FEM in plates

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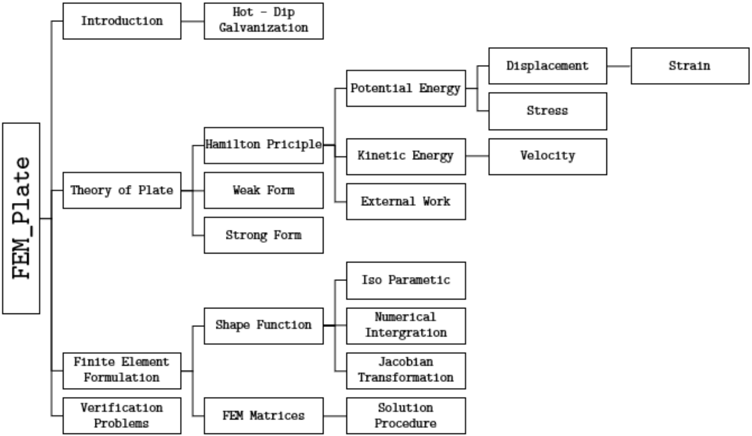
Plan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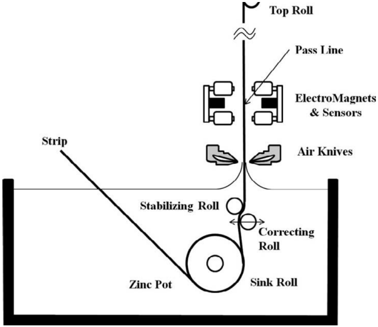
Introduction

Hot-Dip Galvanization Process



- A thin Layer of Zinc is coated to Increase the corrosion resistance of steel
- Air knives control the thickness of the Zinc layer
- Excessive Vibration results in uneven coating.
- Electromagnets are used to control vibration of the strip.

Need For Finite Element Modeling

- Complex behavior of the metal strip.
- Two dimensional domain and Three Dimensional Displacement field.
- Complex and multiple boundary condition.
- Free Control over discretization of the domain.
- Intuitive Solution Procedure.

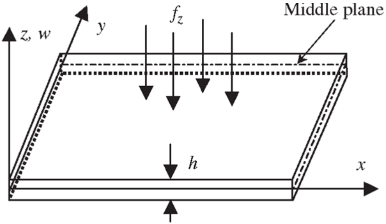
Plan

Theory of Plates

A plate is a flat solid with uniform and smaller thickness than its other dimensions.A middle plane (Z=0) is equidistant from upper and lower faces.

Assumptions

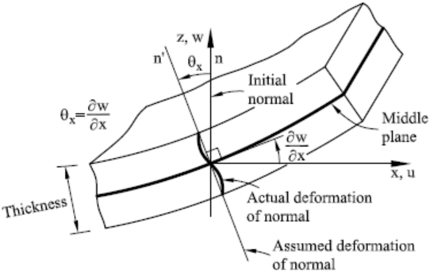
- A point in the middle plane only moves vertically $u = 0$ and $v = 0$
- Thickness does not change during deformation.
- σ_{33} is neglected (plane stress is assumed)



Theory of Plates

Kirchhoff Plate Theory

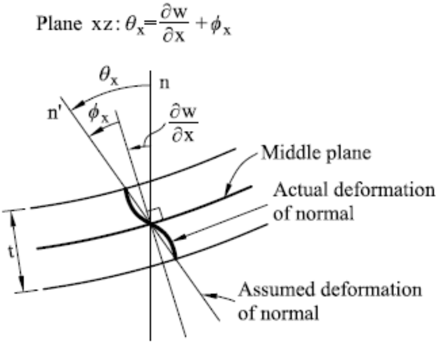
- A line normal to the undeformed middle plane remains straight and normal after deformation.
- Only for **Thin Plates** where $t/a \leq 0.1$.
- σ_{23} and σ_{13} are neglected.



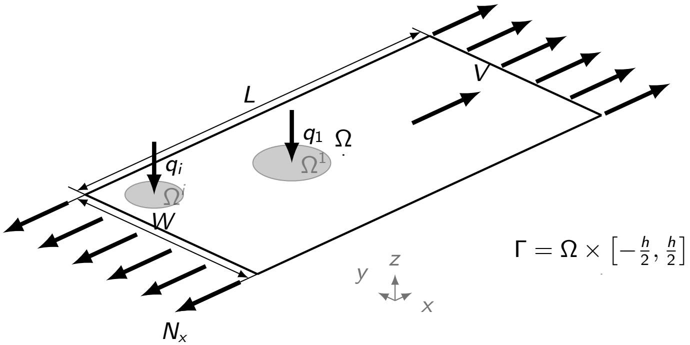
Theory of Plates

Reissner-Mindlin Plate Theory

- A line normal to the undeformed middle plane remains straight and not necessarily normal after deformation.
- For both **Thin and Thin Plates**.
- σ_{23} and σ_{13} are not neglected.



Description of Domain



$$\Omega \in \{x, y\}$$

$$\Gamma \in \{\Omega \times z\}$$

$$z \in \left\{-\frac{t}{2}, \frac{t}{2}\right\}$$

Hamilton principle

Hamilton principle is used to derive the equation of motion.

$$
\begin{aligned}
H &= \int_{t_0}^{t_1} (T - V + W)\, dt \\
\delta H &= \int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt &= 0 \\
\delta u\Big|_{t_0}^{t_1} & &= 0
\end{aligned}
$$

T is the kinetic energy, V in the potential energy and W is the work done to the system.

Potential Energy

$$V = \frac{1}{2}\int\int\int_{\Gamma} \epsilon^{T} \sigma d\Gamma$$

$$V = \frac{1}{2}\int\int\int_{\Gamma} \left(\epsilon^{B}\right)^{T} \sigma^{B} + \left(\epsilon^{S}\right)^{T} \sigma^{S} + \left(\epsilon^{A}\right)^{T} \sigma^{A} d\Gamma$$

$$V = \frac{1}{2}\int\int_{\Omega}\int_{-t/2}^{+t/2} \left(\epsilon^{B}\right)^{T} \sigma^{B} + \left(\epsilon^{S}\right)^{T} \sigma^{S} + \left(\epsilon^{A}\right)^{T} \sigma^{A} dz d\Omega$$

ϵ^{B} is the stress due to bending, ϵ^{S} is the stress due to shear deformation and ϵ^{A} is the axial stress.

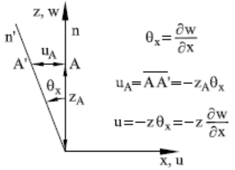
Kinematics

$$u_1(x,y,z,t) = u(x,y,t) - z\theta_x(x,y,t) \qquad \theta_x = \frac{\partial w}{\partial x} + \phi_x$$

$$u_2(x,y,z,t) = v(x,y,t) - z\theta_y(x,y,t) \qquad \theta_y = \frac{\partial w}{\partial y} + \phi_y$$

$$u_3(x,y,z,t) = w(x,y,t)$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_x \\ \theta_y \\ w \end{Bmatrix} = [Z]\,\tilde{u}$$



Strain Definition

The Green - Lagrange strain tensor is given as

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \qquad i,j \in 1,2,3$$

The E_{11} component of the strain tensor is found as

$$E_{11} = -z\frac{\partial w^2}{\partial x^2} + \frac{1}{2}\left(\left[z\frac{\partial \theta_x}{\partial x}\right]^2 + z^2\frac{\partial \theta_y}{\partial x}\frac{\partial \theta_y}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial x}\right)$$

$$E_{11} = -z\frac{\partial w^2}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$E_{i,j} = \begin{bmatrix} -z\dfrac{\partial w^2}{\partial x^2} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 & -z\dfrac{\partial w^2}{\partial x \partial y} + \dfrac{1}{2}\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} - \theta_x\right) \\ & -z\dfrac{\partial w^2}{\partial y^2} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \theta_y\right) \\ symm. & & 0 \end{bmatrix}$$

$$\epsilon^B_{\alpha\beta} = -z \begin{bmatrix} \dfrac{\partial w^2}{\partial x^2} \\ \dfrac{\partial w^2}{\partial y^2} \\ \dfrac{\partial w^2}{\partial x \partial y} \end{bmatrix} = -z\kappa \quad \alpha, \beta \in 1, 2$$

$\kappa = \nabla w$ is the curvature of of a plane.

$$\epsilon^{A}_{\alpha\beta} = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial x}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \left(\frac{\partial w}{\partial y}\right)^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}$$

$$\epsilon^{S}_{3\alpha} = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial x} - \theta_x \\ \frac{\partial w}{\partial y} - \theta_y \end{bmatrix}$$

For Kirchhoff plate

$$\epsilon^{S}_{3\alpha} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For Reissner - Mindlin Plate

$$\epsilon^{S}_{3\alpha} = \frac{1}{2}\begin{bmatrix} -\phi_x \\ -\phi_y \end{bmatrix}$$

Constitute law

For the **linear isotropic** material is considered. Since σ_{33} is not considered the **plane stress** case is considered and the stress - strain relation is given as

$$\sigma^B_{\alpha\beta} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix} = \frac{1}{1-\nu^2} \begin{bmatrix} E & \nu E & 0 \\ \nu E & E & 0 \\ 0 & 0 & (1-\nu^2)G \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

E is the Young's modulus, ν is the Poisson's ratio and G is the shear modulus which is given by $G = E/1 + \nu$.

$$\sigma^B_{\alpha\beta} = \mathbf{D}\epsilon^B_{\alpha\beta}$$

The shear stress and strain relation from the 3D constitutive law is given as

$$\sigma_{3\alpha}^{S} = \begin{bmatrix} 2\sigma_{31} \\ 2\sigma_{32} \end{bmatrix} = G \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\epsilon_{31} \\ 2\epsilon_{32} \end{bmatrix} = \mathbf{D_c} \sigma_{3\alpha}^{S}$$

Axial is stress is provided and it is considered as constant and uniform over the domain.

$$\sigma_{\alpha\beta}^{A} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} = N$$

$$V=\frac{1}{2}\int\int_{\Omega}\int_{-t/2}^{+t/2}\left(\epsilon^{B}\right)^{T}\sigma^{B}+\left(\epsilon^{S}\right)^{T}\sigma^{S}+\left(\epsilon^{A}\right)^{T}\sigma^{A}dzd\Omega$$

$$\begin{aligned}V=\frac{1}{2}\int\int_{\Omega}\left[\int_{-t/2}^{+t/2}z^{2}dz\right]\kappa^{T}D\kappa+\left[\int_{-t/2}^{+t/2}dz\right]\left(\tilde{\epsilon}^{S}\right)^{T}D_{c}\tilde{\epsilon}^{S}\\+\left[\int_{-t/2}^{+t/2}dz\right]\left(\tilde{\epsilon}^{A}\right)^{T}\sigma^{A}d\Omega\end{aligned}$$

$$V=\frac{1}{2}\int\int_{\Omega}\kappa^{T}\tilde{D}\kappa+\left(\tilde{\epsilon}^{S}\right)^{T}\tilde{D}_{c}\tilde{\epsilon}^{S}+\left(\tilde{\epsilon}^{A}\right)^{T}\tilde{\sigma}^{A}d\Omega$$

$$\tilde{D}=\frac{t^{3}}{12}D\quad\tilde{D}_{c}=tD_{c}\quad\tilde{\sigma}^{A}=t\sigma^{A}$$

$$\delta V = \int\int_{\Omega} \kappa^T \tilde{D} \delta\kappa + \left(\tilde{\epsilon}^S\right)^T \tilde{D}_c \delta\tilde{\epsilon}^S + \frac{1}{2}\left(\delta\tilde{\epsilon}^A\right)^T \tilde{\sigma}^A d\Omega$$

$$\frac{1}{2}\left(\delta\tilde{\epsilon}^A\right)^T \tilde{\sigma}^A = w_{,\alpha}\tilde{\sigma}^A \delta w_{,\alpha}$$

Variation of Total Potential Energy

$$\delta V = \int\int_{\Omega} \kappa^T \tilde{D} \delta\kappa + \left(\tilde{\epsilon}^S\right)^T \tilde{D}_c \delta\tilde{\epsilon}^S + w_{,\alpha}\tilde{\sigma}^A \delta w_{,\alpha} d\Omega$$

Kinetic Energy

The kinetic of a material is given as

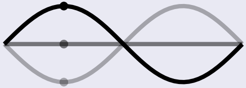
$$T = \frac{1}{2}\int\int\int_{\Gamma} \mathbf{v}^T \rho \mathbf{v} d\Gamma$$

For plate, the integration along thickness is done now.

$$T = \frac{1}{2}\int\int_{\Omega}\left[\int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{v}^T \rho \mathbf{v} dz\right] d\Omega$$

Description of velocity

Lagrangian

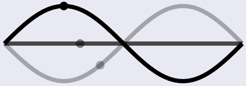


- Material Point moves along with spatial point
- Used for solids

Eulerian

- Material Point moves but spatial point stays
- Used for fluids

Mixed



- Both point moves independently.
- Moving Material

Material Derivative

$$\frac{d(\circ)}{dt} = \frac{\partial(\circ)}{\partial t} + V_i \cdot (\circ)_{,i}$$

$$v_i = \dot{u}_i + V_1 u_{i,1}$$

First the integration along thickness is done

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{v}^T \rho \mathbf{v} dz = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\rho \dot{u}_i \dot{u}_i + 2\rho V_1 \dot{u}_i u_{i,1} + \rho V_1^2 u_{i,1} u_{i,1} \right) dz$$

$$= \rho \dot{\tilde{u}}_i Z_{ij} \dot{\tilde{u}}_i + 2\rho V_1 \dot{\tilde{u}}_i Z_{ij} \tilde{u}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \tilde{u}_{j,1}$$

substituting it in the kinetic energy equation

$$T = \frac{1}{2} \int \int_{\Omega} \left(\rho \dot{\tilde{u}}_i Z_{ij} \dot{\tilde{u}}_i + \rho V_1 \dot{\tilde{u}}_i Z_{ij} \tilde{u}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \tilde{u}_{j,1} \right) d\Omega$$

Variations of Kinetic Energy

$$\delta T = \int \int_{\Omega} \rho \dot{\tilde{u}}_i Z_{ij} \delta \dot{\tilde{u}}_i + \rho V_1 \delta \dot{\tilde{u}}_i Z_{ij} \tilde{u}_{j,1}$$
$$+ \rho V_1 \dot{\tilde{u}}_i Z_{ij} \delta \tilde{u}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \delta \tilde{u}_{j,1} d\Omega$$

$$Z_{ij} = \begin{bmatrix} \frac{t^3}{12} & 0 & 0 \\ 0 & \frac{t^3}{12} & 0 \\ 0 & 0 & t \end{bmatrix}$$

External Work

more summation and write them separately for variations.

$$W = \sum_{i}^{nb} W_i = \sum_{i}^{nb} \int_{\Omega_i} q_i \mathbf{u_i} d\Omega_i$$

Variation of the external work

$$\delta W = \sum_{i}^{nb} \int_{\Omega_i} q_i \delta \mathbf{u_i} d\Omega_i$$

The Hamilton principle

Substituting Everything in Hamilton principle gives

$$\int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt = 0$$

$$\int \int_{\Omega} +\rho \dot{\tilde{u}}_i Z_{ij} \delta \tilde{u}_i + \rho V_1 \delta \tilde{u}_i Z_{ij} \tilde{u}_{j,1} + \rho V_1 \tilde{u}_i Z_{ij} \delta \tilde{u}_{j,1} d\Omega \Big|_{t_0}^{t_1}$$

$$\int_{t_0}^{t_1} \int \int_{\Omega} -\rho \ddot{\tilde{u}}_i Z_{ij} \delta \tilde{u}_i - \rho V_1 \delta \tilde{u}_i Z_{ij} \dot{\tilde{u}}_{j,1} - \rho V_1 \tilde{u}_i Z_{ij} \delta \dot{\tilde{u}}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \delta \tilde{u}_{j,1}$$

$$-\kappa^T \tilde{D} \delta \kappa - \left(\epsilon^S\right)^T \tilde{D}_c \delta \epsilon^S - w_{,\alpha} \tilde{\sigma}^A \delta w_{,\alpha} d\Omega$$

$$+ \sum_{i}^{nb} \int \int_{\Omega_i} q_i \delta \mathbf{u_i} d\Omega_i dt = 0$$

Weak Form

$\int_{t_0}^{t_1} \chi dt = 0$ For this to be true $\chi = 0$ must also be true

Final Weak Form

$$\int \int_{\Omega} \rho \ddot{\tilde{u}}_i Z_{ij} \delta \tilde{u}_i + 2\rho V_1 \delta \tilde{u}_i Z_{ij} \dot{\tilde{u}}_{j,1} - \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \delta \tilde{u}_{j,1}$$

$$+\kappa^T \tilde{D} \delta \kappa + \left(\epsilon^S\right)^T \tilde{D}_c \delta \epsilon^S + w_{,\alpha} \tilde{\sigma}^A \delta w_{,\alpha} d\Omega$$

$$= \sum_{i}^{nb} \int \int_{\Omega_i} q_i \delta \mathbf{u_i} d\Omega_i dt$$

Weak Form

$\int\int_{\Omega} \chi d\Omega = 0$ For this to be true $\chi = 0$ must also be true and by considering $\epsilon^S = 0$, $\tilde{\sigma}^A = N_x$ and $\tilde{u}_1 = \tilde{u}_2 = 0$ we get the strong form.
In most cases a single single force distributed along the area is considered, which is $nb = 1$, $q_i = F$ and $\Omega_i = \Omega$

Final strong Form

$$\rho t \left(\frac{\partial^2 w}{\partial t^2} + 2V_1 \frac{\partial^2 w}{\partial x \partial t} - V_1^2 \frac{\partial^2 w}{\partial x^2} \right) + D\nabla^4 w + N_x t \frac{\partial^2 w}{\partial x^2} = F$$

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial Y^4} \qquad D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

Plan

define the domains! omega gamma d omega omega 1 omega 2 omega i

Shape function of a rectangular element

$$w = \sum_{i=1}^{nN} \left(N_i w_i + \overline{N}_i \theta_{x_i} + \overline{\overline{N}}_i \theta_{y_i} \right)$$

$$N_1 = \frac{1}{4ab} (1-x)(1-y)$$

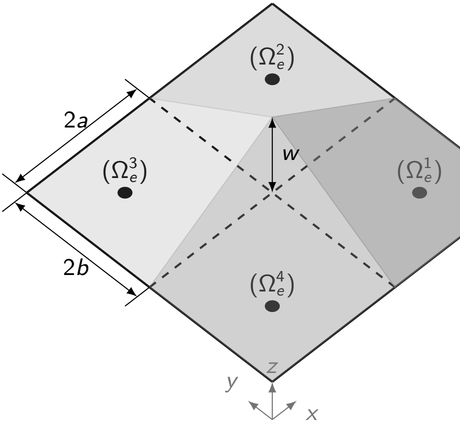
$$N_2 = \frac{1}{4ab} (1+x)(1-y)$$

$$N_3 = \frac{1}{4ab} (1+x)(1+y)$$

$$N_4 = \frac{1}{4ab} (1-x)(1+y)$$

For Ressiner Mindlin element

$$N_i = \overline{N}_i = \overline{\overline{N}}_i$$



Representation of Displacements and Strains in terms of Shape Function.

$$\tilde{\mathbf{u}} = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_{nN} & 0 & 0 \\ 0 & \overline{N}_1 & 0 & \cdots & 0 & \overline{N}_{nN} & 0 \\ 0 & 0 & \overline{\overline{N}}_1 & \cdots & 0 & 0 & \overline{\overline{N}}_{nN} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ w_{nN} \\ \theta_{x_{nN}} \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{N}\tilde{\mathbf{u}}^e$$

similarly

$$\delta\tilde{\mathbf{u}} = \mathbf{N}\delta\tilde{\mathbf{u}}^e \qquad \dot{\tilde{\mathbf{u}}} = \mathbf{N}\dot{\tilde{\mathbf{u}}}^e \qquad \ddot{\tilde{\mathbf{u}}} = \mathbf{N}\ddot{\tilde{\mathbf{u}}}^e \tag{1}$$

Representation of Strains in terms of Shape Function.

$$\kappa = \begin{bmatrix} 0 & \overline{N}_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & \overline{\overline{N}}_{1,2} & \cdots & \overline{\overline{N}}_{nN,2} \\ 0 & \overline{N}_{1,2} & \overline{\overline{N}}_{1,1} & \cdots & \overline{\overline{N}}_{nN,1} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ \theta_{y_{Nn}} \end{Bmatrix} = \mathbf{B}\tilde{\mathbf{u}}^e$$

similarly

$$\tilde{\epsilon}^S = \begin{bmatrix} N_{1,1} & \overline{N}_1 & 0 & \cdots & 0 \\ N_{1,2} & 0 & \overline{\overline{N}}_1 & \cdots & \overline{\overline{N}}_{nN} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ \theta_{y_{Nn}} \end{Bmatrix} = \mathbf{B_S}\tilde{\mathbf{u}}^e \tag{2}$$

$$\tilde{w}_{1,\alpha} = \begin{bmatrix} N_{1,1} & 0 & 0 & N_{2,1} & \cdots & 0 \\ N_{1,2} & 0 & 0 & N_{2,2} & \cdots & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \vdots \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{H_A}\tilde{\mathbf{u}}^e$$

similarly

$$\tilde{w}_{\alpha,1} = \begin{bmatrix} N_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & \overline{N}_{2,1} & 0 & \cdots & 0 \\ 0 & 0 & \overline{\overline{N}}_{3,1} & \cdots & \overline{\overline{N}}_{3,3} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{H_v}\tilde{\mathbf{u}}^e$$

The FE Matrix for the body force is given as

$$\tilde{w} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \vdots \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{N_f} \tilde{\mathbf{u}}^e$$

Weak Form to FE format

The Finite Element Matrix equation is given as

$$\int\int_{\Omega}\left(\rho\,[\mathbf{N}]\,[\mathbf{Z}]\,[\mathbf{N}]\,\{\ddot{\tilde{\mathbf{u}}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}+\left(2\rho V_1\,[\mathbf{N}]\,[\mathbf{Z}]\,[\mathbf{H_v}]\,\{\dot{\tilde{\mathbf{u}}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}$$
$$-\left(\rho V_1^2\,[\mathbf{H_v}]\,[\mathbf{Z}]\,[\mathbf{H_v}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}+\left([\mathbf{B}]\left[\tilde{\mathbf{D}}\right][\mathbf{B}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}$$
$$+\left([\mathbf{B_S}]\left[\tilde{\mathbf{D}}_{\mathbf{S}}\right][\mathbf{B_S}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}+\left([\mathbf{H_A}]\left[\tilde{\mathbf{N}}_{\mathbf{A}}\right][\mathbf{H_A}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}d\Omega$$
$$=\sum_{i}^{nb}\int\int_{\Omega_i}\left(q_i\left[\tilde{\mathbf{N}}_{\mathbf{f}}\right]\right)\delta\tilde{\mathbf{u}}^{e}d\Omega_i$$

FEM matrices

After rearranging them to their respective groups we get.

$$[\mathbf{M}]\{\ddot{\mathbf{u}}\} + [\mathbf{C}]\{\dot{\mathbf{u}}\} + [\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$$

where

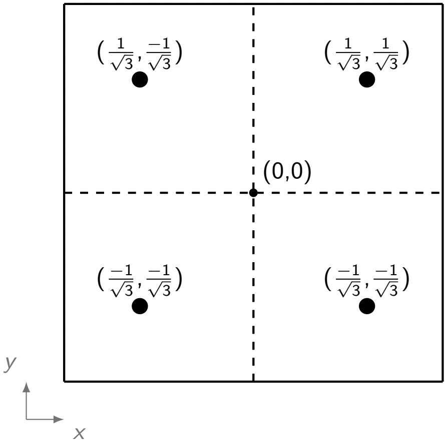
$$[\mathbf{M}] = \rho \int\int_{\Omega} ([\mathbf{N}][\mathbf{Z}][\mathbf{N}])\, d\Omega$$

$$[\mathbf{C}] = 2\rho V_1 \int\int_{\Omega} ([\mathbf{N}][\mathbf{Z}][\mathbf{H_v}])\, d\Omega$$

$$[\mathbf{K}] = -\rho V_1^2 \int\int_{\Omega} ([\mathbf{H_v}][\mathbf{Z}][\mathbf{H_v}])\, d\Omega + \int\int_{\Omega} [\mathbf{B}]\left[\tilde{\mathbf{D}}\right][\mathbf{B}]\, d\Omega$$
$$+ \int\int_{\Omega} [\mathbf{B_S}]\left[\tilde{\mathbf{D}}_{\mathbf{S}}\right][\mathbf{B_S}]\, d\Omega + \int\int_{\Omega} [\mathbf{H_A}]\left[\tilde{\mathbf{N}}_{\mathbf{A}}\right][\mathbf{H_A}]\, d\Omega$$

$$\{\mathbf{F}\} = \sum_{i}^{nb} \int\int_{\Omega_i} q_i \left[\tilde{\mathbf{N}}_{\mathbf{f}}\right] d\Omega_i$$

Gauss Quadrature

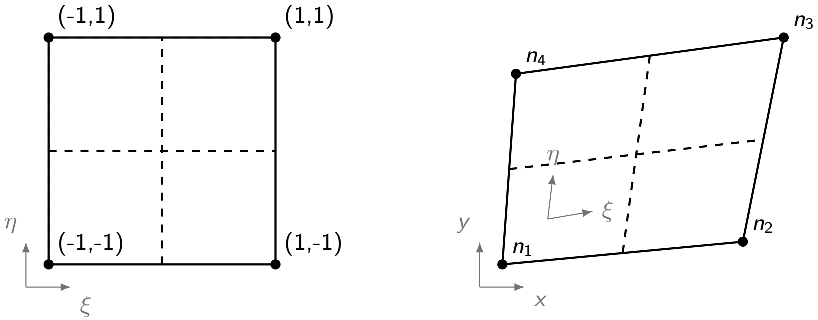


●Gauss Integration points

$$\int\int f(x,y)dxdy = \sum_{i=1}^{nx}\sum_{j=1}^{ny} w_i w_j \cdot f(ix, jy)$$

$$w_i = w_j = 1$$

Iso parametric Shape Function



$$N_1 = \frac{1}{4}(-\xi, -\eta) \quad N_2 = \frac{1}{4}(\xi, -\eta)$$
$$N_3 = \frac{1}{4}(-\xi, \eta) \quad N_4 = \frac{1}{4}(\xi, \eta)$$

Jacobian Transform

$$\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\left\{\begin{array}{c} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{array}\right\} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \left\{\begin{array}{c} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{array}\right\} \quad J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{3}$$

$$\left\{\begin{array}{c} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{array}\right\} = J^{-1} \left\{\begin{array}{c} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{array}\right\} \tag{4}$$

Final FEA matrix

$$[\mathbf{M^e}] = \sum_{i=1}^{ng} \rho \left(w_i \, [\mathbf{N(i)}]^T \, [\mathbf{Z}] \, [\mathbf{N(i)}] \, det(J) \right) d\Omega$$

All the Element mass Matrices $[\mathbf{M^e}]$ are assembled in the final Mass Matrix $[\mathbf{M}]$

Solving them

Modal Analysis

$$\mathbf{M}\ddot{\tilde{\mathbf{x}}} + \mathbf{K}\tilde{\mathbf{x}} = 0$$

$$\tilde{\mathbf{x}} = \overline{\mathbf{x}}e^{i\omega t}$$

$$\left(\mathbf{K} - \omega^2\mathbf{K}\right)\overline{\mathbf{x}} = 0$$

ω is the natural frequency and $\overline{\mathbf{x}}$ is the natural mode. In MATLAB, [V,D]=eig(K,M) function is used to do the modal analysis.

Time Integration

Newmark algorithm

$$R = F_t + \mathbf{M}\left(a_0 u_t + a_2 v_t + a_3 a_t\right) + \mathbf{C}\left(a_1 u_t + a_4 v_t + a_5 a_t\right)$$

$$u_{t+1} = \left[a_0\mathbf{M} + a_1\mathbf{C} + \mathbf{K}\right]^{-1} R$$

$$v_{t+1} = a_1\left(u_{t+1} - u_t\right) - a_4 v_t - a_5 a_t$$

$$a_{t+1} = a_0\left(u_{t+1} - u_t\right) - a_2 v_t - a_3 a_t$$

Integration parameter

$$a_0 = \frac{1}{\alpha h^2} \qquad a_1 = \frac{\theta}{\alpha h} \qquad a_2 = \frac{1}{\alpha h}$$

$$a_3 = \frac{1}{2\alpha} - 1 \qquad a_4 = \frac{\theta}{\alpha} \qquad a_5 = \frac{h}{2}\frac{\theta}{\alpha} - 2$$

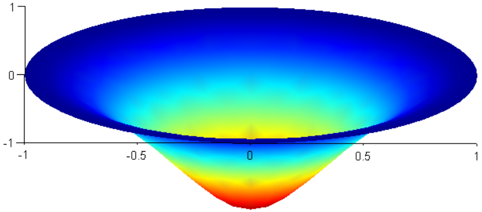
Unconditionally Stable for

$$\theta \geq \frac{1}{2}$$

$$\alpha \geq \frac{1}{4}\left(\frac{1}{2} + \theta\right)^2$$

Plan

TIM69

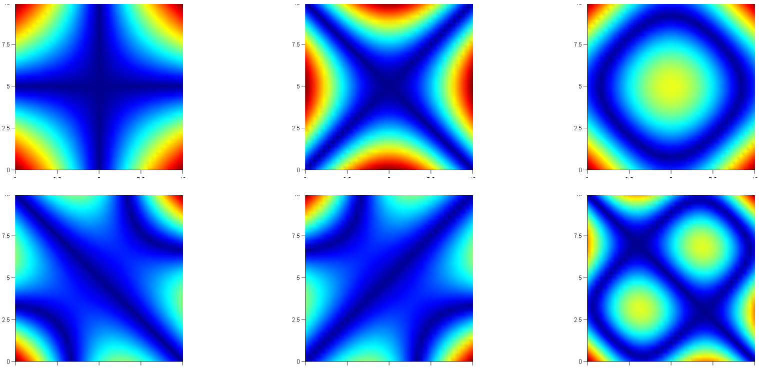


Reference

S.Timoshenko , S . Woinowsky , Theory of Plates and Shells , pg:69, Article : 19 .

Error % = 1.27 %.

VMP09

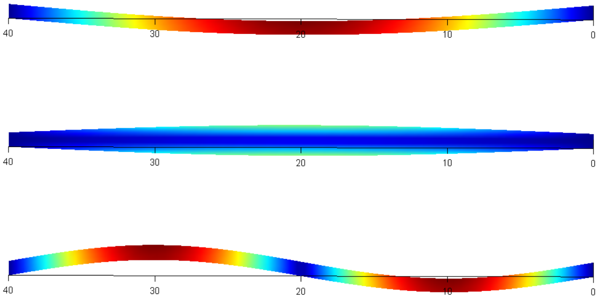


Reference

NAFEMS Manual. Solution Retrieved from Ansys verification problem (VMP09-T12).

Error % = 0.32 %.

NAS227

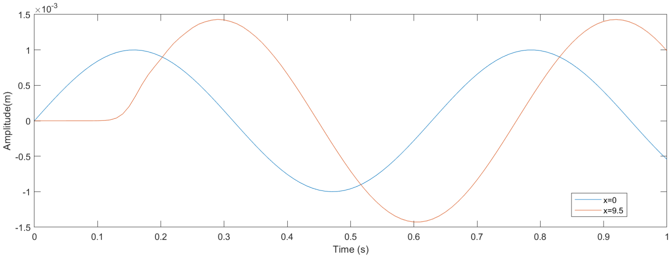


Reference

Arthur W.Leissa ,Vibration of Plates,NASA SP-160, pg:277, Ch:10.2.

Error % $=$ 0.01 %.

Wave Speed

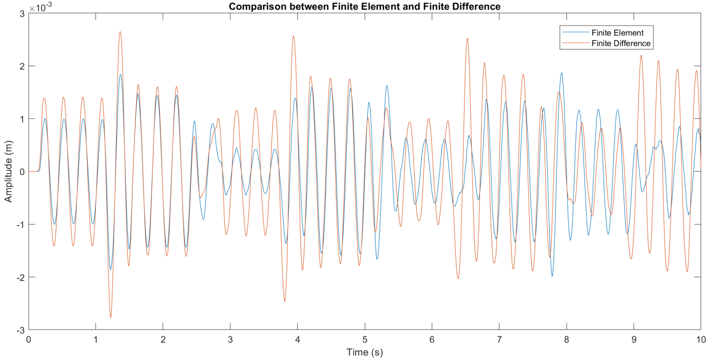


Formula

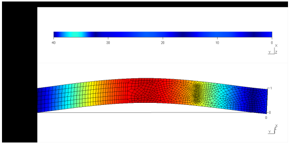
$c = v + sqrt(T/m)$
T = Tension , m = Mass per unit length, v = line speed and c = wave speed.

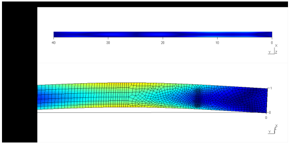
Error % = 0.89 %.

comparison with 1D FD model



Dynamic Analysis





Plan

Conclusion

Advantages of FEM

- More Accurate than many numerical Methods.
- Once coded successfully, It is very easy to implement even for complex geometry and mesh.
- Higher dimensions can be easily modeled.

Disadvantages of FEM

- Computationally expensive.
- Complexity in coding may be overwhelming .
- Suffers from "The curse of dimensionality!".

Thank you for your attention!!!