Quantum Mechanical Bound States of the Yukawa Potential (or some better title)

A Thesis
Presented to
The Division of Mathematics and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

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May 2012

Approved for the Division
(Physics)

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Acknowledgements

People. Things. Shep, the dog who is currently keeping me company.

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Abstract

Math and computers and stuff gave me results!

Introduction

The Yukawa potential describes a force mediated by a massive particle. It is

$$V(r) = -\frac{C}{r}e^{-r/l}, \tag{1}$$

where C and l are constants. C sets the strength of the force; l acts as a length scale. The exponential term provides an effective cutoff once r gets much larger than l, as the exponential term drops off quite rapidly. We are interested in how the bound states change with these constants.

The force described by the potential comes from the exchange of virtual particles. The lenth scale, which limits the range at which the force acts, comes about because the virtual particles exchanged have mass. The mass of these particles then generates the length scale:

$$l = \frac{\hbar}{m_\pi c}. \tag{2}$$

The strength of the force C is only known experimentally and depends on the application of the potential. Most commonly, this is the forces between protons and neutrons in an atomic nucleus.

In general, the time-independent Schrödinger wave equation for some $V(r)$ is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\phi,\theta) + V(r)\psi(r,\phi,\theta) = E\psi(r,\phi,\theta). \tag{3}$$

In this thesis, we work with a system of two particles moving under the influence of the Yukawa potential (in the atomic case, this would be a deuterium nucleus). To simplify the system, it makes the most sense to express this in spherical coordinates as a reduced mass orbiting a center of mass. This is the form used for the wave equation above. With the Yukawa potential, this equation cannot be solved analytically. However, we can solve the more simple system described by the potential without the exponential term, that is,

$$V(r) = -\frac{C}{r}. \tag{4}$$

This potential is the same form as the Coulomb potential in the hydrogen atom. While $r << l$, the Yukawa potential should behave similarly to this, because the $-C/r$ term will dominate. Therefore, we will gain some insight on the Yukawa potential by solving the wave equation for this one. The Schrödinger wave equation with this simpler potential can be solved analytically by separation of variables. While the final

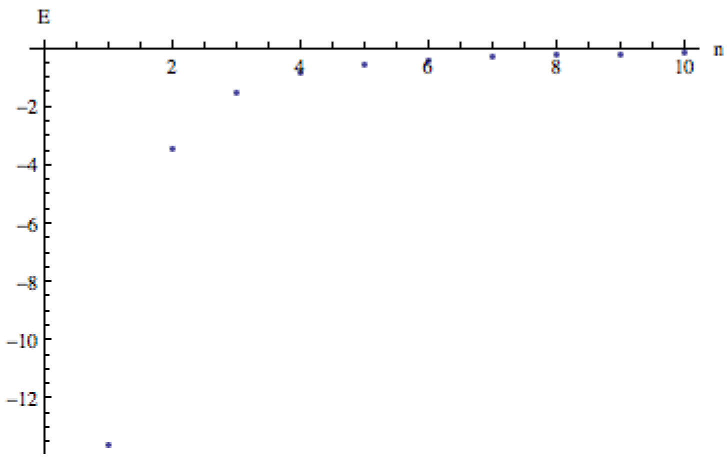

Figure 1: The energies of the first 10 bound states of the electron in the hydrogen atom

wave function will be dependent on all three variables, the energy is just a function of n (the radial quantum number), and is equal to

$$E_n = -\frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2}. \tag{5}$$

This can be rewritten in terms of

$$a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}, \tag{6}$$

the "Bohr radius". A constant with units of length, a_0 is defined in terms of the other constants in the problem. It is also equal to the radius of the smallest orbit in a Bohr hydrogen atom (the ground state). As such, it sets a natural length scale. Substituting a_0 in, the expression for E_n becomes much simpler:

$$E_n = -\frac{\hbar^2}{a_0^2 \mu n^2}. \tag{7}$$

These energies are the energies of the bound states of the electron in the hydrogen atom. As n increases, E_n gets closer and closer to zero, but never quite reaches it – there are an infinite number of bound states possible.

The ground state wave function is spherically symmetric, meaning that ψ there is only a function of r. This function is plotted in Figure 2.

This means the Yukawa potential's range is limited in a way the Coulomb potential's isn't. We expect that this will limit the number of bound states as well, to some number whose rs are less than l.

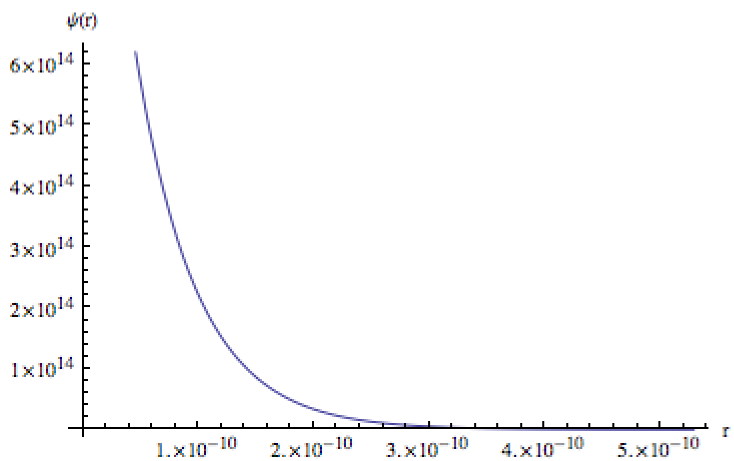

Figure 2: The ground state ($n = 1$, $l = 0$, $m = 0$) wave function for hydrogen