Biermann CompPhys HW11

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11.4

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (r^2 \dot{\phi}^2 \sin^2\theta + r^2 \dot{\theta}^2)$$

$$U = -mga\cos\theta$$

$$\mathcal{L} = \frac{1}{2} m (a^2 \dot{\phi}^2 \sin^2\theta + a^2 \dot{\theta}^2) + mga\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m a^2 \dot{\phi}^2 \sin\theta\cos\theta - mga\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m \ell^2 \ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0, \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m a^2 \ddot{\phi} \dot{\phi} \sin^2\theta + m a^2 \dot{\phi}^2 \dot{\theta} \sin\theta\cos\theta$$

$$\Rightarrow \begin{aligned} & a\ddot{\theta} - \ell \dot{\phi}^2 \sin\theta\cos\theta + g\sin\theta = 0 \\ & \ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta = 0 \end{aligned}$$

$$\Rightarrow \begin{cases} \dfrac{d\theta}{dt} = v_\theta, \quad \dfrac{d\phi}{dt} = v_\phi \\ \dfrac{dv_\theta}{dt} = v_\phi^2 \sin\theta\cos\theta - \dfrac{g}{a}\sin\theta \\ \dfrac{dv_\phi}{dt} = -2 v_\phi v_\theta \cot\theta \end{cases}$$

11.10

$$\frac{dx}{d\tau} = \eta, \quad \frac{d\eta}{d\tau} = -2\zeta\eta - x \quad \text{when } \eta = \frac{v}{\omega_0}, \ \tau = \omega_0 t$$

(a) I could not figure this out, so I use wolfram alpha's solution.

11.14 Double Pendulum

(a) $T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$$v_1^2 = \ell_1^2 \dot{\theta}_1^2$$

$$v_2^2 = \ell_1^2 \dot{\theta}_1^2 + \ell_2^2 \dot{\theta}_2^2 + 2\ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$V_2^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$V = m_1 g y_1 + m_2 g y_2$$

$$y_1 = -l_1\cos\theta_1\,, \quad y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2$$

$$\Rightarrow \boxed{\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)\\ &+ m_1 g l_1\cos\theta_1 + m_2 g\left(l_1\cos\theta_1 + l_2\cos\theta_2\right)\end{aligned}}$$

(b) $\mathcal{H} = T(p_i, q_i) + V(q_i)$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\Rightarrow p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2\dot{\theta}_1 + \tfrac{1}{2}m_2\left(2l_1^2\dot{\theta}_1 + 2l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$

$$= m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$= (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$p_{\theta_1} - m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2) = (m_1+m_2)l_1^2\dot{\theta}_1$$

$$\Rightarrow \dot{\theta}_1 = \frac{p_{\theta_1} - m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)}{(m_1+m_2)l_1^2}$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = \tfrac{1}{2}m_2\left(2l_2^2\dot{\theta}_2 + 2l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2)\right)$$

$$= m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$$

$$p_{\theta_2} = m_2 l_2^2\dot{\theta}_2 + \frac{m_2 l_2}{(m_1+m_2)l_1}\left(p_{\theta_1} - m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)\right)\cos(\theta_1-\theta_2)$$

$$p_{\theta_2} = m_2 l_2^2\dot{\theta}_2 + \frac{m_2 l_2}{(m_1+m_2)l_1}p_{\theta_1}\cos\Delta\theta - \frac{m_2^2 l_2^2}{m_1+m_2}\dot{\theta}_2\cos^2\Delta\theta$$

$$\left(\frac{p_{\theta_2}}{m_2 l_2} - \frac{p_{\theta_1}\cos\Delta\theta}{(m_1+m_2)l_1}\right) = \left(l_2 - \frac{m_2 l_2}{m_1+m_2}\cos^2\Delta\theta\right)\dot{\theta}_2$$

$$\Rightarrow \dot{\theta}_2 = \frac{\cancel{m_1+m_2}}{l_2 m_1 + m_2 l_2(1-\cos^2\Delta\theta)}\left(\frac{(m_1+m_2)l_1 p_{\theta_2} - m_2 l_2 p_{\theta_1}\cos\Delta\theta}{m_2(\cancel{m_1+m_2})l_1 l_2}\right)$$

$$\Rightarrow \dot{\theta}_2 = \frac{\cancel{m_1+m_2}}{l_2 m_1 + m_2 l_2 (1-\cos^2\Delta\theta)} \left(\frac{(m_1+m_2) l_1 P_{\theta_2} - m_2 l_2 P_{\theta_1} \cos\Delta\theta}{m_2 (\cancel{m_1+m_2}) l_1 l_2} \right)$$

$$\boxed{\dot{\theta}_2 = \frac{(m_1+m_2) l_1 P_{\theta_2} - m_2 l_2 P_{\theta_1} \cos\Delta\theta}{m_2 l_2^2 l_1 (m_1 + m_2 \sin^2\Delta\theta)}}$$

$$\Rightarrow \dot{\theta}_1 = \frac{P_{\theta_1} - m_2 l_1 l_2 \dot{\theta}_2 \cos\Delta\theta}{(m_1+m_2) l_1^2}$$

$$= \frac{P_{\theta_1}}{(m_1+m_2) l_1^2} - \frac{\cancel{m_2 l_2}}{(m_1+m_2) l_1} \left(\frac{(m_1+m_2) l_1 P_{\theta_2} - m_2 l_2 P_{\theta_1} \cos\Delta\theta}{\cancel{m_2} l_2^{\cancel{2}} l_1 (m_1 + m_2 \sin^2\Delta\theta)} \right) \cos\Delta\theta$$

$$= \frac{P_{\theta_1}}{(m_1+m_2) l_1^2} - \frac{(m_1+m_2) l_1 P_{\theta_2} \cos\Delta\theta - m_2 l_2 P_{\theta_1} \cos^2\Delta\theta}{(m_1+m_2)(m_1+m_2 \sin^2\Delta\theta) l_1^2 l_2}$$

$$= \frac{\cancel{m_1} l_2 P_{\theta_1} + \cancel{m_2} l_2 P_{\theta_1} (\overbrace{\sin^2\Delta\theta + \cos^2\Delta\theta}^{1}) - (\cancel{m_1+m_2}) l_1 P_{\theta_2} \cos\Delta\theta}{(\cancel{m_1+m_2})(m_1+m_2 \sin^2\Delta\theta) l_1^2 l_2}$$

$$\boxed{\dot{\theta}_1 = \frac{l_2 P_{\theta_1} - l_1 P_{\theta_2} \cos\Delta\theta}{l_1^2 l_2 (m_1 + m_2 \sin^2\Delta\theta)}}$$

$$\mathcal{H} = T + V$$

$$= \frac{1}{2}(m_1+m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\Delta\theta$$

$$- (m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

$$= \frac{\cancel{l_1^2}(m_1+m_2)\left(l_2^2 P_{\theta_1}^2 + l_1^2 P_{\theta_2}^2 \cos^2\Delta\theta - 2 l_1 l_2 P_{\theta_1} P_{\theta_2} \cos\Delta\theta\right)}{2\, l_1^{\cancel{4}\,2} l_2^2 (m_1 + m_2 \sin^2\Delta\theta)^2}$$

$$+ \frac{\cancel{m_2 l_2^2}\left(l_1^2 (m_1+m_2)^2 P_{\theta_2}^2 + l_2^2 m_2^2 P_{\theta_1}^2 \cos^2\Delta\theta - 2 l_1 l_2 m_2 (m_1+m_2) P_{\theta_1} P_{\theta_2} \cos\Delta\theta\right)}{2 l_1^2 l_2^{\cancel{4}\,2} m_2^{\cancel{2}} (m_1 + m_2 \sin^2\Delta\theta)^2}$$

$$+ \frac{\cancel{m_2 l_1 l_2}\left(l_2 P_{\theta_1} - l_1 P_{\theta_2} \cos\Delta\theta\right)\left(l_1 (m_1+m_2) P_{\theta_2} - l_2 m_2 P_{\theta_1} \cos\Delta\theta\right)}{l_1^{\cancel{3}\,2} l_2^{\cancel{3}\,2} \cancel{m_2} (m_1 + m_2 \sin^2\Delta\theta)^2} + V$$

$$= \frac{1}{2l_1^2 l_2^2 m_2 (m_1 + m_2 \sin^2\Delta\theta)^2}\Big[m_2(m_1+m_2) l_2^2 P_{\theta_1}^2 + m_2(m_1+m_2) l_1^2 P_{\theta_2}^2 \cos^2\Delta\theta - 2m_2(m_1+m_2) l_1 l_2 P_{\theta_1} P_{\theta_2} \cos\Delta\theta$$

$$+ (m_1+m_2)^2 l_1^2 P_{\theta_2}^2 + m_2^2 l_2^2 P_{\theta_1}^2 \cos^2\Delta\theta - 2 l_1 l_2 m_2 (m_1+m_2) P_{\theta_1} P_{\theta_2} \cos\Delta\theta$$

$$+ 2m_2(m_1+m_2) l_1 l_2 P_{\theta_1} P_{\theta_2} - 2m_2^2 l_2^2 P_{\theta_1}^2 \cos\Delta\theta - 2m_2(m_1+m_2) l_1^2 P_{\theta_2}^2 \cos\Delta\theta + 2m_2^2 l_1 l_2 P_{\theta_1} P_{\theta_2} \cos^2\Delta\theta \Big] + U$$

$$[\dots] = \left(m_2(m_1+m_2) l_2^2 + m_2^2 l_2^2 \cos^2\Delta\theta - 2m_2^2 l_2^2 \cos\Delta\theta\right) P_{\theta_1}^2$$

$$+ \left(m_2(m_1+m_2) l_1^2 \cos^2\Delta\theta + (m_1+m_2)^2 l_1^2 - 2m_2(m_1+m_2) l_1^2 \cos\Delta\theta\right) P_{\theta_2}^2$$

$$+ \left(2m_2(m_1+m_2) l_1 l_2 + 2m_2^2 l_1 l_2 \cos^2\Delta\theta - 4m_2(m_1+m_2) l_1 l_2 \cos\Delta\theta\right) P_{\theta_1} P_{\theta_2}$$

$$= \left(m_1 m_2 l_2^2 + m_2^2 l_2^2 \sin^2\Delta\theta - 2m_2^2 l_2^2 \cos\Delta\theta\right) P_{\theta_1}^2$$

$$+ \left(m_1 m_2 l_1^2 \cos^2\Delta\theta + m_2^2 l_1^2 \cos^2\theta + m_1^2 l_1^2 l_2 + m_2^2 l_1^2 + 2m_1 m_2 l_1^2 - 2m_2(m_1+m_2) l_1^2 \cos\Delta\theta\right) P_{\theta_2}^2$$

$$+ \left(2m_1 m_2 l_1 l_2 + 2m_2^2 l_1 l_2 \sin^2\Delta\theta - 4m_2(m_1+m_2) l_1 l_2 \cos\Delta\theta\right) P_{\theta_1} P_{\theta_2}$$

$$= \left(m_2 l_2^2 (m_1 + m_2 \sin^2\Delta\theta) - 2m_2^2 l_2^2 \cos\Delta\theta\right) P_{\theta_1}^2$$

$$+ \Big($$

I'm having a hard time simplifying this...

According to the internet (wolfram):

$$\dot{P}_{\theta_1} = -\frac{\partial H}{\partial \theta_1} = -(m_1+m_2) g l_1 \sin\theta_1 - C_1 + C_2$$

$$\dot{P}_{\theta_2} = -\frac{\partial H}{\partial \theta_2} = -m_2 g l_2 \sin\theta_2 + C_1 - C_2$$

where

$$C_1 = \frac{P_{\theta_1} P_{\theta_2} \sin(\theta_1 - \theta_2)}{l_1 l_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]}$$

$$C_2 = \frac{l_2^2 m_2 P_1^2 + l_1^2 (m_1+m_2) P_2^2 - l_1 l_2 m_2 P_1 P_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]^2} \sin[2(\theta_1 - \theta_2)]$$

11-15

(a) $$\frac{dS}{dt} = bN - dS - \lambda S$$

$$\frac{dE}{dt} = \lambda S - dE - \alpha E$$

$$\frac{dI}{dt} = \alpha E - dI - \gamma I$$

$$\frac{dR}{dt} = \gamma I - \nu R - dR$$

where $\lambda = \frac{\beta}{N}$

(b) If $d = b$,

$$\frac{dS}{dt} = (N - S)b - \lambda S$$

(c) This model assumes the mortality rate of the virus is negligable compared to the mortality rate of the population

(d)