Finite Element simulation of 2D metal strip in Hot-Dip galvanization Process

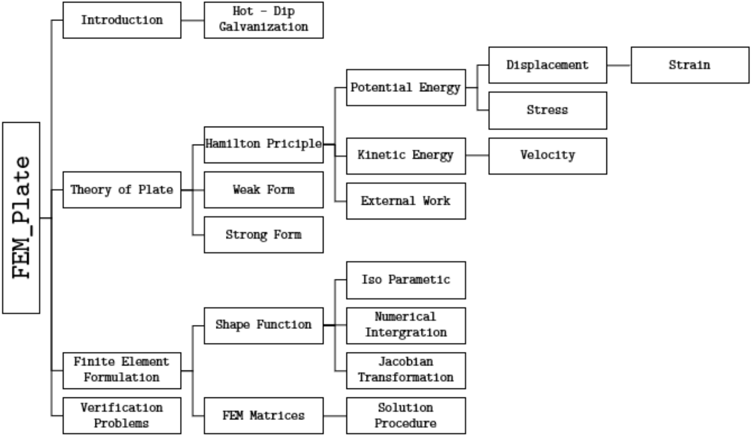
Presented By:
Emayavaramban ELANGO
École centrale De Nantes

Supervisor :
PHAM Van Thang
ArcelorMittal Maizières Research SA

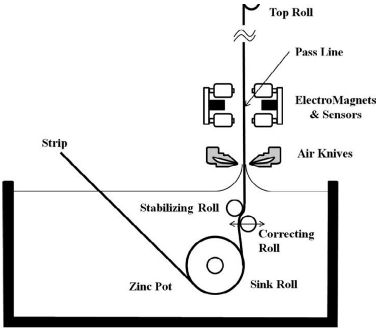
September 13, 2025

Table of Content

Plan


FEM_Plate
Introduction
Hot - Dip Galvanization
Theory of Plate
Hamilton Priciple
Weak Form
Strong Form
Potential Energy
Kinetic Energy
External Work
Displacement
Stress
Velocity
Strain
Finite Element Formulation
Shape Function
FEM Matrices
Iso Parametic
Numerical Intergration
Jacobian Transformation
Solution Procedure
Verification Problems

Hot-Dip Galvanization Process



- A thin Layer of Zinc is coated to Increase the corrosion resistance of steel
- Air knives control the thickness of the Zinc layer
- Excessive Vibration results in uneven coating.
- Electromagnets are used to control the vibration of the strip.

Need For Finite Element Modeling

- Complex behavior of the metal strip.
- Two dimensional domain and Three Dimensional Displacement field.
- Complex and multiple boundary condition.
- Free Control over discretization of the domain.
- Intuitive Solution Procedure.

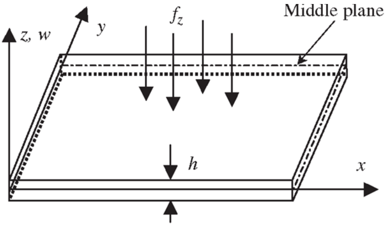
Plan

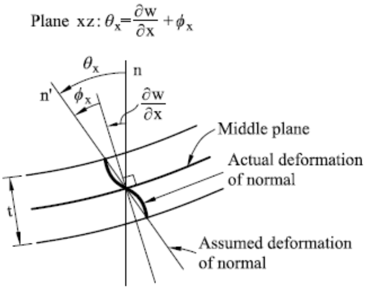
Theory of Plates (Reissner-Mindlin Plate Theory)

A plate is a flat solid with uniform and smaller thickness than its other dimensions.A middle plane (Z=0) is equidistant from upper and lower faces.

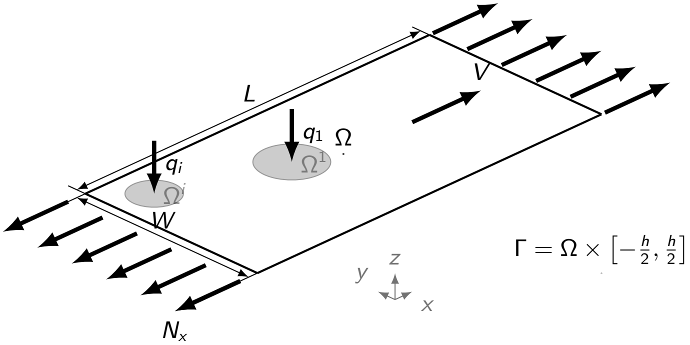
Assumptions (Thin and Thick Plate)

- A point in the middle plane only moves vertically $u = 0$ and $v = 0$
- Thickness does not change during deformation.
- only σ_{33} is neglected (plane stress is assumed)
- A line normal to the undeformed middle plane remains straight and not necessarily normal after deformation.





Description of Domain



- Ω is the two dimensional domain strictly in xy plane
- h is the thickness of the plate
- $\Omega_1 \ldots \Omega_i$ are the sub-domains where pressure forces $q_1..q_i$ are applied
- V is the Line speed and Nx is the tension on the line
- L is the length and W is the width

Hamilton principle

Hamilton principle is used to derive the equation of motion. and the Hamilton is given as

$$H = \int_{t_0}^{t_1} (T - V + W)\, dt$$

The Hamilton Principle states that variation of Hamilton is zero

$$\delta H = \int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt = 0$$

The variation of displacement δu is zero at the beginning and end time.

$$\delta u \Big|_{t_0}^{t_1} = 0$$

T is the kinetic energy, V in the potential energy and W is the work done to the system.

Potential Energy

The Total Potential energy of the domain is given as

$$V = \frac{1}{2} \int\int\int_{\Gamma} \epsilon^{T} \sigma d\Gamma$$

ϵ and σ are strain and stress respectively. The Potential Energy is separated into Bending (B) , Shear (S) and Axial (A) component.

$$V = \frac{1}{2} \int\int\int_{\Gamma} \left(\epsilon^{B}\right)^{T} \sigma^{B} + \left(\epsilon^{S}\right)^{T} \sigma^{S} + \left(\epsilon^{A}\right)^{T} \sigma^{A} d\Gamma$$

Since Thickness is constant and continuous it is integrated now.

$$V = \frac{1}{2} \int\int_{\Omega}\int_{-h/2}^{+h/2} \left(\epsilon^{B}\right)^{T} \sigma^{B} + \left(\epsilon^{S}\right)^{T} \sigma^{S} + \left(\epsilon^{A}\right)^{T} \sigma^{A} dz d\Omega$$

Which gives us

$$V = \frac{1}{2} \int\int_{\Omega} \kappa^{T} \tilde{\sigma}^{B} + \left(\epsilon^{S}\right)^{T} \tilde{\sigma}^{S} + \left(\epsilon^{A}\right)^{T} \tilde{\sigma}^{A} d\Omega$$

Strain

Using the plate theory, each strain components is given as

$$\epsilon^B = -z \begin{bmatrix} \frac{\partial w^2}{\partial x^2} \\ \frac{\partial w^2}{\partial y^2} \\ \frac{\partial w^2}{\partial x \partial y} \end{bmatrix} = -z\kappa = -z\triangle w$$

$$\epsilon^A = \frac{1}{2} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{2} w_{,\alpha} w_{,\beta} \qquad \alpha, \beta \in x, y$$

$$\epsilon^S = \frac{1}{2} \begin{bmatrix} \frac{\partial w}{\partial x} - \theta_x \\ \frac{\partial w}{\partial y} - \theta_y \end{bmatrix} = \frac{1}{2} \left(w_{,\alpha} - \theta_\alpha \right)$$

Stress

For **Linear Isotropic** material the corresponding stress strain relation is given as

$$\tilde{\sigma}^B = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix} = \frac{Eh^3}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \tilde{\mathbf{D}}\kappa$$

The shear stress and strain relation from the 3D constitutive law is given as

$$\tilde{\sigma}^S = \begin{bmatrix} 2\sigma_{31} \\ 2\sigma_{32} \end{bmatrix} = Gh \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\epsilon_{31} \\ 2\epsilon_{32} \end{bmatrix} = \tilde{\mathbf{D}}_c \sigma^S$$

Axial is stress is provided and it is considered as constant and uniform over the domain.

$$\tilde{\sigma}^A = \begin{bmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_{22} \end{bmatrix} = h \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix}$$

E is the Young's modulus, ν is the Poisson's ratio and G is the shear modulus which is given by $G = E/1+\nu$.

variation of Potential Energy

Potential Energy equation is written again

$$V = \frac{1}{2}\int\int_{\Omega} \kappa^T \tilde{\sigma}^B + \left(\epsilon^S\right)^T \tilde{\sigma}^S + \left(\epsilon^A\right)^T \tilde{\sigma}^A d\Omega$$

Now the strain and stress are substituted in the Potential equation

$$V = \frac{1}{2}\int\int_{\Omega} \kappa^T \tilde{D}\kappa + \left(\epsilon^S\right)^T \tilde{D}_c\epsilon^S + w_{,\alpha}\tilde{\sigma}^A w_{,\beta} d\Omega$$

finally taking the variation gives us

Variation of Total Potential Energy

$$\delta V = \int\int_{\Omega} \kappa^T \tilde{D}\delta\kappa + \left(\epsilon^S\right)^T \tilde{D}_c\delta\epsilon^S + w_{,\alpha}\tilde{\sigma}^A \delta w_{,\alpha} d\Omega$$

Kinetic Energy

The kinetic of a material is given as below and the kinetic energy is integrated along thickness first

$$T = \frac{1}{2}\int\int\int_{\Gamma} \mathbf{v}^{T} \rho \mathbf{v} d\Gamma = T = \frac{1}{2}\int\int_{\Omega}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{v}^{T} \rho \mathbf{v} dz\right] d\Omega$$

Mixed Euler - Lagrange formulation is used to find the velocity of the particle. The material derivative of general moving material is provided below.

Material Derivative

$$\frac{d(\circ)}{dt} = \frac{\partial(\circ)}{\partial t} + V_i \cdot (\circ)_{,i}$$

$$v_i = \dot{u}_i + V_1 u_{i,1}$$

First the integration along thickness is done

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{v}^T \rho \mathbf{v} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\rho \dot{u}_i \dot{u}_i + 2\rho V_1 \dot{u}_i u_{i,1} + \rho V_1^2 u_{i,1} u_{i,1} \right) dz$$

$$= \rho \dot{\tilde{u}}_i Z_{ij} \dot{\tilde{u}}_i + 2\rho V_1 \dot{\tilde{u}}_i Z_{ij} \tilde{u}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \tilde{u}_{j,1}$$

substituting it in the kinetic energy equation

$$T = \frac{1}{2} \int \int_{\Omega} \left(\rho \dot{\tilde{u}}_i Z_{ij} \dot{\tilde{u}}_i + \rho V_1 \dot{\tilde{u}}_i Z_{ij} \tilde{u}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \tilde{u}_{j,1} \right) d\Omega$$

Variations of Kinetic Energy

$$\delta T = \int \int_{\Omega} \rho \dot{\tilde{u}}_i Z_{ij} \delta \dot{\tilde{u}}_i + \rho V_1 \delta \dot{\tilde{u}}_i Z_{ij} \tilde{u}_{j,1} + \rho V_1 \dot{\tilde{u}}_i Z_{ij} \delta \tilde{u}_{j,1} + \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \delta \tilde{u}_{j,1} d\Omega$$

$$Z_{ij} = \begin{bmatrix} \frac{h^3}{12} & 0 & 0 \\ 0 & \frac{h^3}{12} & 0 \\ 0 & 0 & h \end{bmatrix}$$

External Work

more summation and write them separately for variations.

$$W = \sum_{i}^{nb} W_i = \sum_{i}^{nb} \int_{\Omega_i} q_i \mathbf{u_i} d\Omega_i$$

Variation of the external work

$$\delta W = \sum_{i}^{nb} \int_{\Omega_i} q_i \delta \mathbf{u_i} d\Omega_i$$

The Weak form

Substituting Everything in Hamilton principle gives

$$\int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt = 0$$

Using integration by parts the following equation is obtained. First term will vanish as the variation of displacement at beginning and end time is zero.

$$\int\int_{\Omega} +\rho\dot{\tilde{u}}_i Z_{ij}\delta\tilde{u}_i + \rho V_1 \delta\tilde{u}_i Z_{ij}\tilde{u}_{j,1} + \rho V_1 \tilde{u}_i Z_{ij}\delta\tilde{u}_{j,1} d\Omega\Big|_{t_0}^{t_1}$$

$$+\int_{t_0}^{t_1}\int\int_{\Omega} -\rho\ddot{\tilde{u}}_i Z_{ij}\delta\tilde{u}_i - \rho V_1\delta\tilde{u}_i Z_{ij}\dot{\tilde{u}}_{j,1} - \rho V_1\tilde{u}_i Z_{ij}\delta\dot{\tilde{u}}_{j,1} + \rho V_1^2\tilde{u}_{j,1}Z_{ij}\delta\tilde{u}_{j,1}$$

$$-\kappa^T\tilde{D}\delta\kappa - \left(\epsilon^S\right)^T\tilde{D}_c\delta\epsilon^S - w_{,\alpha}\tilde{\sigma}^A\delta w_{,\alpha} d\Omega + \sum_i^{nb}\int\int_{\Omega_i} q_i\delta\mathbf{u_i} d\Omega_i dt = 0$$

$\int_{t_0}^{t_1}\chi dt = 0$ For this to be true $\chi = 0$ must also be true . Final Weak Form is

$$\int\int_{\Omega}\rho\ddot{\tilde{u}}_i Z_{ij}\delta\tilde{u}_i + 2\rho V_1\delta\tilde{u}_i Z_{ij}\dot{\tilde{u}}_{j,1} - \rho V_1^2\tilde{u}_{j,1}Z_{ij}\delta\tilde{u}_{j,1}$$

$$+\kappa^T\tilde{D}\delta\kappa + \left(\epsilon^S\right)^T\tilde{D}_c\delta\epsilon^S + w_{,\alpha}\tilde{\sigma}^A\delta w_{,\alpha} d\Omega = \sum_i^{nb}\int\int_{\Omega_i} q_i\delta\mathbf{u_i} d\Omega_i dt$$

Strong Form

The weak form

$$\int\int_{\Omega} \rho \ddot{\tilde{u}}_i Z_{ij} \delta \tilde{u}_i + 2\rho V_1 \delta \tilde{u}_i Z_{ij} \dot{\tilde{u}}_{j,1} - \rho V_1^2 \tilde{u}_{j,1} Z_{ij} \delta \tilde{u}_{j,1}$$

$$+\kappa^T \tilde{D} \delta \kappa + \left(\epsilon^S\right)^T \tilde{D}_c \delta \epsilon^S + w_{,\alpha} \tilde{\sigma}^A \delta w_{,\alpha} d\Omega = \sum_i^{nb} \int\int_{\Omega_i} q_i \delta \mathbf{u_i} d\Omega_i dt$$

$\int\int_{\Omega} \chi d\Omega = 0$ For this to be true $\chi = 0$ must also be true and by considering $\epsilon^S = 0$, $\tilde{\sigma}^A = N_x$ and $\tilde{u}_1 = \tilde{u}_2 = 0$ we get the strong form. Only single force is considered, which is $nb = 1$, $q_i = F$ and $\Omega_i = \Omega$

Final strong Form

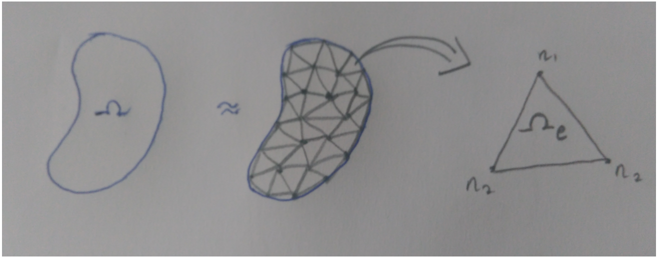
$$\rho h \left(\frac{\partial^2 w}{\partial t^2} + 2V_1 \frac{\partial^2 w}{\partial x \partial t} - V_1^2 \frac{\partial^2 w}{\partial x^2} \right) + D\nabla^4 w + N_x h \frac{\partial^2 w}{\partial x^2} = F$$

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial Y^4} \qquad D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$

Plan

Finite Element Method

In Finite element a continuous domain is discretized into elements. Each elements are connect by nodes.



$$\Omega \approx \sum_{i}^{nE} \Omega_e^i$$

The displacement is represented as a degree of freedom of the system.

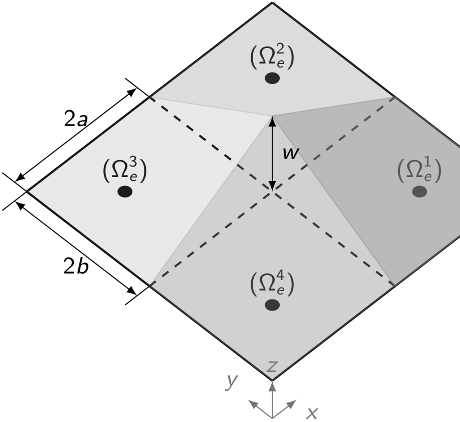
$$\tilde{u} = [w, \theta_x, \theta_y]^T \quad \theta_x = \frac{\partial w}{\partial x} \quad \theta_y = \frac{\partial w}{\partial y}$$

Shape function of a rectangular element

The displacement field over the element is given as the sum of product of shape function and nodal displacements. here nN is the total number of nodes in a element

$$\tilde{\mathbf{u}} \approx \sum_{i=1}^{nN} \left(N_i w_i + \overline{N}_i \theta_{x_i} + \overline{\overline{N}}_i \theta_{y_i} \right)$$

$$N_1 = \frac{1}{4ab}(1-x)(1-y)$$
$$N_2 = \frac{1}{4ab}(1+x)(1-y)$$
$$N_4 = \frac{1}{4ab}(1-x)(1+y)$$
$$N_3 = \frac{1}{4ab}(1-x)(1-y)$$



For Ressiner Mindlin element

$$N_i = \overline{N}_i = \overline{\overline{N}}_i$$

Representation of Displacements and Strains in terms of Shape Function.

The FE approximation

$$\tilde{\mathbf{u}} \approx \sum_{i=1}^{nN} \left(N_i w_i + \overline{N}_i \theta_{x_i} + \overline{\overline{N}}_i \theta_{y_i} \right)$$

is written in matrix format as

$$\tilde{\mathbf{u}} \approx \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_{nN} & 0 & 0 \\ 0 & \overline{N}_1 & 0 & \cdots & 0 & \overline{N}_{nN} & 0 \\ 0 & 0 & \overline{\overline{N}}_1 & \cdots & 0 & 0 & \overline{\overline{N}}_{nN} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ w_{nN} \\ \theta_{x_{nN}} \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{N}\tilde{\mathbf{u}}^e$$

similarly for the velocity and acceleration as they are independent of time.

$$\dot{\tilde{\mathbf{u}}} \approx \mathbf{N}\dot{\tilde{\mathbf{u}}}^e \qquad \ddot{\tilde{\mathbf{u}}} \approx \mathbf{N}\ddot{\tilde{\mathbf{u}}}^e \tag{1}$$

Representation of Strains in terms of Shape Function.

The Bending strain $\kappa = \triangle \tilde{\mathbf{u}}$ as FE matrix is given below. We can see that first column goes to zero since the double derivative shape function is zero.

$$\kappa \approx \begin{bmatrix} 0 & \overline{N}_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & \overline{\overline{N}}_{1,2} & \cdots & \overline{\overline{N}}_{nN,2} \\ 0 & \overline{N}_{1,2} & \overline{\overline{N}}_{1,1} & \cdots & \overline{\overline{N}}_{nN,1} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ \theta_{y_{Nn}} \end{Bmatrix} = \mathbf{B}\tilde{\mathbf{u}}^e$$

similarly for the shear strain is represented as

$$\tilde{\epsilon}^S \approx \begin{bmatrix} N_{1,1} & \overline{N}_1 & 0 & \cdots & 0 \\ N_{1,2} & 0 & \overline{\overline{N}}_1 & \cdots & \overline{\overline{N}}_{nN} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ \theta_{y_{Nn}} \end{Bmatrix} = \mathbf{B_S}\tilde{\mathbf{u}}^e \qquad (2)$$

$N_{1,2}$ represent $\dfrac{dN_1}{dy}$. N_1 is the shape function of w displacement in first node.

$$\tilde{u}_{1,\alpha} \approx \begin{bmatrix} N_{1,1} & 0 & 0 & N_{2,1} & \cdots & 0 \\ N_{1,2} & 0 & 0 & N_{2,2} & \cdots & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \vdots \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{H_A}\tilde{\mathbf{u}}^e$$

$$\tilde{u}_{\alpha,1} \approx \begin{bmatrix} N_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & \overline{N}_{2,1} & 0 & \cdots & 0 \\ 0 & 0 & \overline{\overline{N}}_{3,1} & \cdots & \overline{\overline{N}}_{3,3} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \vdots \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{H_v}\tilde{\mathbf{u}}^e$$

The FE approx for the body force is given as

$$\tilde{w} \approx \begin{bmatrix} N_1 & 0 & 0 & N_2 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ w_2 \\ \vdots \\ \theta_{y_{nN}} \end{Bmatrix} = \mathbf{N_f}\tilde{\mathbf{u}}^e$$

Weak Form to FE format

The Finite Element Matrix equation is given as

$$\begin{aligned}
\int\int_{\Omega} & \left(\rho\,[\mathbf{N}]\,[\mathbf{Z}]\,[\mathbf{N}]\,\{\ddot{\tilde{\mathbf{u}}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e} + \left(2\rho V_1\,[\mathbf{N}]\,[\mathbf{Z}]\,[\mathbf{H_v}]\,\{\dot{\tilde{\mathbf{u}}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e} \\
& - \left(\rho V_1^2\,[\mathbf{H_v}]\,[\mathbf{Z}]\,[\mathbf{H_v}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e} + \left([\mathbf{B}]\left[\tilde{\mathbf{D}}\right][\mathbf{B}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e} \\
& + \left([\mathbf{B_S}]\left[\tilde{\mathbf{D}}_{\mathbf{S}}\right][\mathbf{B_S}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e} + \left([\mathbf{H_A}]\left[\tilde{\mathbf{N}}_{\mathbf{A}}\right][\mathbf{H_A}]\,\{\tilde{\mathbf{u}}^{e}\}\right)\delta\tilde{\mathbf{u}}^{e}\,d\Omega \\
& = \sum_{i}^{nb}\int\int_{\Omega_i}\left(q_i\left[\tilde{\mathbf{N}}_{\mathbf{f}}\right]\right)\delta\tilde{\mathbf{u}}^{e}\,d\Omega_i
\end{aligned}$$

FEM matrices

After rearranging them to their respective groups we get.

$$[\mathbf{M}^e]\{\ddot{\mathbf{u}}^e\} + [\mathbf{C}]\{\dot{\mathbf{u}}^e\} + [\mathbf{K}^e]\{\mathbf{u}^e\} = \{\mathbf{F}^e\}$$

where

$$[\mathbf{M}^e] = \rho \int\int_\Omega ([\mathbf{N}][\mathbf{Z}][\mathbf{N}])\, d\Omega$$

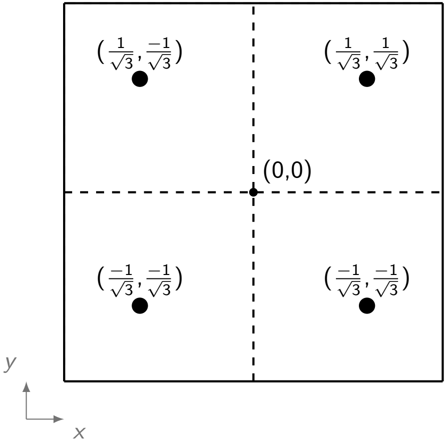
$$[\mathbf{C}^e] = 2\rho V_1 \int\int_\Omega ([\mathbf{N}][\mathbf{Z}][\mathbf{H_v}])\, d\Omega$$

$$[\mathbf{K}^e] = -\rho V_1^2 \int\int_\Omega ([\mathbf{H_v}][\mathbf{Z}][\mathbf{H_v}])\, d\Omega + \int\int_\Omega [\mathbf{B}]\left[\tilde{\mathbf{D}}\right][\mathbf{B}]\, d\Omega$$
$$+ \int\int_\Omega [\mathbf{B_S}]\left[\tilde{\mathbf{D}}_\mathbf{S}\right][\mathbf{B_S}]\, d\Omega + \int\int_\Omega [\mathbf{H_A}]\left[\tilde{\mathbf{N}}_\mathbf{A}\right][\mathbf{H_A}]\, d\Omega$$

$$\{\mathbf{F}^e\} = \sum_i^{nb} \int\int_{\Omega_i} q_i \left[\tilde{\mathbf{N}}_\mathbf{f}\right] d\Omega_i$$

Gauss Quadrature

Gauss Quadrature is used for numerical integration over the element.



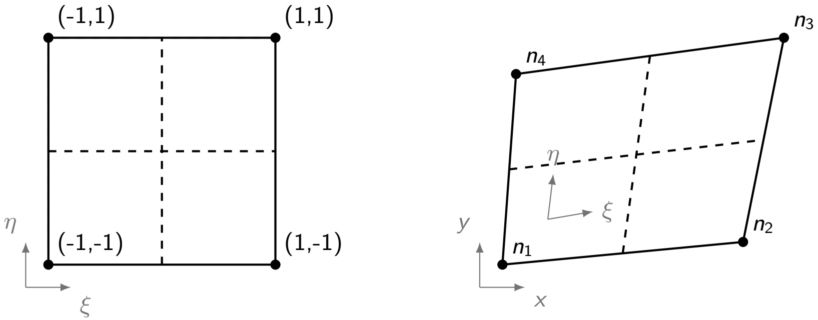
●Gauss Integration points

$$\int\int f(x,y)dxdy = \sum_{i=1}^{nx}\sum_{j=1}^{ny} w_i w_j \cdot f(ix, jy)$$

$$w_i = w_j = 1$$

w_i and w_j are the Gauss weight.
ix and jy are the Gauss Points.

Iso parametric Shape Function



$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

Jacobian Transformation

Derivation of the shape function by the coordinated in parent element using chain rule gives us

$$\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y}{\partial \xi}$$

This relation is written in matrix format which gives us J matrix or jacobian matrix.

$$\left\{ \begin{array}{c} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{array} \right\} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \left\{ \begin{array}{c} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{array} \right\} \quad J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

det(J) must always be greater than zero. det(J) $=0$ means the 2D element disappears into 1D. Jacobian is also a important measure of quality of the mesh.

$$\left\{ \begin{array}{c} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{array} \right\} = J^{-1} \left\{ \begin{array}{c} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{array} \right\}$$

The Inverse relation is employed to find the derivative of shape function from parent element.

Final FEA matrix

$$[\mathbf{M^e}] = \sum_{i=1}^{ng} \rho \left(w_i \, [\mathbf{N(i)}]^T \, [\mathbf{Z}] \, [\mathbf{N(i)}] \, det(J) \right) d\Omega$$

All the Element mass Matrices $[\mathbf{M^e}]$ are assembled in the final Mass Matrix $[\mathbf{M}]$. Which gives us the ODE in terms of FE matrices.

$$[\mathbf{M}] \{\ddot{\mathbf{u}}\} + [\mathbf{C}] \{\dot{\mathbf{u}}\} + [\mathbf{K}] \{\mathbf{u}\} = \{\mathbf{F}\}$$

Solution Procedure

Static Analysis:
To solve a static system

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$$

$u = K\backslash F$ command is used since $u = K^{-1}F$ is a very expensive task. $u = K\backslash F$ command first factorizes the K matrix into upper and lower triangle then solves the system which is a much more efficient process.
Modal Analysis:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0$$

is converted in to a eigenvalue problem

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\bar{\mathbf{u}} = 0 \qquad \mathbf{u} = \bar{\mathbf{u}}e^{i\omega t}$$

ω is the natural frequency and $\bar{\mathbf{u}}$ is the natural mode. In MATLAB, [V,D]=eig(K,M) function is used to do the modal analysis.

Dynamic Analysis:
To solve the dynamic system

$$[\mathbf{M}]\{\ddot{\mathbf{u}}\} + [\mathbf{C}]\{\dot{\mathbf{u}}\} + [\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$$

Newmark time integration scheme is employed.

Newmark algorithm

$$R = F_t + \mathbf{M}\left(a_0 u_t + a_2 \dot{u}_t + a_3 \ddot{u}_t\right) + \mathbf{C}\left(a_1 u_t + a_4 \dot{u}_t + a_5 \ddot{u}_t\right)$$

$$\begin{aligned} u_{t+1} &= \left[a_0\mathbf{M} + a_1\mathbf{C} + \mathbf{K}\right]^{-1} R \\ \dot{u}_{t+1} &= a_1\left(u_{t+1} - u_t\right) - a_4 \dot{u}_t - a_5 \ddot{u}_t \\ \ddot{u}_{t+1} &= a_0\left(u_{t+1} - u_t\right) - a_2 \dot{u}_t - a_3 \ddot{u}_t \end{aligned}$$

a_0 .. a_5 are the integration variables which depends on the Integration parameters θ, α and time step h. u_t,$\dot{u}_t$ and $\ddot{u}_t$ are the displacement , velocity and acceleration of current time step. u_{t+1},$\dot{u}_{t+1}$ and $\ddot{u}_{t+1}$ are the displacement , velocity and acceleration of next time step.

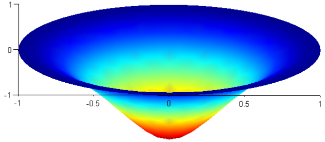
Unconditionally Stable for

$$\theta \geq \frac{1}{2}$$

$$\alpha \geq \frac{1}{4}\left(\frac{1}{2} + \theta\right)^2$$

Plan

TIM69 Static Analysis , Circular Plate with Point load



The target analytically solution given as

$$w = \frac{F_z}{16\pi D}\left[r^2 - a^2\right] + \frac{F_z r^2}{8\pi D}\left[log\,\frac{a}{r}\right] \tag{3}$$

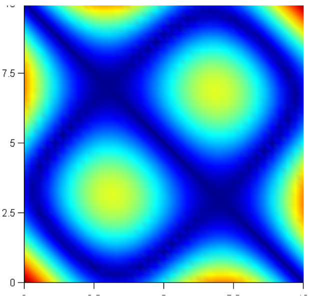
The analytical solution is -0.000434 *in*.
Numerical solution is $-0.000429 in$.
So the Error percentage is 1.26%.

Material Property	
Young's Modulus (E)	5E11 Pa
Poission's Ratio (ν)	0.3
Geometric Data	
Radius (r)	1 m
Thickness(t)	0.01 m
Loading Data	
Point Load (F_z)	-1000 N

Reference

S.Timoshenko , S . Woinowsky , Theory of Plates and Shells , pg:69, Article : 19 .

VMP09 Modal Analysis , SS square Plate



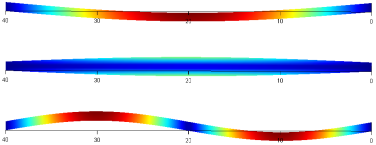
The analytical frequency is 1.632 Hz.
The numerical frequency is 1.626 Hz.
So the Error percentage is 0.32 %

Material Property	
Young's Modulus (E)	25E11 Pa
Poission's Ratio (ν)	0.3
Density(ρ)	8000
Geometric Data	
length (l)	10 m
breath (b)	10 m
Thickness(t)	0.01 m

Reference

S.Timoshenko , S . Woinowsky , Theory of Plates and Shells , pg:69, Article : 19 .

NAS227 modal analysis of thin plate with axial load



Analytical Solution = 77.47
Numerical Solution = 77.45
Error % = 0.01 %.

$$\rho^2\omega_{mn}^2 = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] + N_1\left(\frac{m\pi}{a}\right)^2 + N_2\left(\frac{n\pi}{b}\right)^2$$

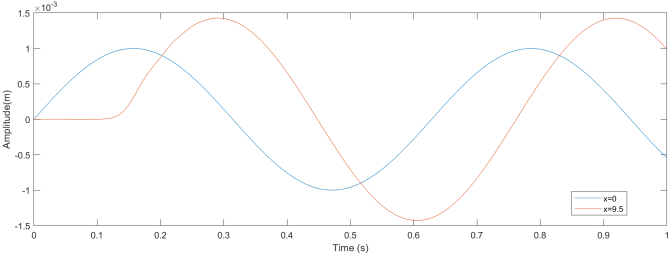
Material Property	
Young's Modulus (E)	1E11 Pa
Poisson's Ratio (ν)	0.3
Density(ρ)	7810
Geometric Data	
length (l)	1 m
breath (b)	40 m
Thickness(t)	0.5 mm
Loading Data	
Axial load (N_x)	6E7 N/m^2

Reference

Arthur W.Leissa ,Vibration of Plates,NASA SP-160, pg:277, Ch:10.2.

Wave Speed in the moving metal strip

The same model used in the previous problem is also used here with additional line speed to simulate moving material.



$$c = v + \sqrt{\frac{T}{m}}$$

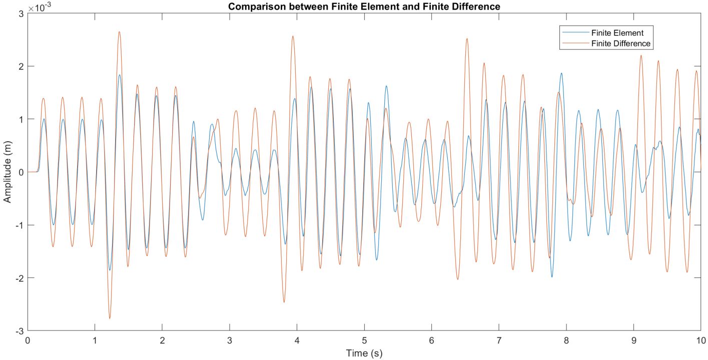
T = Tension , m = Mass per unit length, v = line speed and c = wave speed.

Loading Data	
Line Speed (V_1)	10 m/s

Analytically Solution = 71.977 ms^{-1}, Numerical solution = 71.42 ms^{-1} and The Error % = 0.89 %.

comparison with 1D FD model

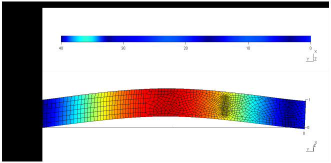
The same model is again employed to compare it with the existing one dimensional finite difference model.



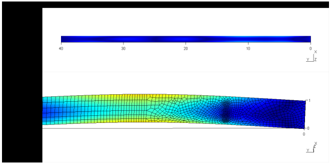
The displacement of the plate at the distance of 10 m from origin is plotted against time.

Solution plot of the Dynamic Analysis

The solution post processing of the previous problem is provided.



The solution post processing of the same problem with additional time varying body forces is provided.



analysis Statistics

nt=250 , nN = 1886 , nE = 1836, ndof = 5640
Solution Time = s

Plan

Conclusion

Advantages of FEM

- Better control over accuracy.
- Once coded successfully, It is very easy to implement even for complex geometry and mesh.
- Higher dimensions can be easily modeled.

Disadvantages of FEM

- Computationally expensive.
- Complexity in coding may be overwhelming .
- Suffers from "The curse of dimensionality!".

Thank you for your attention!!!