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§2. Multipolynomial Resultants

- Previously we saw the resultant of two polynomials has the following properties

#1. $\mathrm{Res}(f,g)$ is an integer polynomial in the coeff. of f, g.
#2. $\mathrm{Res}(f,g)=0 \Leftrightarrow f,g$ have a nontrivial common factor in $\mathbb{K}[x]$.
#3. $\exists A,B \in \mathbb{K}[x]$ s.t. $Af + Bg = \mathrm{Res}(f,g)$. The coeff of A,B are integer polynomials in the coeff of f,g.

Def. A homogeneous polynomial $F \in \mathbb{K}[X_0, X_1, \dots, X_n]$ is a polynomial whose terms have all the same degree.
Ex: $x^2y + y^2 + 1 \leftarrow$ not homog. $\quad x^3 + x^2y + y^3 \leftarrow$ homog.

- Given $F_0, \dots, F_n \in \mathbb{K}[X_0, \dots, X_n]$, homog. of positive degree, we get a system $\begin{cases} F_0 = 0 \\ \vdots \\ F_n = 0 \end{cases}$. $\rightarrow$ Since these eqns are homogeneous, they always have a trivial solution.

QUESTION: Are there nontrivial solutions?
ANSWS: (1) Solutions depend on coefficients. (2) For most choices solutions don't exist.

Example 1: Linear Algebra

$$\begin{cases} F_0 = C_{00}X_0 + \cdots + C_{0n}X_n = 0 \\ \vdots \\ F_n = C_{n0}X_0 + \cdots + C_{nn}X_n = 0 \end{cases} \Rightarrow C = \begin{pmatrix} C_{00} & \cdots & C_{0n} \\ \vdots & \ddots & \\ C_{n0} & \cdots & C_{nn} \end{pmatrix}$$

$C\mathbf{X} = 0$, is a homogeneous system. has nontrivial solution $\Leftrightarrow \det C \neq 0$.

QUESTION: What conditions must the coeff. satisfy in order that $F_0 = \cdots = F_n$ has a nontrivial solution?

Notation: • $d_i = \deg(F_i) \quad \Rightarrow \quad F_i = \sum_{|\alpha| = d_i} C_{i,\alpha} X^\alpha$

• Given $P \in \mathbb{C}[U_{i,\alpha}]$ we let $P(F_0, \dots, F_n)$ denote the number obtained by replacing each variable $U_{i,\alpha} \rightarrow C_{i,\alpha}$.

Theorem 2.3: If we fix positive degrees $d_0, \ldots, d_n$ then there is a unique polynomial $\mathrm{Res} \in \mathbb{Z}[u_{i,\alpha}]$ which has the following properties:

(a) If $F_0, \ldots, F_n \in \mathbb{C}[x_0, \ldots, x_n]$ are homog. of degrees $d_0, \ldots, d_n$, then the equations have a nontrivial solution over $\mathbb{C} \Leftrightarrow \mathrm{Res}(F_0, \ldots, F_n)$.

(b) $\mathrm{Res}(x_0^{d_0}, \ldots, x_n^{d_n}) = 1$.

(c) Res is irreducible even when regarded as a polynomial in $\mathbb{C}[u_{i,\alpha}]$.

Example 1: $\mathrm{Res}_{1,1,\ldots,1}(F_0, \ldots, F_n) = \det(C)$.

Suggested exercises: Ex 1 (p. 85), Ex 2 (p. 87), Ex 3 (p. 89)

Example of a Resultant Calculation (p. 89)

Consider $(*)\begin{cases} F_0 = a_1 x + a_2 y + a_3 z = 0 \\ F_1 = b_1 x + b_2 y + b_3 z = 0 \\ F_2 = c_1 x^2 + c_2 y^2 + c_3 z^2 + c_4 xy + c_5 xz + c_6 yz = 0 \end{cases}$

Want to compute $Res_{1,1,2}$.

Prop 2.10: $Res_{1,1,2}$ is a long polynomial written in page 90.

proof.

Let $R = Res_{1,1,2}$ be the long polynomial.

Suppose (x, y, z) is a solution to $(*)$ $\Rightarrow$ The following six equations also vanish at (x, y, z)

$$x \cdot F_0 = y \cdot F_0 = z \cdot F_0 = y \cdot F_1 = z \cdot F_1 = F_2 = 0$$

This system can also be written as

$$\begin{array}{ccccccccccccc}
a_1 x^2 & + & 0 & + & 0 & + & a_2 xy & + & a_3 xz & + & 0 & = & 0 \\
0 & + & a_2 y^2 & + & 0 & + & a_1 xy & + & 0 & + & a_3 yz & = & 0 \\
0 & + & 0 & + & a_3 z^2 & + & 0 & + & a_1 xz & + & a_2 yz & = & 0 \\
0 & + & b_2 y^2 & + & 0 & + & b_1 xy & + & 0 & + & b_3 yz & = & 0 \\
0 & + & 0 & + & b_3 z^2 & + & 0 & + & b_1 xz & + & b_2 yz & = & 0 \\
c_1 x^2 & + & 0 & + & c_3 z & + & c_4 xy & + & c_5 xz & + & c_6 yz & = & 0
\end{array}$$

$D = \det(\text{matrix of coeff}) = -a_1 R$

12 coefficients. In $\mathbb{C}^{12}$, if $(*)$ has a solution $\Rightarrow$ D vanishes at the coefficients of $(*)$

$\Rightarrow \mathbb{V}(Res_{1,1,2}) \subseteq \mathbb{V}(D) \Rightarrow \mathbf{I}(\mathbb{V}(D)) \subseteq \mathbf{I}(\mathbb{V}(Res_{1,1,2}))$

$\Rightarrow D \in \mathbf{I}(\mathbb{V}(Res_{1,1,2})) = \sqrt{\langle Res_{1,1,2} \rangle}$

$\Rightarrow D \in \langle Res_{1,1,2} \rangle \Rightarrow D = m \cdot Res_{1,1,2} = a_1 R$.

$\Rightarrow Res_{1,1,2} \mid a_1$ or $Res_{1,1,2} \mid R$.

Later we will see that $\deg(Res_{1,1,2}) = 5 \Rightarrow R = Res_{1,1,2}$.

Geometric Interpretation of the Resultant

- $F_0, \dots, F_n \in \mathbb{K}[X_0, \dots, X_n]$ are homogeneous with $\deg(F_i) = d_i$, $0 \le i \le n$.
- For each d_i and each monomial of degree d_i we introduce a variable $u_{i,\alpha}$.

$$\mathbf{F_i} = \sum_{|\alpha| = d_i} u_{i,\alpha} X^{\alpha}$$, $\mathbf{F_i}$ are universal polynomials

- $M = M_{d_0} + M_{d_1} + \cdots + M_d$ where $M_{d_i} = \#$ of variables $u_{i,\alpha}$ for $|\alpha| = d_i$.
- $\mathbb{C}^M$ is an affine space with coordinates $u_{i,\alpha}$, $0 \le i \le n$ and $|\alpha| = d_i$.

$$\mathbb{C}^M = \{ (c_{i,\alpha}) : c_{i,\alpha} \in \mathbb{C} \}$$

- We can take a point $(c_{i,\alpha}) \in \mathbb{C}^M$ and evaluate each $\mathbf{F_i}$ $u_{i\alpha} \mapsto c_{i,\alpha}$ to obtain polynomials $F_1, \dots, F_n \in \mathbb{K}[X_1, \dots, X_n]$.
  ↳ This means that "$\mathbb{C}^M$ parameterizes all possible $(n+1)$-tuples of homog. polyn. of degrees $d_0, \dots, d_n$"

- Use $\mathbb{P}^n$ to keep track of nontrivial solutions

  - $\mathbb{P}^n = \{ (a_0, \dots, a_n) : a_i \in \mathbb{C} \} \setminus \{(0, \dots, 0)\} / \sim$, $(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \iff \exists \lambda \neq 0$ with $(b_0, \dots, b_n) = \lambda (a_0, \dots, a_n)$.

  - For $F(X_0, \dots, X_n)$ homog. $\deg(F) = d$, $(b_0, \dots, b_n) = \lambda (a_0, \dots, a_n) \in \mathbb{P}^n$, $\lambda \neq 0$

    $\Rightarrow F(b_0, \dots, b_n) = \lambda^d F(a_0, \dots, a_n)$.

    Hence $F(p) = 0$ makes sense, $\mathbb{V}(F) := \{ (b_0, \dots, b_n) \in \mathbb{P}^n : F(b_0, \dots, b_n) = 0 \}$

- Consider $\underbrace{\mathbb{C}^M}_{\text{Tuples of } (n+1) \text{ polynomials}} \times \underbrace{\mathbb{P}^n}_{\text{Points in } \mathbb{P}^n}$ $\to (c_{i,\alpha}, a_0, \dots, a_n) \in \mathbb{C}^M \times \mathbb{P}^n$

- The coordinate ring for $\mathbb{C}^M \times \mathbb{P}^n$ is $\mathbb{C}[c_{i,\alpha}][X_0, \dots, X_n]$,
  $\mathbf{F_0}, \dots, \mathbf{F_n} \in \mathbb{C}[c_{i,\alpha}][X_0, \dots, X_n]$
  For the $\mathbf{F_i}$ consider the set $W = \mathbb{V}(\mathbf{F_0}, \dots, \mathbf{F_n})$

$W = \{ (c_{i,\alpha}, a_0, \dots, a_n) \in \mathbb{C}^M \times \mathbb{P}^n : (a_0, \dots, a_n)$ is a nontrivial solution of $F_0 = \cdots = F_n = 0$, where F_i's have coeff. $(c_{i,\alpha}) \}$

$= \{$ All pairs $(c_{i,\alpha}, (a_0, \dots, a_n))$ of equations and a nontrivial solution of the eqns$\}$.

- Consider $\pi : \mathbb{C}^M \times \mathbb{P}^n \longrightarrow \mathbb{C}^M$, $(c_{i,\alpha}, a_0, \dots, a_n) \mapsto (c_{i,\alpha})$ — i.e. project onto the coefficients of the system.
- $\pi(W) = \{ (c_{i,\alpha}) \in \mathbb{C}^M : \exists (a_0, \dots, a_n)$ that is a solution to the system with coeffs. $(c_{i,\alpha}) \}$
  $= \{$ choices of coefficients for the systems s.t. they have a nontriv. solution$\}$

- The main point of Theorem 2.3 is that #1 $\pi(W)$ is an affine algebraic variety i.e $\pi(W) = \mathbb{V}(S)$, where S is a set of polynomials & #2 S consists of exactly one polynomial.

STEPS FOR THE Proof:

STEP 1: Projective extension Theorem: Given a variety $W \subset \mathbb{C}^M \times \mathbb{P}^n$ and the projection map $\pi : \mathbb{C}^M \times \mathbb{P}^n \to \mathbb{C}^M$, the image $\pi(W)$ is a variety in $\mathbb{C}^M$.

★ This is one of the advantages of projective space, the same is not true for affine varieties.

★ Existence of a nontrivial solution of the system is determined by polynomial equations on the coefficients of the system.

STEP 2: Prove that S is exactly ONE polynomial.

- Want to prove that $\dim \pi(W) = M-1$, since $\dim \mathbb{C}^M = M$ and $\pi(W) \subset M \Rightarrow \pi(W)$ is of codim $1 \Rightarrow$ it is "cut out" by one equation.

#1. $\dim \mathbb{C}^M \times \mathbb{P}^n = M + n$

#2. W is defined by n+1 equations $F_0, \ldots, F_n$. Intuitively equations cut dimension by ONE.

$$\Rightarrow \quad \dim W = (M+n) - (n+1) = M-1$$

#3. W is irreducible $\Rightarrow$ $\pi(W)$ is irreducible

#4. Show $W \to \pi(W)$ is 1-1 most of the time.

$$\Rightarrow \quad \dim W = \dim \pi(W) = M-1.$$