I/O STABILITY OF LTI SYSTEMS

- Consider Σ_p as in the previous class

  ↳ ODE $\rightarrow$ $P(s)$ $\leftrightarrow$ $p(t)$

Def Σ_p is I/O or BIBO stable if

$$|u(t)| \le M < +\infty \;\; \forall t \quad \Rightarrow \quad |y_f(t)| \le N < +\infty \;\; \forall t$$

Basic Requirement !!!

Thm Σ_p is BIBO stable IFF

(1) $p(t)$ is absolutely integrable

(2) $P(s) = \dfrac{N_p(s)}{D_p(s)}$ PROPER, (COPRIME)

with D_p HURWITZ :

if $D_p(\hat{s}) = 0 \Rightarrow \mathrm{Re}(\hat{s}) < 0$

Notes : - if P is not proper $\Rightarrow$ IMPULSIVE COMP. in p

$\sim$ DERIVATIVE ACTION

- if $D_p(\lambda) = 0$ and $\mathrm{Re}(\lambda) = 0$, single

The mode $m_1(t) = e^{\lambda t}$ is BOUNDED,

BUT

with input $u(t) = e^{\lambda t}$, bounded

the corresponding $y_f(t)$ has a componte

on:

$$m_2(t) = t\, e^{\lambda t} \qquad \text{UNBOUNDED}$$

- In practice, requesting the controlled system to be BIBO may not be enough

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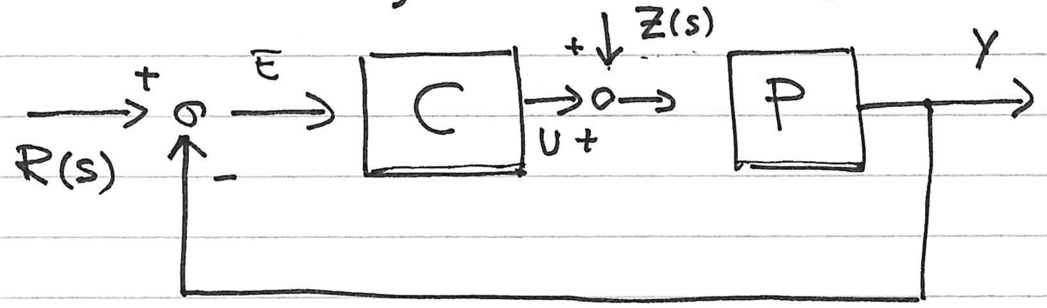


with $P(s) = \frac{s+1}{s-1}$ $\quad C(s) = -2\,\frac{s-1}{s+1}$

!!! UNSTABLE POLE

$$\Rightarrow \quad W = \frac{CP}{1+CP} = \frac{-2}{1-2} = 2$$

However $\quad \frac{Y}{Z} = \frac{P}{1+CP} = \frac{P}{-1} = -\frac{s+1}{s-1}$

!!! UNSTABLE POLE

OR : What if there is a small modeling error

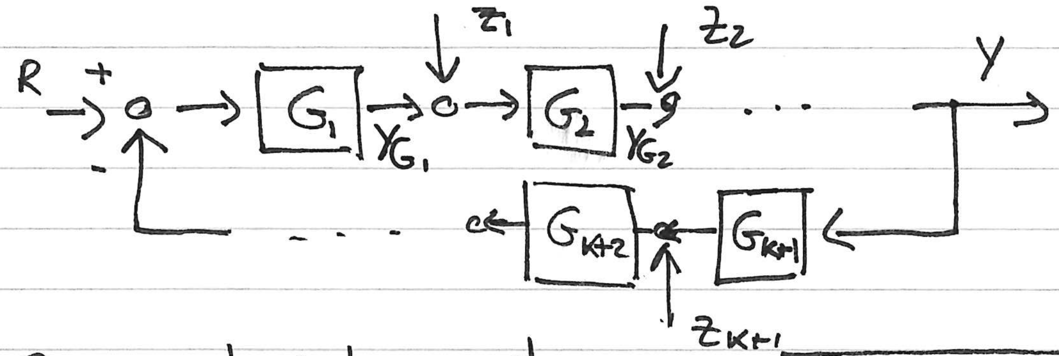
$$P(s) = \frac{s+1}{s-0{,}99} \quad \left[\text{Vs.} \; \frac{s+1}{s-1} \right]$$

$$\Rightarrow W(s) = \frac{-2\,\frac{s-1}{s-0{,}99}}{1-2\,\frac{s-1}{s-0{,}99}} = \frac{2(s-1)}{s - 1{,}01}$$

!!! UNSTABLE

- How to ensure this does not happen?

I/O STABILITY OF INTERCONNECTIONS (INTERNAL STABILITY)



SINGLE FEEDBACK LOOP, with added inputs after each block

G_K ~ physical components;

DEF INTER. STABLE IF $\frac{Y_{G_i}}{Z_j}$ BIBO STABLE $\forall i, j$

THM Let $G_i = \frac{N_i}{D_i}$ be coprime & proper;

the interconnection is STABLE

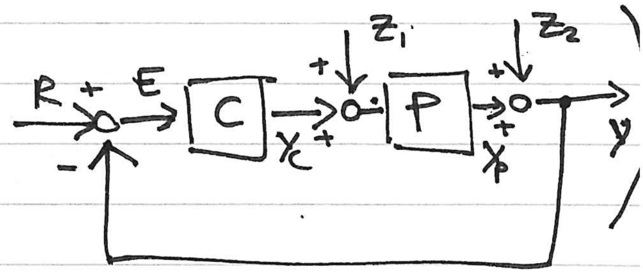
[BIBO stable from each input to each block output]

IFF $D = \prod_i N_i + \prod_i D_i$

(1) is HURWITZ

(2) has the same degree of $\prod_i D_i$

Proof (for the case



$C = \frac{N_C}{D_C}$, $P = \frac{N_P}{D_P}$

- INTERNAL STABILITY MEANS THE FOLLOWING T.F.'s ARE BIBO:

A) $$\frac{Y_P}{R} = W = \frac{CP}{1+CP} = \frac{N_c N_p}{D_c D_p + N_c N_p} = \frac{N_c N_p}{D} \quad \left(= -\frac{Y_P}{Z_2} \right)$$

B) $$\frac{Y_P}{Z_1} = \frac{P}{1+CP} = \frac{N_p D_c}{D} \quad \left(= -\frac{Y_c}{Z_1} \right)$$

C) $$\frac{Y_c}{R} = \frac{C}{1+CP} = \frac{N_c D_p}{D}$$

(1),(2) $\Rightarrow$ A, B, C $\boxed{\text{BIBO STABLE}}$: easy!

(1) D Hurwitz $\Rightarrow$ all poles of A, B, C are stable ($\mathrm{Re}(\lambda)<0$)

(2) $\deg D = \deg D_p D_c \geq \deg N_p N_c,\ \deg N_p D_c,\ \deg N_c D_p$

$\Rightarrow$ A, B, C are PROPER

A, B, C BIBO $\Rightarrow$ (1) : a little more delicate...

- there could be some <u>unstable</u> zero of D that is canceled in each of the A, B, C.

- Let's show it is not possible: [BY CONTRADICTION]:

Assume the pole is $\sim K(s) = s - \lambda$ ← UNSTABLE

- Assume $D(s) = K(s)\,D^{o}(s)$, and that

$K(s)$ is canceled in $W(s) = \dfrac{N_c(s) N_p(s)}{K(s)\,D^{o}(s)}$

- then it must be also

$N_c(s) N_p(s) = \underline{K(s)\, N_c^{o}(s) N_p^{o}(s)}$

- Rewriting in D we have

$D(s) = K(s)\,D^{o}(s) = \underline{K\, N_c^{o} N_p^{o}} + \underline{D_c D_p}$

$\Rightarrow$ it must also be $D_c D_p = K\, D_c^{o} D_p^{o}$

- Since N_p/D_p, N_c/D_c are coprime,

K must be a factor of N_p and D_c <u>OR</u> N_c & D_p,

But NOT Both.

$\Rightarrow$ K would not be canceled in (B) or (C)

<u>CONTRADICTING THE HYPOTHESIS.</u>

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- We are left with A, B, C BIBO $\Rightarrow$ (2):

[By CONTRADICTION] Assume $\deg(D) < \deg(D_c D_p)$;

Since C, P PROPER, it requires $\deg D_c = \deg N_c$, $\deg D_p = \deg N_p$

$\Rightarrow$ A, B, C would not be PROPER $\left(\dfrac{XY}{D}\right)$, NOT BIBO

$\Rightarrow$ CONTRADICTION.

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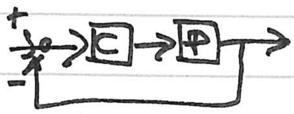
- There is another, INTERESTING characterization

▷ We need to define the SENSITIVITY:

$$S(s) = S_P^W(s) = \lim_{\Delta P \to 0} \frac{\frac{\Delta W}{W}}{\frac{\Delta P}{P}} = \frac{\frac{\partial W}{\partial P}}{\frac{W}{P}}$$



THM Let C, P be COPRIME & PROPER. then the feedback interconnection as above is stable IFF:

(1) $S(s)$ has poles with $Re(s) < 0$ and is PROPER

(2) there are not cancellations of UNSTABLE POLES between C and P

Proof

$$\frac{\partial W}{\partial P} = \frac{\partial}{\partial P}\left(\frac{CP}{1+CP}\right) = \frac{C(1+CP) - CPC}{(1+CP)^2} = \frac{C}{(1+CP)^2}$$

$$\Rightarrow S(s) = \frac{\frac{C}{(1+CP)^2}}{\frac{CP}{(1+CP)P}} = \frac{1}{1+CP} = \frac{D_P D_C}{\underbrace{D_C D_P + N_C N_P}_{D}}$$

the 4th possible T.F.!

- First show

INTERN. STAB $\Rightarrow$ (1), (2)

$\Updownarrow$

D Hurwitz, $\deg(D) = \deg(D_C D_P)$ $\Rightarrow$ (1) OK

- to prove INT. STAB. $\Rightarrow$ (2) :

$\triangleright$ assume by contradiction there is a cancellation of $K(s) = (s-\lambda)$ UNSTABLE

between either N_p and D_c

or N_c and D_p.

$$CP = \frac{N_c N_p}{D_c D_p} = \frac{N_c^\circ N_p^\circ \cancel{K}}{D_c^\circ D_p^\circ \cancel{K}}$$

$$\Rightarrow \quad D = D_c D_p + N_c N_p = \underbrace{\left(D_c^\circ D_p^\circ + N_c^\circ N_p^\circ\right)}_{D^\circ} K$$

$$\Rightarrow \text{ either } \quad \frac{C}{1+CP} = \frac{N_c D_p}{D^\circ K}$$

$$\text{or} \quad \frac{P}{1+CP} = \frac{N_p D_c}{D^\circ K}$$

DO NOT CANCEL K $\Rightarrow$ NOT BIBO

CONTRADICTION

- We are left proving $(\cancel{1}),(2) \Rightarrow$ Int. stab.

(1) implies D Hurwitz, unless there is a cancellation of an UNSTABLE $K(s)$ with $D_p D_s$

- AGAIN, By contradiction :

$$D = K D^\circ \quad ; \quad D_c D_p = K D_c^\circ D_p^\circ$$

$\Rightarrow$ it must be $D = KD^\circ = K D_c^\circ D_p^\circ + N_c N_p$, so

$N_c N_p = K N_c^\circ N_p^\circ$; BUT THEN THERE WOULD A CANCELLATION OF $K(s)$ in CP, CONTRADICTING (2)

- Lastly S PROPER (C, P PROPER) means $\deg D = \deg D_p D_c$.

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SINGLE LTI SYSTEM STABILITY RELATIONS

$\mathrm{Re}(\lambda(A)) < 0$ $\qquad\qquad$ $\mathrm{Re}(\lambda(A)) \leq 0$, if $=0$ $\mu = 1$

$$\text{A.S. FOR STATE} \Longrightarrow \text{SIMPLE STAB. FOR STATE}$$

$$\Downarrow$$

$$\text{A.S. FOR } y_0 \Longleftrightarrow \mathrm{Re}(\lambda(A_{OBS})) < 0$$

(+ PROPER) $\qquad \Downarrow$

$$\text{BIBO ST.} \Longleftrightarrow \mathrm{Re}(\lambda(A_{OBS\text{-}RAG})) < 0$$

NOTE : NL SYSTEM VS LINEARIZATION

$\boxed{\text{VIA LYAPUNOV.}}$

$$\dot{x} = \frac{\partial f}{\partial x}(0)\, x \quad \text{Eq. } x=0 \text{ A.S.} \qquad\qquad \dot{x} = \quad \text{Eq. } x=0 \text{ UNSTABLE}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dot{x} = f(x) \quad \text{Eq. } x=0 \text{ A.S.} \qquad\qquad \text{Eq. } x=0 \text{ UNSTABLE}$$

SIMPLY STABLE? CANNOT SAY

$$\dot{x} = \frac{\partial f}{\partial x}(0)\, x + g(x)$$

This decides stability!

check:

$\dot{x} = 0$

$\dot{x} = -x^3$

$\dot{x} = x^4$

- In order to simplify descriptions in complex interconnections, simple dynamical responses... pole placement information

APPROXIMATION OF STABLE SYSTEMS WITH THEIR DOMINANT POLES

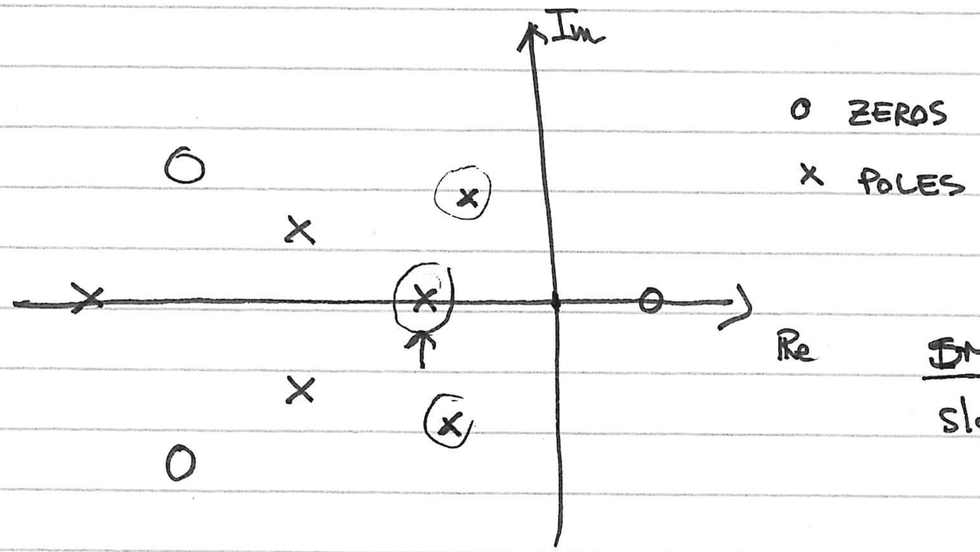
- Consider I/O Representation (ASSUME NO SIMPLIFICATIONS)

$$P(s) = \frac{B(s)}{A(s)}$$

POLES : Roots of $A(s)$, call p_i

- IF $Re(p_i) < 0 \Rightarrow$ SYSTEM BIBO STABLE (ASYMP. STABLE w.r.t. $y_0(t)$)

$$y(t) = \sum_{ik} \underbrace{c_{ik}\, t^{k-1} e^{p_i t}}_{\text{MODES}} + \underbrace{(\ldots)}_{\text{Depends on INPUT}}$$



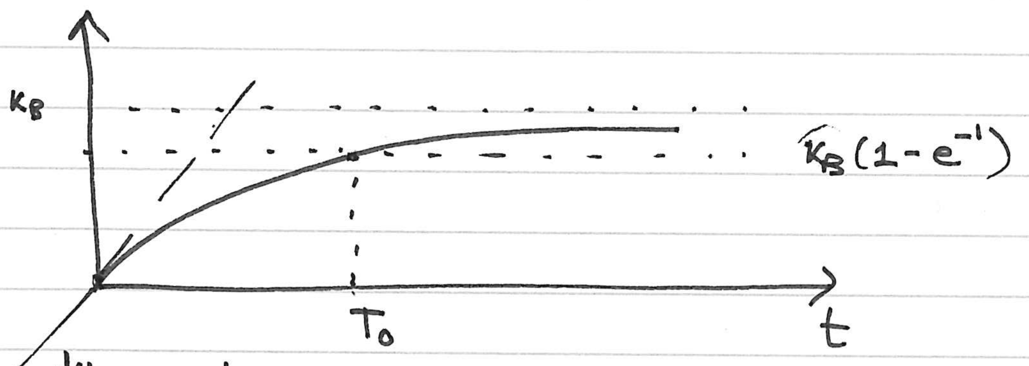
Smallest $|Re(p_i)|$, slowest decaying.

- Assume INPUT is a STEP ; the TRANSIENT behavior is dominated by the SLOWEST DECAYING MODES

I) max $Re(p_i)$ / min $|Re(p_i)|$ attained for a ~~ONE~~ SINGLE REAL POLE p_0

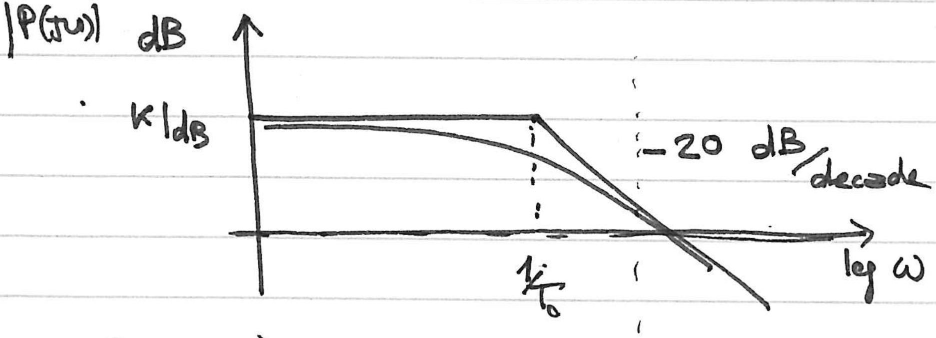
$$P(s) \cong \frac{\alpha}{s - p_0} = K_B \frac{1}{1 + T_0 s}$$

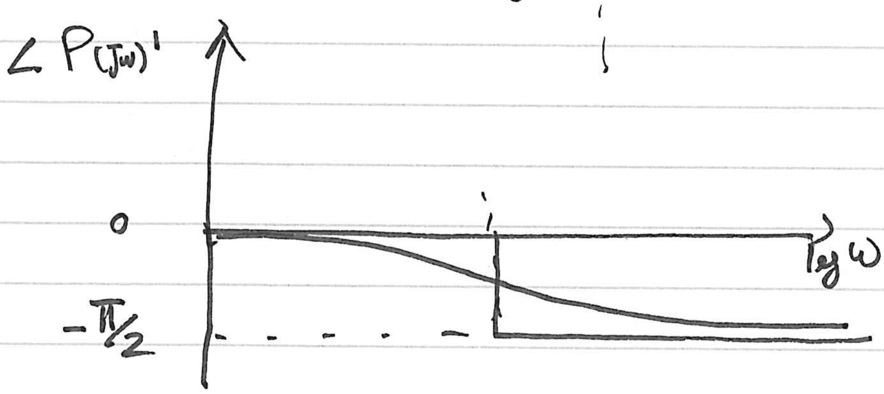
$$T_0 = -\frac{1}{p_0}$$ TIME-CONSTANT OF SYSTEM



$$\frac{dy}{dt}(0) = \frac{1}{T_0}$$

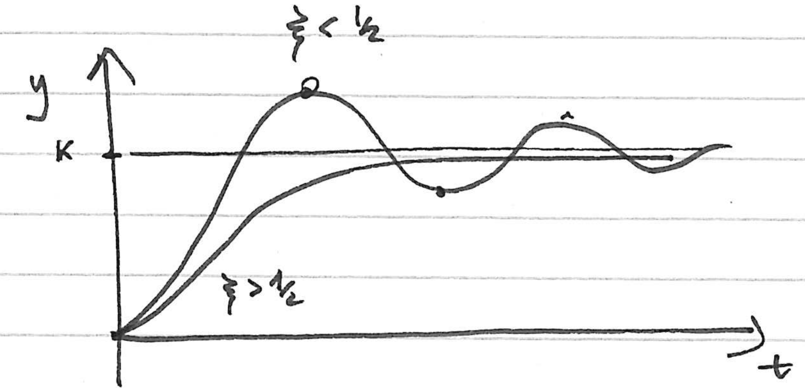
Bode Plots (frequency response)





II) max $Re(p_i)$ attained for a pair of COMPLEX CONJ. POLES $p_{1,2} = \sigma \pm j\omega$

$$P(s) \cong K \frac{1}{1 + 2\xi\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}}, \quad \sigma = -\omega_n \xi$$



Bode