FEDERAL UNIVERSITY DUSTE
FACULTY OF SCIENCE
Computer Science Department

CSC208: DISCRETE STRUCTURES

3 CREDIT UNIT

LECTURE SLIDE

ON GRAPH THEORY

BY

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CSC 208: DISCRETE STRUCTURE

Content

- Graph theory: Directed and Undirected graphs, Graph Isomorphism, Basic Graph Theorems.
- Matrices; Integer and Real matrices, Boolean Matrices, Numerical & Boolean Adjacency matrices.

GRAPH THEORY

A graph G consists of a finite set of vertices V and a finite set of edges E. Mathematically,

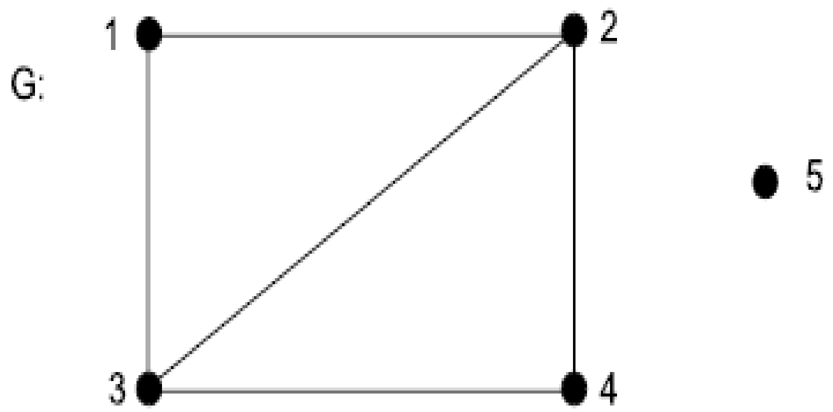
$$G = (V, E)$$

Where, $E = \{(v_i, v_j) \mid v_i, v_j \in V\}$

Let us consider $V = \{1, 2, 3, 4, 5\}$

and $E = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$.

Hence the graph $G = (V, E)$ becomes

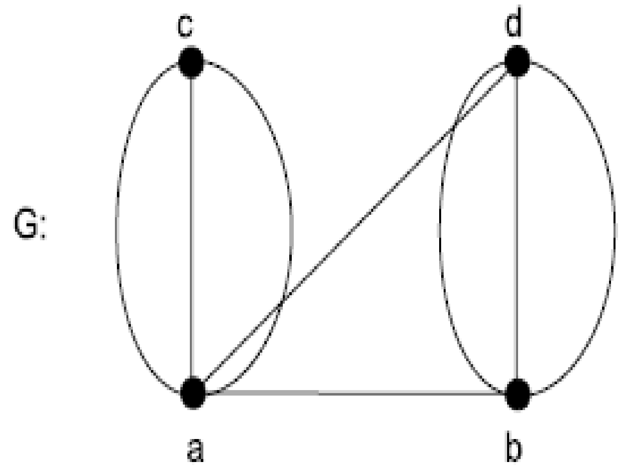


GRAPH THEORY

Order and Size

The number of vertices in a graph G(V, E) is called its order, and the number of edges is its size. That is the order of G is $|V|$ and its size $|E|$

Consider the following graph G



The order of G *i.e.* $|V| = 4$

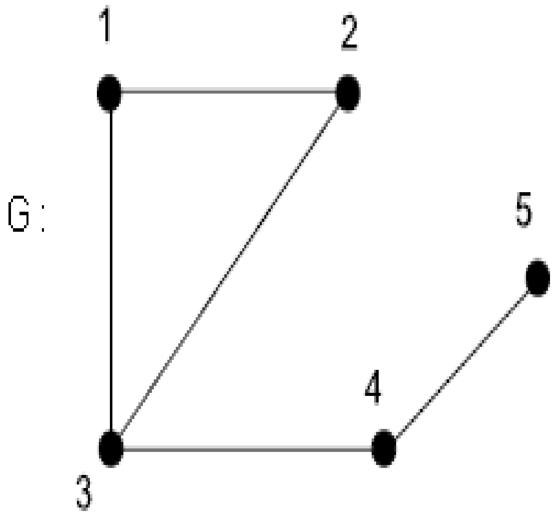
The size of G *i.e.* $|E| = 8$

GRAPH THEORY

Adjacent Vertices

Two vertices v_i and v_j are said to be adjacent if there exists an edge (v_i, v_j) in the graph G(V, E).

Consider the graph G as

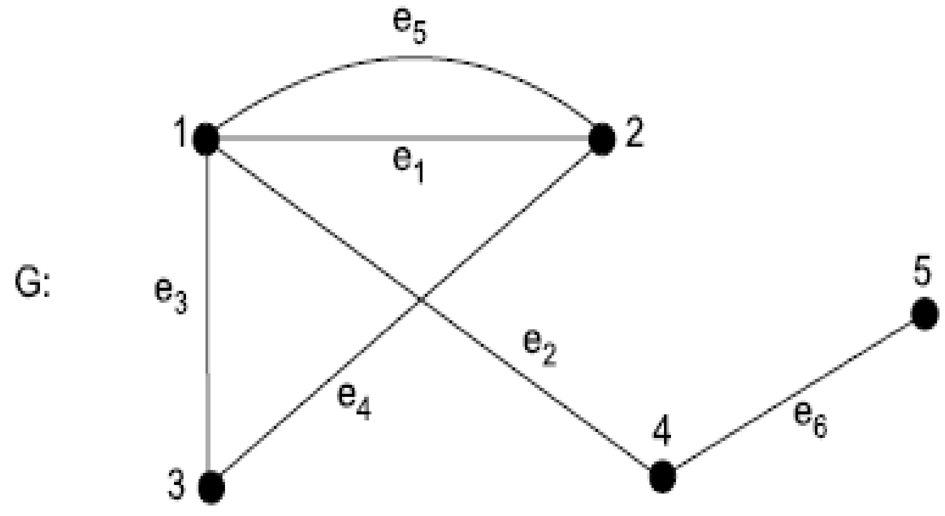


Here the vertices 1 and 2 are adjacent. Similarly, the vertices 1 and 3 are also adjacent.

GRAPH THEORY

Parallel edges

If there is more than one edge between the same pair of vertices, then the edges are termed as parallel edges. Consider the graph G as

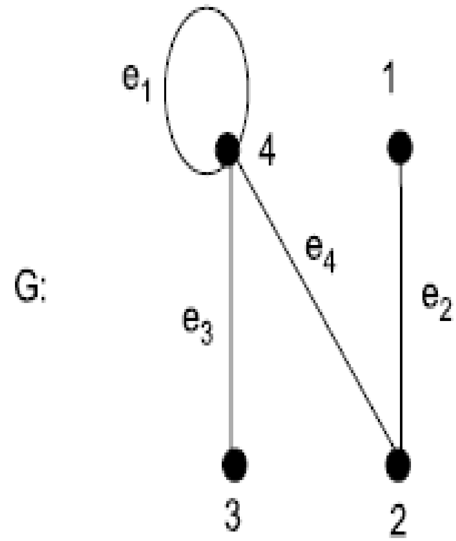


Here the edges e_1 and e_5 are parallel edges.

GRAPH THEORY

Loop

An edge whose starting and ending vertex are same is known as a loop. Mathematically $e = (v_i , v_i)$. Consider the graph G as

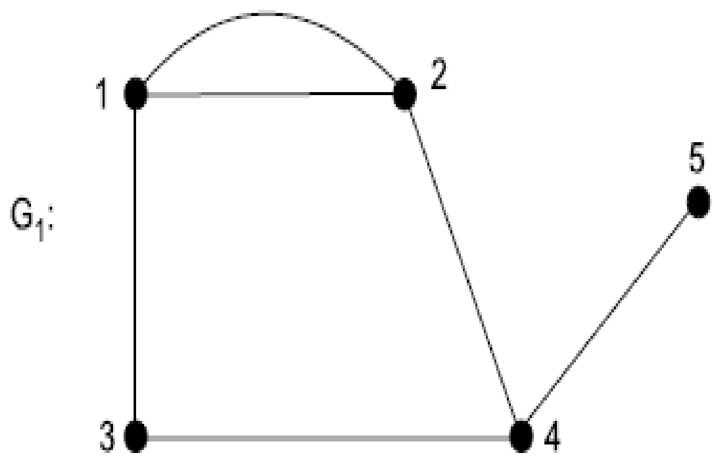


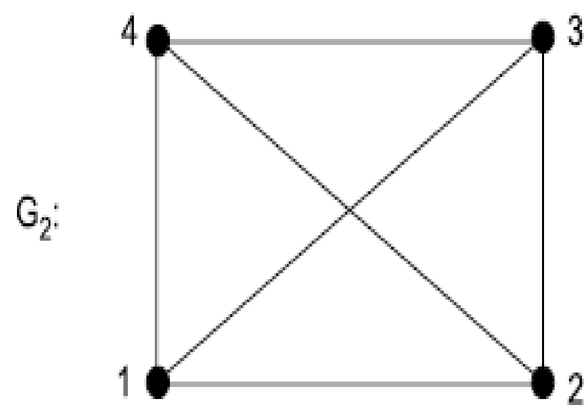
From the graph, it is clear that the edge e_1 is a loop.

KINDS OF GRAPH

Simple Graph

A graph G(V, E) that has no self-loop or parallel edges is called a simple graph. Consider the graphs G_1 and G_2 as



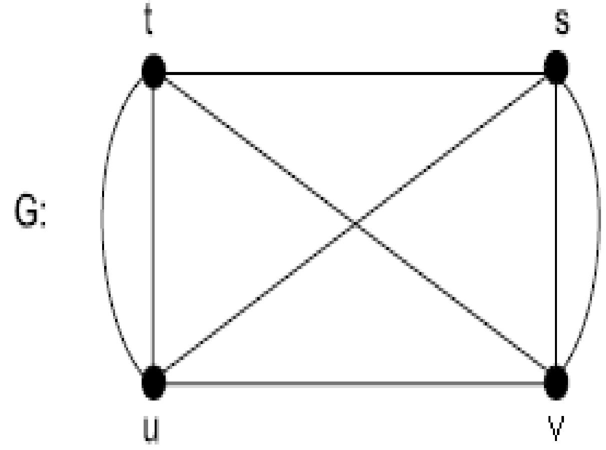


The graph G_1 is not a simple graph because there exists parallel edges between the vertices 1 and 2 whereas the graph G_2 is a simple graph.

KINDS OF GRAPH

Multi Graph

A graph G(V, E) is known as a multi graph if it contains parallel edges, *i.e.* two or more edges between a pair of vertices. It is to be noted that every simple graph is a multi graph but the converse is not true. Consider the graph G as



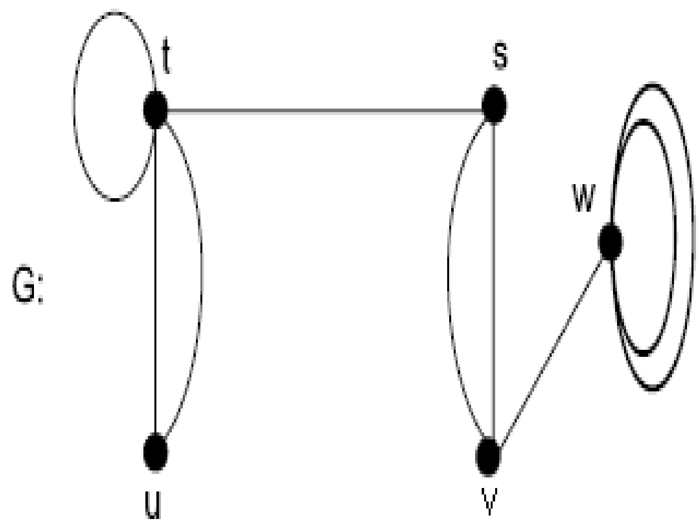
The above graph is a multi graph because there are parallel edges between the vertices u, t and v, s.

KINDS OF GRAPH

Pseudo Graph

A graph G(V,E) is known as a pseudo graph if we allow both parallel edges and loops. It is to be noted that every simple graph and multi graph are pseudo graph but the converse is not true.

Consider the graph G as



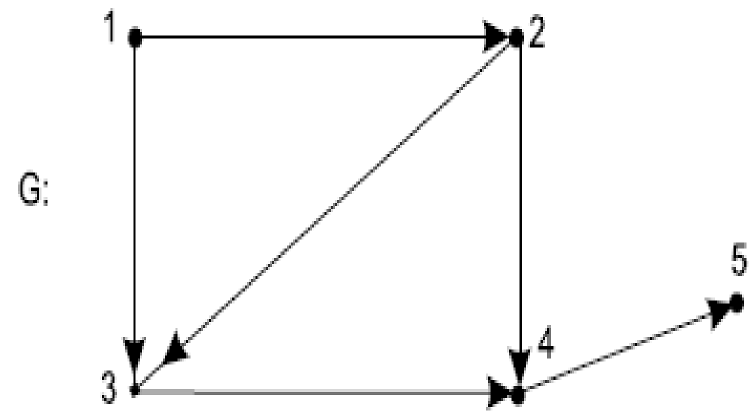
KINDS OF GRAPH

DIGRAPH

A graph G(V, E) where V is the set of nodes or vertices and E is the set of edges having direction. If (v_i, v_j) is an edge, then there is an edge from the vertex v_i to the vertex v_j. A digraph is also called a directed graph. Let us consider

$$V = \{1, 2, 3, 4, 5\} \text{ and } E = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (4, 5)\}$$

Hence, the digraph G becomes



KINDS OF GRAPH

WEIGHTED GRAPH

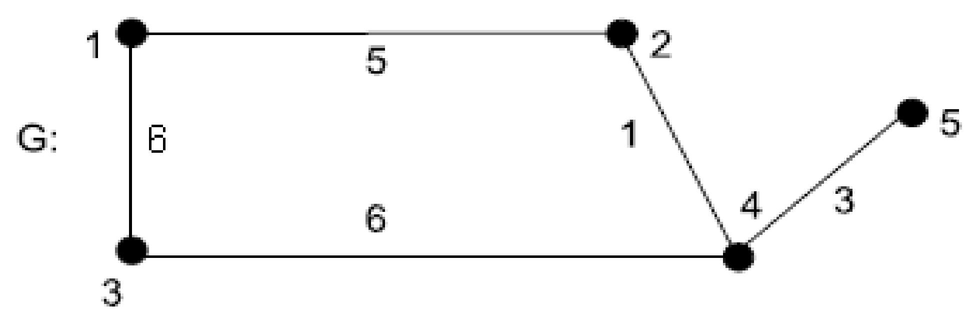
A graph (or digraph) is known as a weighted graph (or digraph) if each edge of the graph has some weights. Let us consider

$$V = \{1, 2, 3, 4, 5\} \text{ and } E = \{e_1, e_2, e_3, e_4, e_5\}$$

Where $e_1 = (1, 2), e_2 = (1, 3), e_3 = (2, 4), e_4 = (3, 4), e_5 = (4, 5)$

and $w(e_1) = 5, w(e_2) = 6, w(e_3) = 1, w(e_4) = 6, w(e_5) = 3$

Hence, the weighted graph G becomes

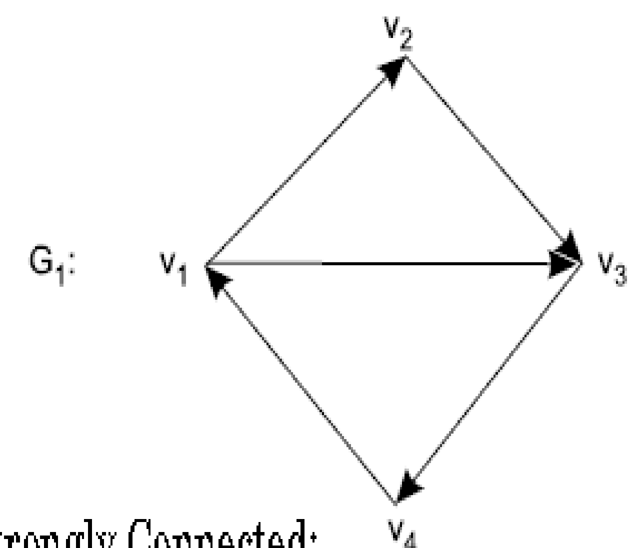


KINDS OF GRAPH

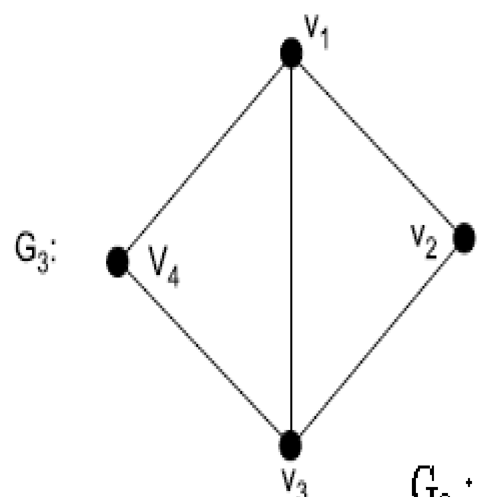
CONNECTED GRAPH

A graph (not digraph) G (V, E) is said to be connected if for every pair of distinct vertices '*u*' and '*v*' in G, there is a path. A directed graph is said to be strongly connected if for every pair of distinct vertices '*u*' and '*v*' in G, there is a directed path from '*u*' to '*v*' and also from '*v*' to '*u*'.

Consider the following graphs



G_1 : Strongly Connected;

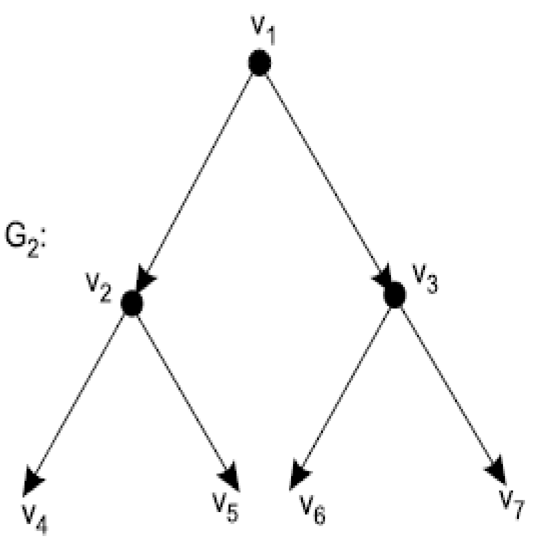


G_3 : Connected;

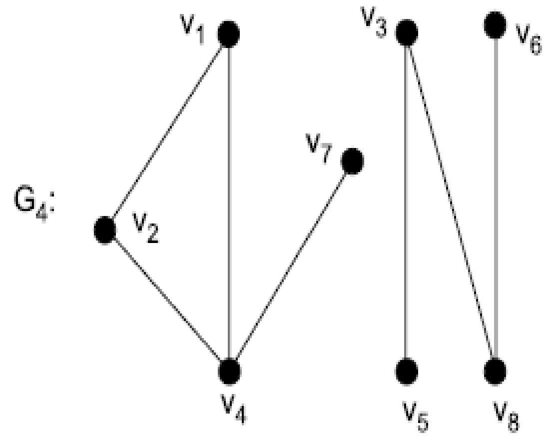
KINDS OF GRAPH

CONNECTED GRAPH

A directed graph is said to be weakly connected if for every pair of distinct vertices, there is a path without taking the direction.



G_2: Weakly Connected



G_4: Disconnected

DEGREE OF A VERTEX

The number of edges connected to the vertex 'v' is known as degree of vertex 'v', generally denoted by degree (v). In case of a digraph, there are two degrees *i.e.* indegree and outdegree.

The number of edges coming to the vertex 'v' is known as indegree of 'v' where as the number of edges emanating from the vertex 'v' is known as outdegree of 'v'. Generally, the indegree is denoted by indegree (v) and the outdegree is denoted by outdegree (v).

Note: In case of a loop, it contributes 2 to the degree of a vertex.

DEGREE OF A VERTEX

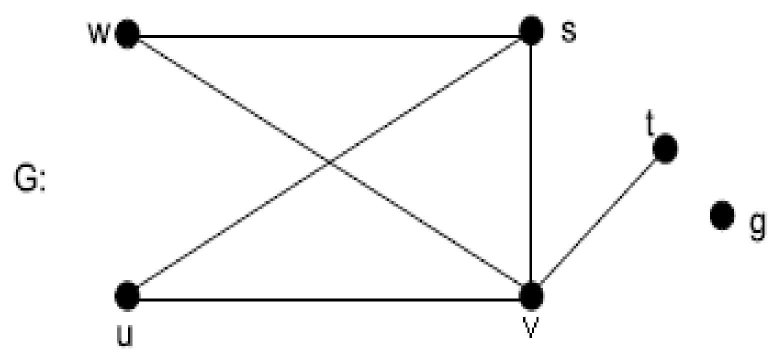
Isolated Vertex

A vertex is said to be an isolated vertex if there is no edge connected from any other vertex to the vertex.

In other words a vertex is said to be an isolated vertex if the degree of that vertex is zero.

i.e. If degree $(v) = 0$, then v is isolated.

Consider the graph G as



Now, degree $(u) = 2$; degree $(v) = 4$; degree $(t) = 1$

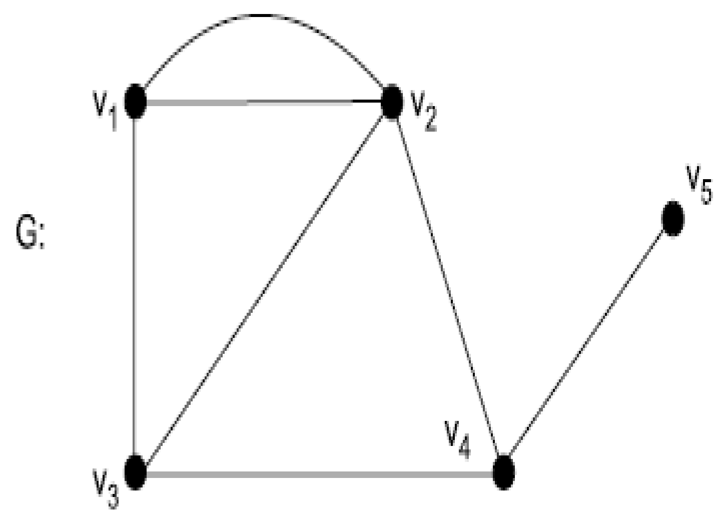
degree $(g) = 0$; degree $(s) = 3$; degree $(w) = 2$

Therefore, it is clear that 'g' is an isolated vertex.

PATH

A path in a graph is a sequence $v_1, v_2, \ldots, v_k$ of vertices each adjacent to the next, and a choice of an edge between each 'v_i' to 'v_{i+1}' so that no edge is chosen more than once.

Consider the graph G as



Here one path is $v_1\, v_2\, v_1\, v_3\, v_4\, v_5$.

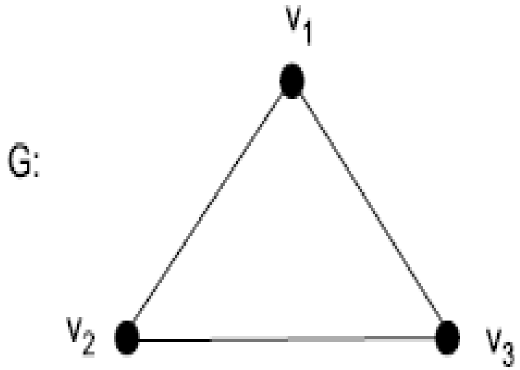
REGULAR GRAPH

A graph G (V, E) is said to be regular if the degree of every vertex are equal. Mathematically, G is denoted as regular if

$$\text{degree}(v_i) = \text{degree}(v_j)\ \forall\ i, j.$$

Where, $v_i, v_j \in G(V, E)$.

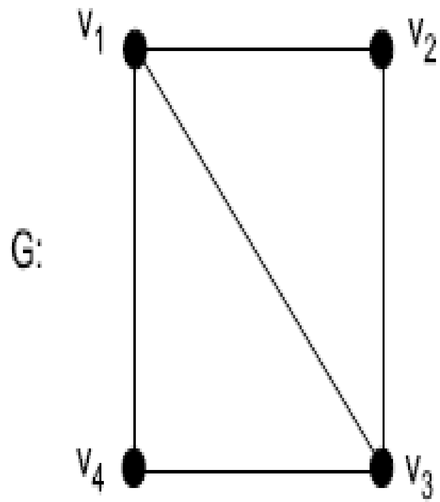
Consider the graph G as



In the above graph, degree (v_1) = degree (v_2) = degree (v_3) = 2. Therefore, the graph G is regular (2 regular). The above graph is also complete.

CYCLIC GRAPH

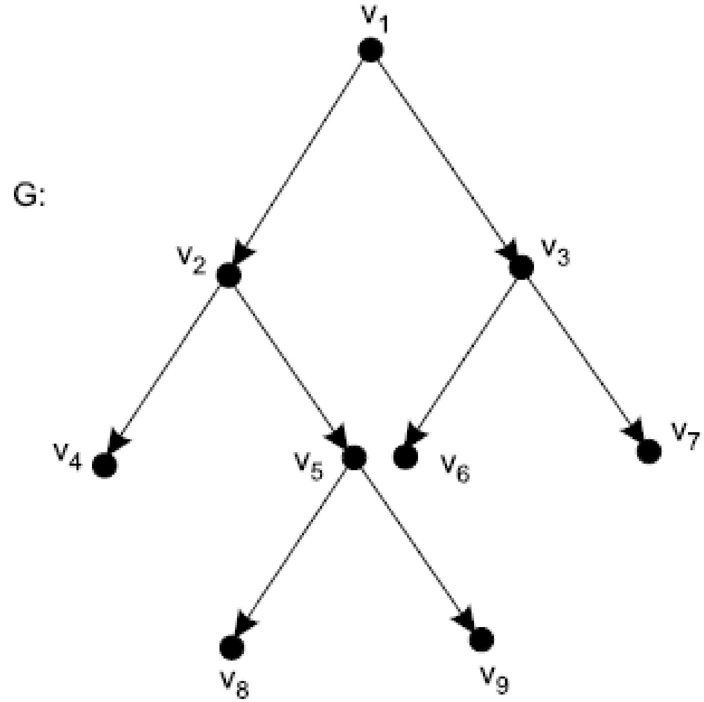
If there is a path containing one or more edges which starts from a vertex 'v' and terminates into the same vertex, then the path is known as a cycle. Consider the graph G as



In the above graph G, one cycle is $v_1 v_2 v_3 v_1$. Similarly, another cycle is $v_1 v_2 v_3 v_4 v_1$.

ACYCLIC GRAPH

A graph (digraph) which does not have any cycle is known as an acyclic graph (digraph).
Consider the graph G as



Here, G is an acyclic graph.

MATRIX REPRESENTATION OF GRAPHS

A matrix is a convenient way to represent a graph. A computer to analyze them can use such a representation.

Adjacency Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

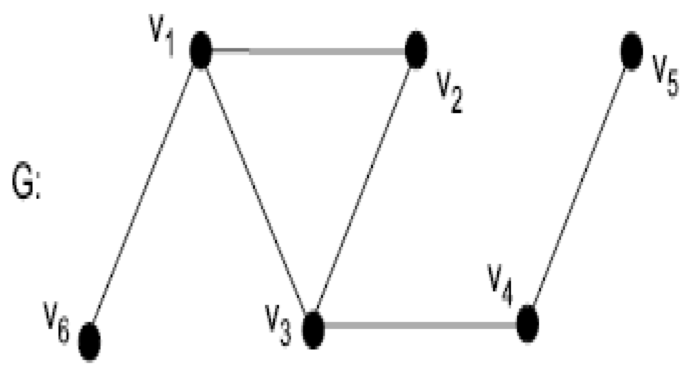
where,

$$a_{ij} = \begin{cases} 1; & \text{if there is an edge from '}v_i\text{' to '}v_j\text{'} \\ 0; & \text{Otherwise} \end{cases}$$

MATRIX REPRESENTATION OF GRAPHS

1 Example

Consider the graph G as



Hence, the adjacency matrix is given as

What will be the matrix representation of the graph?

MATRIX REPRESENTATION OF GRAPHS

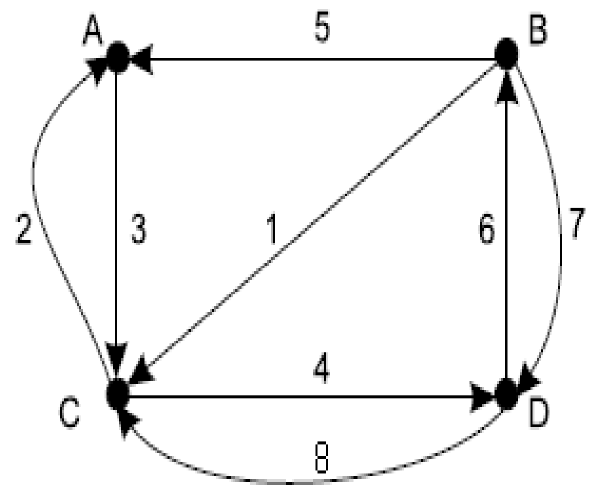
The Adjacency matrix of the above graph is given as

$$
A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

MATRIX REPRESENTATION OF GRAPHS

2 Example

Consider the graph G as



What will be the matrix representation of the graph?

In case of a weighted graph the adjacency matrix can be found out with the relation

$$a_{ij} = \begin{cases} n; & w \text{ is the weight of the edges from } `v_i\text{' to } `v_j\text{'} \\ 0; & \text{Otherwise} \end{cases}$$

MATRIX REPRESENTATION OF GRAPHS

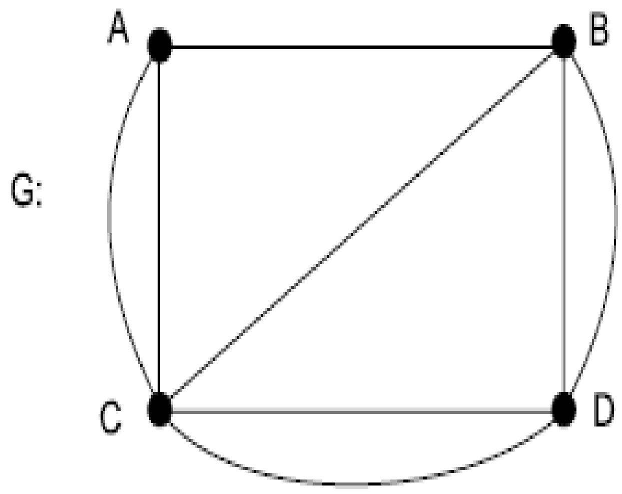
The adjacency matrix of the above graph with respect to the ordering A, B, C and D is given below.

$$A = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 5 & 0 & 1 & 7 \\ 2 & 0 & 0 & 4 \\ 0 & 6 & 8 & 0 \end{bmatrix}$$

MATRIX REPRESENTATION OF GRAPHS

3 Example

Consider the graph G as



In case of a multi graph the adjacency matrix can be found out with the relation.

$$a_{ij} = \begin{cases} n; & n \text{ be the number of edges from } `v_i\text{' to } `v_j\text{'} \\ 0; & \text{Otherwise} \end{cases}$$

MATRIX REPRESENTATION OF GRAPHS

The adjacency matrix of the above graph with respect to the ordering A, B, C and D is given below.

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

MATRIX REPRESENTATION OF GRAPHS

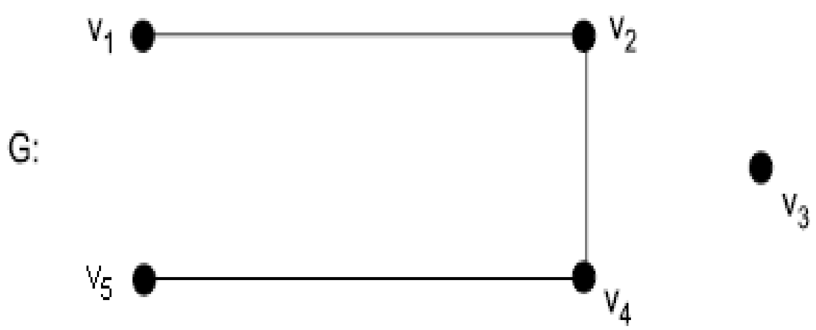
Path Matrix

Suppose that G be simple graph with n-vertices. Then the $(n \times n)$ matrix $P = [P_{ij}](n \times n)$ defined by

$$P_{ij} = \begin{cases} 1; & \text{if there is a path from } v_i \text{ to } v_j \\ 0; & \text{Otherwise} \end{cases}$$

is known as the path matrix or reachability matrix.

Consider the graph G as



MATRIX REPRESENTATION OF GRAPHS

Path Matrix

Therefore, the path matrix of the above graph relative to the ordering v_1, v_2, v_3, v_4, v_5 is given as

$$P = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

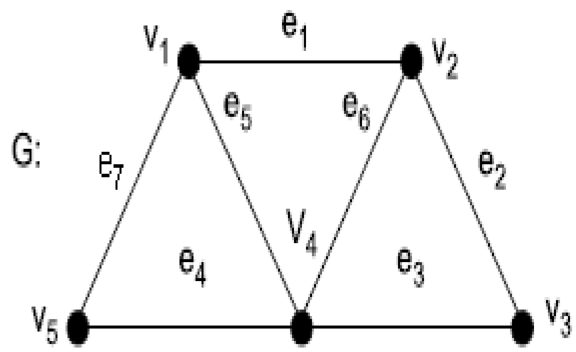
MATRIX REPRESENTATION OF GRAPHS

Incidence Matrix

Suppose that G be a simple undirected graph with m vertices and n edges, then the incidence matrix is a matrix of order $(m \times n)$ where the element a_{ij} is defined as

$$a_{ij} = \begin{cases} 1; & \text{If vertex } i \text{ belongs to edges } j. \\ 0; & \text{Otherwise} \end{cases}$$

Consider the graph G as



MATRIX REPRESENTATION OF GRAPHS

Incidence Matrix

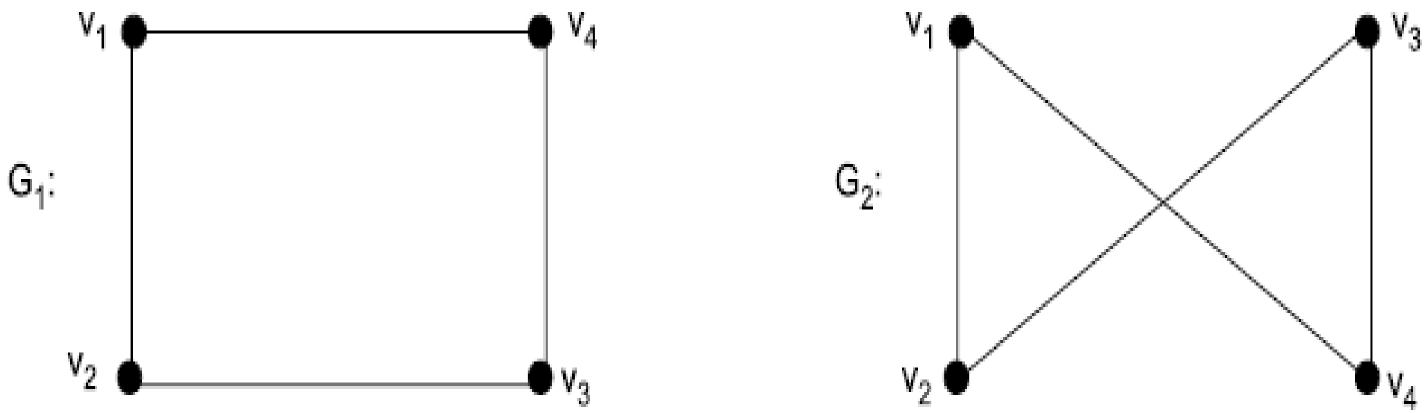
Hence, the incidence matrix of the graph G is of order (5×7). The incidence matrix relative to the ordering v_1, v_2, v_3, v_4, v_5 and $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ is given as below.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

GRAPH ISOMORPHISM

Suppose $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be two graphs. Then the two graphs G_1 and G_2 are said to be isomorphic if there is one to one correspondence between the edges E_1 of G_1 and E_2 of G_2 which indicates that if $(u_1, v_1) \in G_1$ then $(u_1, v_1) \in G_2$.

Such a pair of correspondence is known as graph isomorphism. The different way of representing the same graph is known as graph isomorphism. Consider graph G_1 and G_2 as

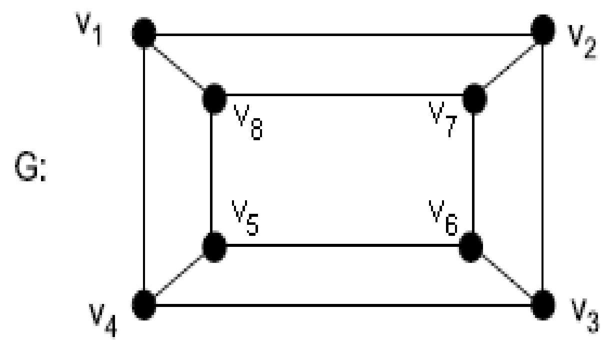


Therefore, the graphs G_1 and G_2 are isomorphic to each other.

BIPARTITIE GRAPH

Suppose that G: (V, E) be the graph. If the vertex set V can be partitioned into two non empty disjoint sets V_1 and V_2 such that each edge of the graph G has one end in V_1 and other end in V_2, then the graph is said to be bipartite graph.

Consider the graph G as



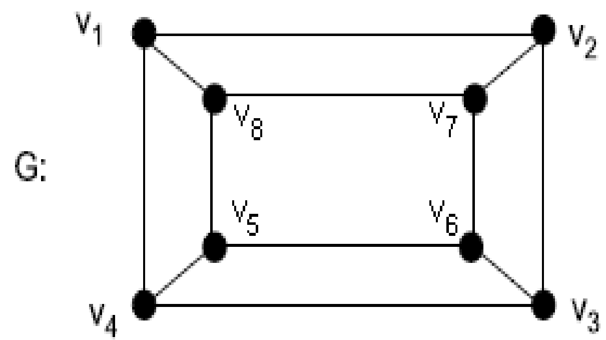
Let $V_1 = \{v_1, v_3, v_5, v_7\}$ and $V_2 = \{v_4, v_2, v_6, v_8\}$

Now, $(V_1 \cap V_2) = \phi$ and each edge of G has one vertex in V_1 and other vertex at V_2. So, G is said to be a bipartite graph.

BIPARTITIE GRAPH

Suppose that G: (V, E) be the graph. If the vertex set V can be partitioned into two non empty disjoint sets V_1 and V_2 such that each edge of the graph G has one end in V_1 and other end in V_2, then the graph is said to be bipartite graph.

Consider the graph G as



Let $V_1 = \{v_1, v_3, v_5, v_7\}$ and $V_2 = \{v_4, v_2, v_6, v_8\}$

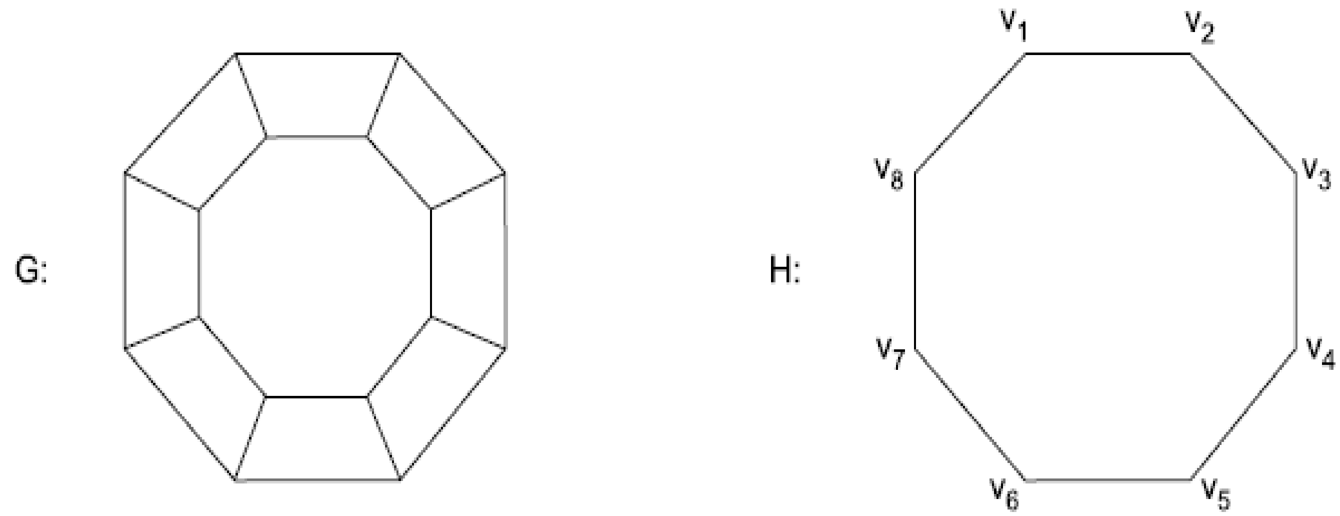
Now, $(V_1 \cap V_2) = \phi$ and each edge of G has one vertex in V_1 and other vertex at V_2. So, G is said to be a bipartite graph.

SUBGRAPH

Suppose that G and H be two graphs with vertex sets V(G) and V(H). Let the edge sets be E(G) and E(H). Now H is said to be subgraph of G if

$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G)$$

Consider two graphs G and H as

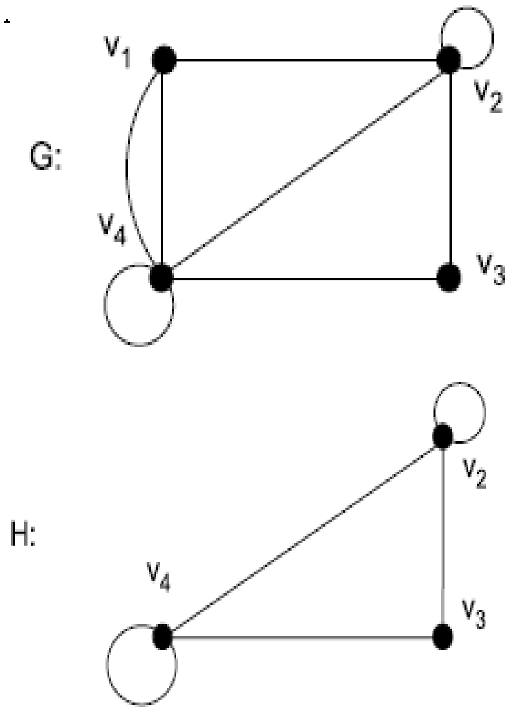


Therefore, it is clear that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. So, H is a subgraph of G.

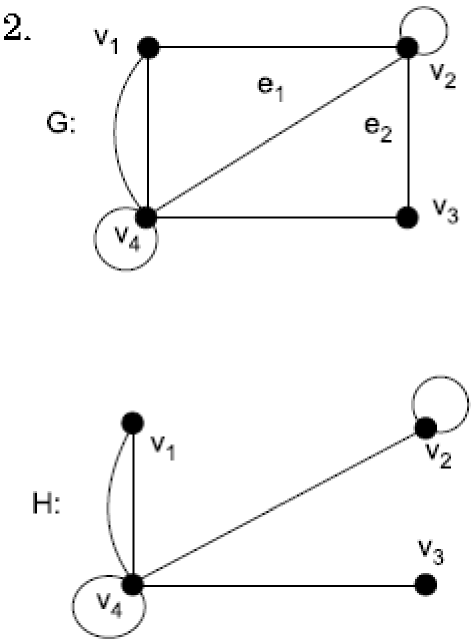
SUBGRAPH

Consider the graph G as

1.



2.



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