Algolab BGL – Maximum Flows

(BGL tutorial 2/3)

Tim Taubner[1]

ETH Zurich

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[1] based on material from Stefano Leucci and Daniel Wolleb-Graf

BGL and Maximum Flow

Last BGL tutorial: BGL Intro

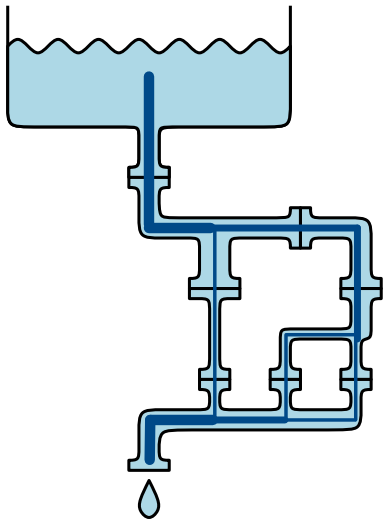
- Graphs in BGL
- Standard algorithms: connected components, dijkstra's shortest paths, etc.

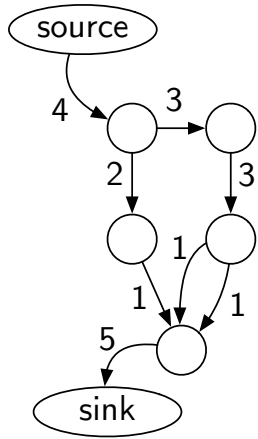
Today' BGL tutorial: Maximum Flows

- what is *maximum flow*: motivation, definition, intuition
- which algorithms exists and how to use them in BGL
- common techniques and examples on how to apply maximum flow

Next BGL tutorial: Minimum Cost Maximum Flow

Network Flow: Example

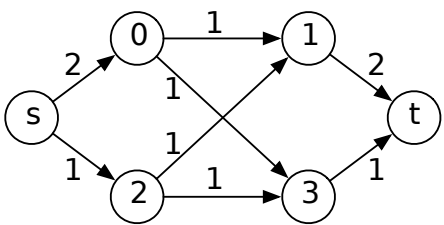




Network Flow: Problem Statement

Input: A flow network consisting of

- directed graph $G = (V, E)$
- *source* $s \in V$ with only out edges
- *sink* $t \in V$ with only in edges
- edge capacity $c : E \to \mathbb{N}$.



Output: Flow function $f : E \to \mathbb{R}$ subject to

- Capacity constraints for every edge e.
  $0 \leq f(e) \leq c(e)$
- Flow conservation at every vertex u.
  $$\sum_{(v,u)\in E} f((v,u)) \quad \textit{in flow}$$
  $$= \sum_{(u,w)\in E} f((u,w)) \quad \textit{out flow}$$
- Maximization of *total flow*.
  $|f| := \sum_{(s,v)\in E} f((s,v))$

Theorem

On a graph with integral capacities, there always exists an integral maximum flow.

BGL flows: Includes

```
#include <iostream>

// BGL include
#include <boost/graph/adjacency_list.hpp>

// BGL flow include *NEW*
#include <boost/graph/push_relabel_max_flow.hpp>
```

Using BGL flows: Typedefs

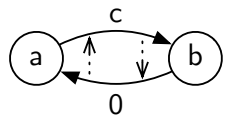
The typedefs now include residual capacities and reverse edges.

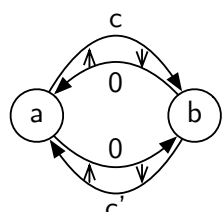
```
// Graph Type with nested interior edge properties for flow algorithms
typedef boost::adjacency_list_traits<boost::vecS, boost::vecS, boost::directedS> traits;
typedef boost::adjacency_list<boost::vecS, boost::vecS, boost::directedS, boost::no_property,
    boost::property<boost::edge_capacity_t, long,
        boost::property<boost::edge_residual_capacity_t, long,
            boost::property<boost::edge_reverse_t, traits::edge_descriptor>>>> graph;

typedef traits::vertex_descriptor vertex_desc;
typedef traits::edge_descriptor edge_desc;
```

BGL flows: Manually creating an edge

Adding a directed edge and its reverse in the residual graph.

```
// Create graph and maps
graph G(4);

auto c_map = boost::get(boost::edge_capacity, G);
auto r_map = boost::get(boost::edge_reverse, G);

vertex_desc a = ...;
vertex_desc b = ...;
long c = ...;

edge_desc e = boost::add_edge(a, b, G).first;
edge_desc rev_e = boost::add_edge(b, a, G).first;
c_map[e] = c;
c_map[rev_e] = 0; // reverse edge has no capacity!
r_map[e] = rev_e;
r_map[rev_e] = e;
```





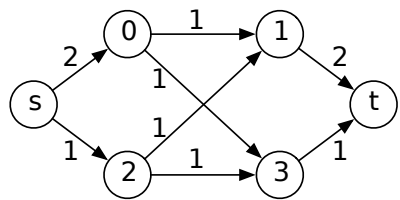
BGL flows: edge_adder helper class

```
class edge_adder {
  graph &G;

 public:
  explicit edge_adder(graph &G) : G(G) {}

  void add_edge(int from, int to, long capacity) {
    auto c_map = boost::get(boost::edge_capacity, G);
    auto r_map = boost::get(boost::edge_reverse, G);
    const auto e = boost::add_edge(from, to, G).first;
    const auto rev_e = boost::add_edge(to, from, G).first;
    c_map[e] = capacity;
    c_map[rev_e] = 0; // reverse edge has no capacity!
    r_map[e] = rev_e;
    r_map[rev_e] = e;
  }
};
```

BGL flows: Creating the graph

Graph creation with edge_adder:

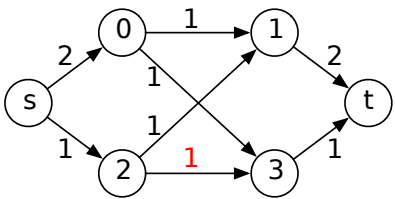
```
graph G(4);
edge_adder adder(G);

// Add some edges using our custom edge adder
adder.add_edge(0, 1, 1); // from, to, capacity
adder.add_edge(0, 3, 1);
adder.add_edge(2, 1, 1);
adder.add_edge(2, 3, 1);

  // Add special vertices source and sink
const vertex_desc v_source = boost::add_vertex(G);
const vertex_desc v_target = boost::add_vertex(G);
adder.add_edge(v_source, 0, 2);
adder.add_edge(v_source, 2, 1);
adder.add_edge(1, v_target, 2);
adder.add_edge(3, v_target, 1);
```



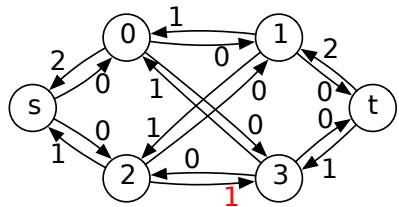
BGL flows: Calling the algorithm

```
// Calculate flow from source to sink
// The flow algorithm uses the interior properties (managed in the edge adder)
// - edge_capacity, edge_reverse (read access),
// - edge_residual_capacity (read and write access).
long flow = boost::push_relabel_max_flow(G, v_source, v_target);

// Retrieve capacity map and reverse capacity map
const auto c_map = boost::get(boost::edge_capacity, G);
const auto rc_map = boost::get(boost::edge_residual_capacity, G);

edge_desc edge = ...;
long flow_through_edge = rc_map[edge] - c_map[edge];
```

Graph with capacities.



Resulting graph with residual capacities.



BGL flows: The different algorithms

Algorithm (BGL-Doc)	by	runtime	use
`boost::push_relabel_max_flow`	Goldberg, Tarjan (1986)	$\mathcal{O}(n^3)$	almost always
`boost::edmonds_karp_max_flow`	Edmonds, Karp (1972)	$\mathcal{O}(\min\{m\|f\|, nm^2\})$	almost never

Use

```
#include <boost/graph/push_relabel_max_flow.hpp>
...
long flow = boost::push_relabel_max_flow(G, source, target);
```

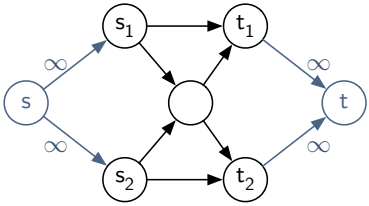
or

```
#include <boost/graph/edmonds_karp_max_flow.hpp>
...
long flow = boost::edmonds_karp_max_flow(G, source, target);
```

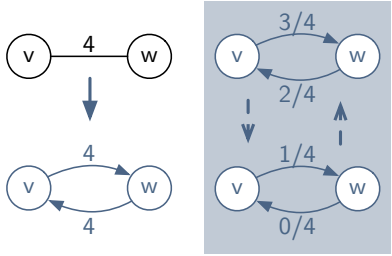
Even for $m, |f| \in \mathcal{O}(n)$, usually push-relabel is at least as fast.
However, it is possible to tailor graphs where Edmonds-Karp beats push-relabel.

Common techniques

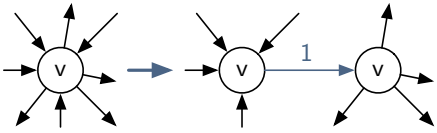
Multiple sources/sinks
with e.g. $\infty \approx \sum_{e \in E} c(e)$



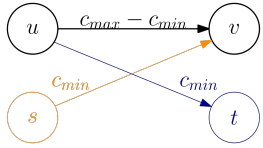
Undirected graphs
antiparallel edges with flow reducable to one direction



Vertex capacities
split into in-vertex and out-vertex

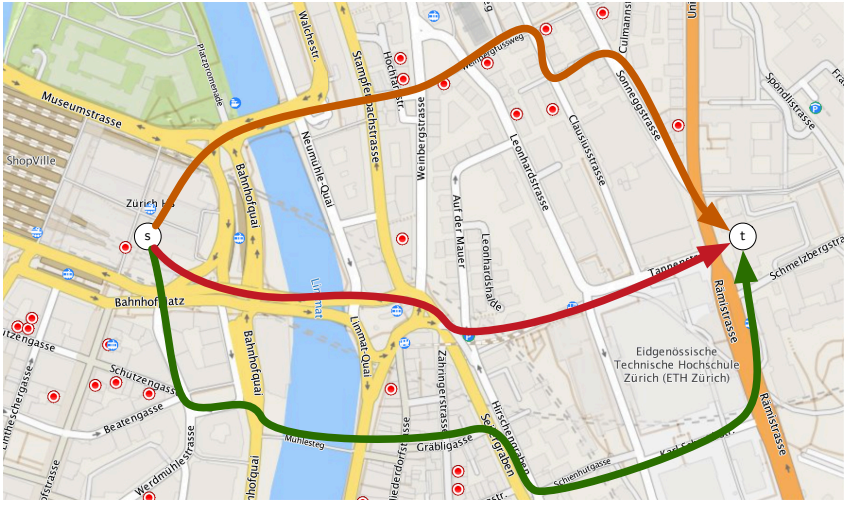


Minimum flow per edge
how to enforce $c_{\min}(e) \leq f(e) \leq c_{\max}(e)$?



Flow Application: Edge Disjoint Paths

How many ways are there to get from HB to CAB without using the same street twice?



Map: search.ch, TomTom, swisstopo, OSM

Flow Application: Edge Disjoint Paths

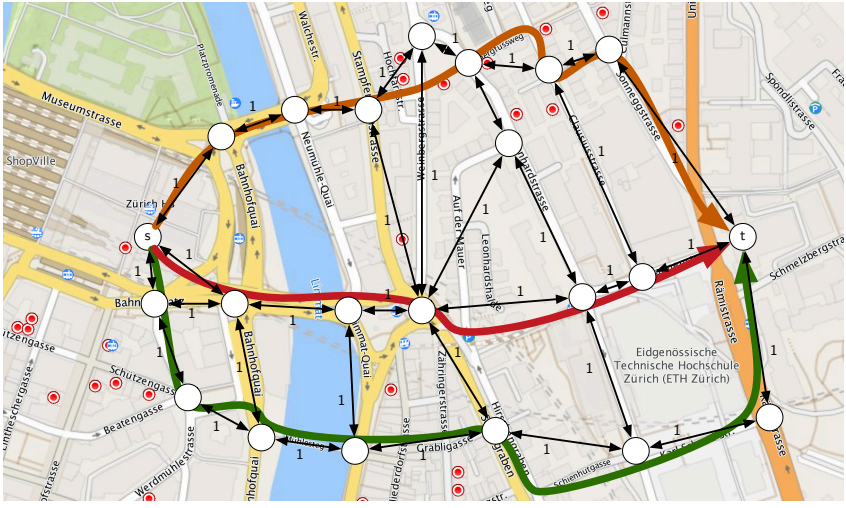
How many ways are there to get from HB to CAB without using the same street twice?

- Does this look like a flow problem? Not immediately.
- Can it be turned into a flow problem? Maybe.
- What we need according to the problem definition
  - $G = (V, E)$, *directed* street graph by adding edges in both directions for each street.
  - $s, t \in V$, intersections of HB and CAB.
  - $c : E \to \mathbb{N}$ with all capacities set to 1.

Lemma (Flow Decomposition)

In a directed graph with unit capacities, the maximum number of edge disjoint s-t-paths is equal to the maximum flow from s to t.

Flow Application: Edge Disjoint Paths



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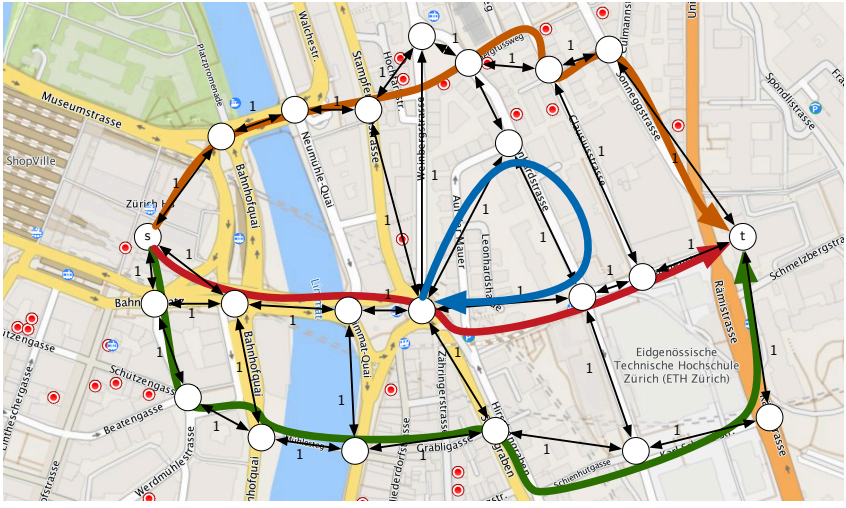
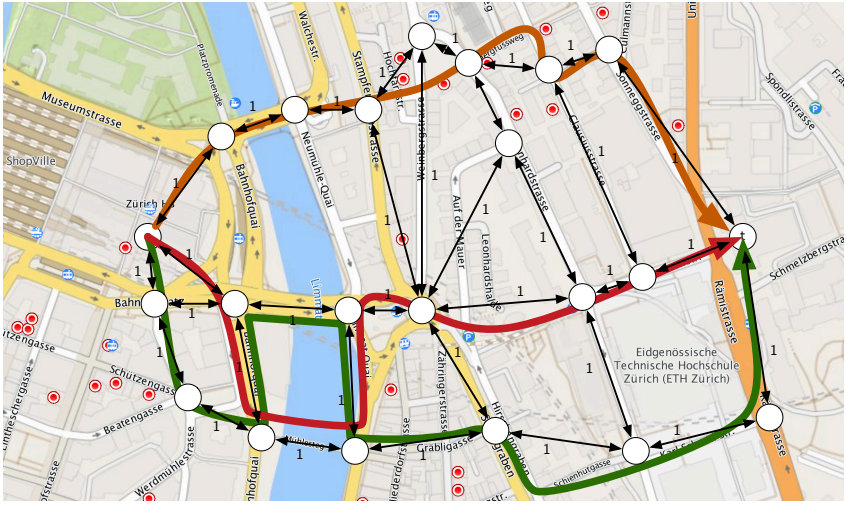
Flow Application: Edge Disjoint Paths: Handle Additional Cycles



Map:
search.ch,
TomTom,
swisstopo,
OSM

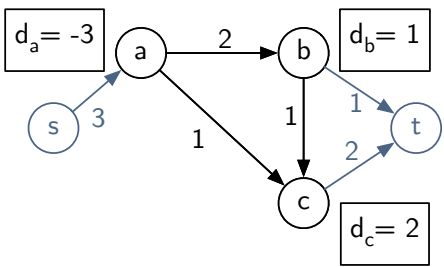
Flow Application: Edge Disjoint Paths: Handle Crossing Paths



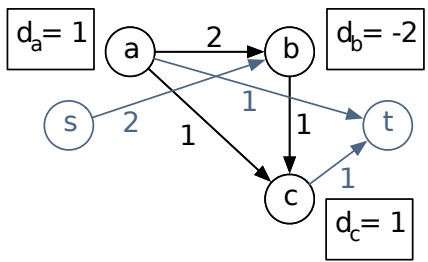
Map: search.ch, TomTom, swisstopo, OSM

Flow Application: Circulation Problem

- Multiple sources with a certain amount of flow to give (**supply**).
- Multiple sinks that want a certain amount of flow (**demand**).
- Model these as negative or positive demand per vertex d_v.
- Question: Is there a feasible flow? Surely not if $\sum_{v \in V} d_v \neq 0$. Otherwise? Add super-source and super-sink to get a maximum flow problem.



feasible flow exists



no feasible flow exists

Tutorial Problem: Soccer Prediction

- Swiss Soccer Championship, two rounds before the end.
- 2 points awarded per game, split 1-1 if game ends in a tie.

Team	Points	Remaining Games
FC St. Gallen (FCSG)	37	FCB, FCW
BSC Young Boys (YB)	36	FCW, FCB
FC Basel (FCB)	35	FCSG, YB
FC Luzern (FCL)	33	FCZ, GCZ
FC Winterthur (FCW)	31	YB, FCSG

- Can FC Luzern still win the Championship? (Assume favourable tie-breaking).
  37 points still possible, so yes?
  But at least one other team will have to score more, so no?
- Does this look like a flow problem? Not immediately.

Tutorial Problem: Modelling

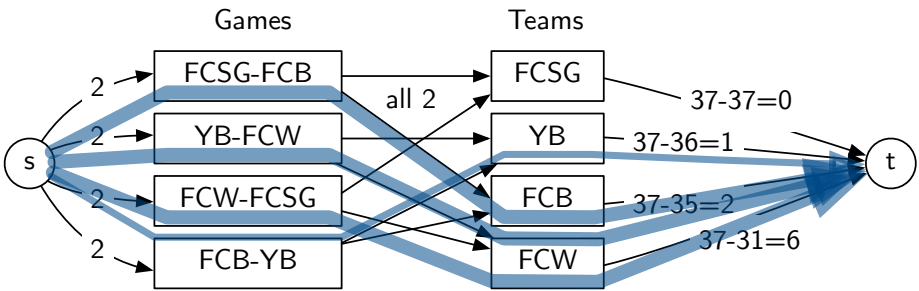
Simplifying assumptions: FC Luzern will win all remaining games.
Flow problem modelling:

- graph with vertices denoting teams and games
- source, target candidates?
  - games *distribute* points
  - teams *receive* points
- edges between games and teams
- capacities:
  - 2 points to distribute per game
  - $37 - x$ limit on the points received by a team with score x

Final question: Can we let the points *flow* from the games to the teams so that all the teams end up with at most 37 points?

Tutorial Problem: Modelling

Final question: Can we let the points *flow* from the games to the teams so that all the teams end up with at most 37 points?



The answer is yes if and only if the maximum flow is $\#points = 2 \cdot \#\text{games} = 8$.
Note: This approach fails for the current system with 3:0, 1:1, 0:3 splits.

Tutorial Problem: Analysis

- Does it look like a flow problem now? Yes.
- What does a unit of flow stand for? A point in the soccer ranking.
- How large is this flow graph?
  For N teams, M games, we have $n = 2 + N + M$ nodes and $m = N + 3M = \mathcal{O}(n)$ edges.
  Flow is at most $2M = \mathcal{O}(n)$.
- What algorithm should we use?
  Push-Relabel runs in $\mathcal{O}(n^3)$.
  Edmonds-Karp runs in $\mathcal{O}(m|f|) = \mathcal{O}(n^2)$.
  Push-Relabel is still faster in practice.

Flow Problems: Summary and Outlook

Summary

What we have seen today:

- "classical" network flow problem
- graph modifications for variations
- solves edge disjoint paths
- solves circulation

What you will see in the problems:

- more techniques and combination with other algorithms like binary search

CGAL-Outlook

Flow-related in the next CGAL tutorial:

- **linear programming** formulation (equivalent, but slower as we use a more powerful tool)

BGL-Outlook

What we will see in the next BGL tutorial:

- relation to **minimum cuts**
- extras for **bipartite graphs**: max matchings, independent sets, **min vertex cover**
- flows with **cost**: Min Cost Max Flow problem, applications and algorithms