# Critical Points

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We start with the nondimensionalized effective potential,

$$V_{eff} = \frac{1}{\rho^2}\left(n^2 - 2\rho e^{-\rho/\lambda}\right) \tag{1}$$

and attempt to find the points at which the minimum of the effective potential is 0 and where the minimum of the effective potential no longer exists.

For the first, we start by taking the derivative of (1) and setting it to 0, obtaining

$$\frac{dV}{d\rho_0} = -\frac{2n^2}{\rho_0^3} + \frac{2}{\rho_0^2}e^{-\rho_0/\lambda} + \frac{2}{\rho_0\lambda}e^{-\rho_0/\lambda} \tag{2}$$

$$0 = -2n^2 + 2\rho_0 e^{-\rho_0/\lambda} + 2\frac{\rho_0^2}{\lambda}e^{-\rho_0/\lambda} \tag{3}$$

where $\rho_0$ is the value of $\rho$ when $V_{eff}$ is minimized. This equation has no general solution, but we can find one for the specific case where $V_{eff}(\rho_0) = 0$ by solving the equation

$$V_{eff}(\rho_0) = 0 = n^2 - 2\rho_0 e^{-\rho_0/\lambda} \tag{4}$$

$$n^2 = 2\rho_0 e^{-\rho_0/\lambda} \tag{5}$$

and plugging it in to the previous, obtaining

$$0 = -2n^2 + n^2 + n^2\frac{\rho_0}{\lambda} \tag{6}$$

$$= -1 + \frac{\rho_0}{\lambda} \tag{7}$$

$$\rho_0 = \lambda. \tag{8}$$

Plugging this back into the equation for $V_{eff}(\rho_0)$ gives us

$$n = \sqrt{\tfrac{2}{e}}\sqrt{\lambda} \tag{9}$$

which is pretty close to the naive $n = \sqrt{\lambda}$.

For the second critical point, the one where $V_{eff}$ has no minimum, we rewrite (3) to be

$$n^2 = \left(\rho + \frac{\rho^2}{\lambda}\right) e^{-\rho/\lambda} \tag{10}$$

The right side of this equation will have some maximum, which will be the last point for which the equation can be solved – after that, the value of $n^2$ will just plain be bigger than the right side. To find this, we differentiate with respect to $\rho$ and set to 0, obtaining

$$\begin{aligned} 0 &= (1 + 2\frac{\rho}{\lambda})e^{-\rho/\lambda} - \frac{1}{\lambda}(\rho + \frac{\rho^2}{\lambda})e^{-\rho/\lambda} && (11) \\ &= 1 + \frac{\rho}{\lambda} + \frac{\rho^2}{\lambda^2}. && (12) \end{aligned}$$

This is a quadratic in $\rho$ and can be solved using the quadratic formula:

$$\begin{aligned} \rho &= \frac{-\frac{1}{\lambda} \pm \sqrt{\left(\frac{1}{\lambda}\right)^2 + 4\frac{1}{\lambda^2}}}{-\frac{2}{\lambda^2}} && (13) \\ &= \frac{\lambda}{2} \mp \frac{\lambda\sqrt{5}}{2} && (14) \end{aligned}$$

As $\rho$ is a real physical quantity that can't be negative, we can discard one solution, obtaining

$$\rho = \frac{\lambda}{2}(1 + \sqrt{5}) \tag{15}$$

We can plug this in to (3) to find the relationship between $\lambda$ and $n$:

$$n^2 = \lambda(1 + \sqrt{5}) \left(\frac{3 + \sqrt{5}}{4}\right) e^{-(1+\sqrt{5})/2} \tag{16}$$