Tutorial 5: Linear optimal control and Kalman Filter

This tutorial will go over the key concepts Linear Quadratic Regulator and Linear Quadratic Gaussian as frameworks to model human sensorimotor control. A MATLAB script has been uploaded onto BlackBoard, which should be used as a starting point to your simulations.

Linear Quadratic Regulator model of sensor-based control

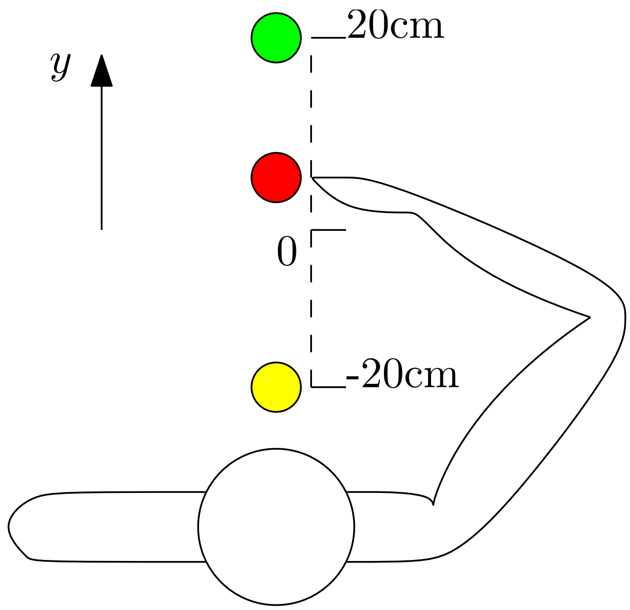

Figure 1: Andy's requirement to move a ball to a target.

Andy has to move a $m = 5g$ frictionless (red) ball along a $20cm$ vertical path to a (green) target in front of him, in $100ms$. The error of the position should be kept within $1cm$ boundary. In some trials the target position, illustrated as yellow target, is shifted during the movement to $20cm$ below the starting point.
To model how Andy is controlling motion, we first assume that the movement is sampled at $1000Hz$, resulting in the following discrete arm dynamics:

$$\begin{aligned} \begin{bmatrix} p_{k+1} \\ v_{k+1} \end{bmatrix} &= \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ v_k \end{bmatrix} + \begin{bmatrix} \frac{(dt)^2}{2m} \\ \frac{dt}{m} \end{bmatrix} (u_k + \eta_k)\,, \quad \eta_k \in N(0, \sigma_\eta) \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ v_k \end{bmatrix} \end{aligned} \tag{1}$$

where $p, v \in \mathbb{R}$ denote the hand's position and velocity in the y direction, k is the time index, $dt = 0.001s$ is the time step, and $u \in \mathbb{R}$ is the force exerted on the ball. Force variability is modelled as zero-mean Gaussian distributed *motor noise* η with covariance $\sigma_\eta^2 = \mathrm{E}[\eta_k^2]$. The y_k line corresponds to perfect measurement of the hand's position available for control. To simplify further algebraic computations, we rewrite equation (1) as

$$\begin{aligned} \mathbf{z}_{k+1} &= \mathbf{A}\mathbf{z}_k + \mathbf{B}\,(u_k + \eta_k) \\ y_k &= \mathbf{C}\mathbf{z}_k\,. \end{aligned} \tag{2}$$

Andy's arm control is modelled as a linear function of the ball's position and velocity feedback

$$u_k = -\mathbf{L}\,(\mathbf{z}_k - \mathbf{z}_r) \tag{3}$$

for a given target $\mathbf{z}_r \equiv [p_r,\, v_r]^T$, where the gain $\mathbf{L}$ has to minimise the cost

$$J = \sum_{k=0}^{\infty} (\mathbf{z}_k - \mathbf{z}_r)^T \mathbf{Q} (\mathbf{z}_k - \mathbf{z}_r) + u_k^T R\, u_k\,. \tag{4}$$

The solution to this *Linear Quadratic Regulator* (LQR) can be computed as

$$\mathbf{L} = (\mathbf{B}^T\mathbf{S}\mathbf{B} + R)^{-1}(\mathbf{B}^T\mathbf{S}\mathbf{A}) \tag{5}$$

where $\mathbf{S}$ is the solution of the *Algebraic Riccati Equation*

$$\mathbf{S} \quad = \mathbf{A}^T[\mathbf{S} - \mathbf{S}\mathbf{B}(\mathbf{B}^T\mathbf{S}\mathbf{B} + R)^{-1}\mathbf{B}^T\mathbf{S}]\mathbf{A} + \mathbf{Q}\,. \tag{6}$$

Question 1: Motion control with perfect sensory feedback (3 marks)

Using the MATLAB command dlqr, design a linear quadratic regulator controller to represent Andy's arm movement, assuming that the force he is exerting with the hand is limited to $10N$. Design the matrix $\mathbf{Q}$ and $\mathbf{R}$ to regulate the maximum force used. Simulate the movement for $1s$. On a single figure, plot:

- Andy's movement to the first target $0.2m$ ahead of the start.
- The hand movement when the target shifted to the yellow circle 30 milliseconds after movement start.
- The force trajectory for the entire movement.

Add a vertical line showing the time when the target change happened. Values for variables not indicated here are already defined in the MATLAB script.

Linear Quadratic Gaussian filtering

When Andy forgets his glasses, the measurement of his hand is no longer perfect, but blurred. This can be modelled as

$$y_k = \mathbf{C}\mathbf{z}_k + \omega_k\,, \quad \omega \in N(0, \sigma_\omega) \tag{7}$$

where the *measurement noise* ω is zero-mean Gaussian distributed with covariance $\sigma_\omega^2 = \mathrm{E}[\omega_k^2]$. We assume that Andy compensates for his blurred vision by using a *forward model*

$$\widehat{\mathbf{z}}_{k+1} = \mathbf{A}\widehat{\mathbf{z}}_k + \mathbf{B}u_k + \mathbf{K}(y_k - \mathbf{C}\widehat{\mathbf{z}}_k), \tag{8}$$

where $(\hat{\cdot})$ represents an estimate of the actual variable, which is used to determine the motor command

$$u_k = -\mathbf{L}\,(\widehat{\mathbf{z}}_k - \mathbf{z}_r) \tag{9}$$

This equation is known as a *linear quadratic Gaussian* (LQG) controller.

Question 2: Influence of sensory noise (3 marks)

Using

$$\mathbf{Q} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\,, \quad R \equiv 0.005\,, \tag{10}$$

plot Andy's performance in both force and position in reaching the target when $\mathbf{K} \equiv [0.5\ 145]$ and $\mathbf{K} \equiv [0.004\ 0.005]$. Interpret the different behaviour in the two cases. (You will need to plot the actual position which is **free** from the sensory noise ω.)

Kalman Filtering

Traditional filter will simply weight the most recent observations more heavily than those before without consider the statistics of the noisy signal it is filtering. However, the gains $\mathbf{K}$ can be selected in order to optimally use the statistical information of the signal that is being fed to it. In particular, the *Kalman filter* (KF) takes advantage of a model of the signal's statistics to best estimate the "true" noiseless signal. The KF is an algorithm that must be run iteratively at every time step, where (i) it uses the dynamics model to predict what the state and the uncertainty of its predicted state are, (ii) which are used to estimate the current state in a stochastically optimal manner:

$$
\begin{aligned}
\widehat{\mathbf{z}}_{k+1} &= \mathbf{A}\widehat{\mathbf{z}}_k + \mathbf{B}u_k + \mathbf{K}_k\left(y_k - \mathbf{C}\widehat{\mathbf{z}}_k\right), \quad \mathbf{K}_k = \mathbf{A}\mathbf{P}_k\mathbf{C}^T\left(\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \sigma_\omega^2\right)^{-1}, \\
\mathbf{P}_{k+1} &= \mathbf{A}\left[\mathbf{P}_k - \mathbf{P}_k\mathbf{C}_k^T(\mathbf{C}_k\mathbf{P}_k\mathbf{C}_k^T + \sigma_\eta^2)^{-1}\mathbf{C}\mathbf{P}_k\right]\mathbf{A}^T + \sigma_\eta\,. \\
\widehat{\mathbf{z}}_0 &\equiv E[\mathbf{z}_0]\,, \quad \mathbf{P}_0 \equiv E[\mathbf{z}_0\,\mathbf{z}_0^T]\,.
\end{aligned}
\tag{11}
$$

Question 3: LQG with Kalman filtering (6 marks)

a. (3 marks) Simulate the movement of Question 2 while using a Kalman filter. On a new figure, add this version to the plot you made in Question 2. Analyse the differences between the new plot and the plot of Question 2 and interpret what the Kalman Filter predicts with regards to Andy's behaviour.

b. (3 marks) It turns out that Andy's motor noise has a covariance $\sigma_\eta^2 \equiv 10^{-4}(N^2)$ while his visual noise a covariance $\sigma_\omega^2 \equiv 0.01(m^2)$. On a single figure plot:

- Andy's performance when using the Kalman filter and
- Andy's performance for the two cases of using the two $\mathbf{K}s$ in Question 2.

Describe the effect of the change in the covariance compared to that of Question 3 part a) and interpret what the Kalman Filter now predicts with regards to Andy's behaviour.