Homework 2: Inverse kinematics

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1 Introduction

In this homework, we consider a two-link planar arm with a prismatic joint and a revolute joint, whose structure is shown in the figure below. The reference systems for each of the links are also indicated in the same figure (they are assigned according to Denavit-Hartenberg convention). All relevant physical quantities are assumed to be known, and in formulating the model, as required, all forms of friction are neglected.

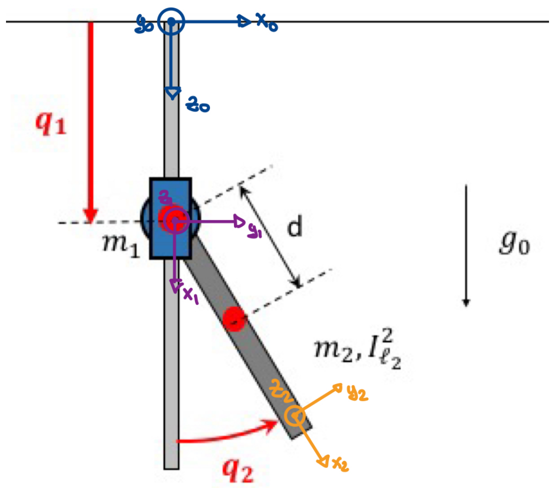

Figure 1: PR planar arm.

2 Dynamic derivation

In this first section, we are going to derive the model dynamic in the form

$$B(q)\ddot{q} + C(q, \dot{q}) + g(q) = \tau \tag{1}$$

In fig. 1 we can appreciate the reference frames we assigned to the robot links we refer to in the calculations. Notice, also, that q_1 and q_2 are the joint variables, hence

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{2}$$

First of all, we have to determine all relevant quantities that are needed in the following calculations. We have that the position of the centre of mass of the links with respect to the base frame are

$$p_{l_1} = \begin{bmatrix} 0 \\ 0 \\ q_1 \end{bmatrix} \quad p_{l_2} = \begin{bmatrix} d \sin q_2 \\ 0 \\ q_1 + d \cos q_2 \end{bmatrix} \tag{3}$$

while the rotation matrices of interest are

$$R_1 = R_1^0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad R_2^1 = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

and

$$R_2 = R_0^1 R_2^1 \begin{bmatrix} \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \\ \cos q_2 & -\sin q_2 & 0 \end{bmatrix} \tag{5}$$

Now we can start evaluating the partial Jacobians relative to the links. Since Joint 1 is prismatic and Joint 2 is revolute, we have

$$J_P^{(l_1)} = \begin{bmatrix} J_{P_1}^{(l_1)} & 0 \end{bmatrix} = \begin{bmatrix} z_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{6}$$

$$J_O^{(l_1)} = \begin{bmatrix} J_{0_1}^{(l_1)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

and

$$J_P^{(l_2)} = \begin{bmatrix} J_{P_1}^{(l_2)} & J_{P_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} z_0 & z_1 \times (p_{l_1} - p_1) \end{bmatrix} = \begin{bmatrix} 0 & d \cos q_2 \\ 0 & 0 \\ 1 & -d \sin q_2 \end{bmatrix} \tag{8}$$

$$J_O^{(l_2)} = \begin{bmatrix} J_{0_1}^{(l_2)} & J_{0_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} 0 & z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{9}$$

The following step is to evaluate the total kinetic energy of the system, by exploiting the relation

$$\mathcal{T} = \frac{1}{2} \dot{q}^T B(q) \dot{q} \tag{10}$$

where

$$B(q) = \sum_{i=1}^{2} m_i {J_{P_i}^{(l_i)}}^T J_{P_i}^{(l_i)} + {J_{O_i}^{(l_i)}}^T R_i I_{l_i}^{l_i} R_i^T J_{O_i}^{(l_i)} \tag{11}$$

Let's evaluate the various contributions. Considering Joint 1 and the fact that $J_{O_1}^{(l_1)} = \mathbf{0}$, it follows

$$m_1 {J_{P_1}^{(l_1)}}^T J_{P_1}^{(l_1)} = m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{12}$$

and

$$J_{O_1}^{(l_1)^T} R_1 I_{l_1}^{l_1} R_1^T J_{O_1}^{(l_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{13}$$

Looking at Joint 2, we have that

$$m_2 J_{P_2}^{(l_2)^T} J_{P_2}^{(l_2)} = m_2 \begin{bmatrix} 1 & -d\sin q_2 \\ -d\sin q_2 & d^2 \end{bmatrix} \tag{14}$$

and

$$J_{O_2}^{(l_2)^T} R_2 I_{l_2}^{l_2} R_2^T J_{O_2}^{(l_2)} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{I} \end{bmatrix} \tag{15}$$

Where $\bar{I} = I_{l_2}^2$ (the only component of the inertia tensor that matters). Summing up all the contributions, it follows that the **inertia matrix** is given by

$$\boxed{B(q) = \begin{bmatrix} m_1 + m_2 & -dm_2\sin q_2 \\ -dm_2\sin q_2 & d^2 m_2 + \bar{I} \end{bmatrix}} \tag{16}$$

Now we have to derive the matrix $C(q, \dot{q})$ and we do this step by exploiting *Christoffel symbols* of the first type. Applying the definition we have that

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right) \tag{17}$$

and

$$c_{ij} = \sum_k c_{ijk}\dot{q}_k \tag{18}$$

Here we report the all the intermediate results, only highlighting the non-zero terms:

$$c_{111} = 0 \tag{19}$$

$$c_{112} = 0 \tag{20}$$

$$c_{121} = 0 \tag{21}$$

$$c_{122} = \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_2}\right) = -dm_2\cos q_2 \tag{22}$$

$$c_{211} = 0 \tag{23}$$

$$c_{212} = \frac{1}{2}\left(\frac{\partial b_{21}}{\partial q_2} - \frac{\partial b_{12}}{\partial q_2}\right) = 0 \tag{24}$$

$$c_{221} = \frac{1}{2}\left(\frac{\partial b_{21}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_2}\right) = 0 \tag{25}$$

$$c_{221} = 0 \tag{26}$$

Putting results (19)-(26) together, the final matrix is

$$\boxed{C(q, \dot{q}) = \begin{bmatrix} 0 & -dm_2\dot{q}_2\cos q_2 \\ 0 & 0 \end{bmatrix}} \tag{27}$$

It is easy to verify that, with such choice of $B(q)$, $C(q,\dot{q})$, the matrix

$$N(q,\dot{q}) = \dot{B}(q) - 2C(q,\dot{q}) = \begin{bmatrix} 0 & dm_2\dot{q}_2\cos q_2 \\ -dm_2\dot{q}_2\cos q_2 & 0 \end{bmatrix} \tag{28}$$

is skew symmetric (this is granted by the choice of $C(q,\dot{q})$ given by Christoffel symbols).

Finally we have to evaluate the gravitational terms of the links. We remember that

$$g_i(q) = \frac{\partial \mathcal{U}}{\partial q_i} = -\sum_{j=0}^{2} m_j g_0^T J_{P_j}^{(l_j)} \tag{29}$$

where, in our case, $g_0^T = \begin{bmatrix} 0 & 0 & g \end{bmatrix}$. By substitution if follows that

$$g_1(q) = \begin{bmatrix} -gm_1 \\ 0 \end{bmatrix} \tag{30}$$

and

$$g_2(q) = \begin{bmatrix} -gm_2 \\ dgm_2\sin q_2 \end{bmatrix} \tag{31}$$

In conclusion we have that the **gravitational term** is given by

$$\boxed{g(q) = \begin{bmatrix} -g(m_1+m_2) \\ dgm_2\sin q_2 \end{bmatrix}} \tag{32}$$

Summarizing all the matrices found till now, we have that the **model** in the required form is given by

$$\begin{bmatrix} m_1+m_2 & -dm_2\sin q_2 \\ -dm_2\sin q_2 & d^2m_2+\bar{I} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & -dm_2\dot{q}_2\cos q_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} -g(m_1+m_2) \\ dgm_2\sin q_2 \end{bmatrix} = \tau$$

3 PD with gravity compensation

As seen during *lecture 25*, in order to guarantee global asymptotic stability to the system, we can exploit a PD controller with constant gravity compensation. Such controller can be expressed in the following form

$$u = g(q_d) + K_P\,(q_d - q) - K_D\dot{q} \tag{33}$$

By substituting the final desired position q_d with the given value $(0,\pi)$ we obtain

$$u = \begin{bmatrix} -g(m_1+m_2) \\ 0 \end{bmatrix} + K_P\begin{bmatrix} -q_1 \\ \pi - q_2 \end{bmatrix} - K_D\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Now, assuming that K_D, K_P are diagonal matrices, in order to guarantee asymptotic stability, they must assume some minimum values. First, we have to impose that

$$K_D > 0$$

Secondly, we have some extra constraints on K_P, to guarantee robustness with respect to approximations of the gravitational term. Since $g(q)$ contains only trigonometric or linear terms in q, the following structural properties holds

$$\exists \alpha > 0 \ : \left\| \frac{\partial g}{\partial q} \right\| \leqslant \alpha \ \forall q \tag{34}$$

By substitution, it easily follows that

$$\left\| \frac{\partial g}{\partial q} \right\| = \left\| \begin{bmatrix} 0 & 0 \\ 0 & dgm_2 \cos q_2 \end{bmatrix} \right\| = \max\{0, dgm_2 \cos q_2\} \leqslant dgm_2 = \bar{\alpha} \tag{35}$$

Hence, to *guarantee global asymptotic stability* we must chose K_P such that

$$K_P > \bar{\alpha} I_{2\times 2}$$

4 Parameter estimation

By doing a few calculations, we can provide a linear parametrization of the dynamic model of the type

$$Y\left(q, \dot{q}, \ddot{q}\right) \pi = \tau \tag{36}$$

where π has the form reported in the homework assignment. In fact, just by imposing

$$Y\left(q, \dot{q}, \ddot{q}\right) = \begin{bmatrix} \ddot{q}_1 - g & -\ddot{q}_2 \sin q_2 - \dot{q}_2{}^2 \cos q_2 & 0 \\ 0 & g \sin q_2 - \ddot{q}_1 \sin q_2 & \ddot{q}_2 \end{bmatrix}$$

we can say that such parametrization exists.

5 Adaptive controller

In order to obtain an adaptive control law, we need to define some extra quantities. First of of all we define a new reference

$$\dot{q}_r = \dot{q}_d + \Lambda(q_d - q) \tag{37}$$

where Λ is a suitable positive definite square matrix. Furthermore, we also define

$$\sigma = (\dot{q}_d - \dot{q}) + \Lambda(q_d - q) \tag{38}$$

At this point we are ready to define the desired **control law**, which is in the form

$$u = Y\left(q, \dot{q}, \dot{q}_r, \ddot{q}_r\right) \hat{\pi} + K_D \sigma$$

where K_D is a positive definite square matrix and $\hat{\pi}$ represents the available estimate on the parameter, governed according to the following **update rule**

$$\dot{\hat{\pi}} = K_\pi^{-1} Y^T\left(q, \dot{q}, \dot{q}_r, \ddot{q}_r\right) \sigma$$

In this case we can chose K_π as a positive definite matrix (by changing its value we can modify the estimator performances). Now we just have to specify the structure of $Y\left(q,\dot{q},\dot{q}_r,\ddot{q}_r\right)$, that is different form the one seen in the previous section. In fact, after some calculation, it follows that

$$Y\left(q,\dot{q},\dot{q}_r,\ddot{q}_r\right)=\begin{bmatrix} \ddot{q}_{1r}-g & -\ddot{q}_{2r}\sin q_2-\dot{q}_{2r}\dot{q}_2\cos q_2 & 0 \\ 0 & g\sin q_2-\ddot{q}_{1r}\sin q_2 & \ddot{q}_{2r} \end{bmatrix} \tag{39}$$

Such matrix is obtained by doing similar calculations to the ones of the previous section in which we obtained $Y(q,\dot{q},\ddot{q})$. By applying the control and the update law we get a possible adaptive control law. By acting on the different parameters, such as K_D, K_π and Λ we can tune the performances of our adaptive control law in order to satisfy the desired requirements.