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§ 4. Computing Resultants

- Last time we saw the theorem of existence of the resultant, some of its properties and how to do a concrete computation of the resultant. Today we generalize this construction and obtain two formulas to compute the resultant.

PREVIOUSLY: In p. 89, we computed a resultant by associating to the system another system of linear equations and finding the determinant of the coeff. matrix of the linear system.

Main idea: Multiply equations by monomials to get a square system and take determinant of that.

- $F_0 = F_1 = \cdots = F_n = 0$, $F_i \in \mathbb{C}[X_0, \dots, X_n]$ $\deg(F_i) = d_i$, $d := \sum_{i=0}^{n} (d_i - 1) = \sum_{i=0}^{n} d_i - n$.
- Monomials of degree d satisfy that each of them is divisible by $X_i^{d_i}$ for at least one i, $0 \le i \le n$.
  $\hookrightarrow$ $X_0^{d_0-1} X_1^{d_1-1} \cdots X_n^{d_n-1}$ by the pigeon-hole principle.

- Consider the set $\{X^\alpha : |\alpha| = d\}$ ← These is the set of monomials of total degree d. and partition it into $n+1$ sets as follows:

$$S_0 = \{ X^\alpha : |\alpha| = d, \ X_0^{d_0} \mid X^\alpha \}$$
$$S_1 = \{ X^\alpha : |\alpha| = d, \ X_0^{d_0} \nmid X^\alpha, \ X_1^{d_1} \mid X^\alpha \}$$
$$\vdots$$
$$S_n = \{ X^\alpha : |\alpha| = d, \ X_0^{d_0}, \dots, X_{n-1}^{d_{n-1}} \nmid X^\alpha, \ X_n^{d_n} \mid X^\alpha \}$$

NOTE: Every element in $\{X^\alpha : |\alpha| = d\}$ is in one of $S_0, \dots, S_n$ and $S_i \cap S_j = \emptyset$.

- If $X^\alpha \in S_i \Rightarrow X^\alpha = X_i^{d_i} \cdot X^\alpha / X_i^{d_i}$ $\quad \deg(X^\alpha / X_i^{d_i}) = d - d_i$

Example: (p. 89)

Want to compute $Res_{1,1,2}$

$$(*)\begin{cases} F_0 = a_1 x + a_2 y + a_3 z = 0 \\ F_1 = b_1 x + b_2 y + b_3 z = 0 \\ F_2 = c_1 x^2 + c_2 y^2 + c_3 z^2 + c_4 xy + c_5 xz + c_6 yz = 0 \end{cases}$$

We write down all monomials of total degree d and S_0, S_1, S_2.

$d = (1-1) + (1-1) + (2-1) + 1 = 2$.

$R = k[x,y,z] \Rightarrow$ Monomials of degree $2 = \{x^2, xy, y^2, yz, z^2, xz\}$

$x_0^{d_0} = x \Rightarrow S_0 = \{x^2, xy, xz\}$ $\qquad S_0/x = \{x, y, z\}$

$x_1^{d_1} = y \Rightarrow S_1 = \{y^2, yz\}$ $\qquad S_1/y = \{y, z\}$

$x_2^{d_2} = z^2 \Rightarrow S_2 = \{z^2\}$ $\qquad S_2/z = \{1\}$

From the given system $F_0 = \cdots = F_n = 0$ we write down a new and bigger system by using the monomials in $S_0, \dots, S_n$

New system from $(*)$ using $S_0, \dots, S_n$

$$(**)\begin{cases} x^\alpha / x_0^{d_0} \cdot F_0 = 0 & \forall x^\alpha \in S_0 \\ \vdots & \\ x^\alpha / x_n^{d_n} \cdot F_n = 0 & \forall x^\alpha \in S_n \end{cases}$$

Since $\deg(F_i) = d_i$, $\deg\left(x^\alpha / x_i^{d_i}\right) = d - i$

$\Rightarrow \deg\left(x^\alpha / x_i^{d_i} \cdot F_i\right) = d - d_i + d_i = d$.

Each polynomial in $(**)$ is a linear combination of monomials of degree d.

$N =$ # of monomials of total degree d $\qquad N =$ # of equations in $(**)$ $= \#(S_0 \cup \cdots \cup S_n)$

$\Rightarrow$ Get a system of N linear eqns in N unkowns. The unknowns are the monomials of total degree d.

Def: The determinant of the coefficient matrix of the system $(**)$ is denoted D_n.

Example: For $(*)$ the coeff matrix is $D = \begin{pmatrix} a_1 & 0 & 0 & a_2 & a_3 & 0 \\ 0 & a_2 & 0 & a_1 & 0 & a_3 \\ 0 & 0 & a_3 & 0 & a_1 & a_2 \\ 0 & b_2 & 0 & b_1 & 0 & b_3 \\ 0 & 0 & b_3 & 0 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}$

$D_2 = \det D$

Important properties of D_n:

EX#6: (a) Since D_n is the coefficient matrix of $(*\_*)$, it is clearly a polynomial in the coeff. of F_i.

(b) D_n has degree $d_0 \cdots d_{n-1}$ as a polynomial in the coefficients of F_n.

Ex#7: D_n is divisible by the resultant

$$D_n = \text{Res} \cdot \text{Extraneous factors}.$$

- The extraneous factor doesn't involve the coefficients of F_n, it only uses the ones in $F_0, \dots, F_{n-1}$.

(4.6) Proposition: The extraneous factor in (4.5) is an integer polynomial in the coefficients of $\overline{F}_0, \dots, \overline{F}_{n-1}$, where $\overline{F}_i = F_i(x_0, \dots, x_{n-1}, 0)$

- Method 1: Compute resultant by factoring D_n into a product of irreducibles. The only factor in which all variables appear, is the resultant.
- The sets $S_0, \dots, S_n$ and the polyn. D_n depend on the ordering of the variables. D_n means X_n is the last variable.
- Fix $0 \leq i \leq n-1$ and order $X_1, \dots, X_n$ so X_i is last $\Rightarrow$ the sets $S_0, \dots S_n$ will change. Let D_i be the determinant of the coefficient matrix of this system.

(4.7) Proposition: When $\mathbf{F_0, \dots, F_n}$ are universal polynomials, as before, the resultant is the GCD of $D_0, \dots, D_n$ in $\mathbb{Z}[u_{i,\alpha}]$, i.e.

$$\text{Res} = \pm \, \text{GCD}(D_0, \dots, D_n)$$

Definition: Let $d_0, \dots, d_n$ and d be as usual

(a) A monomial X^α of total degree d is reduced if $X_i^{d_i}$ divides X^α for exactly one i.

(b) $D_n' :=$ subdeterminant of coeff matrix obtained by deleting all rows and columns corresponding to reduced monomials

Example:

$$S_0 \begin{cases} x^2 \\ xy \\ xz \end{cases} \rightarrow \; S_0/x \; \begin{cases} x \cdot F_0 = 0 \\ y \cdot F_0 = 0 \\ z \cdot F_0 = 0 \end{cases}$$

$$S_1 \begin{cases} y^2 \\ yz \end{cases} \rightarrow \; S_1/y \; \begin{cases} y \cdot F_1 = 0 \\ z \cdot F_1 = 0 \end{cases}$$

$$S_2 \{ z^2 \rightarrow \; S_2/z^2 \; \{ 1 \cdot F_2 = 0$$

$$\begin{array}{cccccc} x^2 & y^2 & z^2 & xy & xz & yz \end{array}$$
$$\begin{pmatrix} a_1 & 0 & 0 & a_2 & a_3 & 0 \\ 0 & a_2 & 0 & a_1 & 0 & a_3 \\ 0 & 0 & a_3 & 0 & a_1 & a_2 \\ 0 & b_2 & 0 & b_1 & 0 & b_3 \\ 0 & 0 & b_3 & 0 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}$$

- The extraneous factor for $D_n = \mathrm{Res} \cdot E_n$ is exactly D_n' up to sign.

(4.9) <u>Theorem:</u> When $F_0, \ldots, F_n$ are universal polynomials, the resultant is given by
$$\mathrm{Res} = \pm \frac{D_n}{D_n'}$$
Further, if K is any field and $F_0, \ldots, F_n \in K[x_0, \ldots, x_n]$, then the above formula for Res holds whenever $D_n' \neq 0$.

These formulas have some disadvantages, use with caution.