REVIEW & SMALL EXTENSION OF LINEAR QUADRATIC OPTIMAL CONTROL

so far : CONTROL SPECS. → STABILITY

→ perf. indexes on transient

→ asympt. beh.

NOW : OPTIMAL CONTROL → STABILITY

→ MINIMIZE COST FUNCTION

↳ QUADRATIC FUNC. of $x(t)$, $u(t)$

↳ TIME TO REACH TARGET

↳ DISTANCE FROM TARGET @ GIVEN T

▷ Let us FORMALIZE :

$$(\Sigma) \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad x(0) = x_0$$

→ solve :

$$\text{minimize } J(x(\cdot),u(\cdot)) = \int_0^{+\infty} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$

$Q \geq 0$, $R > 0$

$Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$

$Q^T = Q$

Notice →

$$J = J_Q(x) + J_R(u)$$

Quadratic forms

→ if $Q = C^T Q_0 C$, $D = 0$ for simplicity

$$J_Q(x) = \int_0^{+\infty} x^T(t) \underbrace{Q}_{C^T Q_0 C} x(t)\, dt = \int_0^{+\infty} y^T(t) Q_0 y(t)\, dt$$

What is the form of the solution?

$$\underline{u}(t) = -K_\infty\, x(t)$$

LINEAR STATIC FEEDBACK

↳ sol. of the LQ in the infinite-time horizon

↳ we will <u>not</u> rederive solution

↳ Review some properties & Facts ...

(1) How do we obtain $x(t) \longrightarrow x_\infty$

$u(t) \longrightarrow u_\infty$

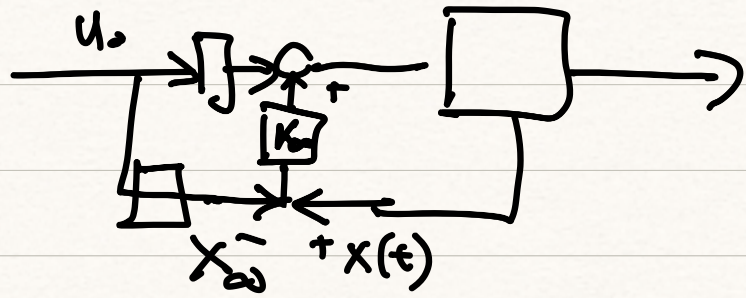
(so that $A x_\infty + B u_\infty = 0$ (Δ)) to satisfy asympt. tracking of steps.

→ Define $x_\Delta(t) = x(t) - x_\infty$

$u_\Delta(t) = u(t) - u_\infty$

$$\dot{x}_\Delta = \dot{x} - \underbrace{\dot{x}_\infty} = Ax + Bu - \overset{(\Delta)}{Ax_\infty - Bu_\infty}$$

$$\Rightarrow J_\Delta(x_\Delta(\cdot), u_\Delta(\cdot)) = \int_0^{+\infty} \left[\underline{x_\Delta}^T Q \underline{x_\Delta} + \underline{u_\Delta}^T R \underline{u_\Delta} \right] dt$$



② How to find K_∞?

Recall : (A,B) is STABILIZABLE if

↳ REACHABLE, or

↳ UNREACH. MODES ARE A.S.

$$A = \left[\begin{array}{c|c} A_R & * \\ \hline 0 & \underline{A_{NR}} \end{array}\right]$$

Dual concept : (A,C) is DETECTABLE if

↳ OBSERVABLE, OR

↳ UNOBSERVABLE MODES are A.S.

$\sqrt{Q}^T U^T U \sqrt{Q}$

THM Let $\sqrt{Q}$ be s.t. $Q = \sqrt{Q}^T \sqrt{Q}$

② the Algebraic Riccati Eq. (ARE)

$$\underline{A^T P + PA - PBR^{-1}B^T P + Q} = 0$$

has a unique solution $P_\infty \geq 0$

if pair $[A, \sqrt{Q}]$ is detectable
$[A, \sqrt{Q}]$ is observable $P_\infty > 0$

② Define $K_\infty = R^{-1} B^T P_\infty$

$u(t) = - K_\infty x(t)$

→ solves the LQ problem $\left(\begin{array}{l}\min J \\ \text{s.t. } \dot{x} = \dots\end{array}\right)$

→ stabilizes the system ($A - BK_\infty$ is A.S.)

iff $\left[\begin{array}{l}(A, B) \text{ stabilizable} \\ (A, \sqrt{Q}) \text{ is detectable.}\end{array}\right.$

OBSERVE : ▷ K_∞ : does not depend on x_0, t_0

▷ MATLAB LQR

$Q > 0 \Rightarrow Q = Q^T \Rightarrow \exists U, \; U^T U = I$

$$Q = U^T D_Q U$$

$\Rightarrow$ choose $\sqrt{Q} = U^T \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \end{bmatrix} U$, $D_Q = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

$\Rightarrow$ ... $\left[\quad \ddots \quad + d_n \right]$ $\quad d_i \in \mathbb{R}_+$

Notice : Rewrite

$$J(x,u) = \int_0^t [x^T \vdots u^T] \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & R \end{array}\right] \begin{bmatrix} x \\ \cdots \\ u \end{bmatrix} dt$$

<u>Generalize</u> : we can introduce off-diag "cost" N

$$J_N(x,u) = \int_0^t [x^T \; u^T] \underbrace{\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix}}_{\geq 0} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

$\Rightarrow$ <u>IDEA</u> : Transfer N from J to the dynamics ...

Minimize $\tilde{J}(\tilde{x}, v) = \int_0^{+\infty} [\tilde{x}^T, v^T] \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{R} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ v \end{bmatrix} dt$

subject to $\dot{\tilde{x}} = (A - BR^{-1}N^T)\tilde{x} + Bv$

where $\begin{cases} v = u + R^{-1}N^T x(t) \\ \tilde{Q} = Q - NR^{-1}N^T \\ \tilde{R} = R \end{cases}$

$\Rightarrow$ if I solve for $\tilde{J}$, $\tilde{K}_\infty$?

How do I get a solution for $\underline{u}$?

① solve a New ARE

$(A - BR^{-1}N^T)^T P + P(A - BR^{-1}N^T) - PBR^{-1}BP + Q = 0$

$\Rightarrow$ find P_∞, positive if $(A, \sqrt{\tilde{Q}})$ observable.

② optimal $v(t) = -\underbrace{R^{-1}B^T P_\infty}_{K_{\infty,v}}$

③ use the def. of $v(t)$:

$$u(t) = v(t) - R^{-1}N^T x(t)$$

$$= -\underbrace{R^{-1}\left[B^T P_\infty + N^T\right]}_{K_{\infty,u}} x(t)$$

STABILIZING IF :

$(A - BR^{-1}N^T, B)$ STABILIZABLE

$(\underbrace{A - BR^{-1}N^T}_{\tilde{A}}, \sqrt{\tilde{Q}})$ DETECTABLE

So far : → Pb definition
→ Pb generalization
→ How to compute solutions
(→ How to apply to Regulation)

OPEN QUESTION How do we choose Q, R?

First observation :

↳ Q : associated to STAB. PERF.
HOW FAST $x(t) \to 0$

↳ R : associated to cost of CONTROL

NOTE :

① $J_1 \sim Q, R$

$J_2 \sim \lambda_1 Q, \lambda_2 R \qquad \lambda_i \geq 0, \in \mathbb{R}$

$\Rightarrow$ The corresponding $K_{\infty,1}, K_{\infty,2}$

are different iff $\lambda_1 \neq \lambda_2$

② if $Q = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix} \qquad R = \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{bmatrix}$

$\rightarrow$ q_i weights $|x_i|^2$

r_i " $|u_i|^2$

$$J_{Q,R}(x,u) = \int_0^t \left(\sum_i q_i |x_i(t)|^2 + r_i |u_i(t)|^2 \right) dt$$

BRYSON'S RULE : choose Q, R diagonal

call $\bar{x}_i$, $\bar{u}_i$: maximum acceptable values of $|x_i|, |u_i|$

Define :

$$Q = \begin{bmatrix} q_{11} & & 0 \\ & \ddots & \\ 0 & & q_{nn} \end{bmatrix} \qquad q_{ii} = \frac{1}{\bar{x}_i^{\,2}}$$

$$R = \begin{bmatrix} r_{11} & & 0 \\ & \ddots & \\ 0 & & r_{pp} \end{bmatrix} \qquad r_{ii} = \frac{1}{\bar{u}_i^{\,2}}$$

Good starting point for TRIAL & ERROR

▷ Pb : we want minimum guaranteed convergence rate for $x(t)$

$$x(t) \xrightarrow[t \to +\infty]{} 0$$

$u = -K_\infty x$: stabilizing, ok

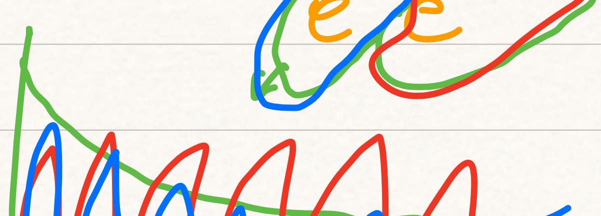
$A - BK_\infty$

min. speed?

$$e^{-\alpha t}(\dots)$$

modes $e^{\lambda t}$ $\quad \lambda = \sigma + i\gamma$

$$= e^{\sigma t} e^{i\gamma t}$$



$$\mathrm{eig}(\overset{J_A}{A}) = \{\lambda_i\}$$

$$\mathrm{eig}(\underset{J_A}{A} + \alpha \underset{I}{I}) = \{\lambda_i + \alpha\}$$

$\Rightarrow$ Requiring $\mathrm{Re}(\mathrm{eig}(A-BK)) < -\alpha < 0$

is equivalent $\mathrm{Re}(\mathrm{eig}(A-BK+\alpha I)) < 0$

$\Rightarrow$ • We can solve LQ for

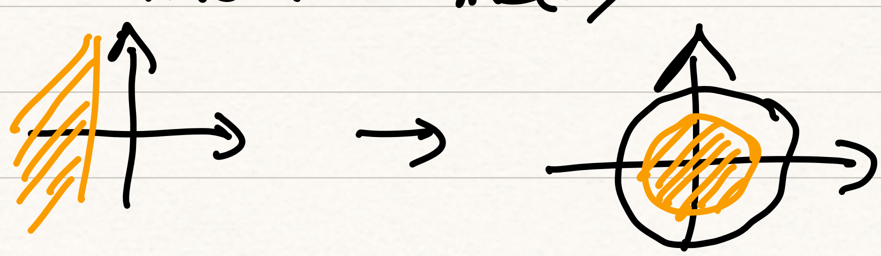
$$\dot{x} = (A+\alpha I)x + Bu \quad (\text{same } J)$$

• Find stabilizing K_∞

• the same K_∞ ensures

$$\mathrm{Re}(\mathrm{eig}(A-BK)) < -\alpha$$

DISCRETE TIME? $\mathrm{Re}(\lambda) < -\alpha \rightarrow |\lambda| < \rho < 1$



TRICK : Define
$$\begin{cases} \tilde{A} = \frac{1}{\rho} A \\ \tilde{B} = \frac{1}{\rho} B \end{cases}$$

STABILIZING LQ GAIN K_∞ ensures

$$|\operatorname{eig}(\underbrace{\tilde{A} - \tilde{B} K_\infty}_{\frac{1}{\rho}(A - BK_\infty)})| < 1$$

$\operatorname{eig}(X) = \{\lambda_i\}$

$\operatorname{eig}(\rho X) = \{\rho \lambda_i\}$

$$\Rightarrow |\operatorname{eig}(A - BK_\infty)| < \rho$$

▷ if we let LQ alg. choose eigs(A-BK)

where does it allocate them?

SISO case:

minimize $J_r(y,u) = \int_0^{+\infty} \left(y^2(t) + r\, u^2(t) \right) dt$

ONLY Rel. magnitude matters!

subj. to $\dot{x} = Ax + Bu$

$y = Cx$

→ Compute asymp. optimal gain K_r

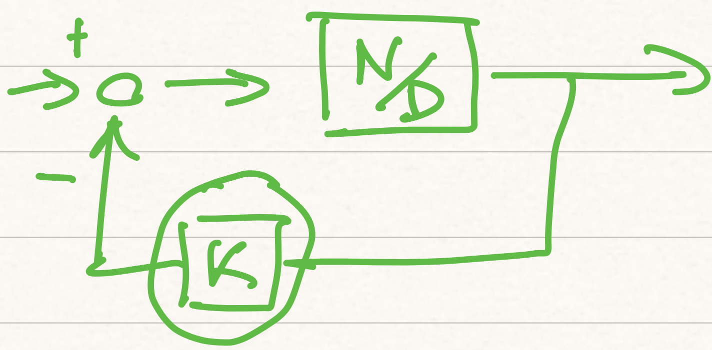
THM (SYMM. ROOT LOCUS)

the closed-loop eig.s ($\mathrm{eig}(A-BK_r)$) are the STABLE ROOTS of

$$p(s) = D(-s)\,D(s) + r^{-1} N(-s)N(s)$$

where $\frac{N(s)}{D(s)} = C(sI-A)^{-1}B$

Recall



$$W = \frac{N}{D+KN}$$

IN OUR CASE: $D(-s)D(s) + \frac{1}{r}N(-s)N(s)$

→ has twice as many roots than $D(s)$

→ is symmetric with respect to the imaginary axis

⇒ HALF OF ROOT LOCUS in $\mathrm{Re}(s)<0$

Two extreme cases:

A) $r \to +\infty$ "EXPENSIVE" CONTROL

Poles in closed loop $\to$ stable roots of $D(-s)D(s)$

B) $r \to 0$ "CHEAP" CONTROL

$\to$ STABLE ROOTS of $N(-s)N(s)$ + asymptotes.

Ex DC MOTOR

$$P(s) = \frac{b}{s(s+p)} \quad \text{(simplified form)}$$

No ZEROS

$$G(s) = P(s)P(-s) = \frac{N(s)N(-s)}{D(s)D(-s)} = \frac{b^2}{-s^2(s+p)(s-p)}$$