Non-dimensionalization

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The energy of a particle moving under the influence of the Yukawa potential is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} - \frac{c}{r}e^{-r/l}. \tag{1}$$

To non-dimensionalize this expression for further treatment this year, we first define a length scale

$$\alpha \equiv \frac{\hbar^2}{cm}. \tag{2}$$

This serves the same function in this problem as the Bohr radius does in the Coulomb problem. We then use this to nondimensionalize r, defining

$$\rho \equiv \frac{r}{\alpha}. \tag{3}$$

Finally, we define a time scale

$$\beta \equiv \frac{c^2 m}{\hbar^3}. \tag{4}$$

We can now substitute in ρ, α, and $\tau = \beta t$:

$$E = \frac{1}{2}m\left(\frac{d\rho}{d\tau}\alpha\beta\right)^2 + \frac{1}{2}\frac{c}{\rho^2\alpha}(n^2 - \rho e^{-(\rho\alpha)/l}) \tag{5}$$

$$= \frac{1}{2}\frac{c}{\alpha}\rho'^2 + \frac{1}{2}\frac{c}{\alpha}\frac{1}{\rho^2}(n^2 - \rho e^{-(\rho\alpha)/l}) \tag{6}$$

$$\frac{2\alpha}{c}E = e = \rho'^2 + \frac{1}{\rho^2}(n^2 - \rho e^{-\rho/\lambda}) \tag{7}$$

In the last line, we have used $\lambda = \frac{l}{\alpha}$, relating the strength of the force and the length scale of the problem. We can now use

$$V_{eff} = \frac{1}{\rho^2}(n^2 - \rho e^{-\rho/\lambda}) \tag{8}$$

as a dimensionless effective potential for further analysis.