Formulas reference sheet

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$$t_r \approx \frac{1.8}{\omega_n}, \qquad t_{s,5\%} = \frac{3}{\delta\omega_n}, \qquad t_{s,1\%} = \frac{4.6}{\delta\omega_n} \tag{1}$$

$$M_p = e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}} \tag{2}$$

$$\omega_B\, t_r \approx 1.8 \tag{3}$$

$$\omega_B\, t_{s,5\%} \approx 3/\delta \qquad \text{and} \qquad \omega_B\, t_{s,1\%} \approx 4.6/\delta \tag{4}$$

$$\delta = \frac{\log(1/M_p)}{\sqrt{\pi^2 + \log^2(1/M_p)}} \tag{5}$$

$$\omega_{gc} \approx \omega_B \qquad \varphi_m = \operatorname{atan} \frac{2\delta}{\sqrt{\sqrt{1+4\delta^4} - 2\delta^2}} \tag{6}$$

$$\begin{cases} K_P = \Delta K \cos\Delta\varphi & (7) \\ T_D = \dfrac{\tan\Delta\varphi + \sqrt{(\tan(\Delta\varphi))^2 + 4/\alpha}}{2\,\omega_{gc}} & (8) \\ T_I = \alpha T_D & (9) \end{cases}$$

$$\begin{cases} K_P = \Delta K \cos\Delta\varphi \\ K_D = \dfrac{1}{\omega_{gc}} \Delta K \sin\Delta\varphi \end{cases} \qquad \text{(PD controller)} \tag{10}$$

$$\begin{cases} K_P = \Delta K \cos\Delta\varphi \\ K_I = -\omega_{gc}\, \Delta K \sin\Delta\varphi \end{cases} \qquad \text{(PI controller)} \tag{11}$$

$$\text{Error Space}: \begin{bmatrix} e^{(1)} \\ e^{(2)} \\ e^{(3)} \\ \vdots \\ e^{(m)} \\ \hline \dot{\boldsymbol{\xi}} \end{bmatrix} = \underbrace{\left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & \mathbf{0} \\ 0 & 0 & 1 & \cdots & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \mathbf{0} \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{m-1} & \boldsymbol{C} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{A} \end{array}\right]}_{\triangleq \boldsymbol{A}_z} \underbrace{\begin{bmatrix} e \\ e^{(1)} \\ e^{(2)} \\ \vdots \\ e^{(m-1)} \\ \hline \boldsymbol{\xi} \end{bmatrix}}_{\triangleq \boldsymbol{z}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline \boldsymbol{B} \end{bmatrix}}_{\triangleq \boldsymbol{B}_z} u_\xi \tag{12}$$

$$u_\xi = -\boldsymbol{K}_z\, \boldsymbol{z} = -\left[\begin{array}{cccc|c} k_0 & k_1 & \cdots & k_{m-1} & \boldsymbol{K}_\xi \end{array}\right] \begin{bmatrix} e \\ e^{(1)} \\ \vdots \\ e^{(m-1)} \\ \hline \boldsymbol{\xi} \end{bmatrix} \tag{13}$$

$$\tilde{U}(s) \;=\; -\frac{\sum_{i=0}^{m-1} k_i\, s^i}{\sum_{i=0}^{m} \alpha_i\, s^i} E(s) \;=\; -\underbrace{\frac{k_{m-1}\, s^{m-1} \,+\, \cdots \,+\, k_1\, s \,+\, k_0}{s^m \,+\, \alpha_{m-1}\, s^{m-1} \,+\, \cdots \,+\, \alpha_1\, s \,+\, \alpha_0}}_{H(s)} E(s) \tag{14}$$

$$\text{Extended estimator} \;:\; \begin{cases} \begin{bmatrix} \dot{\boldsymbol{x}}_\rho \\ \dot{\boldsymbol{x}}' \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} \boldsymbol{A}_\rho & \boldsymbol{0} \\ \boldsymbol{B}\boldsymbol{C}_\rho & \boldsymbol{A} \end{bmatrix}}_{\triangleq \boldsymbol{A}_e} \underbrace{\begin{bmatrix} \boldsymbol{x}_\rho \\ \boldsymbol{x}' \end{bmatrix}}_{\triangleq \boldsymbol{x}_e} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B} \end{bmatrix}}_{\triangleq \boldsymbol{B}_e} u & (15) \\ e \;=\; \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{C} \end{bmatrix}}_{\triangleq \boldsymbol{C}_e} \begin{bmatrix} \boldsymbol{x}_\rho \\ \boldsymbol{x}' \end{bmatrix} & (16) \end{cases}$$

$$\text{Reduced Estimator (discrete)} \;:\; \begin{cases} \boldsymbol{z}[k+1] \;=\; \boldsymbol{\Phi}_o\, \boldsymbol{z}[k] \,+\, \boldsymbol{\Gamma}_o\, [\boldsymbol{u}[k],\; \boldsymbol{y}[k]]^T & (17) \\ \hat{\boldsymbol{x}}[k] \;=\; \boldsymbol{H}_o\, \boldsymbol{z}[k] \,+\, \boldsymbol{J}_o\, [\boldsymbol{u}[k],\; \boldsymbol{y}[k]]^T & (18) \end{cases}$$

$$\begin{aligned} \boldsymbol{\Phi}_o \;&=\; \boldsymbol{\Phi}'_{22} - \boldsymbol{L}\,\boldsymbol{\Phi}'_{12} \\ \boldsymbol{\Gamma}_o \;&=\; \left[\boldsymbol{\Gamma}'_2 - \boldsymbol{L}\,\boldsymbol{\Gamma}'_1\,,\;\; (\boldsymbol{\Phi}'_{22} - \boldsymbol{L}\,\boldsymbol{\Phi}'_{12})\,\boldsymbol{L} + \boldsymbol{\Phi}'_{21} - \boldsymbol{L}\,\boldsymbol{\Phi}'_{11}\right] \end{aligned} \tag{19}$$

$$\boldsymbol{H}_o \;=\; \boldsymbol{T} \begin{bmatrix} \boldsymbol{0}_{p\times(n-p)} \\ \boldsymbol{I}_{(n-p)\times(n-p)} \end{bmatrix}, \qquad \boldsymbol{J}_o \;=\; \boldsymbol{T} \begin{bmatrix} \boldsymbol{0}_{p\times m} & \boldsymbol{I}_{p\times p} \\ \boldsymbol{0}_{(n-p)\times m} & \boldsymbol{L} \end{bmatrix} \tag{20}$$

where $\boldsymbol{T}$ is the state transformation that partitions the state vector $\boldsymbol{x}$ and

$$\boldsymbol{\Phi}' \;=\; \boldsymbol{T}^{-1}\,\boldsymbol{\Phi}\,\boldsymbol{T} \;=\; \begin{bmatrix} \boldsymbol{\Phi}'_{11} & \boldsymbol{\Phi}'_{11} \\ \boldsymbol{\Phi}'_{21} & \boldsymbol{\Phi}'_{22} \end{bmatrix}, \qquad \boldsymbol{\Gamma}' \;=\; \boldsymbol{T}^{-1}\,\boldsymbol{\Gamma} \;=\; \begin{bmatrix} \boldsymbol{\Gamma}'_1 \\ \boldsymbol{\Gamma}'_2 \end{bmatrix} \tag{21}$$

Forward Euler	Backward Euler	Trapezoidal	Exact
$C\left(\dfrac{z-1}{T}\right)$	$C\left(\dfrac{z-1}{T\,z}\right)$	$C\left(\dfrac{2}{T}\,\dfrac{z-1}{z+1}\right)$	$\left(1-z^{-1}\right)\,\mathcal{Z}\left\{\dfrac{C(s)}{s}\right\}$

Table 1: Discretization of a continuous–time transfer function $C(s)$.

	Forward Euler	Backward Euler	Trapezoidal	Exact
$\boldsymbol{\Phi}$	$\boldsymbol{I} + \boldsymbol{A}\,T$	$(\boldsymbol{I} - \boldsymbol{A}\,T)^{-1}$	$\left(\boldsymbol{I} + \dfrac{\boldsymbol{A}\,T}{2}\right)\left(\boldsymbol{I} - \dfrac{\boldsymbol{A}\,T}{2}\right)^{-1}$	$\boldsymbol{e}^{\boldsymbol{A}\,T}$
$\boldsymbol{\Gamma}$	$\boldsymbol{B}\,T$	$(\boldsymbol{I} - \boldsymbol{A}\,T)^{-1}\,\boldsymbol{B}\,T$	$\left(\boldsymbol{I} - \dfrac{\boldsymbol{A}\,T}{2}\right)^{-1}\boldsymbol{B}\,\sqrt{T}$	$\int_0^T \boldsymbol{e}^{\boldsymbol{A}\,\tau}\boldsymbol{B}\,d\tau$
$\boldsymbol{H}$	$\boldsymbol{C}$	$\boldsymbol{C}\,(\boldsymbol{I} - \boldsymbol{A}\,T)^{-1}$	$\sqrt{T}\,\boldsymbol{C}\left(\boldsymbol{I} - \dfrac{\boldsymbol{A}\,T}{2}\right)^{-1}$	$\boldsymbol{C}$
$\boldsymbol{J}$	$\boldsymbol{D}$	$\boldsymbol{D} + \boldsymbol{C}\,(\boldsymbol{I} - \boldsymbol{A}\,T)^{-1}\,\boldsymbol{B}\,T$	$\boldsymbol{D} + \boldsymbol{C}\left(\boldsymbol{I} - \dfrac{\boldsymbol{A}\,T}{2}\right)^{-1}\dfrac{\boldsymbol{B}\,T}{2}$	$\boldsymbol{D}$

Table 2: Discretization of a continuous–time state–space model $\Sigma_c = (\boldsymbol{A},\,\boldsymbol{B},\,\boldsymbol{C},\,\boldsymbol{D})$.