FREQUENCY - SHAPE LQ DESIGN

Previously : STANDARD LQ

$$\text{minimize } J(x,u) = \int_0^{+\infty} \underbrace{\left(x^T Q x + u^T R u\right)}_{Q,R} dt$$

$$\text{subject to } \left.\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x_0 \end{aligned}\right\} \text{DATA}$$

- J is determined by Q, R
- Typical choice : Q, R diagonal

$$Q_{ii} \sim 1/|x_i|^2$$

$$R_{ii} \sim 1/|u_i|^2$$

→ FREQUENCY SHAPING : useful
- ↳ Limit high-freq
- ↳ avoid resonances
- ↳ ....

→ Notice : J quadratic

$\sim$ ENERGY of $x(t)$, $u(t)$

$\sim \| \ \|_2$ - NORM ....

⟹ Parseval's Thm :

TRANSLATE $J(x(t), u(t))$

$\rightarrow J(X(j\omega), U(j\omega))$

PARSEVAL'S THM.

$$\int_{-\infty}^{+\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y(j\omega)| \, d\omega$$

$\mathcal{F}(y)$

- CONSIDER $Q = S_Q^T S_Q$, $R = S_R^T S_R$

$$\int_0^{+\infty} \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt =$$

$$= \int_0^{+\infty} \left[\underbrace{(x^T(t) S_Q^T)}_{x_Q^T(t)} (S_Q x(t)) + \underbrace{(u^T(t) S_R)}_{u_R^T(t)} (S_R u(t)) \right] dt$$

sum $\|x_Q\|_2^2 + \|u_R\|_2^2$

Pars.

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[X_Q^T(j\omega) X_Q(j\omega) + U_R^T(j\omega) U_R(j\omega) \right] d\omega$$

$x_Q(t) = S_Q x(t)$, $\mathcal{F}$ Linear $\Rightarrow X_Q(j\omega) = S_Q X(j\omega)$

$u_R(t) = S_R u(t)$ " $\Rightarrow U_R(j\omega) = S_R U(j\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Big[X^T(j\omega) \underbrace{\underbrace{S_Q^T S_Q}_{Q} X(j\omega) + U^T(j\omega) \underbrace{S_R^T S_R}_{R}}\, U(j\omega) \Big] d\omega$$

→ Now we want to add freq. dep. on Q, R !

$$Q \longrightarrow \underline{Q(j\omega)} = H_Q^*(j\omega)\, H_Q(j\omega)$$

Rational, causal, ...

$$R \longrightarrow R(j\omega) = H_R^*(j\omega)\, H_R(j\omega)$$

Define
$$\begin{cases} Y_Q(s) = H_Q(s)\, X(s) \\ Y_R(s) = H_R(s)\, U(s) \end{cases}$$

$$J_\omega(x,u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Big[X^T(j\omega) \underbrace{Q(j\omega)}_{H_Q^* H_Q} X(j\omega) + U^T(j\omega) \underbrace{R(j\omega)}_{H_R^* H_R} U(j\omega) \Big] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Big[Y_Q^T(j\omega)\, Y_Q(j\omega) + Y_R^T(j\omega)\, Y_R(j\omega) \Big] d\omega$$

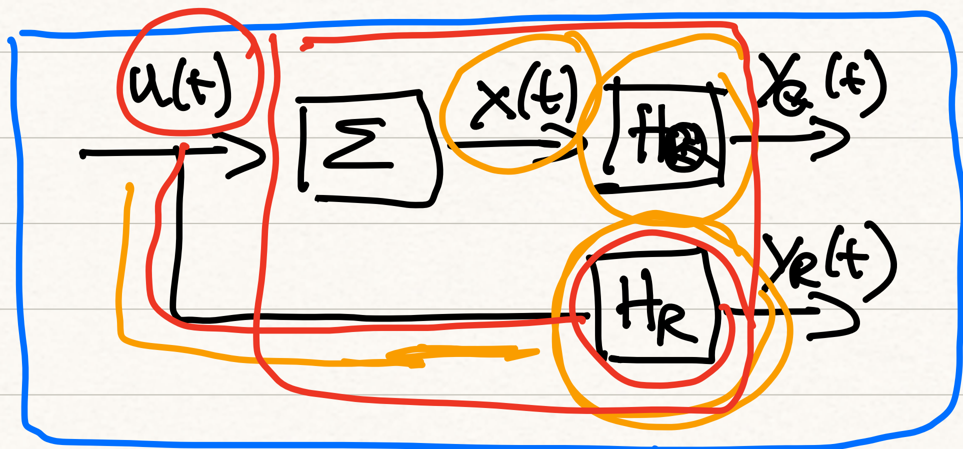
$-\infty$

$$= \int_0^{+\infty} \left[y_Q^T(t)\, y_Q(t) + y_R^T(t)\, y_R(t) \right] dt$$

contain the Freq. dom. "weights"

Before: $\xrightarrow{u(t)} \boxed{\Sigma} \xrightarrow{x(t)}$ $\qquad J(\cancel{x}, u)$

Now:



$J_w(y_Q, y_R)$

New dynamics ∇_0

IDEA: ▷ Construct STATE-SPACE REAL. FOR $H_Q(s)$, $H_R(s)$

▷ Construct an extended ST. SP model $x_A = \begin{bmatrix} x \\ x_Q \\ x_R \end{bmatrix}$, $y_A = \begin{bmatrix} y_Q \\ y_R \end{bmatrix}$

$$H_Q(s) \xrightarrow{\text{REAL}} \begin{cases} \dot{x}_Q = A_Q x_Q + B_Q x \\ y_Q = C_Q x_Q + D_Q x \end{cases}$$

$$H_R(s) \longrightarrow \begin{cases} \dot{x}_R = A_R x_R + B_R u \\ y_R = C_R x_R + D_R u \end{cases}$$

▷ CREATE AN EXTENDED MODEL FOR THE INTERCONN. ABOVE

$$x_A = \begin{bmatrix} x \\ x_Q \\ x_R \end{bmatrix} \qquad A_A = \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline B_Q & A_Q & 0 \\ \hline 0 & 0 & A_R \end{array}\right]$$

$$B_A = \left[\begin{array}{c} B \\ \hline 0 \\ \hline B_R \end{array}\right]$$

$$y_A = \left[\begin{array}{c} y_Q \\ \hline y_R \end{array}\right] \qquad C_A = \left[\begin{array}{c:c:c} D_Q & C_Q & 0 \\ \hdashline 0 & 0 & C_R \end{array}\right] \qquad D_A = \left[\begin{array}{c} 0 \\ \hline D_R \end{array}\right]$$

$$\begin{cases} \dot{x}_A = A_A x_A + B_A u \\ y_A = C_A x_A + D_A u \end{cases}$$

EXT. MODEL CONTAINING FREQ. SHAPING FILTERS.

$$\begin{bmatrix} y_Q \\ y_R \end{bmatrix} = \begin{bmatrix} C_A & \vdots & D_A \end{bmatrix} \begin{bmatrix} x_A \\ \cdots \\ u \end{bmatrix}$$

Let's get back to $J_\omega(y_Q, y_R) \to J_\omega(x_A, u)$

$$J_\omega(y_Q, y_R) = \int_0^{+\infty} \begin{bmatrix} y_Q^T & y_R^T \end{bmatrix} \begin{bmatrix} I & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & I \end{bmatrix} \begin{bmatrix} y_Q \\ y_R \end{bmatrix} dt$$

$$= \int_0^{+\infty} \begin{bmatrix} x_A^T & u^T \end{bmatrix} \begin{bmatrix} C_A^T \\ D_A^T \end{bmatrix} \begin{bmatrix} C_A & D_A \end{bmatrix} \begin{bmatrix} x_A \\ u \end{bmatrix} dt$$

$$= \int_0^{+\infty} \begin{bmatrix} x_A^T & u^T \end{bmatrix} \begin{bmatrix} C_A^T C_A & C_A^T D_A \\ D_A^T C_A & D_A^T D_A \end{bmatrix} \begin{bmatrix} x_A \\ u \end{bmatrix} dt$$

$\to Q_A$

$\to R_A := D_A^T D_A$

sbj. to $\dot{x}_A = A_A x_A + B_A u$

LQ PROBLEM with OFF-DIAG. TERMS!

OPTIMAL SOLUTION : $u(t) = -K\, x_A(t)$

$$K = R_A^{-1}\left(B^T P_\infty + N_A^T\right)$$

where P_∞ : solution of A.R.E. with $A_A - B_A R^{-1} N_A^T$ (instead of A_A)

"Algorithm" for FREQ. SHAPED DESING

→ $Q(j\omega)$, $R(j\omega)$ chosen

→ $H_Q(j\omega)$ $H_R(j\omega)$ derive

→ (A_Q, B_Q, C_Q, D_Q), (A_R, B_R, C_R, D_R) derive

→ construct A_A, B_A, C_A, D_A

→ compute Q_A, R_A, N_A

→ solve with LQ.

$(\tilde{A} = A_A - B_A R^{-1} N_A^T)$

IF NEEDED : USE A MIXED APPROACH.

$$Q = \left[\begin{array}{c|c} Q_1(j\omega) & 0 \\ \hline 0 & Q_2 \end{array}\right]$$

$$R = \left[\begin{array}{c|c} R_1(j\omega) & 0 \\ \hline 0 & R_2 \end{array}\right]$$

$$\Rightarrow \quad y_A = \begin{bmatrix} y^1_{Q,R} \\ y^2_{Q,R} \end{bmatrix}$$

Need add. dynamics

STATIC function of x, u

$$H_{Q,R}(s) = \left[\begin{array}{c|c} H^1_{Q,R}(s) & 0 \\ \hline 0 & \underline{H_{Q,R}} \end{array}\right]$$