Literature Review of ARCH and GARCH Processes

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1 Introduction

Time Series Analysis is a field within Mathematics and Statistics which concerns itself with the various characteristics of random variables with respect to time. In order to fulfill the need to analyze these characteristics, several rudimentary models were developed, such as the Autoregressive (AR) Model and the Moving Average (MA) Model, which were later combined into the Autoregressive Moving Average (ARMA) Model. As time progressed, however, the linear constraints of these models became more apparent when applied to financial and economic data, which is often times non-linear, and thus more models were developed to remove this linear constraint and allow for dynamic, non-linear processes to be measured properly. These newer models incorporated aspects such as conditional variance and non-linear terms to stochastic processes, which proved to be vital in fields that require quantitative analysis of data over time, most notably in finance and economics. In this paper, we will be conducting a literature review regarding two of these models: the Autoregressive Conditional Heteroskedasticity (ARCH) Model, developed by Engle (1982) and the Generalized Auto Regressive Conditional Heteroskedasticity (GARCH) Model, developed by Bollerslev (1986) as an expansion of Engle's original Model. Since being published, these models have became some of the most cited papers in economics and finance, as these models were some of the first with the ability to model the non-linear dynamics commonly found in financial and monetary data. These two original models have since also gone on to become the basis for many more models, such as the EGARCH, APARCH, Spatial GARCH Models, among others (these will not be elaborated on in this paper). The purpose of this paper will be to provide fundamental information of ARCH and GARCH Models, and some time series concepts, while providing examples of applications of both models using real-world data.

2 Preliminaries

2.1 Notation

We will denote a time series, i.e. a sequence of random variables, by $(X_k)_k$, for all $k \in \mathbb{Z}$. ε_k will be used to denote random errors which follow the properties of white noise, which pertains to both models. At times, the error terms fulfill $\varepsilon_k \sim N(0, \sigma^2)$, that is, ε_k is distributed normally with a zero mean and σ^2 variance. This is also referred to as "Gaussian White Noise". The models themselves are denoted as ARCH(p) and GARCH(p,q), with p representing the degree of the ARCH terms in the formula. Likewise, for the GARCH Model, p is the degree of the ARCH term, while q is the degree of the GARCH term in the formula. "Degree" in this sense is how many instances of p or q will be summed together in the models. These specific terms will be more thoroughly defined in a later section of this paper.

2.2 Basic concepts in Time Series Analysis

In order to go further in depth with our discussion of ARCH and GARCH Models, we must first establish some basic concepts in time series analysis. These basic concepts serve as preconditions for allowing these

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models, among others, to be used to describe time series data.

The first, and arguably most, important precondition to establish is *stationarity*:

Definition 2.1 (Stationarity). *A time series* $(X_k)_{k\in\mathbb{Z}}$ *is called* (strictly) stationary *if for all* $h \in \mathbb{Z}$, $t_1, ..., t_n \in \mathbb{Z}$ *and* $n \in \mathbb{N}$ *holds*

$$(X_{t_1}, ..., X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, ..., X_{t_n+h}). \tag{2.1}$$

More intuitively speaking, any vector of values of $(X_k)_{k\in\mathbb{Z}}$ (that is, a vector that contains values of the time series at any points) must have the same distribution. Following from this definition, when a process is independently and identically distributed (denoted as "iid", $\stackrel{iid}{\sim}$), it is a strictly stationary process. For example, $(X_k)_{k\in\mathbb{Z}} \stackrel{iid}{\sim} N(\mu, \sigma^2)$ is a strictly stationary process, as it is independently and identically distributed normally with mean μ and variance σ^2.

Following from *(strict) stationarity*, we introduce the concept of *weak stationarity.*

Definition 2.2 (Weak Stationarity). *A time series* $(X_k)_{k\in\mathbb{Z}}$ *is called* (weakly) stationary *if the following three conditions hold:*

(i) $E(X_k^2) < \infty, \quad \forall\, k \in \mathbb{Z}$

(ii) $E(X_k) = \mu \in \mathbb{R}, \quad \forall\, k \in \mathbb{Z}$

(iii) $\mathrm{Cov}(X_k, X_l) = \mathrm{Cov}(X_0, X_{l-k}), \quad \forall\, k, l \in \mathbb{Z}$

In other words, a time series must possess a finite variance, a constant mean, and constant covariances at all points in the time series. As the conditions for strict stationarity may not apply to time series data, we work with weakly stationary processes when deciding which models are appropriate to use. Introducing the concept of stationarity is extremely important for modeling time series data, as most models (especially ARCH and GARCH) assume the processes are stationary (weak and/or strict). It is also important to note that there is a significant overlap in the conditions for weak and strictly stationary processes, however there exists cases when processes are one or the other. For example, a time series $(X_k)_{k\in\mathbb{Z}} \stackrel{iid}{\sim}$ *Cauchy* is strictly but not weakly stationary, as the first moments do not exist, and the second moment, its variance, is infinite. Likewise, consider a scenario where $(X_k)_{k\in\mathbb{Z}} \sim Binomial(n, p)$, where the sample is made without replacement. In this case, since the outcomes are not independent, but the first moment *np*, second moment *npq*, and constant covariances exist, then $(X_k)_{k\in\mathbb{Z}}$ is a weakly but not strictly stationary process.

Moving on from stationary processes, we introduce what are called *white noise.*

Definition 2.3 (Weak White Noise). *A time series* $(X_k)_{k\in\mathbb{Z}}$ *is called* (Weak) White Noise *if the following three conditions hold:*

(i) $0 < E(X_k^2) < \infty, \quad \forall\, k \in \mathbb{Z}$

(ii) $E(X_k) = 0, \quad \forall\, k \in \mathbb{Z}$

(iii) $\mathrm{Cov}(X_k, X_l) = 0, \quad$ *if* $k \neq l$

White noises often are characterized through accounting for unknown differences, and are commonly associated with the error terms ε_k of time series models. It is important to note that because of the similar conditions, weak white noises are very similar to weakly stationary processes, but have more specific conditions, these being a mean of zero and covariances equal to zero at all points in the time series process. Also similarly to stationary processes, an independently and identically distributed white noise is considered a *strong white noise.* The aforementioned ε_k, $\varepsilon \stackrel{iid}{\sim} N(0, \sigma^2)$ (Gaussian white noise) process is an example of a strong white noise process, as they are independently and identically normally distributed error terms, with a mean of zero.

3 Models

In this section of the paper, we define the ARCH and GARCH Models in depth, along with interpretations. It must first be noted that in more traditional statistics, heteroskedasticity is viewed as something that ought to be remedied. However in time series analysis, and primarily in the use of the ARCH and GARCH Models, heteroskedasticity is seen as another type of variance that ought to be measured. More intuitively speaking, it is a "varying variances" which helps predict the future variance of what is being measured with the model.

3.1 The ARCH Model

The Autoregressive Conditional Heteroskedasticity (ARCH) model defines values X_k of a process $(X_k)_k$ as a function of the standard deviation term σ_k multiplied by the error term ε_k. From this definition, the ARCH Model is then set as a function of the conditional variance term σ_k^2. The ARCH Model can be intuitively described as the conditional variance being represented as a function of current variances.

Definition 3.1 (ARCH Model)**.** *An ARCH(p) process with $p \in \mathbb{N}$ is a time series $(X_k)_{k\in\mathbb{Z}}$ which satisfies:*

$$X_k = \sigma_k \varepsilon_k, \quad \sigma_k^2 = \delta + \sum_{i=1}^{p} \alpha_i X_{k-i}^2, \ \forall k \in \mathbb{Z}, \tag{3.1}$$

where $p \in \mathbb{N} \ \geq 0$, $\delta > 0$, $\alpha_i \geq 0$, for $i = 1, ..., p$, and $\alpha_p \neq 0$

In this model, it is said that the error terms ε_k, $\varepsilon \overset{iid}{\sim} N(\mu, \sigma^2)$ follow a Gaussian white noise, meaning that the errors are independently and identically normally distributed.

Following from the original ARCH Model, we introduce the GARCH Model, which includes an extra term to assist in the fitting of the model to the time series data.

3.2 The GARCH Model

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is an extension of the previously defined ARCH Model, where the output $X_k \in (X_k)_{k\in\mathbb{Z}}$ as a function of the standard deviation term σ_k multiplied by the error term ε_k. From this definition, the GARCH model (like the ARCH model) is then set as a function of the conditional variance term σ_k^2. However in this model, a second term is added; the GARCH term $\sum_{j=1}^{q} \beta_j \sigma_{k-i}^2$, which is the inclusion of the lagged conditional variances. The inclusion of lagged conditional variances into the ARCH model is how the model was "Generalized" into GARCH. As Bollerslev describes it in his original paper on GARCH Processes, the GARCH term "acts as a sort of adaptive learning mechanism", incorporating the past volatility to help produce more accurate forecasts on volatility.

Definition 3.2 (GARCH Model)**.** *A GARCH(p,q) process with $p, q \in \mathbb{N}$ is a time series $(X_k)_{k\in\mathbb{Z}}$ which satisfies:*

$$X_k = \sigma_k \varepsilon_k, \sigma_k^2 = \delta + \sum_{i=1}^{p} \alpha_i X_{k-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{k-i}^2, \forall k \in \mathbb{Z}, \tag{3.2}$$

where $p \geq 0$, $q \geq 0$, $\delta > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, for $i = 1, ..., p$, $j = 1, ..., q$

In this model, it is said that the error terms ε_k follow a white noise, meaning that the errors have mean zero and zero covariances throughout.

An important remark to make regarding both models is that of the error variances. In the ARCH and GARCH models, the error variances are assumed to be modeled by the Autoregressive (AR) and Autoregressive Moving Average Models (ARMA) respectively. It is possible for this to be shown as in both cases, it is possible to interpret the AR and ARMA Models as squared ARCH and GARCH processes. A full proof of this is provided in the appendix. The ARCH and GARCH processes are squared in order to be more descriptive of time series, as they are a higher order function when squared, and will fit the data far better. In this case, the higher order function is used to describe the error variances, as elaborated on above.

3.3 Example: ARCH and GARCH in the First Degree

Since we have now introduced the ARCH(p) and GARCH(p,q) Models in their general form, we will also provide a commonly used variation of both models. In much of the literature surrounding these topics, many examples are conducted with the simple ARCH(1) and GARCH(1,1) models, meaning the ARCH Model with one degree, and the GARCH Model with one degree of ARCH, and one degree of GARCH.

Definition 3.3 (First Degree ARCH). *An ARCH(1) Model is a process for the time series* $(X_k)_{k\in\mathbb{Z}}$ *with degree 1 which follows as:*

$$X_k = \sigma_k \epsilon_k, \sigma_k^2 = \delta + \alpha_1 X_{k-1}^2, \ \ \forall k \in \mathbb{Z} \tag{3.3}$$

Definition 3.4 (First Degree GARCH). *A GARCH(1,1) Model is a process for the time series* $(X_k)_{k\in\mathbb{Z}}$ *with degree 1 which follows as:*

$$X_k = \sigma_k \epsilon_k, \sigma_k^2 = \delta + \alpha_1 X_{k-1}^2 + \beta_1 \sigma_{k-1}^2, \ \ \forall k \in \mathbb{Z} \tag{3.4}$$

Using their first degree forms, we can now decompose the GARCH(1,1) model to more accurately show the specific terms:

(i) The part of variance, represented by $\alpha_1 X_{k-1}^2$, is the ARCH term in the first degree. This term is comprised of the ARCH coefficient α_1 multiplied by the squared lagged output value X_{k-1}^2. Since this is in the first degree, it is the value immediately previous to X_k^2

(ii) The part given by the previous variances, written as $\beta_1 \sigma_{k-1}^2$, is the GARCH component in the first degree. This term is comprised of the is GARCH regression coefficient β_1 multiplied by the lagged conditional variance value σ_{k-1}^2, which is the conditional variance of previous value in the time series process.

(iii) Finally, the δ term is the intercept, shown in both models.

It is important to note here that unlike in other models, the error term ε_k is not included in the formulas for ARCH and GARCH, as the error terms are already established in the model for the overall output $X_k = \sigma_k \varepsilon_k$. Since the ARCH and GARCH Models are the formulas for the conditional variance, and not the output, it would be erroneous to place the error terms in their formulas.

4 Applications

In this section, we display the use of the ARCH and GARCH models, with respect to real-world data, to showcase how standardized the use of both models have become in finance and economics.

4.1 Example 1: ARCH and GARCH applied to US stock market data

The first example comes from a paper written by Engle (2001), which is an exemplification of both models, using financial data taken from the US stock market from 1990 to 2000. In this paper, Engle applies both ARCH and GARCH Models to a hypothetical investment portfolio comprised up of stocks from the NASDAQ, Dow Jones, and US Treasury Bonds markets, with the data showing long term investments over a span of ten years. Below is Figure 1, which displays the volatility over time for each of the three markets.

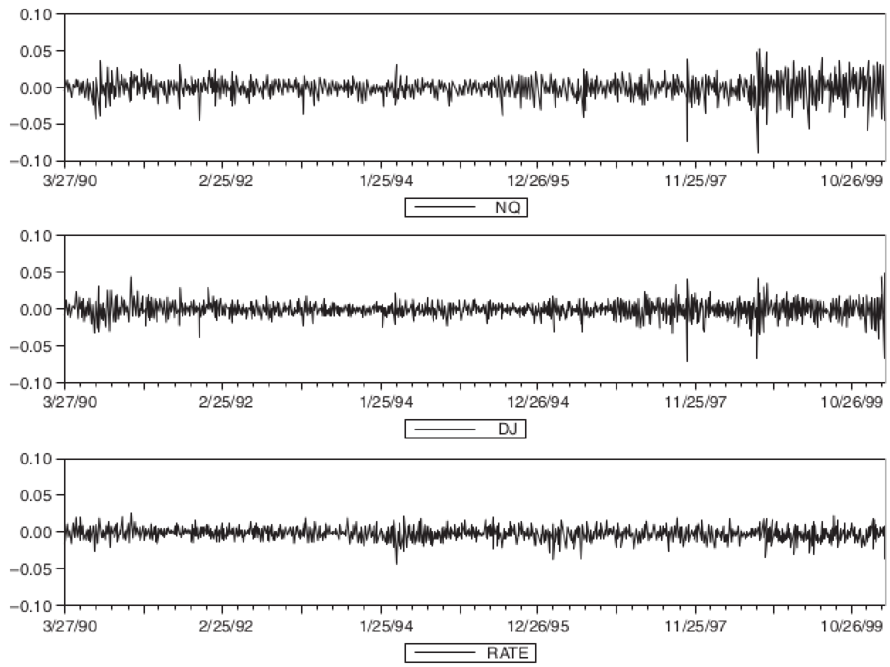

Figure 1: Volatility of the NASDAQ (NQ), Dow Jones (DJ), and US Treasuries (RATE) from 1990-2000

It can be seen visually here that the first two graphs, indicating the volatility of the NASDAQ and Dow Jones respectively, are quite heteroskedastic, with periods of low variance (around 1992 to 1995) preceded by a period of higher volatility and succeeded by even higher volatility until the year 2000 is reached, where we can see there was high volatility in both markets, presumably being the lead-up to the Dotcom Crash. Conversely, we can see that the volatility for the US Treasury Bonds is a far more homoskedastic process, with the conditional variance being more consistent than the other two markets throughout the ten year period. It can also be noted that all three graphs display weakly stationary processes, as they have constant means which are close to zero and finite variances. However this is not definitive, as depending on the specific period of time indices, the time series process can be not stationary at all, or display traits likening it to other types of stationarity. It is heavily subjective to the time indices we choose to analyze, but for the entire period of the ten years, we can assume that the data is weakly stationary.

Figures 2 and 3 below display descriptive statistics regarding each of the three markets, and the ARCH and GARCH Coefficients:

Sample: 3/23/1990 3/23/2000				
	NQ	**DJ**	**RATE**	**PORT**
Mean	0.0009	0.0005	0.0001	0.0007
Std. Dev.	0.0115	0.0090	0.0073	0.0083
Skewness	−0.5310	−0.3593	−0.2031	−0.4738
Kurtosis	7.4936	8.3288	4.9579	7.0026

Figure 2: Portfolio Statistics

Dependent Variable: PORT
Sample (adjusted): 3/26/1990 3/23/2000
Convergence achieved after 16 iterations
Bollerslev-Wooldrige robust standard errors and covariance

Variance Equation				
C	0.0000	0.0000	3.1210	0.0018
ARCH(1)	0.0772	0.0179	4.3046	0.0000
GARCH(1)	0.9046	0.0196	46.1474	0.0000

S.E. of regression	0.0083	Akaike info criterion	−6.9186
Sum squared resid	0.1791	Schwarz criterion	−6.9118
Log likelihood	9028.2809	Durbin-Watson stat	1.8413

Figure 3: GARCH(1,1) Results.

As seen in the table displaying the results of running the GARCH(1,1) Process on the data, which produces a result in the form:

$$\sigma_k^2 = 0 + 0.0772X_{k-1}^2 + 0.9046\sigma_{k-1}^2, \quad \forall k \in \mathbb{Z} \tag{4.1}$$

Since the coefficients for the ARCH and GARCH terms sum to a number close to 1 (0.9818, to be specific), they indicate the presence of what is called a “mean-reverting variance process”, meaning that the mean reverts to the long term mean of the time series processes after periods of high volatility. This means that after the period of high volatility that is seen as the data approaches the year 2000, the values will inevitably return to their long run means, which were seen to be close to zero, as per Figure 2. The table in Figure 3 also provides the p-values for the ARCH and GARCH terms, which are seen to be extremely low. This indicates that there are ARCH and GARCH effects present in the data.

4.2 Example 2: GARCH and Variants Applied to Nigerian Naira/US Dollar Exchange rates

The next example is sourced from the paper Agya et al. (2021) from the Federal University of Wukari in Nigeria. I have decided to use this study as an example in this paper for two main reasons. Firstly, it provides an easily understandable example of how the GARCH(1,1) model can be used with real-world data. Secondly, and most importantly, Nigeria is a rapidly growing country in both population and significance, as it possesses the largest population and economy on the African Continent. The potential for it to be a world power is partially reliant on how its economy either flourishes or fails, and an indicator of this is its exchange rates with the (current) worlds most widely used currency, the US Dollar. This example uses the GARCH model, and some of its variants, to model volatility in the Nigerian Naira - US Dollar exchange rates over time, and contains three sources of data regarding the currency exchange rates in Nigeria for the US Dollar, those being: the Official Nigerian Government’s Rate, the Interbank Rate (the rate at which banks exchange money amongst themselves), and the Bureau de Change’s Rate (BDC), with the data spanning from January 2004 to September 2020. The descriptive statistics regarding this data can be seen below.

Descriptive Statistics and Autocorrelation of Naira exchange rate (Raw)

Statistics	Official Rate	Interbank rate	Bureau de change (BDC)
Mean	1.0003	1.0005	1.0007
Median	1.0000	0.9999	1.0000
Maximum	1.0293	1.0274	1.0303
Minimum	0.9936	0.9929	0.9839
Std. Dev.	0.0033	0.0038	0.0051
Skewness	6.1456	3.9111	1.9787
Kurtosis	51.1674	25.2299	13.8430
Jarque-Bera	13694.42	3077.602	738.3367
	(0.000)	(0.000)	(0.000)
Observations	133	133	133
Ljung Box Q Statistics			
Q(1)	0.399**	0.488**	0.371**
	(0.000)	(0.000)	(0.000)
Q(5)	0.009**	-0.026**	0.024**
	(0.001)	(0.000)	(0.000)
Q(10)	-0.019**	-0.063**	-0.061**
	(0.001)	(0.000)	(0.000)

Figure 4: Descriptive statistics of rates for exchanges between the Nigerian Naira to US Dollar (2004-2020)

From these statistics, we can gather that the Bureau de Change has the highest volatility out of the three rates, followed by the Interbank Rate, followed by the Nigerian Government's Official Exchange rate. It is worth noting that governments tend to have far more stable currency exchange rates, as it is often decided artificially, as opposed to privately-owned banks, who's exchange rates are more responsive to supply and demand. A good example of this today is Lebanon, where (as of March 22nd, 2023, and according to lira-rate.com), the Lebanese Government's Lebanese Lira to US Dollar exchange rate is 15,000 LL for 1 USD, while the black market rates are around 110,000 LL for 1 USD. Figures 5 and 6 below has been provided for a more visual representation of the volatility of the three rates:

Volatility Clustering of Interbank Exchange Rate Return

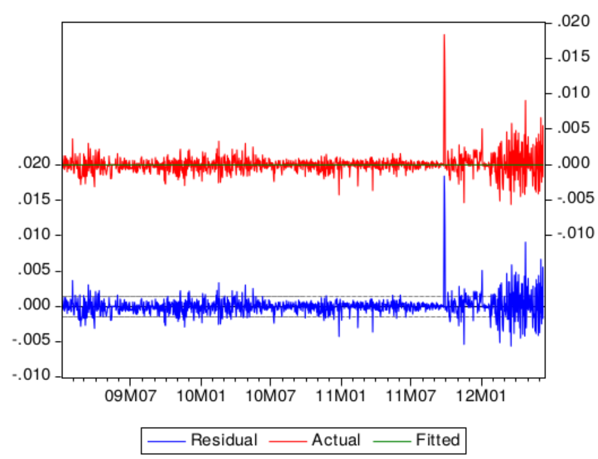

Figure 5: Visualized Volatility for the Interbank exchanges rate (2004-2020).

Volatility Clustering of Bureau de Change Exchange Rate Return

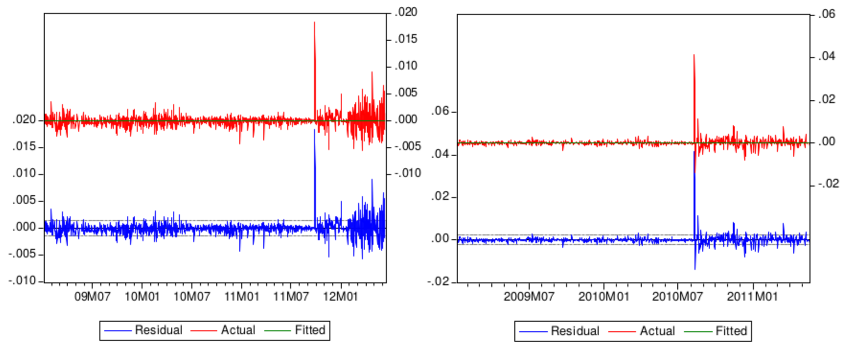

Figure 6: Visualized Volatility for the BDC and Official Gov. exchanges rates (2004-2020).

Now that we have visualized the situation and provided background statistics, we see the results of the GARCH estimations on each of the three exchange rates:.

Estimates of GARCH Models Official Rate Return, January 2004 –September 2020

	GARCH	GJR-GARCH	EGARCH	APARCH	IGARCH	TS-GARCH
Mean equation						
C	0.999 (2.280)	0.999 (1.890)	0.999 (2.790)	0.999 (2.550)	0.999 (1.905)	0.999 (2.490)
Variance Equation						
ϖ	2.750 (1.310)	2.130 (1.250)	-3.271 (0.561)	1.950 (1.060)		1.310 (1.610)
α	0.701 (0.025)	0.812 (0.054)	0.762 (0.041)	0.811 (0.009)	0.061 (0.026)	0.712 (0.036)
β	0.148 (0.054)	0.024 (0.001)	0.201 (0.038)	0.069 (0.042)	0.311 (0.026)	0.116 (0.073)
γ		5.941* (3.102)	-1.535* (0.885)	0.226* (0.110)		6.994 (5.340)
δ				0.146 (0.100)		
V	2.406 (0.193)	2.208 (0.935)	2.012 (0.013)	2.084 (0.139)	2.677 (0.130)	2.138 (0.177)
LL	827.679	835.758	9344.422	842.539	805.763	842.857

Figure 7: Values for The Official GARCH model coefficients.

Estimates of GARCH Models Interbank Rate Return, January 2004-September 2020

	GARCH	GJR-GARCH	EGARCH	APARCH	IGARCH	TS-GARCH
Mean equation						
C	1.000	1.000	0.999	1.000	1.000	1.000
	(0.000)	(0.002)	(0.000)	(0.000)	(6.570)	(0.001)
Variance Equation						
ϖ	4.120	3.206	-2.977	1.690		4.620
	(2.640)	(1.514)	(0.0521)	(0.000)		(6.740)
α	0.209	0.890	0.521	0.156	-0.009	0.950
	(0.561)	(0.403)	(0.206)	(0.316)	(0.002)	(0.140)
β	0.511	0.021	0.219	0.656	1.001	-0.160
	(0.022)	(0.019)	(0.046)	(0.134)	(0.012)	(0.007)
γ		2.613*	-4.160*	0.057*		1.312
		(1.215)	(2.434)	(0.010)		(2.388)
δ				0.441		
				(0.049)		
V	2.139*	2.145*	2.005*	2.223*	2.261*	2.123*
	(0.134)	(0.407)	(0.006)	(0.154)	(0.047)	(0.204)
LL	1083.049	1009.647	1016.967	1077.605	977.876	1002.049

Figure 8: Values for The Interbank GARCH model coefficients.

Estimates of GARCH Models for Bureau de Change Rate Return, January 2004-September 2020

	GARCH	GJR-GARCH	EGARCH	APARCH	IGARCH	TS-ARCH
Mean equation						
C	0.999	1.000	0.999	0.999	0.999	0.999
	(0.002)	(0.001)	(0.000)	(0.007)	(0.000)	(0.051)
Variance Equation						
Π	0.000	4.130	-1.762	5.580		0.000
	(0.013)	(6.690)	(0.317)	(0.004)		(0.014)
A	0.417	0.952	0.022	0.801	0.618	0.802
	(0.272)	(0.267)	(0.107)	(0.410)	(0.024)	(0.340)
B	0.355	-0.107	0.859	0.026	0.361	-0.024
	(0.081)	(0.059)	(0.025)	(0.007)	(0.024)	(0.003)
Γ		0.027*	0.579*	-0.667*		1.520*
		(0.000)	(0.178)	(0.213)		(0.206)
Δ				0.470		
				(0.078)		
V	2.001*	2.340*	2.349*	2.223*	3.397*	2.001*

Figure 9: Values for The Bureau De Change's GARCH model coefficients.

Using a GARCH(1,1) process on each of the three exchange rates, the authors of this study found that the Naira to USD exchange, has a mean reverting variance process, as the coefficients add to a value less than one, for each of the exchange rates (much like in the prior example). This assures that the volatility will almost inevitably die down, and that each of the exchange rates will return to their long-run means.

5 Conclusions

In summary, Time Series Analysis is a field of study which concerns itself with measuring time-indexed data, and the use of models and methods to measure characteristics of the data. The use of these models and methods is determined through the stationarity of the time series, whether it is strict, weak, or not stationarity. When stationarity is present in the time series data, it is possible to use the ARCH and GARCH Models, among others. The ARCH and GARCH Models measure changing conditional variances given prior squared values and other variables (i.e. conditional heteroskedasticity) to forecast volatility, which is hugely important in the fields of finance and economics. This forecasting is possible as ARCH and GARCH models (and the field of time series analysis) do not treat heteroskedasticity as a problem, like in more traditional statistics, and rather seek to measure it as a type of variance. Due to these properties, both models have become some of the most important papers in economics and finance due to their ability to model volatility. Subsequently, ARCH and GARCH Models have been the basis for many other models to be created, each of which is tailored to specific conditions, such as spatial GARCH, which takes geographic locations into account. See Real-world applications of ARCH and GARCH were discussed in this paper, with the first example being in the field of finance and the second being in the field of economics.

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A Appendix

This section will be used to show proofs for concepts explained in the essay.

AR(1) interpreted as a Squared ARCH(1) process:

$$\sigma_k^2 = \delta + \alpha_1 X_{k-1}^2, \ \ \forall k \in \mathbb{Z}, \tag{A.1}$$

$$set \ w_k = X_k^2 - \sigma_k^2, \tag{A.2}$$

$$\Rightarrow \sigma_k^2 + X_k^2 - \sigma_k^2 = \delta + \alpha_1 X_{k-1}^2 + w_k, \tag{A.3}$$

$$\Rightarrow X_k^2 = \delta + \alpha_1 X_{k-1}^2 + w_k. \tag{A.4}$$

The above equation is an AR(1) with X^2 in lieu of X

ARMA(1) interpreted as a Squared GARCH(1,1) process:

$$\sigma_k^2 = \delta + \alpha_1 X_{k-1}^2 + \beta_1 \sigma_{k-1}^2, \ \ \forall k \in \mathbb{Z}, \tag{A.5}$$

$$set \ \ w_k = X_k^2 - \sigma_k^2, \tag{A.6}$$

$$\Rightarrow \sigma_k^2 + X_k^2 - \sigma_k^2 = \delta + \alpha_1 X_{k-1}^2 + \beta_1 \sigma_{k-1}^2 + w_k, \tag{A.7}$$

$$\Rightarrow X_k^2 = \delta + \alpha_1 X_{k-1}^2 + \beta_1 \sigma_{k-1}^2 + w_k, \tag{A.8}$$

$$now, \ \ \sigma_{k-1}^k = X_{k-1}^2 - w_{k-1}, \tag{A.9}$$

$$\Rightarrow X_k^2 = \delta + \alpha_1 X_{k-1}^2 + \beta_1 (X_{k-1}^2 - w_{k-1}) + w_k, \tag{A.10}$$

$$\Rightarrow X_k^2 = \delta + (\alpha_1 + \beta_1)(X_{k-1}^2 - \delta) - \beta_1 w_{k-1} + w_k. \tag{A.11}$$

The above equation is an ARMA(1,1) with X^2 in lieu of X