Partial Differential Equations

Forward Equations

Beltrami's Equations (Generalized Laplacian Equations)

$$\frac{\partial}{\partial s^j}\left(\sqrt{g^s}g_s^{jk}\frac{\partial \xi^i}{\partial s^k}\right)=0, \quad i,j,k=1,...,n\,.$$

Diffusion Equations

$$\frac{\partial}{\partial s^j}\left(w(s)g_s^{jk}\frac{\partial \xi^i}{\partial s^k}\right)=0, \quad i,j,k=1,...,n\,.$$

The diffusion equations are equivalent to Beltrami's equations if

$$w(s)=\sqrt{g^s}\,,$$

where g^s is the determinant of the covariant

$$g^s=\det\left(g_{ij}^s\right)\,.$$

Inverted Equations

General forms using index notation

$$w(s)g_\xi^{km}\frac{\partial^2 s^i}{\partial \xi^k \partial \xi^m}=\frac{\partial}{\partial s^j}\left(w(s)g_s^{ji}\right), \quad i,j,k,m=1,...,n.$$

Writing out terms for $n=2$

$$w(s)\left(g_\xi^{11}\frac{\partial s^i}{\partial \xi^1\partial \xi^1}+2g_\xi^{12}\frac{\partial s^i}{\partial \xi^1\partial \xi^2}+g_\xi^{22}\frac{\partial s^i}{\partial \xi^2\partial \xi^2}\right)=\frac{\partial \xi^1}{\partial s^1}\frac{\partial}{\partial \xi^1}\left(wg_s^{i1}\right)+\frac{\partial \xi^2}{\partial s^1}\frac{\partial}{\partial \xi^2}\left(wg_s^{i1}\right)+\frac{\partial \xi^1}{\partial s^2}\frac{\partial}{\partial \xi^1}\left(wg_s^{i2}\right)+\frac{\partial \xi^2}{\partial s^2}\frac{\partial}{\partial \xi^2}\left(wg_s^{i2}\right)$$

Form multiplied by g^ξ and simplifying with $g_{ij}^\xi g_\xi^{jk}=\delta_i^k$, the LHS becomes

$$g^\xi LHS=w(s)\left(g_{22}^\xi\frac{\partial s^i}{\partial \xi^1\partial \xi^1}-2g_{12}^\xi\frac{\partial s^i}{\partial \xi^1\partial \xi^2}+g_{11}^\xi\frac{\partial s^i}{\partial \xi^2\partial \xi^2}\right)\,.$$

Using the identity $g^\xi=g^sJ^2$, where J is the Jacobian

$$J=det\left(\frac{\partial s^i}{\partial \xi^j}\right)\,,$$

the RHS becomes

$$\begin{aligned}g^\xi RHS&=g^sJ\left[J\frac{\partial \xi^1}{\partial s^1}\frac{\partial}{\partial \xi^1}\left(wg_s^{i1}\right)+J\frac{\partial \xi^2}{\partial s^1}\frac{\partial}{\partial \xi^2}\left(wg_s^{i1}\right)+J\frac{\partial \xi^1}{\partial s^2}\frac{\partial}{\partial \xi^1}\left(wg_s^{i2}\right)+J\frac{\partial \xi^2}{\partial s^2}\frac{\partial}{\partial \xi^2}\left(wg_s^{i2}\right)\right]\\&=g^sJ\left[\frac{\partial s^2}{\partial \xi^2}\frac{\partial}{\partial \xi^1}\left(wg_s^{i1}\right)-\frac{\partial s^2}{\partial \xi^1}\frac{\partial}{\partial \xi^2}\left(wg_s^{i1}\right)-\frac{\partial s^1}{\partial \xi^2}\frac{\partial}{\partial \xi^1}\left(wg_s^{i2}\right)+\frac{\partial s^1}{\partial \xi^1}\frac{\partial}{\partial \xi^2}\left(wg_s^{i2}\right)\right]\,.\end{aligned}$$

This form of the inverted diffusion equation becomes

$$g_{22}^\xi\frac{\partial s^i}{\partial \xi^1\partial \xi^1}-2g_{12}^\xi\frac{\partial s^i}{\partial \xi^1\partial \xi^2}+g_{11}^\xi\frac{\partial s^i}{\partial \xi^2\partial \xi^2}=\frac{g^sJ}{w}\left[\frac{\partial s^2}{\partial \xi^2}\frac{\partial}{\partial \xi^1}\left(wg_s^{i1}\right)-\frac{\partial s^2}{\partial \xi^1}\frac{\partial}{\partial \xi^2}\left(wg_s^{i1}\right)-\frac{\partial s^1}{\partial \xi^2}\frac{\partial}{\partial \xi^1}\left(wg_s^{i2}\right)+\frac{\partial s^1}{\partial \xi^1}\frac{\partial}{\partial \xi^2}\left(wg_s^{i2}\right)\right]\,.$$

Instead, multiplying the general form of the inverted diffusion equation by J^2 yields

$$
\begin{aligned}
J^2 LHS &= wJ^2 \left(g_\xi^{11} \frac{\partial s^i}{\partial \xi^1 \partial \xi^1} + 2 g_\xi^{12} \frac{\partial s^i}{\partial \xi^1 \partial \xi^2} + g_\xi^{22} \frac{\partial s^i}{\partial \xi^2 \partial \xi^2} \right) \\
&= wJ^2 \left(g_s^{jk} \frac{\partial \xi^1}{\partial s^j} \frac{\partial \xi^1}{\partial s^k} \frac{\partial s^i}{\partial \xi^1 \partial \xi^1} + 2 g_s^{jk} \frac{\partial \xi^1}{\partial s^j} \frac{\partial \xi^2}{\partial s^k} \frac{\partial s^i}{\partial \xi^1 \partial \xi^2} + g_s^{jk} \frac{\partial \xi^2}{\partial s^j} \frac{\partial \xi^2}{\partial s^k} \frac{\partial s^i}{\partial \xi^2 \partial \xi^2} \right) \\
&= w \left(L_{11} \frac{\partial s^i}{\partial \xi^1 \partial \xi^1} + 2 L_{12} \frac{\partial s^i}{\partial \xi^1 \partial \xi^2} + L_{22} \frac{\partial s^i}{\partial \xi^2 \partial \xi^2} \right) \\
J^2 RHS &= J \left[\frac{\partial s^2}{\partial \xi^2} \frac{\partial}{\partial \xi^1} \left(w g_s^{i1} \right) - \frac{\partial s^2}{\partial \xi^1} \frac{\partial}{\partial \xi^2} \left(w g_s^{i1} \right) - \frac{\partial s^1}{\partial \xi^2} \frac{\partial}{\partial \xi^1} \left(w g_s^{i2} \right) + \frac{\partial s^1}{\partial \xi^1} \frac{\partial}{\partial \xi^2} \left(w g_s^{i2} \right) \right] .
\end{aligned}
$$

Hence this form of the inverted diffusion equations become

$$
L_{11} \frac{\partial s^i}{\partial \xi^1 \partial \xi^1} + 2 L_{12} \frac{\partial s^i}{\partial \xi^1 \partial \xi^2} + L_{22} \frac{\partial s^i}{\partial \xi^2 \partial \xi^2} = \frac{J}{w} \left[\frac{\partial s^2}{\partial \xi^2} \frac{\partial}{\partial \xi^1} \left(w g_s^{i1} \right) - \frac{\partial s^2}{\partial \xi^1} \frac{\partial}{\partial \xi^2} \left(w g_s^{i1} \right) - \frac{\partial s^1}{\partial \xi^2} \frac{\partial}{\partial \xi^1} \left(w g_s^{i2} \right) + \frac{\partial s^1}{\partial \xi^1} \frac{\partial}{\partial \xi^2} \left(w g_s^{i2} \right) \right] ,
$$

where

$$
\begin{aligned}
L_{11} &= J^2 g_s^{jk} \frac{\partial \xi^1}{\partial s^j} \frac{\partial \xi^1}{\partial s^k} \\
&= J^2 \left(g_s^{11} \frac{\partial \xi^1}{\partial s^1} \frac{\partial \xi^1}{\partial s^1} + 2 g_s^{12} \frac{\partial \xi^1}{\partial s^1} \frac{\partial \xi^1}{\partial s^2} + g_s^{22} \frac{\partial \xi^1}{\partial s^2} \frac{\partial \xi^1}{\partial s^2} \right) \\
&= g_s^{11} \frac{\partial s^2}{\partial \xi^2} \frac{\partial s^2}{\partial \xi^2} - 2 g_s^{12} \frac{\partial s^2}{\partial \xi^2} \frac{\partial s^2}{\partial \xi^1} + g_s^{22} \frac{\partial s^2}{\partial \xi^1} \frac{\partial s^2}{\partial \xi^1} \\
L_{12} &= J^2 g_s^{jk} \frac{\partial \xi^1}{\partial s^j} \frac{\partial \xi^2}{\partial s^k} \\
&= J^2 \left(g_s^{11} \frac{\partial \xi^1}{\partial s^1} \frac{\partial \xi^2}{\partial s^1} + g_s^{12} \frac{\partial \xi^1}{\partial s^1} \frac{\partial \xi^2}{\partial s^2} + g_s^{21} \frac{\partial \xi^1}{\partial s^2} \frac{\partial \xi^2}{\partial s^1} + g_s^{22} \frac{\partial \xi^1}{\partial s^2} \frac{\partial \xi^2}{\partial s^2} \right) \\
&= -g_s^{11} \frac{\partial s^2}{\partial \xi^2} \frac{\partial s^2}{\partial \xi^1} + g_s^{12} \frac{\partial s^2}{\partial \xi^2} \frac{\partial s^1}{\partial \xi^1} + g_s^{21} \frac{\partial s^1}{\partial \xi^2} \frac{\partial s^2}{\partial \xi^1} - g_s^{22} \frac{\partial s^1}{\partial \xi^2} \frac{\partial s^1}{\partial \xi^1} \\
L_{22} &= J^2 g_s^{jk} \frac{\partial \xi^2}{\partial s^j} \frac{\partial \xi^2}{\partial s^k} \\
&= J^2 \left(g_s^{11} \frac{\partial \xi^2}{\partial s^1} \frac{\partial \xi^2}{\partial s^1} + 2 g_s^{12} \frac{\partial \xi^2}{\partial s^1} \frac{\partial \xi^2}{\partial s^2} + g_s^{22} \frac{\partial \xi^2}{\partial s^2} \frac{\partial \xi^2}{\partial s^2} \right) \\
&= g_s^{11} \frac{\partial s^2}{\partial \xi^1} \frac{\partial s^2}{\partial \xi^1} - 2 g_s^{12} \frac{\partial s^2}{\partial \xi^1} \frac{\partial s^1}{\partial \xi^1} + g_s^{22} \frac{\partial s^1}{\partial \xi^1} \frac{\partial s^1}{\partial \xi^1} .
\end{aligned}
$$