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STELLAR PERTURBATIONS OF ORBITS OF LONG-PERIOD COMETS AND THEIR SIGNIFICANCE FOR COMETARY CAPTURE

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Approximate expressions for the heliocentric impulse gained by a long-period comet from stellar passages at different distances are derived. The frequency of stellar passages and its dependence on the passage distance is investigated. The relative importances of different passage distances over very long time intervals are discussed. Expected values of the transverse component of the impulse gained from stellar perturbations during one revolution are estimated for different values of the aphelion distance. The resulting dispersions of the perihelion distance and inclination are computed. They are found to influence the initial stages of cometary capture from the Oort cloud but have a very small effect on the capture efficiency.

Возмущение орбит долгопериодических комет от звезды и его значение для кометного захвата

Определяются приблизительные выражения для гелиоцентрического импульса присбретаемого долгопериодической кометой вследствие прохождения звезды на различных расстояниях. Исследуется частота этих прохождений и ее зависимость от расстояния. Обсуждаются долговременные значения прохождений на различных расстояниях. Для различных расстояний афелия сделаны оценки ожидаемых значений поперечного компонента импульса приобретаемого кометой от возмущений в течение одного оборота. Вычисляются окончательные значения дисперсии расстояния перигелия и наклона орбиты. Найдено, что эти возмущения оказывают влияние на захват кометы из облака Оорта, но их воздействие на эффективность захвата очень мало.

1. Introduction

The majority of comets observed so far move in very elongated orbits with aphelia far outside the planetary system. During the passages of these comets through the planetary region very important changes of the orbits are caused by planetary perturbations. Thus, for some comets the total energies may decrease so that the aphelia are displaced into the planetary system, and occasionally the comets may be „captured" into the Jupiter family.

The distribution of orbital energies, or inverse semimajor axes $(1/a)$, for the long-period comets seems to peak at $a \sim 10^5$ a.u. according to Oort (1950). This is interpreted as a „cometary reservoir" (commonly called „the Oort cloud"), which is often identified as the direct origin of all the observed comets. However, very little is known both about the age and origin of the Oort cloud and about its size and structure. One may note that the primary observable characteristic of the cloud, namely its radius as evidenced by the clustering of semimajor axes, has been subject to drastic revision. Oort's determination was based upon comets of all observed perihelion distances (q) with no consideration of non-gravitational forces. Marsden and Sekanina (1973), however, made a new estimate based only on comets with $q > 3$ a.u., for which the non-gravitational forces are believed to be negligible. The result implies a reduction of the cloud radius by a factor of four to $a \approx 25\,000$ a.u.

The concept of „the Oort cloud" must therefore be used with care, and in this work it will not be used at all. In any case, however, it is obvious that a large number of comets move in strongly elongated, elliptical orbits with perihelia in the inner planetary system and aphelia considerably more than 100 a.u. from the sun. Such orbits may be imagined to represent different stages of evolution in a capture process where certain comets are transferred from the long-period reservoir to the short-period group. A comet in such an orbit spends most of the time far from the sun, where planetary perturbations are inconspicuous, but during this time the motion of the comet relative to the sun is instead influenced by the gravitation of nearby stars, and on some occasions the comet may gain a large impulse from a star passing through the close vicinity.

Some obvious questions are the following:

How strong are the stellar perturbations, i.e. how large an impulse per revolution in the heliocentric frame of reference does a comet gain from the stars?

How does this impulse depend on the aphelion distance (Q) of the comet?

How does the impulse influence the orbital elements of the comet, and how can the capture process thereby be affected?

Which stars contribute the largest fraction of the impulse: the rare stellar passages very close to the

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comet, or the abundance of stars all the time present at larger distances?

2. Choice of a Dynamical Model

The cometary orbits under consideration are characterized by small perihelion distances ($q \lesssim 10$ a.u.) and large aphelion distances ($Q \gg 100$ a.u.), and the eccentricities are consequently close to unity. The orientations of the orbits are considered to be random. This is very nearly true for the observed long-period comets. These comets fulfil the condition of observational selection, which is roughly: $q < 2.5$ a.u. The diffusion of comets toward smaller aphelion distances, caused by planetary perturbations, is characterized by a gradual clustering of the orbital planes towards the plane of the ecliptic for decreasing aphelion distances (Shteins, 1972). Therefore it may happen that the growing lack of isotropy of aphelion directions for comet groups of decreasing aphelion distance is much more pronounced for the non-observed comets with perihelia near the orbit of Jupiter (Delsemme, 1973), for which the diffusion takes place at a higher rate (Shteins, 1972; Everhart, 1972, 1974). However, this possibility does not influence our conclusions in a significant way, and isotropically distributed aphelion directions may be assumed for all the cometary orbits under consideration.

In order to compute the expected impulse per revolution as a function of the distances of the perturbing stars, one may proceed in two different ways.

1) One considers a static space distribution of stars at rest relative to the sun. Each spherical shell centered at the sun contains a certain number of stars. The resultant gravitational attraction of these stars can be statistically estimated, as well as the differential (tidal) acceleration of the comet relative to the sun. By integrating the tidal acceleration with respect to time over the revolution period of the comet, one obtains the heliocentric impulse gained by the comet while moving once around its orbit. In order just to compare the importance of different shells, however, it suffices to compare the tidal accelerations, and the integration with respect to time need not be executed for this purpose.

2) One computes the tidal impulse (I) for a „complete passage" by each star, as it moves along its orbit for a very long time (T) before and after the occasion of closest approach to the comet and the sun. If P is the orbital period of the comet, the impulse per revolution is obtained, if I is multiplied by $\frac{1}{2}P/T$. The impulses gained separately by the comet and the sun can easily be computed approximately (cf. below), and each of them is determined by the distance and direction of the closest point on the stellar orbit. For the majority of passages (the ones excluded are the closest ones), the point of closest approach to the sun alone effectively characterizes each passage. One may therefore consider these points as representative of the passages and having a quasi-static space distribution, and the importance of different spherical shells can then be investigated in the same way as under 1).

The dynamical problem, generally formulated is a three-body problem where one of the masses is infinitesimal and starts in an eccentric orbit about one of the finite masses, while the relative speed[1]) of the two finite masses is in general quite large. Therefore, if the sun is fixed at the origin, both the massless comet and the massive star move at different speeds. Corresponding to the two methods described above, there are two different ways of simplifying the problem.

1) Relative to the sun, the comet moves while the star is at rest (Fesenkov, 1951; Shteins, 1955; Makover, 1964).

2) Relative to the sun, the comet is at rest while the star moves (Öpik, 1932; Oort, 1950; Shteins and Sture, 1962; Sekanina, 1968b).

The geometrical situation is shown in Fig. 1, and if G is the gravitational constant and M_* the mass of the star, the impulse per unit mass (I) of the comet in the heliocentric frame of reference is given by:

$$(1) \quad I = GM_* \left\{ \int_{-T}^{T} \frac{\boldsymbol{r}_* - \boldsymbol{r}_c}{|\boldsymbol{r}_* - \boldsymbol{r}_c|^3} \, dt - \int_{-T}^{T} \frac{\boldsymbol{r}_*}{r_*^3} dt \right\}$$

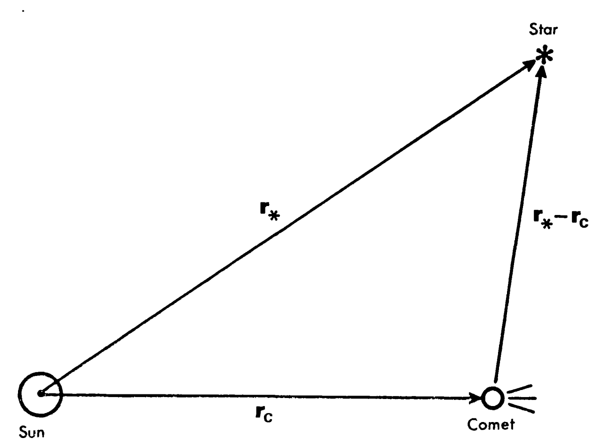


Fig. 1. The position vectors of the comet and the star in the heliocentric frame of reference ($\boldsymbol{r}_\odot = \boldsymbol{0}$).

[1]) Throughout this work a strict distinction will be kept between the *velocity*, which is a vector, and the *speed*, which is the absolute value of the velocity.

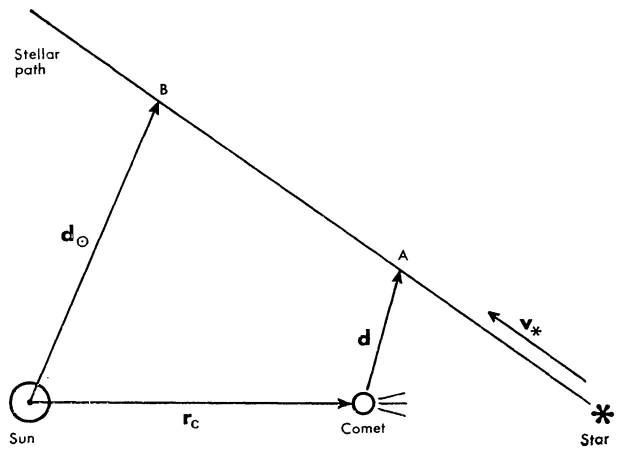


Fig. 2. The dynamical situation in the case of moving star and fixed comet. Points A and B are the points of closest approach of the star to the comet and the sun, respectively.

The approximation consists in the first case of putting $\boldsymbol{r}_* = \mathbf{const}$ while in the second case one puts $\boldsymbol{r}_c = \mathbf{const}$.

The speeds of the nearby stars relative to the sun are of the order of 10 km s^{-1}, but the comets move at such high speeds only near perihelion, where stellar perturbations are inconspicuous (the two integrals in (1) are nearly equal, when $\boldsymbol{r}_c$ is very small). During most of the time they move much slower. The second type of computation, where the comet is at rest, consequently seems most realistic for the actual purpose. This means that the dependence of the impulse on stellar distance also should be estimated according to the latter of the two methods described.

3. The Impulse Formula

The impulse $\boldsymbol{I}$ will be computed from equation (1) under the assumption $\boldsymbol{r}_c$ = constant. This situation is described further in Fig. 2. The assumption of course implies that both the closest points (A and B) on the stellar path are well-defined. The time-scale is chosen such that $t = 0$ as the star finds itself at point A.

In fact, the essential simplifications to be used may be summarized in three points.

1) The cometary radius vector $\boldsymbol{r}_c$ is constant.

2) The time interval of the perturbation is centered at $t = 0$.

3) The stellar velocity $\boldsymbol{v}_*$ is constant.

Regarding the third simplification one may state that it is invalid only for extremely close and therefore exceedingly rare stellar passages. If the deflection (Ψ) of the stellar path due to the sun's gravitation is computed, one gets the approximate formula:

$$(2) \qquad \Psi \approx \frac{2G(M_* + M_\odot)}{d_\odot v_*^2} \; (\Psi \text{ in radians})$$

which is valid for small values of Ψ. $M_\odot$ is the solar mass and $d_\odot$ is the minimum distance from the sun to the star (Fig. 2). If $M_* = M_\odot$ is taken as a representative stellar mass and $v_* = 30 \text{ km s}^{-1}$ as a representative stellar speed, one gets:

$$(3) \qquad \Psi \approx \frac{4 \text{a.u.}}{d_\odot}$$

Such a close passage as $d_\odot = 100$ a.u., which would take place only once in about 10^{11} years with the stellar speeds and number density of the present solar neighbourhood, involves a deflection of only two degrees. The validity of the remaining two simplifications is discussed below.

The vector $\boldsymbol{r}_* - \boldsymbol{r}_c$ involved in equation (1) can be written:

$$(4) \qquad \boldsymbol{r}_* - \boldsymbol{r}_c = \boldsymbol{d} + t\,\boldsymbol{v}_*$$

Accordingly, the first term in equation (1) can be subdivided into components parallel and perpendicular to the stellar motion:

$$(5) \qquad \int_{-T}^{T} \frac{\boldsymbol{r}_* - \boldsymbol{r}_c}{|\boldsymbol{r}_* - \boldsymbol{r}_c|^3}\, dt = \int_{-T}^{T} \frac{\boldsymbol{d}\, dt}{(d^2 + v_*{}^2 t^2)^{3/2}} + \int_{-T}^{T} \frac{\boldsymbol{v}_* t\, dt}{(d^2 + v_*{}^2 t^2)^{3/2}}$$

The second integral on the right hand side vanishes. It corresponds to the force component parallel to the stellar motion, for which any two positions of the star symmetrical with respect to A exactly outweigh each other. We therefore find that the impulse gained by the comet ($\boldsymbol{I}_c$) is directed towards A, and after evaluation of the first integral on the right hand side of (5), its magnitude is given by:

$$(6) \qquad I_c = \frac{2GM_*}{v_* d} \cdot \sin\left[\arctan\left(\frac{v_* T}{d}\right)\right].$$

The rapid convergence of $\sin[\arctan(v_* T/d)]$ towards unity for $v_* T \gg d$ implies that, provided T is very large:

1) the centering of the time interval exactly at $t = 0$ is not essential (this is confirmed also by the fact that the error arising from the second term in (5) due to

a displacement of the center of the time interval approaches zero rapidly, as $T \to \infty$),

2) instead of (6) one may use the simpler equation:

$$I_c = \frac{2GM_*}{v_* d} \tag{7}$$

Considering the second term in equation (1), one can make the subdivision into components in a completely analogous way by writing:

$$\mathbf{r}_* = \mathbf{d}_\odot + (t - t_0)\, \mathbf{v}_* \tag{8}$$

where t_0 is the time when the star passes point B. By the same argument as above we conclude that, again provided that T is very large, the impulse gained by the sun ($\mathbf{I}_\odot$) is directed towards B, and its magnitude is given by:

$$I_\odot = \frac{2GM_*}{v_* d_\odot} \tag{9}$$

One may note that equations (7) and (9) correspond to equivalent formulae given by e.g. Öpik (1932), Oort (1950), Ogorodnikov (1958) and Nezhinskij (1972).

The first of the three simplifications, as stated above, is based on the fact that the speeds of the stars relative to the sun are in general much larger than those of the comets in question. The errors that arise due to a variation of $\mathbf{r}_c$ in the above treatment can be described partly through a variation of $\mathbf{d}$ and partly through a motion of the point A. These errors are considerable only for very slow and close stellar passages. It is true that these passages contribute relatively large impulses, but they are at the same time rare. We can not ignore the close stellar passages, but we can ignore the risk of a close stellar passage which is also extremely slow.

Equation (1) for the heliocentric impulse of the comet now takes the form:

$$\mathbf{I} = \frac{2GM_*}{v_*} \left\{ \frac{\hat{\mathbf{d}}}{d} - \frac{\hat{\mathbf{d}}_\odot}{d_\odot} \right\} \tag{10}$$

where $\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}$ are unit vectors.

4. Comparison of Different Passage Distances

With the aid of equation (10) one may derive approximate expressions for the magnitudes of impulses corresponding to passages at different distances. Two limiting cases are:

1) extremely close passages, for which:

$$I = \frac{2GM_*}{v_* d} \tag{11a}$$

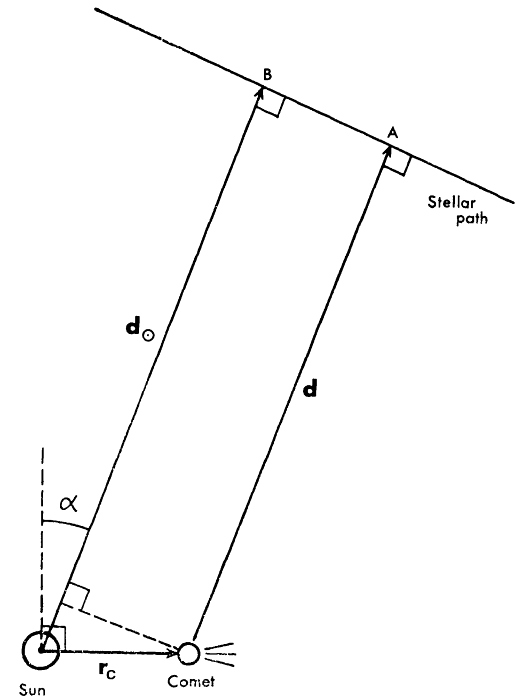


Fig. 3. The geometry of a distant stellar passage.

or

$$I = \frac{2GM_*}{v_* d_\odot} \tag{11b}$$

corresponding to a passage extremely close to the comet or the sun respectively.

2) distant passages (cf. Fig. 3). The comet may be placed at some typical distance r_c from the sun. If both d and $d_\odot$ are much greater than r_c, the angle between $\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}_\odot$ is very small, and the two unit vectors may be considered as parallel. In this case equation (10) simplifies to:

$$\mathbf{I} = \frac{2GM_*}{v_*} \cdot \frac{d_\odot - d}{d d_\odot} \cdot \hat{\mathbf{d}}_\odot \tag{12}$$

If α is the angle between $\hat{\mathbf{d}}_\odot$ and the plane perpendicular to $\mathbf{r}_c$ ($-\pi/2 \leqq \alpha \leqq \pi/2$), equation (12) may be written:

$$\mathbf{I} = \frac{2GM_* r_c \sin\alpha}{v_* d_\odot^{\,2}} \cdot \hat{\mathbf{d}}_\odot \tag{13}$$

considering also that $d \approx d_\odot$. The magnitude of the impulse thus depends on the direction to the point of closest approach of the star, the greatest impulses for a certain distance $d_\odot$ corresponding to stellar motions for which the point of closest approach lies on the line joining the sun and the comet. Furthermore, if $\hat{\mathbf{d}}_\odot$ makes an angle greater than $\pi/2$ with $\mathbf{r}_c$, α is negative and the impulse has the direction of $-\hat{\mathbf{d}}_\odot$.

Therefore, even if the points of closest approach are isotropically distributed, the impulse $\boldsymbol{I}$ always makes an angle smaller than $\pi/2$ with $\boldsymbol{r}_c$, and thus the radial impulse is always positive in the approximation made (the vectors $\hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{d}}_\odot$ are parallel).

Obviously the magnitude of the impulse per stellar passage decreases rapidly with the distance $d_\odot$. This decrease it to some extent compensated by the increasing number of passages at increasing distances. The increase of the passage frequency with distance depends on the space distribution of the points of closest approach ("passage points"). If the stellar passages during a certain time interval are considered, the number of passage points inside the sphere of radius D, centered at the sun or the comet, may be approximated by the simple formula:

$$n(D) = n_0 D^z \,. \tag{14}$$

Both the factor n_0 and the exponent z deserve special discussion. Equation (14) holds true exactly under certain assumptions, namely:

1) the stars of the solar neighbourhood are uniformly distributed in space at any instant of time;

2) the distribution of stellar velocities relative to the sun does not vary from place to place in the solar neighbourhood.

At the beginning of a time interval dt, every volume element dV contains an equal number of stars. Those which are to make their passages during the interval dt are characterized by nearly transverse motions. The constant velocity distribution is not isotropic, mainly because of the sun's peculiar motion, and thus the fraction of stars moving transversely with respect to

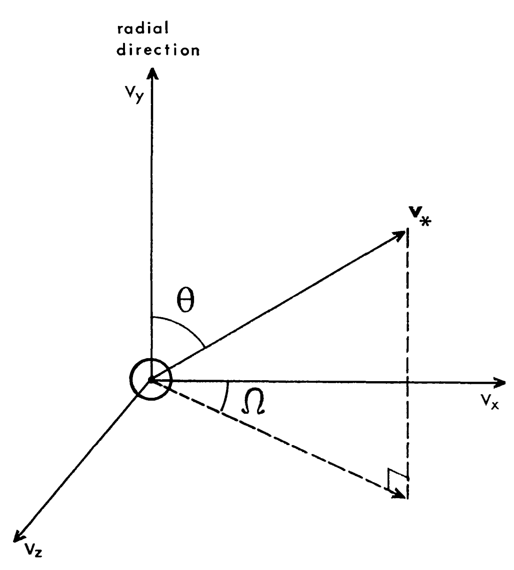


Fig. 4. Illustration of the angles Θ and Ω defining the direction of the stellar velocity $\boldsymbol{v}_*$ relative to the sun.

the sun is greatest near the plane perpendicular to the apex-antapex direction.

However, in any given direction in space, the distribution of the directions of motion as referred to the transverse plane (the polar angles Θ and Ω defined in fig. 4) is independent of the distance r. The condition for a passage during the time dt is for a star moving at speed v:

$$\tfrac{1}{2}\pi \leqq \Theta \leqq \tfrac{1}{2}\pi + \frac{v\,.\,\mathrm{d}t}{r}\,. \tag{15}$$

Thus if the fraction f of stars at a place in the given direction make their passages during the time interval in question, f varies as r^{-1}. Hence the space density of passage points also varies as r^{-1}, and by integrating over the inside of a sphere with radius D, one finds: $n(D) = n_0 D^2$. This corresponds to equation (14) with $z = 2$. It should be noticed that the exponent is independent of the stellar kinematics of the solar neighbourhood, as long as the conditions above are fulfilled. The relative sizes of the sun's peculiar speed and the velocity dispersion of the surrounding stars determine the degree of concentration of the passage points towards the plane perpendicular to the apex-antapex direction, but the exponent is not influenced by this.

One may ask how well this idealized model represents reality. The interesting region for stellar passages extends out to about 10 pc from the sun, and at any instant of time the number of stars within this region is rather small, so the isofropy assumptions are not valid momentarily. On the other hand, it would be difficult to find convincing arguments for a more refined model of the average situation over very long periods of time.

As regards the factor n_0 appearing in equation (14), it can be determined in several different ways. It represents the number of passages within unit distance during a certain time interval, and is thus a function of the number density of stars in the solar neighbourhood and the peculiar motion of the sun. A simple expression for this dependence is:

$$n_0 = \pi \bar{v} \nu_* \, \Delta t \tag{16}$$

where $\bar{v}$ is the average speed of the nearby stars relative to the sun, ν_* is the number density of stars and Δt is the time interval. In order to find ν_* one can count the stars in the Catalogue of Nearby Stars (Gliese, 1969). This gives the density of stars at present observed in the solar neighbourhood, and the result is: 64 star systems (83 stars) within 6 pc distance, where of 24 systems (33 stars) within 4 pc. These counts yield the number densities: 0·07 pc^{-3} within 6 pc and 0·09 pc^{-3} within 4 pc. This minor difference may depend on statistical fluctuations as well as on a somewhat increased likelihood for a star to remain undetected, as the distance increases from 4 pc to 6 pc. We may use the average value of 0·08 star systems per pc^3.

If the peculiar speed of the sun is taken as 20 km s^{-1} and the velocity dispersion of nearby stars as 30 km s^{-1}, then $\hat{v}$ = 50 km s^{-1} and we obtain: $n_0 = 12{\cdot}5\ pc^{-2}$ for a time interval $\Delta t = 10^6$ years. Leaving the question open whether this value is a good approximation to the average value of n_0 over very long time periods, it is at least only a lower limit for the present n_0-value, because it refers to only stars which are at present known to be nearby. The mass density corresponding to these stars is: $\varrho = 0{\cdot}06\ M_\odot$ density corresponding to these stars is: $\varrho = 0{\cdot}06\ M_\odot\ pc^{-3}$ according to Allen (1973), so the average mass of the nearby star systems is: $\overline{M} = 0{\cdot}8\ M_\odot$, which is close to the value given by Sekanina (1968a). The choice of a mean stellar mass and speed will be discussed further below.

It must be recognized that a significant fraction of the star systems contain several components with comparable masses. One then encounters a dynamical problem concerning the close stellar passages, if one uses the mean mass and passage frequency of the star systems. An error occurs when the different components give rise to significantly different impulses on the comet, in which case the total impulse from the star system may not be well approximated by treating a single star with the total mass of the system. However, in most cases the separation of the components is so small that this error is negligible except for extremely close and therefore rare stellar passages. Treating the components separately would yield a number density of 0·10 stars per pc^3 with a mean mass of 0·4 $M_\odot$, but one would then have to take account of the fact that the stars do not arrive randomly in space and time, but very often arrive in pairs or triplets. This would imply a statistical problem concerning the chance for a comet of experiencing at least one stellar passage per orbital revolution at a certain distance.

Another method of determining n_0 was used by Sekanina (1968b). The source of data was in this case the list of encounters of the sun with stars in the past by Makover (1964). All the stars of this list are at present contained in a "sphere of investigation" with a radius of 20 pc around the sun. The distribution of minimum distances from the sun was derived, including all passages during the previous million years. This distribution must of course be corrected for the incompleteness of the data, which mainly depends on two factors.

1) All the stars at present situated in the sphere of investigation have not been detected, and the incompleteness rises with the distance of the stars. As shown by Sekanina (1968a), the space distribution of the detected stars is uniform only out to 5–6 pc.

2) Many stars which passed inside the sphere of investigation during the 10^6 years considered have already moved out of the sphere, and thus they would not enter the statistics even if the detection of stars were complete out to the distance of 20 pc.

To correct appropriately for both these circumstances is a very complicated matter, and the correction can not be made very accurately. This is true especially for the larger passage distances, where the detection of stars may be far from complete. On the other hand, restricting the discussion to a sphere of investigation with a radius of only 5–6 pc would yield a statistically insignificant material. The result found by Sekanina was $n_0 = 13{\cdot}5\ pc^{-2}$ for $\Delta t = 10^6$ years, which agrees well with the value derived above. Considering however the uncertainties of both methods, we should give only one significant figure, and thus use the value: $n_0 = 10\ pc^{-2}$ for $\Delta t = 10^6$ years.

In order to investigate the relative importances of different passage distances, one may roughly divide the circumsolar space into four different regions [(I)–(IV)] according to Fig. 5. During a very long time interval T a large number of passage points will be distributed inside these regions, and the numbers can

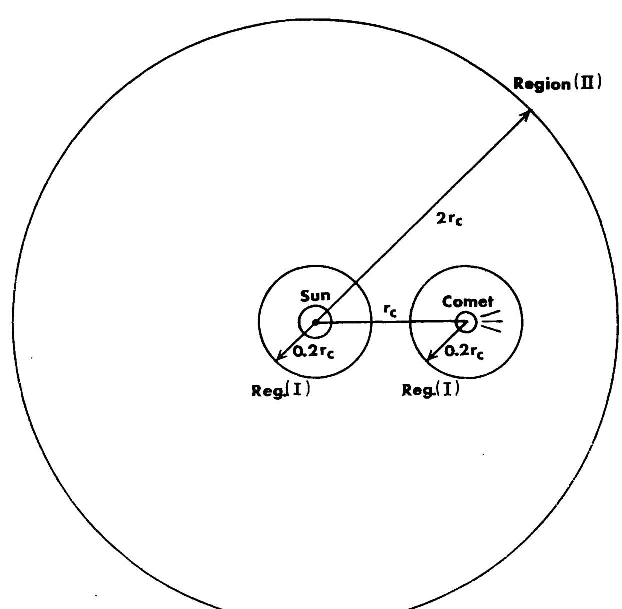
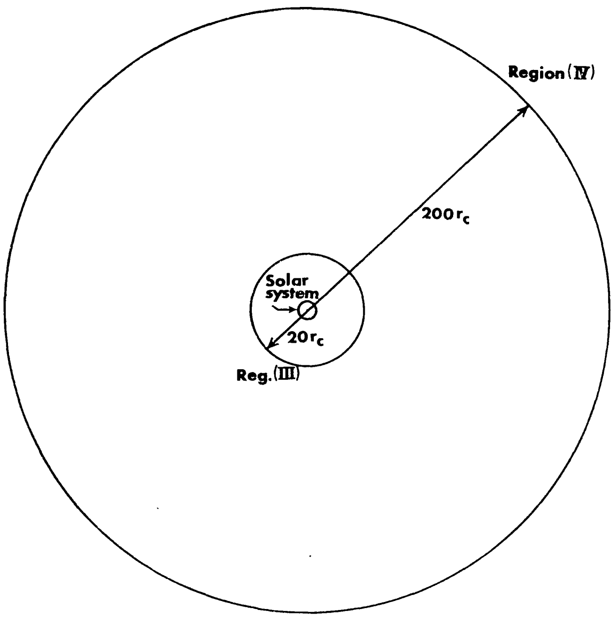


Fig. 5a. Cross-section of regions (I) and (II) containing the vector $\boldsymbol{r}_c$.



5b. Cross-section of regions (III) and (IV).

be estimated according to formula (14). The directions to the passage points are not distributed isotropically, as mentioned above, but show a certain asymmetry due to the peculiar motion of the sun. This means that the impulse vectors ($\boldsymbol{I}$) to a certain extent are concentrated towards a plane perpendicular to the apex-antapex direction. Comets in different parts of the circumsolar system thus experience different stellar perturbations, but in order to estimate the mean perturbation averaged over all directions of $\boldsymbol{r}_c$, one may consider the directions to the points of closest approach as isotropically distributed because of the isotropy of the system of long-period comets.

The magnitudes of the impulses for passages in regions (I)–(IV) are estimated according to equations (11) and (13), and for each region a mean value with respect to $d_\odot$ (or d) is used. In this connection two points should be considered.

1) Only the transverse impulse, i.e. the component of $\boldsymbol{I}$ perpendicular to $\boldsymbol{r}_c$, will be taken into account. The reason for this is that the perturbations of perihelion distance and inclination are determined by this component, as shown below. Another advantage of this restriction is that if, in accordance with the argument above, one considers the directions to the points of closest approach to be isotropically distributed, the directions of the transverse impulses are uniformly distributed in the transverse plane (perpendicular to $\boldsymbol{r}_c$).

2) For distant passages the factor $\sin \alpha$ in equation (13) should not be averaged, but instead taken into account in the averaging over different passage directions.

If the magnitudes of the impulses for region (j) have the mean value $I_{(j)}$ and this region contains $n_{(j)}$ passage points, the expectation of the magnitude of the total transverse impulse is:

$$I^t_{(j)} = I_{(j)} \cdot \sqrt{n_{(j)}} \cdot \sqrt{\pi/6} \tag{17}$$

if $I_{(j)}$ is independent of the direction as in equations (11), while for $I_{(j)} = I^0_{(j)} \sin \alpha$ as in equation (13) one gets:

$$I^t_{(j)} = I^0_{(j)} \cdot \sqrt{n_{(j)}} \cdot \sqrt{\pi/30} \tag{18}$$

A derivation of equations (17) and (18) is found in appendix A. For estimating $I^t_{(\mathrm{I})}$ through $I^t_{(\mathrm{IV})}$, the following methods are used:

Region (I). This corresponds to stellar passages within $0{\cdot}2r_c$ from either the sun or the comet, and $I_{(\mathrm{I})}$ is computed from equation (11) using a mean value for d or $d_\odot$ equal to $0{\cdot}1r_c$. The number of passages is:

$$n_{(\mathrm{I})} = 2 \cdot n_0 \cdot (0{\cdot}2r_c)^2 \cdot T/10^6 \text{ yrs} \tag{19}$$

and $I^t_{(\mathrm{I})}$ is computed with the aid of equation (17).

Region (II). This corresponds to stellar passages outside region (I), but still within $2r_c$ from the sun. Those passages are the most complicated ones to treat accurately, because the comet and the sun gain comparable impulses, the direction of which need not coincide. For a rough estimate of the magnitude of the impulses one may extrapolate the validity of equations (11) to this case. One then computes only the impulse gained by the sun, and considers the impulse gained by the comet as either increasing or decreasing the magnitude of the impulse with equal probabilities. In this case $I_{(\mathrm{II})}$ is computed from equation (11b) using a mean value for $d_\odot$ equal to r_c. The number of passages is roughly:

$$n_{(\mathrm{II})} = n_0 \cdot (2r_c)^2 \cdot T/10^6 \text{ yrs} \tag{20}$$

and $I^t_{(\mathrm{II})}$ is computed with the aid of equation (17). This procedure will give a slight overestimate of $I^t_{(\mathrm{II})}$, mainly because the impulses gained by the comet and the sun are more often parallel than antiparallel, and thus the relative impulse is most frequently smaller than the impulse gained separately by the sun. The magnitudes of the impulses are also somewhat dependent upon the direction $\hat{\boldsymbol{d}}_\odot$, the greatest impulses being the radial ones.

Region (III). This corresponds to stellar passages between $2r_c$ and $20r_c$ from the sun. Also in this case some of the passages are difficult to treat accurately, but an estimate is provided by regarding all passages as distant ones in the sense that equations (13) and (18) are applicable. Thus $I^0_{(\mathrm{III})}$ is computed from:

$$(21) \qquad I^0_{(\mathrm{III})} = \frac{2GM_*}{121 v_* r_c}$$

using a mean value for $d_\odot$ equal to $11r_c$. The number of passages is approximately:

$$(22) \qquad n_{(\mathrm{III})} = n_0 \, . \, (20r_c)^2 \, . \, T/10^6 \text{ yrs}$$

and $I^t_{(\mathrm{III})}$ is computed with the aid of equation (18).

Region (IV). This corresponds to stellar passages between $20r_c$ and $200r_c$ from the sun. $I^0_{(\mathrm{IV})}$ is computed from

$$(23) \qquad I^0_{(\mathrm{IV})} = \frac{2GM_*}{1{\cdot}21 \, . \, 10^4 v_* r_c}$$

using a mean value for $d_\odot$ equal to $110r_c$. The number of passages is approximately:

$$(24) \qquad n_{(\mathrm{IV})} = n_0 \, . \, (200r_c)^2 \, . \, T/10^6 \text{ yrs}$$

and $I^t_{(\mathrm{IV})}$ is computed from equation (18).

One may notice that a typical value of r_c for the comets under consideration is $r_c = 10^4$ a.u., and in this case the outer limits of regions (III) and (IV) are 1 pc and 10 pc from the sun, respectively.
If $I^t_{(\mathrm{I})}$ is set equal to unity, the values of $I^t_{(j)}$ are then:

$$I^t_{(\mathrm{I})} = 1{\cdot}0$$
$$I^t_{(\mathrm{II})} = 0{\cdot}7$$
$$I^t_{(\mathrm{III})} = 0{\cdot}03$$
$$I^t_{(\mathrm{IV})} = 0{\cdot}003$$

Thus, viewed over an extremely long period of time, the distant passages in regions (III) and (IV) contribute with only about two percent of the total impulse, and they are therefore relatively unimportant. Of course the situation is different if one considers a short interval, when there may be a very small probability of a close stellar passage in region (I) or (II).

5. The Transverse Impulse per Revolution

We want to estimate quantitatively the transverse impulses per revolution for different comets. This quantity is strongly dependent on the aphelion distance (Q) for two reasons. For the cometary orbits under consideration the aphelion distance is approximately equal to twice the semimajor axis (a) and thus determines the period (P) according to:

$$(25) \qquad P \approx (Q/2)^{3/2}$$

where P is measured in years and Q in astronomical units. Therefore the expected number of stellar passages per revolution within a certain volume is proportional to $Q^{3/2}$. At the same time the average distance from the comet to the sun increases with the aphelion distance of the comet, and therefore the region of space, where stellar passages are important, also grows with Q.

A proper choice of a mean value of r_c is important both for estimating the expected number of stellar passages per revolution and for determining the influence of the transverse impulse on the perihelion distance and inclination of the comet. In the following, somewhat arbitrarily, we consider only the „outer half" of the cometary orbit, defined by eccentric anomalies between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and also characterized by the condition $r_c > 0{\cdot}5Q$. The time-averaged median value of r_c in this part of the orbit is: $\overline{r_c} \approx 0{\cdot}89Q$ (i.e. 50% of the corresponding time is spent at $r_c > \overline{r_c}$). The interval of mean anomaly which corresponds to $r_c > 0{\cdot}5Q$ is $\Delta M \approx 5{\cdot}14$ rad., and thus the time spent by the comet at heliocentric distances greater than $0{\cdot}5Q$ during each revolution is: $\Delta t = P \, . \, 5{\cdot}14/(2\pi) \approx 0{\cdot}82P$.

Thus the number of stellar passages per revolution is estimated on the assumption that the comet is situated at a heliocentric distance $r_c = 0{\cdot}89Q$ during a time interval $\Delta t = 0{\cdot}82P$. Using the adopted value $n_0 = 10 \text{ pc}^{-2}$ for $\Delta t = 10^6$ years in equation (19) one finds the number of passages per revolution in region (I):

$$(26) \qquad N_{(\mathrm{I})} = \frac{0{\cdot}82P}{10^6 \text{ yrs}} \, . \, 20 \, . \left(\frac{0{\cdot}2r_c}{1\text{pc}}\right)^2 .$$

If Q_4 is the aphelion distance expressed in 10^4 a. u., equation (26) is equivalent to:

$$(27) \qquad N_{(\mathrm{I})} = 4{\cdot}3 \, . \, 10^{-4} \, . \, Q_4^{3{\cdot}5} \, .$$

The numbers $N_{(\mathrm{II})}$, $N_{(\mathrm{III})}$ and $N_{(\mathrm{IV})}$ in regions (II) – (IV) are obtained correspondingly from equations (20), (22) and (24) as: $N_{(\mathrm{II})} = 50N_{(\mathrm{I})}$, $N_{(\mathrm{III})} = 100N_{(\mathrm{II})}$ and $N_{(\mathrm{IV})} = 100N_{(\mathrm{III})}$. Values of $N_{(\mathrm{I})}$ through $N_{(\mathrm{IV})}$ for different values of Q are given in table 1.

One may once again compute typical impulses for regions (I)–(IV) with the aid of equations (11) and (13) as described in the previous section. It must now be noticed that equations (17) and (18) are based on the central limit theorem (Freund, 1958) and thus apply approximately for large values of $n_{(j)}$

Table 1

Numbers of stellar passages in regions (I) — (IV) per revolution for different values of the aphelion distance.

Q (a. u.)	$N_{(\mathrm{I})}$	$N_{(\mathrm{II})}$	$N_{(\mathrm{III})}$	$N_{(\mathrm{IV})}$
5.10^3	4.10^{-5}	2.10^{-3}	2.10^{-1}	2.10^1
1.10^4	4.10^{-4}	2.10^{-2}	2.10^0	2.10^2
2.10^4	5.10^{-3}	2.10^{-1}	2.10^1	2.10^3
5.10^4	1.10^{-1}	6.10^0	6.10^2	6.10^4
1.10^5	1.10^0	7.10^1	7.10^3	7.10^5
2.10^5	2.10^1	8.10^2	8.10^4	8.10^6

only. On the other hand, from table 1 it is obvious that the values of $N_{(j)}$ are in most cases not large, and one should instead compute the expected transverse impulse per stellar passage. This is given by:

$$ {}^1I^t_{(j)} = I_{(j)} \,.\, \pi/4 \tag{28}$$

for regions (I) and (II), and:

$$ {}^1I^t_{(j)} = I^0_{(j)} \,.\, \tfrac{1}{3} \tag{29}$$

for regions (III) and (IV). The result is e.g. for region (I):

$$ {}^1I^t_{(\mathrm{I})} = \frac{2GM_*}{v_* \,.\, 0{\cdot}1 r_c} \,.\, \frac{\pi}{4} \tag{30}$$

A derivation of equations (28) and (29) is found in appendix B. ${}^1I^t_{(\mathrm{I})}$ corresponds to the change of the comet's transverse velocity, but since the perturbations of the orbital elements are determined by the relative change of the transverse velocity, one should also compute the transverse speed, v_t, and divide ${}^1I^t_{(\mathrm{I})}$ by v_t. v_t is given by:

$$ v_t = \frac{GM_\odot}{v_\oplus r_c} \sqrt{\frac{2q}{r_\oplus}} \tag{31}$$

if $v_\oplus$ and $r_\oplus$ are the earth's orbital speed and heliocentric distance, respectively. Equations (30) and (31) then give:

$$ \frac{{}^1I^t_{(\mathrm{I})}}{v_t} = \frac{M_*}{\mathrm{M}_\odot} \,.\, \frac{v_\oplus}{v_*} \,.\, 10 \,.\, \pi/2 \,.\, \sqrt{\frac{r_\oplus}{2q}} \tag{32}$$

The corresponding equations for regions (II)–(IV) differ from (32) by constant factors only. For numerical estimates the following values are used:

$$ M_* = \mathrm{M}_\odot \,, \quad v_* = v_\oplus \approx 30\ \mathrm{km\,s^{-1}} \,, \quad q = 5\ \mathrm{a.u.} $$

The relative transverse impulses per stellar passage in regions (I)–(IV) are then:

region (I) : 5·0
region (II) : 0·50
region (III) : 0·0017
region (IV) : 0·000017

These values are independent of Q, and together with the figures in table 1 they provide the result that we want, namely, the transverse impulse per revolution as a function of the aphelion distance. Several important facts can be established:

1) In spite of the fact that the close stellar passages over extremely long periods of time are the most important ones, they occur so rarely that they can often be ignored over time intervals comparable with the revolution periods of the comets in question. For example, the passages in region (I), which are the closest ones in comparison with r_c, become important only for aphelion distances on the order of 10^5 a.u. which is already near the stability limit of the circumsolar cometary system (Oort, 1950; Chebotarev, 1964).

2) The distant passages, corresponding to region (IV), never have any decisive importance. Their effect is always smaller than the one produced by the closer passages except for comets with aphelion distances smaller than 10^4 a.u., and for such small aphelion distances the expected total relative transverse impulse is $\lesssim 10^{-4}$, which is quite negligible.

3) If one considers comets in order of increasing aphelion distances, one can regard a certain passage region as becoming important when the number of passages per revolution approaches unity. If the relative transverse impulse per stellar passage in region (j) is denoted by $\Delta'_{(j)}$ and $N_{(j)}$ is the number of passages per revolution, one expects a total relative transverse impulse per revolution:

$$ \Delta_{(j)} = \Delta'_{(j)} \,.\, N_{(j)} \tag{33}$$

if $N_{(j)} \lesssim 1$, and:

$$ \Delta_{(j)} \approx \Delta'_{(j)} \,.\, \sqrt{N_{(j)}} \tag{34}$$

if $N_{(j)} \gg 1$. $\Delta_{(j)}$ can now be estimated for each of the aphelion distances in table 1, and the total relative transverse impulse per revolution for all the regions together is given by:

$$ \Delta = \sum_{j=1}^{4} \Delta_{(j)} \tag{35}$$

In Fig. 6 the computed values of Δ are plotted against Q.

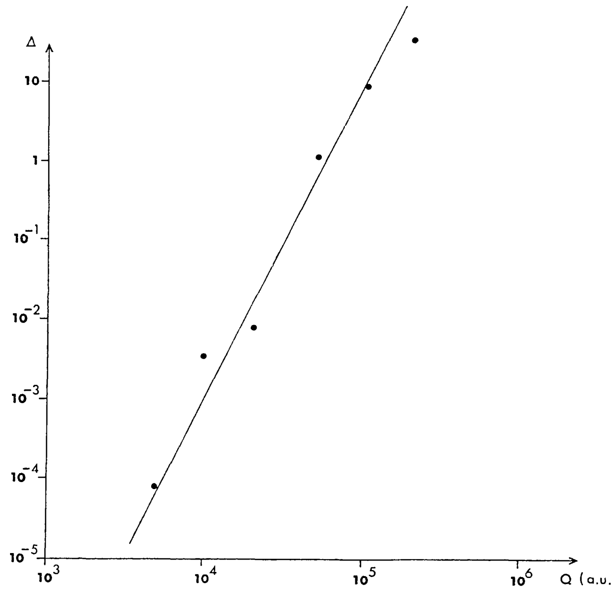


Fig. 6. The stellar perturbation of the transverse velocity in units of the transverse speed, as a function of the aphelion distance. The values corresponding to the six points are computed using the estimates of $\Delta'_{(j)}$ per stellar passage and the numbers in table 1.

From Fig. 6 it is evident that stellar perturbations of the transverse velocity are not important for comets with aphelion distances $\lesssim 10^4$ a.u. For $Q \approx \approx 5 \cdot 10^4$ a.u. Δ is nearly unity, and then the perihelion distance and inclination are subject to very large perturbations at each aphelion passage, as will be shown below. This corresponds to the present opinion about the size of the Oort cloud (Marsden and Sekanina, 1973). For $Q \gtrsim 6 \cdot 10^4$ a.u. Δ is greater than 1, and the perihelion distance and inclination are not at all stable against stellar perturbations. One may suspect that the stability limit of the solar system is then also reached. According to the present investigation this limit may thus be situated closely outside the Oort cloud with the recent value of the cloud radius.

It must be emphasized that the line drawn in Fig. 6 is constructed visually as a reasonably good fit to the six points, which are based on very rough estimates, as should be obvious from the foregoing. Among these estimates are the mean values adopted for the stellar speeds and masses. Concerning the stellar masses, the adopted value $M_* = M_\odot$ may be a slight overestimate, since the mean value for the stellar systems within a distance of 6 pc is $M_* = 0{\cdot}7 M_\odot$ according to Sekanina (1968a). However, the figures given in this paper are in any case very rough, and there is also a possibility that the additional effect of a number of low-mass and low-speed stars could compensate for the reduction of the mean stellar mass. Considering next the stellar speeds, there is some divergence among mean values given by different authors. Sekanina (1968b) gave the value 60 km s^{-1}, while Yabushita (1972) used the value 20 km s^{-1}. The former figure corresponds to the mean speed of observed red dwarfs in the solar neighbourhood. Since nearby stars are identified by virtue of large proper motions, one often fails to detect low-speed stars or stars moving nearly parallel to the line of sight, and this figure is therefore probably an overestimate. On the other hand, the latter figure coincides with the generally accepted value of the peculiar speed of the sun, and this implies a certain underestimate, because one neglects the velocity dispersion of the stars in the solar neighbourhood. Thus, apparently, the mean value $v_* = 30$ km s^{-1} can be used with some confidence.

6. Perturbations of the Perihelion Distance and Inclination

For long-period comets passing through the planetary system the perihelion distance (q) and the inclination (i) are orbital elements of primary dynamical interest, at least as far as capture processes are concerned. Both these elements depend on the transverse velocity of the comet at a certain point in its orbit (e.g. the aphelion point), and thus they are affected by the transverse impulses caused by stellar passages.

For the perihelion distance the relation is simple. If r_c is the distance from the comet to the sun and v_t the transverse speed of the comet, the perihelion distance is given by:

$$q = \frac{r_c v_t^2}{2GM_\odot} \tag{36}$$

Thus if q and v_t are perturbed by the amounts Δq and Δv_t at a certain point in the orbit:

$$\Delta q = \frac{2q}{v_t} \Delta v_t \tag{37}$$

which is correct for small perturbations. As Öpik (1932) pointed out, there is a progressive tendency towards increasing q-values especially for the smallest perihelion distances, because both v_t and q must remain positive, and the distribution of Δq is therefore asymmetric.

For the inclination the situation is somewhat more complicated. If $\mathbf{v}_t(Q)$ is the transverse velocity of the comet at aphelion (heliocentric distance Q), the inclination is determined by the angle Φ between $\mathbf{v}_t(Q)$ and the ecliptic plane together with the argument of perihelion, $\tilde{\omega}$, through the equation:

$$\sin i = -\frac{\sin \Phi}{\cos \tilde{\omega}} \tag{38}$$

For the distant parts of very elongated orbits $\mathbf{v}_t(r_c)$ is everywhere approximately parallel to $\mathbf{v}_t(Q)$, and thus equation (38) is generally approximately true if Φ is the angle between $\mathbf{v}_t$ and the ecliptic plane. If $\mathbf{v}_t$ is perturbed by the amount $\Delta\mathbf{v}_t$ at a certain point in the orbit, both i, $\tilde{\omega}$ and Φ are in general perturbed by the amounts Δi, $\Delta\tilde{\omega}$ and $\Delta\Phi$, which are related through the equation:

$$\cos \Phi \, \Delta\Phi = \sin i \sin \tilde{\omega} \, \Delta\tilde{\omega} - \cos i \cos \tilde{\omega} \, \Delta i \tag{39}$$

From equation (39) Δi can be obtained as a function of $\Delta\Phi$ by the aid of an additional restraint on the perturbations Δi and $\Delta\tilde{\omega}$. The latitude of aphelion (β) of the comet is given by:

$$\sin \beta = -\sin \tilde{\omega} \sin i \tag{40}$$

Since the direction of aphelion is not very sensitive to stellar perturbations, as long as q remains small, β can be regarded as a constant, and thus:

$$\Delta\tilde{\omega} = -\Delta i \cot i \tan \tilde{\omega} \tag{41}$$

Equations (39) and (41) yield:

$$\Delta i = -\Delta\Phi \frac{\cos \Phi \cos \tilde{\omega}}{\cos i} \tag{42}$$

and specifically, in the case of low inclination:

$$\Delta i \approx -\Delta\Phi \cos \tilde{\omega} \tag{43}$$

When the inclination is small, the transverse plane is roughly perpendicular to the ecliptic plane, and as an approximation one may consider the angle Φ to be situated in the transverse plane. Then the quantities $\Delta v_t/v_t$ and $\Delta\Phi$ appearing in equations (37) and (43) can both be expressed in terms of Δ in order to compute the expected perturbations of q and i per revolution. They correspond to the component of the transverse impulse parallel to and perpendicular to the transverse velocity, respectively (cf. Fig. 7).

Evidently, if λ is the angle between $\mathbf{v}_t$ and $\Delta\mathbf{v}_t$:

$$\frac{\Delta v_t}{v_t} = \frac{|\Delta\mathbf{v}_t|}{v_t} \cos \lambda \tag{44}$$

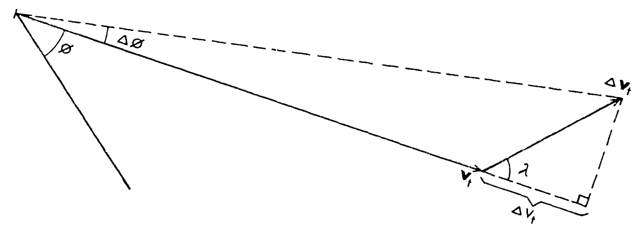


Fig. 7. Illustration of the perturbations Δv_t and $\Delta\Phi$.

$$\Delta\Phi = \frac{|\Delta\mathbf{v}_t|}{v_t} \sin \lambda \tag{45}$$

Equations (44) and (45) are correct for small perturbations, and since λ is rectangularly distributed between 0 and 2π, the expectations of $\Delta v_t/v_t$ and $\Delta\Phi$ are both zero, as well as the expectations of Δq and Δi according to (37) and (43). We are, however, more interested in the absolute values of Δq and Δi. From equations (44) and (45) we easily find that the expectations of the absolute values of $\Delta v_t/v_t$ and $\Delta\Phi$ are both equal to the expectation of $|\Delta\mathbf{v}_t|/v_t$ divided by $\sqrt{2}$. In the previous section the expectation of $|\Delta\mathbf{v}_t|/v_t$ per revolution by the comet was denoted by Δ, and thus the expectations of the magnitudes of Δq and Δi per revolution are given by:

$$\mathrm{E}|\Delta q| = \sqrt{(2)}\, q\Delta \tag{46}$$

$$\mathrm{E}|\Delta i| = \tfrac{1}{2}\Delta \tag{47}$$

for a rectangular distribution of $\tilde{\omega}$ between 0 and 2π, which follows from the assumption of isotropy of the circumsolar cometary system.

7. Implications for Capture of Comets

What influence may the stellar perturbations have on the process of cometary capture? In its usual sense the expression „capture of comets" denotes the transfer of comets from a long-period cometary reservoir into the Jupiter family by means of planetary perturbations. Therefore in the treatments of capture of comets only planetary perturbations are taken into account, and this is justified inasmuch as the necessary change of orbital energy can be accomplished only by these. However, the perturbations of perihelion distance and inclination caused by passing stars are obviously considerable for comets with aphelion distances greater than a few tens of thousands of astronomical units. Therefore, if the capture efficiency depends on perihelion distance and inclination, the stellar perturbations may also affect the capture efficiency provided that cometary orbits with the required aphelion distances are important in the capture process.

Several important facts concerning capture of comets from original parabolic orbits were established by Everhart (1972). Captures by Jupiter occurred almost exclusively for original orbits with perihelion distances between 4 a.u. and 6 a.u. and inclinations smaller than 9°. The corresponding region of the (q, i)-plane is called „the capture region". Long-period comets inside the capture region thus have a greater chance of being captured than other long-period comets. The capture efficiency in the capture region can be estimated with the aid of Everhart's results by dividing the number of captures found by the number of perihelion passages followed, and in this way one obtains the capture probability $p_c = 2{\cdot}5 \,.\, 10^{-4}$ per perihelion passage (Vaghi, 1973; Delsemme, 1973). For comets outside the capture region the capture probability is much smaller than this value.

Table 2

Scatters of q and i per revolution for low-inclination comets with $q = 5$ a. u. computed with equations (46) and (47).

Q (a. u.)	Δq (a. u.)	$\Delta i(°)$
$2 \,.\, 10^4$	0·1	0·5
$3 \,.\, 10^4$	0·6	3
$4 \,.\, 10^4$	2	8
$5 \,.\, 10^4$	5	20

In table 2 the expected scatter of q and i per revolution are given for different values of the aphelion distance of comets in Jupiter's capture region. It is seen that the scatters of the orbital elements attain the same oder of magnitude as the dimensions of the capture region for aphelion distances around $4 \,.\, 10^4$ a.u. The cometary reservoir from which the capture takes place is identified with the Oort cloud, and thus the original orbits are characterized by aphelion distances of this magnitude. For any single comet the capture process is unaffected by stellar perturbations as soon as the aphelion distance is $\lesssim 2 \,.\, 10^4$ a.u. according to table 2. However, if the first passages through the planetary system do not bring the aphelion distance below this limit, the comet may well be thrown out of the capture region by stellar perturbations, and this would delay the capture process. On the other hand, of course, comets are also thrown into the capture region by stellar perturbations, so the situation may be described in the following way.

For the largest aphelion distances ($Q \gtrsim 4 \,.\, 10^4$ a.u.) there are no comets definitely belonging to the capture region or the region outside it. Instead the capture process for such comets involves a larger region of the (q, i)-plane, defined by the possibility for single comets to move into or out of Everhart's capture region over the time scale of one revolution period. This tends to increase the number of comets available for capture and thus also the number of captures expected as compared with the estimates by Vaghi and Delsemme. However, at the same time the probability of a significant reduction of the aphelion distance is averaged over a larger region of the (q, i)-plane, and this tends to decrease the probability of capture per comet.

One should not expect the stellar perturbations to have any deep influence on the capture efficiency, because reductions of the aphelion distance by a factor of two for long-period comets at their passages through the planetary system are not at all rare events even for the observed comets, which in general have perihelion distances and inclinations far outside the capture region. Such a reduction would at once remove an Oort cloud comet from the influence of stellar perturbations, as we have seen. Concerning the estimates of capture efficiency from Everhart's results one is probably justified in stating that they can not be very much in error, because the reduction of the aphelion distance below the limit of importance of stellar perturbations is essentially a single-stage process, which does not depend on multiple encounters with Jupiter all the time with q and i inside the capture region. Therefore in a certain number of captures found in Everhart's work the comet should probably have been lost by stellar perturbations at an early stage, but ignoring this circumstance is equivalent to stating that the lost comet may be exchanged with another one, which in a similar way is brought into the capture region.

Appendix A

Derivation of Equations (17) *and* (18)

First we consider a population of $n_{(j)}$ impulse vectors with equal lengths $I_{(j)}$. $n_{(j)}$ is assumed to be a large number, and the vectors are isotropically distributed. We choose a rectangular system of coordinates x, y, z such that the x-axis points in the radial direction. If the polar angles α, β are defined in such a way that α is the angle between the vector and the (y, z)-plane $(-\pi/2 \leqq \alpha \leqq \pi/2)$ the coordinates of the vectors are given by:

(A 1)
$$x = I_{(j)} \sin \alpha$$
$$y = I_{(j)} \cos \alpha \cos \beta$$
$$z = I_{(j)} \cos \alpha \sin \beta$$

The relative frequency function of α and β is:

(A 2)
$$\varphi(\alpha, \beta) = \frac{1}{4\pi} \cos \alpha$$

With the aid of equations (A 1) and (A 2) the expectations of the rectangular coordinates can easily be computed, and the result is:

(A 3)
$$\mathrm{E}x = \mathrm{E}y = \mathrm{E}z = 0$$

The expectations of the squares of the coordinates are found to be:

(A 4)
$$\mathrm{E}x^2 = \mathrm{E}y^2 = \mathrm{E}z^2 = \tfrac{1}{3} I_{(j)}^2$$

consistently with the fact that the expectation of the square of the length of the vector must be:

(A 5)
$$\mathrm{E}(x^2 + y^2 + z^2) = I_{(j)}^2$$

According to (A 4) the variances of x, y and z are all equal to $1/3\, I_{(j)}^2$, and according to (A 3) their mean values are zero. The sum of all $n_{(j)}$ vectors has coordinates X, Y and Z, and according to the central limit theorem (Freund, 1958), as $n_{(j)} \to \infty$, the probability distributions of X, Y and Z approach a normal distribution with mean value equal to zero and variance equal to:

(A 6)
$$\mathrm{Var}\, X = \mathrm{Var}\, Y = \mathrm{Var}\, Z = \frac{n_{(j)}}{3} I_{(j)}^2$$

Since we are interested in the total transverse impulse we want to compute the expectation of $\sqrt{(Y^2 + Z^2)}$. According to (A 6) we can write:

(A 7)
$$X = \sqrt{\frac{n_{(j)}}{3}}\, I_{(j)} X_1$$
$$Y = \sqrt{\frac{n_{(j)}}{3}}\, I_{(j)} Y_1$$
$$Z = \sqrt{\frac{n_{(j)}}{3}}\, I_{(j)} Z_1$$

where X_1, Y_1 and Z_1 have a normal distribution with mean value equal to zero and variance equal to unity.

Thus the expectation of the magnitude of the total transverse impulse is:

(A 8)
$$I_{(j)}^t = \mathrm{E}\sqrt{(Y^2 + Z^2)} = I_{(j)} \sqrt{\frac{n_{(j)}}{3}}\, \mathrm{E}\sqrt{(Y_1^2 + Z_1^2)}$$

where:

(A 9)
$$\mathrm{E}\sqrt{(Y_1^2 + Z_1^2)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sqrt{(Y_1^2 + Z_1^2)}\, \frac{1}{2\pi} \times$$
$$\times \exp\left[-\tfrac{1}{2}(Y_1^2 + Z_1^2)\right] \mathrm{d}Y_1\, \mathrm{d}Z_1 = \sqrt{\frac{\pi}{2}}$$

and consequently:

(A 10)
$$I_{(j)}^t = I_{(j)} \sqrt{n_{(j)}} \sqrt{\frac{\pi}{6}}$$

which is the same as equation (17).

Next we consider a population with the same statistical characteristics, except that the lengths of the vectors depend on the direction through the formula:

(A 11)
$$I_{(j)} = I_{(j)}^0 \sin \alpha$$

where $I_{(j)}^0$ is a constant. The coordinates are in this case given by:

(A 12)
$$x = I_{(j)}^0 \sin^2 \alpha$$
$$y = I_{(j)}^0 \sin \alpha \cos \alpha \cos \beta$$
$$z = I_{(j)}^0 \sin \alpha \cos \alpha \sin \beta$$

and φ is again given by equation (A 2). The expectations of x, y and z are found to be:

(A 13)
$$\mathrm{E}x = \tfrac{1}{3} I_{(j)}^0$$
$$\mathrm{E}y = \mathrm{E}z = 0\,.$$

The expectations of the squares of the coordinates are:

(A 14)
$$\mathrm{E}x^2 = \tfrac{1}{5} I_{(j)}^{02}$$
$$\mathrm{E}y^2 = \mathrm{E}z^2 = \tfrac{1}{15} I_{(j)}^{02}$$

According to (A 14) the variances of y and z are both equal to $\frac{1}{15} I_{(j)}^{02}$, and according to (A 13) their mean values are zero. Again applying the central limit theorem we find that, as $n_{(j)} \to \infty$, the probability distributions of Y and Z approach a normal distribution with mean value equal to zero and variance equal to:

(A 15)
$$\mathrm{Var}\, Y = \mathrm{Var}\, Z = \tfrac{1}{15} n_{(j)} I_{(j)}^{02}$$

In analogy with the above procedure we find that the expectation of the magnitude of the total transverse impulse is:

(A 16)

$$I^t_{(j)} = I^0_{(j)} \sqrt{\frac{n_{(j)}}{15}} \, \mathrm{E} \sqrt{(Y_1^2 + Z_1^2)} = I^0_{(j)} \sqrt{n_{(j)}} \sqrt{\frac{\pi}{30}}$$

which is the same as equation (18).

Appendix B

Derivation of Equations (28) *and* (29)

From the populations of impulse vectors considered in appendix A we now choose at random one of the members, and we are interested in the expectation of the magnitude of its projection of the (y, z)-plane. In the case of equal lengths $I_{(j)}$ of all vectors equations (A 1) and (A 2) still apply, and we find:

(B 1)

$$\mathrm{E} \sqrt{(y^2 + z^2)} = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} I_{(j)} \cos \alpha \frac{1}{4\pi} \cos \alpha \, \mathrm{d}\alpha \, \mathrm{d}\beta =$$

$$= I_{(j)} \int_0^{\pi/2} \cos^2 \alpha \, \mathrm{d}\alpha = I_{(j)} \frac{\pi}{4}$$

Thus the expected transverse component of each impulse vector is:

(B 2) $$^1I_{(j)} = \mathrm{E} \sqrt{(y^2 + z^2)} = \tfrac{1}{4}\pi I_{(j)}$$

which is the same as equation (28).

In the other case, when the lengths of the vectors are determined by equation (A 11), we use equations (A 12) and (A 2) to find:

(B 3) $$\mathrm{E} \sqrt{(y^2 + z^2)} =$$

$$= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} I^0_{(j)} |\sin \alpha| \cos \alpha \frac{1}{4\pi} \cos \alpha \, \mathrm{d}\alpha \, \mathrm{d}\beta =$$

$$= I^0_{(j)} \int_0^{\pi/2} \cos^2 \alpha \sin \alpha \, \mathrm{d}\alpha = \tfrac{1}{3} I^0_{(j)} \, .$$

In this case the expected transverse component of each impulse vector is:

(B 4) $$^1I^t_{(j)} = \mathrm{E} \sqrt{(y^2 + z^2)} = \tfrac{1}{3} I^0_{(j)} \, .$$

which is the same as equation (29).

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