Quantum Mechanical Bound States of the Yukawa Potential (or some better title)

A Thesis
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People. Things. Shep, the dog who is currently keeping me company.

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Abstract

Math and computers and stuff gave me results!

Introduction

The Yukawa potential governs the force between protons and neutrons in the nucleus of the atom. It is

$$V(r) = -\frac{C}{r}e^{-r/l}, \tag{1}$$

where C and l are constants. C sets the strength of the force; l acts as a length scale. The exponential term provides an effective cutoff once r gets much larger than l, as the exponential term drops off quite rapidly. The simplest system in which the potential can be studied is the deuterium nucleus. The system of a single neutron and proton can be expressed as a reduced mass orbiting the center of mass.

The force comes from the exchange of virtual pions between the two nucleons. The limited range is due to the fact that pions have mass. The mass of the pion then generates the length scale:

$$l = \frac{\hbar}{m_\pi c} \approx 1.41 \times 10^{-15} \text{ m} \tag{2}$$

The strength of the force C is only known experimentally; it has been found to be about 2.96×10^{-6} eV*m(1)[1].

In general, the time-independent Schrödinger wave equation for some $V(r)$ is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\phi,\theta) + V(r)\psi(r,\phi,\theta) = E\psi(r,\phi,\theta). \tag{3}$$

Because we are working with nuclei in atoms, it makes the most sense to express this in spherical coordinates, as above. With the Yukawa potential, this equation cannot be solved analytically. However, we can solve the more simple system described by the potential without the exponential term, that is,

$$V(r) = -\frac{C}{r}. \tag{4}$$

Plugging this into the wave equation, we have

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\phi,\theta) - \frac{C}{r}\psi(r,\phi,\theta) = E\psi(r,\phi,\theta). \tag{5}$$

[1] Get a better citation for this than Eisberg and Resnick

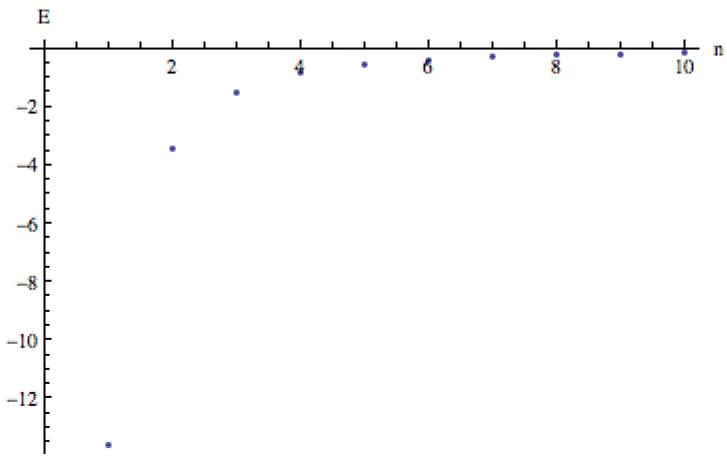

Figure 1: The energies of the first 10 bound states of the electron in the hydrogen atom

This equation can be solved analytically by separation of variables.[2] While the final wave function will be dependent on all three variables, the energy is just a function of n (the radial quantum number), and is equal to

$$E_n = -\frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2}. \tag{6}$$

This can be rewritten in terms of

$$a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}, \tag{7}$$

the "Bohr radius". A constant with units of length, a_0 is defined in terms of the other constants in the problem. It is also equal to the radius of the smallest orbit in a Bohr hydrogen atom (the ground state). As such, it sets a natural length scale. Substituting a_0 in, the expression for E_n becomes much simpler:

$$E_n = -\frac{\hbar^2}{a_0^2 \mu n^2}. \tag{8}$$

These energies are the energies of the bound states of the electron in the hydrogen atom. As n increases, E_n gets closer and closer to zero, but never quite reaches it – there are an infinite number of bound states possible.

The ground state wave function is spherically symmetric, meaning that ψ there is only a function of r. This function is plotted in Figure 2.

This means the Yukawa potential's range is limited in a way the Coulomb potential's isn't. We expect that this will limit the number of bound states as well, to some number whose rs are less than l.

[2]How much of this should I write out? After writing below this, probably more than this.

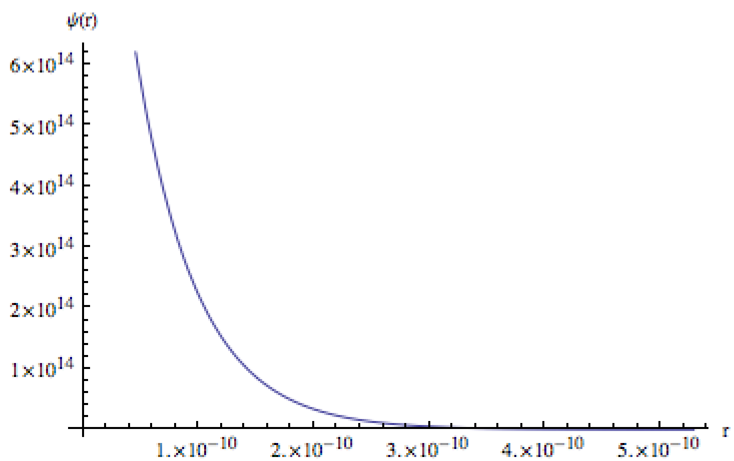

Figure 2: The ground state ($n = 1$, $l = 0$, $m = 0$) wave function for hydrogen