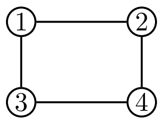
Exercises Discussion Session 1

Toric Varieties, Exponential Families, Examples of graphical models.

(1) Let (X_1, X_2, X_3) be a vector of binary random variables. Consider the following two graphs.

Write a polynomial parametrization of the model $\mathcal{M}(G_1)$ and write down the matrix A that defines the monomial parametrization of the model $\mathcal{M}(G_2)$. Find the implicit equations that define these models either by hand or by using a computer algebra software like Macaulay2. How do the two polynomial parametrizations compare?

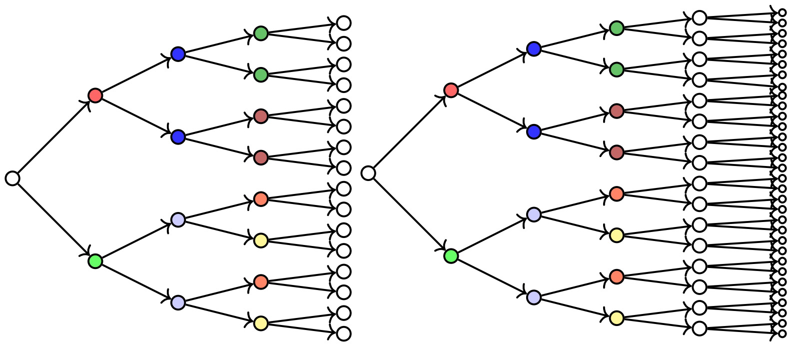
(2) Write down the matrix A for the undirected graphical model of the graph shown below and find its defining equations either by hand or using a computer algebra software like Macaulay2. Here the nodes of the graph represent discrete binary random variables.



(3) Prove that the family of pdfs $\{f(x|\mu, \sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}, f \text{ is a normal density}\}$ is an exponential family.

(4) Why is it that in Algebraic Statistics Δ°_{r-1} is identified with $\mathbb{P}^{r-1}_{\mathbb{C}}$

(5) Consider the following two staged trees $\mathcal{T}, \mathcal{T}'$. Prove that the tree on the left is balanced. Now consider $\mathcal{T}'$ on the left. What are all possible choices of stages you can make in the second to last level in such a way that $\mathcal{T}'$ is also balanced.



(4) Prove Proposition 1.3 in the slides, or read the proof from Sullivant (2018) Proposition 6.2.4., or ask a friend.

Extra remarks

- For a quick intro to ideals, polynomials and varieties check out Kahle u. a. (2018) available at the arxiv https://arxiv.org/pdf/1705.07411.pdf.
- The computer algebra package Macaulay2 is great to find implicit descriptions of varietes/statistical models. You can try it out here https://www.unimelb-macaulay2.cloud.edu.au/#home

```
-- Ring for the distributions
R=QQ[p00,p01,p10,p11]
-- Ring for the parameters
S=QQ[s0,s1,t0,t1]
-- A matrix for the model
A=matrix{{1,1,0,0},
         {0,0,1,1},
 {1,0,1,0},
 {0,1,0,1}}
needsPackage("FourTiTwo")
toricMarkov(A,R)
---
-- Homogeneous version
restart
SS=QQ[z,s0,s1,t0,t1]
-- sum to one condition ideal
sto1=ideal(s0+s1-1,t0+t1-1)
-- quotient ring
S=SS/sto1
R=QQ[p00,p01,p10,p11]
mons=z*{s0*t0,s0*t1,s1*t0,s1*t1}
phiG=map(S,R,mons)
I=ker(phiG)
```

- Some references: The most popular references in Algebraic Statistics are Drton u. a. (2008), Sullivant (2018), Pachter und Sturmfels (2005). For specific papers in maximum likelihood estimation and MLdegrees check: Huh (2013b), Huh (2013a), Huh (2014), Duarte u. a. (2021). For toric varieties: Cox u. a. (2011). The authority book on graphical models is Lauritzen (1996).

Exercises Discussion Session 2

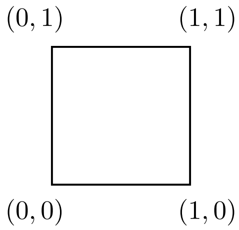
1. Find the MLE of the independence model $\mathcal{M}_{X \perp\!\!\!\perp Y}$ where X, Y are binary random variables.

2. Let $u = (u_{000}, \cdots, u_{111})$ be a vector of counts.

   (a) Find the MLE for the models in Exercise 1, from the Discussion Session 1. Use the formulas for the MLE of a Bayesian network and a decomposable model to check your answers. You can find the general formulas in `https://arxiv.org/pdf/1903.06110.pdf` page 8.

   (b) Write the Horn matrices for the MLEs of the two models from part (a).

3. Let $\mathcal{M}_{X \perp\!\!\!\perp Y}$ be the independence model on two binary random variables. This model is described by the projective toric variety associated to the unit square.



   (a) Write down a hyperplane description of the square with inward pointing facing normals.

   (b) Make a picture of the normal fan Σ of the square and compute its primitive collections. A primitive collection is a subset c of the 1-dimensional rays in the fan such that the rays in c do not span a cone in Σ but such that every subset does span a cone.

   (c) Construct the following matrix $H_+ = (h_{ij})$: Index the columns by the lattice points in the square i.e $\{(0,0), (1,0), (0,1), (1,1)\}$ and index the rows by the facets of the square. The entry h_{ij} is the lattice distance function from the point corresponding to column j to the facet corresponding to row i. Now construct a matrix H_- in the following way: The rows of H_- are indexed by the primitive collections of c. To write a row in H_-, take a primitive collection c and add the rows in H_+ that correspond to faces whose normal vectors belong to c this gives you a row vector, put negatives in front of that vector.

   (d) Check that the matrix block matrix $H = H_+ | H_-$ is a Horn matrix, what is λ in this case? Write down the function $\varphi_{H,\lambda} : \mathbb{R}^4 \to \mathbb{R}^4$ where $(u_{00}, u_{01}, u_{10}, u_{11}) \mapsto \lambda * (Hu^T)^H$

   (e) Find the MLE of $u = (u_{00}, u_{01}, u_{10}, u_{11})$ in the model $\mathcal{M}_{X \perp\!\!\!\perp Y}$. Compare with $\varphi_{H,\lambda}$.

4. **Open Question**: For which polytopes P in any dimension, does the construction from the previous exercise give a Horn matrix of a satistical model?

References

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