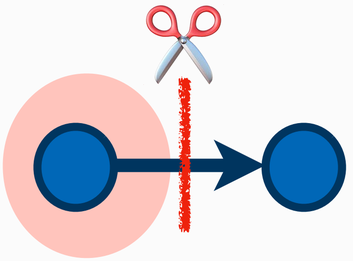
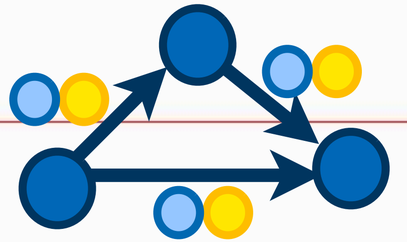
Max Flow Min Cut = MFMC



Minimum Cut, Bipartite Matching and Minimum Cost Maximum Flow with BGL

Min Cost Max Flow = MCMF



Pascal Su [1]

November 13, 2019

ETH Zürich,

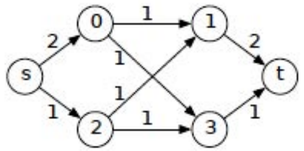
[1] *based on material from Daniel Graf and Andreas Bärtschi*

Recap: Basic Network Flows – What did we see last time?

One problem to rule them all...

Network Flow: Problem Statement

Input: A flow network consisting of
- directed graph $G = (V, E)$
- *source* $s \in V$ with only out edges
- *sink* $t \in V$ with only in edges
- edge capacity $c : E \to \mathbb{N}$.



Output: Flow function $f : E \to \mathbb{R}$ subject to
- Capacity constraints for every edge e.
  $0 \leq f(e) \leq c(e)$
- Flow conservation at every vertex u.
  $\sum_{(v,u)\in E} f((v,u))$ *in flow*
  $= \sum_{(u,w)\in E} f((u,w))$ *out flow*
- Maximization of *total flow*.
  $|f| := \sum_{(s,v)\in E} f((s,v))$

Theorem
On a graph with integral capacities, there always exists an integral maximum flow.

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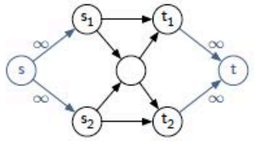
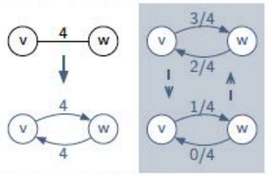
Common techniques

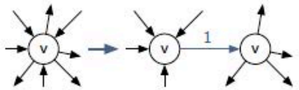
Multiple sources/sinks
with e.g. $\infty \approx \sum_{e\in E} c(e)$



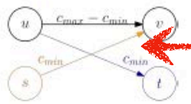
Undirected graphs
antiparallel edges with flow reducable to one direction



Vertex capacities
split into in-vertex and out-vertex



Minimum flow per edge
how to enforce $c_{min}(e) \leq f(e) \leq c_{max}(e)$?



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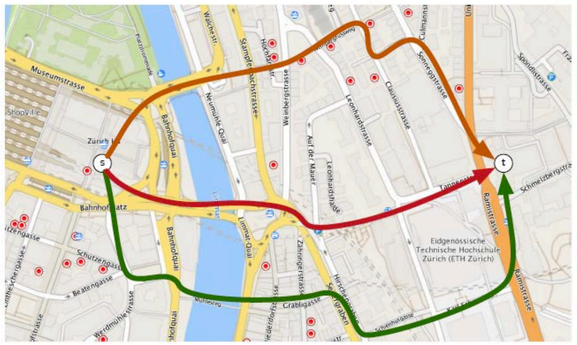
see addition to slides week 5

Flow Application: Edge Disjoint Paths

How many ways are there to get from HB to CAB without using the same street twice?



Map: search.ch, TomTom, swisstopo, OSM

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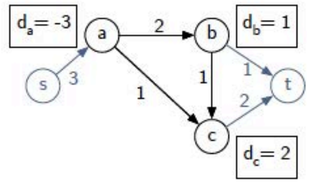
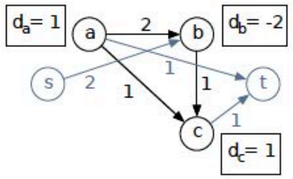
Flow Application: Circulation Problem

- Multiple sources with a certain amount of flow to give (**supply**).
- Multiple sinks that want a certain amount of flow (**demand**).
- Model these as negative or positive demand per vertex d_v.
- Question: Is there a feasible flow? Surely not if $\sum_{v\in V} d_v \neq 0$. Otherwise? Add super-source and super-sink to get a maximum flow problem.



feasible flow exists



no feasible flow exists

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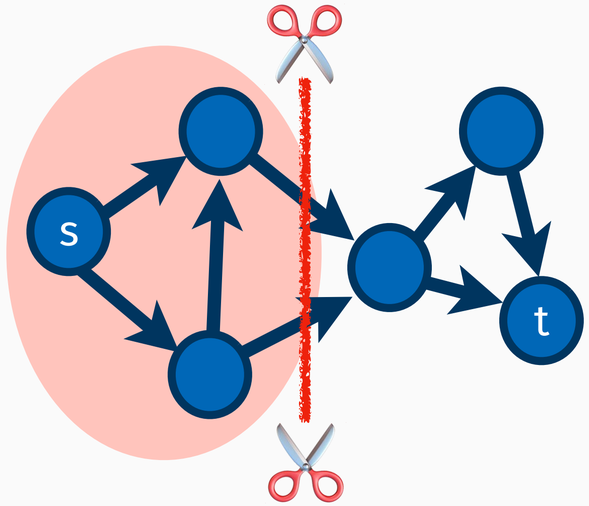
Today: Advanced Network Flow – What else are flows useful for?

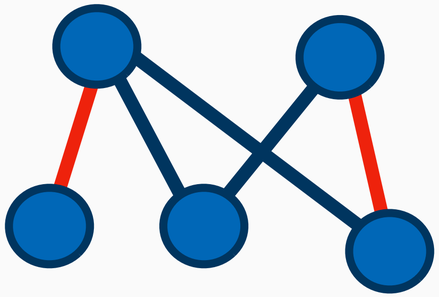
Minimum Cuts

- How to disconnect t from s cheaply?



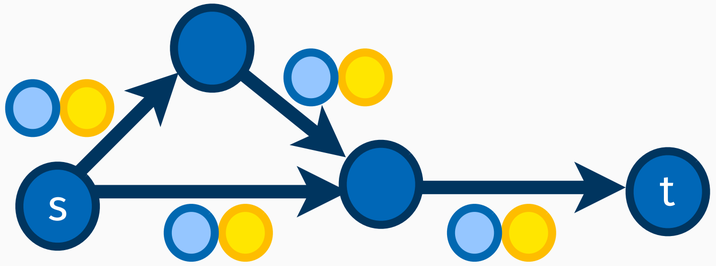
Bipartite Matching

- How to assign A's to B's effectively?



Flows with Costs

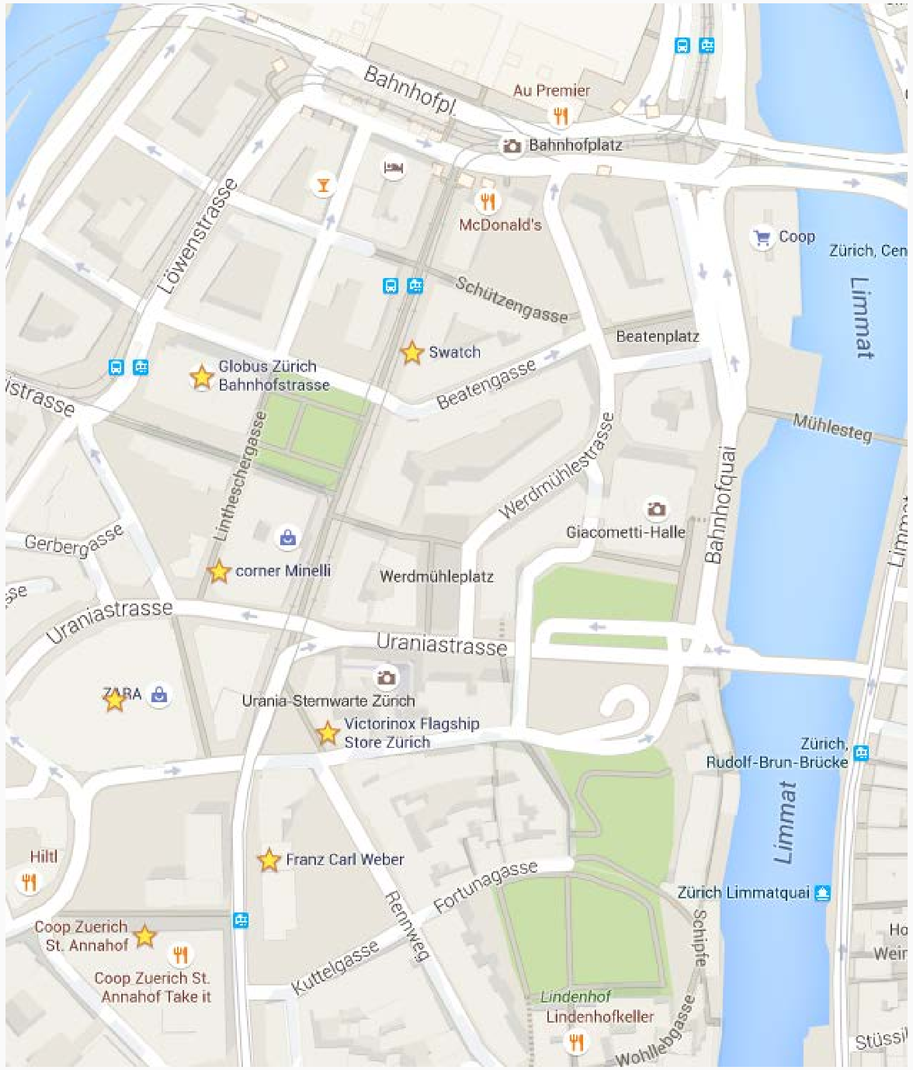
- What if sending flow comes with a price?



Minimum Cut

Minimum Cut: Shopping Trip

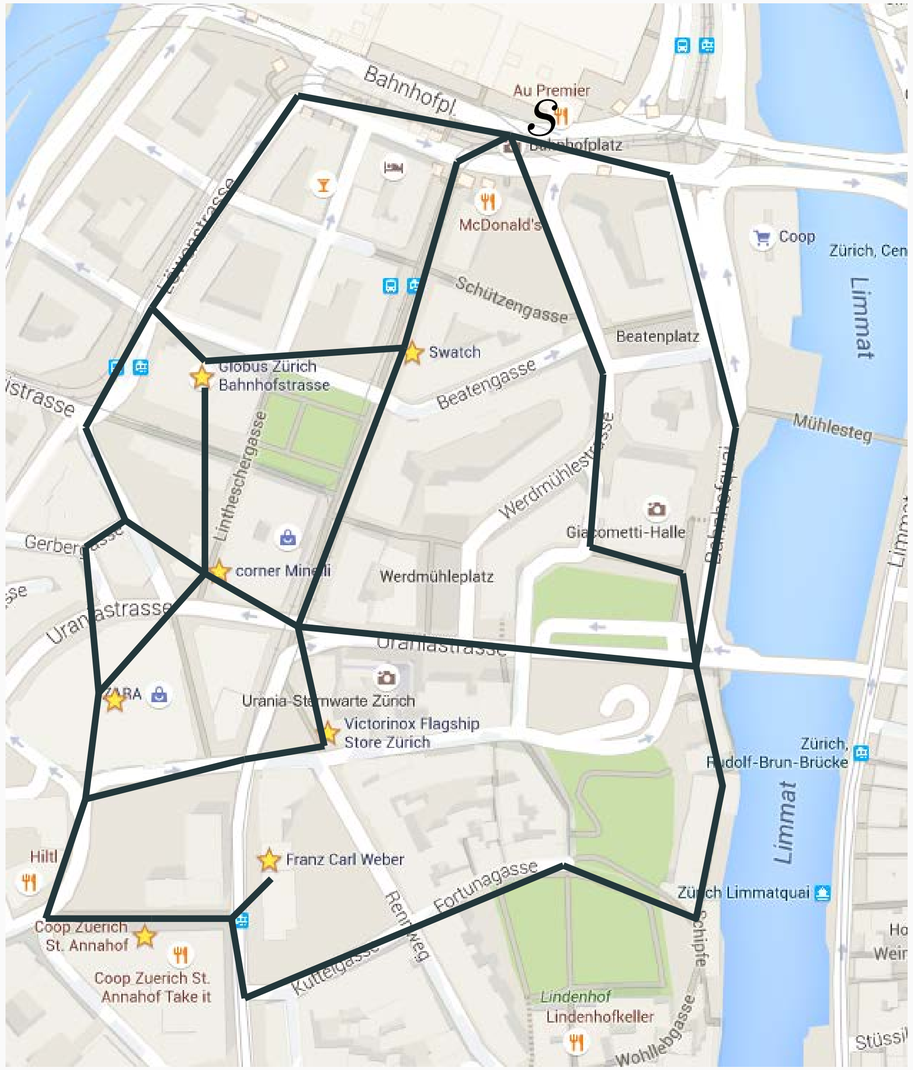
Minimum Cut: Shopping Trip



Start from HB:

- Visit as many shops as possible.
- Return to HB after each shop.

Minimum Cut: Shopping Trip



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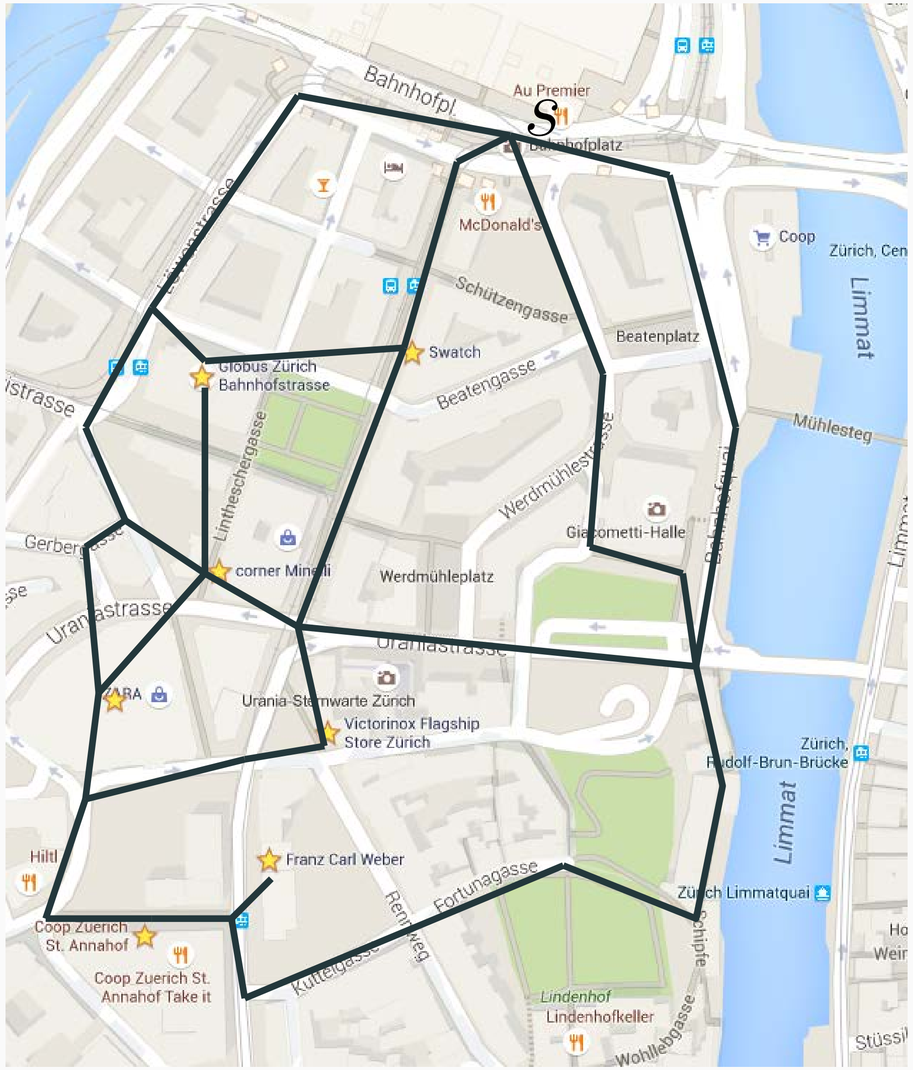

Start from HB:

- Visit as many shops as possible.
- Return to HB after each shop.

Condition: Use each road on at most one trip.

Minimum Cut: Shopping Trip

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Minimum Cut: Shopping Trip

Start from HB:

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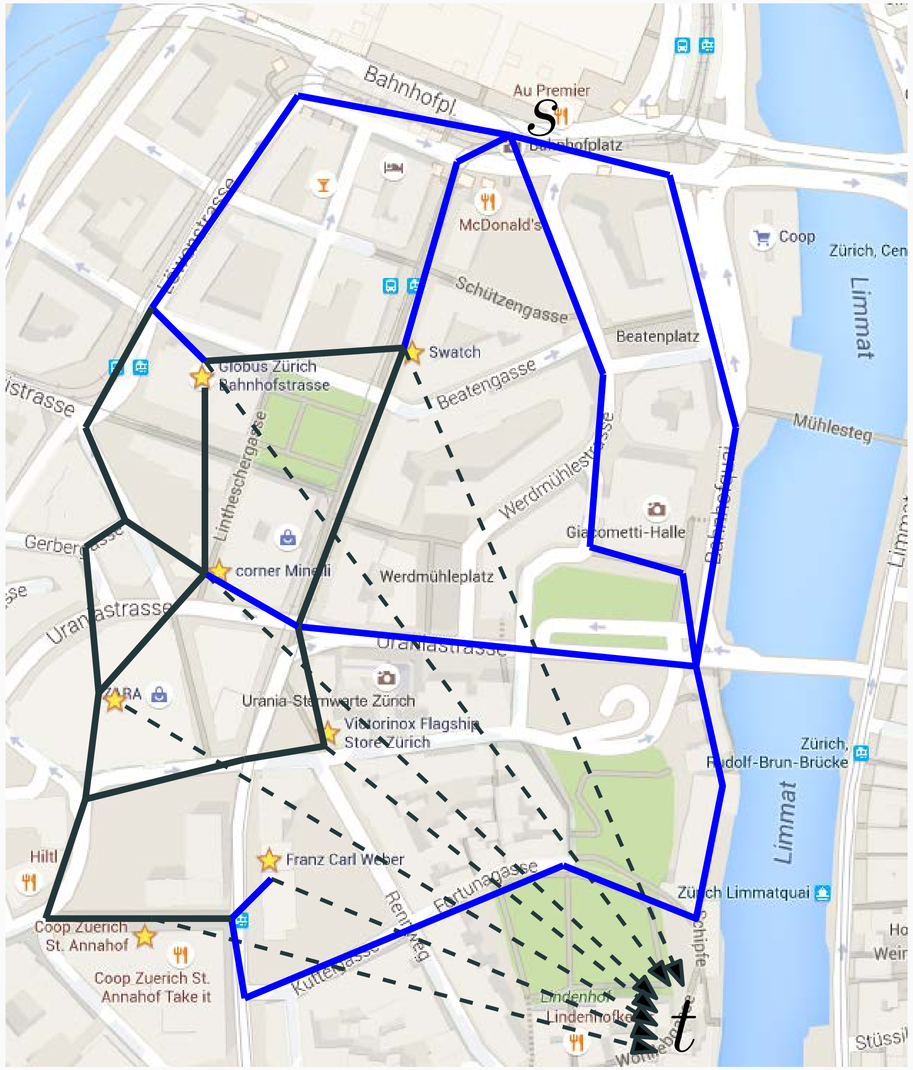
Condition: Use each road on at most one trip.

Compute **the bottleneck**, i.e. the number of **edge-disjoint paths**.

Minimum Cut: Shopping Trip



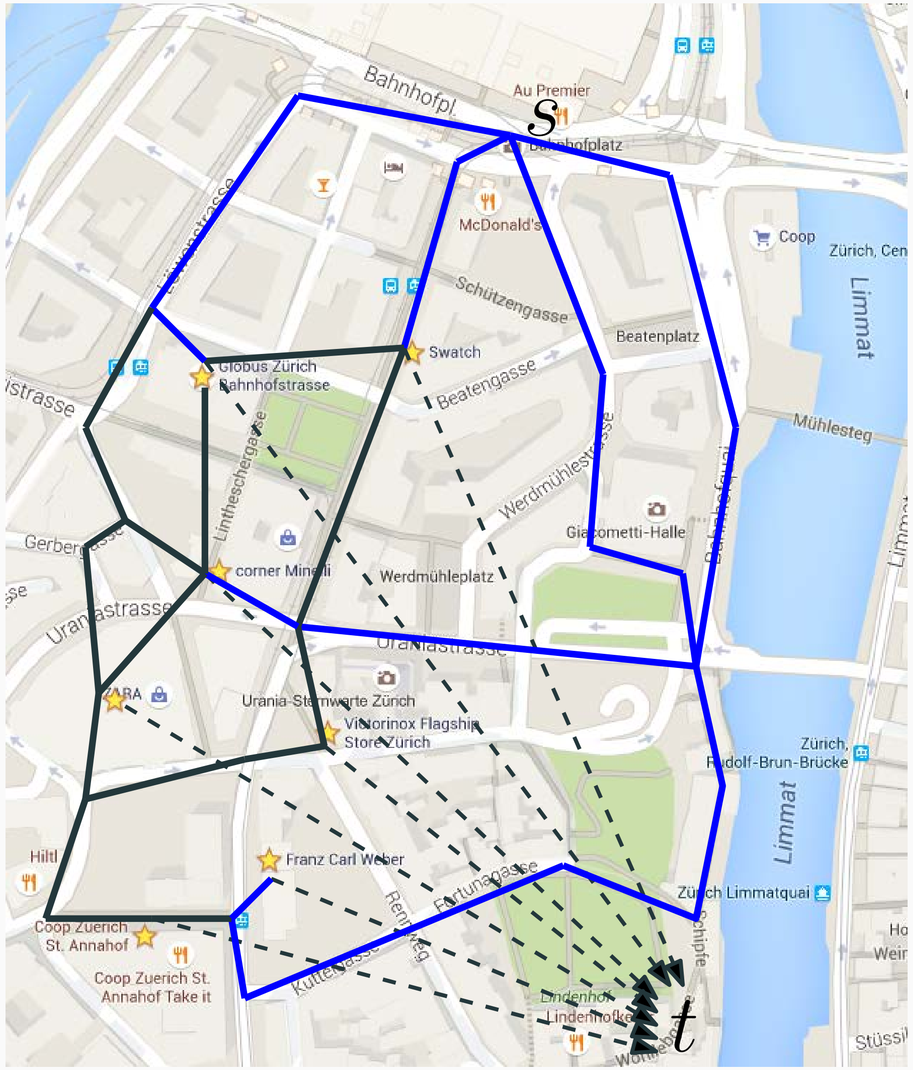
Start from HB:

- Visit as many shops as possible.
- Return to HB after each shop.

Condition: Use each road on at most one trip.

Compute **the bottleneck**, i.e. the number of **edge-disjoint paths**. $\Rightarrow$ Four shops.

Minimum Cut: Shopping Trip



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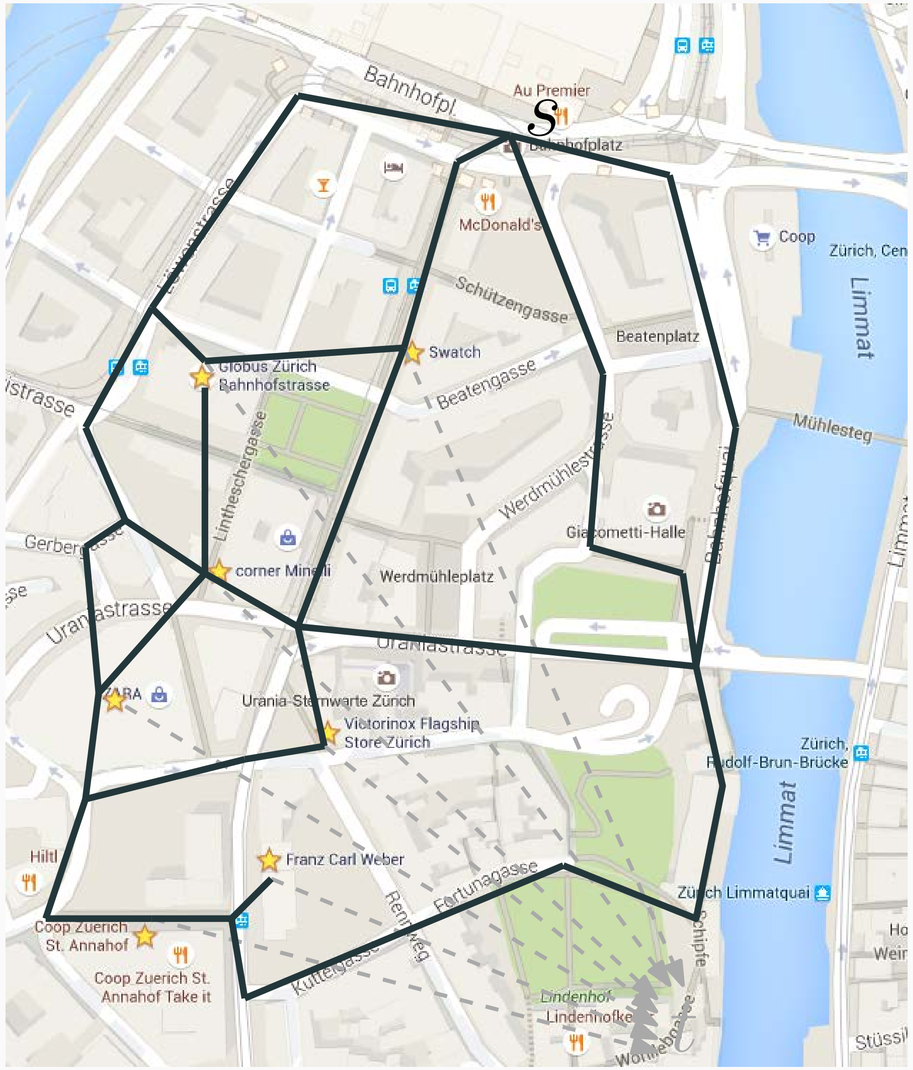
Condition: Use each road on at most one trip.

Compute **the bottleneck**, i.e. the number of **edge-disjoint paths**. $\Rightarrow$ Four shops.

Unrealistic condition!

(There are interesting streets in Zürich.)

Minimum Cut: Shopping Trip



Start from HB:

- Visit as many shops as possible.
- Return to HB after each shop.

Condition: Use beautiful roads more often.

Minimum Cut: Shopping Trip

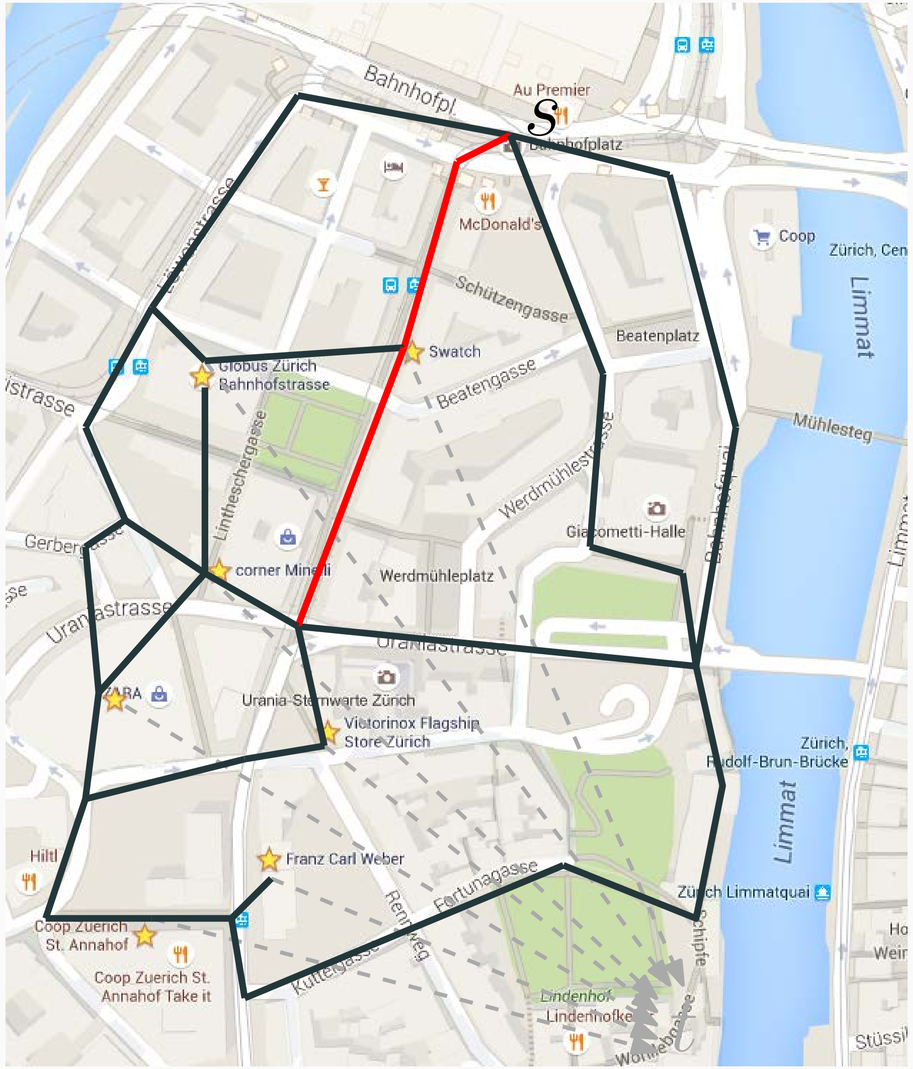
Start from HB:

- Visit as many shops as possible.
- Return to HB after each shop.

Condition: **Use beautiful roads more often.**

Use route via Lindenhof at most twice.

Minimum Cut: Shopping Trip



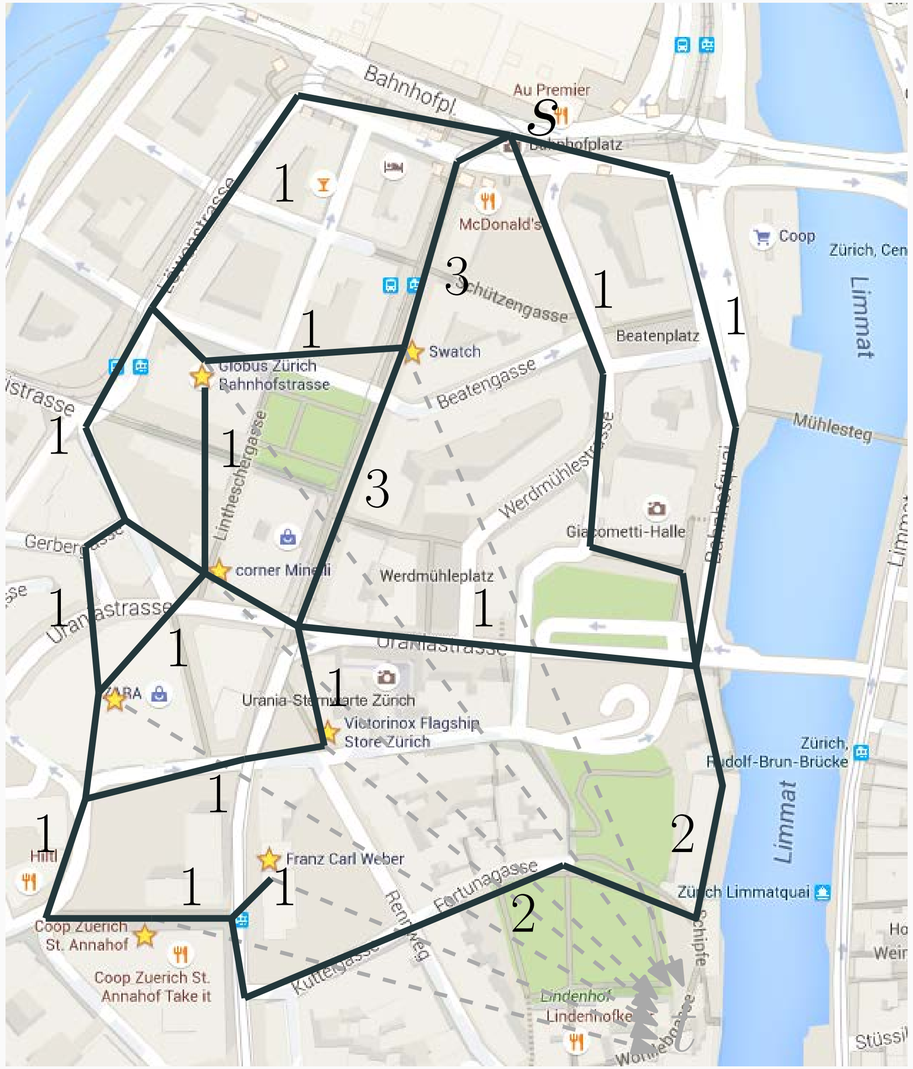
Start from HB:

- Visit as many shops as possible.
- Return to HB after each shop.

Condition: **Use beautiful roads more often.**

Use Bahnhofstrasse up to three times.

Minimum Cut: Shopping Trip



Start from HB:

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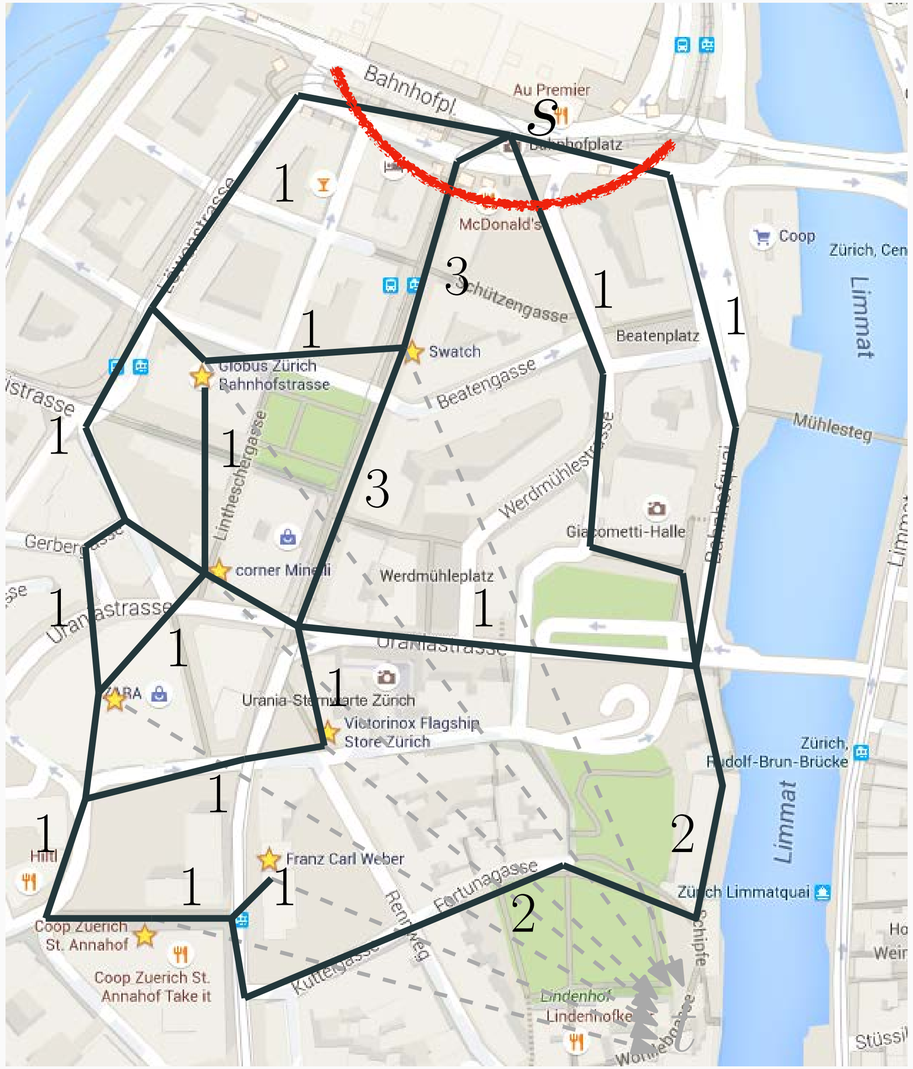
Compute **the weighted bottleneck**, i.e. the **minimum cut between s and t**.

Minimum Cut: Shopping Trip



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- Visit as many shops as possible.
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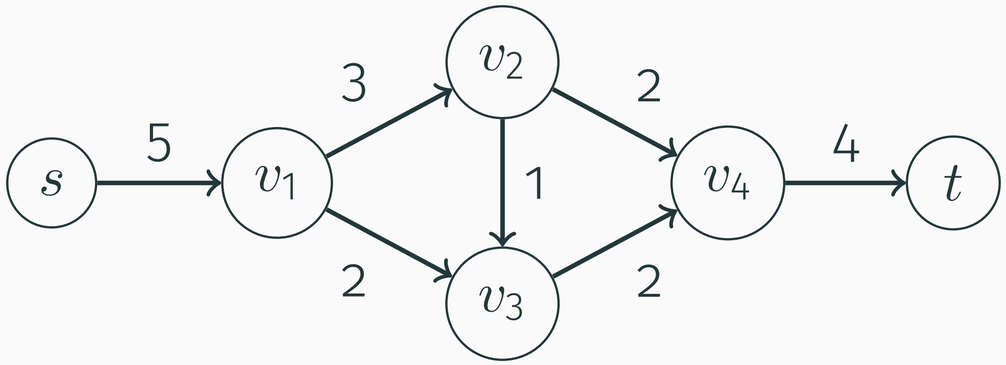
Condition: **Use beautiful roads more often.**

Use Bahnhofstrasse up to three times.

Compute **the weighted bottleneck**, i.e. the **minimum cut between s and t.** $\Rightarrow$ 6 shops.

Minimum Cut: Cuts and Flows

$G = (V, E, s, t)$ a flow network. $S \subset V$ s.t. $s \in S, t \in V \setminus S$,



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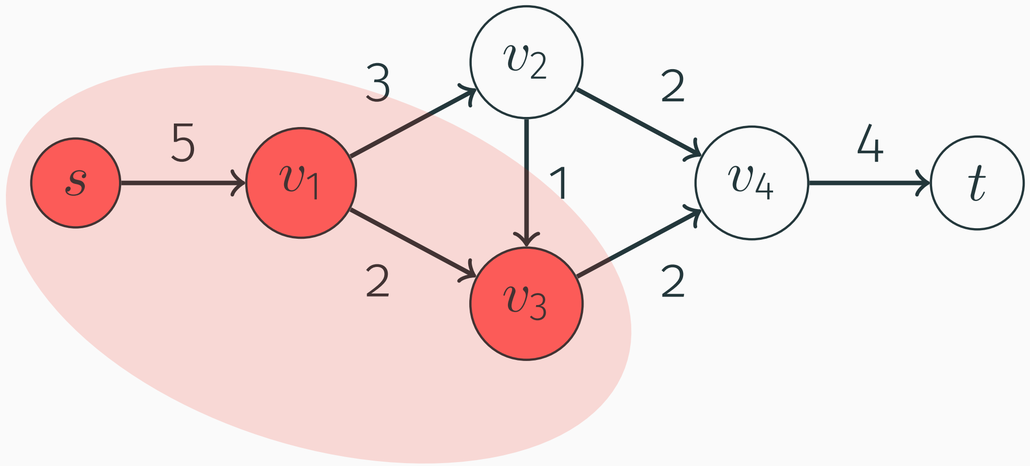

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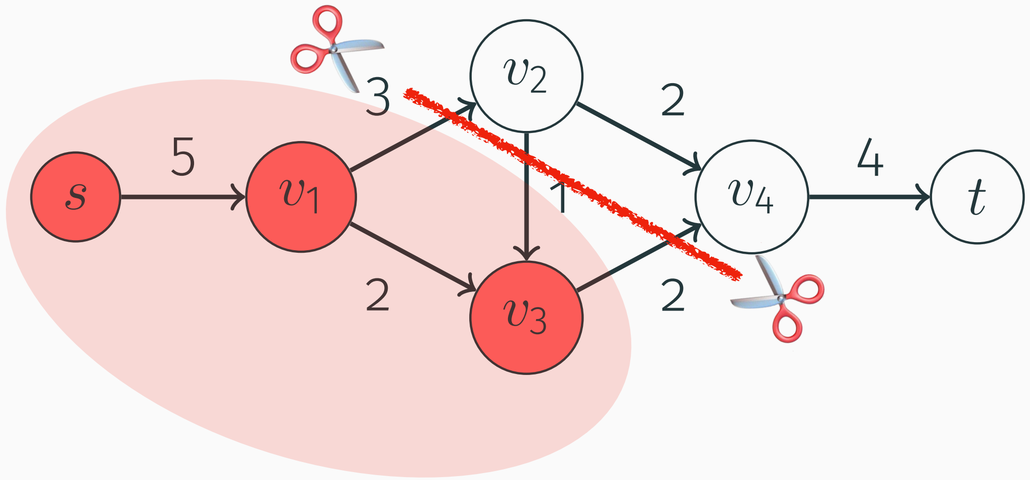
The value of the $(S, V \setminus S)$-cut is

$$\text{cap}(S, V \setminus S) := \text{outgoing capacity}$$
$$= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{cap}(e)$$



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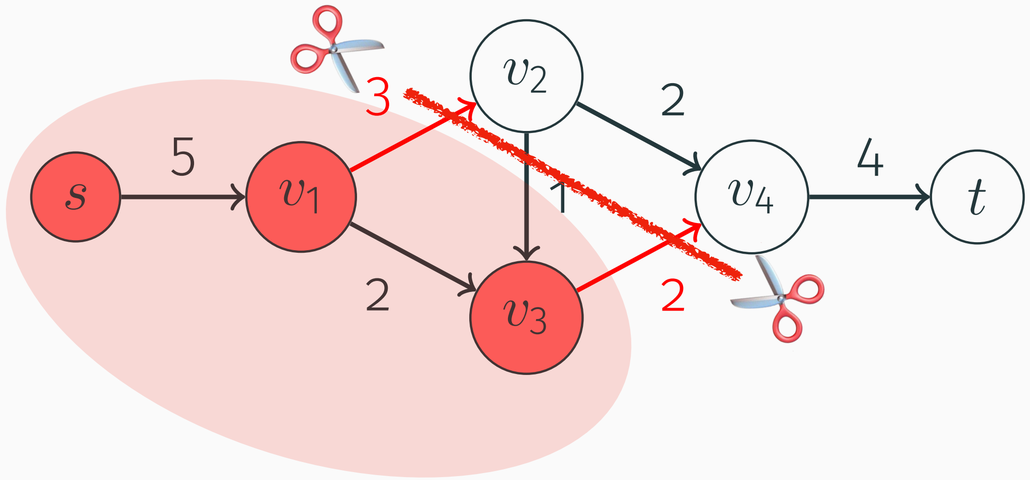

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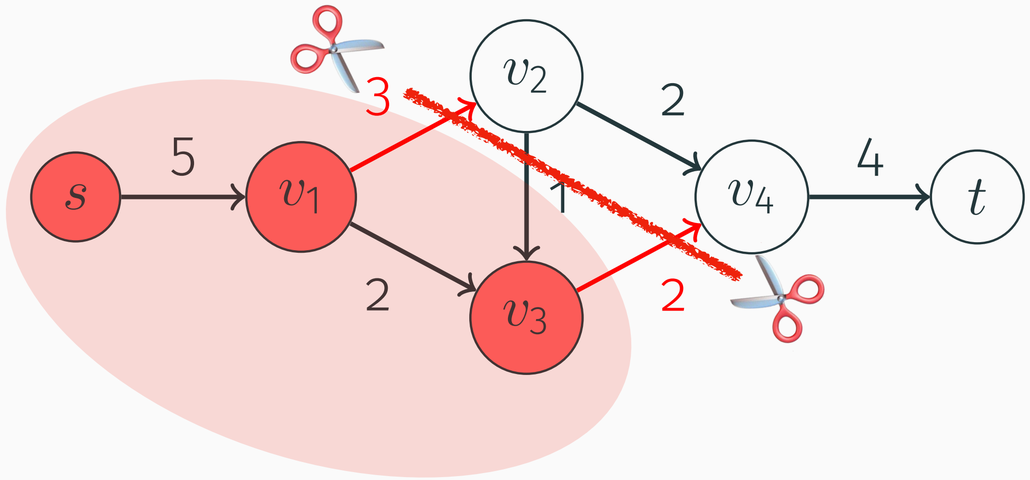
$$\begin{aligned} \text{cap}(S, V \setminus S) &:= \text{ outgoing capacity} \\ &= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{cap}(e) \\ &= 3 + 2 = 5. \end{aligned}$$



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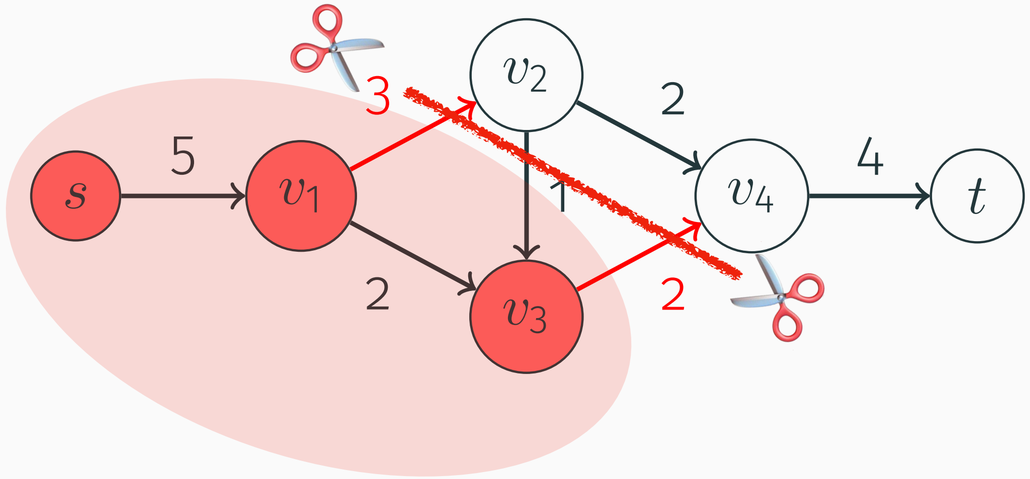
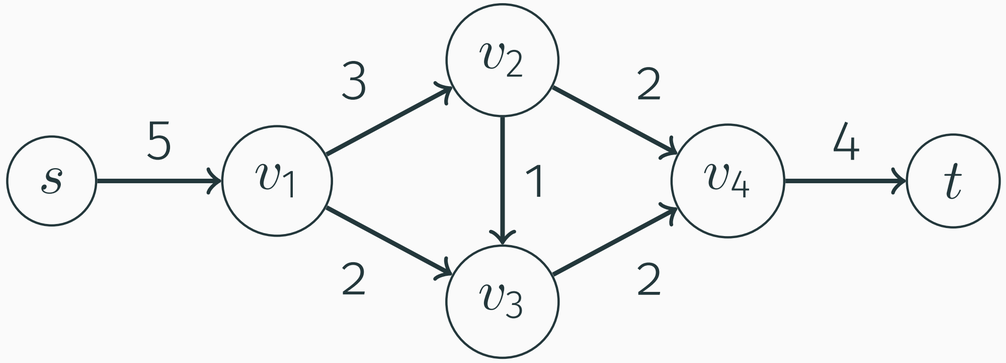


The value of the flow f from S to $V \setminus S$ is

$f(S, V \setminus S) :=$ outgoing flow $-$ incoming flow



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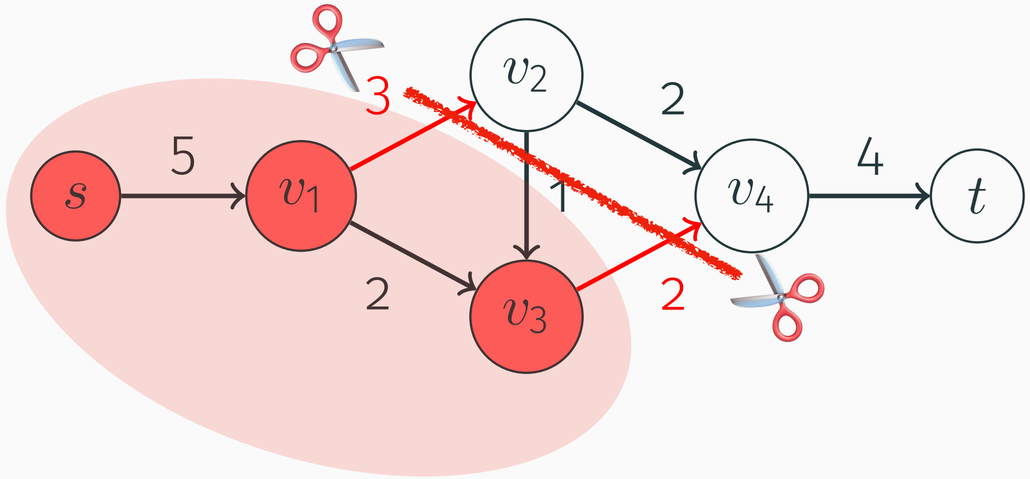

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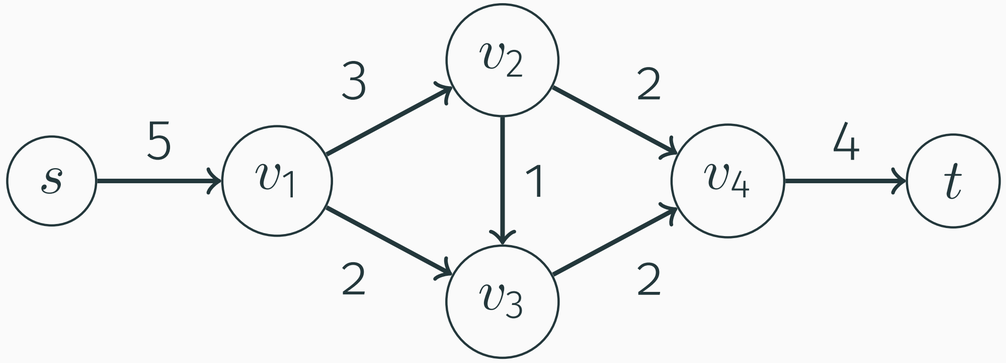

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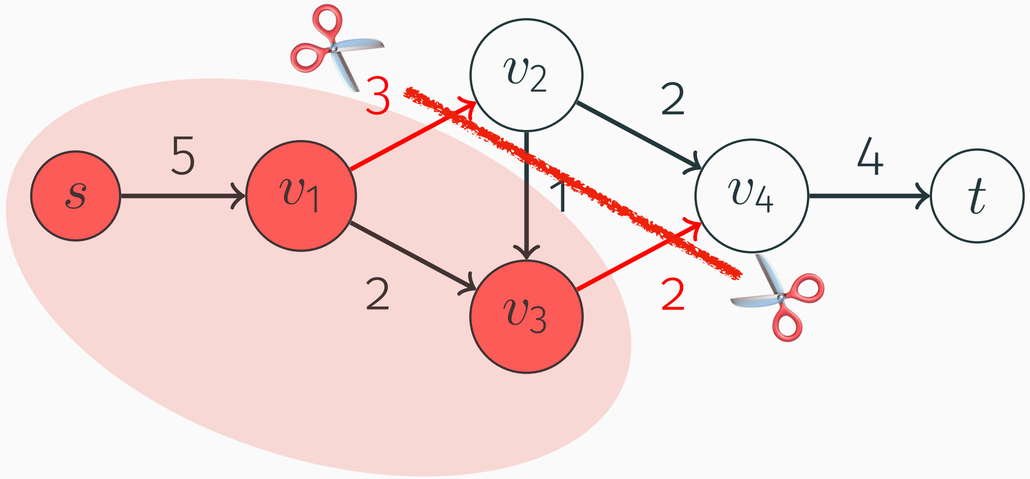

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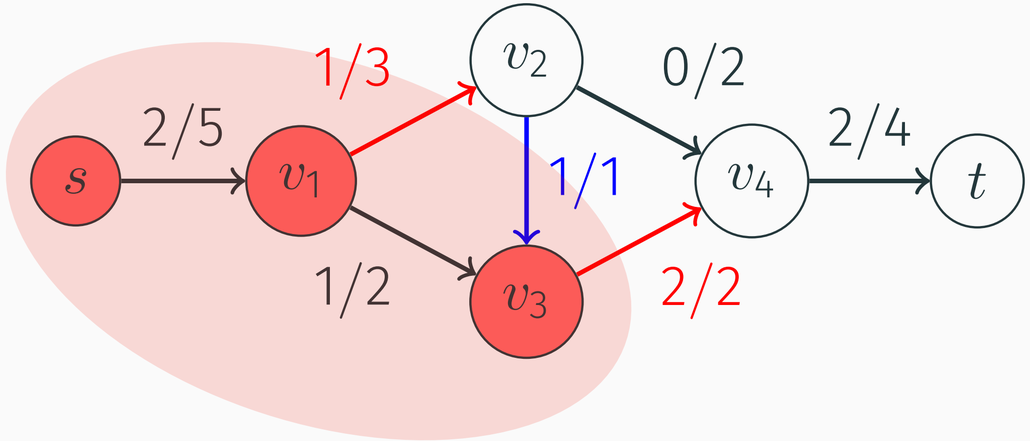

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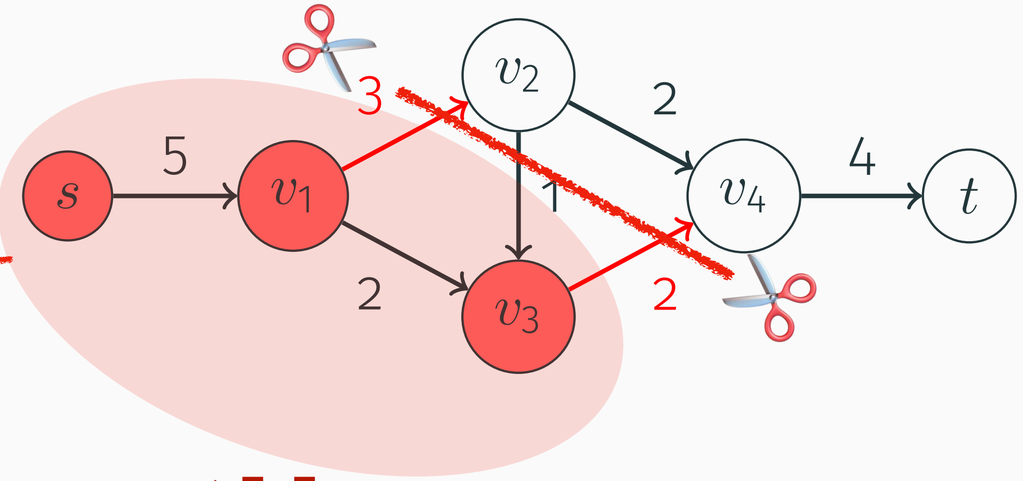
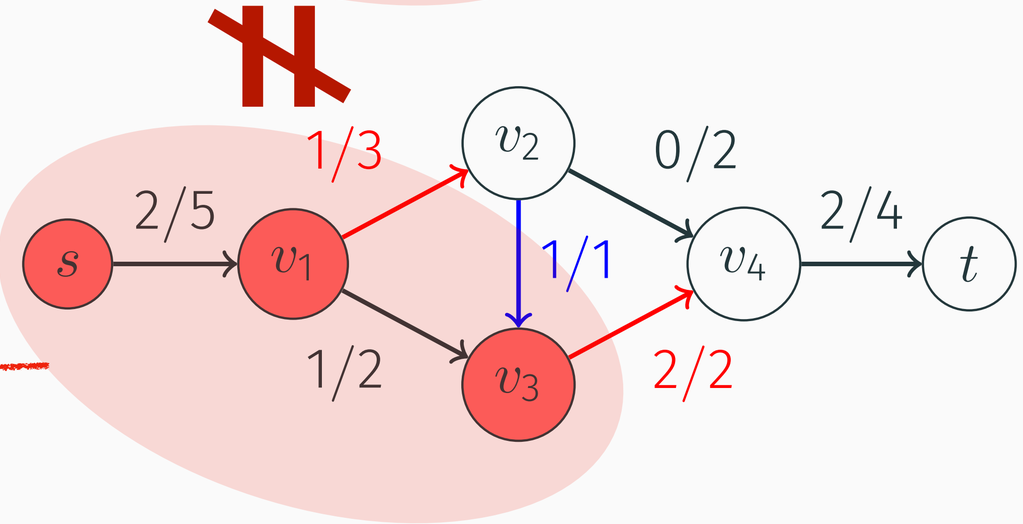


The value of the flow f from S to $V \setminus S$ is

$$\begin{aligned} f(S, V \setminus S) &:= \text{ outgoing flow} - \text{ incoming flow} \\ &= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{flow}(e) - \sum_{\substack{e=(v,u) \\ u \in S, v \in V \setminus S}} \text{flow}(e) \\ &= 1 + 2 - 1 = 2. \end{aligned}$$



Minimum Cut: Maxflow-Mincut-Theorem

Theorem (Maxflow-Mincut-Theorem)

Let f be an s-t-flow in a graph G. Then f is a maximum flow if and only if

$$|f| = \min_{S\colon\, s\in S, t\notin S} cap(S, V \setminus S).$$

This allows us to easily find a minimum s-t-cut:

- Construct the residual graph $G_f := (V, E_f)$. For each edge $(u, v) \in G$ we have:
  - An edge $(u, v) \in G_f$ with capacity $\text{cap}(e) - f(e)$, if $\text{cap}(e) - f(e) > 0$.
  - An edge $(v, u) \in G_f$ with capacity $f(e)$, if $f(e) > 0$.
- Since f is a maximum flow, there is no s-t path in the residual graph G_f.
- Take S to be all vertices in G_f reachable from s.
  $\Rightarrow (S, V \setminus S)$ is a minimum s-t-cut.

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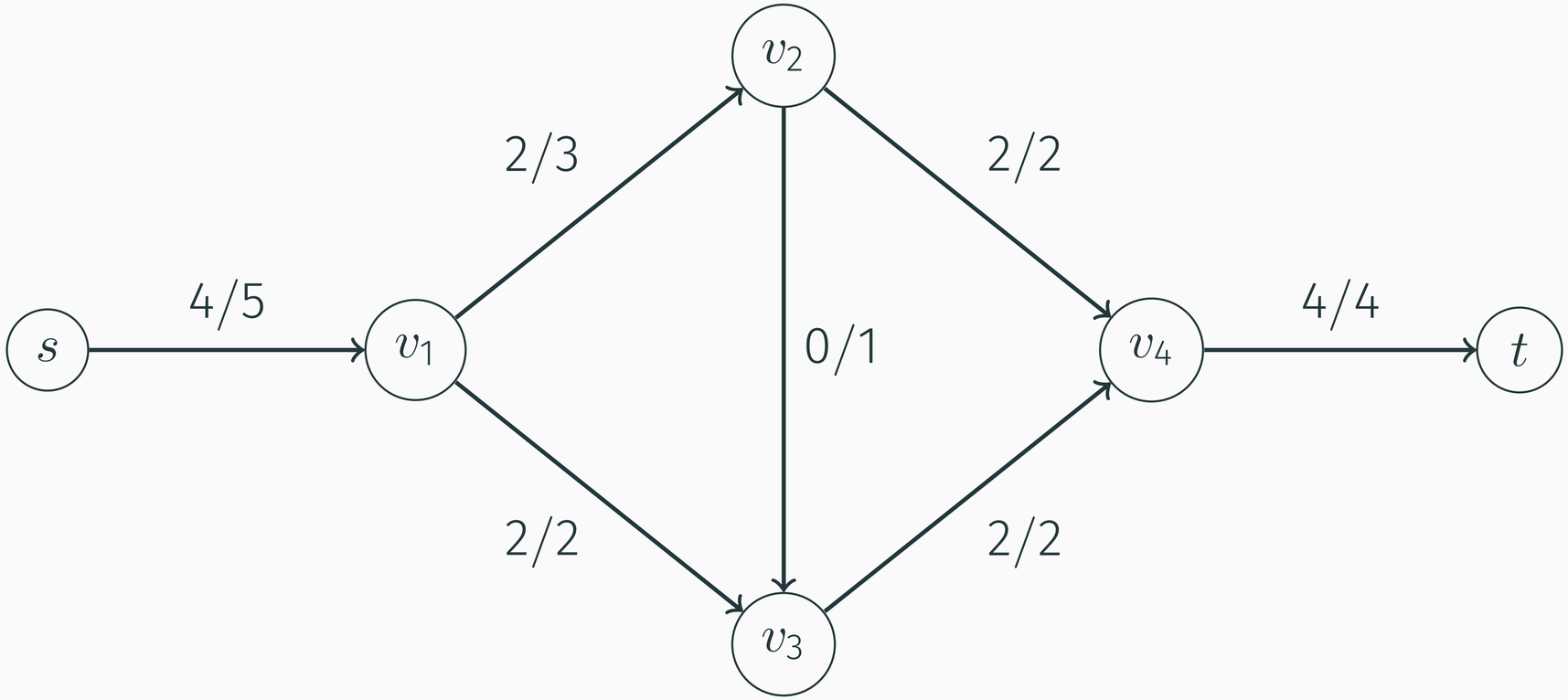
Minimum Cut: Example

Graph G and a maximum flow f.



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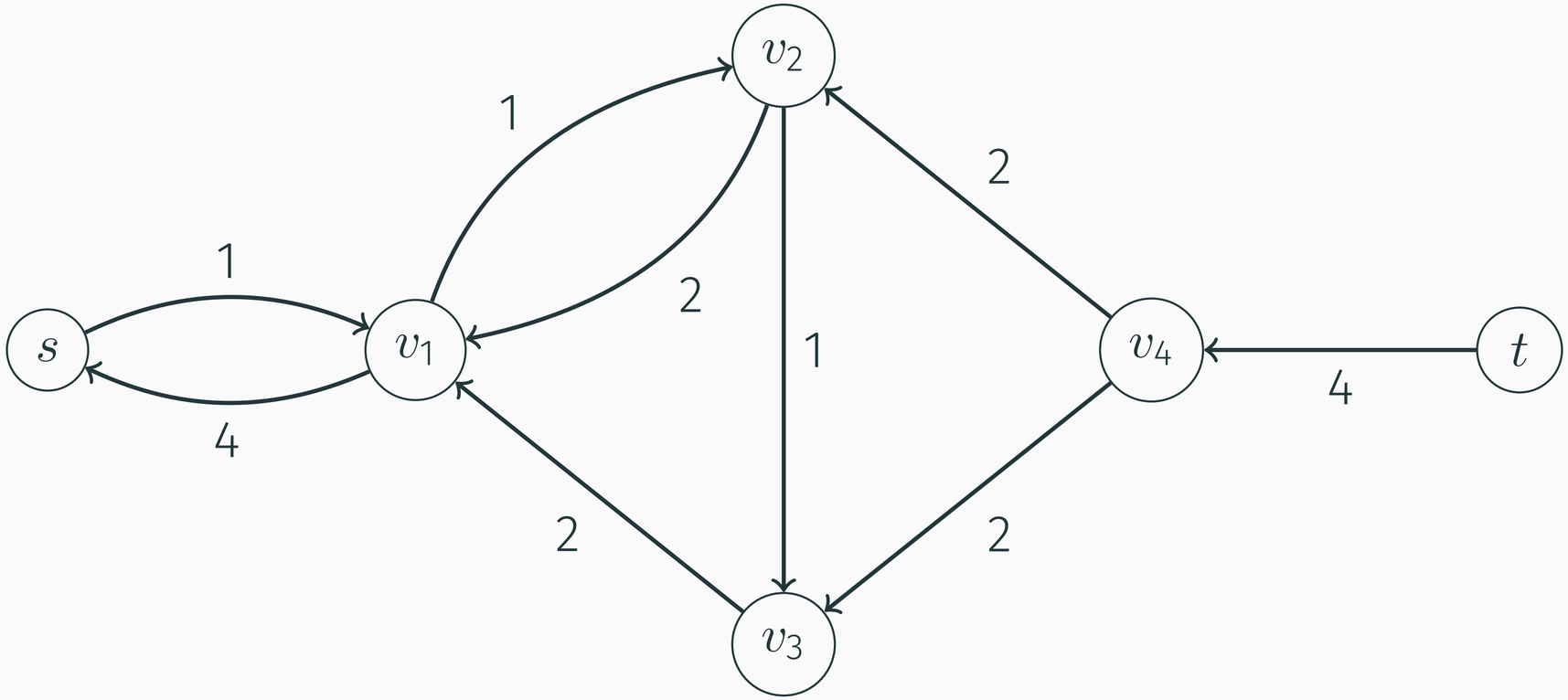
Residual graph G_f.



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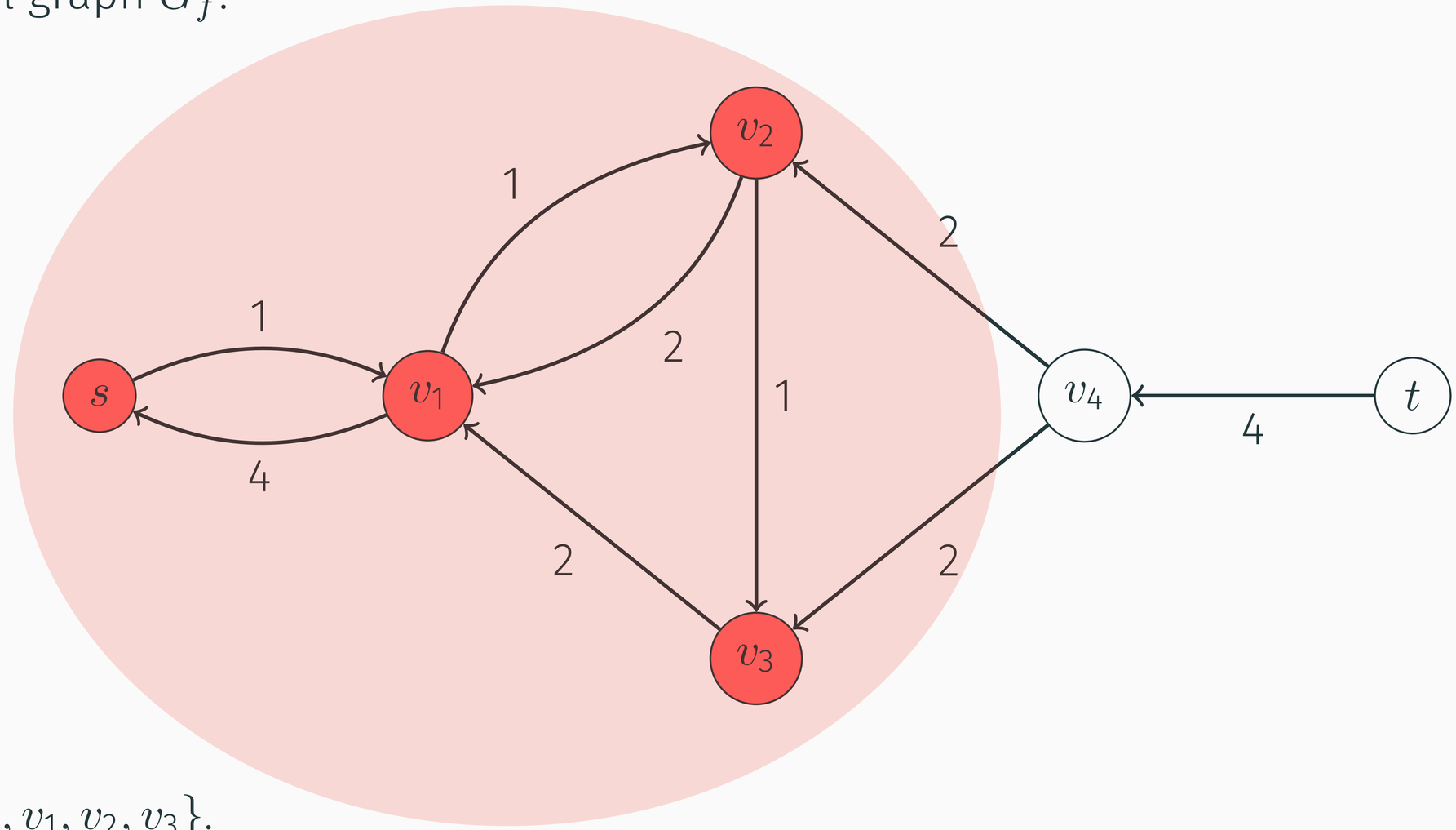

$|S| = \{s, v_1, v_2, v_3\}$.

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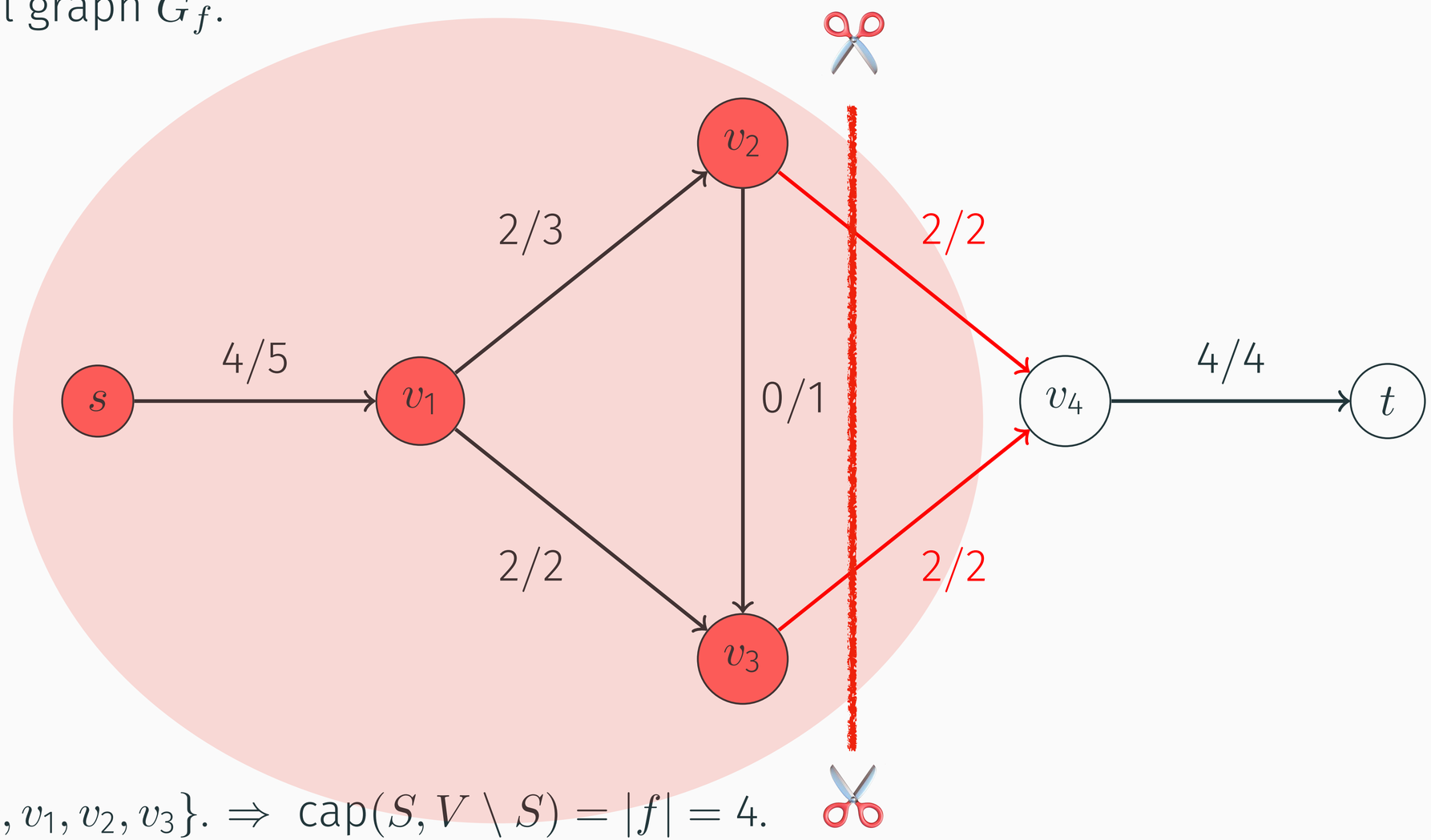


$|S| = \{s, v_1, v_2, v_3\}. \Rightarrow \text{ cap}(S, V \setminus S) = |f| = 4.$

Minimum Cut: Code

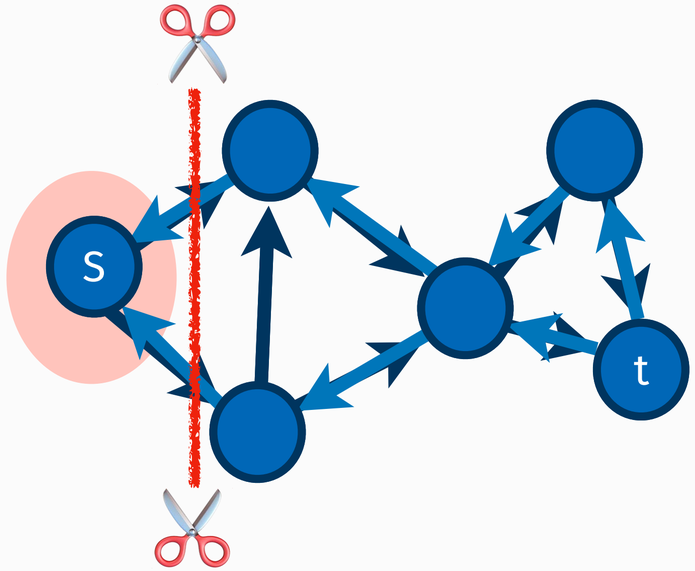
Example code: BFS on the residual graph G_f. → bgl_residual_bfs.cpp

```
90  // BFS to find vertex set S
91  std::vector<int> vis(N, false); // visited flags
92  std::queue<int> Q; // BFS queue (from std:: not boost::)
93  vis[src] = true; // Mark the source as visited
94  Q.push(src);
95  while (!Q.empty()) {
96      const int u = Q.front();
97      Q.pop();
98      out_edge_it ebeg, eend;
99      for (boost::tie(ebeg, eend) = boost::out_edges(u, G); ebeg != eend; ++ebeg) {
100         const int v = boost::target(*ebeg, G);
101         // Only follow edges with spare capacity
102         if (rc_map[*ebeg] == 0 || vis[v]) continue;
103         vis[v] = true;
104         Q.push(v);
105     }
106 }
```

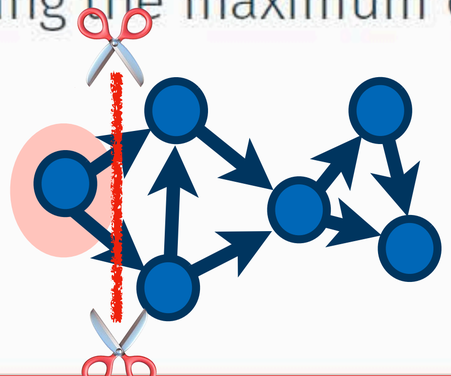
Minimum Cut: Algorithm

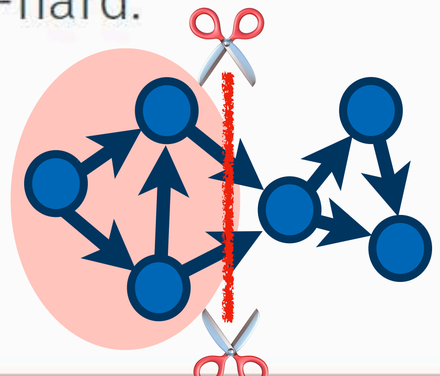
Summary of what you need to do to find a minimum cut:

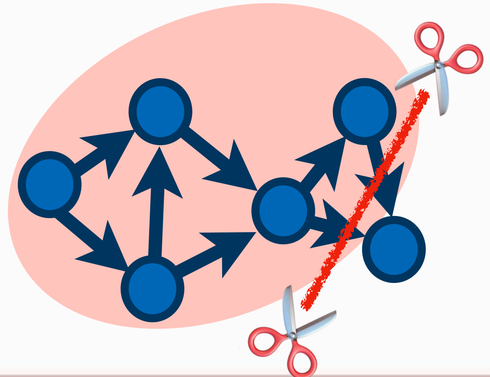
1. Compute maximum flow f and the residual graph G_f.
2. Compute the set of vertices S:
   - S is reachable from the source s in G_f.
   - BFS on edges with residual capacity > 0.
3. Output (depending on the task):
   - All vertices in S.
   - All edges going from S to $V \setminus S$.



Note: Minimum cuts are not necessarily unique. But earliest and latest min-cuts are. Also computing the maximum cut is NP-hard.





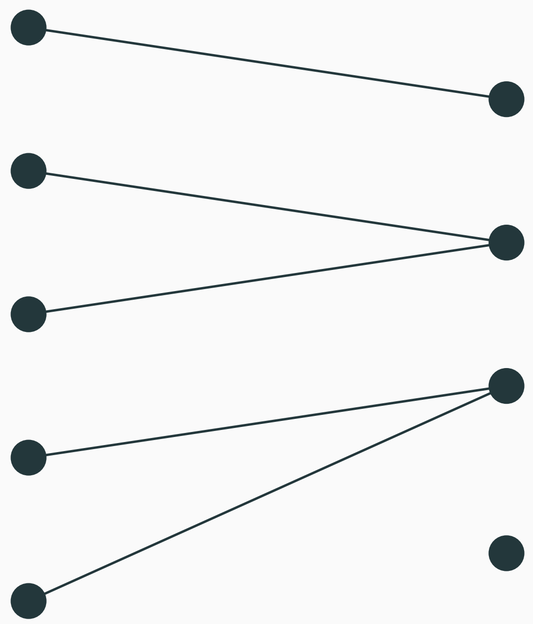


Bipartite Matchings

Maximum Matchings: Bipartite Graphs

Maximum Matching = pick as many non-adjacent edges as possible

Flow formulation through circulation / vertex capacities / edges-disjoint paths:



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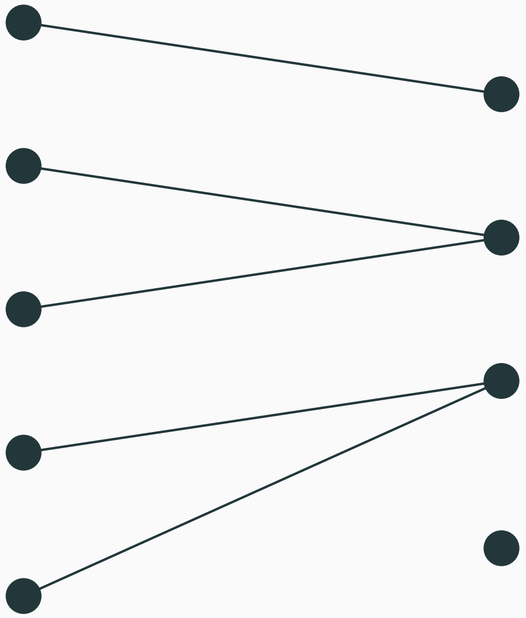

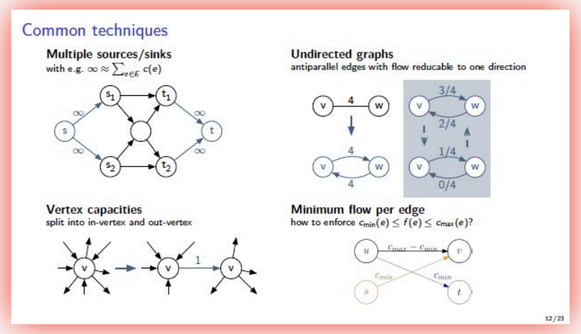

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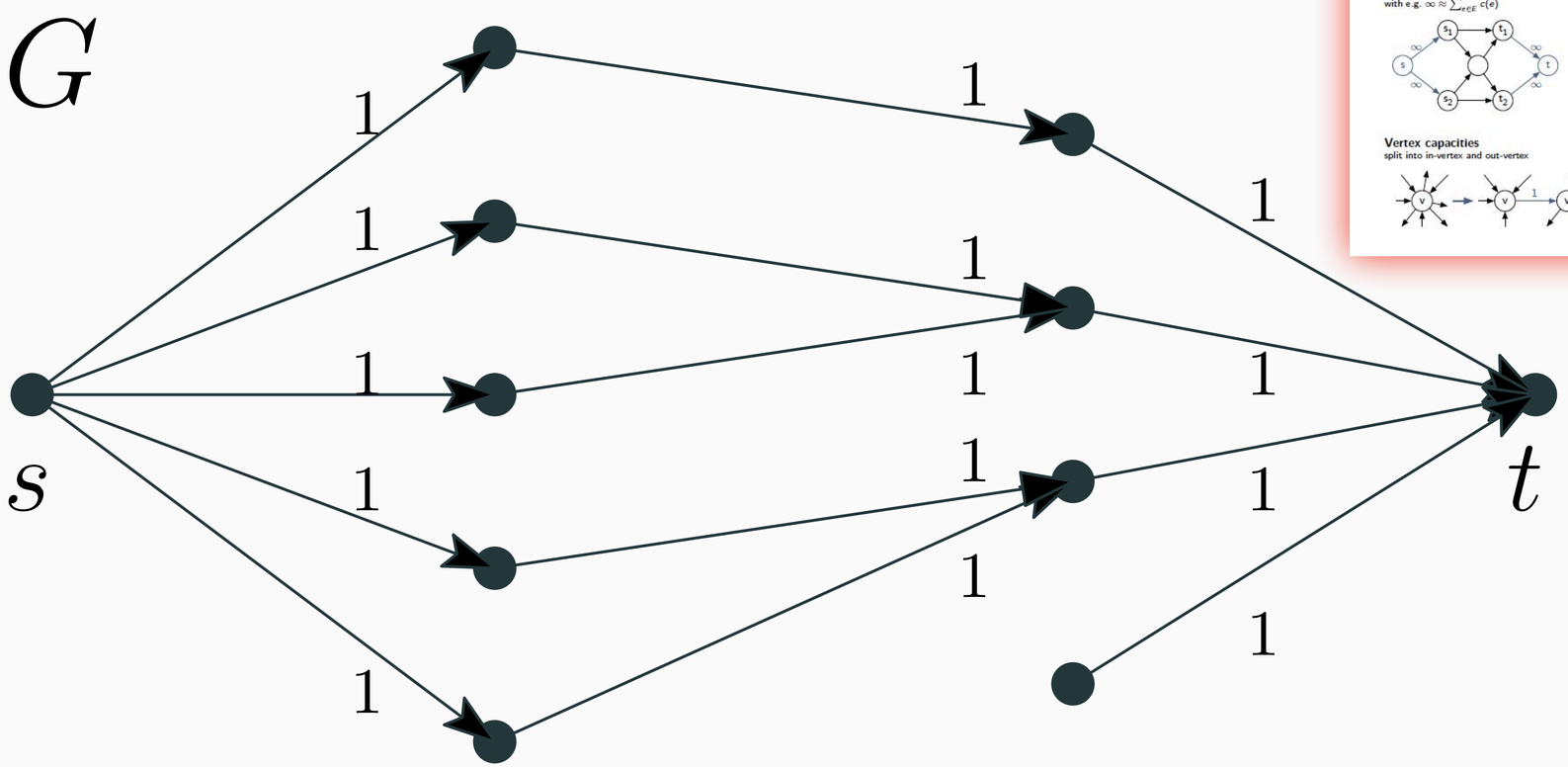

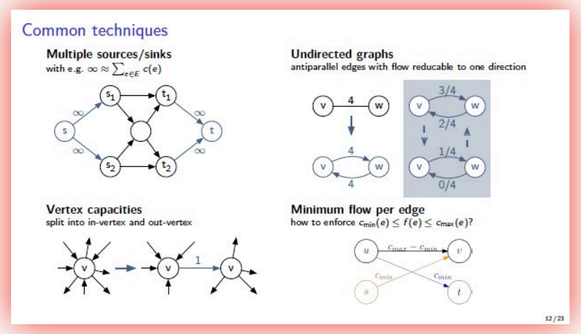

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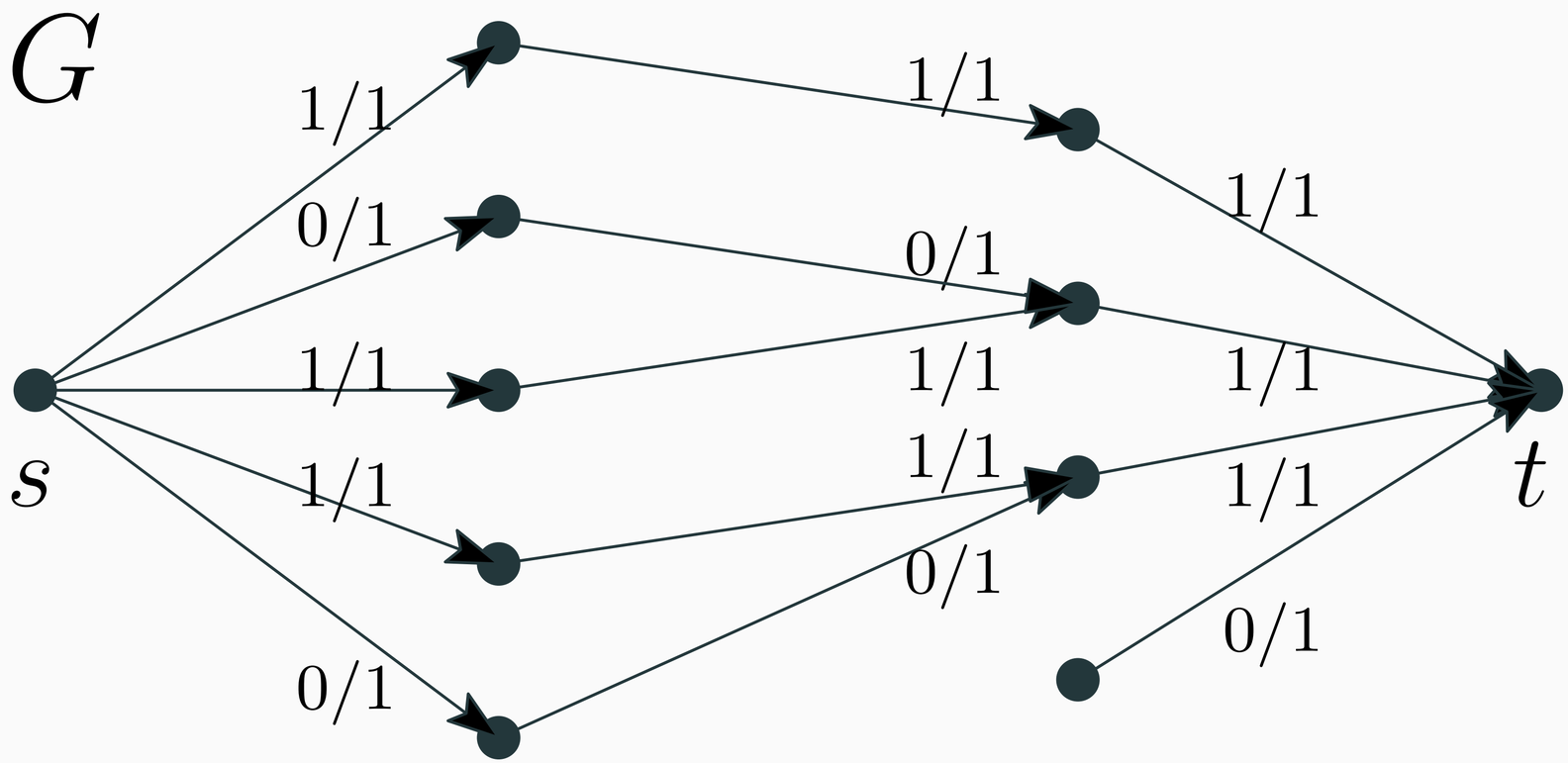

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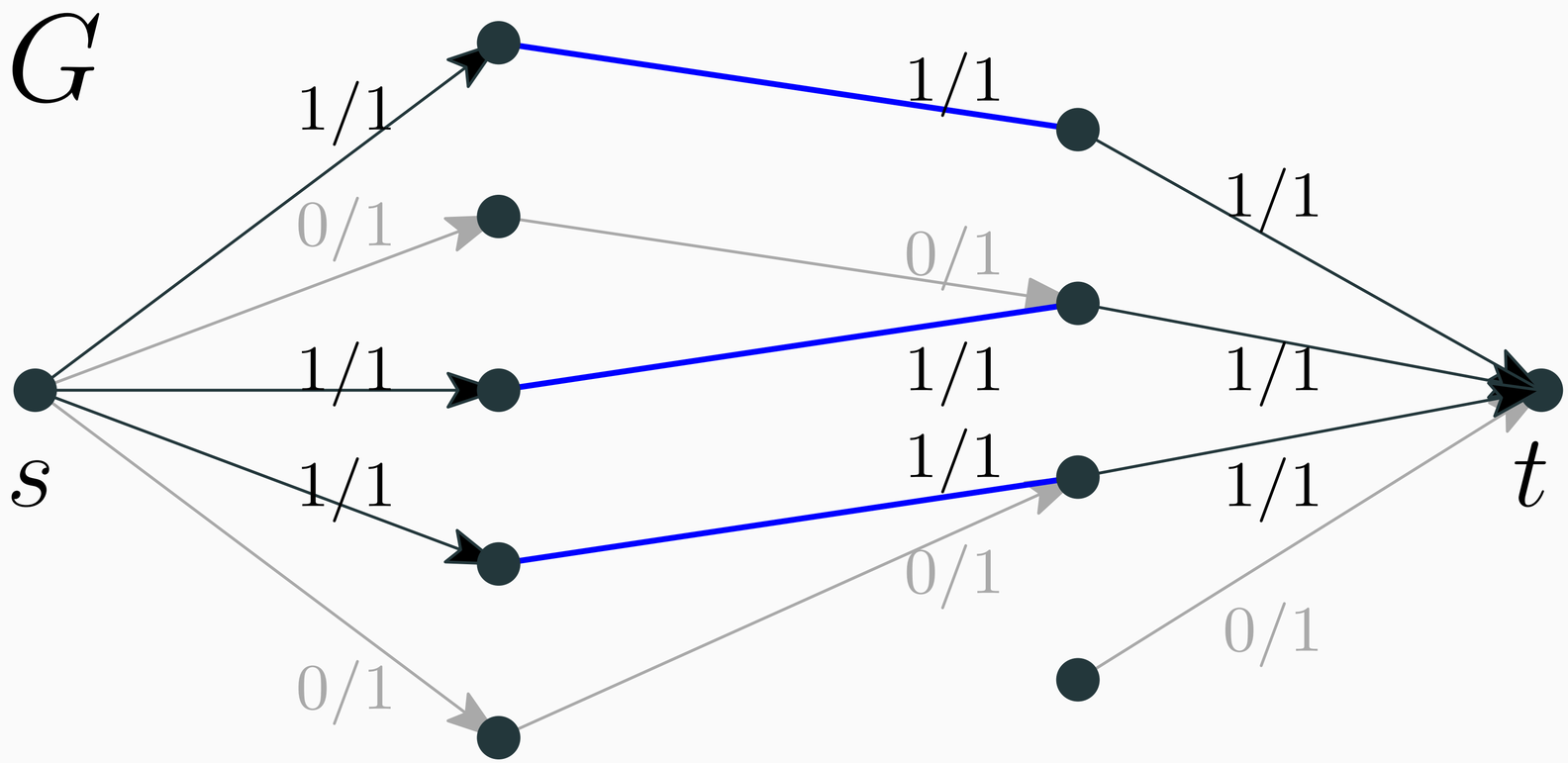
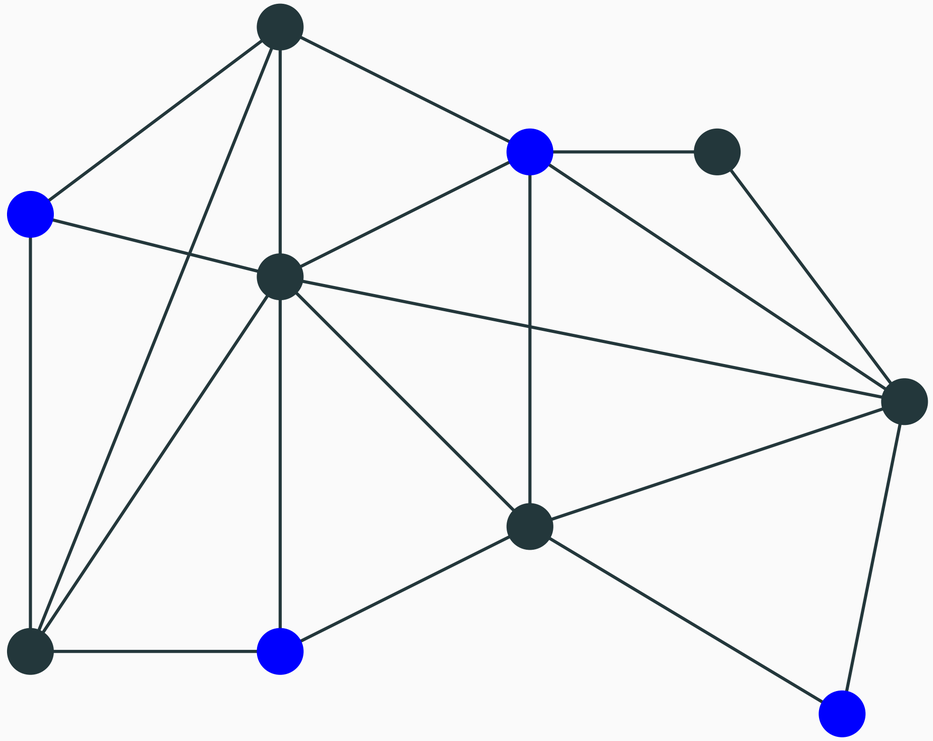


Vertex Cover and Independent Set: General Graphs

- **Maximum independent set (MaxIS)**
  Largest $T \subseteq V$, such that
  $\nexists u, v \in T : (u, v) \in E$.
- **Minimum vertex cover (MinVC)**
  Smallest $S \subseteq V$, such that
  $\forall (u, v) \in E : u \in S \vee v \in S$.
- These problems are complementary!
  MaxIS $= V \setminus$ MinVC



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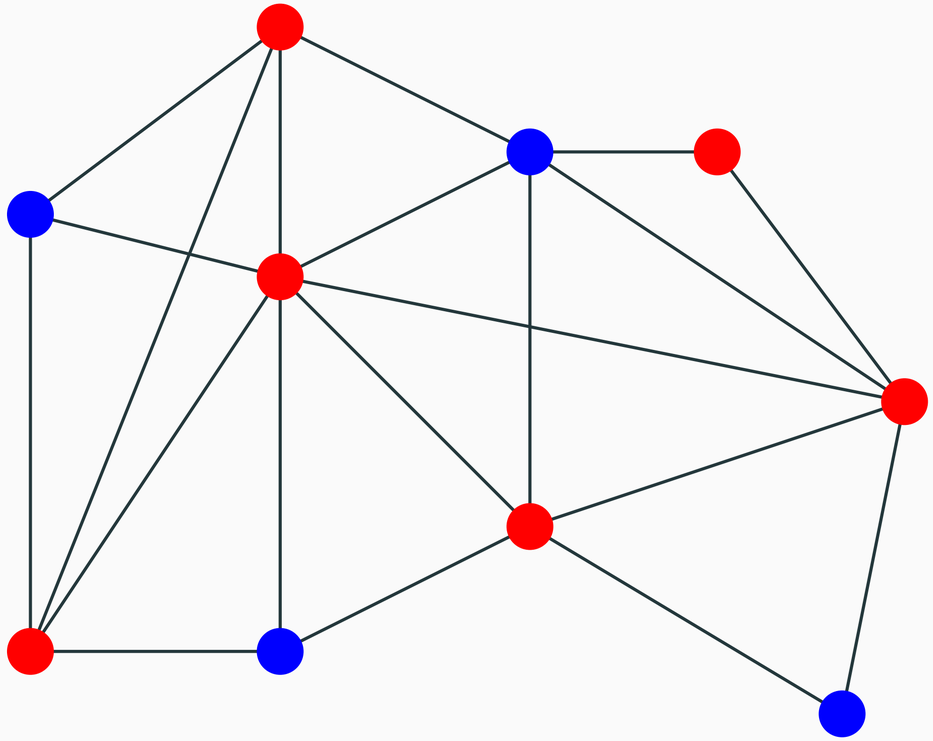

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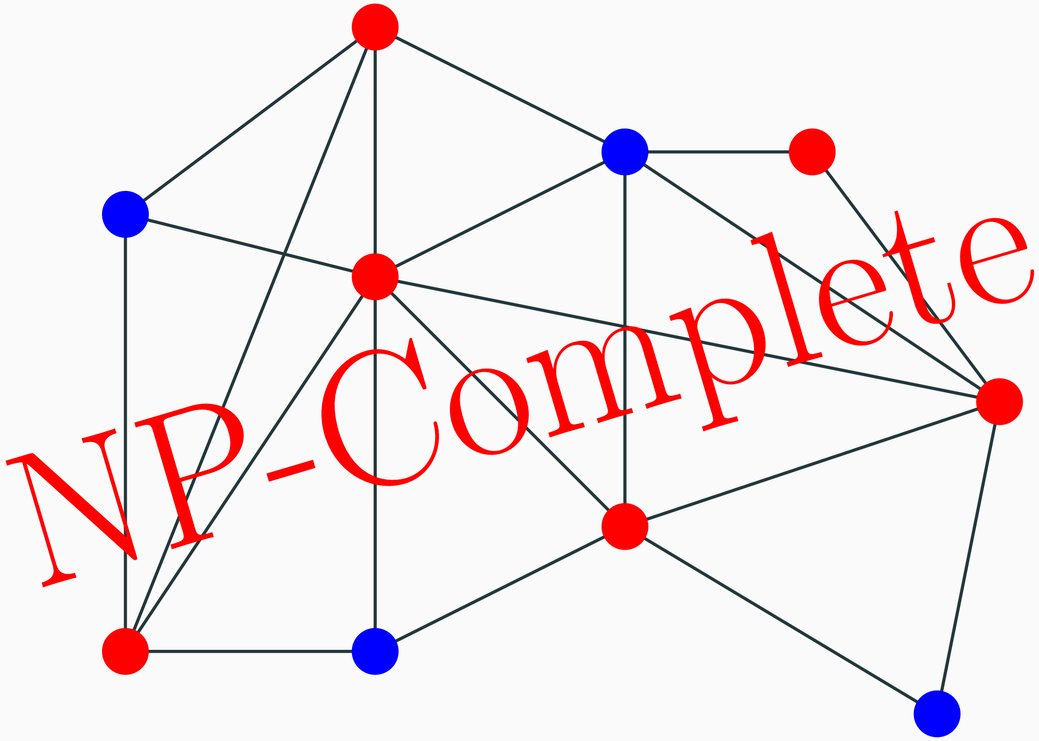

Brief Excursion: Options for Runtime Analysis

By now, we know many ways of deciding whether an algorithm is fast or slow:

- look at the input size (the classical way)
- look at the output size (e.g. fast as the answer is guaranteed to be small)
  If you know that MaxIS is very small, then it might be tractable while computing a MinVC directly is too slow
- look at some special input restrictions (e.g. Attack of the clones)
- look at detailed structure of the input (e.g. all graphs are trees)

Vertex Cover and Independent Set: Bipartite Graphs

Theorem (König: MinVC and MaxIS is simpler on bipartite graphs!)

In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Proof: See Wikipedia for a nice and short proof.

Algorithm:

1. Maximum matching M, $V = L \cup R$. Find all unmatched vertices in L, label them as visited.
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Careful! Step 2 can take several rounds.
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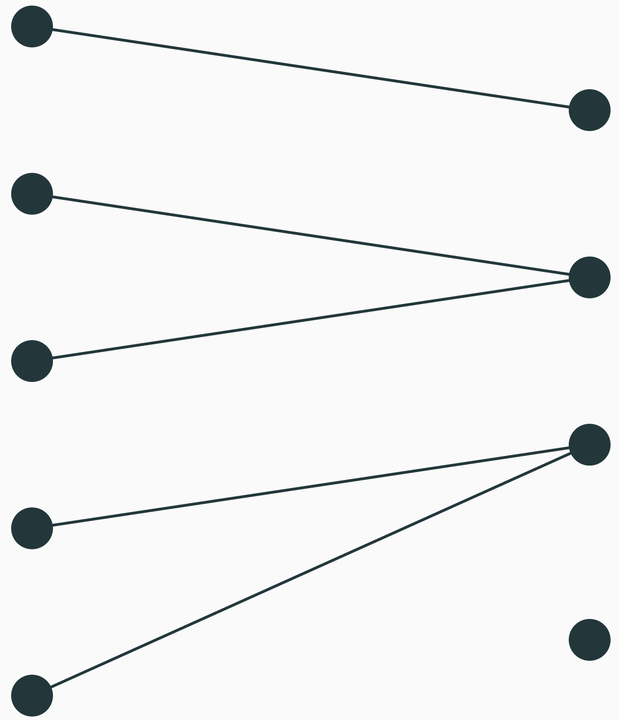

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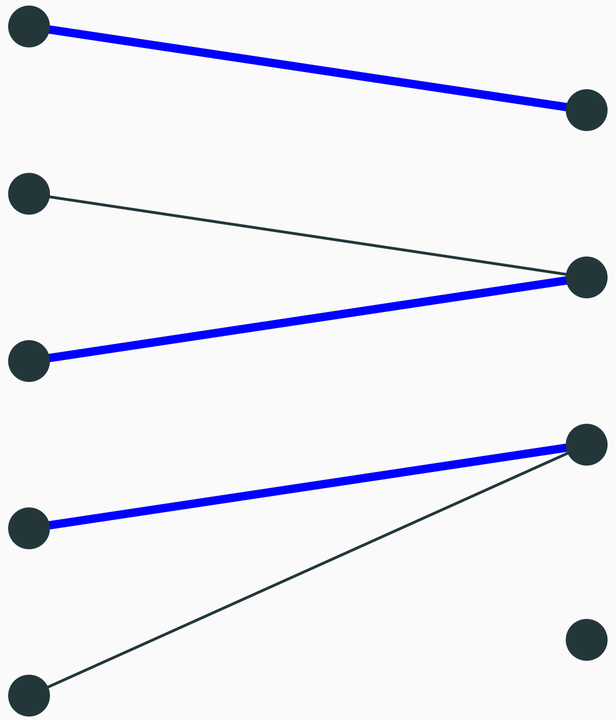

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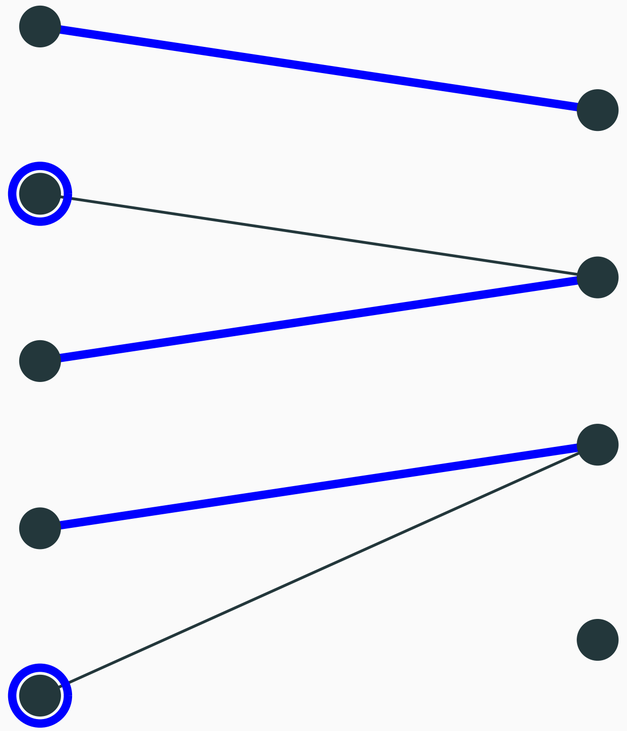

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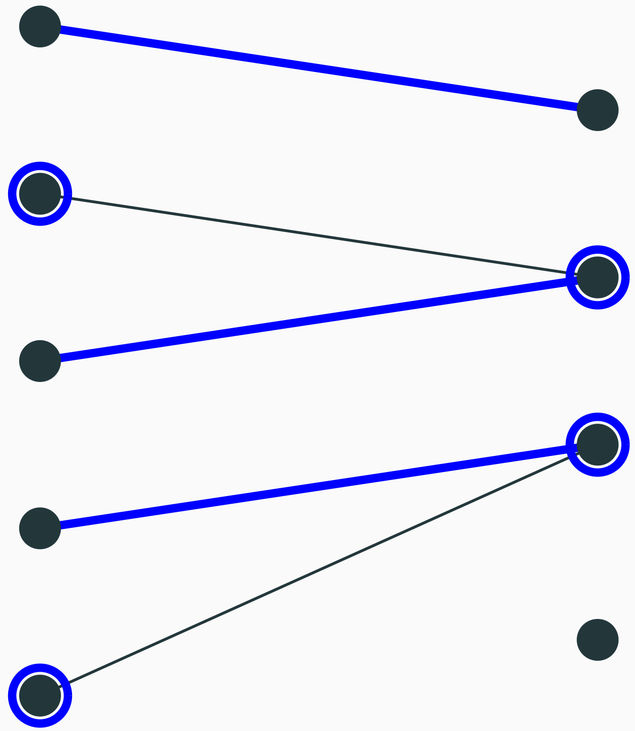

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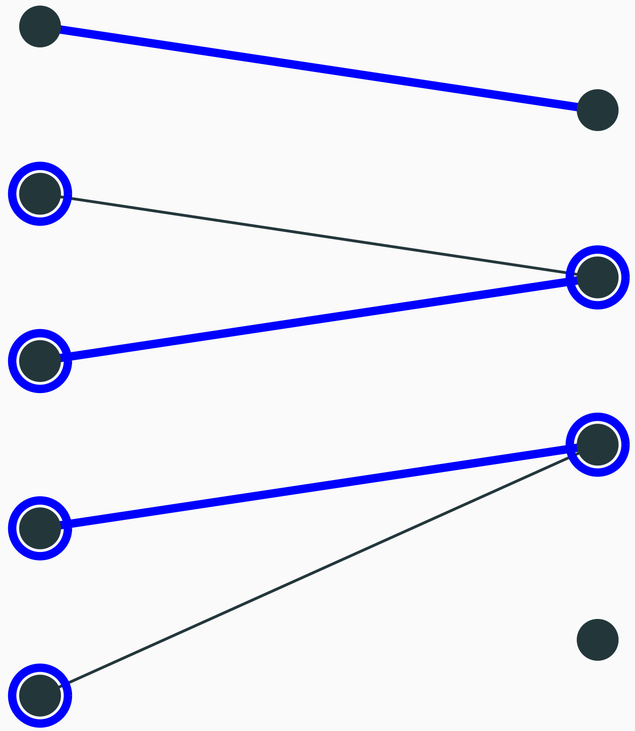

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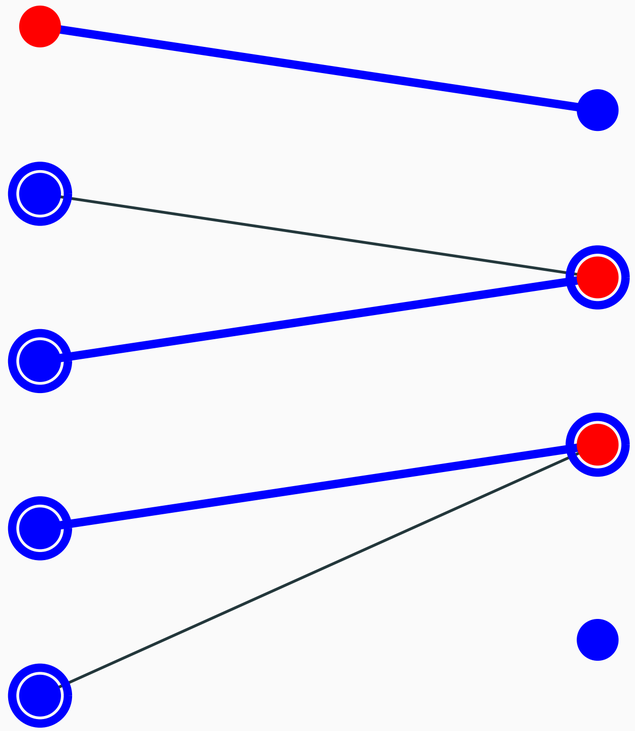

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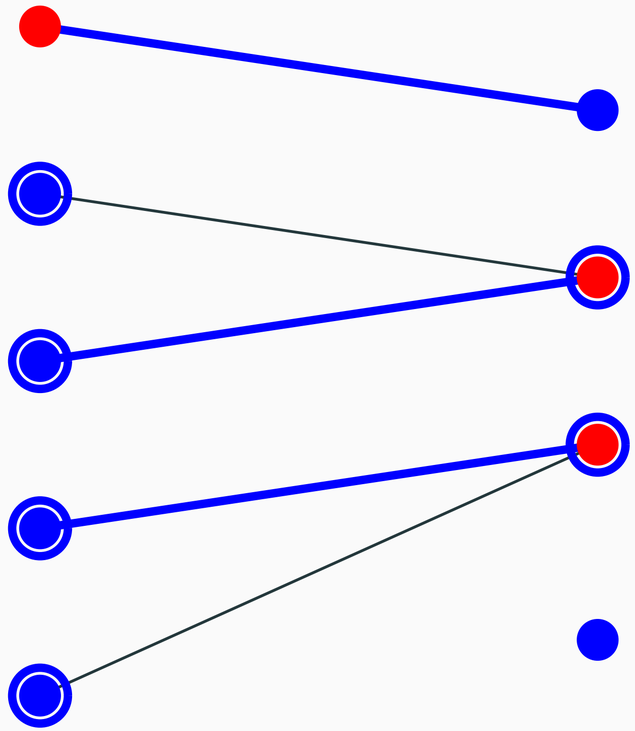

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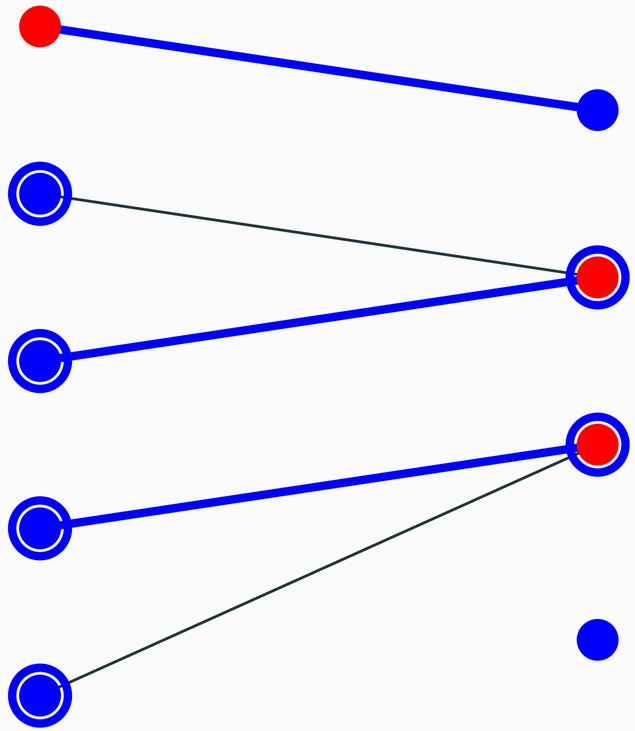

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Finding a MinVC or MaxIS in bipartite graphs: step by step

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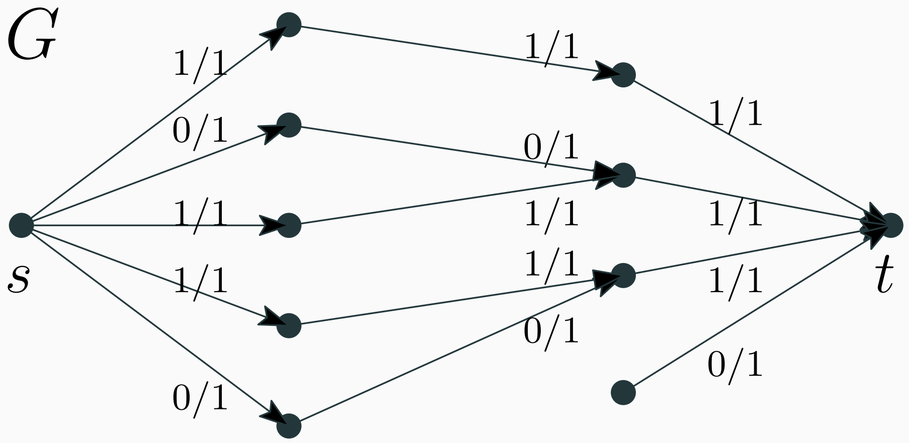

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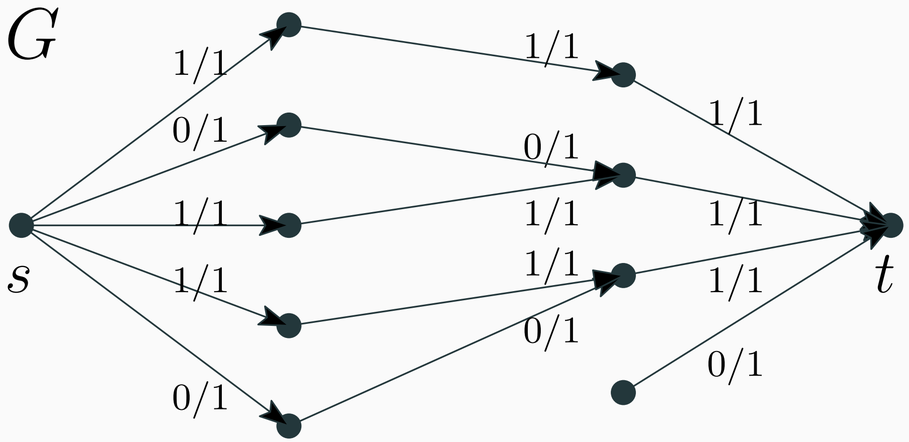
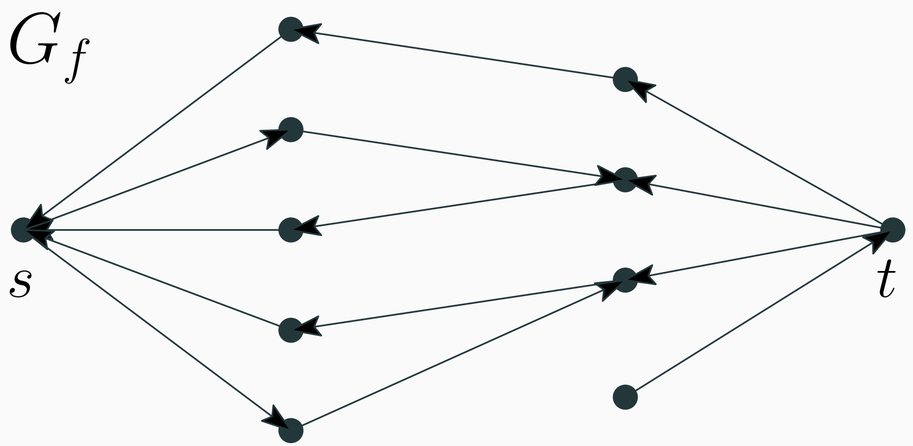


2) Compute the residual graph G_f:



Finding a MinVC or MaxIS in bipartite graphs: step by step

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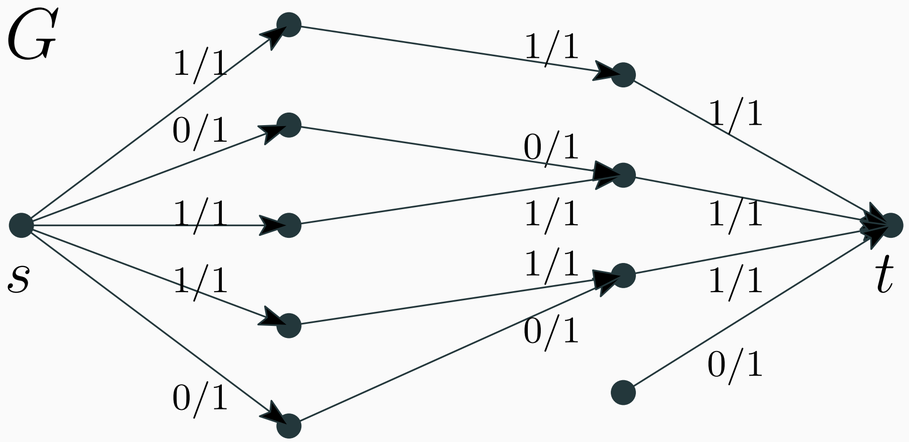

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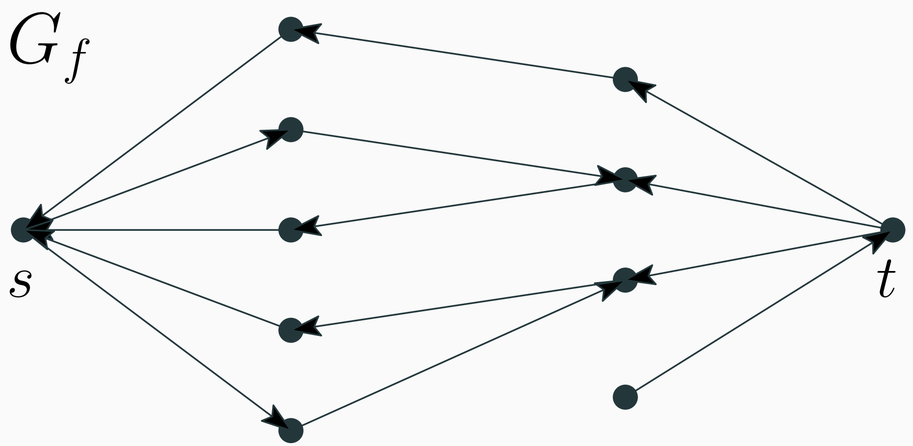
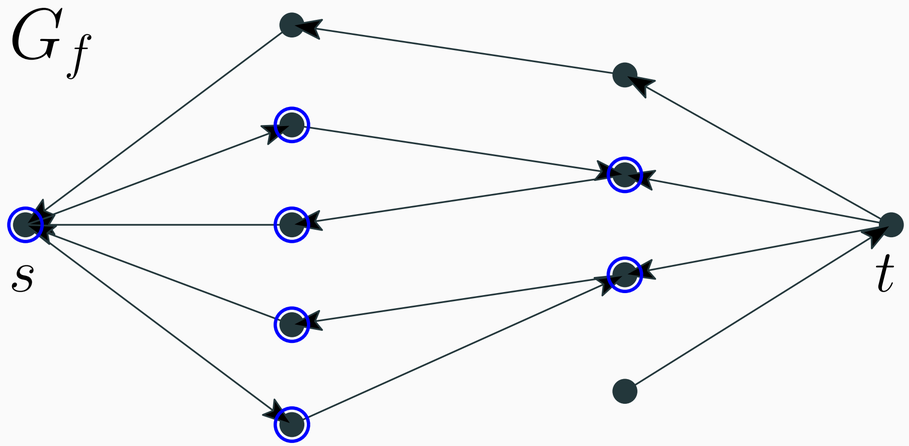


3) Mark reachable vertices from s with BFS:



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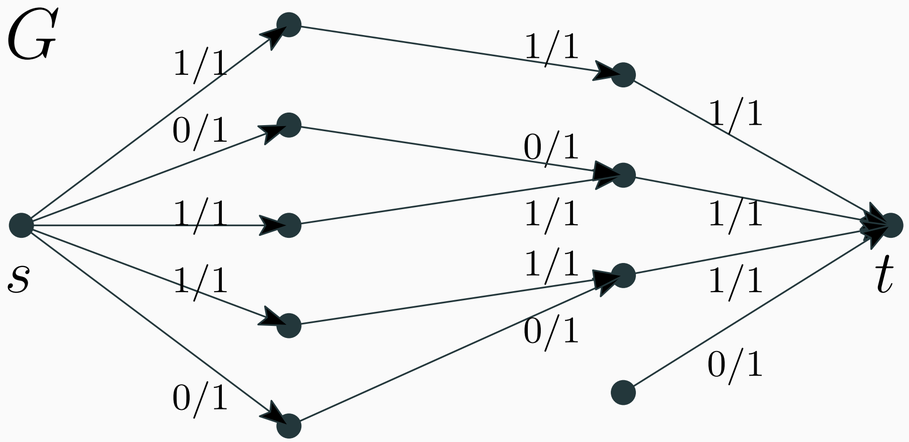

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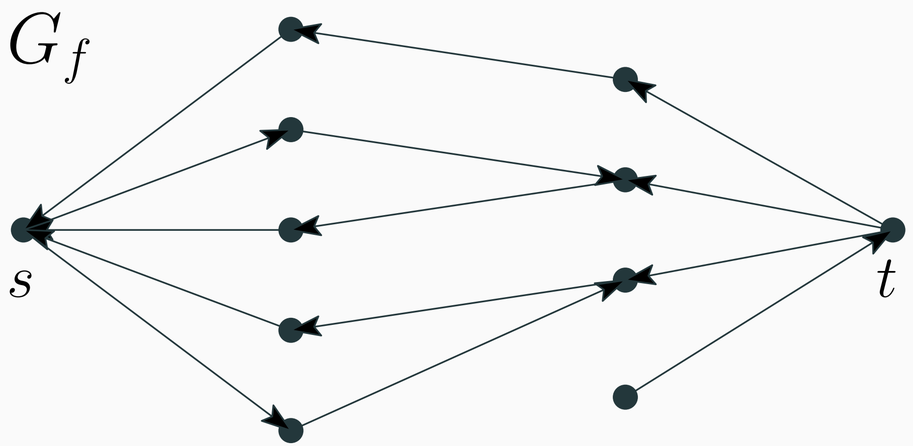

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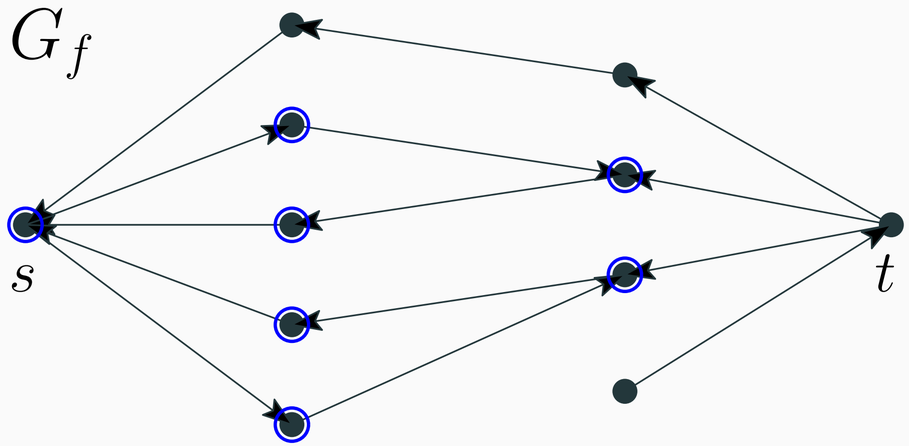
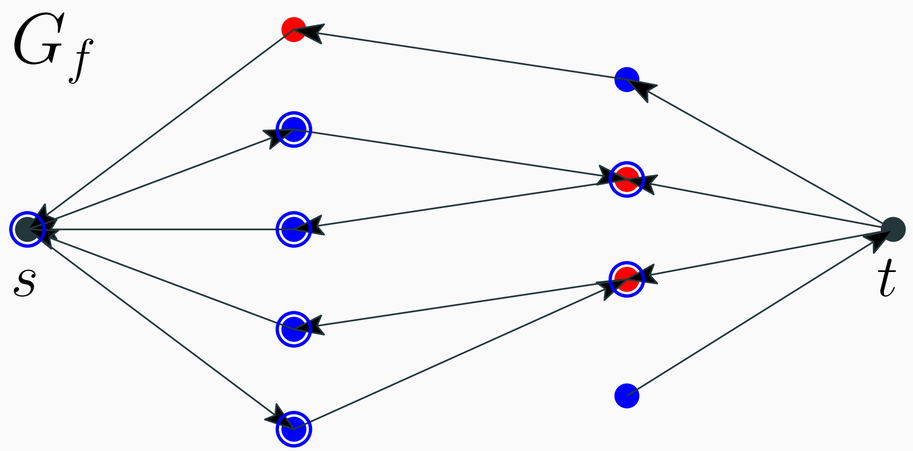


4) Read the MinVC or MaxIS from the marks:



Summary: MaxFlowMinCut and Bipartite Matching

What you should remember:

Minimum Cut

- Theorem: maximum amount of any s-t-flow = minimum capacity of any s-t-cut
- Finding the cut: BFS/DFS on residual graph starting from s.

Vertex Cover

- Minimum vertex cover and maximum independent set are hard problems.
- Bipartite graphs allow fast MinVC and MaxIS (both on top of maximum matching).
- Finding the minimum vertex cover: BFS/DFS on residual graph from s.

Min Cost Max Flow

Minimum Cost for a Bipartite Matching

How to pick the *best* maximum matching? How to break ties among equally large ones?

E.g.: Which set of marriages is the most stable?

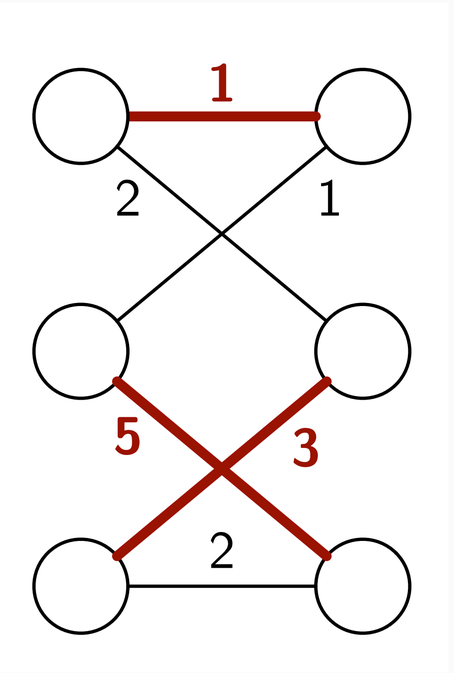
Or: Which consumer/producer pairing is the most effective?

What if the edges in a matching also have costs associated?

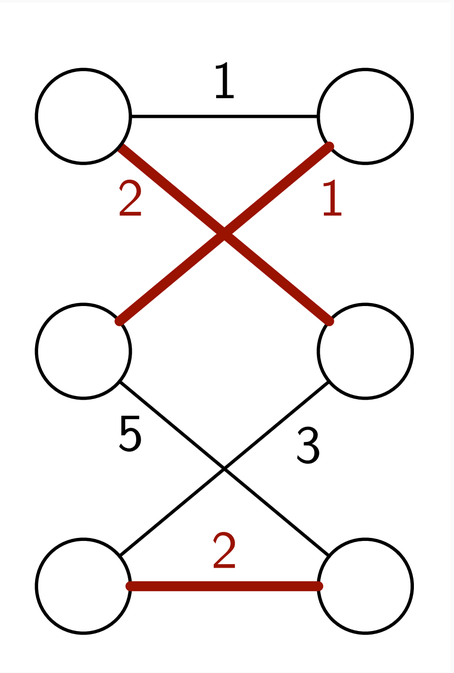
- cardinality of the matching is no longer the only objective
- second priority:
  minimize the total cost

We search for the cheapest among all maximum matchings.

two maximum matchings of different cost:



$1 + 5 + 3 = 9$



$2 + 1 + 2 = 5$

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This does not fit into our model of flows.

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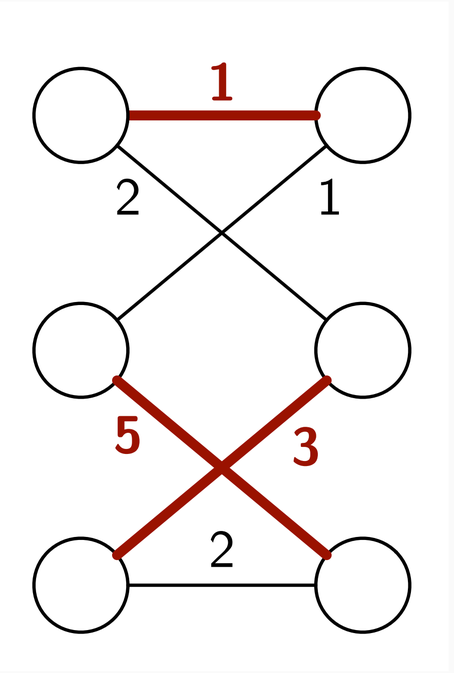

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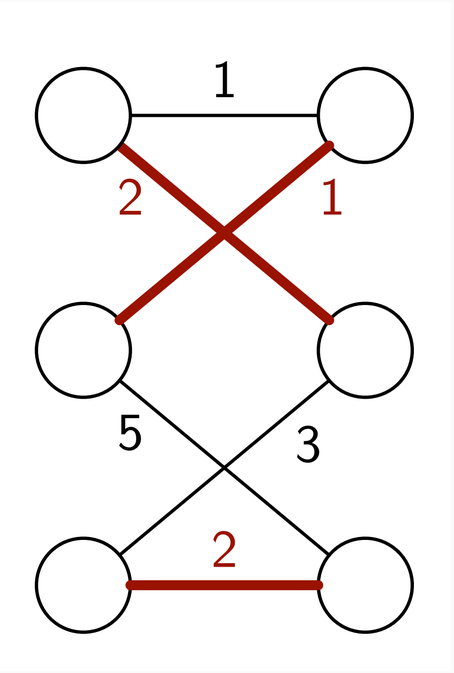


$2 + 1 + 2 = 5$

Problem Landscape

More General Model: Minimum Cost Maximum Flow

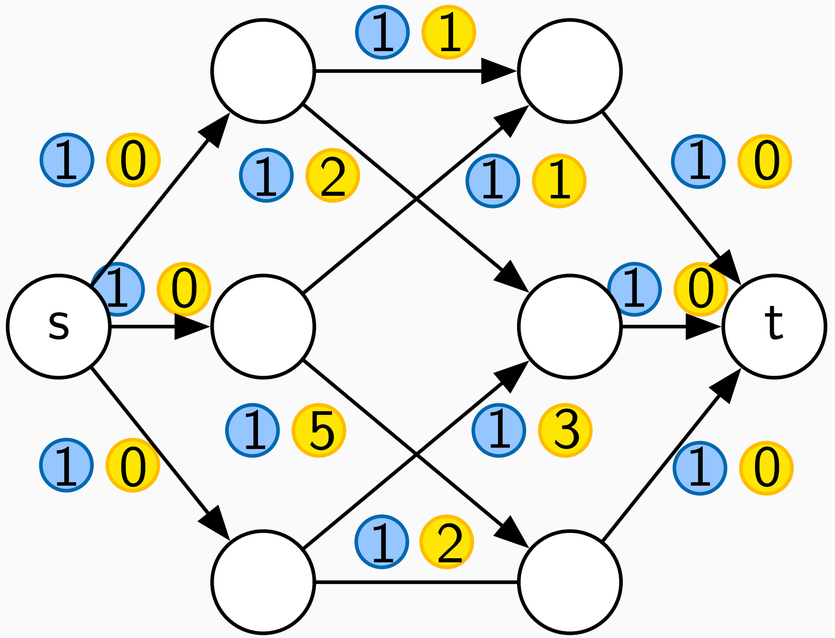
We extend the network flow problem by allowing edge costs that occur per unit of flow.

Input: A flow network consisting of

- directed graph $G = (V, E)$
- source and sink $s, t \in V$
- edge capacity $cap : E \to \mathbb{N}$
- **edge cost** $cost : E \to \mathbb{Z}$.

Output: A flow function f with minimal $cost(f) = \sum_{e \in E} f(e) \cdot cost(e)$ among all flows with maximal $|f|$.

This can model much more than just minimum cost bipartite matching.



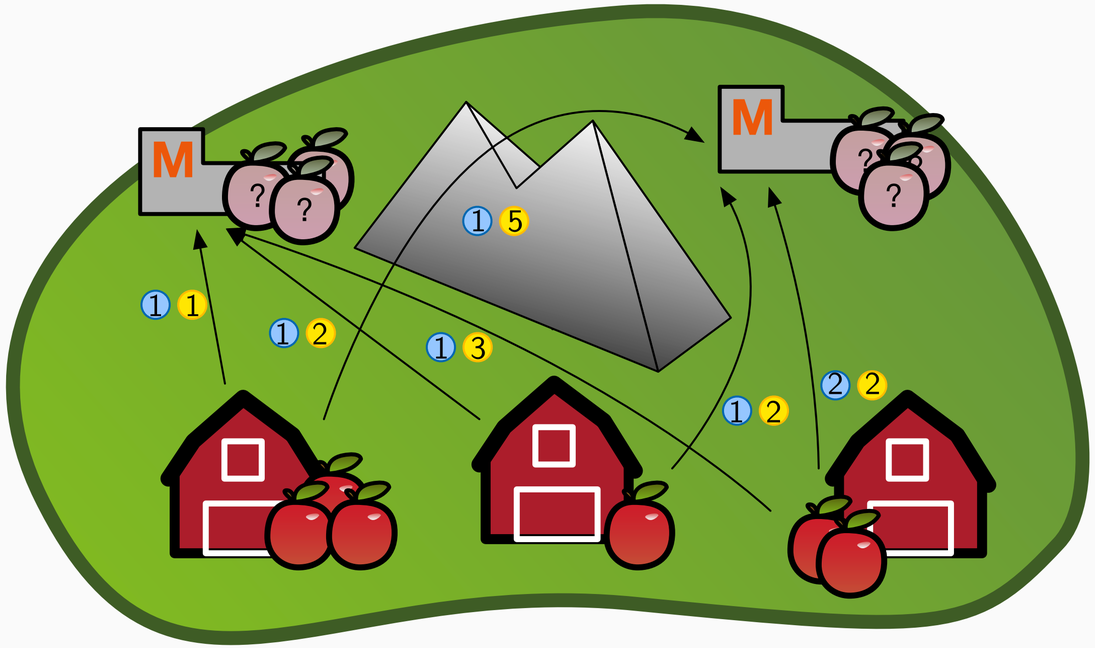
Legend: capacity vs cost

Example: Fruit Delivery

Migros wants to schedule fruit deliveries from their farmers to their shops.

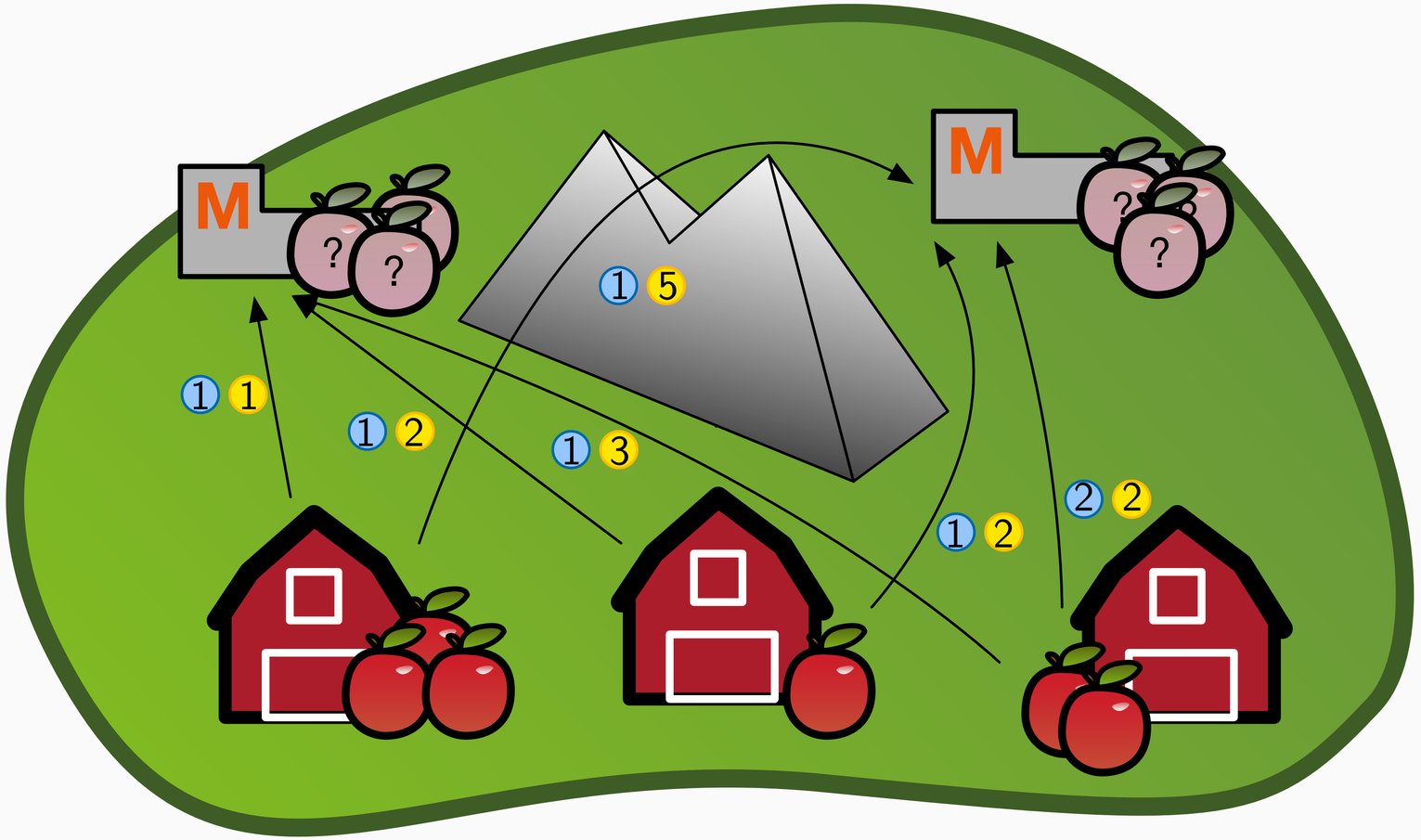
They know all important parameters;

- production per farm [in kg]
- demand per shop [in kg]
- transportation capacity [in kg] and transportation cost [in Fr. pro kg] for every farm-shop pair



Note: This is not just a bipartite matching, even though the graph is bipartite.
One farm might deliver to multiple shops (and vice versa).

Example: Fruit Delivery



Flow: $2 + 1 + 2 = 5$

Cost: $1 \cdot 1 + 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 2 = 12$

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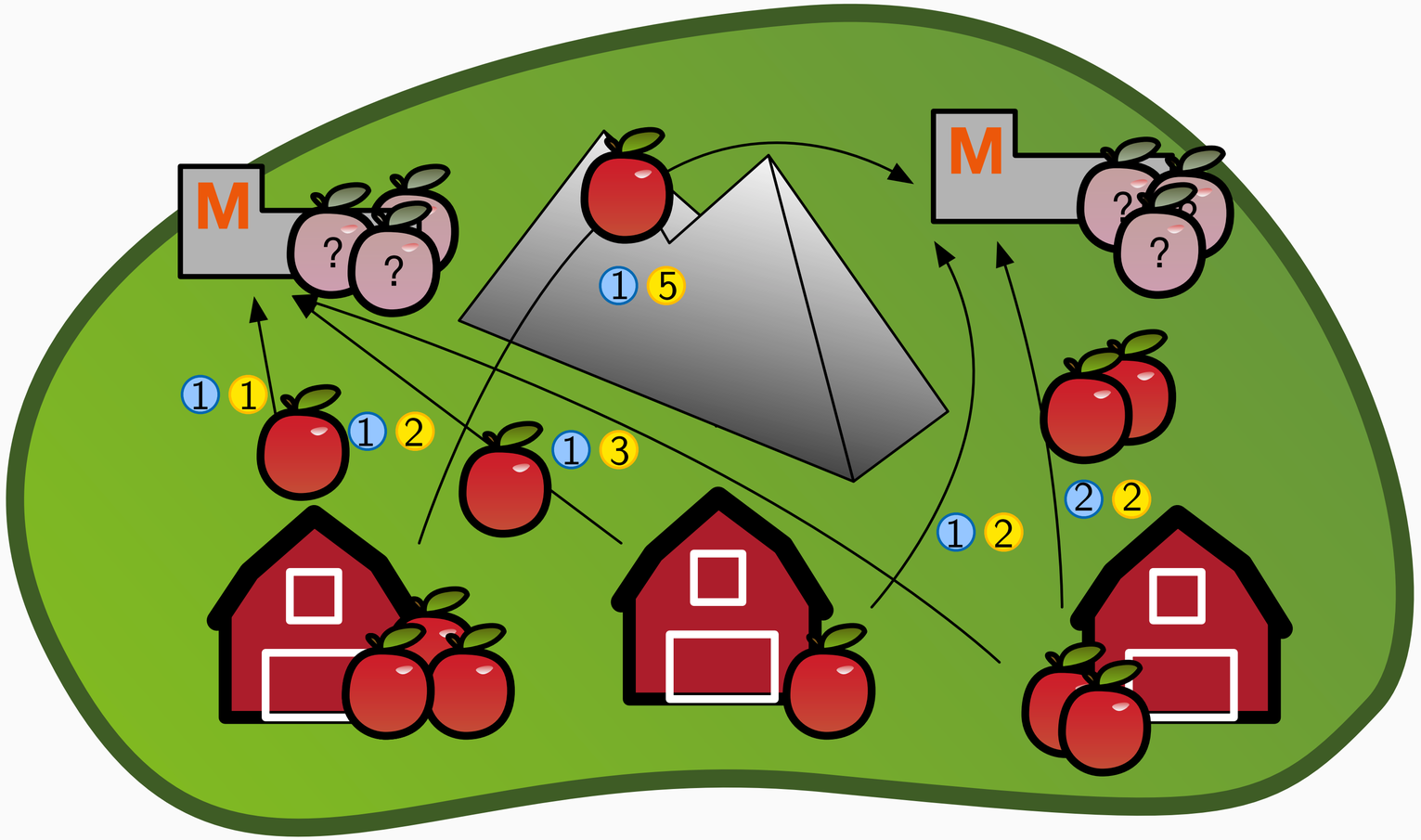

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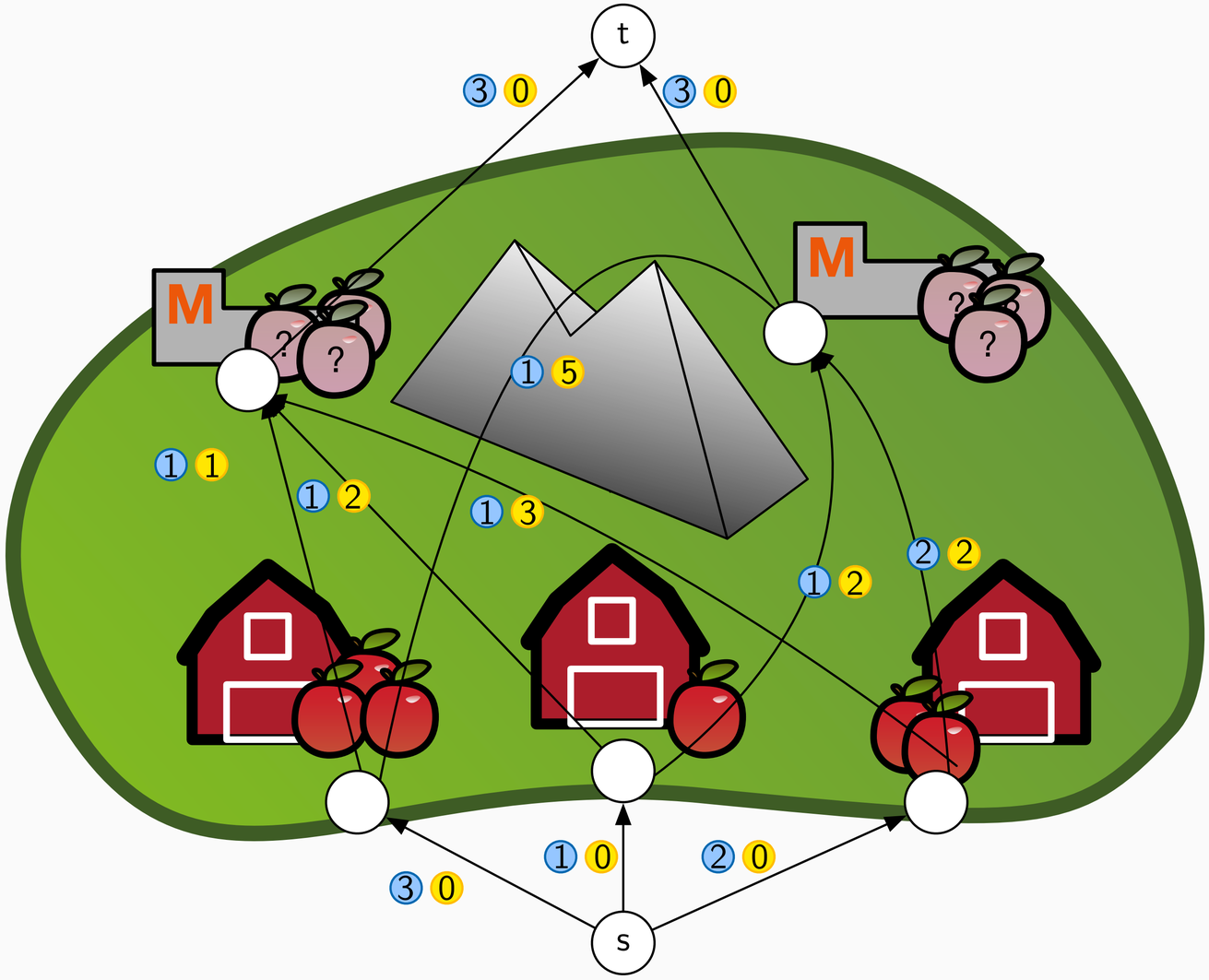
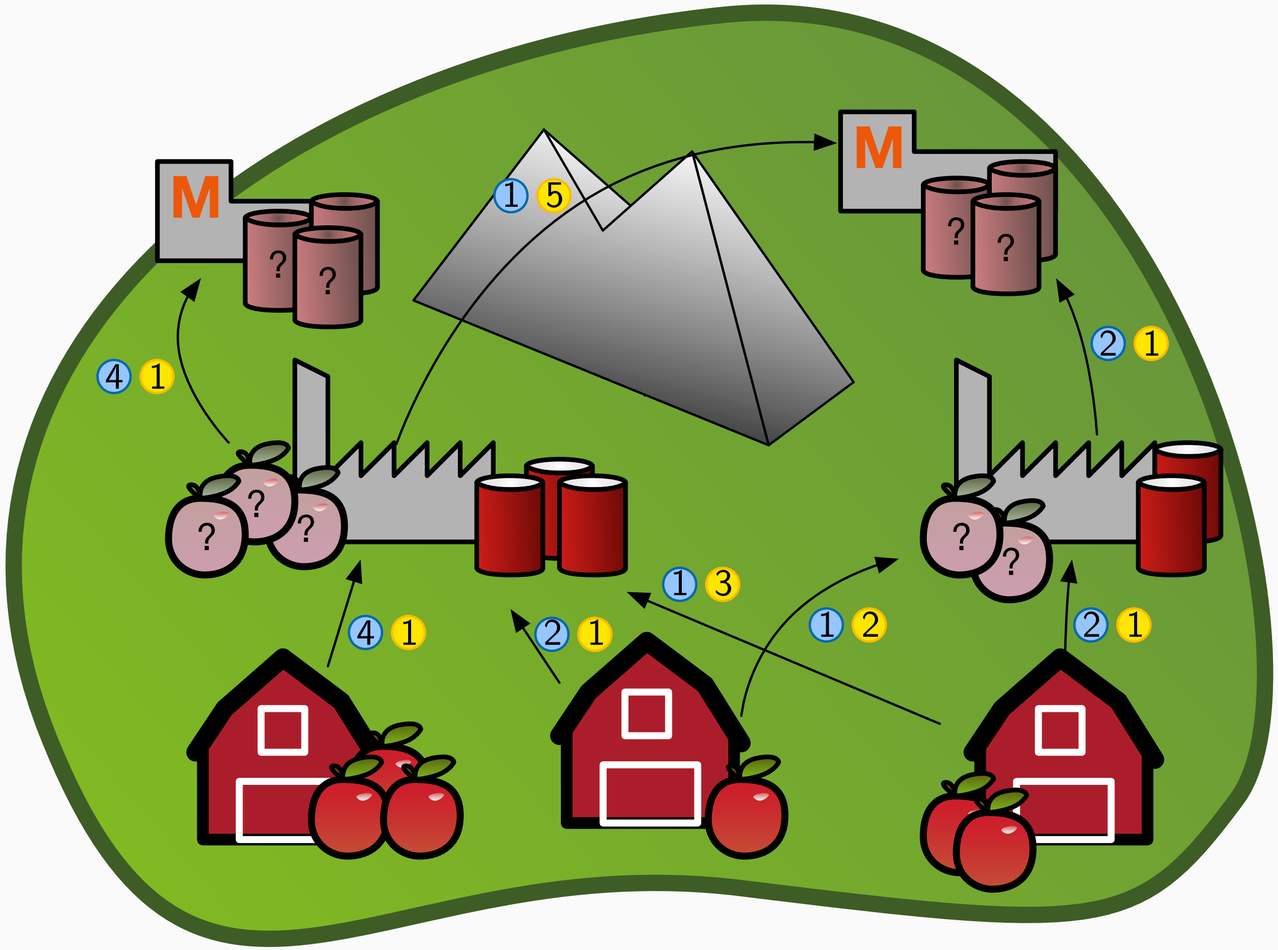


Extended Example: Canned Fruit Delivery

Extension: Canned fruit requires transportation to and from a canning factory.



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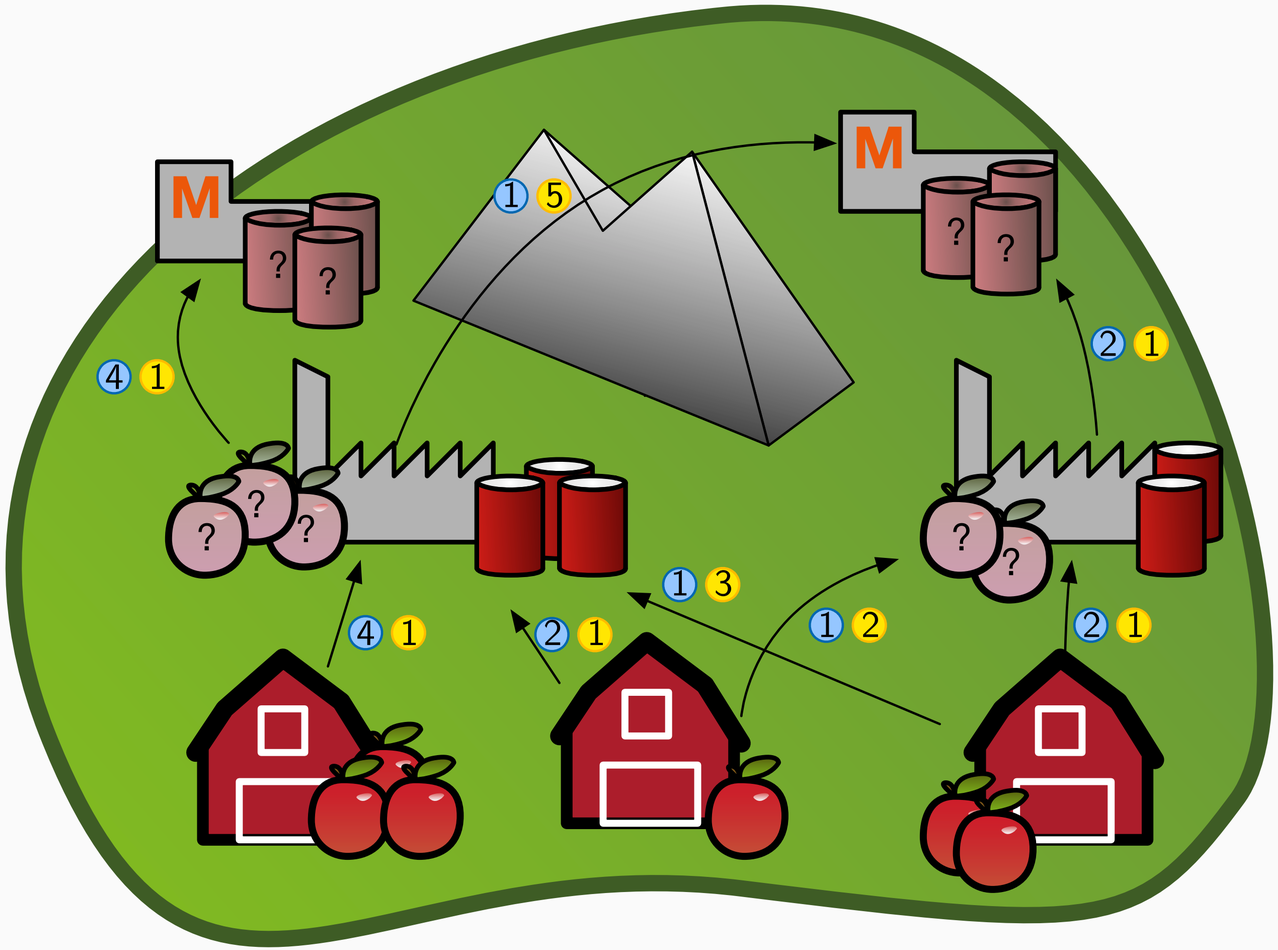

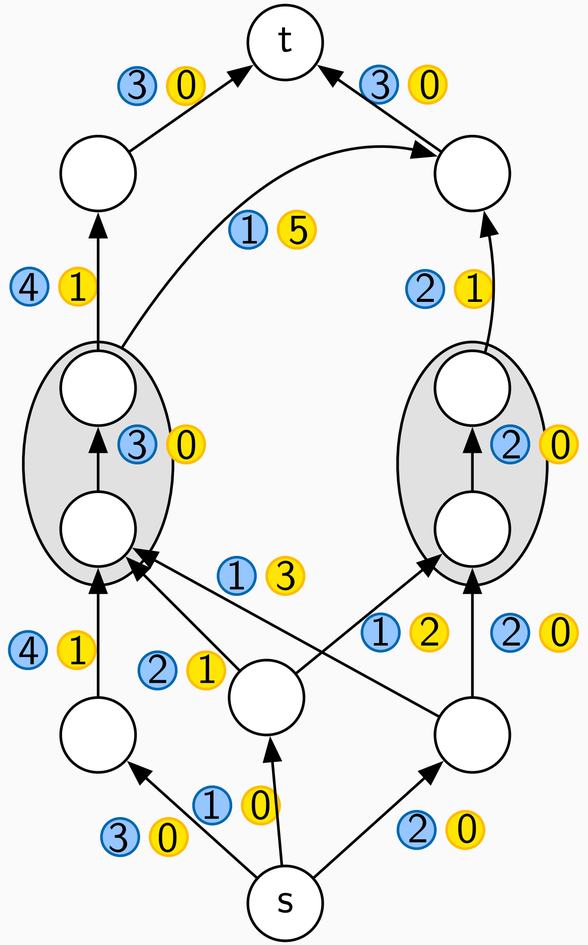

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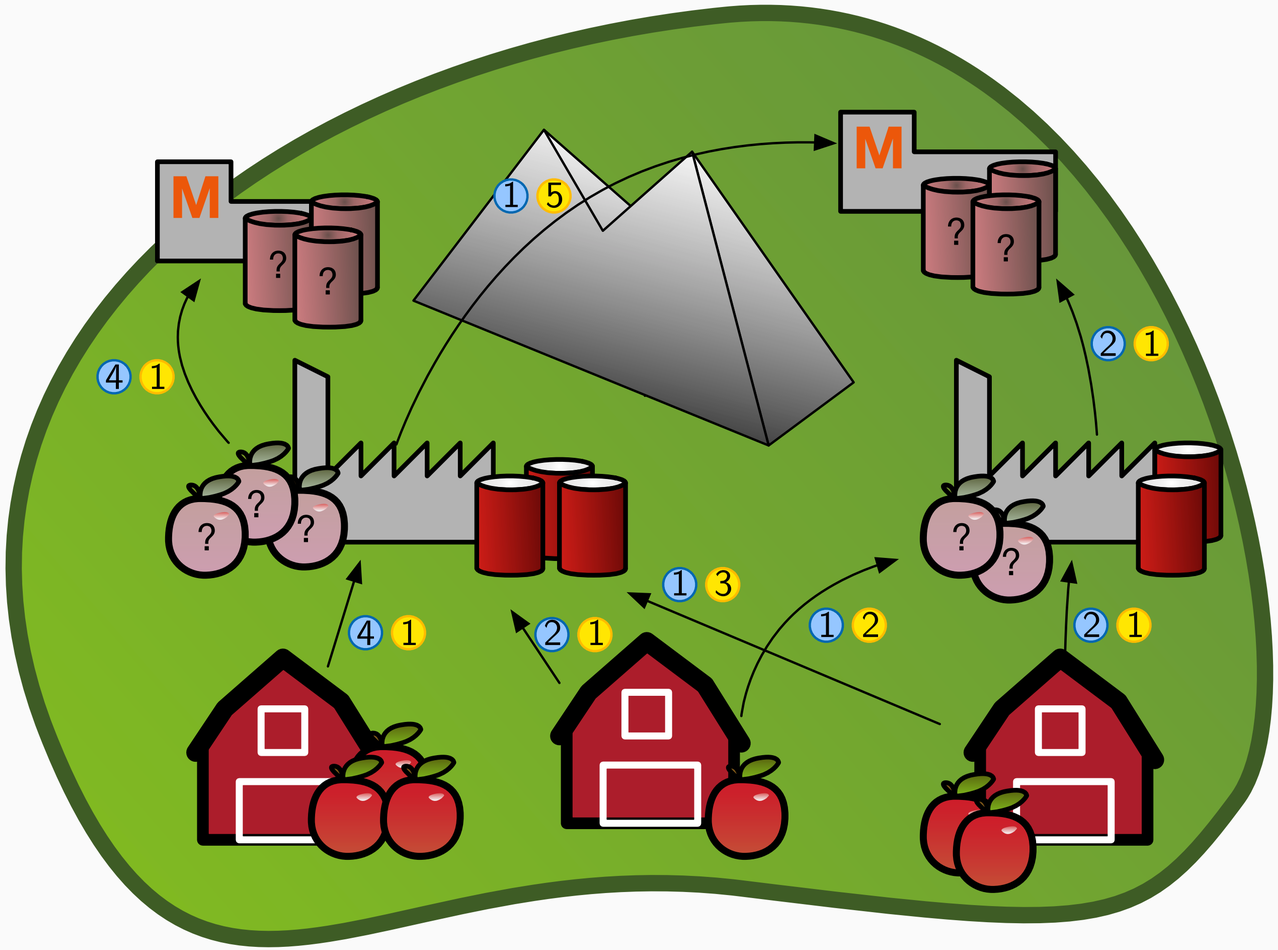

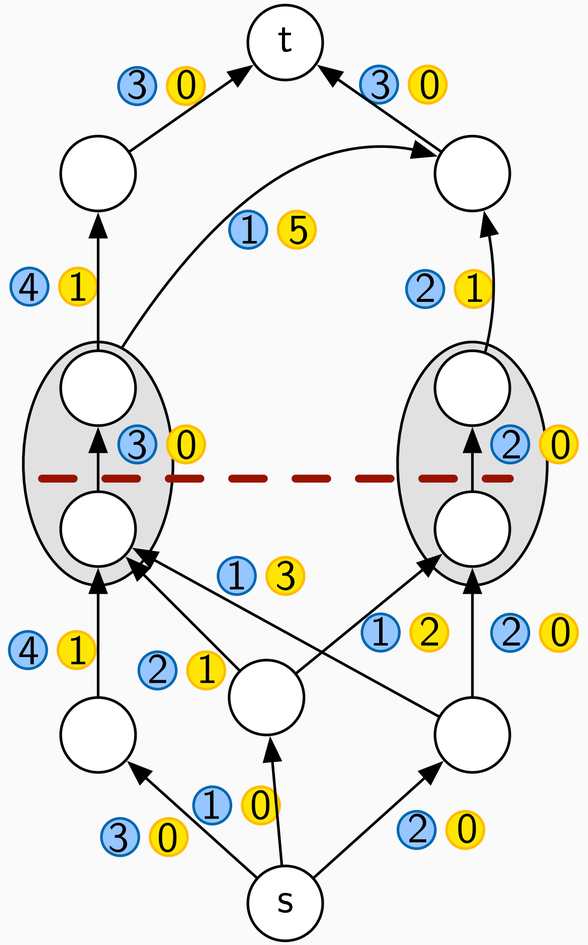

Min Cost Max Flow with BGL

There are two algorithms available in BGL (available in BGL v1.55+):

- `cycle_canceling()`
  - slow, but can handle negative costs
  - needs a maximum flow to start with (call e.g. `push_relabel_max_flow` before)
  - runtime $\mathcal{O}(C \cdot (nm))$ where C is the cost of the initial flow
  - [BGL documentation], [BGL example].

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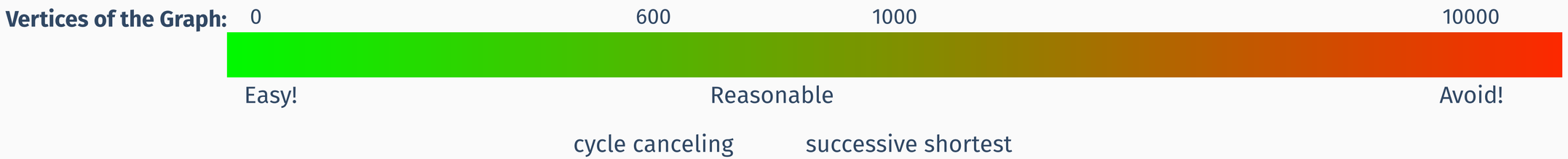

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Useful things to know:

- costs implemented as `edge_weight_t` property
- call `find_flow_cost()` to compute the cost of the flow

Min Cost Max Flow with BGL

Weights and capacities, just one more nesting level in the typedefs:

```
16 // Graph Type with nested interior edge properties for Cost Flow Algorithms
17 typedef boost::adjacency_list_traits<boost::vecS, boost::vecS, boost::directedS> traits;
18 typedef boost::adjacency_list<boost::vecS, boost::vecS, boost::directedS, boost::no_property,
19     boost::property<boost::edge_capacity_t, long,
20         boost::property<boost::edge_residual_capacity_t, long,
21             boost::property<boost::edge_reverse_t, traits::edge_descriptor,
22                 boost::property <boost::edge_weight_t, long> > > > > graph; // new!
23
24 typedef boost::graph_traits<graph>::edge_descriptor            edge_desc;
25 typedef boost::graph_traits<graph>::out_edge_iterator  out_edge_it; // Iterator
```

Code file: → bgl_mincostmaxflow.cpp

Min Cost Max Flow with BGL

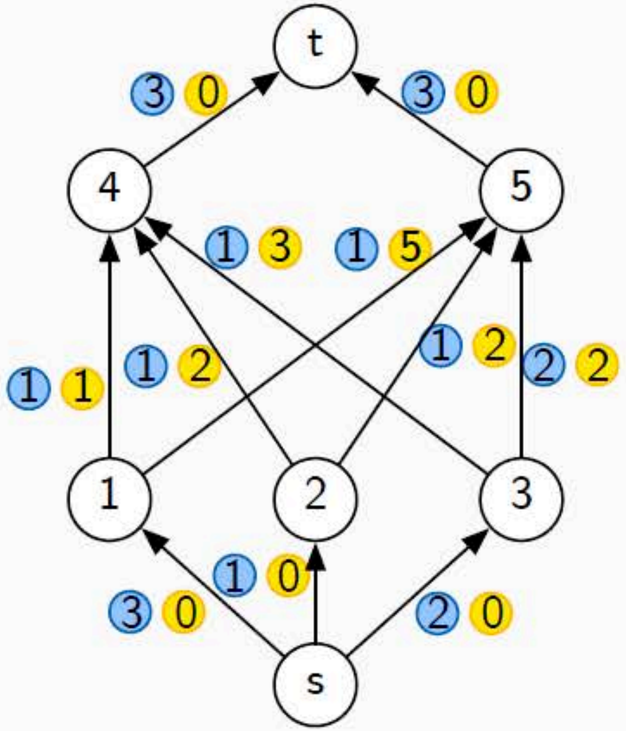
Extending the edge adder:

```
28 class edge_adder {
29   graph &G;
30
31  public:
32   explicit edge_adder(graph &G) : G(G) {}
33   void add_edge(int from, int to, long capacity, long cost) {
34     auto c_map = boost::get(boost::edge_capacity, G);
35     auto r_map = boost::get(boost::edge_reverse, G);
36     auto w_map = boost::get(boost::edge_weight, G); // new!
37     const edge_desc e = boost::add_edge(from, to, G).first;
38     const edge_desc rev_e = boost::add_edge(to, from, G).first;
39     c_map[e] = capacity;
40     c_map[rev_e] = 0;
41     r_map[e] = rev_e;
42     r_map[rev_e] = e;
43     w_map[e] = cost;   // new assign cost
44     w_map[rev_e] = -cost;   // new negative cost
45   }
46 };
```

Min Cost Max Flow with BGL

Building the graph

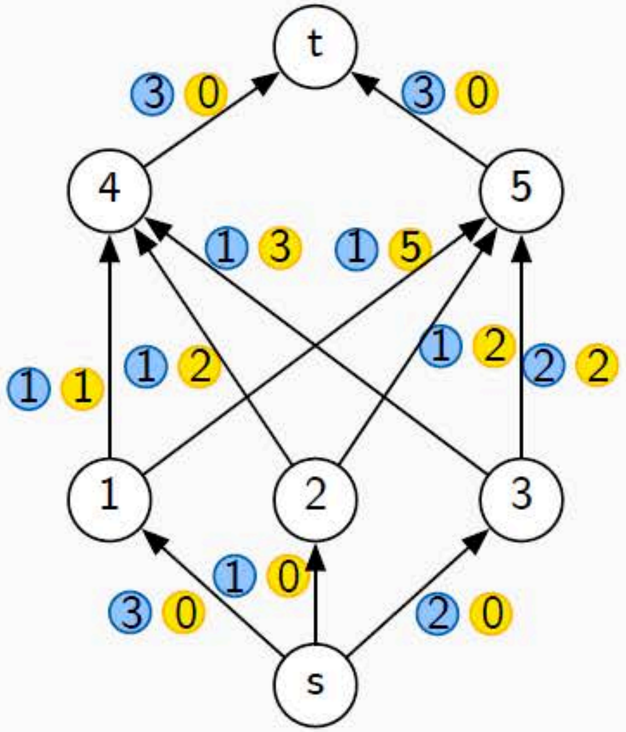
```
50 const int N=7;
51 const int v_source = 0;
52 const int v_farm1 = 1;
53 const int v_farm2 = 2;
54 const int v_farm3 = 3;
55 const int v_shop1 = 4;
56 const int v_shop2 = 5;
57 const int v_target = 6;
58
59 // Create graph, edge adder class and propery maps
60 graph G(N);
61 edge_adder adder(G);
62 auto c_map = boost::get(boost::edge_capacity, G);
63 auto r_map = boost::get(boost::edge_reverse, G);
64 auto rc_map = boost::get(boost::edge_residual_capacity, G);
```



Min Cost Max Flow with BGL

Add the edges:

```
66 // Add the edges
67 adder.add_edge(v_source, v_farm1, 3, 0);
68 adder.add_edge(v_source, v_farm2, 1, 0);
69 adder.add_edge(v_source, v_farm3, 2, 0);
70
71 adder.add_edge(v_farm1, v_shop1, 1, 1);
72 adder.add_edge(v_farm1, v_shop2, 1, 5);
73 adder.add_edge(v_farm2, v_shop1, 1, 2);
74 adder.add_edge(v_farm2, v_shop2, 1, 2);
75 adder.add_edge(v_farm3, v_shop1, 1, 3);
76 adder.add_edge(v_farm3, v_shop2, 2, 2);
77
78 adder.add_edge(v_shop1, v_target, 3, 0);
79 adder.add_edge(v_shop2, v_target, 3, 0);
```



Min Cost Max Flow with BGL

Running the algorithm:

```
83 // Option 1: Min Cost Max Flow with cycle_canceling
84 int flow1 = boost::push_relabel_max_flow(G, v_source, v_target);
85 boost::cycle_canceling(G);
86 int cost1 = boost::find_flow_cost(G);
87 std::cout << "flow " << flow1 << "\n"; // 5
88 std::cout << "cost " << cost1 << "\n"; // 12
```

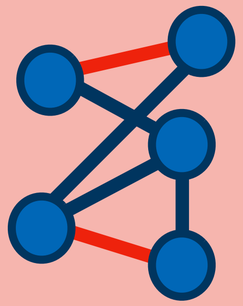
Min Cost Max Flow with BGL

Running the algorithm:

```
92  // Option 2: Min Cost Max Flow with successive_shortest_path_nonnegative_weights
93  boost::successive_shortest_path_nonnegative_weights(G, v_source, v_target);
94  int cost2 = boost::find_flow_cost(G);
95  std::cout << "cost " << cost2 << "\n"; // 12
96  // Iterate over all edges leaving the source to sum up the flow values.
97  int s_flow = 0;
98  out_edge_it e, eend;
99  for(boost::tie(e, eend) = boost::out_edges(boost::vertex(v_source,G), G); e != eend; ++e)
100     s_flow += c_map[*e] - rc_map[*e];
101 std::cout << "s-out flow " << s_flow << "\n"; // 5
102 // Or equivalently, you can do the summation at the sink, but with reversed edge.
103 int t_flow = 0;
104 for(boost::tie(e, eend) = boost::out_edges(boost::vertex(v_target,G), G); e != eend; ++e)
105     t_flow += rc_map[*e] - c_map[*e];
106 std::cout << "t-in flow " << t_flow << "\n"; // 5
```

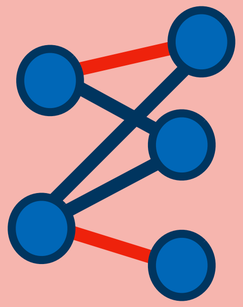
Problem Landscape

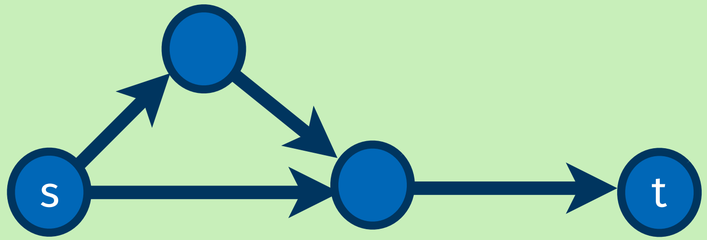
General Matching



Edmonds Maximum Cardinality Matching

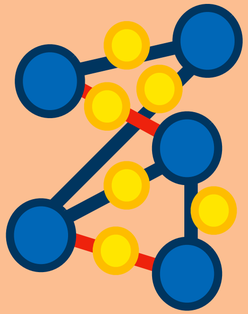
Bipartite Matching





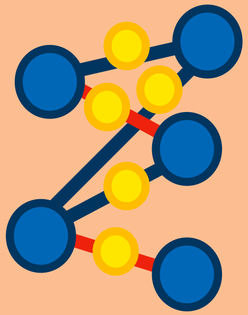
Maximum Flow

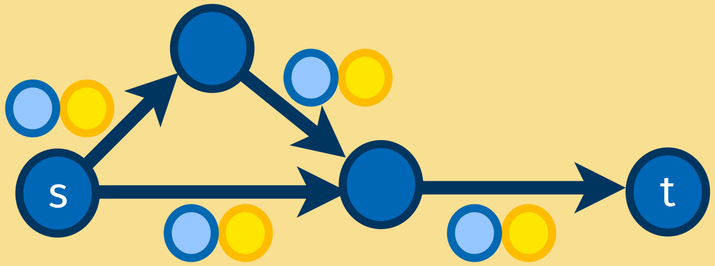
General Cost Matching



???
not in BGL

Bipartite Cost Matching





Min Cost Max Flow

Summary: Min Cost Max Flow with BGL

What you should remember from this part:

Minimum Cost Maximum Flow

- is a powerful and versatile modeling tool.
- is a tiebreaker among several maximum flows (but might still not be unique).
- = *maximum* cost maximum flow with negated costs.
- can often be reformulated without negative costs which allows us to use a faster algorithm in BGL (key step in many problems).
- can easily be implemented when starting with our template on the Judge.
- is not harder to use in BGL than the regular network flow algorithms, see BGL Docs.

A little bit of a conclusion...

No new theory and no new tools past this point

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But be prepared to combine all your skills:

- On top of a flow problem, do binary search for the answer
- LP formulation vs. flow formulation?
- Some graph problems can be solved greedily (e.g. MST), others not (e.g. flow)
- Dijkstra and MinCostMaxFlows are just “special” dynamic programs
- Find a Min Cost Max Flow formulation where greedy fails for non-unit weights
- Do BFS on Delaunay triangulation or do Union-Find on Euclidean MST
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Starting next week:

- How to balance reading, solving, coding, debugging under time constraints
- No more problems labeled by topic – figure it out yourself
- In-class exercises on Wednesdays