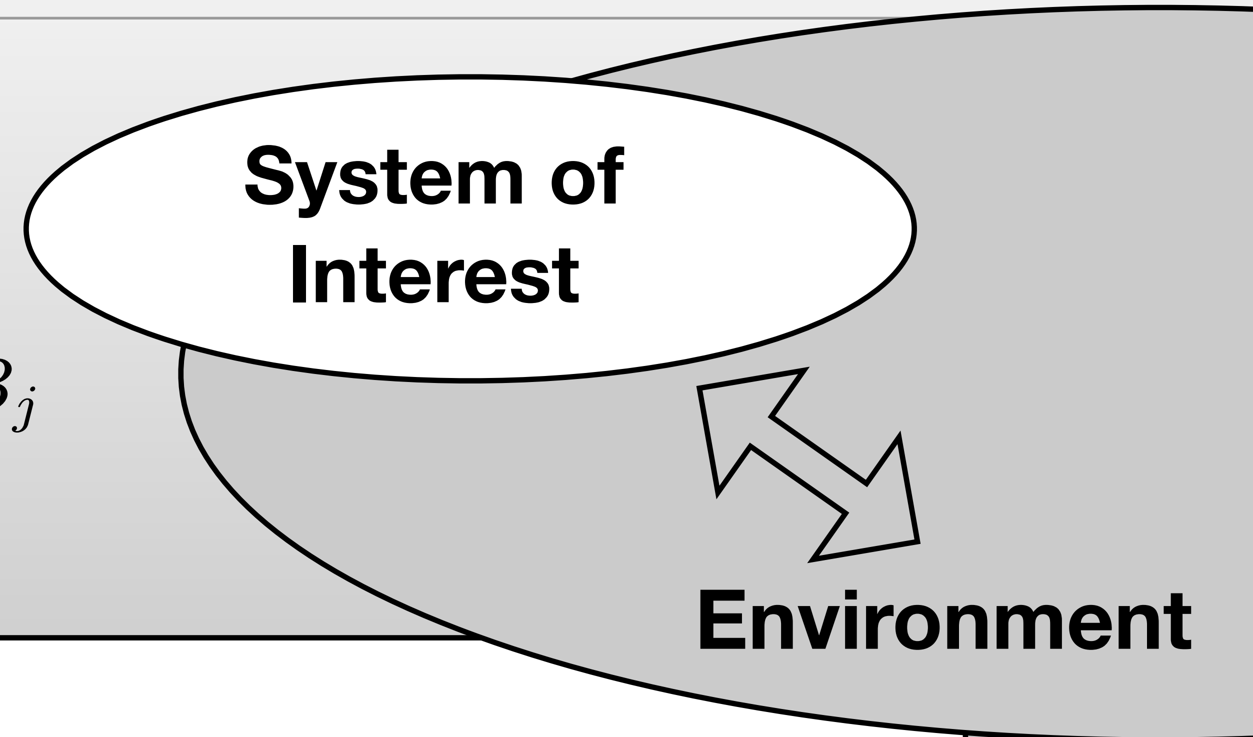
- **Adiabatic passage techniques (STIRAP and similar).**
  Steering the state (population) from one energy level to another using adiabatic eigenvalues crossing. Good for NMR. [Sugny, Boscain,…]

- **Optimal control for state transfer and gate realization.**
  Krotov method (sequential update), GRAPE (parallel update), CRAB (based on suitable rep. of control functions).

- **Pulse design (NMR style).**
  Fast/impulsive sequences of laser controls for state manipulation, and optimization of readout in resonance-type techniques.

Real Systems are not Isolated!

- Consider a bipartite system-environment:



$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

$$H_{tot} = H_S \otimes I_E + I_S \otimes H_E + \sum_j S_j \otimes B_j$$

- Isolated SE - **Schroedinger's Equation:**

$$\frac{d}{dt} U_t = -i H_{tot} U_t \Longrightarrow \boxed{\begin{array}{c} \frac{d}{dt}\rho_{SE}(t) = -i[H_{tot}, \rho_{SE}(t)] \\ \rho_{SE}(t) = U_t \rho_{SE}(t_0) U_t^\dagger \quad U_t = e^{-iH_{tot}t} \end{array}}$$

- **Problem:** (full) information on the environment may be not available: Under proper assumptions a **reduced model** for the system can be derived by computing expectations:

CPTP: Q-equivalent to Stochastic Maps

$$\boxed{\frac{d}{dt}\rho(t) = \int_{t_0}^{t} \mathcal{K}(t-u)\rho(u)\, du\ , \quad \rho(t_0) = \rho_0\ , \Longrightarrow \rho_t = \mathcal{E}_{t,t_0}(\rho_0)}$$

If the environment "memoryless": Markov dynamics…

The Role of the Environment

- **Two Prevailing & Complementary Approaches:**

I. **Environment as Enemy: we want to "remove" the coupling.**
Noise avoidance or suppression methods, including hardware engineering, *noiseless subsystem encoding, quantum error correction*, ***dynamical decoupling***;

II. **Environment as Resource: we want to "engineer" the coupling.**
Needed for state preparation, open-system simulation, and much more...

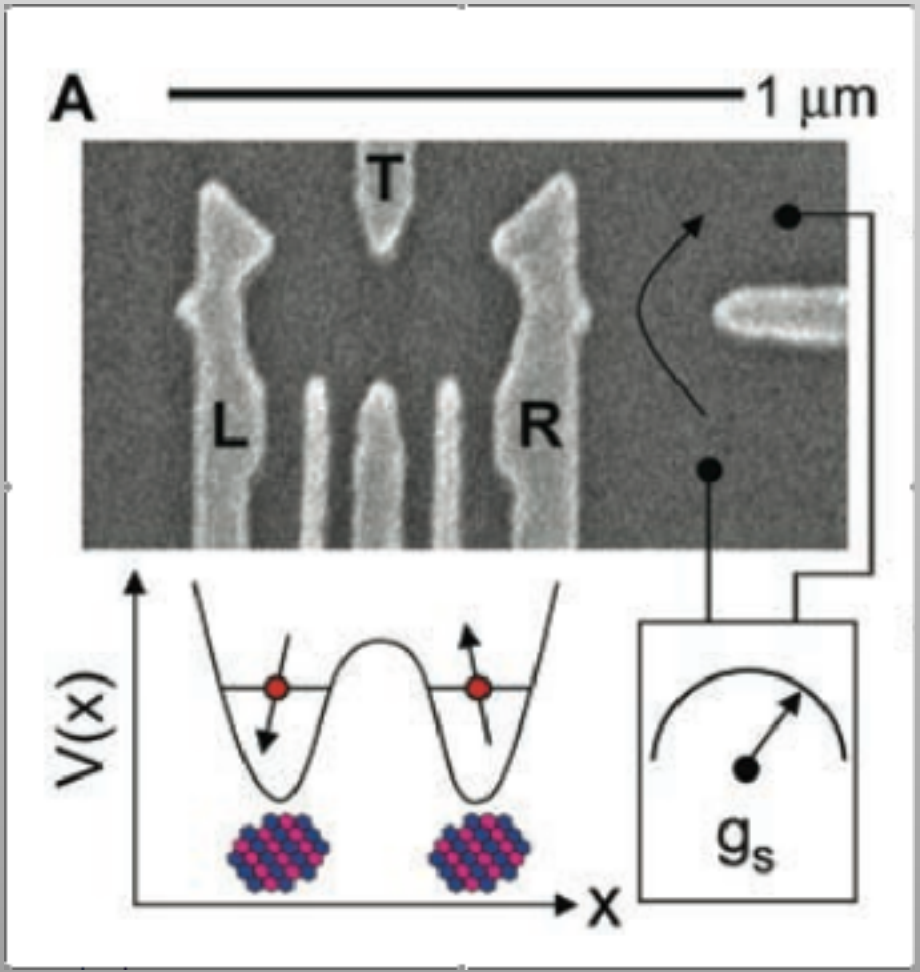
How long can QI survive in a real environment?

- Every real quantum system is immersed in a (quantum) environment:
- Loss of quantum superpositions: Only “classical” information survives.

Example: [Petta & al., Science 05]:

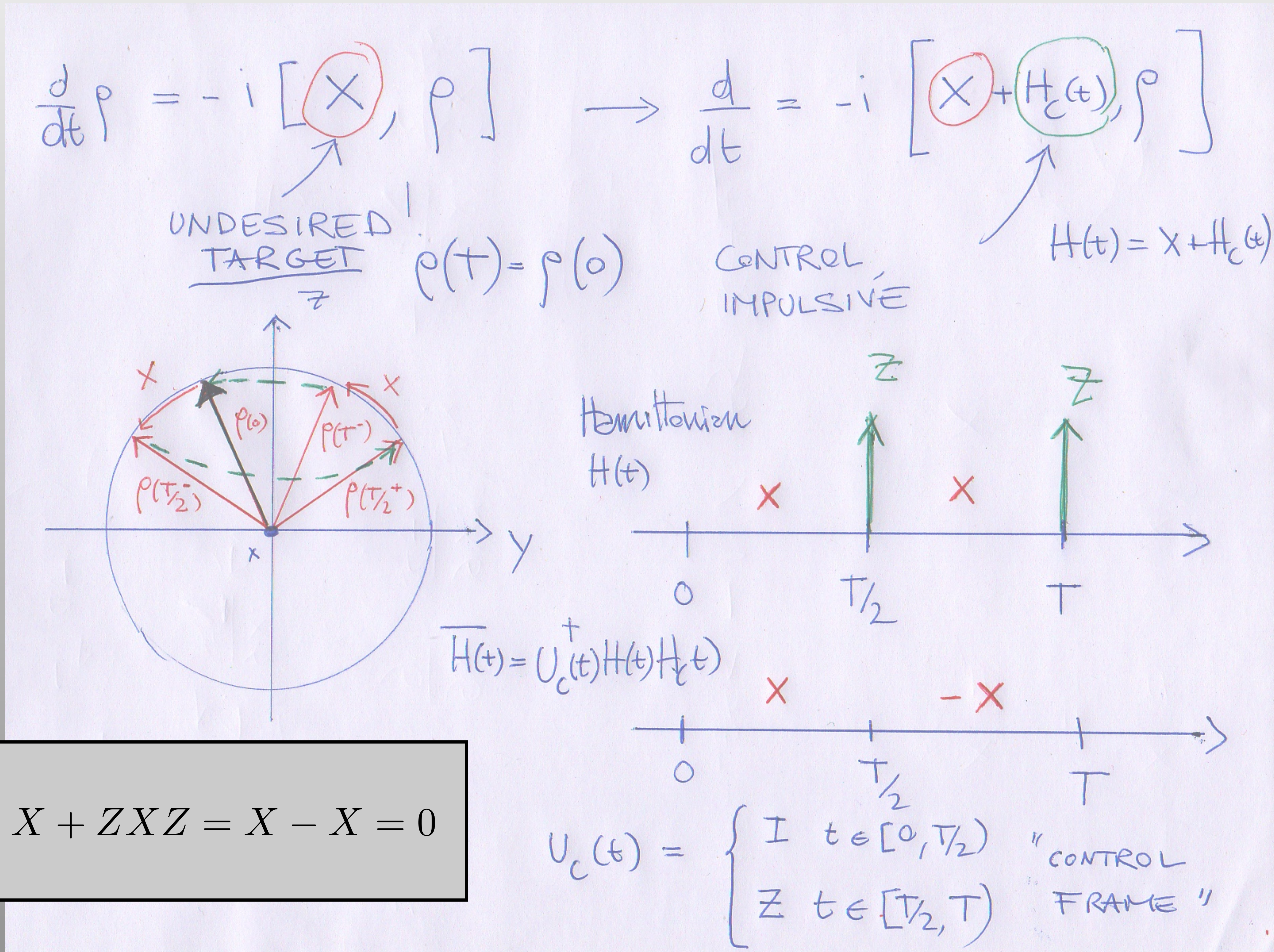
- GaAs quantum dots.
- Confined electrons interact with on the order of 10^6 spin-3/2 nuclei through the hyperfine interaction;
- Coherence Time-Scale: ~ 10 ns.

- With Decoherence Control: ~1μs



Basic Idea: Spin Re-Phasing with Pulses

- Consider a qubit, in an arbitrary (pure) state, depicted on the bloch ball.



$$\hat{H}(T) = X + ZXZ = X - X = 0$$

Decoherence Control: Decoupling the System

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$$H_0 \quad = \quad H_{\mathcal{S}} \otimes I_{\mathcal{B}} + I_{\mathcal{S}} \otimes H_{\mathcal{B}} + H_{\mathcal{SB}} = \sum_{i=1}^{p} S_i \otimes B_i;$$

- Problem 1 *"Decoupling":*
  suppress <u>any Hamiltonian coupling</u> between the system and the environment by controlling only the system;

- Problem 2 *"Selective Decoupling":*
  suppress <u>a known Hamiltonian coupling</u> between the system and the environment by controlling only the system;

It can be seen as a kind of *disturb rejection* problem, with strict constraints on the control choice.

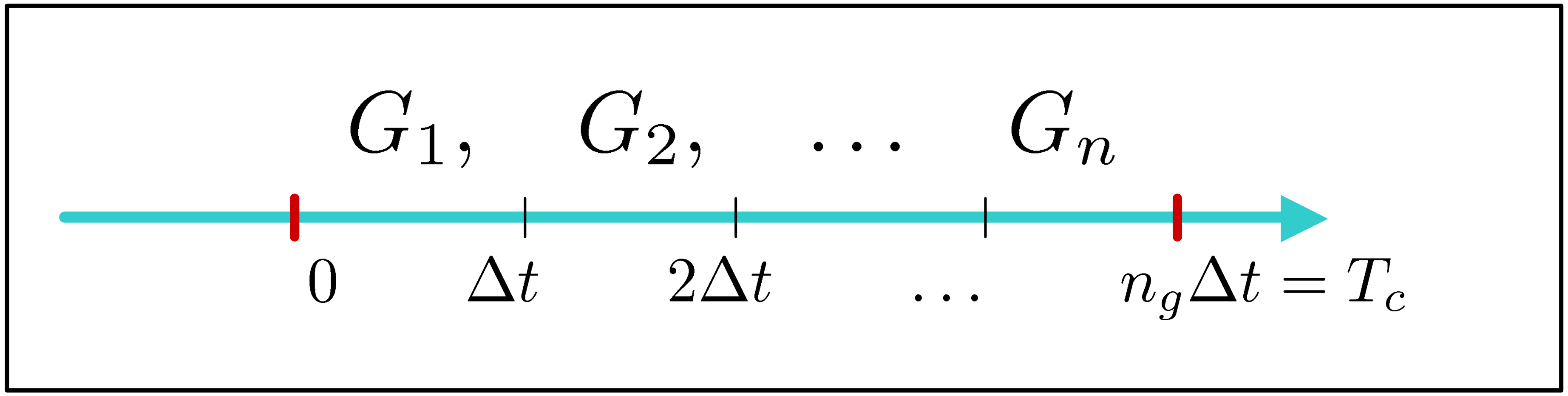
Dynamical Decoupling (I)

It is a Decoupling Control Strategy that:

- is Open Loop (no measurement is needed);
- is Cyclic (the propagator $U_c(t)$ is periodic);
- relies on a first order approximation.

Sketch: Divide each control period T_c in n_g time intervals;
Consider a piecewise constant control action:

$$U_c(t) \equiv G_j, \quad j\Delta t \leq t \leq (j+1)\Delta t, \quad j = 0, \dots, n_g - 1;$$



Dynamical Decoupling (II)

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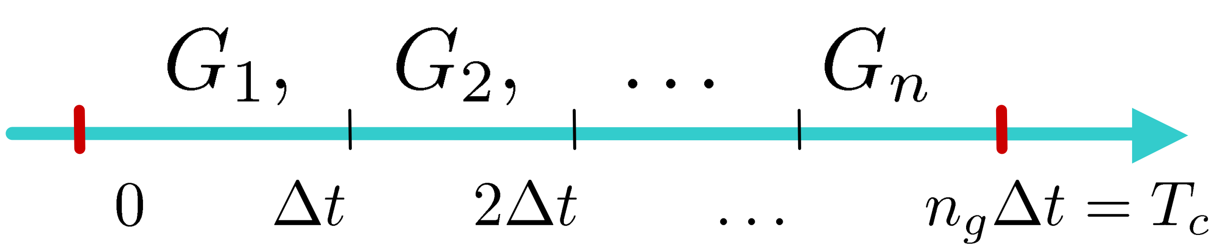
- If the $T_c \rightarrow 0$ we can consider only the average effect, and then evolution on the interaction term results in:



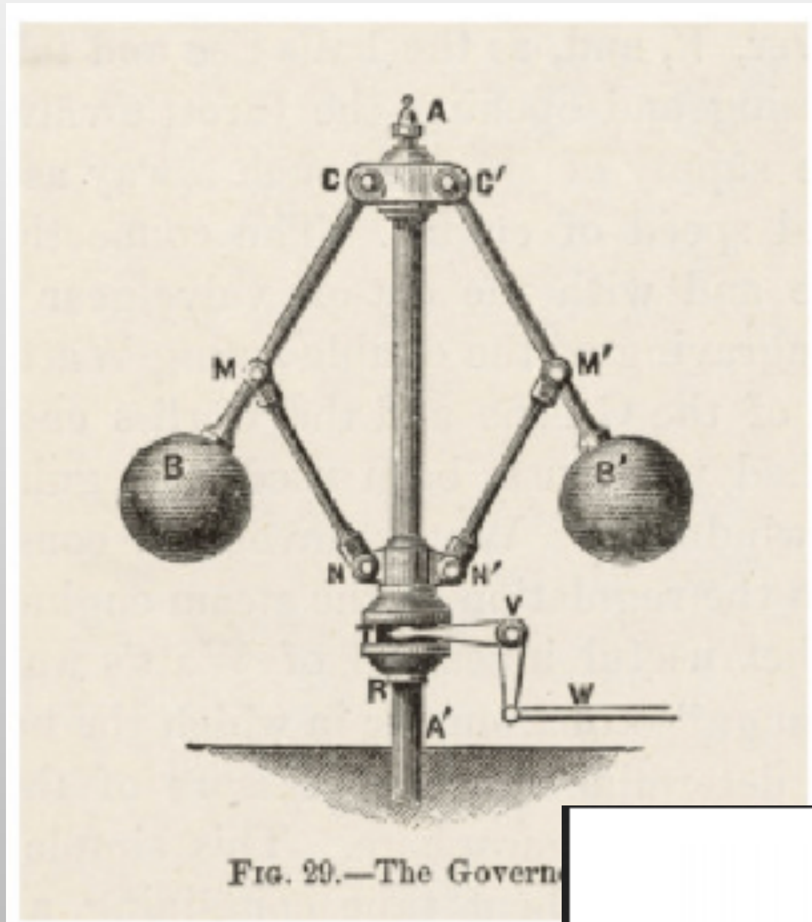
$$S_i(T_c) \approx \sum_{j=1}^{n_g} G_j^\dagger S_i(0) G_j;$$

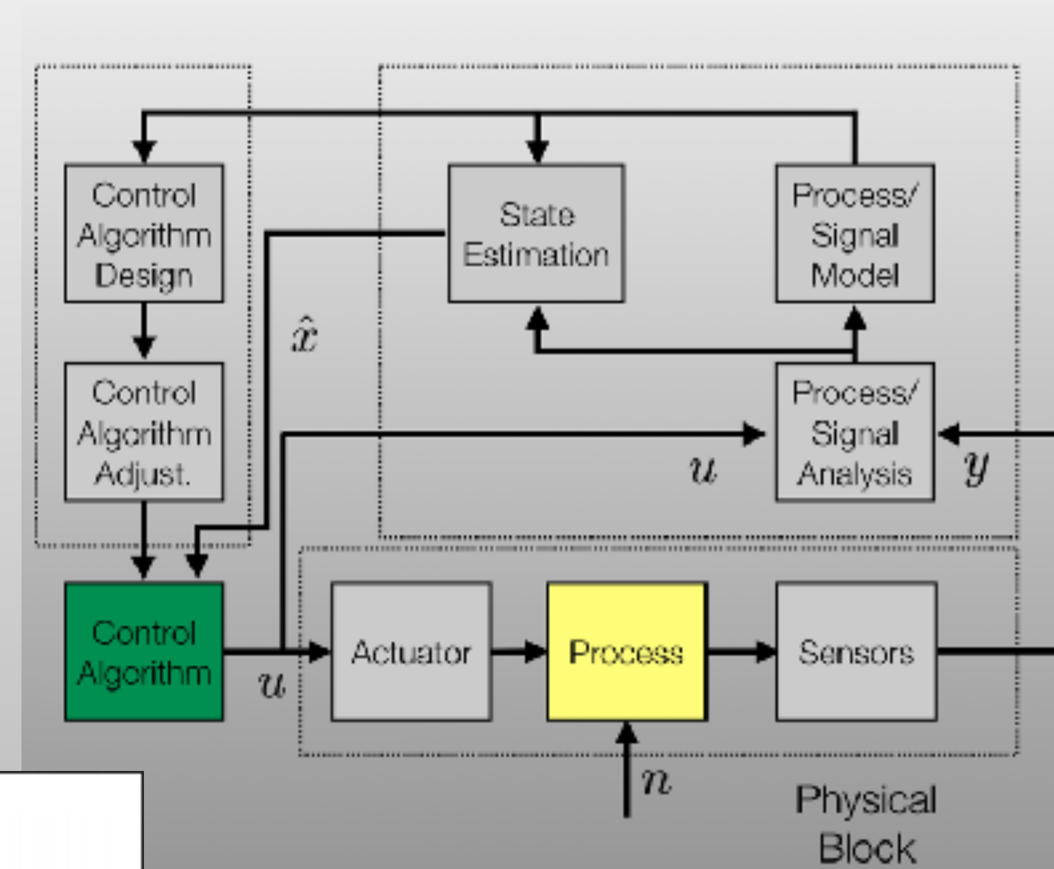
- [Problem 1] If we choose the G_i to be a group and the generators of the G_i to generate all the bounded Hermitian opts., then decoupling is attained: [L.Viola-S.Lloyd, '99].

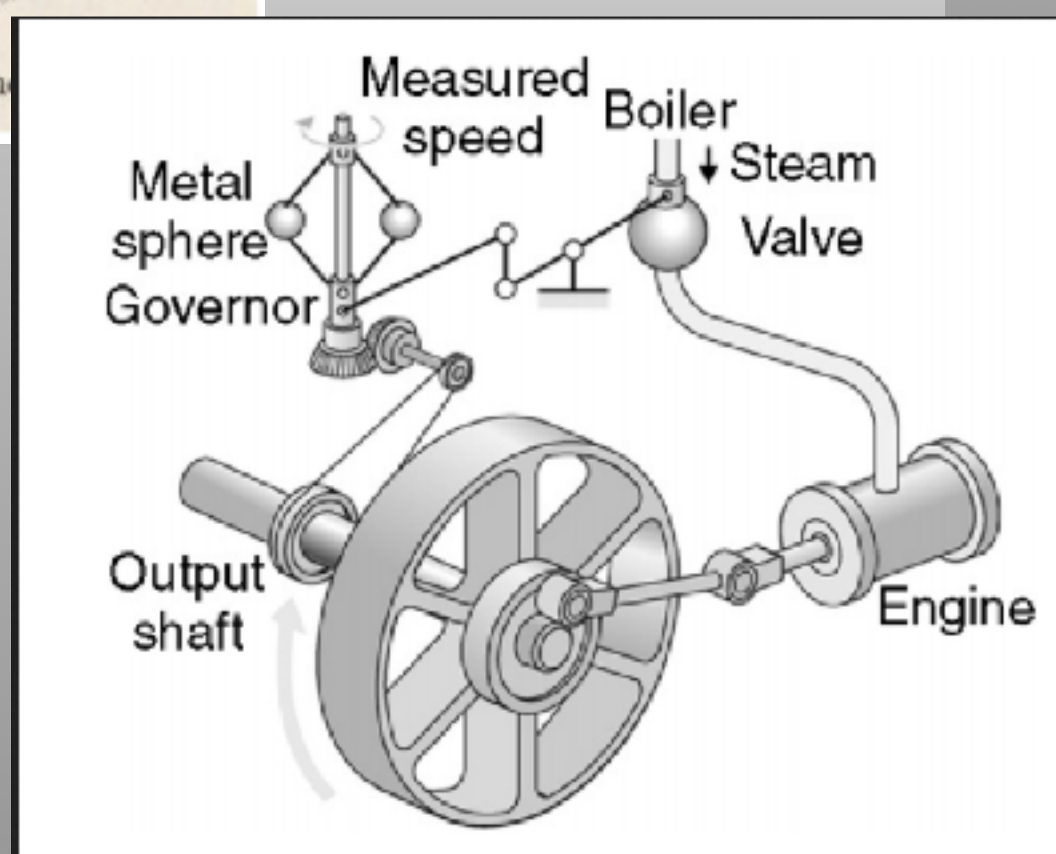
$$\sum_{j=1}^{n_g} G_j^\dagger S_i(0) G_j \approx \lambda I;$$

Many, sophisticated variations:
selective, high-order, nested, feedback,
continuous, randomized...

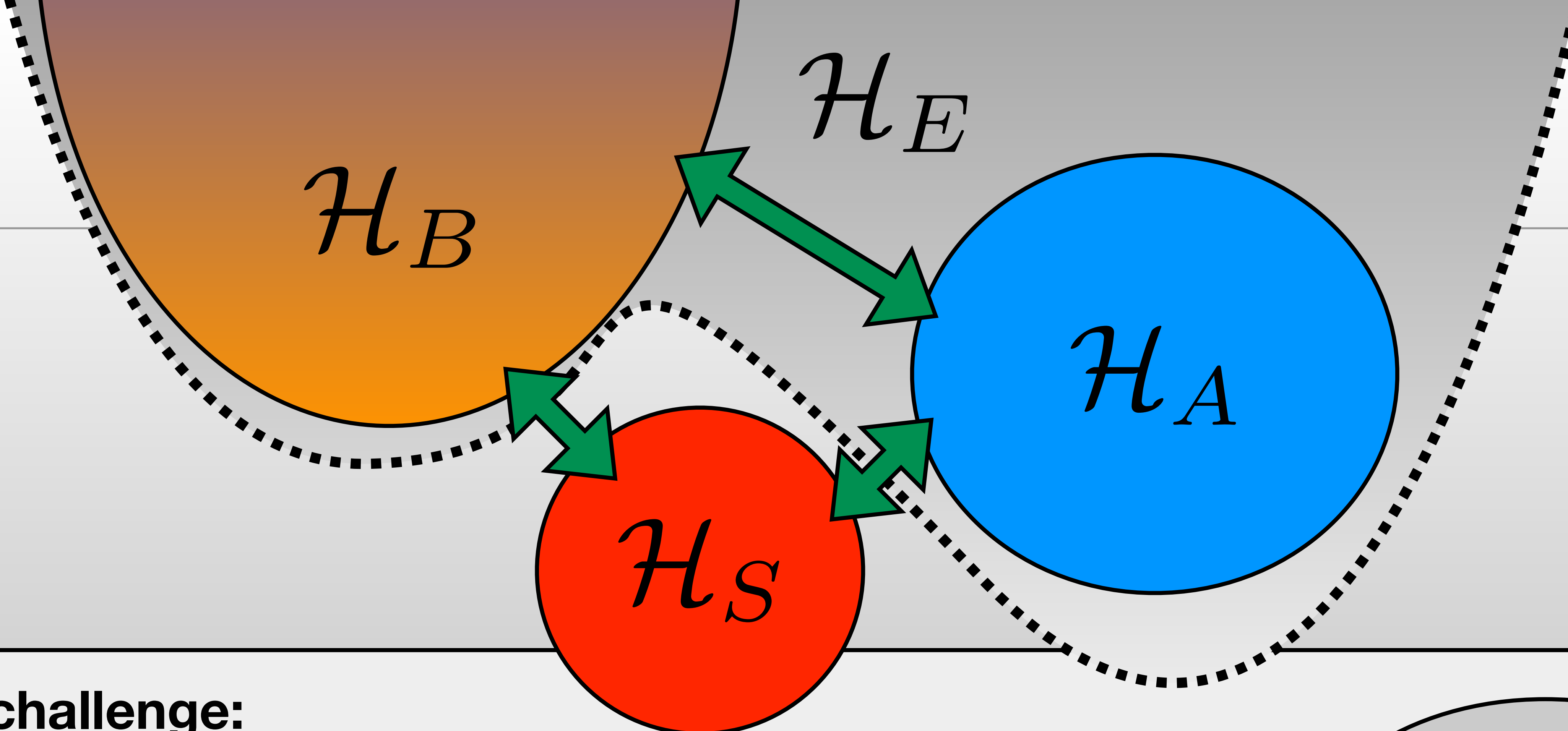
The Core of Control Theory: Feedback!







From Modern Control System,
9th ed., R.C. Dorfand R.H. Bishop,
Prentice-Hall, 2001.



New challenge:

Engineering of open quantum dynamics

Recent work:
What resources do we need for obtaining arbitrary desired dynamics

S: system of interest;

E: environment, including possibly:

- *B*: uncontrollable environment
- *A*: auxiliary, engineered system (quantum and/or classical controller)

Full description via Joint Hamiltonian:

$$H = (H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_{SE}) + H_c(t)$$

Reduced description with control:

$$\frac{d}{dt}\rho(t) = \int_{t_0}^{t} \mathcal{K}(t-u)\rho(u)\, du \ , \quad \rho(t_0) = \rho_0 \ , \Longrightarrow \rho_t = \mathcal{E}_{t,t_0}(\rho_0)$$

Open Dynamics for Information Engineering

- **Dissipation for QIP**
- **State stabilization**
- **Entanglement Generation**
- **Open System simulation**

nature physics LETTERS
PUBLISHED ONLINE: 20 JULY 2009 | DOI: 10.1038/NPHYS1342

Quantum computation and quantum-state engineering driven by dissipation

Frank Verstraete[1]*, Michael M. Wolf[2] and J. Ignacio Cirac[3]*

ARTICLE
doi:10.1038/nature09801

An open-system quantum simulator with trapped ions

Julio T. Barreiro[1]*, Markus Müller[2,3]*, Philipp Schindler[1], Daniel Nigg[1], Thomas Monz[1], Michael Chwalla[1,2], Markus Hennrich[1], Christian F. Roos[1,2], Peter Zoller[2,3] & Rainer Blatt[1,2]

Entanglement Generated by Dissipation and Steady State Entanglement of Two Macroscopic Objects

Hanna Krauter,[1] Christine A. Muschik,[2] Kasper Jensen,[1] Wojciech Wasilewski,[1,*] Jonas M. Petersen,[1] J. Ignacio Cirac,[2] and Eugene S. Polzik[1,†]

LETTER
doi:10.1038/nature12802

Autonomously stabilized entanglement between two superconducting quantum bits

S. Shankar[1], M. Hatridge[1], Z. Leghtas[1], K. M. Sliwa[1], A. Narla[1], U. Vool[1], S. M. Girvin[1], L. Frunzio[1], M. Mirrahimi[1,2] & M. H. Devoret[1]

LETTER
doi:10.1038/n

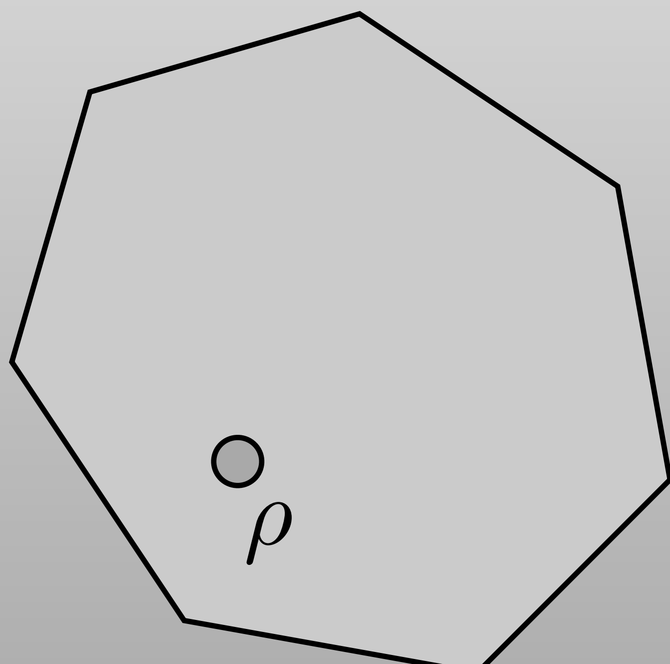
Deterministic entanglement of superconducting qubits by parity measurement and feedback

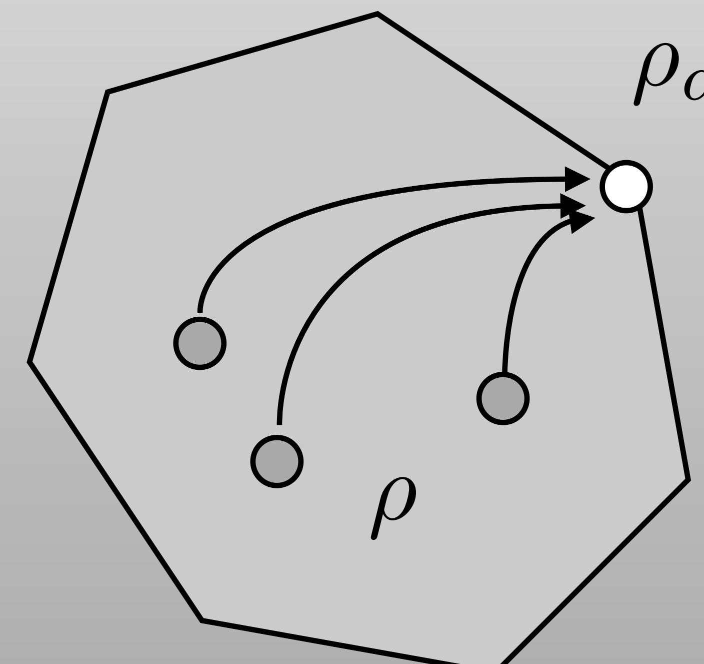
D. Ristè[1], M. Dukalski[1], C. A. Watson[1], G. de Lange[1], M. J. Tiggelman[1], Ya. M. Blanter[1], K. W. Lehnert[2], R. N. Schouten[1] & L. DiCarlo[1]

Stabilization for Quantum Systems

Consider a finite-dimensional quantum system;
General states (*density operators*) form a convex set,
extreme points are **pure states** (*rank-one orth. projections*):

$$\rho \in \mathfrak{D}(\mathcal{H}) := \{\rho = \rho^\dagger > 0,\ \mathrm{trace}(\rho) = 1\}$$





Stabilization Task:

Design dynamics that
1) prepare a given state from any initial condition, (asymptotycally or in finite time) and
2) leave it *invariant*.

Why Pure State Preparation?

- **The problem is key to (e.g.)**:

  I. **Quantum Information Processing:**

  a. Initialization of the “quantum register” for an algorithm (Di Vincenzo’s requirements);
     Initialization in a protected quantum code;

  b. Initialization of an entangled state for one-way quantum computation;

  c. Write in a quantum memory;

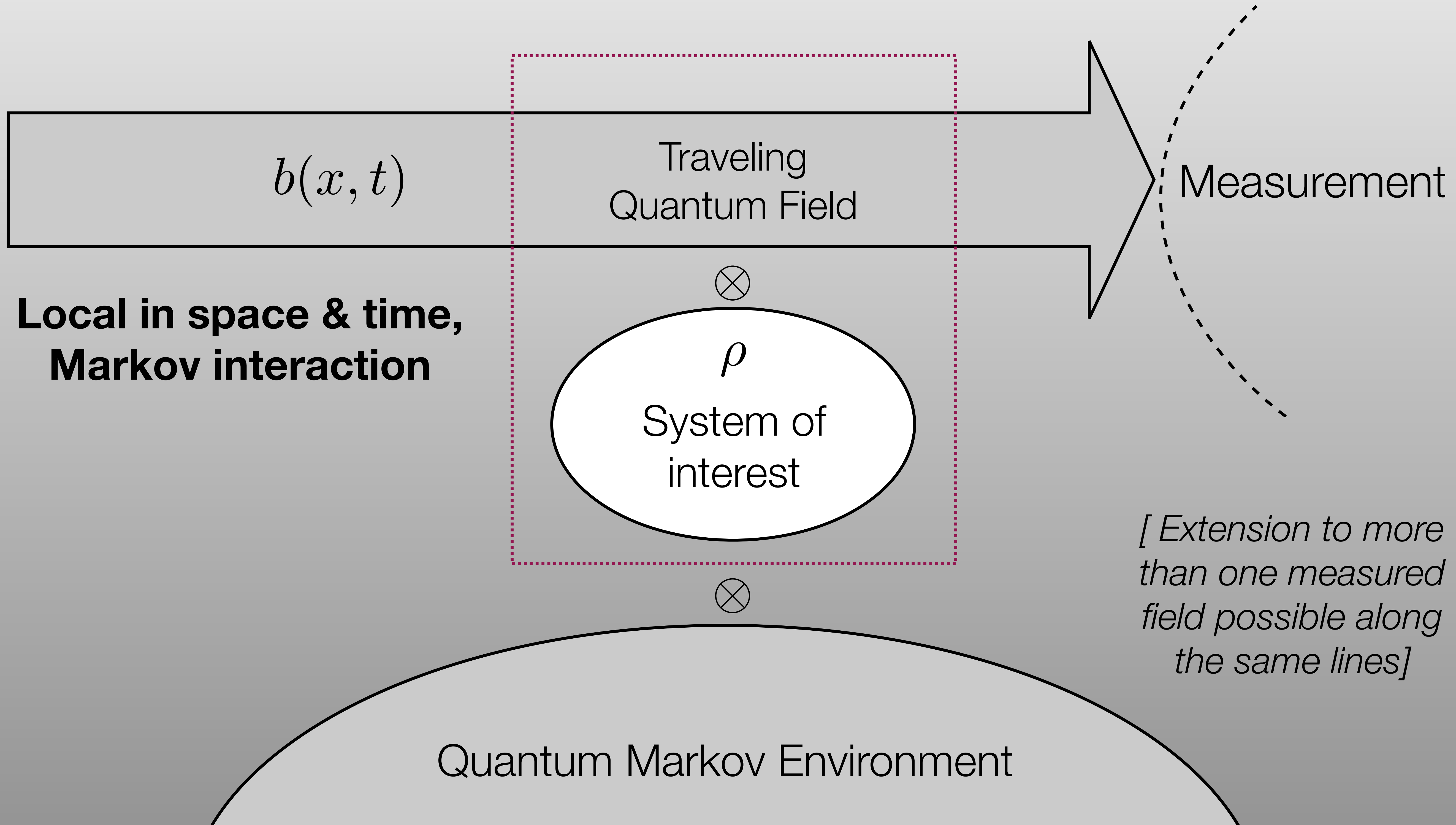
  d. Create steady entangled states*;*

  II. **Cooling to minimum energy eigenstate** *(optical, nano-mechanical, electro-mechanical);*

  III. **Initialization for other control algorithms** *(optimal, adiabatic).*

System, Continuous Measurement & Environment

In order to allow for continuous feedback,
we need continuous measurements. Consider:



Local in space & time,
Markov interaction

[Extension to more than one measured field possible along the same lines]

Quantum Markov Dynamics in Continuous Time

- **Dynamics with continuous homodyne-type measurements** (and estimation-based feedback): Belavkin's filtering equation, a.k.a. ***Stochastic Master Equation (SME)***

$$d\rho_t = \Big(-i[H,\rho_t] + \sum_{k=0}^{K}\mathcal{D}(L_k,\rho_t)\Big)dt + \mathcal{G}(L_0,\rho_t)dW_t,$$

$$\mathcal{D}(L,\rho) := L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L) \quad \mathcal{G}(L,\rho) := \sqrt{\eta}(L\rho + \rho L^\dagger - \mathrm{Tr}((L+L^\dagger)\rho)\rho)$$

Driven by Wiener process: (*innovation process*)

$$dW_t = dy_t - \sqrt{\eta}\frac{1}{2}\mathrm{Tr}(\rho_t(L_0 + L_0^\dagger))dt$$

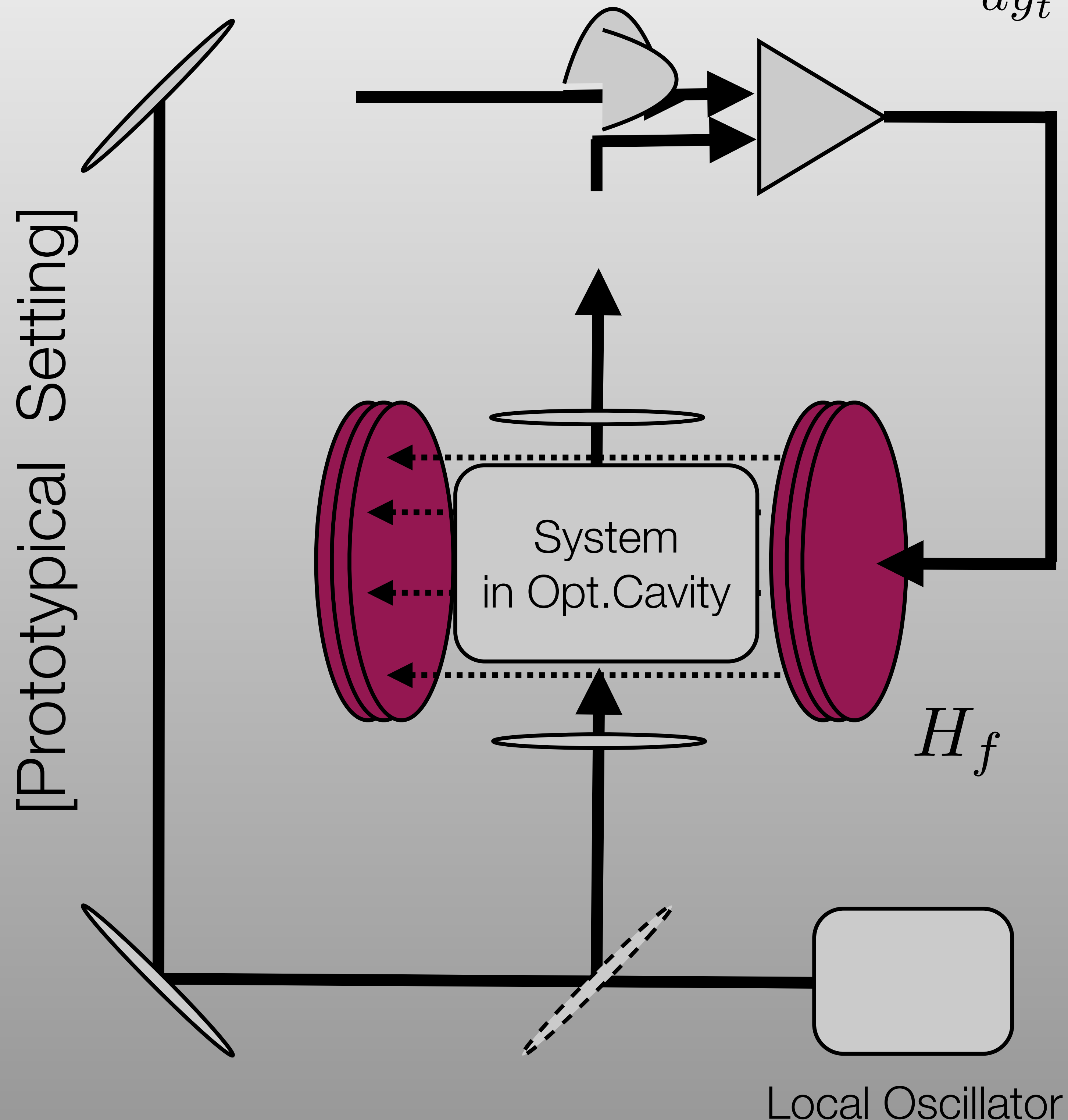
If time-independent, average over trajectories (including output feedback): Generator of the associated Markov semigroup or ***Master Equation (ME)***

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)) = -i[H,\rho(t)] + \sum_k \mathcal{D}(L_k,\rho(t)).$$

We proved this actually is enough for convergence in probability to pure states.

Output Feedback

$$dy_t = \sqrt{\eta}\frac{1}{2}\mathrm{Tr}(\rho_t(L_0 + L_0^\dagger))dt + dW_t,$$



Controlled Hamiltonian:

$$H_{tot} \approx H_0 + H_c + dy_t H_f$$

Allows to derive a
Feedback ME
(average over trajectory)
we characterized
which states can be stabilized

Feedback Master Equation

- Hamiltonian and Feedback Control [Wiseman-Milburn Feedback ME]:

$$\dot{\rho}_t = -i[H + H_c, \rho_t] + L_f \rho_t L_f^\dagger - \frac{1}{2}\{L_f^\dagger L_f, \rho_t\}$$

where $L_f = L_0 - iF$

- THEOREM: $\rho = \Pi_S$ can be rendered **invariant and attractive** via

feedback control **if and only if** :

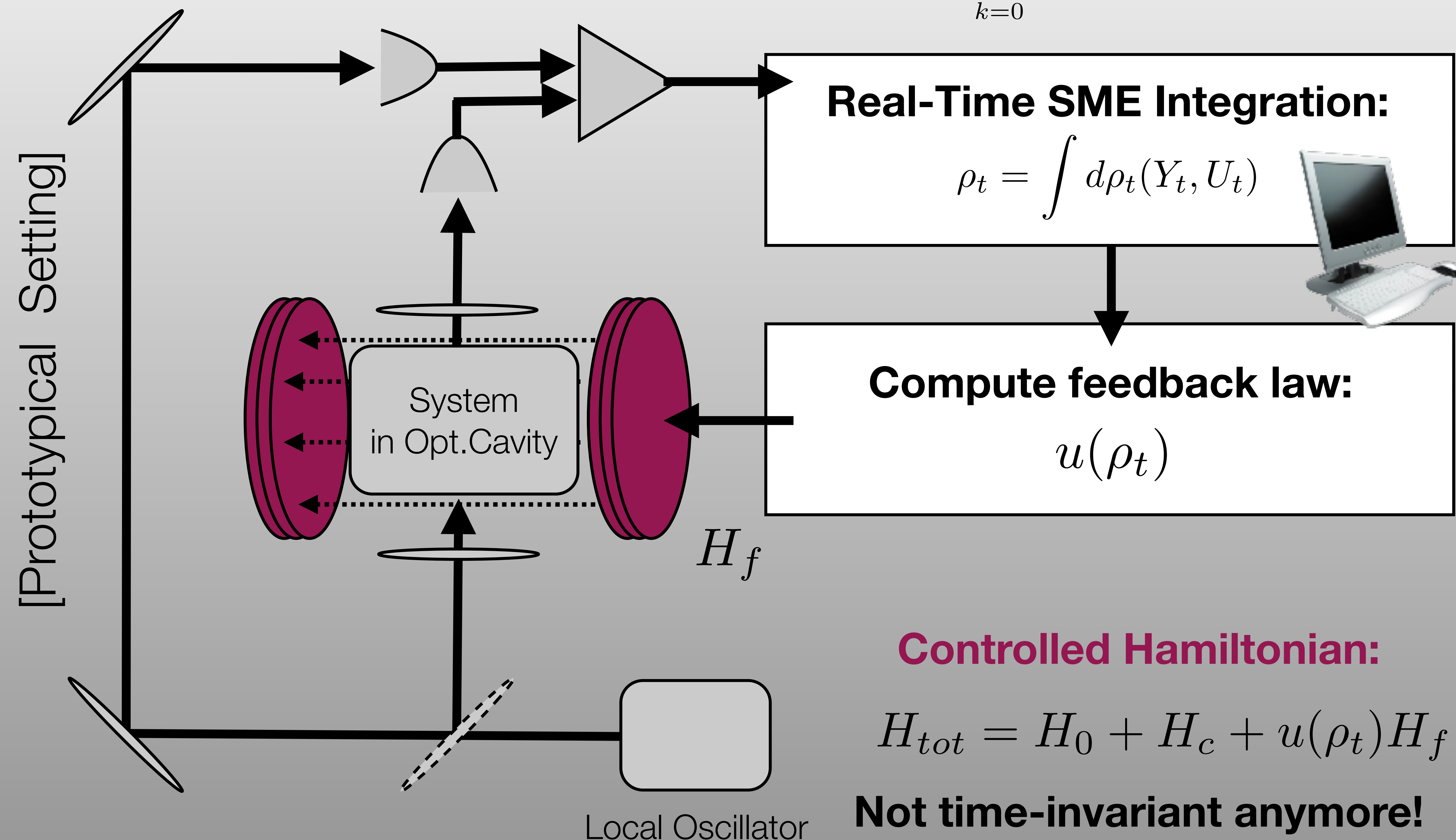
$$[\rho, (L_0 + L_0^\dagger)] = \rho(L_0 + L_0^\dagger) - (L_0 + L_0^\dagger)\rho \neq 0.$$

- **Good:** *Constructive proof*, can generate invariant states, some *freedom in the choice of the control Hamiltonian*.

  Bad: Instantaneous feedback assumption (infinite bandwidth), single noise channel (or compatible ones), perfect detection.

Estimation-Based Feedback

$$dy_t = \sqrt{\eta}\frac{1}{2}\mathrm{Tr}(\rho_t(L_0 + L_0^\dagger))dt + dW_t, \quad \Rightarrow \quad d\rho_t = \Big(-i[H,\rho_t] + \sum_{k=0}^{K}\mathcal{D}(L_k,\rho_t)\Big)dt + \mathcal{G}(L_0,\rho_t)dW_t,$$



Controlled Hamiltonian:

$$H_{tot} = H_0 + H_c + u(\rho_t)H_f$$

Not time-invariant anymore!

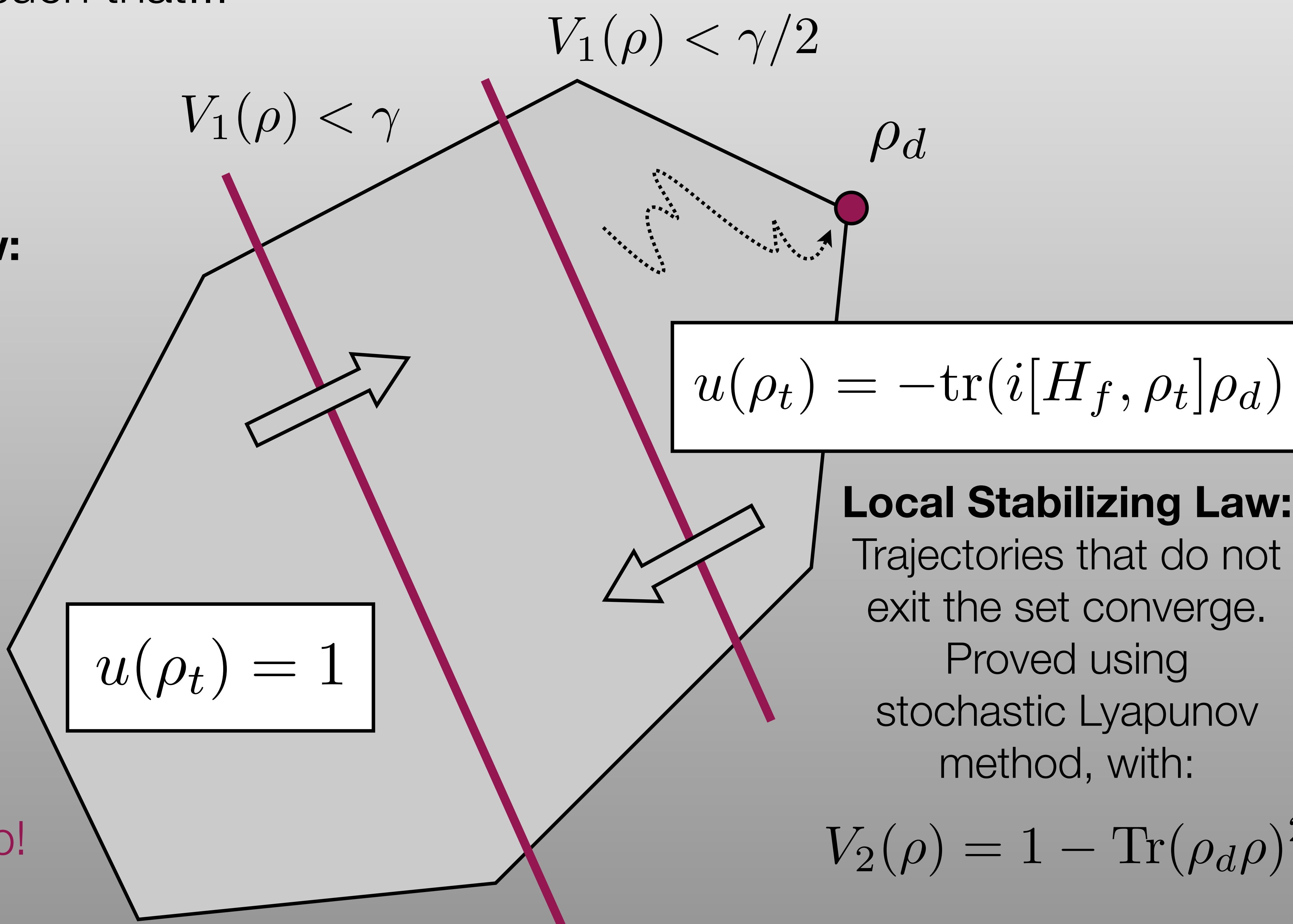
Switching Feedback Controller

Define: $V_1(\rho) = 1 - \mathrm{Tr}(\rho_d \rho)$

Exists $\gamma > 0$ such that...

Following [Mirrahimi-van Handel SIAM Cont. Opt. 2007]

De-Stabilizing Law:
Trajectories exit the set in finite time in expectation. Proved using Support Theorem.



Local Stabilizing Law:
Trajectories that do not exit the set converge. Proved using stochastic Lyapunov method, with:

$$V_2(\rho) = 1 - \mathrm{Tr}(\rho_d \rho)^2$$

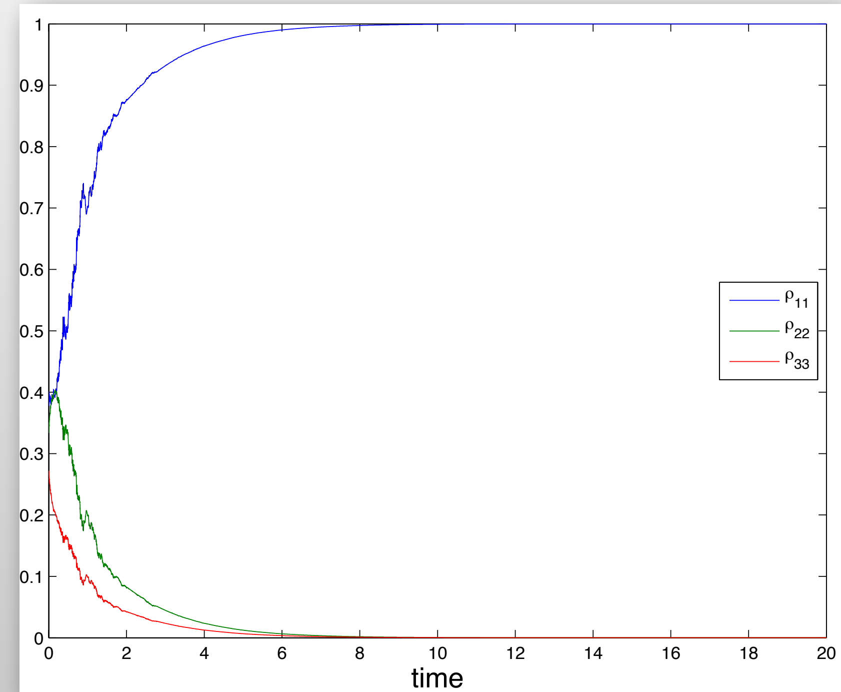
Open loop control does most of the job!

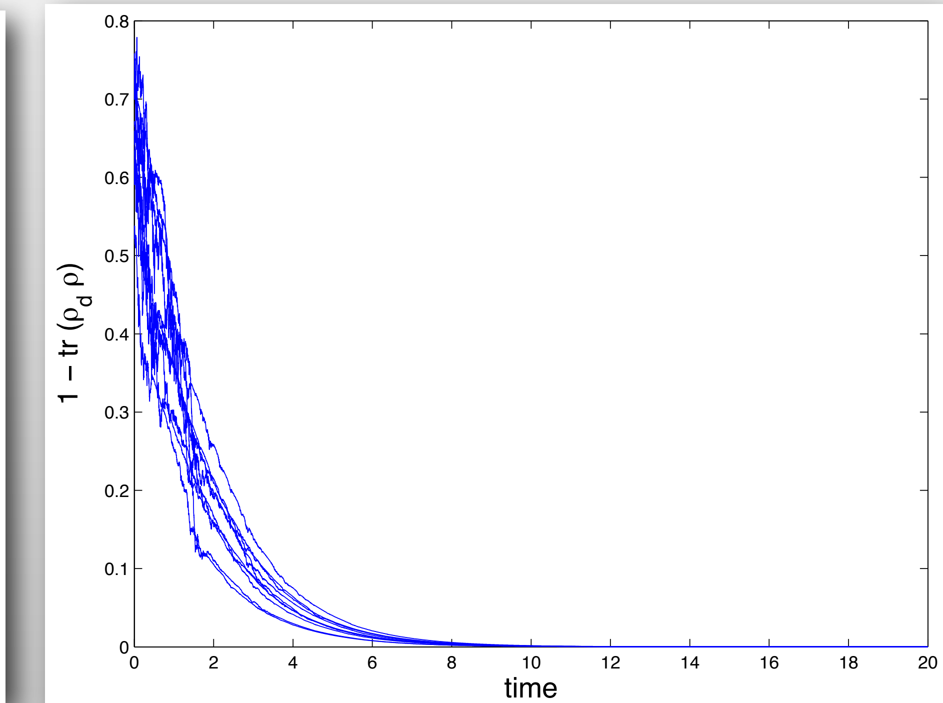
Numerical Comparison: 3-Level System

Environment-Assisted Stabilization (or output feedback)

$$L_{P,0} \neq 0$$

$$\rho = \left[\begin{array}{c|cc} \rho_{11} & * & * \\ \hline * & \rho_{22} & * \\ * & * & \rho_{33} \end{array}\right]$$

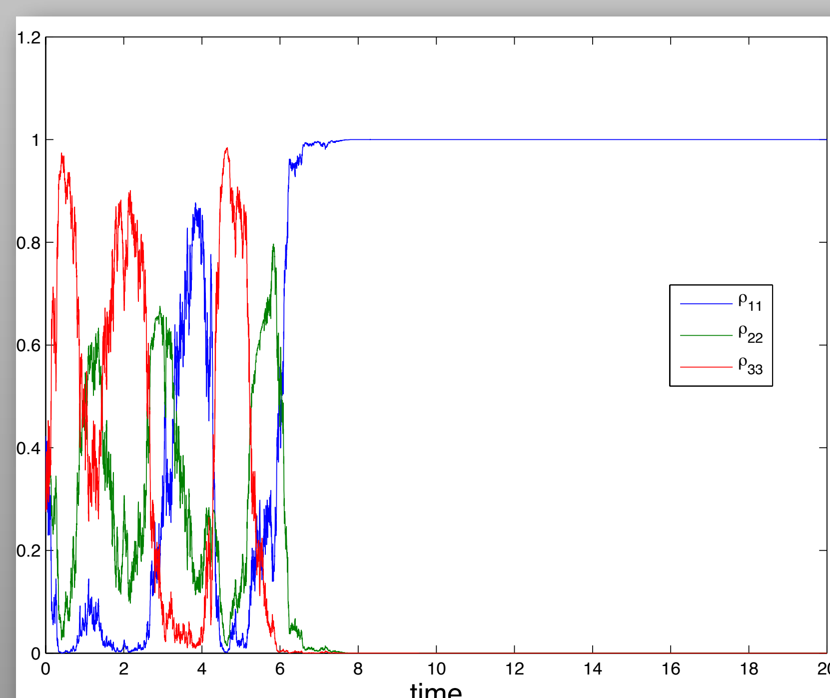


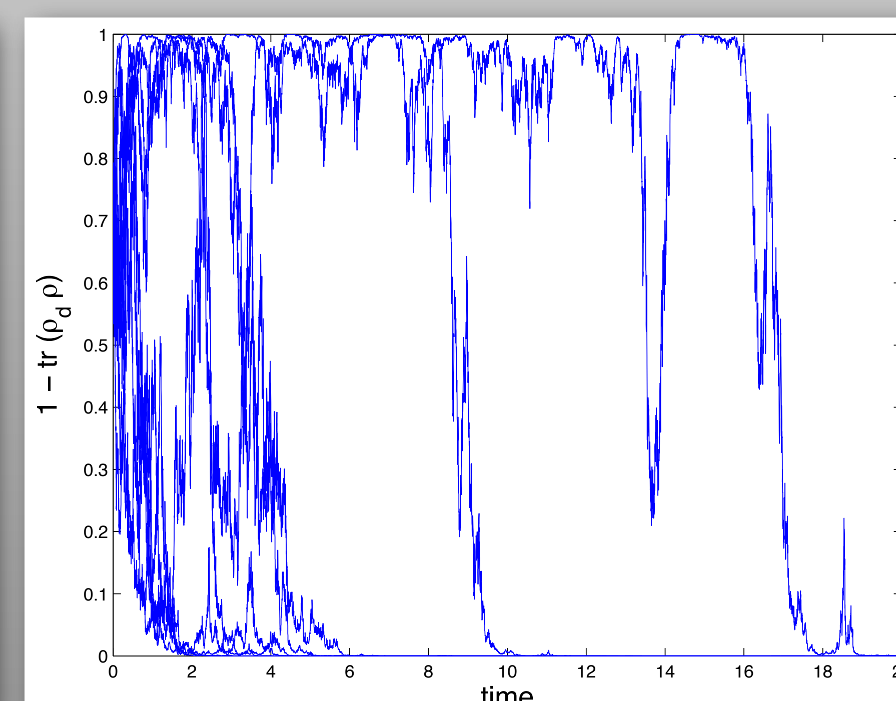


$$V_1(\rho) = 1 - \mathrm{Tr}(\rho_d \rho)$$

Estimation-based Feedback Stabilization

$$L_{P,0} = 0$$





Prototypical Entangled State

- **Bell State:** finite-dimensional version of Schroedinger's Cat + atom state
- **System -** two "qubits"

$$\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathrm{span}\{v_0, v_1\} \otimes \mathrm{span}\{v_0, v_1\}$$

- **State:**

$$\rho_{\mathrm{BELL}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(v_0 \otimes v_0 + v_1 \otimes v_1)(v_0^T \otimes v_0^T + v_1^T \otimes v_1^T)$$

- It is positive-semidefinite, unit trace, spectrum {1,0,0,0}.
  One-dimensional projector -> Pure state (~delta probability function).
- It cannot be factorized or separated:
  ENTANGLED!

$$\rho_{\mathrm{BELL}} \neq \sum_j \lambda_j \rho_{1,j} \otimes \rho_{2,j}$$

- **The state is in a superposition of (0,0) and (1,1).**
  Equiprobable outcomes, but it is not classical ignorance!
  Can be generalized to n subsystems -> GHZ states.

Switching Environments for Faster Convergence

- Target state:

$$\rho_{\mathrm{BELL}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(v_0 \otimes v_0 + v_1 \otimes v_1)(v_0^T \otimes v_0^T + v_1^T \otimes v_1^T)$$

Define: $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$ and two generators:

1) **Dissipation:** $M = \sigma_z \otimes I - i\sigma_y \otimes \sigma_x.$

$$\mathcal{L}_1(\rho) = M\rho M^\dagger - \frac{1}{2}(M^\dagger M\rho + \rho M^\dagger M)$$

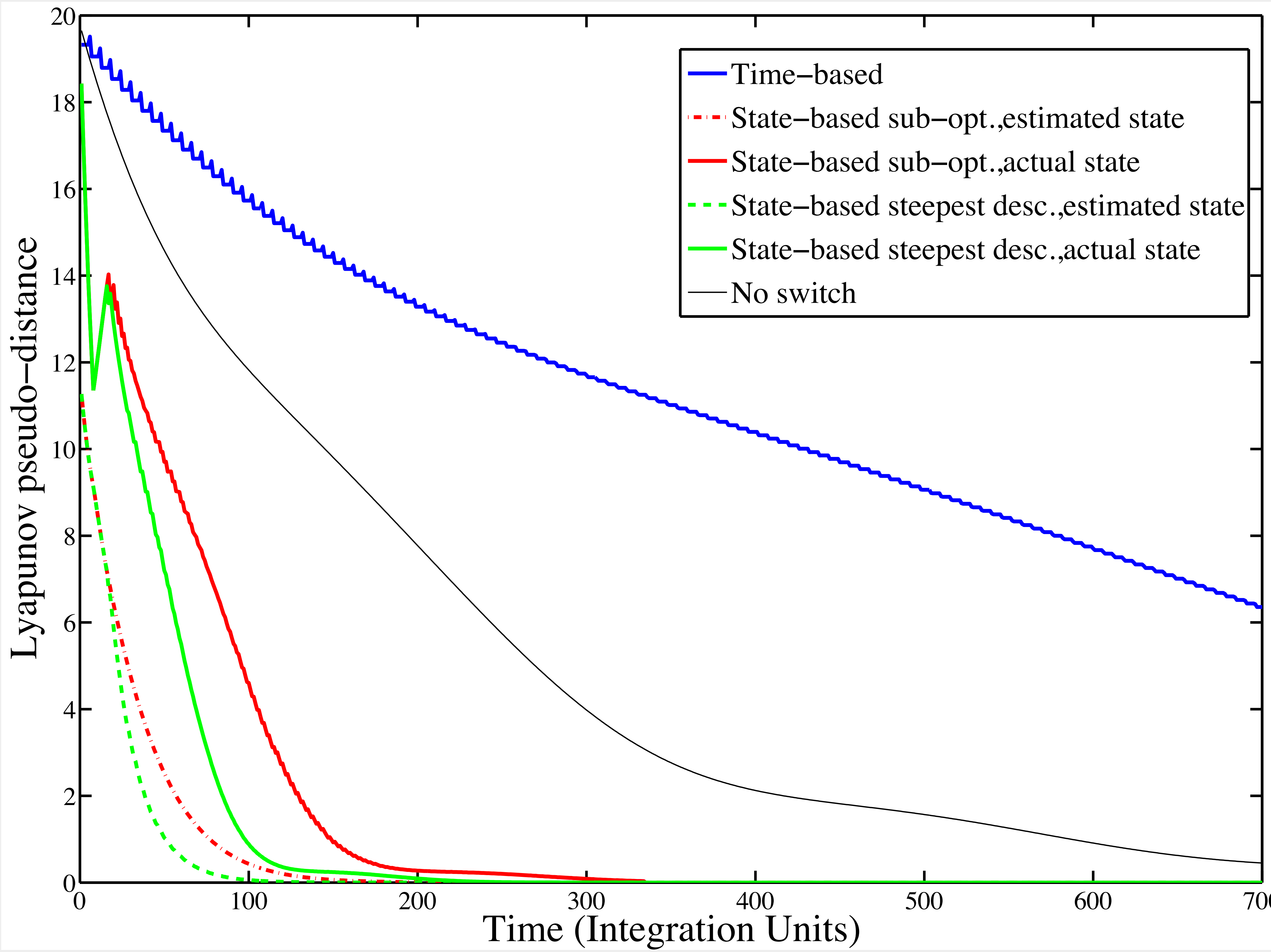
2) **Hamiltonian:** $H_c = \sigma_y \otimes I + I \otimes \sigma_y$

$$\mathcal{L}_2(\rho) = -i[H_c, \rho]$$

Can be obtained by feedback;
but alternating between the two (optimizing) is better than using them together!

- The target is an invariant state, but not the unique one, for both. It is uniquely invariant for any non-trivial convex combination of both.
This implies the associated ME stabilizes it [T-Viola, Automatica 2009].

Numerical Simulations



First Part - Summary

- **Dissipation is needed for state preparation;**
- **Quantum Markov dynamics are the natural models for this task;**
- **Switching generators provide faster convergence;**
  Feedback is however needed in certain situation.
- **Speed of convergence can be improved by switching different components of the generator *if the initial state is known*;**
- ***Rich theory by Gough-James on feedback networks, includes hybrid quantum-classical situation*;**

- **Open questions:**

★ **Can quantum controllers do better than classical ones for general quantum control tasks?**
We saw indication of this in speed/type of convergence;
results by Wang-Jacobs indicate it is true in other scenarios;

★ **Can quantum information processing be used to improve classical control???** G-J theory suggests we do at least as good.

Where is the Challenge?

- **Mathematical methods for Analysis (Controllability, Stability) and Design of dynamics for a single system well established;** Direct ties to:
  (i) Quantum Information Protection (codes as stable manifolds);
  (ii) Gate design (control problems for the propagator);
  (iii) Initialization (as state preparation);
- **Analytic tools for networks of interacting/multipartite systems less so!**

# Systems	Object	Topics
N = 2	Quantum controller for Quantum System	Q.controller feedback; State and Dynamical Controllability
N ~ 10-100	Small Quantum “Networks”	Q. simulation; synchronization, symmetrization; control networks; proof-of-principles; robustness
N –> infinity	Large Quantum “Networks”	Advantage of Q. computation; scalability of dissipative methods.

Can I use discrete-time dynamics
to stabilize states in **complex systems**?

State stabilization under locality constraints

[Zuccato,T,Johnson,Viola, TAC 2017;
Johnson-T-Viola, *Phys.Rev.A, 2017*]

Discrete-time counterpart of

[T-Viola, *Phil. Trans. R. Soc. A, 2012; T-Viola, QIC, 2013*]

Quantum Discrete-Time Markov Dynamics

- We consider sequences of quantum channels [CPTP maps]:

$$\{\mathcal{E}_t\}_{t\geq 0}$$

- At each time step:

$$\rho_{t+1} = \mathcal{E}_t(\rho_t) = \sum_{k=1}^{p} M_{t,k}\rho_t M_{t,k}^\dagger$$

$$\sum_{k=1}^{p} M_{t,k}^\dagger M_{t,k} = I, \quad \forall t \geq 0$$

- The dynamical flow is a two-parameter discrete semigroup:

$$\rho_t = \mathcal{E}_{t,0}(\rho_0) = \mathcal{E}_t \circ \ldots \circ \mathcal{E}_1(\rho_0)$$

- **Convergence features and invariant sets:**
  - **Well understood for time-homogenous dynamics (iterated maps):** Operator theory [Petz,Takesaki et al], Spectral methods [Wolf et al]; Linear-algebraic/Frobenius [T-Cirillo, JPA 2015 et al.]
  - **Few results time-inhomogeneous case.** [Reeb et al, JMP 2009]

Dynamical Engineering for a Single System

- **Naive/abstract answer: YES!**
  Simple stabilizing dynamics (in finite time!) always exist:
  - **General states state - projection map:**

$$\mathcal{E}(\rho) = \rho_{\text{target}}\, \text{trace}(\rho)$$

Classical analogue:

$$\mathcal{P} = \mathbb{P}_{\text{target}} \otimes \mathbf{1}^{\mathbf{T}}$$

  - **Pure state: "Simple" maps are enough for finite time stabilization**
    [Bolognani-T, IEEE T.A.C., 2010]

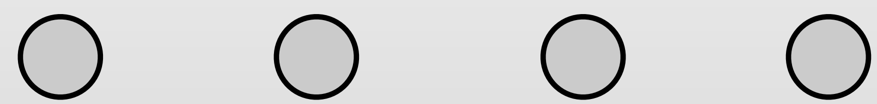
$$\mathcal{E}(\rho) = M_1 \rho M_1 + M_2 \rho M_2, \quad M_1 = \text{diag}(1, 0, \dots 0),\ M_2 = \text{upper diagonal}.$$

- **Can I obtain it with experimentally-available, constrained control capabilities? Typically not.**
- We need to take into account:
  - The control method [open-loop, switching, feedback, coherent feedback,...]
  - Limits on speed and strength of the control actions;
  - **Faulty controls, uncertain states,...**
  - **Locality constraints, scalability…**

key limitation for multipartite entanglement generation

Multipartite Systems and Locality Constraints

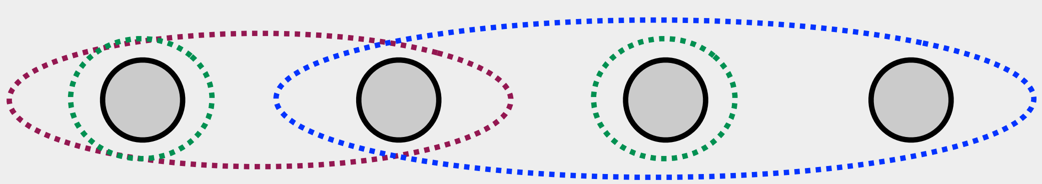
- Consider *n* finite-dimensional systems, indexed:



$a = 1 \quad 2 \quad 3 \quad \cdots$

$$\mathcal{H}_Q = \bigotimes_{a=1}^{n} \mathcal{H}_a$$

- **Locality notion:** from the start, we specify *subsets of indexes,* or ***neighborhoods,*** corresponding to group of subsystems:



$$\mathcal{N}_1 = \{1, 2\}$$
$$\mathcal{N}_2 = \{1, 3\}$$
$$\mathcal{N}_3 = \{2, 3, 4\}$$

*...*on which "**we can act simultaneously**": how?

- An *operator* is said Quasi-Local (QL) if $M_k = M_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$

Note: definitions depend on the neighborhoods!

This framework encompasses different notions: graph-induced locality, N-body locality,

Quasi-Local Stabilizable State

- Assume (for now) that we can engineer **any QL channel**;

- **Definition:** A state ρ is **Quasi-Local Stabilizable (QLS)** if there exist a **sequence of QL channels** $\{\mathcal{E}_t\}_{t\geq 0}$ such that:

I) $$\forall\, t \geq 0 \quad \mathcal{E}_t(\rho) = \rho$$ *[invariance]*

II) $$\forall\, \rho_0, \lim_{t\to\infty} \|\rho_t - \rho\| = 0$$ *[convergence]*

where: $$\rho_t = \mathcal{E}_t \circ \ldots \circ \mathcal{E}_1(\rho_0)$$

- *Ensures QL dynamics, and convergence irrespective of the initial state.*

- **Which states are QLS? Is there a way to test a target state?**

- **How fast and robust are the stabilizing dynamics?**

Other State Preparation Protocols

- **[Standard Preparation + Unitary Circuit]**
  Assume that we can engineer *an universal set of QL unitaries and initialize a factorized pure state;*
  We first initialize (*dissipatively*) and then rotate (*unitarily*):

$$\rho_0 \to |\psi_0\rangle\langle\psi_0|^{\otimes n} \to U_k \cdots U_1 (|\psi_0\rangle\langle\psi_0|)^{\otimes n} U_1^\dagger \cdots U_k^\dagger$$

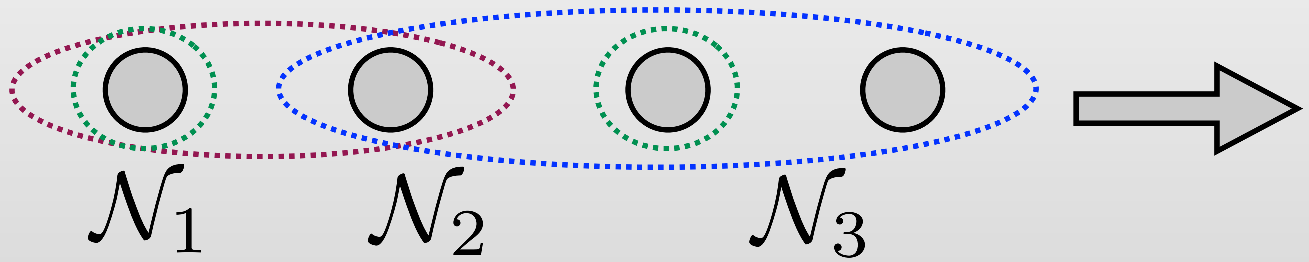
- **[Sequential Generation]** [Schon et al, PRL 2005]
  Given a *matrix product state representation*, we can associate CPTP maps to QL tensors and obtain the state dissipatively.

$$\rho_F = \mathcal{E}_t \circ \ldots \circ \mathcal{E}_1(\rho_0)$$

Both these methods:
1) **Do not stabilize,** namely do not leave the target invariant;
2) If errors occur, the whole procedure must **re-run to correct** them;
3) **The order of the operations is given** and in general cannot change.

QLS Stabilization of Pure States

- For each *neighborhood* compute the reduced states (marginals);



$$\rho_{\mathcal{N}_1},\ \rho_{\mathcal{N}_2},\ \rho_{\mathcal{N}_3}$$

- For each neighborhood calculate the *support* of the reduced state times the identity on the rest:

$$\mathcal{H}_{\mathcal{N}_k} = \mathrm{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k})$$

- ***Theorem [ZTJV-TAC, 2017]:***

$$\mathcal{H}_\rho := \bigcap_k \mathcal{H}_{\mathcal{N}_k} = \mathrm{supp}(\rho)$$

if and only if a pure ρ is QLS;

Sufficiency: The support is “where the probability is”: I try to prepare it locally. Lyapunov method (similar to *alternating projection algorithms*).
Necessity: It can be shown that everything in the support must be invariant. With more than one state in the intersection, no convergence.

QLS, Or Not? Physical Interpretation

- **Equivalent characterization:** $\rho = |\psi\rangle\langle\psi|$ is QLS **if and only if it is the unique ground state of a Frustration-Free QL Hamiltonian**, that is:
  - There exists a QL Hamiltonian for which $|\psi\rangle$ is the unique ground state and

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

such that $\langle\psi|H_k|\psi\rangle = \min \sigma(H_k), \quad \forall k.$

Proof: It suffices to choose $H_k = \Pi^{\perp}_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$, $\Pi^{\perp}_{\mathcal{N}_k}$ projects on $\mathrm{supp}(\rho_{\mathcal{N}_k})^{\perp}$.

- **We retrieve the FF Hamiltonian - exactly as in continuous time!**
- **Similar for full-rank states, with FF Lindbladians instead of FF Hamiltonians.**
- Interesting connection to physically-relevant cases, and previous work by Verstraete, Perez-Garcia, Cirac, Wolf, B. Kraus, Zoller and co-workers.
- **Differences:** In their setting, the proper locality notion is induced by the target state. In our setting, *the locality is fixed a priori. We also prove necessity of the condition.*

Is Dissipation Enough?

- **Which states are DQLS? Using our test, it turns out that...**
  - *GHZ states (maximally entangled) and W states are never QLS*
    (unless we have neighborhoods that cover the network - star/complete graphs);
  - *Any* ***graph state*** *is QLS* with respect to **the locality induced by the graph;**
    To each node is assigned a neighborhood, which contain all the nodes connected by edges.
  - *Generic (injective) MPS/PEPS are QLS* for **some locality definition**...
    neighborhood size may be big! `[see work by Peres-Garcia, Wolf, Cirac and co-workers]`
  - Some **Dicke states** *that are not graph* can be stabilized! E.g. on linear graph:

$$\frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |0110\rangle + |0101\rangle + |0011\rangle + |1001\rangle)$$

Good, but not great. Important states are left out. Can we do better?
YES, with a two-step protocol, and conditional stabilization.

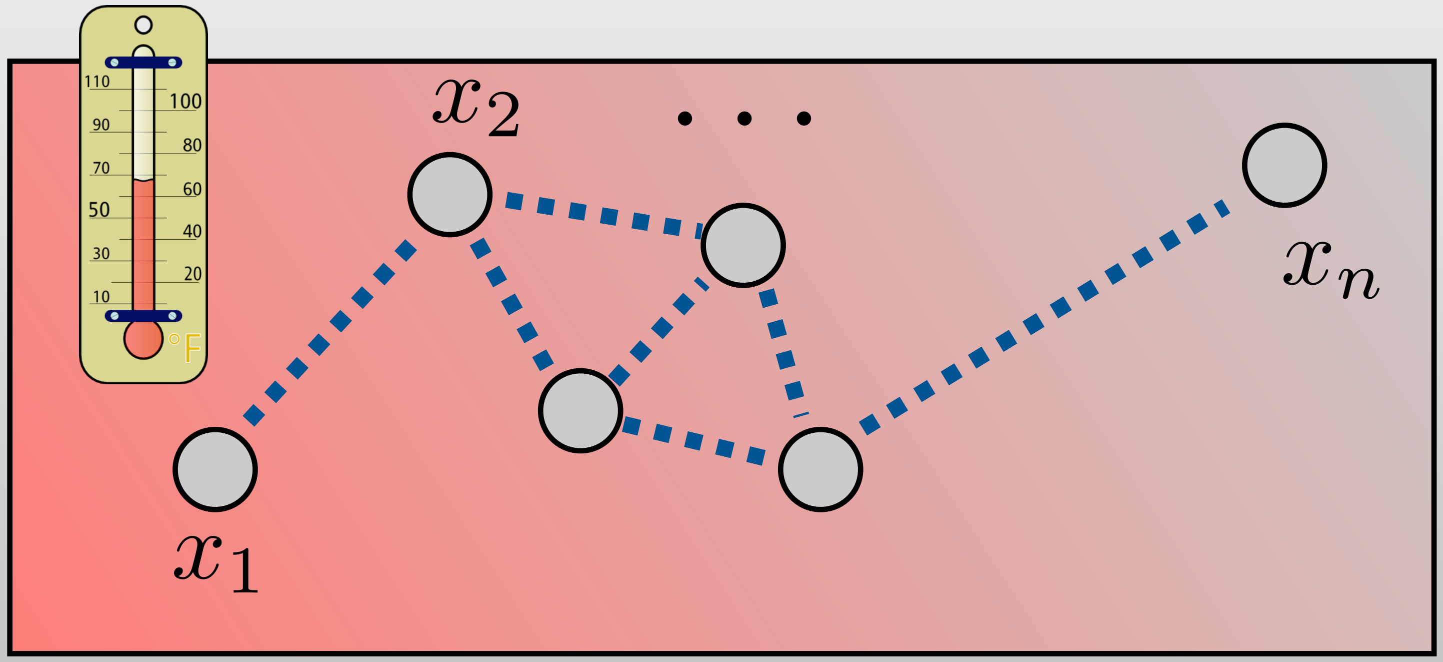
Can I use QL dynamics
to globally symmetrize the network?

Quantum Consensus & Group Symmetrization

[Mazzarella, Sarlette, T, IEEE TAC 15 & SICON 15 & MTNS 2014]

Inspiring Classical Application

- **Consider n sensors (agents) probing the temperature of a room:**



- Each agent records a (different) value, x_j, due to errors and temp. gradient;
- **Then it broadcasts the value on a *limited* communication network;**
- **The task:** have the sensor **agree** on the initial mean value (*average consensus*);

$$\forall, j, k \;\; x_j(\infty) = x_k(\infty) = \bar{x}(0) := \sum_j x_j(0)$$

How to (asymptotically) reach it?

- Can the protocol be made robust w.r.t. size, sequence, initial values....?

Rewriting the Gossip Algorithm

- **A simple Gossip algorithm:**
  - At each t, one communication link (graph) edge is selected, say **(j,k)**;
  - The two agents share their values and compute the (weighted) mean:

$$
\begin{aligned}
x_j(t+1) &= x_j(t) + \alpha(t)(x_k(t) - x_j(t)) \\
x_k(t+1) &= x_k(t) + \alpha(t)(x_j(t) - x_k(t))
\end{aligned}
$$

  - Under **weak assumptions** on how they are selected, it converges.

- Let us rewrite the interaction as **the convex sum of two permutations:**

$$(x_j(t+1),\ x_k(t+1)) = (1\text{-}\alpha(t))\ (x_j(t),\ x_k(t))\ +\ \alpha(t)\ (x_k(t),\ x_j(t))$$

or, including all readings in one vector: $x(t) = (x_1(t), \ldots, x_n(t))$

$$x(t+1) = ((1-\alpha(t))I + \alpha(t)P_{jk})\, x(t)$$

Distributed,
Robust,
Unsupervised,
Scalable

Classical Consensus as Symmetrization

- Thus, the evolution up to time t can be always written as

$$x(t) = \left(\sum_{\pi \in \mathfrak{P}} p_\pi(t) P_\pi \right) x(0)$$

$p_\pi(t)$ **:** non-negative weights, sum up to one.

π **:** runs over the finite group of permutation of n elements;

- **Consensus is obtained if the state is permutation invariant, and the average is preserved;**

- In order for this to happen asymptotically, we need:

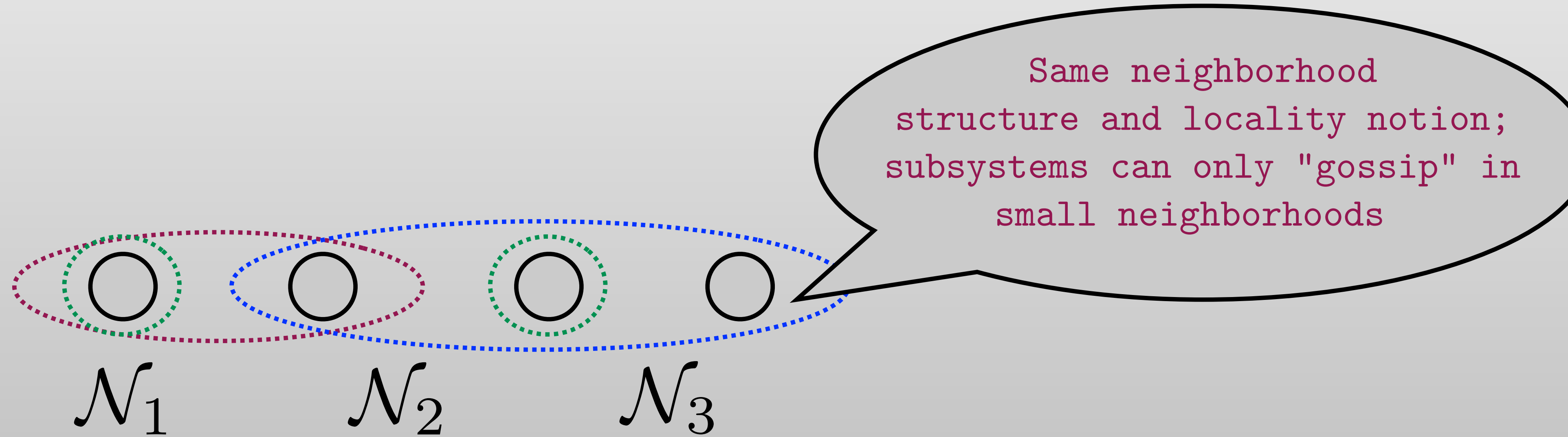
$$x(\infty) = \frac{1}{n!} \sum_{\pi \in \mathfrak{P}} P_\pi x(0)$$

`Projection on the group-symmetric configuration!`

- **Under very weak assumptions on the selection rule, it converges!**

Quantum Consensus

- We can then use the same ideas to **symmetrize a quantum network.**
- Quantum Network of *n* subsystems with dynamics subject to QL constraints.



- Consider subsystem permutation operators defined on a factorized basis (and extended by linearity) by:

$$U_\pi^\dagger (X_1 \otimes \ldots \otimes X_m) U_\pi = X_{\pi(1)} \otimes \ldots \otimes X_{\pi(m)}$$

- We want to generate a **uniform convex sum of subsystem permutations. Now (unital) CPTP map of the form:**

$$\bar{\mathcal{E}}(\rho) = \frac{1}{n!} \sum_{\pi \in \mathfrak{P}} U_\pi^\dagger \rho U_\pi$$

[projection onto symmetric states, centralizer]

Quantum Gossip Algorithm

- **Consider maps of the form:**

$$\mathcal{E}_t(\rho) = \alpha(t)\rho + (1-\alpha(t))U_{j,k}(t)\rho U^{\dagger}_{j,k}(t),$$

where $\{U_{j,k}\}$ are QL "swap" operators, and $\alpha(t) \in [0,1]$.

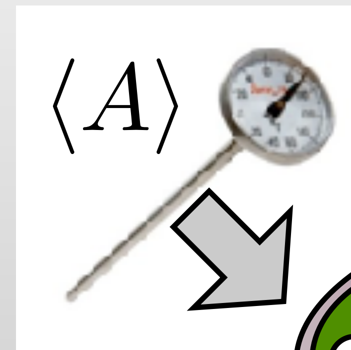
- **Same form of the classical case!**
- **Proposition:** The fixed points are the permutation invariant states if and only if the $\{U_{j,k}\}$ generate the full permutation group.
- **Idea:** The dynamics is unital, i.e. preserves the identity. Fixed point for unital semigroups are the commutant of the algebra generated by the $\{U_k\}$.
- **Theorem:** Under the same weak assumptions of the classical case, a switching dynamics with QL operators converges!
- **In fact, there is a deeper underlying reason why these similar algorithms both converge - they symmetrize the p(t) [SICON paper for details].**

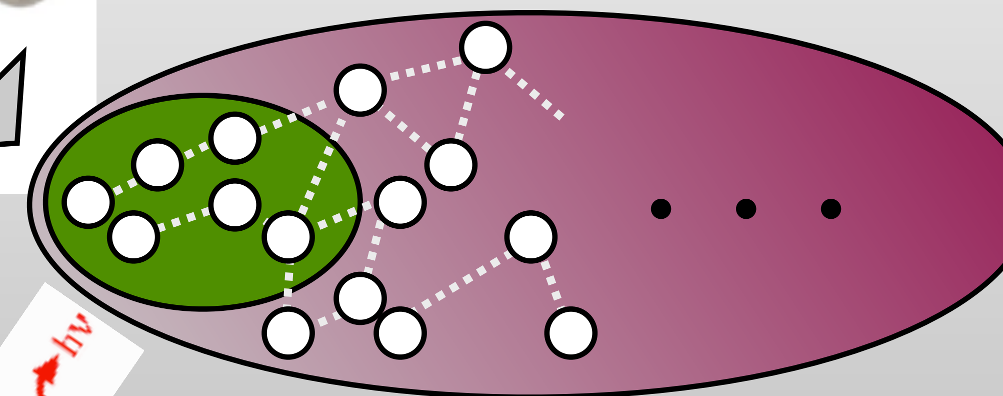
Applications of Quantum Consensus

- **Mixing for Partial Sampling:**

  Average observable estimation

$$A = \frac{1}{n} \sum_j A^{(j)}$$





Gossip algorithm on the whole network; Complete control & measurements on a portion

- **“Cooling” with Localized Control:**

  -> Purify a locally accessible sample;

  -> Use gossip to mix;

  -> Iterate;

- **Size Estimation of a Quantum Network:**

  -> Prepare the network in a pure state (as above) $|\phi\rangle^{\otimes n}$;

  -> Prepare the sample in an **orthogonal state** $|\psi\rangle^{\otimes m}$**;**

  -> Use gossip to mix;

  -> Estimate the average expectation of A as above, with

$$A = \frac{1}{n} \sum_j |\psi\rangle\langle\psi|^{(j)}$$

The Message...

- Using system-theoretic methods, we derived a framework for
  - **(Stability analysis of Markov dynamics on multipartite systems);**
  - **Robust symmetrization methods;**
  - **Tests for checking pure QLS states;**
  - **Constructive results for pure entangled state preparation under locality constraints; Robust methods;**
  - **Scalable protocols for conditional preparation of GHZ states;**

➡ **Also done:**
Consensus and (some) purification together; Stronger Consensus; Mixed state preparation and thermalization;

➡ **Open problems:**
Better *classification* of QLS states;
Speed of convergence (when the system size grows - scalability);
Robustness w.r.t. control errors; Non-Markov models;

Quantum Estimation Problems

- **State and Process Estimation (*Tomography*) are ubiquitous problems in quantum information and control, e.g. if one wants to:**

I. **Verify state preparation** for quantum information processing;

II. **Detect output** of quantum communications;

III. **Estimate noise** “principal components” to design quantum error correction protocols;

IV. **Verify accuracy** in quantum gate engineering;

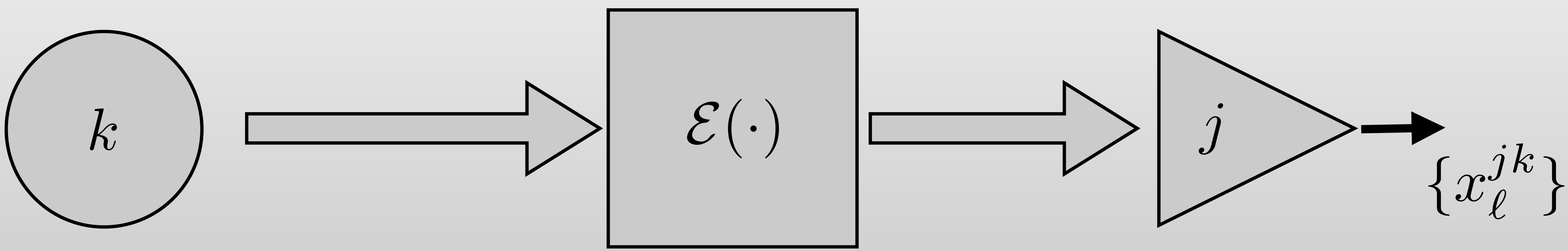
V. **Check validity of noise threshold** theorems for concatenated error correction;

VI. **Build models from data** for quantum control;

VII. **Adaptive methods** for control, encoding, etc....

Typical Experimental Setting for Channel ID

- Source of probe states; Channel; Measurements.



$\{\rho_k\}_{k=1}^{L}$ $\{\Pi_j\}_{j=1}^{M}$

- Iterate N times for each combination (j,k);

- The data we have available are the frequencies: i.e. estimates of the theoretical probabilities:

$$f_{jk} := \frac{1}{N}\sum_{l=1}^{N} x_l^{jk}$$

$$p_{jk} = \mathrm{trace}(\mathcal{E}(\rho_k)\Pi_j)$$

- *Ideally,* we want an estimate of $\hat{\chi}$ such that the constraints are satisfied:

$$f_{jk} = \mathrm{trace}(\mathcal{E}_{\hat{\chi}(\rho_k)}\Pi_j)$$

Convex methods: General framework

- Parametrize: $$\chi(\underline{\theta}) = d^{-1} I_{d^2} + \sum_{\ell} \theta_\ell Q_\ell$$
- Choose a **strictly convex functional** $J(\underline{\theta})$ that quantifies the distance from the (possibly non-positive) data fitting solution;

- Solve: $$\hat{\underline{\theta}} = \arg\min_{\underline{\theta}} J(\underline{\theta})$$

  subject to: $$\underline{\theta} \in \mathcal{A}_+ = \{\underline{\theta} \mid \chi(\theta) \geq 0\}$$

- Then the solution is unique *iff* $\mathcal{B} = \mathcal{S}_{\mathrm{TP}}$;
- ML functional:

$$J(\underline{\theta}) = \sum_{j,k} f_{jk} \log[\mathrm{tr}(\chi(\underline{\theta})(\Pi_j \otimes \rho_k^T))] + (1 - f_{jk}) \log[1 - \mathrm{tr}(\chi(\underline{\theta})(\Pi_j \otimes \rho_k^T))]$$