Software Development (cs2500)

Lecture 27: Recursion

M. R. C. van Dongen

November 25, 2013

Outline

- We study recursion:
  - Methods that call themselves;
  - Definitions that are defined in terms of themselves.
- We start with some easy/recreative applications:
  - We study a recursive method for computing factorials.
  - We study the recursive breeding habits of rabits.
  - We study the famous towers of hanoi.
- We end with some practical real-world applications:
  - Binary search;
  - Quicksort.

Recursion: Merriam Says

Function: noun

Etymology: Late Latin recursion-, recursio, from recurrere

Date: 1616

1. return
2. the determination of a succession of elements (as numbers or functions) by operation on one or more preceding elements according to a rule or formula involving a finite number of steps
3. a computer programming technique involving the use of a procedure, subroutine, function, or algorithm that calls itself one or more times until a specified condition is met at which time the rest of each repetition is processed from the last one called to the first

Recursion

- Many concepts in computer science and mathematics are defined or computed *recursively*, i.e. using *recursion*.
- The idea is to define a complicated concept in terms of itself.

Recursion: Base Case

Base Case: Simple computation.

- We don't have to call the method itself.

Recursive Computation: Complicated computation involving:

- Simple computations.
- Lower order computation(s).

Recursion: Recursive Computation

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Recursive Computation: Complicated computation involving:

- Simple computations.
- Lower order computation(s).

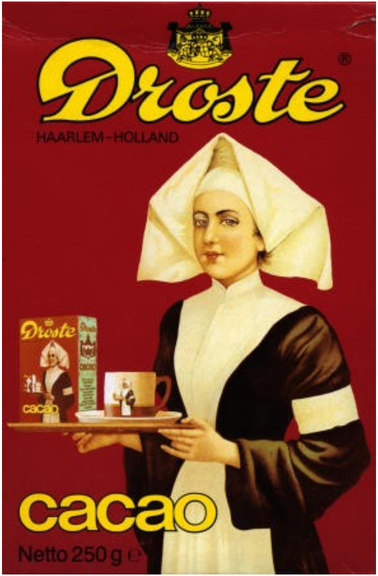
Recursive Algorithm: Dictionary Search

Dictionary Contains all Possible Words: One Word per Page

To *search* for the word given n pages do the following:

- If there's only one page ($n = 1$): We've found the word.
- Otherwise ($n > 1$):
  - Find the page in the "middle."
  - Read the word on the middle page.
  - If that word is our word: We've found the word.
  - If our word is smaller: *search* to the left.
  - Otherwise: *search* to the right.

Recursive Picture



Recursive Picture

Pitfalls

- Recursive computations involve themselves.
- If we're not careful we may get an infinite chain of computations.
- For example, we may be
  - Computing what's on Box 1 with Box 2 on it, which involves
  - Computing what's on Box 2 with Box 3 on it, which involves
  - Computing what's on Box 3 with Box 4 on it, which involves
  - ....
- Each recursive computation should eventually terminate.
- This only happens if they all reach some base case condition.
  - (The base conditions may be different.)

Controlling the Size

Guaranteeing Termination

- Each computation should have a *size:* a non-negative integer.
- The size should depend on one or several method parameters.
- Base-case computations have small fixed sizes.
- Recursive sub-computations should get smaller and smaller.
- Using an induction argument this guarantees termination.

How does this Work?

Dictionary Search Revisited

- Let's call the top computation C_0.
- Let C_1 be the recursive computation of C_0,
- Let C_2 be the recursive computation of C_1, and so on.
- Finally, let S_i be the size of C_i.
  - By nature of the algorithm we have $S_i > S_{i+1}$.
- Let's assume an infinite chain of computations C_0, C_1, C_2, ...
- Then we have an infinite chain of *integers* $S_0 > S_1 > S_2 > \cdots$.
- But this is impossible since $S_i \geq 0$, for all i.

Computing Factorials

- Let n be a positive integer.
- The *factorial* of n, denoted $n!$, is defined as follows:

$$n! = 1 \times 2 \times \cdots \times (n-1) \times n\,.$$

- Using the product notation we may write this as follows:

$$n! = \prod_{i=1}^{n} i\,.$$

Computing Factorials: An Iterative Solution

Java

```
public static int factorial( int n ) {
    int product = 1;
    for (int i = 1; i != n; i ++) {
        product = product * i;
    }
    return product;
}
```

Computing Factorials: A Recursive Solution

Base Case: Clearly $1! = 1$.

Recursion: The recursion may be found by noticing that

$$\prod_{i=1}^{n} i = n \times \prod_{i=1}^{n-1} i\,.$$

This gives us

$$n! = (n-1)! \times n\,.$$

Combining the Base Case and Recursive Case

$$n! = \begin{cases} 1 & \text{if } n = 1\,; \\ (n-1)! \times n & \text{if } n > 1\,. \end{cases}$$

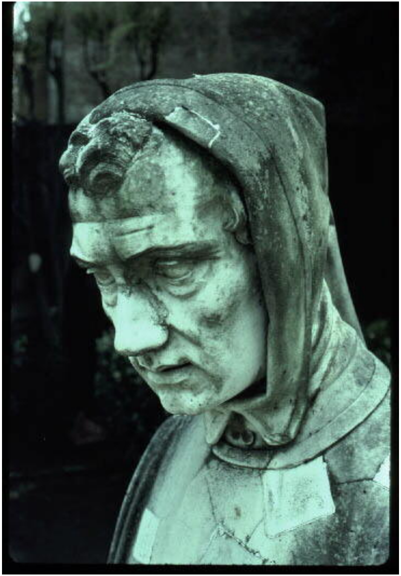
Java

```java
public static int factorial( int n ) {
    final int result;

    if (n == 1) {
        result = 1;                          // Base Case
    } else {
        result = factorial( n - 1 ) * n; // Recursion
    }

    return result;
}
```

Simulating a Computation

```
factorial( 4 ) = ( factorial( 3 ) * 4 ) )
               = ( ( factorial( 2 ) * 3 ) * 4 )
               = ( ( ( factorial( 1 ) * 2 ) * 3 ) * 4 )
               = ( ( ( 1 * 2 ) * 3 ) * 4 )
               = ( ( 2 * 3 ) * 4 )
               = ( 6 * 4 )
               = 24.
```

Yer Man



Facts about Fibonacci

- Born: about 1175 AD.
- Died: 1250 AD.
- Famous mathematician.
- Introduced the Decimal System into Europe.
- Well known for many of his problems.

Rabbits

A pair of rabbits are put in a field and, if rabbits take a month to become mature and then produce a new pair every month after that, how many pairs will there be in twelve months time?

Rabbits do not Escape and Don't Die

A pair of rabbits are put in a field and, if rabbits take a month to become mature and then produce a new pair every month after that, how many pairs will there be in twelve months time?

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1			

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0		

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2			

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1		

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	

Fibonacci's Solution


Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3			

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1		

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
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0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	

Fibonacci's Solution

Month (n)	Pairs of Rabbits Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4			

Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2		

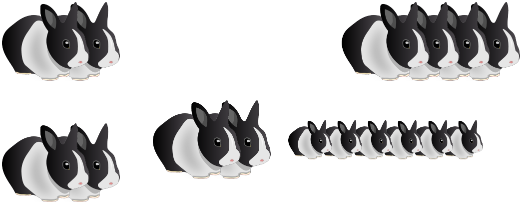
Fibonacci's Solution

Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	

Fibonacci's Solution

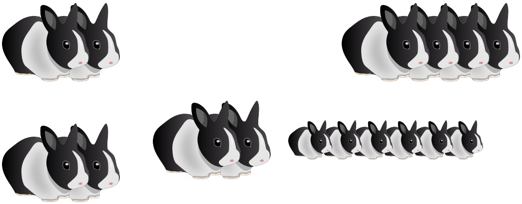
Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5

Fibonacci's Solution



Month (n)	Pairs of Rabbits: Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5
5			

Fibonacci's Solution



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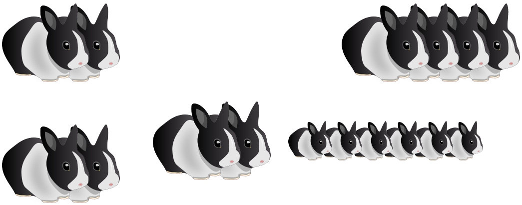
4	2	3	5
5	3		

Fibonacci's Solution



Month (n)	Pairs of Rabbits		
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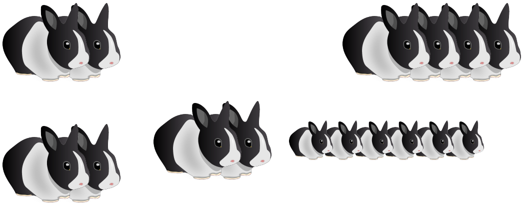
4	2	3	5
5	3	5	

Fibonacci's Solution



Month (n)	Pairs of Rabbits		
	Babies	Mature	Total (F_n)
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5
5	3	5	8

The Fibonacci Sequence

- Fibonacci's solution involves the series of numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

- Given the first two we can compute the remaining numbers:

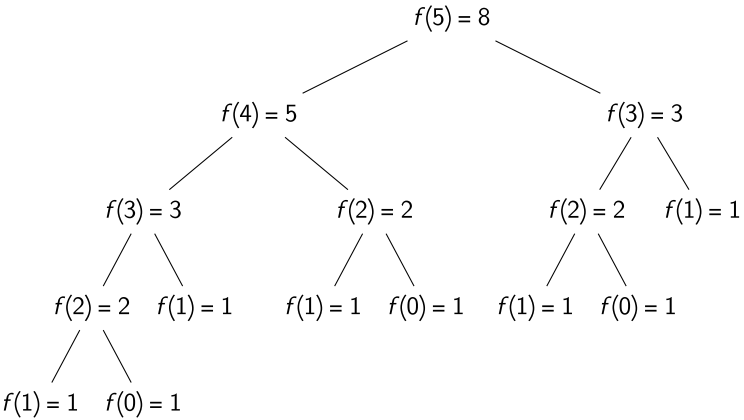
$$F_n = \begin{cases} 1 & \text{if } n = 0\,; \\ 1 & \text{if } n = 1\,; \\ F_{n-1} + F_{n-2} & \text{if } n > 1\,. \end{cases}$$

Back to Java

Java

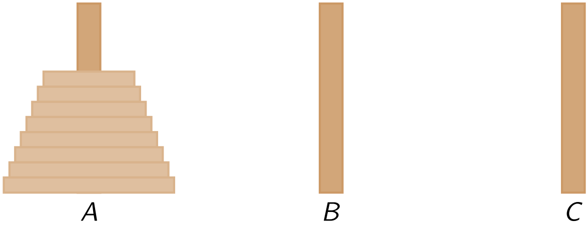
```
public static int fibonacci( int n ) {
    final int result;

    if (n <= 1) { /* Base Case */
        result = 1;
    } else {      /* Recursion */
        result = fibonacci( n - 1 ) + fibonacci( n - 2 );
    }

    return result;
}
```

Tracing the Calls

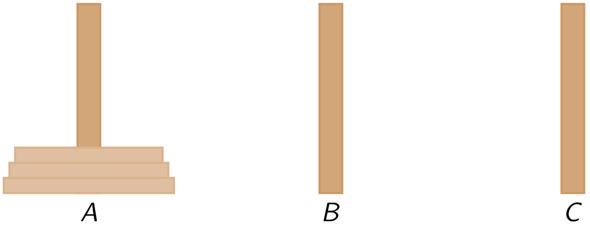


The Towers of Hanoi

- We're given a tower of 8 disks and three pegs: A, B, and C.
- Each disk has a hole in the centre.
- Initially, the disks are stacked in decreasing size on Peg A.
- The objective is to transfer the stack to a different peg, but
  - We're only allowed to stack disks on pegs,
  - We're only allowed to move one disk at a time, and
  - We can only stack a smaller disk on top of a larger disk.



Simulation



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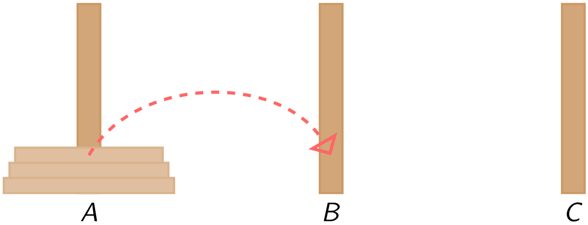

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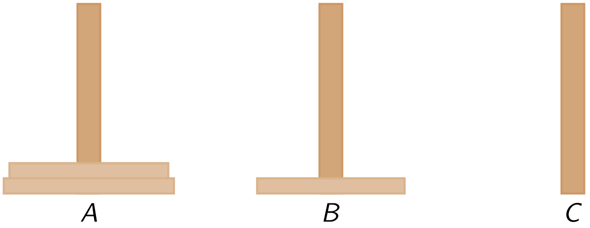

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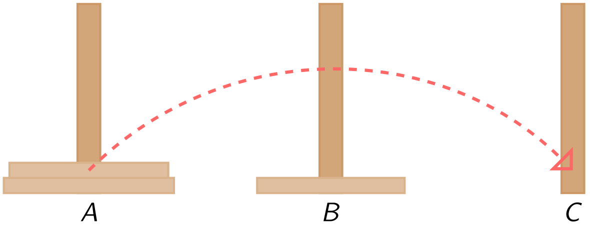

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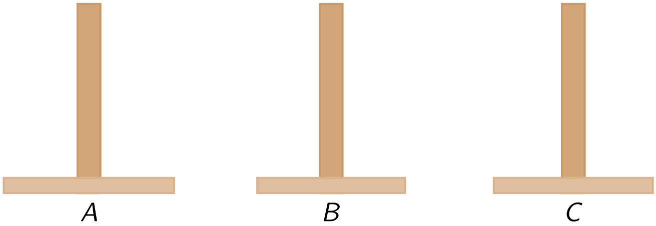

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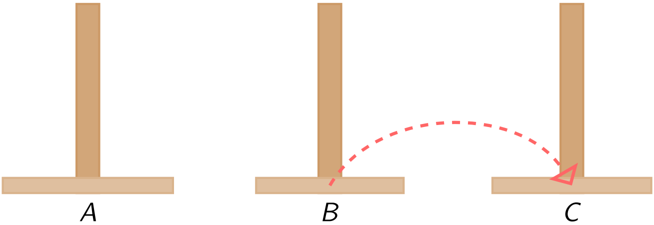

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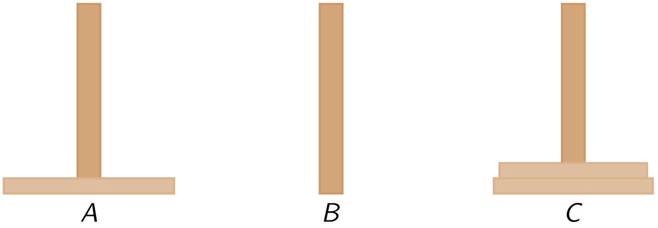

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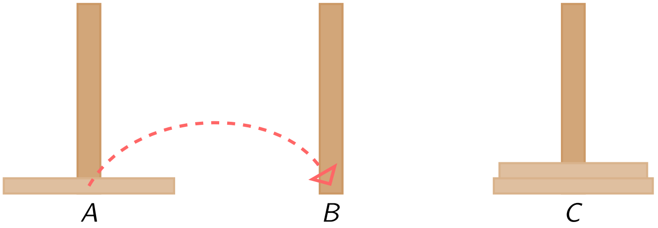

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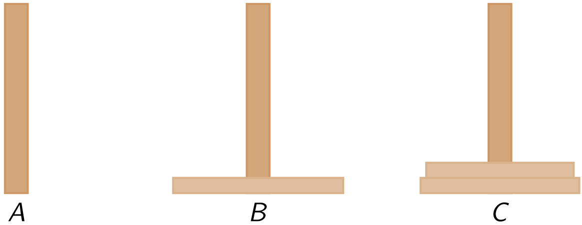

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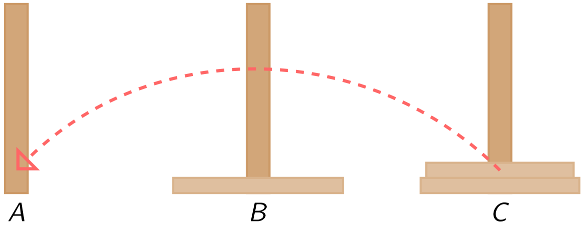

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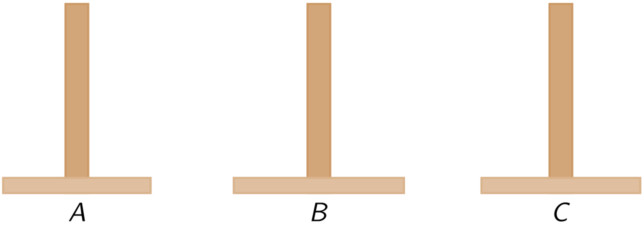

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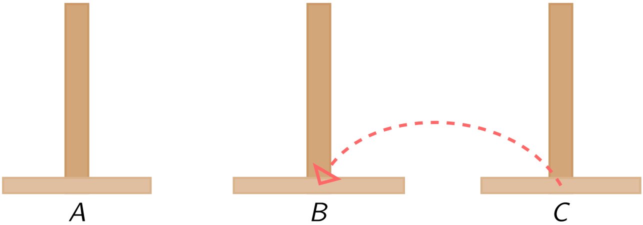

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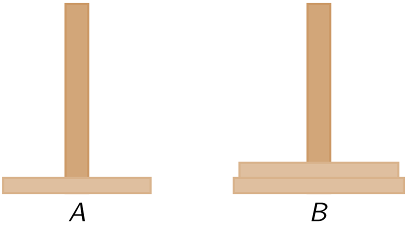

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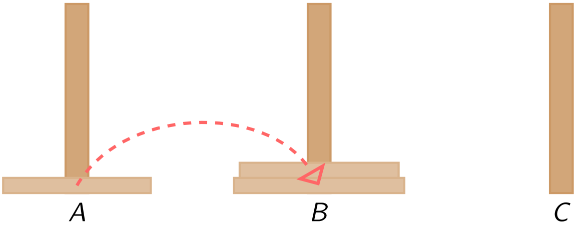

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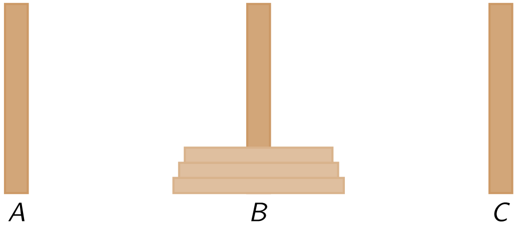
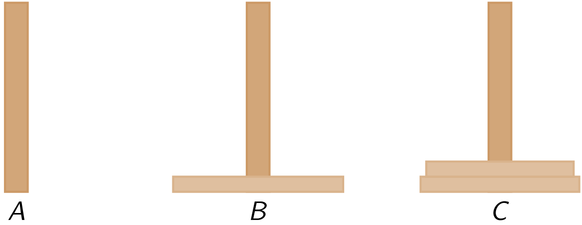


Intermediate State of the 3-Disk Version



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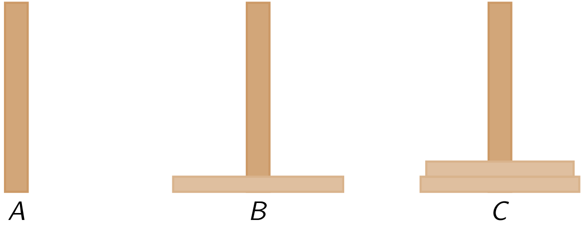
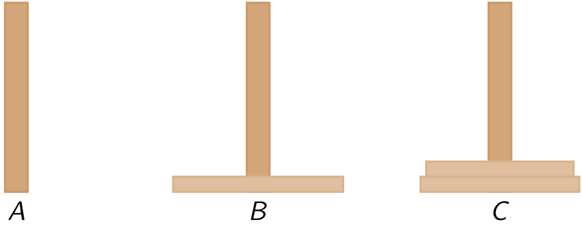


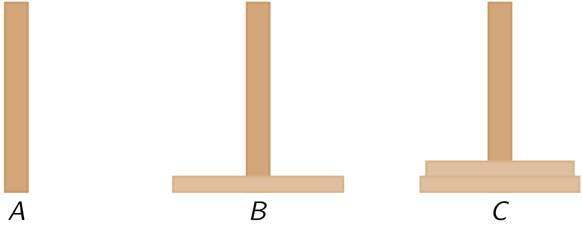
- Here we *recursively* moved disks from C to B and were done!

Intermediate State of the 3-Disk Version



- Here we *recursively* moved disks from C to B and were done!
- So, how did we arrive at the intermediate state?
- If we can solve this subproblem, we can solve the whole problem:

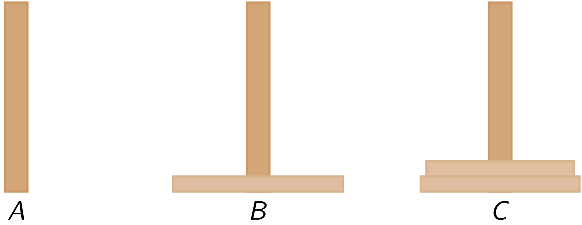
Intermediate State of the 3-Disk Version



- Here we *recursively* moved disks from C to B and were done!
- So, how did we arrive at the intermediate state?
- If we can solve this subproblem, we can solve the whole problem:

  1. Start at initial state.

Intermediate State of the 3-Disk Version



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- If we can solve this subproblem, we can solve the whole problem:

  1. Start at initial state.
  2. Solve the sub-problem to arrive at the intermediate state.

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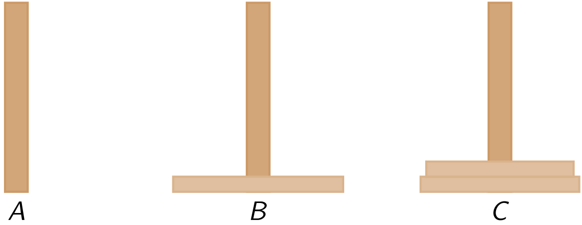

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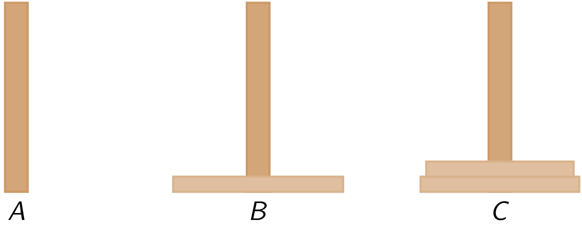
- So, how did we arrive at the intermediate state?
- If we can solve this subproblem, we can solve the whole problem:

1. Start at initial state.
2. Solve the sub-problem to arrive at the intermediate state.
3. Use recursion to go from the intermediate to the target state.

Intermediate State of the 3-Disk Version



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  1. Start at initial state.
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- So, how did we get at the intermediate state?

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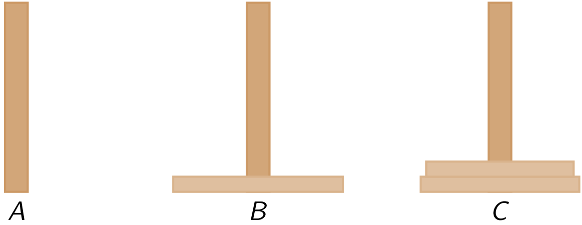

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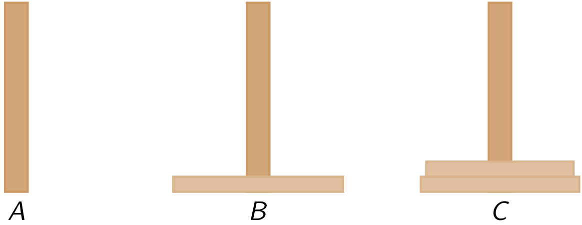
  1. Start at initial state.
  2. Solve the sub-problem to arrive at the intermediate state.
  3. Use recursion to go from the intermediate to the target state.
- So, how did we get at the intermediate state?
  1. We started with all disks stacked on Peg A.
  2. We moved all disks except for the largest one from A to C.

Intermediate State of the 3-Disk Version



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  1. We started with all disks stacked on Peg A.
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  3. We moved the largest disk to Peg B.

Intermediate State of the 3-Disk Version

- Here we *recursively* moved disks from C to B and were done!
- So, how did we arrive at the intermediate state?
- If we can solve this subproblem, we can solve the whole problem:

  1. Start at initial state.
  2. Solve the sub-problem to arrive at the intermediate state.
  3. Use recursion to go from the intermediate to the target state.
- So, how did we get at the intermediate state?
  1. We started with all disks stacked on Peg A.
  2. We moved all disks except for the largest one from A to C.
     - But this is just the 2-disk version: move 2 disks from A to C.
  3. We moved the largest disk to Peg B.

Designing the Algorithm

Base case: If $n = 1$:

1. Move disk n to target peg.

Recursion: If $n > 1$:

1. Move disks $1, \ldots, n - 1$ to intermediate peg.
2. Move disk n to target peg.
3. Move disks $1, \ldots, n - 1$ to target peg.

Designing the Algorithm

Base case: If $n = 1$:

1. Move disk n to target peg.

Recursion: If $n > 1$:

1. Move disks $1, \ldots, n - 1$ to intermediate peg.
2. Move disk n to target peg.
3. Move disks $1, \ldots, n - 1$ to target peg.

Alternative solution

- If $n \geq 1$ then
  1. Move disks 1, ..., $n - 1$ from source to intermediate peg.
  2. Move disk n to target disk.
  3. Move disks 1, ..., $n - 1$ from intermediate to target peg.

Back to Java

Java

```
/**
 * @param n Number of disks.
 * @param source The source peg: should be 0, 1, or 2.
 * @param target The target peg: should be 0, 1, or 2.
 * <PAR> {@code source} and {@code target} should be different.</PAR>
 */
private static
void hanoi( int n, int source, int target ) {
    if (n >= 1) {
        // Compute the number of the intermediate peg:
        final int intermediate = 3 - source - target;
        hanoi( n - 1, source, intermediate );
        moveDisk( n, source, target );
        hanoi( n - 1, intermediate, target );
    }
}

public static
void hanoi( int n ) {
    // move n disks from Peg 0 to Peg 1.
    hanoi( n, 0, 1 );
}
```

Back to Java

Java

```
/**
 * @param n Number of disks.
 * @param source The source peg: should be 0, 1, or 2.
 * @param target The target peg: should be 0, 1, or 2.
 * <PAR> {@code source} and {@code target} should be different.</PAR>
 */
private static
void hanoi( int n, int source, int target ) {
    if (n >= 1) {
        // Compute the number of the intermediate peg:
        final int intermediate = 3 - source - target;
        hanoi( n - 1, source, intermediate );
        moveDisk( n, source, target );
        hanoi( n - 1, intermediate, target );
    }
}

public static
void hanoi( int n ) {
    // move n disks from Peg 0 to Peg 1.
    hanoi( n, 0, 1 );
}
```

Back to Java

Java

```
/**
 * @param n Number of disks.
 * @param source The source peg: should be 0, 1, or 2.
 * @param target The target peg: should be 0, 1, or 2.
 * <PAR> {@code source} and {@code target} should be different.</PAR>
 */
private static
void hanoi( int n, int source, int target ) {
    if (n >= 1) {
        // Compute the number of the intermediate peg:
        final int intermediate = 3 - source - target;
        hanoi( n - 1, source, intermediate );
        moveDisk( n, source, target );
        hanoi( n - 1, intermediate, target );
    }
}

public static
void hanoi( int n ) {
    // move n disks from Peg 0 to Peg 1.
    hanoi( n, 0, 1 );
}
```

Back to Java

Java

```
private static void moveDisk( int disk, int source, int target ) {
    final String pegNames[] = { "A", "B", "C" };
    System.out.println( "Move disk " + disk
                        + " from " + pegNames[ source ]
                        + " to " + pegNames[ target ] );
}
```

Binary Search

- *Binary search* is an algorithm that:
  - Determines whether a given item is in a sorted list, and
  - If it is, returns the position of that element in the list.
- It works like the "dictionary search" algorithm.
- It repeatedly halves the number of elements.
  - It is a typical case of a *divide and conquer* algorithm.
  - Because of the halving it is sometimes called *dichotomic*.
- Requires (worst-case) time that is logarithmic in size of the input.

The Basic Idea

- Before studying the algorithm let's define its main task.

  Input: The input of the algorithm consists of:
  - An item; and
  - A list of items sorted in non-decreasing order.
  - For simplicity the items in list are unique.

  Output: The output of the algorithm is an `int`.

  The output depends on one of the following cases.

  Item is in list: The index of item in the list.

  Item is not in list: A negative number.
- For simplicity we'll assume that all items are `int`s.
- Furthermore, we'll assume that the list is presented as an array.

The Algorithm

`binSearch( item, items, lo, hi )`

lo > hi: Return -1.

lo <= hi:

1. Determine “the” middle index.
   - We implement this as `mid = (lo + hi) / 2`.
2. Compare `item` and `items[ mid ]`.
   - **`item == items[ mid ]`**:
     - Return `mid`.
   - **`item < items[ mid ]`**:
     - Return `binSearch( item, items, lo, mid - 1 )`.
   - **`item > items[ mid ]`**:
     - Return `binSearch( item, items, mid + 1, hi )`.

The Algorithm

`binSearch( item, items, lo, hi )`

`lo > hi`: Return -1.

`lo <= hi`:

1. Determine "the" middle index.
   - We implement this as `mid = (lo + hi) / 2`.
   - Unfortunately, this is not correct due to overflow.
   - You can fix this by implementing it as
     - '`mid = lo + (hi - lo) / 2`' or as
     - '`mid = (hi + lo) >>> 1`'.
2. Compare `item` and `items[ mid ]`.
   - **`item == items[ mid ]`**:
     - Return `mid`.
   - **`item < items[ mid ]`**:
     - Return `binSearch( item, items, lo, mid - 1 )`.
   - **`item > items[ mid ]`**:
     - Return `binSearch( item, items, mid + 1, hi )`.

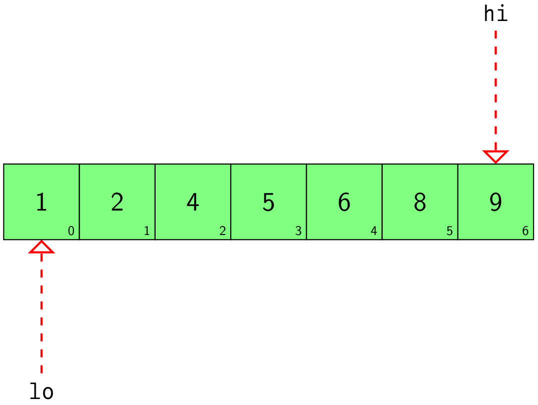
Implementation in Java

Java

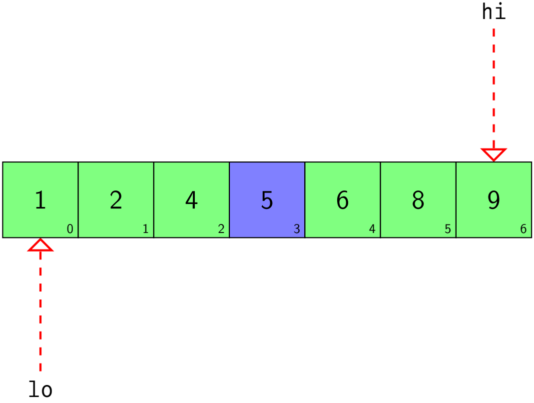
```java
public static int binSearch( int item, int[] items ) {
    return binSearch( item, items, 0, items.length - 1 );
}

public static int binSearch( int item, int[] items, int lo, int hi ) {
    final int result;

    if (lo > hi) {
        result = - 1;
    } else {
        int mid = (lo + hi) / 2;
        if (item == items[ mid ]) {
            result = mid;
        } else if (item < items[ mid ]) {
            result = binSearch( item, items, lo, mid - 1 );
        } else {
            result = binSearch( item, items, mid + 1, hi );
        }
    }

    return result;
}
```

binSearch(4, {1,2,4,5,6,8,9}, 0, 6)

Intial Situation

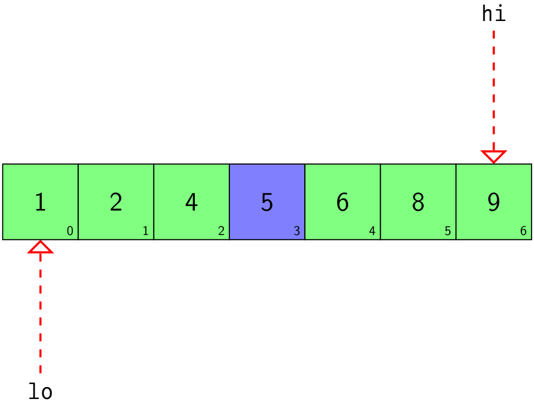


```
binSearch( 4, {1,2,4,5,6,8,9}, 0, 6 )
mid = (lo + hi) / 2
```



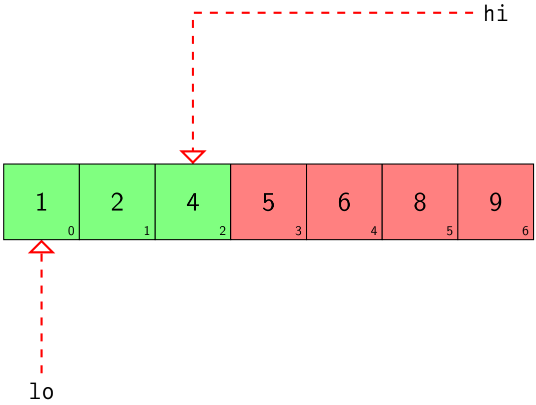
```
binSearch( 4, {1,2,4,5,6,8,9}, 0, 6 )
```

```
item < item[ mid ]
```



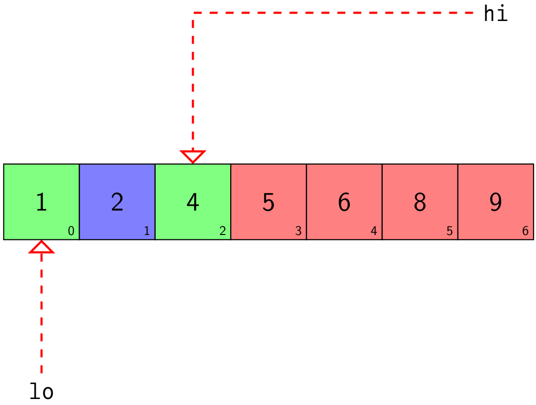
binSearch(4, {1,2,4,5,6,8,9}, 0, 6)

Search to Left of `mid`



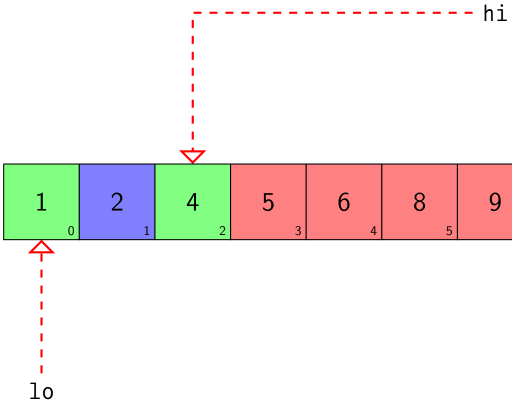
binSearch(4, {1,2,4,5,6,8,9}, 0, 6)

mid = (lo + hi) / 2



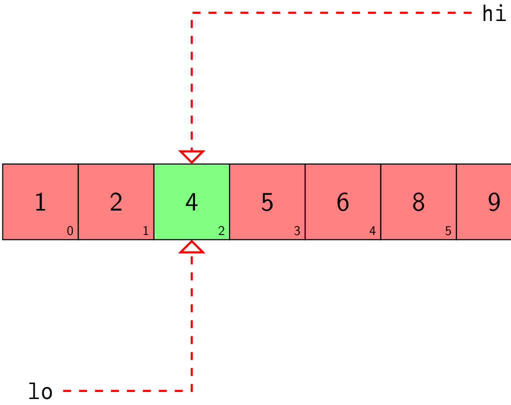
`binSearch( 4, {1,2,4,5,6,8,9}, 0, 6 )`

`item > item[ mid ]`



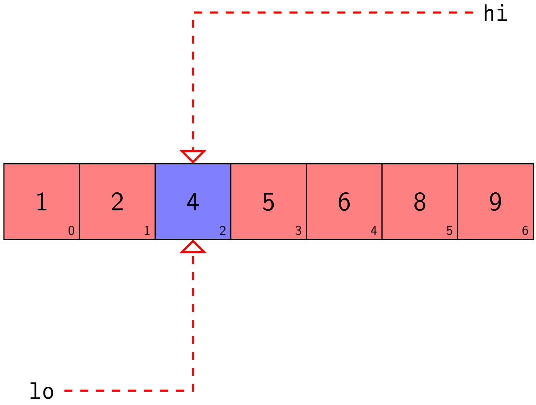
binSearch(4, {1,2,4,5,6,8,9}, 0, 6)

Search to Right of mid



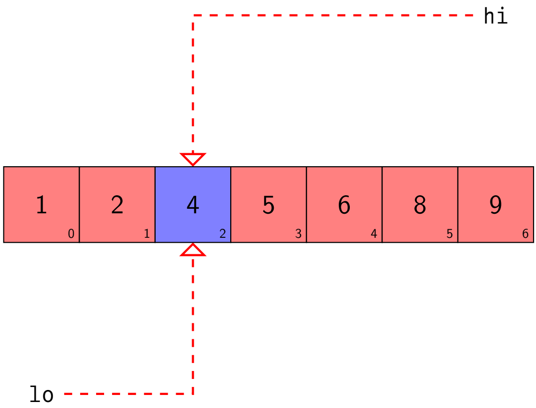
`binSearch( 4, {1,2,4,5,6,8,9}, 0, 6 )`

`mid = (lo + hi) / 2`



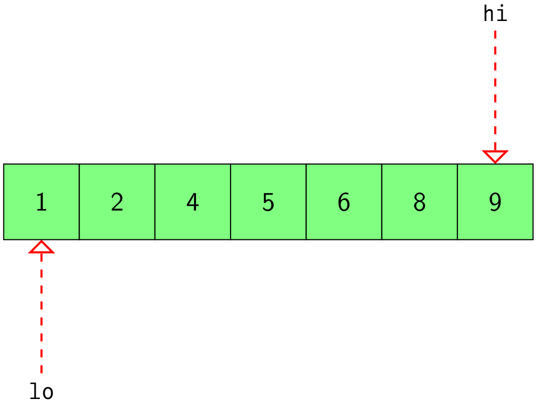
binSearch(4, {1,2,4,5,6,8,9}, 0, 6)

Celebration



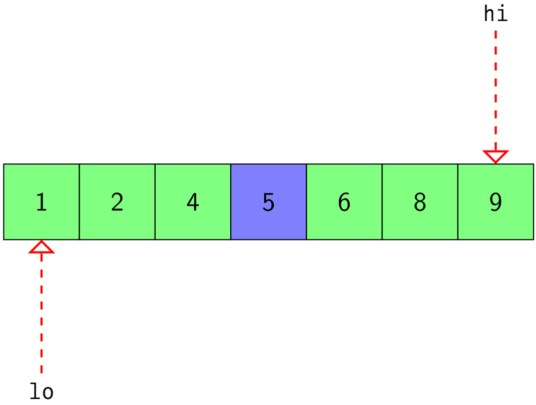
binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

Intial Situation



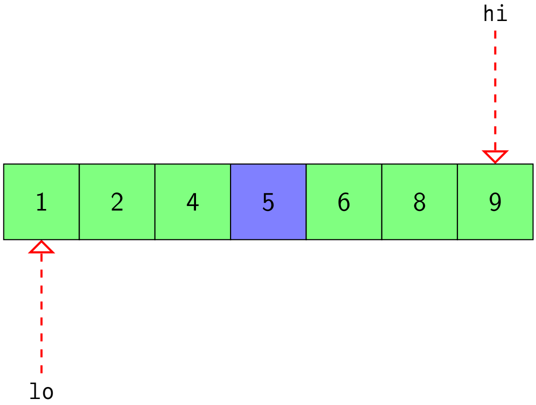
binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

mid = (lo + hi) / 2



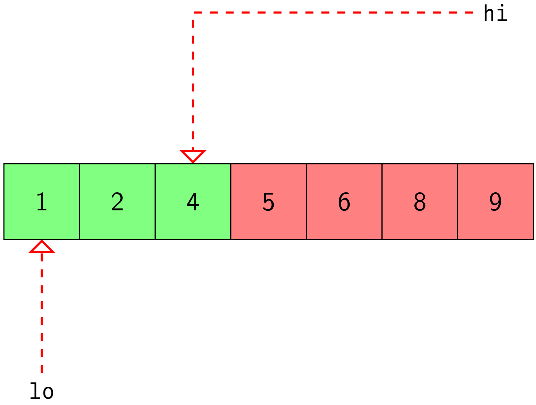
binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

`item < item[ mid ]`



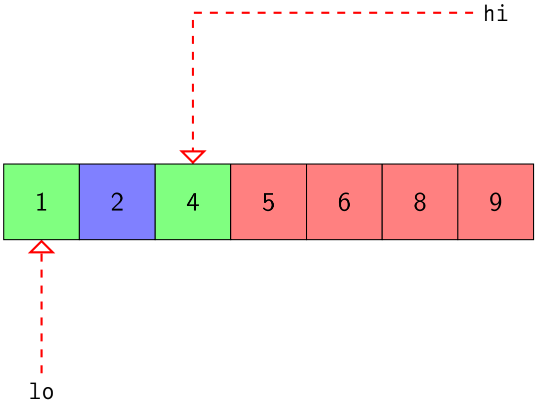
binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

Search to Left of mid



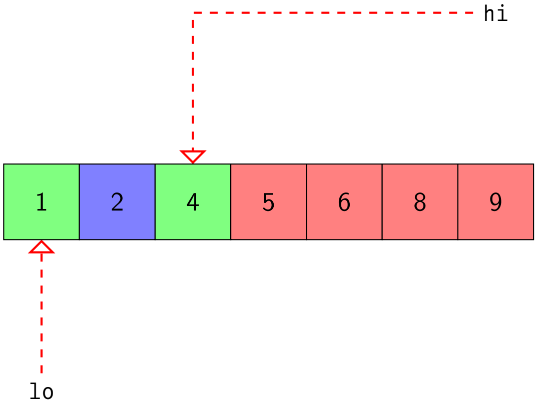
binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

mid = (lo + hi) / 2



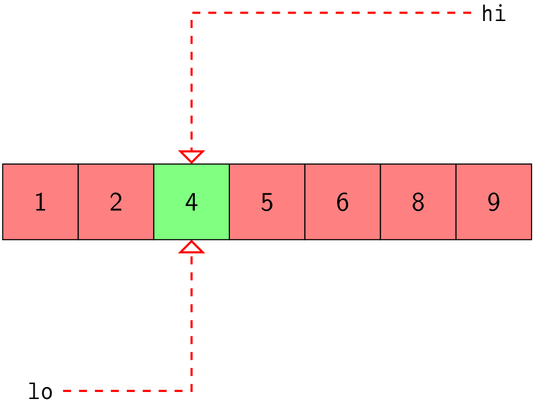
`binSearch( 3, {1,2,4,5,6,8,9}, 0, 6 )`

`item > item[ mid ]`

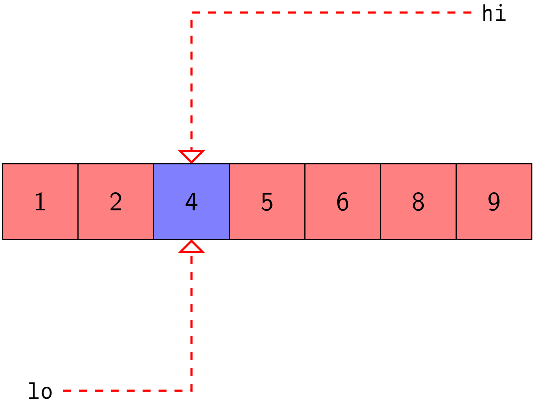


binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

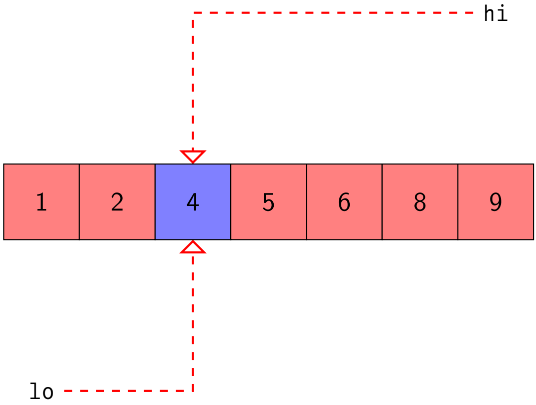
Search to Right of mid



```
binSearch( 3, {1,2,4,5,6,8,9}, 0, 6 )
mid = (lo + hi) / 2
```

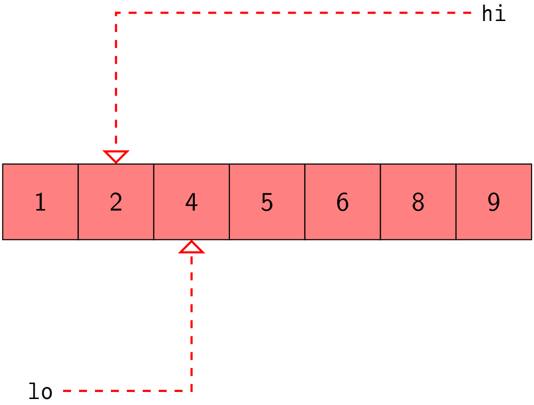


```
binSearch( 3, {1,2,4,5,6,8,9}, 0, 6 )
item < item[ mid ]
```



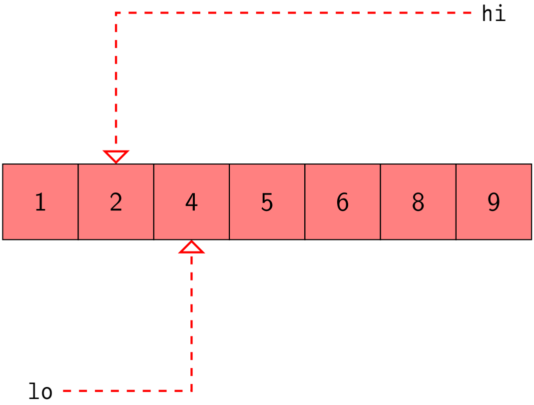
binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

Search to Left of mid



binSearch(3, {1,2,4,5,6,8,9}, 0, 6)

Bummer



The `Comparable` Interface

- We've seen how to use binary search for `int`s.
- We should be able to generalise it for other *comparable* things.
- *Implementing* an *interface* is almost the same as extending a class.
  - If class B implements interface A, B behaves as A.
- A class *implements* the `Comparable` *interface* if it overrides

```
int compareTo( Object that )
```

- Many classes implement the `Comparable` interface:
  - `Integer`,
  - `Double`,
  - `String`,
  - ....

A Comparable-Compatible Version

Java

```
public static int binSearch( Comparable item, Comparable[] items, int lo, int hi ) {
    final int result;

    if (lo > hi) {
        result = - 1;
    } else {
       int mid = (lo + hi) / 2;
       int compare = item.compareTo( items[ mid ] );
       if (compare == 0) {
           result = mid;
       } else if (compare < 0) {
           result = binSearch( item, items, lo, mid - 1 );
       } else {
           result = binSearch( item, items, mid + 1, hi );
       }
    }

    return result;
}
```

The `Quicksort` Algorithm

- Sorting algorithms are a very important class of algorithms.
  - Sorting efficiently is crucial to many applications.
- `Quicksort` is a simple but efficient sorting algorithm.
- Given n *random* items its requires $O(n \log n)$ comparisons (on average).
- But, it requires $O(n^2)$ comparisons in the worst case.
- If the input is given as an array, we can sort the array in-situ.
- The algorithm was invented by C. A. R. Hoare in 1962.
- For simplicity we shall study the version for sorting `int` arrays.
- `Arrays` defines several `quicksort`-based sorting methods.

Main Idea

Base case: If $n \leq 1$ then the input is sorted.

Recursion: If $n > 1$:

1. Select any item from the input.
2. Partition remaining items into classes L and G.
   - L are the items less than or equal to the pivot.
   - G are the remaining items.
3. Members of L should end up before those of G.
4. Put the pivot between L and G.
5. Recursively sort L and G.

The Wrapper

Java

```java
public static void qsort( int[] items ) {
    qsort( items, 0, items.length - 1 );
}
```

Main Algorithm

Java

```java
// Sorts items[ lo .. hi ] in non-descending order.
private static void qsort( int[] items, int lo, int hi ) {
   if (hi - lo >= 1) {
      int pivotPosition = partition( items, lo, hi );
      qsort( items, lo, pivotPosition - 1 );
      qsort( items, pivotPosition + 1, hi );
   }
}
```

Main Algorithm: Any Sorting to Do?

Java

```
// Sorts items[ lo .. hi ] in non-descending order.
private static void qsort( int[] items, int lo, int hi ) {
    if (hi - lo >= 1) {
        int pivotPosition = partition( items, lo, hi );
        qsort( items, lo, pivotPosition - 1 );
        qsort( items, pivotPosition + 1, hi );
    }
}
```

Main Algorithm: Partition

Divide

Java

```java
// Sorts items[ lo .. hi ] in non-descending order.
private static void qsort( int[] items, int lo, int hi ) {
    if (hi - lo >= 1) {
        int pivotPosition = partition( items, lo, hi );
        qsort( items, lo, pivotPosition - 1 );
        qsort( items, pivotPosition + 1, hi );
    }
}
```

Main Algorithm: Sort Items to Left of Pivot

Divide and Conquer

Java

```
// Sorts items[ lo .. hi ] in non-descending order.
private static void qsort( int[] items, int lo, int hi ) {
    if (hi - lo >= 1) {
        int pivotPosition = partition( items, lo, hi );
        qsort( items, lo, pivotPosition - 1 );
        qsort( items, pivotPosition + 1, hi );
    }
}
```

Main Algorithm: Sort Items to Right of Pivot

Divide and Conquer

Java

```
// Sorts items[ lo .. hi ] in non-descending order.
private static void qsort( int[] items, int lo, int hi ) {
    if (hi - lo >= 1) {
        int pivotPosition = partition( items, lo, hi );
        qsort( items, lo, pivotPosition - 1 );
        qsort( items, pivotPosition + 1, hi );
    }
}
```

Partition

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Pivot Selection and Exchange

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Partitioning of Lower Elements

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Move Item to Left?

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Move Item to Left? Exchange

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Move Item to Left? Adjust Destination

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Loop Invariant

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Consequence of Loop Invariant

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Move Pivot to Destination

Java

```java
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Final Invariant

Java

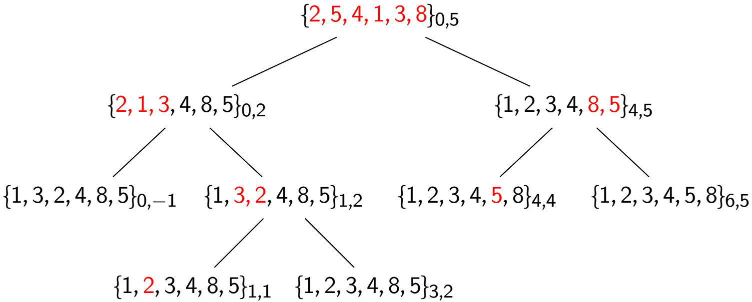
```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Partition: Return Pivot Position

Java

```
private static int partition( int[] items, int lo, int hi ) {
   int destination = lo;
   swop( items, (hi + lo) >>> 1, hi );
   // The pivot is now stored in items[ hi ].
   for (int index = lo; index != hi; index ++) {
      if (items[ hi ] >= items[ index ]) {
         // Move current item to start.
         swop( items, destination, index );
         destination ++;
      }
      // items[ i ] <= items[ hi ] if lo <= i < destination.
      // items[ i ] >  items[ hi ] if destination <= i <= index.
   }
   // items[ i ] <= items[ hi ] if lo <= i < destination.
   // items[ i ] >  items[ hi ] if destination <= i < hi.
   swop( items, destination, hi );
   // items[ i ] <= items[ destination ] if lo <= i <= destination.
   // items[ i ] >  items[ destination ] if destination < i <= hi.
   return destination;
}
```

Call Trace of `qsort( {2,5,4,1,3,8}, 0, 5 )`



For Wednesday

- Study [Horstmann 2013, Sections 12.1–12.2].

Acknowledgements

- This lecture corresponds to [Horstmann 2013, Sections 12.1–12.2].

About this Document

- This document was created with `pdflatex`.
- The LaTeX document class is `beamer`.