Quantum Mechanical Bound States of the Yukawa Potential (or some better title)

A Thesis
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Bachelor of Arts

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People. Things. Shep, the dog who is currently keeping me company.

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Abstract

Math and computers and stuff gave me results!

Introduction

0.1 The hydrogen atom

Before we talk about the Yukawa potential, let's look at a simpler system: the Coulomb potential in the hydrogen atom. Here, a single negatively charged electron is bound to the positively charged nucleus, consisting of a single proton. The time-independent wave equation for that electron is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r,\phi,\theta) - \frac{e^2}{4\pi\epsilon_0 r}\psi(r,\phi,\theta) = E\psi(r,\phi,\theta). \tag{1}$$

This equation can be solved analytically by separation of variables.[1] While the final wave function will be dependent on all three variables, the energy is just a function of n (the radial quantum number), and is equal to

$$E_n = -\frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2}. \tag{2}$$

This can be rewritten in terms of

$$a_0 \equiv \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}, \tag{3}$$

the "Bohr radius". A constant with units of length, a_0 is defined in terms of the other constants in the problem. It is also equal to the radius of the smallest orbit in a Bohr hydrogen atom (the ground state). As such, it sets a natural length scale. Substituting a_0 in, the expression for E_n becomes much simpler:

$$E_n = -\frac{\hbar^2}{a_0^2\mu n^2}. \tag{4}$$

These energies are the energies of the bound states of the electron in the hydrogen atom. As n increases, E_n gets closer and closer to zero, but never quite reaches it – there are an infinite number of bound states possible.

The ground state wave function is spherically symmetric, meaning that ψ there is only a function of r. This function is plotted in Figure 2.

[1] How much of this should I write out? After writing below this, probably more than this.

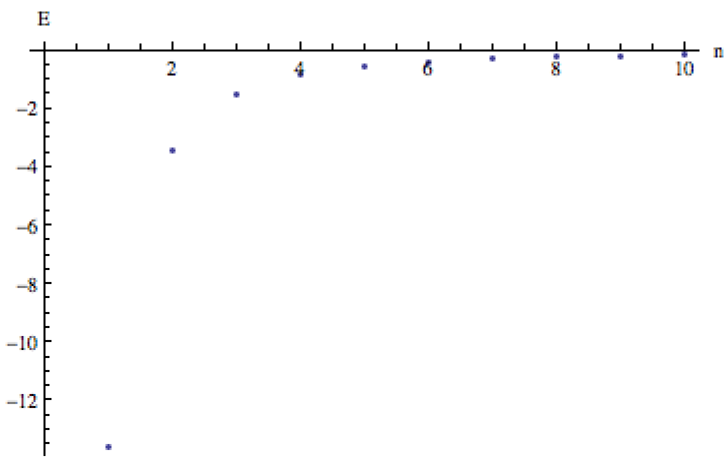

Figure 1: The energies of the first 10 bound states of the electron in the hydrogen atom

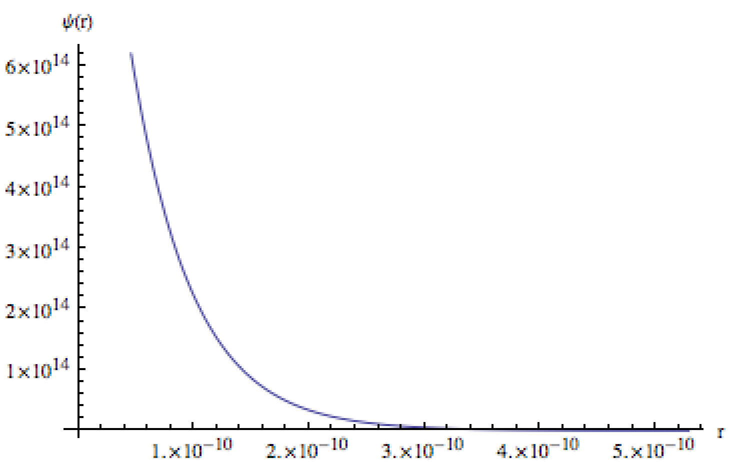

Figure 2: The ground state ($n = 1$, $l = 0$, $m = 0$) wave function for hydrogen

0.2 Comparison with the Yukawa potential

The Yukawa potential governs the force between protons and neutrons in the nucleus of the atom. It is

$$V(r) = -\frac{C}{r}e^{-r/l}. \tag{5}$$

When considering this potential, the analogous system to hydrogen is the deuterium atom. One proton and one neutron means you can view the system as a reduced mass orbiting the center of mass. The exponential term provides an effective cutoff once r gets much larger than l, as the exponential term drops off quite rapidly. This means the Yukawa potential's range is limited in a way the Coulomb potential's isn't. We expect that this will limit the number of bound states as well, to some number whose rs are less than l.

The force comes from the exchange of virtual pions between the two nucleons. The limited range is due to the fact that pions have mass; the Coulomb force is mediated by massless virtual photons. The mass of the pion then generates the length scale:

$$l = \frac{\hbar}{m_\pi c} \approx 1.41 \times 10^{-15} \text{ m} \tag{6}$$

The strength of the force C is only known experimentally; it has been found to be about 2.96×10^{-6} eV*m[1][2]. These values give us an idea of how the force will work in an actual atom, and provide something we can plug in to test our model.

[2]Get a better citation for this than Eisberg and Resnick

References

[1] Robert Eisberg and Robert Resnick. *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles*. Wiley, 1985.