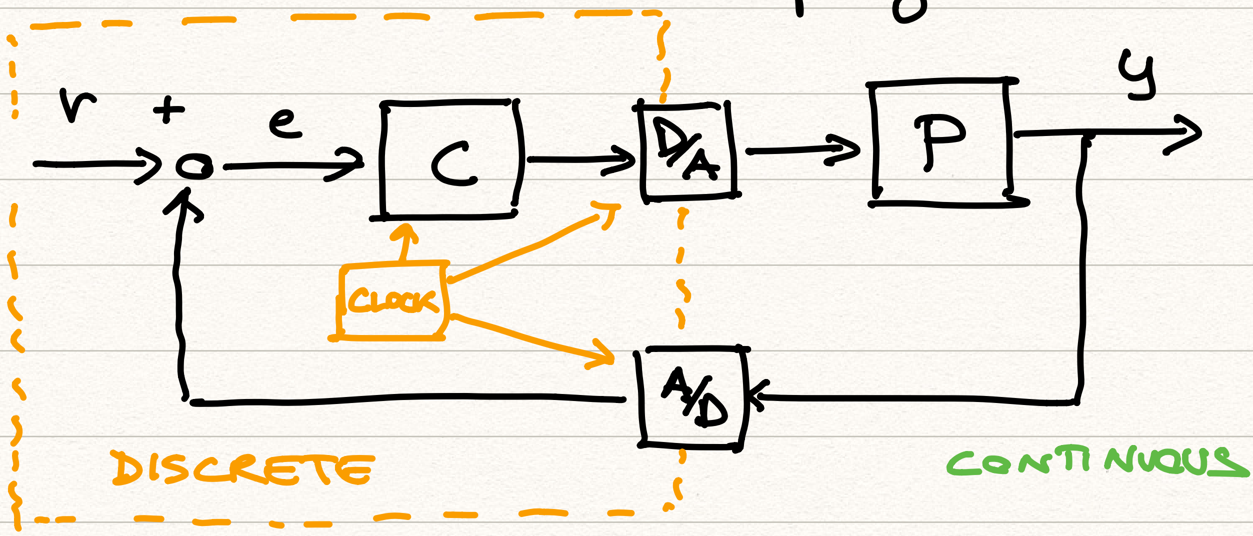
NEW CHAPTER : DIGITAL CONTROL

↳ IN LAB #2 : Test different approaches for control of sampled-data systems

BASIC SETUP & ASSUMPTIONS :

① Digital signals ~ discrete-time signals with UNIFORM, SYNCRONOUS sampling

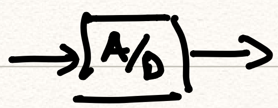


▷ WHAT ABOUT QUANTIZATION ?

↳ Neglect in design / Consider as noise

↳ SIMULATE IN LAB'S SIMULINK SCHEMES

② SAMPLING : VIA PULSE-AMPLITITUDE MODULATION



↳ Preserves LINEARITY

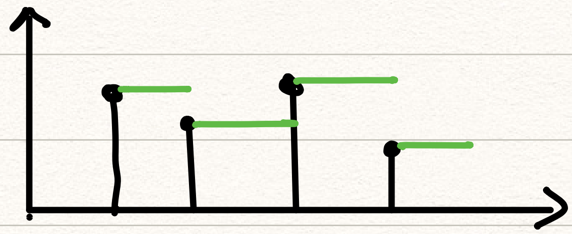
↳ ALLOWS US TO FIND RELATIONSHIP

$\mathcal{L} \longleftrightarrow \mathcal{Z}$

③ D/A CONVERSION : ZERO-ORDER HOLD

$\rightarrow \boxed{D/A} \rightarrow$ $\qquad \bar{x}(k) \longrightarrow x(t) = \bar{x}(k)$

$$t \in [kT_s, (k+1)T_s)$$



▷ WE WILL SEE TWO APPROACHES TO CONTROL DESIGN

I) EMULATION : a) DESIGN $C(s)$ IN CONT. TIME

b) approx. TRANSLATE

$$C(s) \longrightarrow \tilde{C}(z) \leftarrow \text{DISCRETE}$$

CRITICAL PARAMETER : SAMPLING TIME T_s

How to choose?

↳ it needs to be shorter than system's TIMESCALES

↳ rule of thumb

$$\frac{t_r}{1000} < T_s < \frac{t_r}{20}$$

II) DIRECT DIGITAL DESIGN : a) TRANSLATE $P(s) \rightarrow P(z)$ EXACT!

b) DESIGN $C(z)$

$$\underbrace{\rightarrow \boxed{D/A} \rightarrow \boxed{P(s)} \rightarrow \boxed{A/D} \rightarrow}_{P(z)}$$

MORE ROBUST!

QUICK REVIEW: LANGUAGE & TOOLS FOR DISCRETE-TIME SIGNALS & SYSTEMS

Signals : (see above)
↳ sequences of real numbers/vectors

$u(k)$, $y(k)$, $x(k)$

~ samples
$y(k) = y(kT_s)$

Systems : (Remember continuous Time)

(1) DIFFERENCE EQUATIONS

$$\sum_{i=0}^{n} a_i \cdot y(k-i) = \sum_{i=0}^{m} b_i \, u(k-i)$$

- typically obtained from ODE by approx. - e.g.

$$\frac{d}{dt} y(t) \cong \frac{y(kT_s) - y((k-1)T_s)}{T_s}$$

- LINEAR, CAUSAL, TI, FINITE MEMORY

(2) I/O MODEL (IN $\mathbb{C}$-VARIABLE)

TRANSFER FUNCTION

CONTINUOUS $\mathcal{L}$ (UNILATERAL) $\longrightarrow$ DISCRETE $\mathcal{Z}$

$\mathcal{Z}$ - TRANSFORM :

LINEAR MAP ,

$$f(k) \longrightarrow F(z) = \mathcal{Z}[f](z) = \sum_{k=0}^{+\infty} f(k) z^{-k}$$

complex function $z \in \mathbb{C}$

Converges OUTSIDE of a circle of radius ρ

VICEVERSA :

if $F(z)$ (from $\mathbb{C}$ to $\mathbb{C}$) can be expanded in series around ∞ :

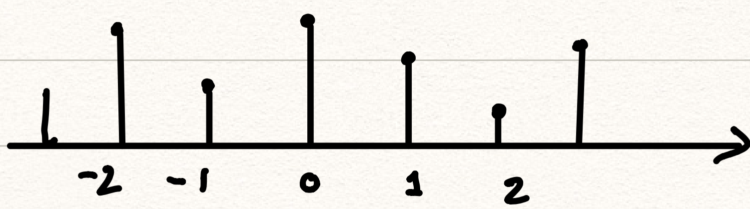
$$F(z) = \sum_{k=0}^{+\infty} f_k z^{-k}$$

I can call

$$f(k) := f_k \quad , \quad f(\cdot) = \mathcal{Z}^{-1}(F(\cdot))$$

KEY PROPERTY : if $\mathcal{Z}(f(k)) = F(z)$

(RIGHT TIME SHIFT)
$$\mathcal{Z}[f(k-1)] = z^{-1} F(z) + f(-1)$$



(LEFT TIME SHIFT)
$$\mathcal{Z}[f(k+1)] = z F(z) - z f(0)$$

IN GENERAL
$$\mathcal{Z}[f(k-r)] = z^{-r} F(z) + N_r(z^{-1})$$

$\in$ Poly. in z^{-1}

① → ②

Z - TRANSFORM the DIFFERENCE EQUATION :

$$\sum_i a_i z^{-i} Y(z) + \hat{N}(z) = \sum_i b_i z^{-i} U(z) + \emptyset$$

Assuming $U(k) = 0$ if $k < 0$

where $Y(z) = \mathcal{Z}[y(k)]$

$U(z) = \mathcal{Z}[u(k)]$

$$= A(z) Y(z) + \hat{N}(z) = B(z) U(z)$$

$$\Rightarrow Y(z) = \underbrace{\frac{B(z)}{A(z)}}_{P(z)} U(z) - \frac{\hat{N}(z)}{A(z)}$$

$P(z)$: TRANSFER FUNCTION in z.

③ STATE SPACE MODEL

$$\begin{cases} x(k+1) = A\,x(k) + B\,u(k) \\ y(k) = C\,x(k) + D\,u(k) \end{cases} \qquad X(0) = X_0$$

$x \in \mathbb{R}^n$, $y, u \in \mathbb{R}$ $(\mathbb{R}^m, \mathbb{R}^p)$

again, Z - transform & TIMESHIFT

$$z X(z) - z X(0) = A X(z) + B U(z)$$

2nd

$$Y(z) = C X(z) + D U(z)$$
$$= \left[C (zI - A)^{-1} B + D \right] U(z)$$
$$+ \; z C (zI - A)^{-1} x_0$$

▷ We showed ③ → ② ; VICEVERSA, REALIZATION. Theory

• How do we STUDY STABILITY ?
as in continuous time - MODAL ANALYSIS

I/O: Decompose $P(z) = \frac{N_p(z)}{D_p(z)}$ in simple fractions

Assume relative degree $r \geq 0$

$$\frac{N(z)}{D(z)} = z^{-r} F_0(z)$$
$$= z^{-r} \cdot z \left(\frac{F_0(z)}{z} \right)$$

p_i : poles $F_1(z)$

$$= z^{-r} \cdot z \sum_{i=0}^{q} \sum_{l=0}^{\eta_j - 1} \frac{A_{il}}{(z-p_i)^l}$$

$$= z^{-r} \sum_i \sum_l A_{il} \frac{z}{(z-p_i)^l}$$

Z^{-1}

r-STEP SHIFT

Z^{-1}

MODES

$$f_0(z) = \sum_i \sum_l \frac{A_{il}}{p_i^l} \binom{k}{l} p_i^k$$

MODES BEHAVIOR : $\sim p_i^k$

- CONVERGENT IF $|p_i| < 1$
- BOUNDED, NOT CONV. if $|p_i| = 1$, $l = 0$
- DIVERGENT OTHERWISE

$\Rightarrow$ STABILITY I/O

BIBO $\iff$ p_i of $D_p(z)$ have $|p_i| < 1$

A.S. $\iff$ p_i of $A(z)$ have $|p_i| < 1$

OBSERVE : CONT. TIME, SIMPLE MODE $y(t) = ce^{pt}$

IF SAMPLED : $y(k) = y(kT_s) = c\,e^{pT_s k}$

$\Rightarrow$ POLES are mapped as $p_{disc} = e^{T_s p_{CONT}}$

$$\mathrm{Re}(p_{CONT}) < 0 \Rightarrow |p_{disc}| < 1$$

- FOR STATE SPACE : MODES $\sim$ eigs(A)
- NEXT : We want to recall some sampling ideas...

▷ SAMPLING – Can I model it in CONT. TIME?

↳ PRODUCT WITH A TRAIN OF PULSES

$$y(t) \longrightarrow \bigotimes \longrightarrow \quad y_\delta(t) = \sum_k \underbrace{y(t)\,\delta(t-kT_s)}_{\text{EXTRACTS } y(kT_s)}$$

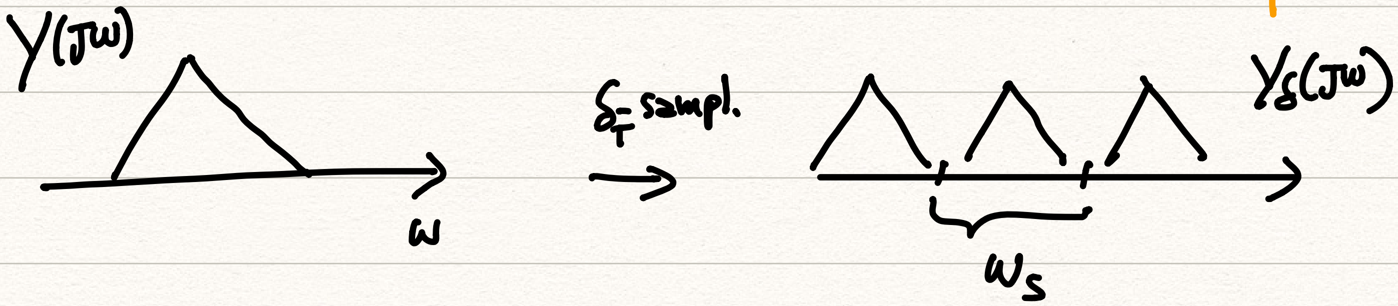
$$\downarrow \delta_T(t)$$

$$\delta_T(t) = \sum_k \delta(t-kT_s)$$

▷ USING $\mathcal{L}$ – TRANSF. PROPERTIES, $\omega_s := \frac{2\pi}{T_s}$

$$Y_\delta(s) = \frac{1}{T_s}\sum_k \mathcal{L}\left[e^{jk\omega_s t}\, y(t)\right]$$

$$= \frac{1}{T_s}\sum_k \underbrace{Y(s - jk\omega_s)}_{\text{TRANSLATED COPIES OF } Y(s)}$$



• ON THE OTHER HAND:

$$Y_\delta(s) = \int_0^{+\infty} \underline{y(\tau)\,\delta(\tau-kT_s)}\, e^{-s\tau}\, d\tau$$

RELATION:

$Y(s) \to Y_\delta(s)$

$\Updownarrow$

$Y(z)$

$$= \sum_k y(kT_s)\, e^{-sT_s k}$$

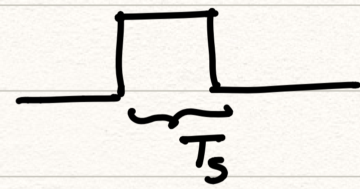
$$= \sum_k y(kT_s)\left(e^{sT_s}\right)^{-k} = \mathcal{Z}[y(k)](e^{sT_s})$$

▷ CAN I MODEL ZOH (D/A) IN CONT. TIME?

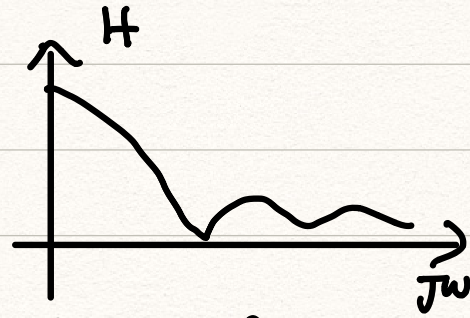
As a filter from $u(k) \approx u_\delta(t) \rightarrow \tilde{u}(t)$

CONVOLUTION WITH IMPULSE RESPONSE:

$$h_0(t) = 1(t) - 1(t - T_s)$$

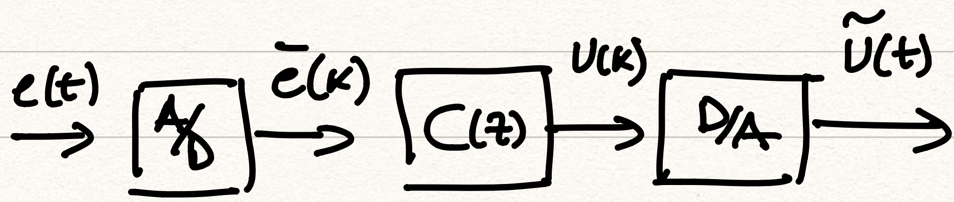


$$\Rightarrow H_0(s) = 1/s - \frac{e^{sT_s}}{s} = \frac{1 - e^{sT_s}}{s}$$

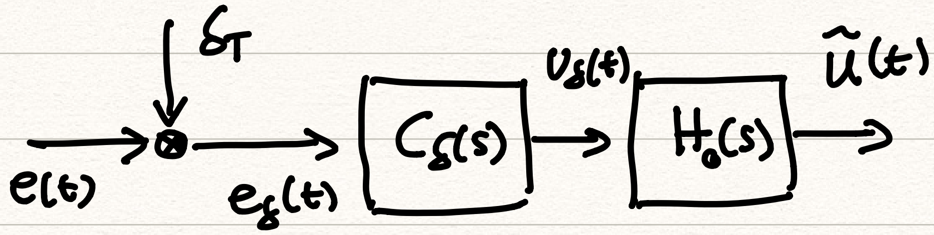


Low Pass ≈ RECONSTR. FILTER IN SAMPLING

▷ what do we do with this?



THIS IS WHAT THE PLANT "SEES"!



$$\Rightarrow \frac{\tilde{U}(s)}{E(s)} = \frac{1}{T_s} H_0(s)\, C_\delta(s) \leftrightarrow \bar{C}(z) \leftrightarrow$$

$$= \frac{1 - e^{-sT_s}}{T_s\, s} \bar{C}(e^{sT})$$