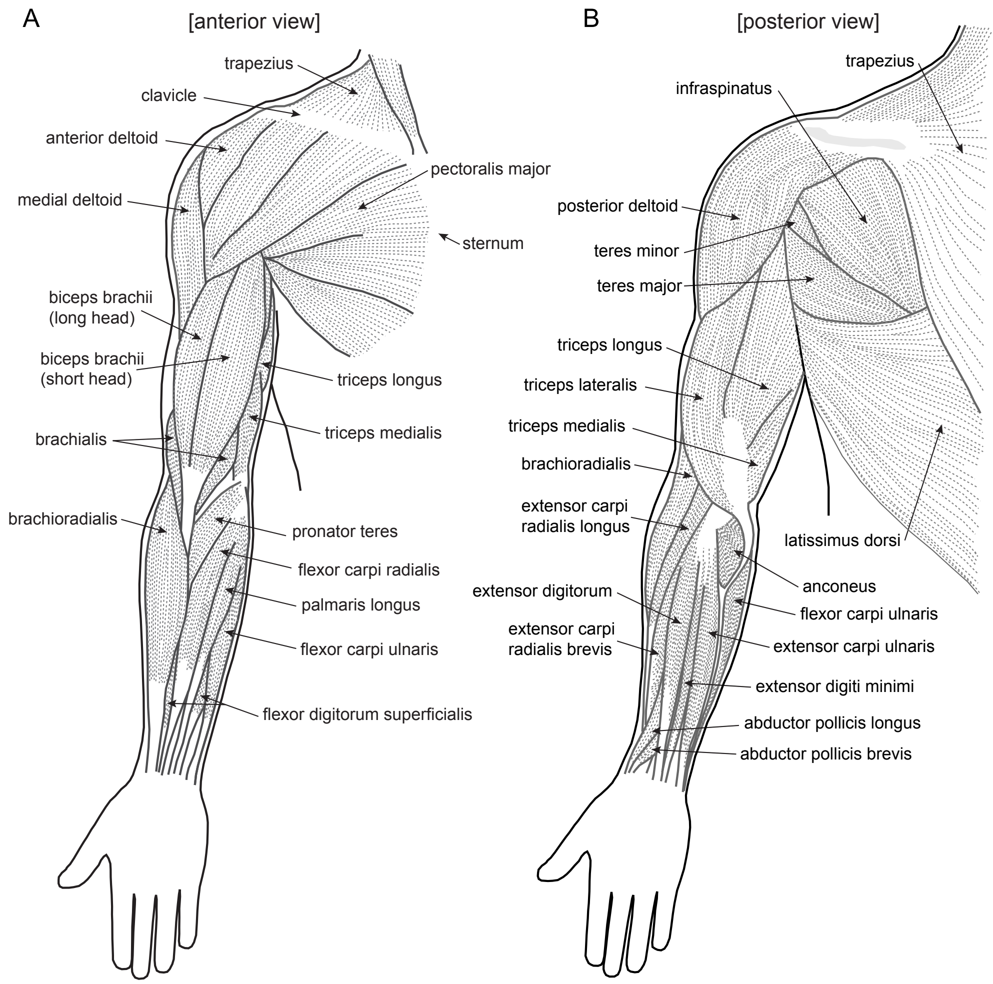
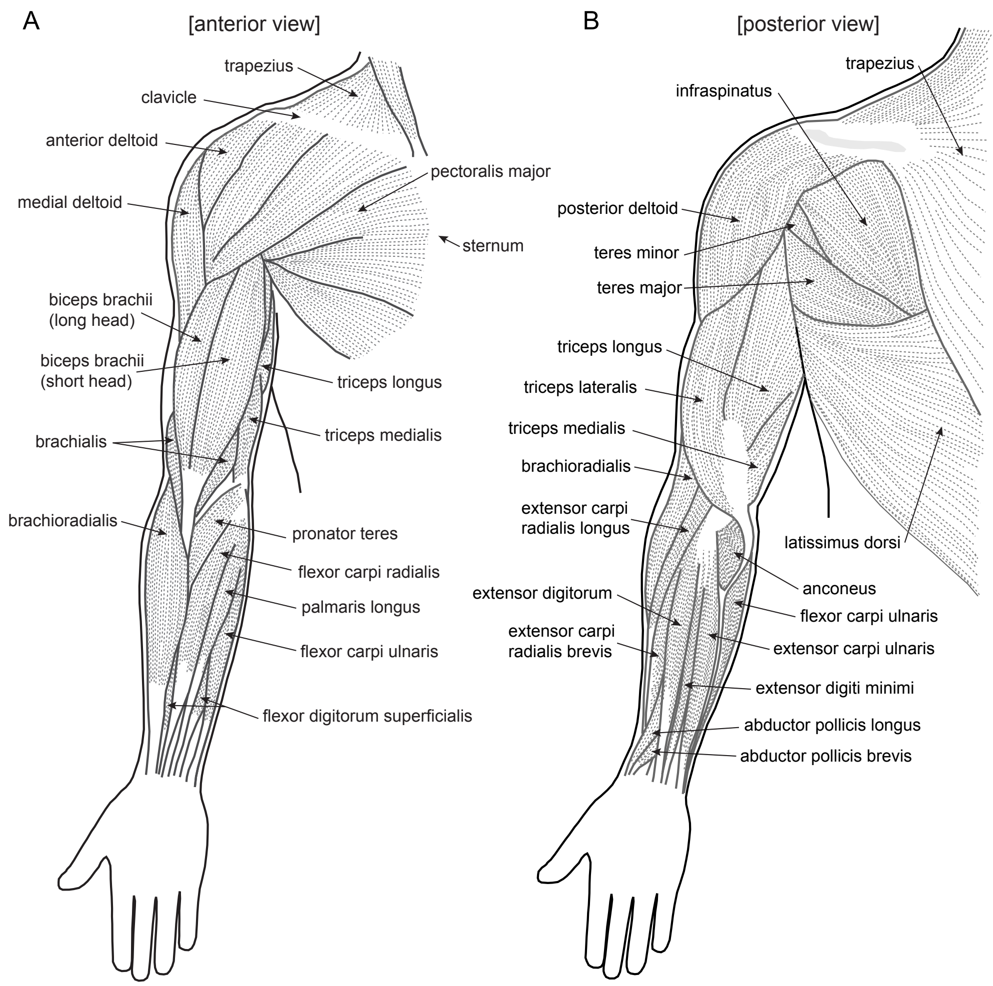
MULTI-JOINT MULTI-MUSCLE KINEMATICS

MULTI-MUSCLE CONTROL (1)



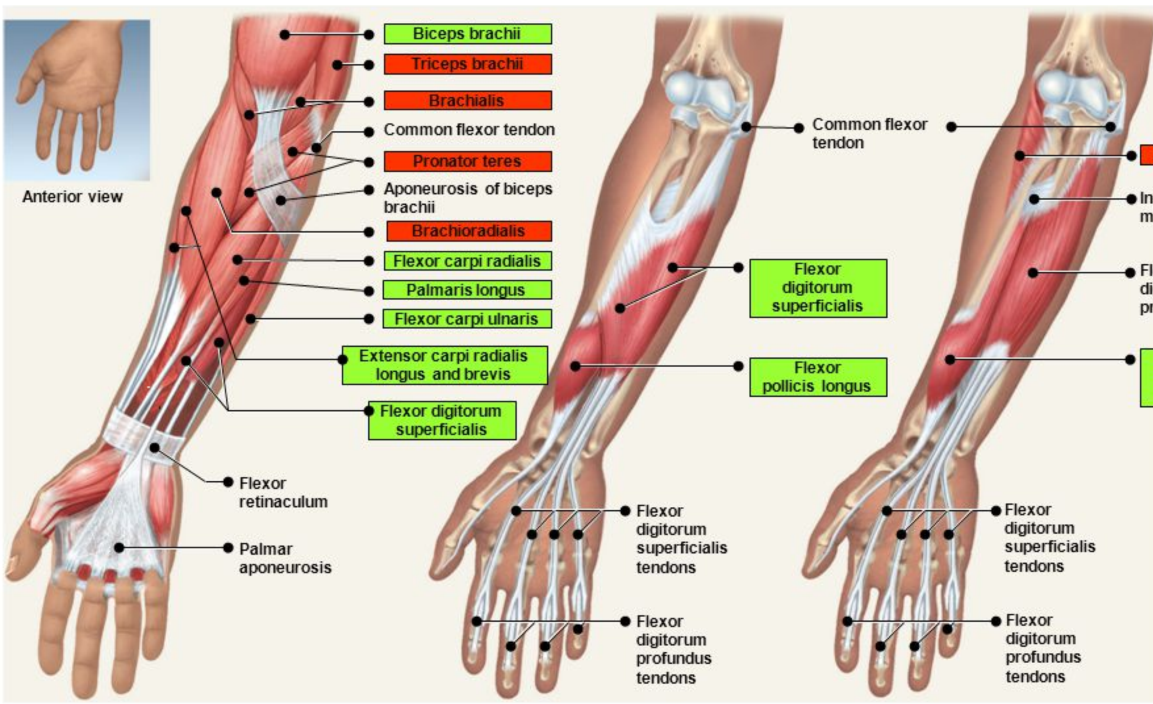
whether a muscle acts as an agonist, antagonist or stabiliser will depend on kinetic factors such as joint angles, link lengths, moment arms and force directions

MULTI-MUSCLE CONTROL (2)



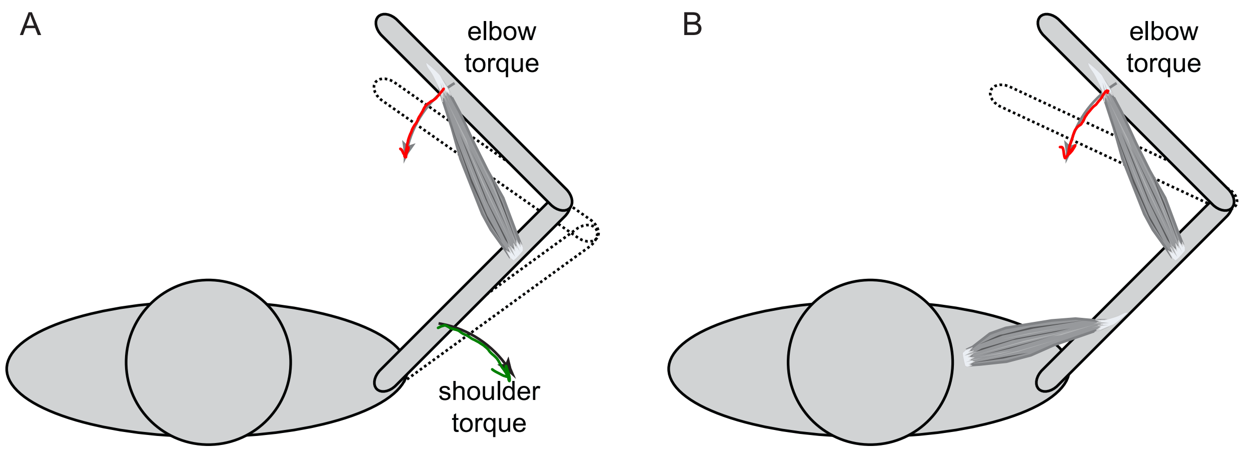
many muscles or their tendons span more than one joint, e.g. biceps brachii and triceps longus are biarticular muscles

MULTI-MUSCLE CONTROL (3)



- the tendons of some finger flexors and extensors cross up to four joints, so most of the muscle mass is removed from the hand, reducing the inertia and allowing rapid response

MULTI-MUSCLE AND LIMB CONTROL (1)



- with the coupling between joints, muscles produce torques directly on the joints they cross (red arrows) but also at remote joints (green arrows)

- motion control requires the coordination of muscles acting at several joints. Some muscles act as movers, some as stabilisers and others as brakes

MULTI-MUSCLE AND LIMB CONTROL (2)

- the limbs and muscles form a highly redundant system with nonlinear and coupled dynamics

- it is not a straightforward matter to predict the activation pattern of a set of muscles when these muscles act on multiple limb segments or about multiple axes of rotation

→ **it is necessary to use kinematics, dynamics and control to describe the trajectories, endpoint force and muscles activation**

KINEMATIC DESCRIPTION

- limbs are open kinematic chains (linkages) consisting of limb segments and joints

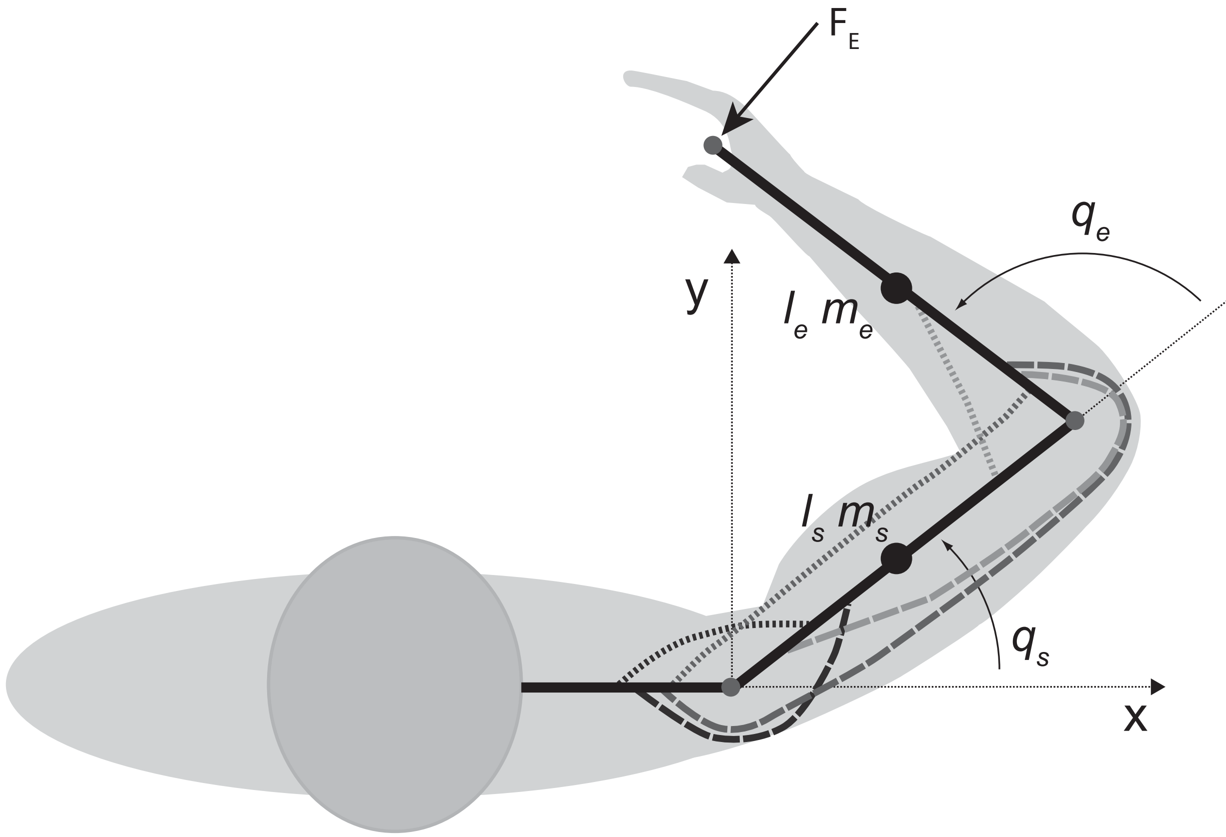
- it is often desirable to simplify the representation of the joints which comprise a limb to avoid the complexity that would arise if an attempt was made to be anatomically correct

- **degrees-of-freedom** (DOF): set of independent displacements or rotations specifying the change in position and orientation of the limb

- arm: 7DOF, can be simplified to 2DOF for horizontal movements → still nonlinear and redundant

3 in shoulder
1 in elbow
3 in wrist (2 in wrist, 1 in forearm).

2 JOINT 6 MUSCLE MODEL



	shoulder	biarticular	elbow
flexor	*pectoralis major*	*biceps*	*brachioradialis*
extensor	*posterior deltoid*	*triceps longus*	*triceps lateralus*

KINEMATICS

- convention: italic scalars s, bold vectors $\mathbf{v}$ and bold large matrices $\mathbf{M}$

- hand or task space $(x_1, x_2, x_3; o_1, o_2, o_3)^T$
  - moving in the Cartesian space
  - $\mathbf{x}$: position, $\mathbf{o}$: orientation
  - related to visual space via visuo-motor coordination

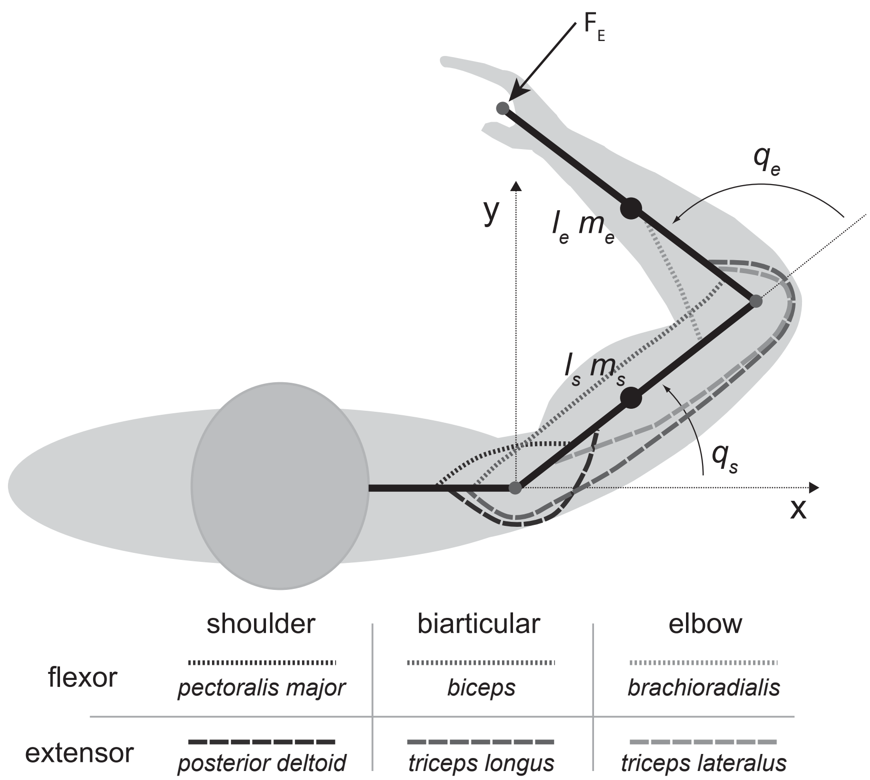
- joint space $\mathbf{q} = (q_1, q_2, \ldots, q_N)^T$ (arm: $N = 7$)

- muscle space $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)^T$: muscles length (many muscles)

KINEMATICS

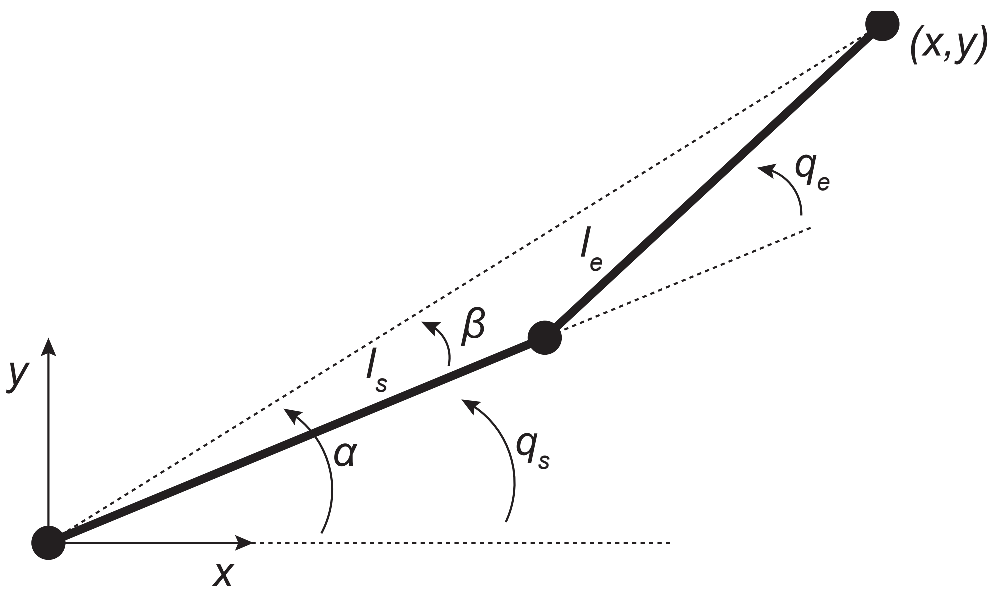
- **forward kinematics**: $\mathbf{x}(\mathbf{q})$ or $\mathbf{x}(\lambda)$
  - transformation from joint or muscle space to hand (Cartesian) space
  - realize a movement
- **inverse kinematics**: $\mathbf{q}(\mathbf{x})$ or $\boldsymbol{\lambda}(\mathbf{x})$
  - from hand to joint or to muscle space
  - how to realize a desired movement?

FORWARD KINEMATICS OF HORIZONTAL ARM MOTION



$$
\begin{aligned}
x &= l_s \cos(q_s) + l_e \cos(q_s + q_e) \\
y &= l_s \sin(q_s) + l_e \sin(q_s + q_e)
\end{aligned}
$$

INVERSE KINEMATICS



$$x^2 + y^2 = l_s^2 + l_e^2 - 2\, l_s l_e \cos(\pi - q_e)$$

$$\rightarrow q_e = \cos^{-1}\left(\frac{x^2 + y^2 - l_s^2 - l_e^2}{2\, l_s l_e}\right)$$

$$q_s = \alpha - \beta\,, \qquad \alpha = \arctan 2\left(\frac{y}{x}\right)$$

$$l_e^2 = x^2 + y^2 + l_s^2 - 2\, l_s \sqrt{x^2 + y^2} \cos\beta$$

$$\beta = \cos^{-1}\left(\frac{x^2 + y^2 + l_s^2 - l_e^2}{2\, l_s \sqrt{x^2 + y^2}}\right)$$

DIFFERENTIAL KINEMATICS

- difficult to find $\mathbf{q}(\mathbf{x})$ and $\mathbf{q}(\boldsymbol{\lambda})$
  (no solution or multiple solutions)

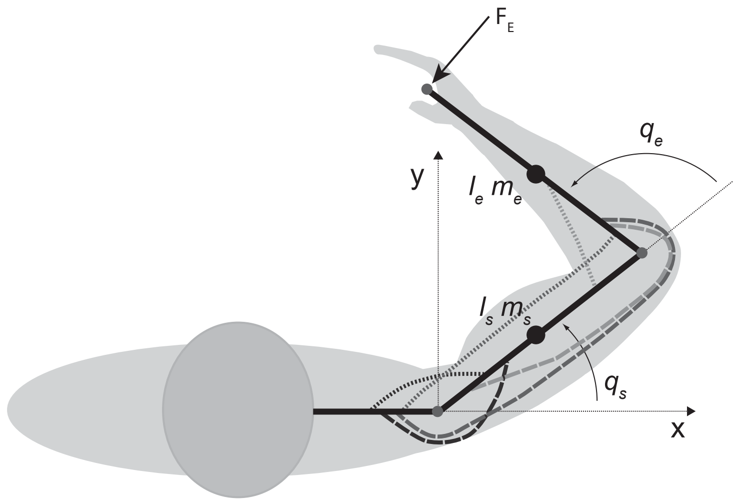
- solution: **differential kinematics** $\dot{\mathrm{x}}(\dot{\mathrm{q}})$

$$\frac{dx_i}{dt} = \sum_j \frac{\partial x_i}{\partial q_j}\frac{dq_j}{dt} \quad \forall i\,, \quad \text{i.e.}$$

$$\dot{\mathrm{x}} = \mathbf{J}(\mathbf{q})\,\dot{\mathrm{q}}\,, \quad \mathbf{J}(\mathbf{q}) = \left(\frac{\partial x_i}{\partial q_j}\right) \textbf{Jacobian}$$

i.e. we have linearized the kinematics

DIFFERENTIAL KINEMATICS OF TWO-JOINT ARM MODEL (1)



$$
\begin{aligned}
x &= l_s \cos(q_s) + l_e \cos(q_s + q_e) \\
y &= l_s \sin(q_s) + l_e \sin(q_s + q_e)
\end{aligned}
$$

$$
\mathbf{J}(\mathbf{q}) = \left(\frac{\partial x_i}{\partial q_j}\right) = \begin{pmatrix} -l_s\, s_s - l_e\, s_{se} & -l_e\, s_{se} \\ l_s\, c_s + l_e\, c_{se} & l_e\, c_{se} \end{pmatrix},
$$

$$
\begin{aligned}
s_s &= \sin(q_s)\,, \;\; s_{se} = \sin(q_s + q_e) \\
c_s &= \cos(q_s)\,, \;\; c_{se} = \cos(q_s + q_e)
\end{aligned}
$$

DIFFERENTIAL KINEMATICS OF TWO-JOINT ARM MODEL (2)

it is also possible to derive a linear relationship between the joint and muscle spaces:

$$\dot{\boldsymbol{\lambda}} = \mathbf{J}_\mu \, \dot{\mathbf{q}} \, , \qquad \mathbf{J}_\mu(\boldsymbol{\rho}) \equiv \left(\frac{\partial \lambda_i}{\partial q_j} \right)$$

$$\mathbf{J}_\mu = \begin{bmatrix} \rho_{s_+} & 0 \\ -\rho_{s_-} & 0 \\ \rho_{bs_+} & \rho_{be_+} \\ -\rho_{bs_-} & -\rho_{be_-} \\ 0 & \rho_{e_+} \\ 0 & -\rho_{e_-} \end{bmatrix}$$

$\boldsymbol{\rho} = (\rho_{s_+}, \rho_{s_-}, \rho_{bs_+}, \rho_{bs_-}, \rho_{be_+}, \rho_{be_-}, \rho_{e_+}, \rho_{e_-})^T$ are the **moments arms** of the shoulder, bi-articular and elbow antagonist muscles pairs

FORCE TRANSFORMATIONS

- let $\mathbf{F}$ be the Cartesian force vector

- the power

$$(\mathbf{F}, \dot{\mathbf{x}}) = \mathbf{F}^T \dot{\mathbf{x}} = \sum_{i=1}^{n} F_i \, \dot{x}_i$$

does not depend on the coordinates:

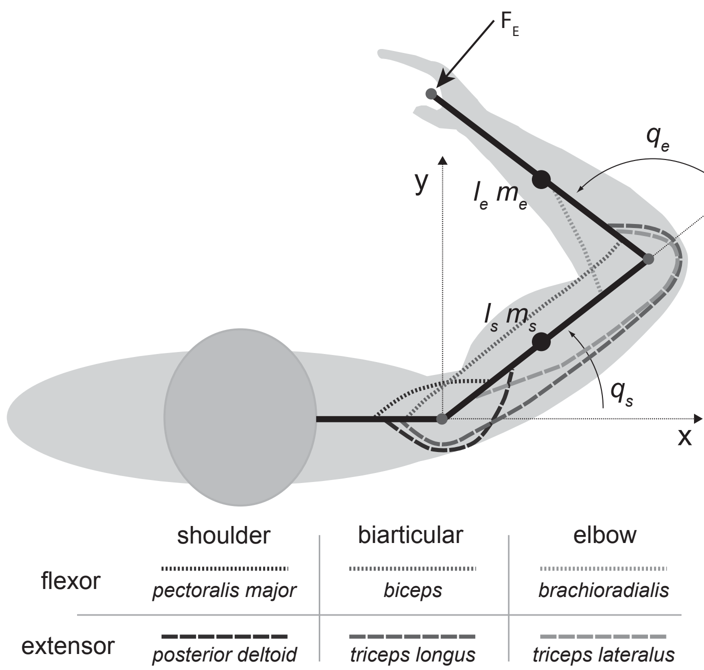
$$(\boldsymbol{\tau}, \dot{\mathbf{q}}) = (\mathbf{F}, \dot{\mathbf{x}}) = (\mathbf{F}, \mathbf{J}\dot{\mathbf{q}}) = (\mathbf{J}^T\mathbf{F}, \dot{\mathbf{q}}) \quad \forall \dot{\mathbf{q}} \, ,$$

thus

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}$$

- this relation is used to compute the torque which must be exerted at the joints to produce a required force at the hand

TRANSFORMATIONS FROM MUSCLE TO JOINT SPACES



let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)^T$ be the **vector of muscles tension**

$$(\boldsymbol{\tau}, \dot{\mathrm{q}}) \equiv (\boldsymbol{\mu}, \dot{\boldsymbol{\lambda}}) = (\boldsymbol{\mu}, \mathbf{J}_\mu \dot{\mathrm{q}}) = (\mathbf{J}_\mu^T \boldsymbol{\mu}, \dot{\mathrm{q}}) \quad \forall \dot{\mathbf{q}}\,,$$

thus

$$\boldsymbol{\tau} = \mathbf{J}_\mu(\boldsymbol{\rho})^T \boldsymbol{\mu}$$

MECHANICAL IMPEDANCE (1)

- a major role of the motor system is to interact with the environment

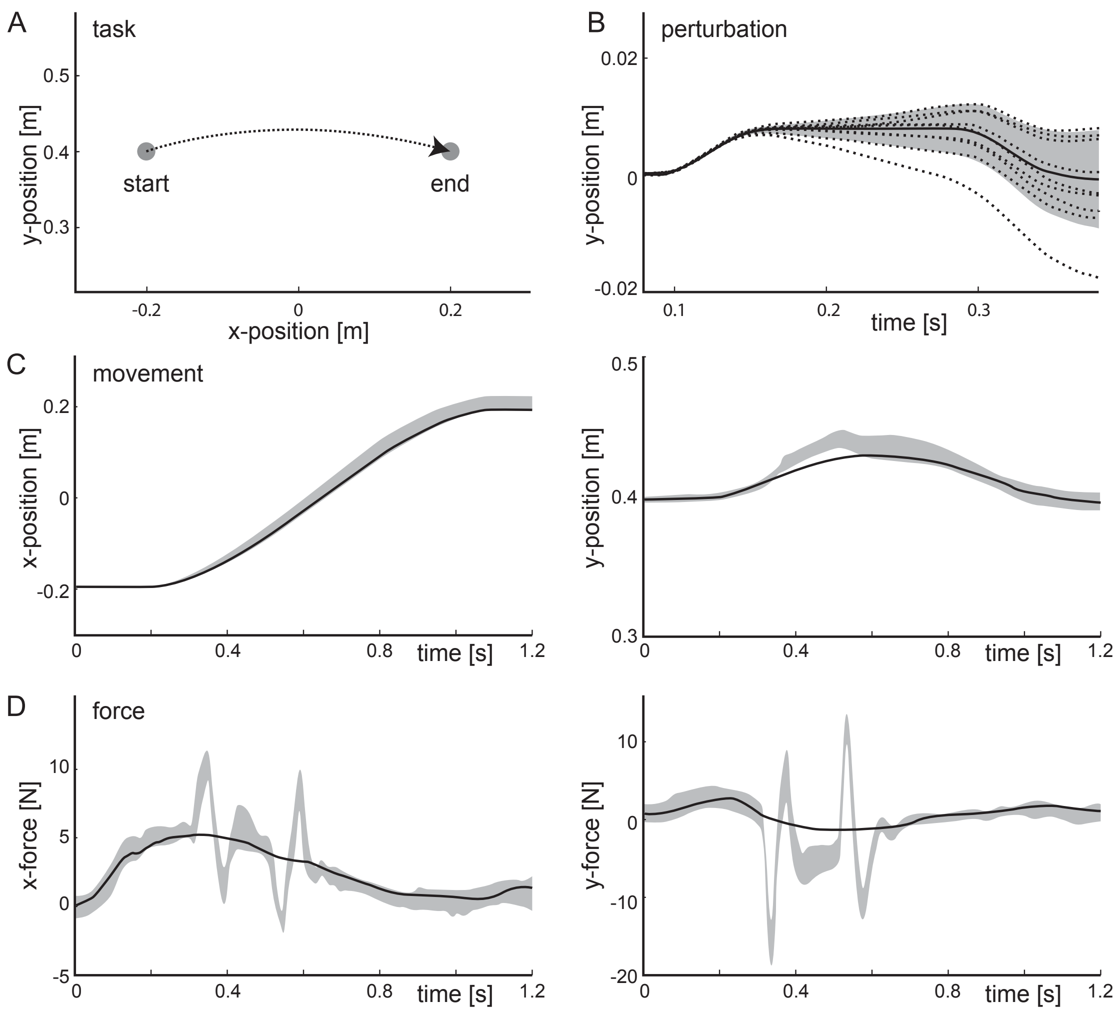
- this interaction is two way: action = reaction

- it is characterised by the response to imposed motion or **mechanical impedance**

- when the arm is slightly perturbed during movement it tends to return to its undisturbed trajectory, as if it is connected to a spring moving along a planned trajectory

- this springlike property stems mainly from muscle elasticity and the stretch reflex

ESTIMATE IMPEDANCE



- stiffness identified from the change of force and the displacement
- 0 change in velocity: viscosity not identified

MECHANICAL IMPEDANCE (2)

- most mechanical interactions take place at the endpoint of the limb

- **mechanical impedance**: resistance to a perturbation of the state

$$\delta\mathbf{F}(\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}},\mathbf{u}(\mathbf{x},\dot{\mathbf{x}})) = \mathbf{K}_x\,\delta\mathbf{x} + \mathbf{D}_x\,\delta\dot{\mathbf{x}} + \mathbf{I}_x\,\delta\ddot{\mathbf{x}}$$

$$\begin{aligned}
\mathbf{K}_x &\equiv \left(\frac{\partial F_i}{\partial x_j} + \sum_k \frac{\partial F_i}{\partial u_k}\frac{\partial u_k}{\partial x_j}\right) && \textbf{stiffness}\\
\mathbf{D}_x &\equiv \left(\frac{\partial F_i}{\partial \dot{x}_j} + \sum_k \frac{\partial F_i}{\partial u_k}\frac{\partial u_k}{\partial \dot{x}_j}\right) && \textbf{viscosity}\\
\mathbf{I}_x &\equiv \left(\frac{\partial F_i}{\partial \ddot{x}_j}\right) && \textbf{inertia}
\end{aligned}$$

- in this sense impedance is composed of stiffness, damping and inertia

MECHANICAL IMPEDANCE (3)

- motion stability depends on muscle elasticity and reflexes

- determined by measuring the restoring elastic force to small displacements of the hand

- cannot isolate the activation-dependent parts (u terms), i.e., reflex feedback, from muscle intrinsic viscoelastic properties

- measurements normally combine both components, corresponding to the above definitions of $\mathbf{K}_x$ and $\mathbf{D}_x$

MECHANICAL IMPEDANCE (4)

- linearize endpoint force as a function of position, velocity and acceleration:

  $$\delta\mathbf{F}(\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}},\mathbf{u}(\mathbf{x},\dot{\mathbf{x}})) = \mathbf{K}_x\,\delta\mathbf{x} + \mathbf{D}_x\,\delta\dot{\mathbf{x}} + \mathbf{I}_x\,\delta\ddot{\mathbf{x}}\,,$$

- Laplace transform

  $$\begin{aligned} L[\mathbf{F}](s) &= \mathbf{K}_x\mathbf{X} + \mathbf{D}_x\,s\mathbf{X} + \mathbf{I}_x\,s^2\mathbf{X} \\ &= (\mathbf{K}_x + s\mathbf{D}_x + s^2\mathbf{I}_x)\,\mathbf{X}(s) \end{aligned}$$

- **impedance = transfer function**

  $$\left(\frac{L[F_i]}{L[x_j]}\right)(s) = \mathbf{K}_x + s\mathbf{D}_x + s^2\mathbf{I}_x$$

STIFFNESS TRANSFORMATION

from

$$\begin{aligned}
\mathbf{K} &\equiv \left(\frac{\partial \tau_i}{\partial q_j}\right) = \left(\frac{\partial (\mathbf{J}^T \mathbf{F})_i}{\partial q_j}\right) \\
&= \left(\sum_k \frac{\partial (\mathbf{J}^T)_{ik}}{\partial q_j} F_k\right) + \mathbf{J}^T \left(\frac{\partial F_i}{\partial q_j}\right) \\
&= \left(\sum_k \frac{\partial (\mathbf{J}^T)_{ik}}{\partial q_j} F_k\right) + \mathbf{J}^T \sum_k \left(\frac{\partial F_i}{\partial x_k}\right) \left(\frac{\partial x_k}{\partial q_j}\right)
\end{aligned}$$

follows

$$\mathbf{K} = \frac{d\mathbf{J}^T}{d\mathbf{q}} \mathbf{F} + \mathbf{J}^T \mathbf{K}_x \, \mathbf{J}$$

similarly

$$\mathbf{K} = \frac{d\mathbf{J}_\mu^T}{d\mathbf{q}} \boldsymbol{\mu} + \mathbf{J}_\mu^T \mathbf{K}_\mu \, \mathbf{J}_\mu$$

2 JOINT 6 MUSCLE EXAMPLE

assumptions: no external force, unit moment arms, no heteronymous reflexes

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \kappa_{s+} & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa_{s-} & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_{b+} & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_{b-} & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_{e+} & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_{e-} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \kappa_{s+} + \kappa_{s-} + \kappa_{b+} + \kappa_{b-} & \kappa_{b+} + \kappa_{b-} \\ \kappa_{b+} + \kappa_{b-} & \kappa_{e+} + \kappa_{e-} + \kappa_{b+} + \kappa_{b-} \end{bmatrix} \equiv \begin{bmatrix} \kappa_s + \kappa_b & \kappa_b \\ \kappa_b & \kappa_e + \kappa_b \end{bmatrix} \quad ((
$$

- without heteronymous reflexes the stiffness matrix and thus the couplings between the joints are symmetric

- heteronymous reflexes could allow the CNS to produce asymmetric stiffness or impedance

2 JOINT 6 MUSCLE EXAMPLE

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \kappa_{s+} & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa_{s-} & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_{b+} & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_{b-} & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_{e+} & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_{e-} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \kappa_{s+} + \kappa_{s-} + \kappa_{b+} + \kappa_{b-} & \kappa_{b+} + \kappa_{b-} \\ \kappa_{b+} + \kappa_{b-} & \kappa_{e+} + \kappa_{e-} + \kappa_{b+} + \kappa_{b-} \end{bmatrix} \equiv \begin{bmatrix} \kappa_s + \kappa_b & \kappa_b \\ \kappa_b & \kappa_e + \kappa_b \end{bmatrix} \qquad (($$

- monoarticular muscles and reflexes influence the diagonal elements of the stiffness matrix and enable modulation of stiffness at the endpoint along the axes determined by the transformation represented by the Jacobian

- biarticular muscles produce cross terms enabling modification of the stiffness geometry along different axes

IMPEDANCE TRANSFORMATION

$$\begin{aligned}
\mathbf{D} &= \mathbf{J}^T \mathbf{D}_x \mathbf{J} \\
\mathbf{I} &= \mathbf{J}^T \mathbf{I}_x \mathbf{J}
\end{aligned}$$

for example

$$\begin{aligned}
\mathbf{D} &\equiv \left(\frac{\partial \tau_i}{\partial \dot{q}_j}\right) = \left(\frac{\partial (\mathbf{J}^T \mathbf{F})_i}{\partial \dot{q}_j}\right) \\
&= \left(\sum_k \frac{\partial (\mathbf{J}(\mathbf{q})^T)_{ik}}{\partial \dot{q}_j} F_k\right) + \mathbf{J}^T \left(\frac{\partial F_i}{\partial \dot{q}_j}\right) \\
&= \mathbf{0} + \mathbf{J}^T \sum_k \left(\frac{\partial F_i}{\partial \dot{x}_k}\right) \left(\frac{\partial \dot{x}_k}{\partial \dot{q}_j}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \dot{x}_k}{\partial \dot{q}_j} &= \sum_l \frac{\partial}{\partial \dot{q}_j} J_{kl} \dot{q}_l \\
&= \sum_l \frac{\partial J_{kl}(\mathbf{q})}{\partial \dot{q}_j} \dot{q}_l + \sum_l J_{kl} \frac{\partial \dot{q}_l}{\partial \dot{q}_j} \\
&= 0 + J_{kj}
\end{aligned}$$

TRANSFORMATIONS TABLE

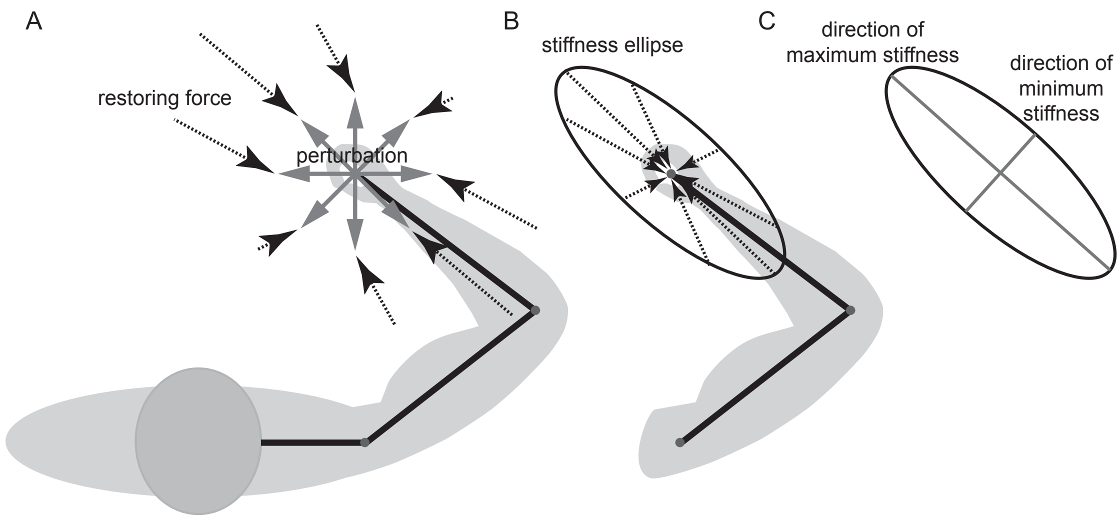
hand		joint		muscle	
$\mathbf{x}$	$\longleftarrow$	$\mathbf{q}$	$\longrightarrow$	λ	position
$\dot{\mathbf{x}}$	$\xleftarrow{\mathbf{J}}$	$\dot{\mathbf{q}}$	$\xrightarrow{\mathbf{J}_\mu(\rho)}$	$\dot{\lambda}$	velocity
$\mathbf{F}$	$\xrightarrow{\mathbf{J}^T}$	$\boldsymbol{\tau}$	$\xleftarrow{\mathbf{J}_\mu(\rho)^T}$	$\boldsymbol{\mu}$	force
$\mathbf{K}_x$	$\xrightarrow{\mathbf{J}^T\mathbf{K}_x\mathbf{J}+\frac{d\mathbf{J}^T}{d\mathbf{q}}\mathbf{F}}$	$\mathbf{K}$	$\xleftarrow{\mathbf{J}_\mu^T\mathbf{K}_\mu\mathbf{J}_\mu+\frac{d\mathbf{J}_\mu^T}{d\mathbf{q}}\boldsymbol{\mu}}$	$\mathbf{K}_\mu$	stiffness
$\mathbf{D}_x$	$\xrightarrow{\mathbf{J}^T\mathbf{D}_x\mathbf{J}}$	$\mathbf{D}$	$\xleftarrow{\mathbf{J}_\mu^T\mathbf{D}_\mu\mathbf{J}_\mu}$	$\mathbf{D}_\mu$	damping

2-link 6-muscle model:

- $\mathbf{x}, \mathbf{q}, \boldsymbol{\tau}, \mathbf{F}$: 2x1 vectors, $\boldsymbol{\lambda}, \boldsymbol{\mu}$: 6x1 vectors

- $\mathbf{K}_x, \mathbf{K}$ are 2x2 matrices, $\mathbf{K}_\mu$ is a 6x6 matrix

STIFFNESS AND STABILITY

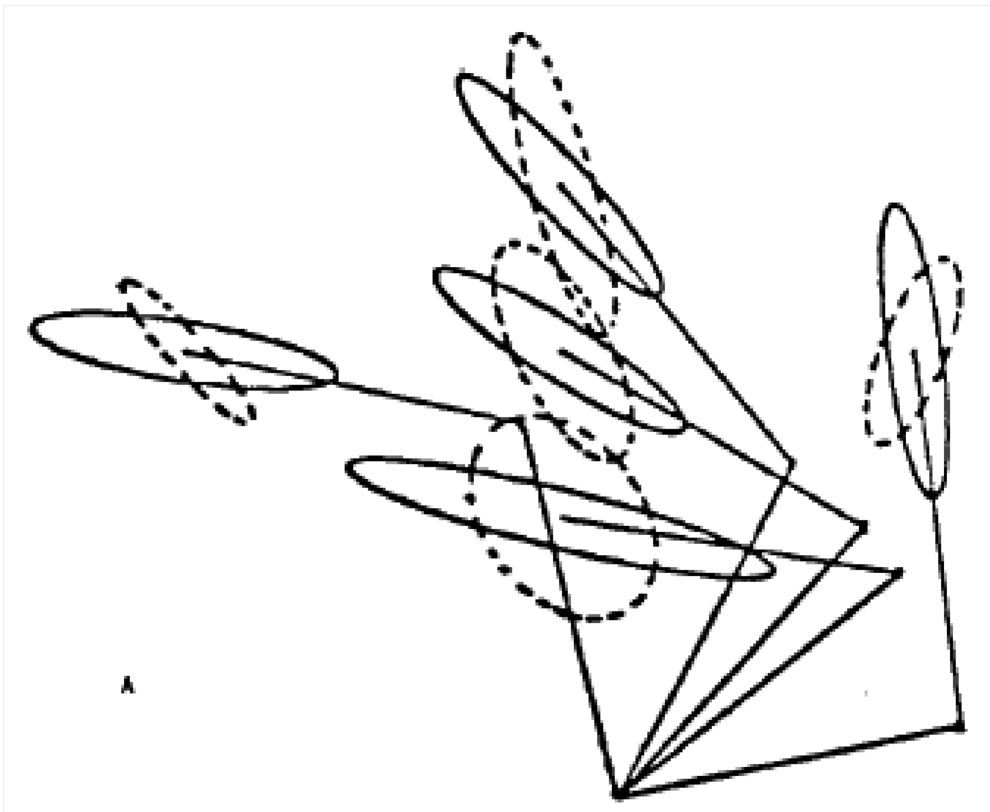


- motion stability depends on muscle elasticity and reflexes, i.e. impedance

- determined by measuring the restoring elastic force to small displacements of the hand

- stiffness ellipse $\mathbf{K}_x \frac{\mathbf{x}}{||\mathbf{x}||}$, i.e. force corresponding to a unit displacement, can be plotted to visualize stiffness geometry

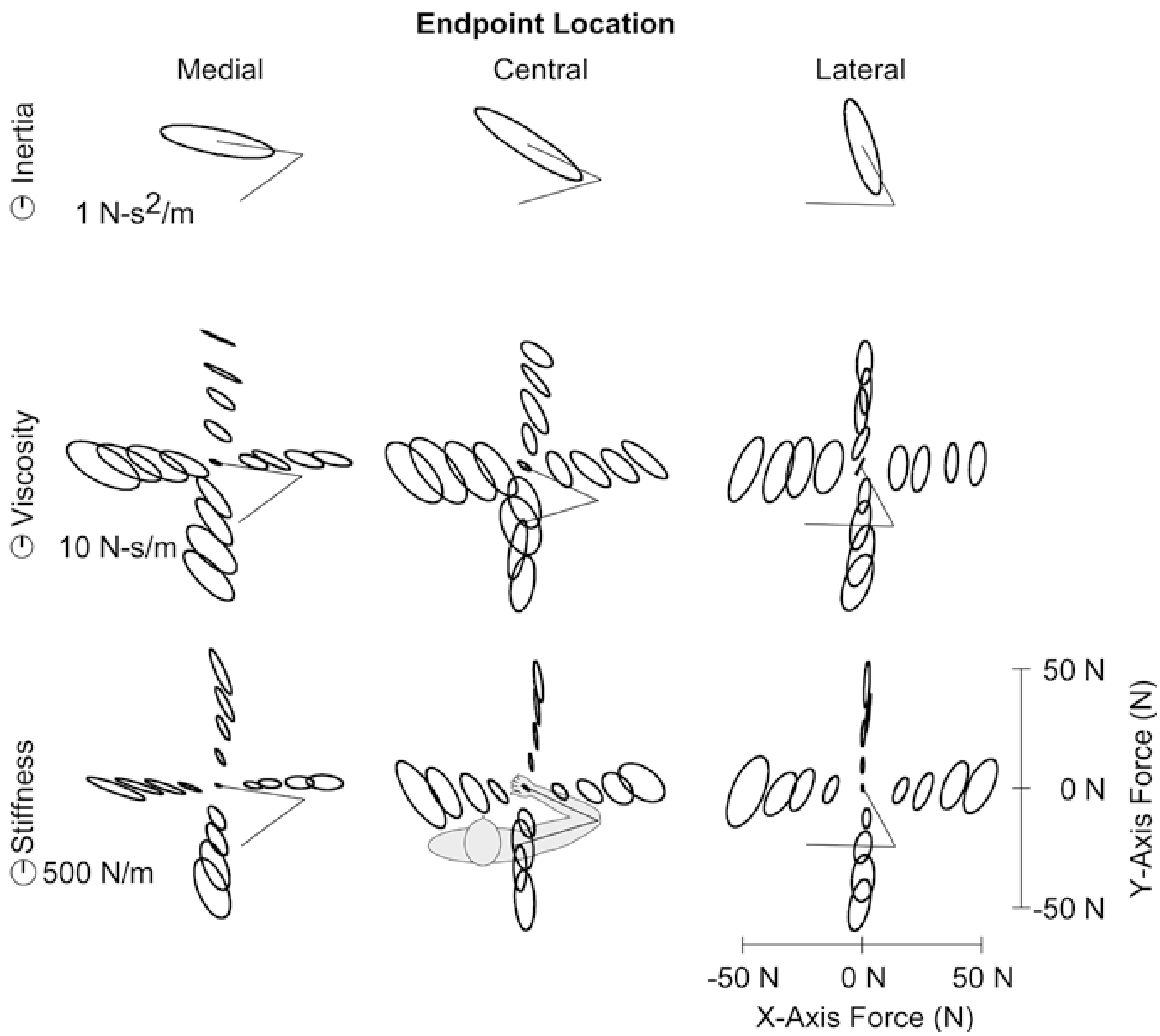
IMPEDANCE GEOMETRY DEPENDS ON POSITION



The static endpoint stiffness and inertia ellipses of the relaxed arm vary in shape, orientation and size when the hand is displaced at different locations in the horizontal plane

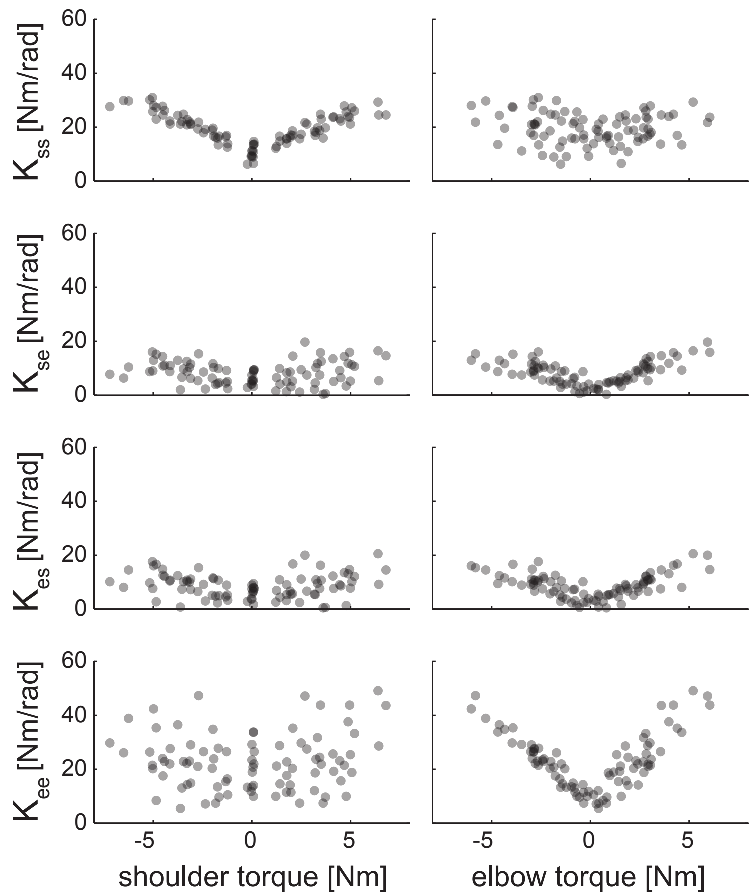
[Mussa-Ivaldi et al. Journal of Neuroscience 1985]

IMPEDANCE GEOMETRY DEPENDS ON APPLIED FORCE



[Perreault et al. Experimental Brain Research 2004]

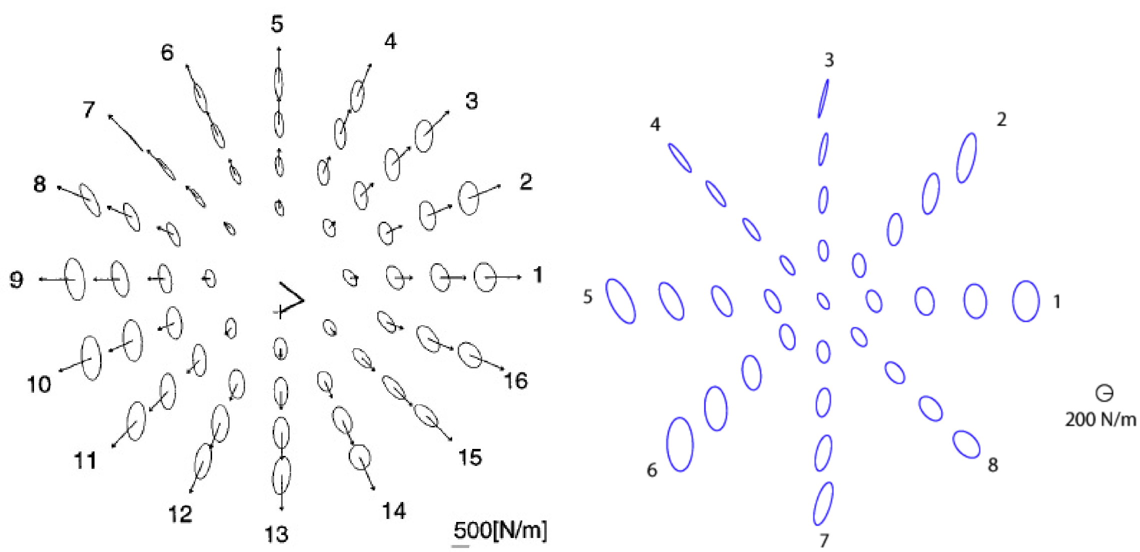
STIFFNESS DEPENDS ON THE TORQUE MAGNITUDE (1)



the coefficients of the joint stiffness matrix grow linearly with the torque magnitude

[Gomi and Osu Journal of Neuroscience 1998]

STIFFNESS DEPENDS ON THE TORQUE MAGNITUDE (2)



for planar arm

$$\mathbf{K} = \begin{pmatrix} 10.8 + 3.18|\tau_s| & 2.83 + 2.15|\tau_e| \\ 2.51 + 2.34|\tau_e| & 8.67 + 6.18|\tau_e| \end{pmatrix} \frac{Nm}{rad}$$

[Gomi and Osu Journal of Neuroscience 1998]

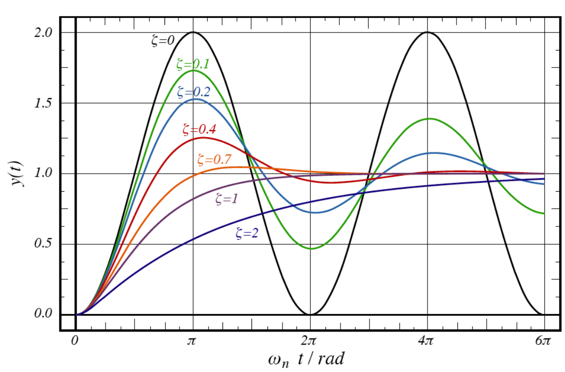
DAMPING DEPENDS ON THE TORQUE MAGNITUDE (3)

- the coefficients of the joint damping matrix grow linearly with the square root of the torque magnitude

- damping ratio

$$\zeta = \frac{D_x}{2\sqrt{K_x\, I_x}} \sim 0.26$$

- muscle mechanics is underdamped



[Perreault et al. Experimental Brain Research 2004]

STATIC STABILITY

for planar arm

$$\mathbf{K} = \begin{pmatrix} 10.8 + 3.18|\tau_s| & 2.83 + 2.15|\tau_e| \\ 2.51 + 2.34|\tau_e| & 8.67 + 6.18|\tau_e| \end{pmatrix} \frac{Nm}{rad}$$

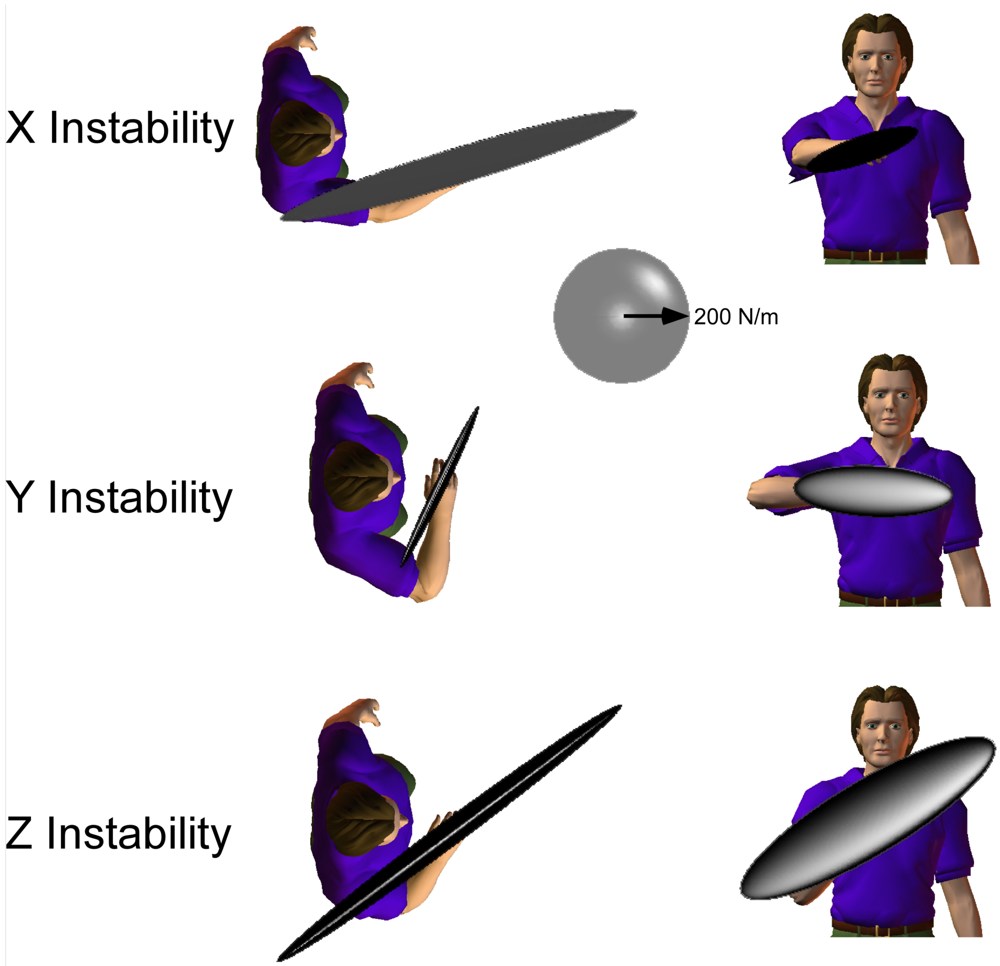
- this nearly symmetric stiffness matrix corresponds to the signature of spring-like conservative forces which can be defined by a potential function

- a system which has spring-like properties is guaranteed to interact in a stable manner with passive environments

- this may explain some of the amazing capabilities humans have for stable interaction with the environment

[Colgate and Hogan J Control 1988]

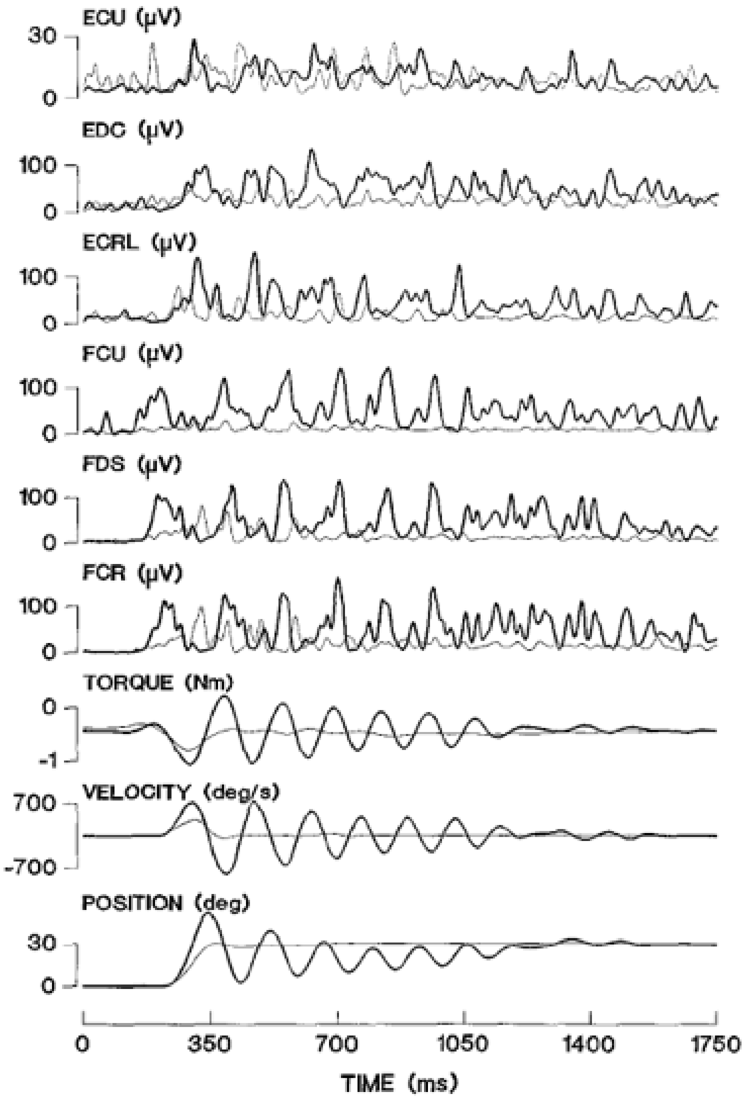
TASK ADAPTED POSTURE



- subjects adapt posture against instability
- in general: it is possible to change the stiffness, damping and inertia by simply changing the kinematic configuration of the limb

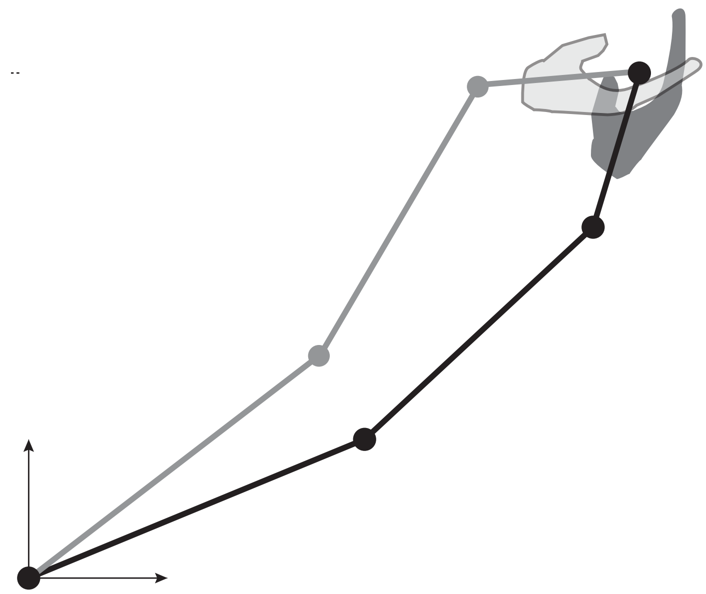
[Perreault et al. PlosONE 2009]

REDUNDANCY (1)



wrist flexion/extension involves several flexor and several extensor muscles which could be combined in an infinite number of ways

REDUNDANCY (2)



a system is **redundant** when there are more DOF than required by the task

in a redundant mechanism, the joint can be combined in various ways, for example to take a ping-pong ball on a table

if the task requires a certain orientation, for example to take a mobile phone on a table, the mechanism is not redundant

REDUNDANCY (3)

- the human arm has 7 DOF, though performing movements in space requires only 6 DOF
  kinematic redundancy

- most movements can be performed using various muscle combinations
  muscle redundancy

- in reaching movements, for example, infinitely many path can be used to the target, and the muscles can also be used in infinite many different sequences
  trajectory redundancy

REDUNDANCY (4)

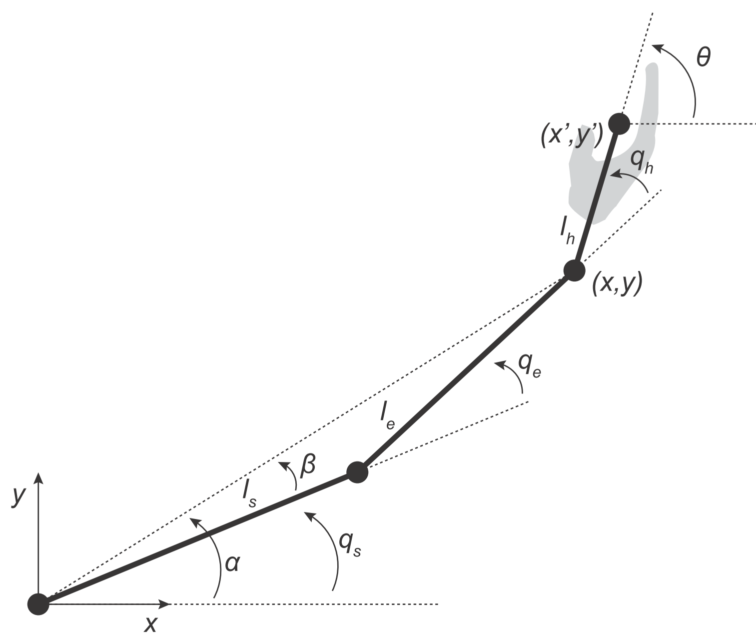
- the control of the musculo-skeletal system is characterised by redundancy (in the muscles, in the joints, and in the trajectory), which the CNS needs to solve

- conversely this redundancy may allow us to avoid uncomfortable postures or to minimise effort

- humans generally repeat a task using always the same movement patterns and coordinate the limbs in a specific way

- how can this be modelled?

REDUNDANCY IN THE PLANE (1)

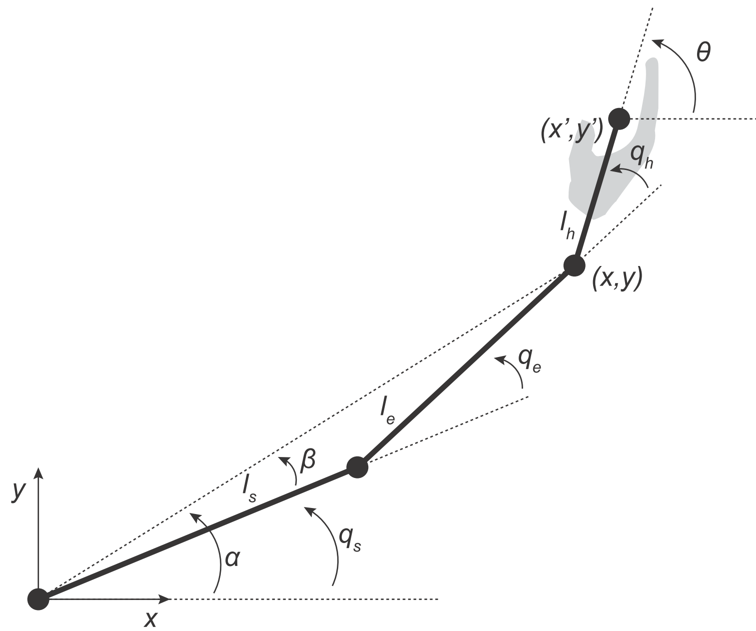


$$
\begin{aligned}
x' &= l_s\, c_s + l_e\, c_{se} + l_h\, c_{seh} \\
y' &= l_s\, s_s + l_e\, s_{se} + l_h\, s_{seh}\,,
\end{aligned}
$$

$s_s = \sin(q_s), c_s = \cos(q_s)$
$s_{se} = \sin(q_s + q_e), c_{se} = \cos(q_s + q_e),$
$s_{seh} = \sin(q_s + q_e + q_h), c_{seh} = \cos(q_s + q_e + q_h)$

there are many possibilities to reach a given position $(x', y')^T$

REDUNDANCY IN THE PLANE (2)



- one can reach this target with any orientation of the hand

Introduce an extra parameter, to make jacobian square.

- orientation $\theta = q_s + q_e + q_h$, then

$$\begin{aligned} x &\equiv x' - l_h\, c_\theta = l_s\, c_s + l_e\, c_{se} \\ y &\equiv y' - l_h\, s_\theta = l_s\, s_s + l_e\, s_{se} \end{aligned}$$

can be solved as before

REDUNDANCY IN THE PLANE (3)

$$\begin{bmatrix} \dot{x}' \\ \dot{y}' \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}, \quad J_{ij} \equiv \frac{\partial x_i}{\partial q_j}$$

add a constraint to solve the redundancy

if we specify the orientation $\theta = q_1 + q_2 + q_3$

Square jacobian.

$$\begin{bmatrix} \dot{x}' \\ \dot{y}' \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \equiv \mathbf{J}\dot{\mathbf{q}}$$

then $\quad \dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})\,\dot{\mathbf{x}}$

(if $\mathbf{J}^{-1}(\mathbf{q})$ is defined, e.g. if $\det \mathbf{J}(\mathbf{q}) \neq 0$)

REDUNDANCY IN ROBOTS

- to reduce redundancy: minimizing speed $||\dot{\mathbf{q}}||$ under the constraints $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}}$

- e.g. by hydraulic manipulators the oil consumption is roughly proportional to speed

- Lagrange undetermined multipliers method:

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\dot{\mathbf{x}}\,, \quad \mathbf{J}^{\dagger} = \mathbf{J}^{T}(\mathbf{J}\mathbf{J}^{T})^{-1}\,,\ \det \mathbf{J}\mathbf{J}^{T} \neq 0$$

- **pseudo-inverse** $\mathbf{J}^{\dagger}$ is right-inverse of $\mathbf{J}$:

$$\mathbf{J}\mathbf{J}^{\dagger} = \mathbf{J}\underset{M^{-1}}{\mathbf{J}^{T}}(\mathbf{J}\underset{M^{-1}}{\mathbf{J}^{T}})^{-1} = \mathbf{1}$$

inverse mass matrix

This addition gives the weighted pseudo-inverse.

EXAMPLE OF REDUNDANCY REDUCTION

one joint mechanism actuated by two rotary actuators placed in series

joint angle velocity

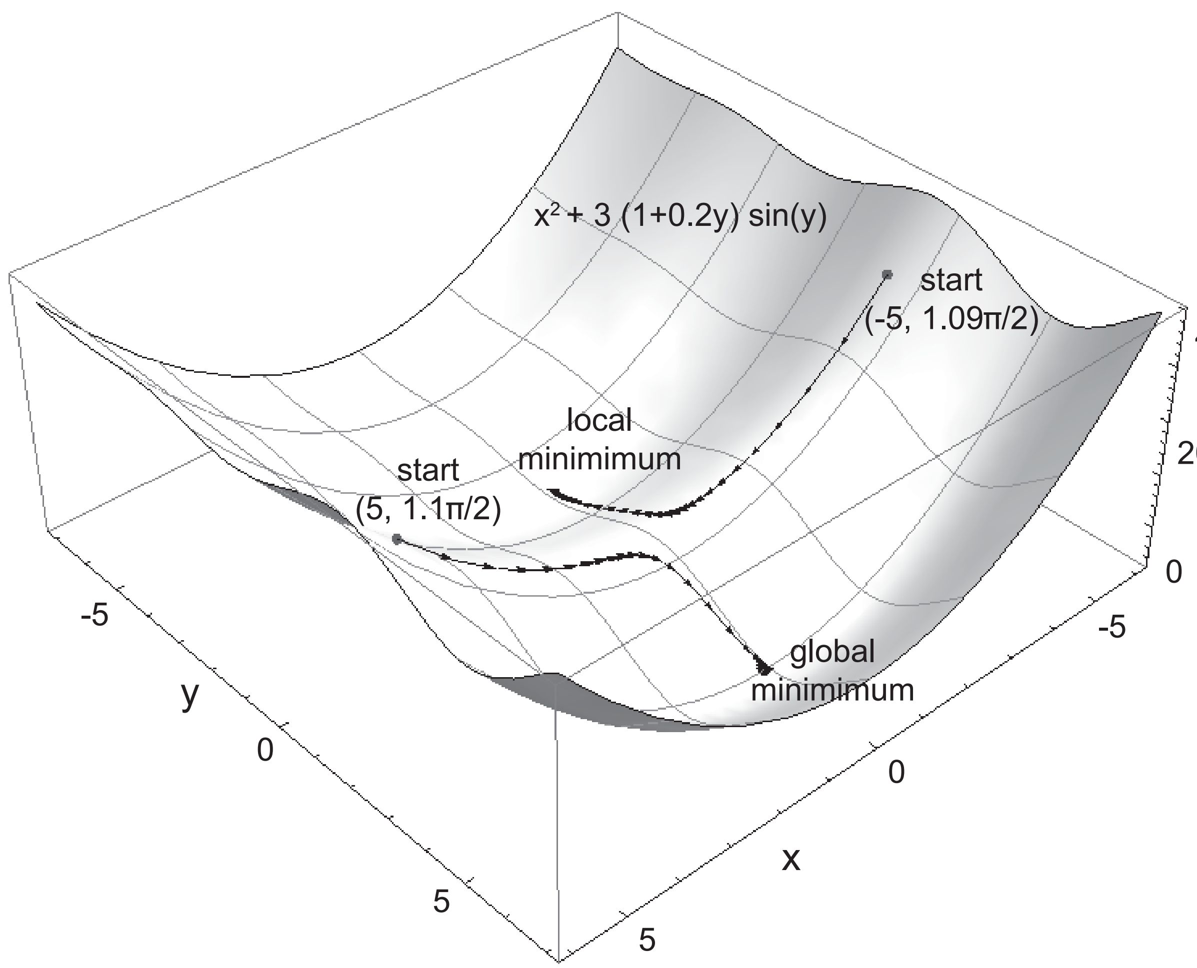
$$\dot{\theta} = \dot{q}_1 + \dot{q}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \equiv \mathbf{J}\,\dot{\mathbf{q}}$$

$$\begin{aligned} \mathbf{J}\mathbf{J}^T &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\,, \quad \text{thus} \\ \mathbf{J}^\dagger &= \mathbf{J}^T \left(\mathbf{J}\mathbf{J}^T\right)^{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{2} \end{aligned}$$

actuation minimizing speed $\dot{q}_1^2 + \dot{q}_2^2$:

$$\begin{bmatrix} \dot{q}_1^* \\ \dot{q}_2^* \end{bmatrix} = \mathbf{J}^\dagger \dot{\theta} = \begin{bmatrix} \dot{\theta}/2 \\ \dot{\theta}/2 \end{bmatrix}$$

GRADIENT DESCENT MINIMISATION



parameter is shifted in the direction of the steepest descent of the cost function

REDUNDANCY IN ROBOTS (2)

- redundancy to reconfigure the manipulator structure without changing the end-effector position or orientation

- using 'internal motions' inside the **nullspace**

$$N \equiv \{\dot{\mathbf{q}}, \mathbf{J}\dot{\mathbf{q}} = \mathbf{0}\}$$

to reduce some cost $V(\mathbf{q})$

- if $\dot{\mathbf{q}}_o$ is a particular solution of $\dot{\mathbf{x}}_o = \mathbf{J}\dot{\mathbf{q}}_o$, then for all $\dot{\mathbf{q}} \in N$ $\dot{\mathbf{q}}_o + \dot{\mathbf{q}}$ is also solution

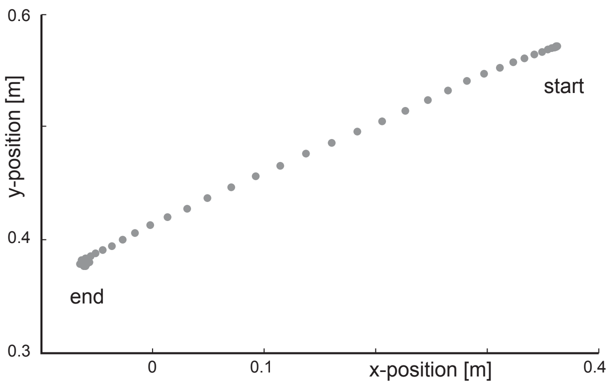
- the cost V can be reduced using the gradient descent

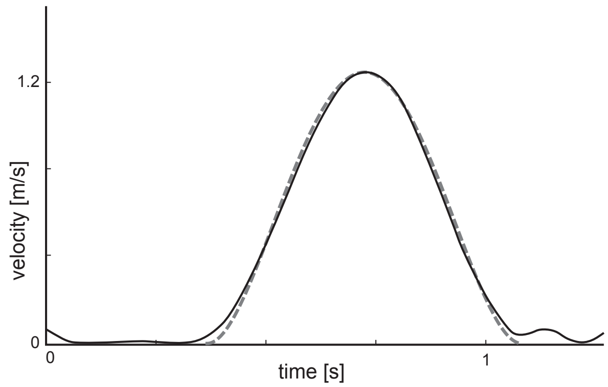
$$\dot{\mathbf{q}} = -\zeta \nabla_{\mathbf{n}} V \, \mathbf{n} = -\zeta \left(\frac{dV}{d\mathbf{q}} \cdot \mathbf{n} \right) \mathbf{n} \,, \quad \zeta > 0 \,, \quad \mathbf{n} \in N$$

- to avoid joint limits $\mathbf{q}^-$ and $\mathbf{q}^+$:

$$V(\mathbf{q}) = (\triangle\mathbf{q})^T \triangle\mathbf{q} \,, \quad \triangle\mathbf{q} = \mathbf{q} - \frac{\mathbf{q}^+ + \mathbf{q}^-}{2}$$

PLANNED TRAJECTORY





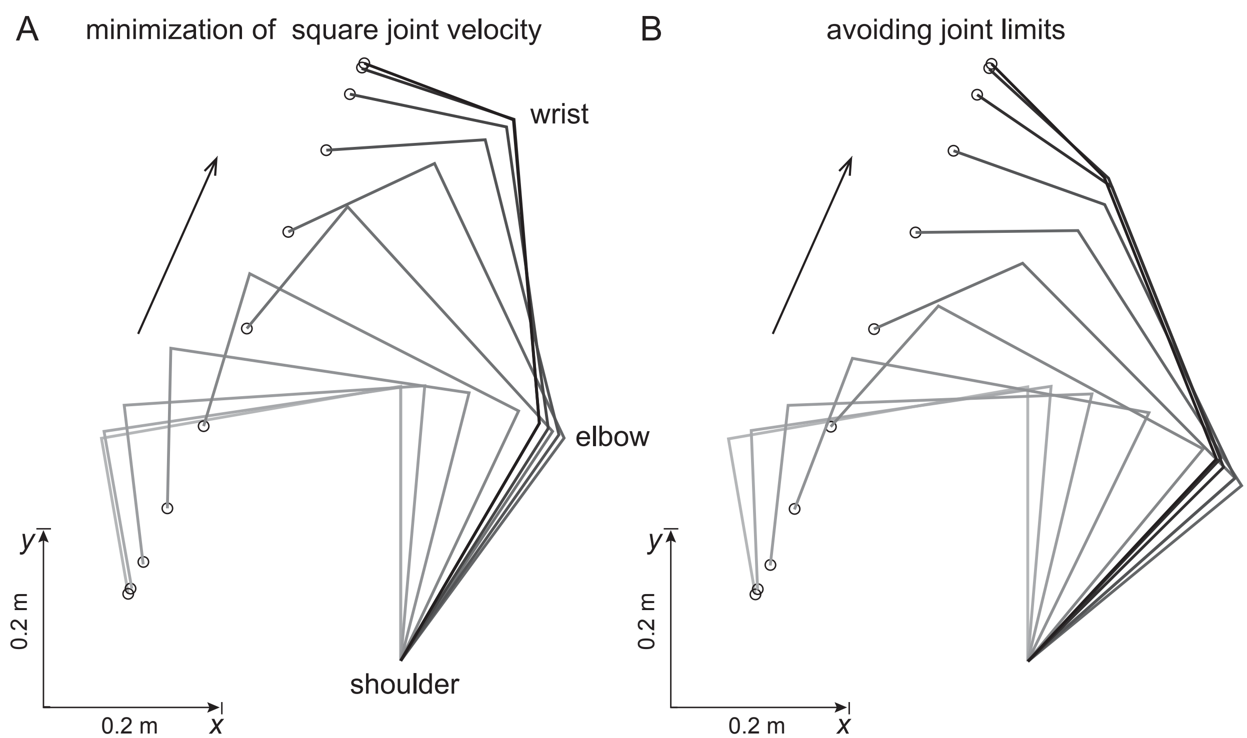
- horizontal reaching arm movements are generally straight with bell-shaped velocity profile

- modelled as planed straight line movement with smooth velocity profile

$$v(t) = (t_n^2 - 2\,t_n + 1)\frac{30At_n^2}{T}, \quad t_n = \frac{t}{T}$$

A: movement amplitude
t: the time, T the movement duration

TO REDUCE REDUNDANCY



Cruse proposed that the CNS *i*) follows roughly straight paths with the hand, *ii*) coordinates the movement in the redundant joints and *iii*) use “comfortable” postures, e.g. avoid joint limits

[Cruse et al. Journal of Motor Behavior 1993]

SUMMARY (1)

- analysis of complex multi-joint multi-muscle systems in motor behaviour is facilitated by mathematical tools such as infinitesimal kinematics

- three coordinate systems are involved in motor control, representing the task, the joint kinematics and the muscle geometry

- mechanical interaction with the environment is controlled through mechanical impedance, the resistance to imposed motion

- in a linear model impedance can be represented by three components: stiffness, viscosity and inertia, which depend on the body geometry.

SUMMARY (2)

- we have learned how to transform position, velocity, force, stiffness, damping and inertia between these spaces

- we have also learned what redundancy means for these motions and how optimization can be used to solve it

- in a static posture, stiffness increases linearly with the applied torque and the damping ratio is constant