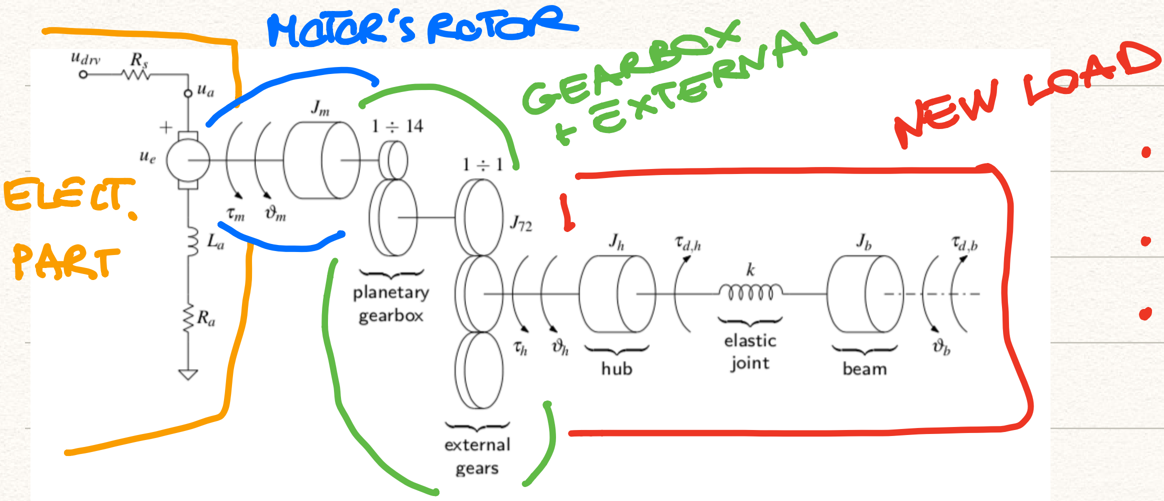


Figure 1: DC gearmotor with elastic joint: lumped–element diagram.

- HUB
- SPRINGS
- BEAM

EQUATIONS:

ELECTRICAL: $$L_a \frac{di_a}{dt} + (R_a + R_s) i_e = u_{drv} - u_e$$

MECHANICAL: $$J_m \frac{d\omega_m}{dt} + B_m \omega_m = \tau_m - \underbrace{\tau_h'}_{\text{EFFECT OF LOAD AT MOTOR'S SIDE}}$$

HUB: $$J_h \frac{d\omega_h}{dt} + B_h \omega_h = \tau_h - \underbrace{\tau_e}_{\text{BEAM}} - \underbrace{\tau_{d,h}}_{\text{STATIC FRICT.}}$$

$\text{sign}(\omega_m)\tau_{SF}$

BEAM: $$J_b \frac{d\omega_b}{dt} + B_b \omega_b = \tau_e - \tau_{d,b}$$

$\tau_{d,b} = 0$ for now

IN ADDITION : "CONNECTING" EQUATIONS

MOTOR : $\begin{cases} u_e = K_e\, \omega_m \\ \tau_m = K_\tau\, i_a \end{cases}$ $\Longleftrightarrow$ Electrical

GEARBOX : $\begin{cases} \omega_h = \omega_m / N \\ \tau_h = N\, \tau_h' \end{cases}$ $\Longleftrightarrow$ MOTOR

ELASTIC TORQUE : $\tau_e = K\,(\vartheta_h - \vartheta_b)$

DRIVER DYN. : $T_{DRV} \dfrac{dU_{drv}}{dt} + U_{drv} = K_{drv}\, u$

EQUIV. INERTIA : $J_{eq} = J_m + \dfrac{J_h}{N^2}$

(at motor's side)

EQUIV. FRICTION : $B_{eq} = B_m + \dfrac{B_h}{N^2}$

(")

NEXT : COMBINE ALL EQTS.

1) Let's START : MOTOR + HUB using GEARBOX,

EQUIV. FRICTIONS

↳ FROM HUB : $\tau_R = \dots$

↳ SUBS IN MOTOR ; "TRANSLATED" $\tau_R' = \frac{1}{N} \dots$

↳ USE DEFS. FOR J_{eq}, B_{eq}.

$$(\Delta)\quad J_{eq}\frac{d\omega_m}{dt} + B_{eq}\,\omega_m = \tau_m - \frac{1}{N}\tau_{dR} - \frac{1}{N}\tau_e$$

2) COUPLE BEAM TO (Δ) USING ELASTIC TORQUE

$$\tau_e = K(\vartheta_R - \vartheta_b)$$

$$\begin{cases} (\square)\quad J_{eq}\ddot{\vartheta}_m + B_{eq}\dot{\vartheta}_m + \frac{K}{N}\underbrace{\vartheta_R}_{=\frac{\vartheta_m}{N}} - \frac{K}{N}\vartheta_b = \tau_m - \frac{1}{N}\tau_{dR} \\ (\circ)\quad J_b\ddot{\vartheta}_b + B_b\dot{\vartheta}_b + K\vartheta_b - \vartheta_m = 0 \end{cases}$$

3) COUPLE ELECTRICAL + DRIVER : $R_{eq} = R_s + R_a$

$$L_a\frac{di_a}{dt} + R_{eq}\,i_a = U_{drv} - k_e\frac{d\vartheta_m}{dt}$$

$$T_{DRV}\frac{dU_{drv}}{dt} + U_{drv} = K_{DRV}\,u$$

$$(\square)\quad (\dots) = K_T\, i_a$$

$$(\Delta)\quad (\dots) = 0 \qquad \text{SAME AS ABOVE}$$

NOTE : • Now as a bad we have TWO RIGID BODIES.

HUB $\leftrightarrow \vartheta_h$; BEAM $\leftrightarrow \vartheta_b$

• THIS TYPE OF "ARTICULATED" SYSTEMS ARE CALLED:

→ COLLOCATED : if ACTUATOR RIGIDLY CONNECTED TO OUTPUT (SENSOR)

→ NON-COLLOCATED : if ACTUATOR IS ON A DIFFERENT RIGID BODY w.r.t. OUTPUT

FOR US : ACTUATION $\leftrightarrow \tau_m$

if OUTPUT : $\vartheta_h \rightarrow$ COLLOCATED (OUR FOCUS)

$\vartheta_b \rightarrow$ NON-COLLOCATED (SEE HANDOUT)

→ LET US COMPUTE TRANSFER FUNCTIONS

$P_{\tau_m \rightarrow \vartheta_h}$; $P_{\tau_m \rightarrow \vartheta_b}$

• Laplace transform of (□) , (○)

WRITTEN AT HUB SIDE

$$N^2 J_{eq} \ddot{\vartheta}_h + N^2 B_{eq} \dot{\vartheta}_h + K \vartheta_h - K \vartheta_b = N \tau_m + (\dots)$$

$$\Rightarrow \begin{cases} \left(N^2 J_{eq} s^2 + N^2 B_{eq} s + K\right) \underbrace{\Theta_R(s)}_{\mathcal{L}[\vartheta_R]} - K \underbrace{\Theta_b(s)}_{\mathcal{L}[\vartheta_b]} = N \underbrace{T_m(s)}_{\mathcal{L}[\tau_m]} \\ \left(J_b s^2 + B_b s + K\right) \Theta_b(s) = k\, \Theta_R(s) \end{cases}$$

- NOW SUBSTITUTING THE SECOND IN THE FIRST

$$\left(N^2 J_{eq} s^2 + N^2 B_{eq} s + K\right)\left(J_b s^2 + B_b s + K\right) \frac{\Theta_b(s)}{K} - K\, \Theta_b(s) = N\, T_m(s)$$

$$\Rightarrow \Theta_b(s) = \frac{K}{N^2} \frac{1}{\left(T_{eq} s^2 + B_{eq} s + \frac{K}{N^2}\right)\left(J_b s^2 + B_b s + K\right) - \frac{K^2}{N^2}} N\, T_m(s)$$

$$= \underbrace{\frac{K}{N} \frac{1}{s\, D'(s)}}_{=: \underbrace{P_{\tau_m \to \vartheta_b}(s)}} T_m(s)$$

$\Rightarrow$ FROM THE BEAM DYNAMICS, $\mathcal{L}$-TRANSF.

$$\Theta_h(s) = \left(\frac{J_b s^2 + B_b s + K}{K}\right) \Theta_b(s)$$

$$= \underbrace{\frac{1}{K}\left(J_b s^2 + B_b s + K\right) \underbrace{P_{\tau_m \to \vartheta_b}(s)}} T_m(s)$$

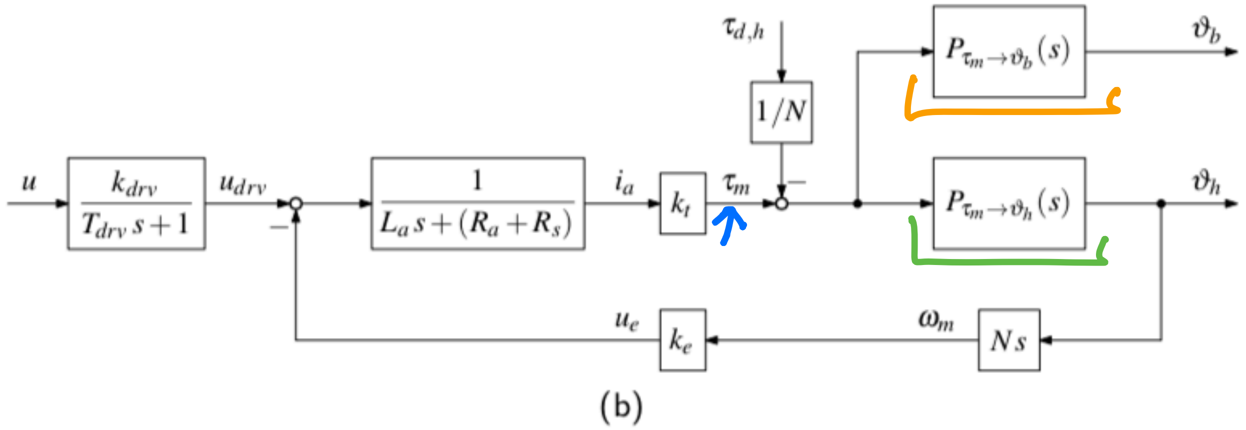


Figure 3: DC gearmotor with elastic joint: block diagram.

$$\Rightarrow \quad P_{v \to \vartheta_h}(s) = \left(\frac{K_{DRV}}{T_{DRV}\,s+1}\right) \frac{\frac{K_T}{L_a s + R_{eq}} P_{\tau_m \to \vartheta_h}}{1 + \frac{K_T K_e N s}{L_a s + R_{eq}} P_{\tau_m \to \vartheta_h}}$$

$$= (\dots) \frac{K_T \left(J_b s^2 + B_b s + K\right)}{(L_a s + R_{eq}) D'(s) + K_T K_e \left(J_b s^2 + B_b s + K\right)}$$

$\hookrightarrow$ WITH USUAL SIMPLIFICATIONS (L_a, T_{DRV}):

$$(\text{DENOMINATOR}) = R_{eq} D'(s) + K_T K_e \left(J_b s^2 + B_b s + K\right)$$

WHAT ABOUT $P_{v \to \vartheta_b}(s)$?

SAME DENOMINATOR ; NUMERATOR $\underline{\underline{K K_T}}$

(NO ZEROS)

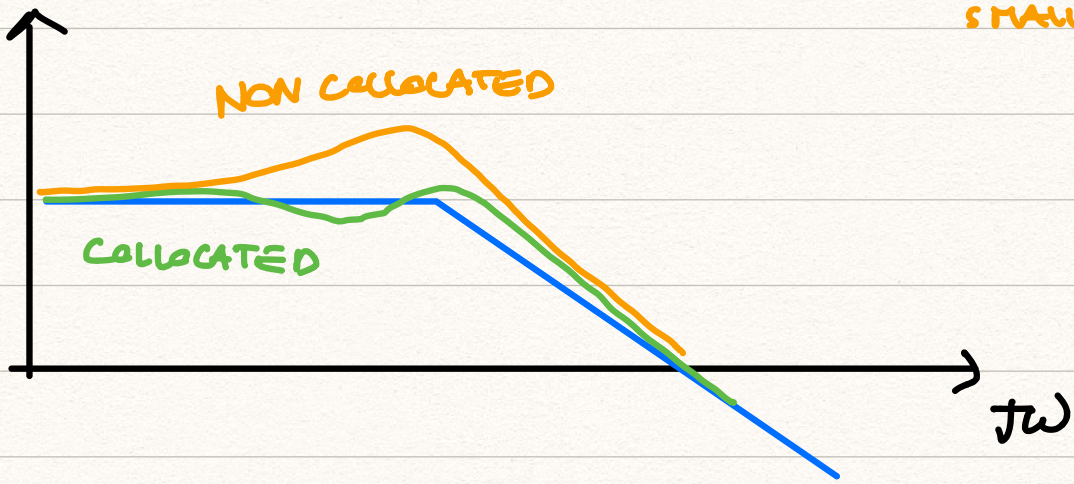
NOTE: IN THE COLLOCATED SYSTEMS, ZEROS HELP!

↳ ZEROS COMPENSATE RESONANCE FOR SMALL $B_{eq} \sim B_b \to 0$

ZEROS : $\to \pm j\sqrt{\frac{K}{J_b}}$

POLES : $\to \pm j\sqrt{\frac{K}{J_B} + \underbrace{\frac{K}{J_{eq} N^2}}_{\text{SMALLER}}}$

Bode



STATE SPACE MODEL
with simplifying assumtions ($L_a \sim 0$, $T_{dev} \sim 0$)
ODE's become:

$$\begin{cases} N^2 J_{eq} \ddot{\vartheta}_l + N^2\left(B_{eq} + \frac{K_T K_e}{R_{eq}}\right)\dot{\vartheta}_l + K\vartheta_l - K\vartheta_b = \frac{N K_T K_{drv}}{R_{eq}} u \quad (+\text{frict.}) \\ J_{eq} \ddot{\vartheta}_b + B_b \dot{\vartheta}_b + K\vartheta_b - K\vartheta_l = 0 \end{cases}$$

MECH. SYSTEM : $x = [\vartheta_l, \vartheta_b, \dot{\vartheta}_l, \dot{\vartheta}_b]^T$

FOR STATE SPACE MODEL :

↳ SOLVE ODE'S FOR $\ddot{\vartheta}_h$, $\ddot{\vartheta}_e$;

↳ WRITE TERMS IN LAST TWO LINES OF A ;

$$\Rightarrow \boxed{\dot{x} = A\,x + B u + \underbrace{B_d\, t_{d,h}}_{\text{FRICTION}}}$$

$$A = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -\frac{K}{N^2 J_{eq}} & \frac{K}{N^2 J_{eq}} & -\frac{1}{J_{eq}}\left(B_{eq} + \frac{K_T K_e}{R_e}\right) & 0 \\ \frac{K}{J_b} & -\frac{K}{J_b} & 0 & -\frac{B_b}{J_b} \end{array}\right]$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{K_T K_{DRV}}{N J_{eq} R_{eq}} \\ 0 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{N^2 J_{eq}} \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

for ϑ_h