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INSTITUTE OF TECHNOLOGY

SCHOOL OF ELECTROMECHANICAL ENGINEERING

INTRODUCTION TO CONTROL SYSTEM

MODULE

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CHAPTER ONE

Introduction to Control Systems

A control system consist s of subsystems and processes (or plants) assembled for the purpose of obtaining a desire d output with desire d performance, given a specified input. Figure 1.1 shows a control system in its simplest form, where the input represents a desired output.

Figure 1.1: Simplified description of Control System

Examples of Control Systems:

a) **Speed Control System:** The basic principle of a Watt's speed governor for an engine is illustrated in the schematic diagram of Figure 1.2. The amount of fuel admitted to the engine is adjusted according to the difference between the desired and the actual engine speeds.

The sequence of actions may be stated as follows: The speed governor is adjusted such that, at the desired speed, no pressured oil will flow into either side of the power cylinder. If the actual speed drops below the desired value due to disturbance, then the decrease in the centrifugal force of the speed governor causes the control valve to move downward, supplying more fuel, and the speed of the engine increases until the desired value is reached. On the other hand, if the speed of the engine increases above the desired value, then the increase in the centrifugal force of the governor causes the control valve to move upward. This decreases the supply of fuel, and the speed of the engine decreases until the desired value is reached.

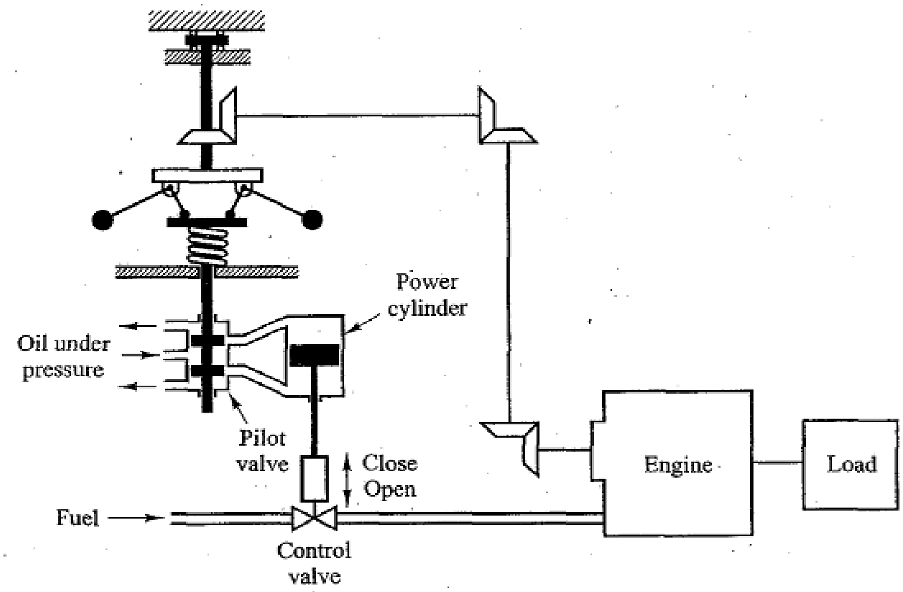


Figure 1.2: Speed Control System

Basic terminologies in control systems

Controlled Variable and Manipulated Variable. The controlled variable is the quantity or condition that is measured and controlled. The manipulated variable is the quantity or condition that is varied by the controller so as to affect the value of the controlled variable. Normally, the controlled variable is the output of the system. Control means measuring the value of the controlled variable of the system and applying the manipulated variable to the system to correct or limit deviation of the measured value from a desired value.

In studying control engineering, we need to define additional terms that are necessary to describe control systems.

Plants: A plant may be a piece of equipment, perhaps just a set of machine parts functioning together, the purpose of which is to perform a particular operation. In this book, we shall call any physical object to be controlled (such as a mechanical device, a heating furnace, a chemical reactor, or a spacecraft) a plant. Processes. The Merriam-Webster Dictionary defines a process to be a natural, progressively continuing operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead toward a particular result or end; or an artificial or voluntary, progressively continuing operation that consists of a series of controlled actions or movements systematically directed toward a particular result or end. In this book we shall call any operation to be controlled a process. Examples are chemical, economic, and biological processes.

Types of Control System

i) **Feedback Control Systems:** A system that maintains a prescribed relationship between the output and the reference input by comparing them and using the difference as a means of control is called a feedback control system. An example would be a room temperature control system. By measuring the actual room temperature and comparing it with the reference temperature (desired temperature), the thermostat turns the heating or cooling equipment on or off in such a way as to ensure that the room temperature remains at a comfortable level regardless of outside conditions.

   Feedback control systems are not limited to engineering but can be found in various Non engineering fields as well. The human body, for instance, is a highly advanced feedback control system. Both body temperature and blood pressure are kept constant by means of physiological feedback. In fact, feedback performs a vital function: It makes the human body relatively insensitive to external disturbances, thus enabling it to function properly in a changing environment.

ii) **Closed-Loop Control Systems:** Feedback control systems are often referred to as closed-loop control systems. In practice, the terms feedback control and closed-loop control are used interchangeably. In a closed-loop control system the actuating error signal, which is the difference between the input signal and the feedback signal (which may be the output signal itself or a function of the output signal and its derivatives and/or integrals), is fed to the controller so as to reduce the error and bring the output of the system to a desired value. The term closed-loop control always implies the use of feedback control action in order to reduce system error.

iii) **Open-Loop Control Systems:** Those systems in which the output has no effect on the control action are called open-loop control systems. In other words, in an open loop control system the output is neither measured nor fed back for comparison with the input. One practical example is a washing machine. Soaking, washing, and rinsing in the washer operate on a time basis. The machine does not measure the output signal, that is, the cleanliness of the clothes.

In any open-loop control system the output is not compared with the reference input. Thus, to each reference input there corresponds a fixed operating condition; as a result, the accuracy of the system depends on calibration. In the presence of disturbances, an open-loop control system will not perform the desired task. Open-loop control can be used, in practice, only if the relationship between the input and output is known and if there are neither internal nor external disturbances. Clearly, such systems are not feedback control systems. Note that any control system that operates on a time basis is open loop. For instance, traffic control by means of signals operated on a time basis is another example of open-loop control.

Closed-Loop versus Open-Loop Control Systems

An advantage of the closed-loop control system is the fact that the use of feedback makes the system response relatively insensitive to external disturbances and internal variations in system parameters. It is thus possible to use relatively inaccurate and inexpensive components to obtain the accurate control of a given plant, whereas doing so is impossible in the open-loop case.

From the point of view of stability, the open-loop control system is easier to build because system stability is not a major problem. On the other hand, stability is a major problem in the closed-loop control system, which may tend to overcorrect errors and thereby can cause oscillations of constant or changing amplitude.

It should be emphasized that for systems in which the inputs are known ahead of time and in which there are no disturbances it is advisable to use open-loop control. Closed- loop control systems have advantages only when unpredictable disturbances and/or un- predictable variations in system components are present. Note that the output power rating partially determines the cost, weight, and size of a control system. The number of components used in a closed-loop control system is more than that for a corresponding open-loop control system. Thus, the closed-loop control system is generally higher in cost and power. To decrease the required power of a system, open-loop control may be used where applicable. A proper combination of open-loop and closed-loop controls is usually less expensive and will give satisfactory overall system performance.

Advantages and disadvantages of open-loop control systems

a) **Advantages**

1. Simple construction and ease of maintenance.

2. Less expensive than a corresponding closed-loop system.

3. There is no stability problem.

4. Convenient when output is hard to measure or measuring the output precisely is economically not feasible. (For example, in the washer system, it would be quite expensive to provide a device to measure the quality of the washer's output, cleanliness of the clothes.)

b) Disadvantages

1. Disturbances and changes in calibration cause errors, and the output may be different from what is desired.

2. To maintain the required quality in the output, recalibration is necessary from time to time.

Feedback systems has the following features:

- reduced effect of nonlinearities and distortion
- Increased accuracy
- Increased bandwidth
- Less sensitivity to variation of system parameters
- Tendency towards oscillations
- Reduced effects of external disturbances

The general block diagram of a control system is shown below.

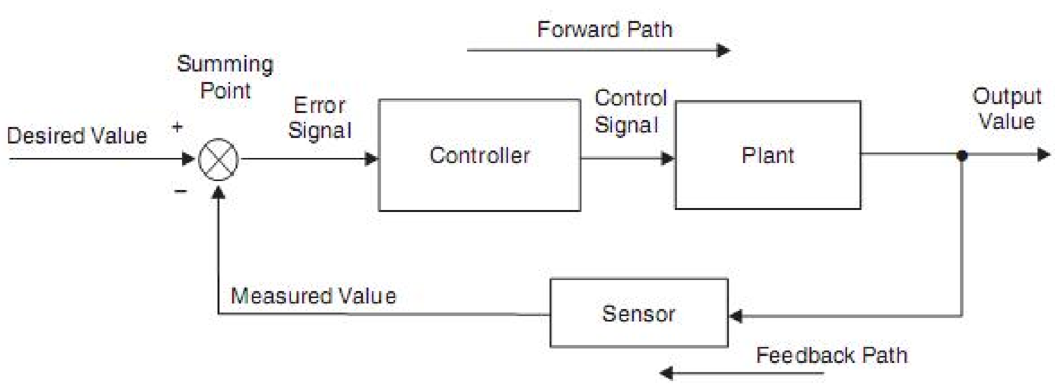


Figure: Closed-loop control system

Some Definitions

Reference input – It is the actual signal input to the control system.

Output (Controlled variable) – It is the actual response obtained from a control system.

Actuating error signal – It is the difference between the reference input and feedback signal.

Controller – It is a component required to generate control signal to drive the actuator.

Control signal – The signal obtained at the output of a controller is called control signal.

Actuator – It is a power device that produces input to the plant according to the control signal, so that output signal approaches the reference input signal.

Plant – The combination of object to be controlled and the actuator is called the plant.

Feedback Element – It is the element that provides a mean for feeding back the output quantity in order to compare it with the reference input.

Servomechanism – It is a feedback control system in which the output is mechanical position, velocity, or acceleration.

Example of Control Systems

Toilet tank filling system:

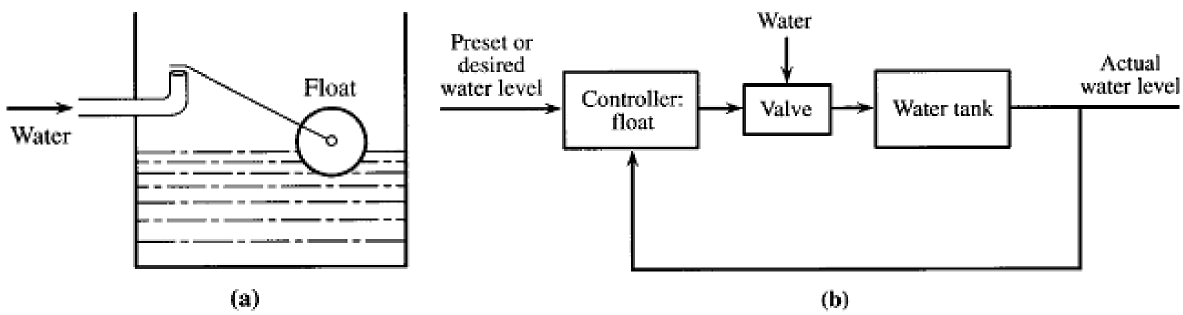


Figure: Toilet tank filling system

CHAPTER TWO

MODELING AND REPRESENTATION OF CONTROL SYSTEMS

Introduction

In studying control systems the reader must be able to model dynamic systems and analyze dynamic characteristics. A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately or, at least, fairly well. Note that a mathematical model is not unique to a given system. A system may be represented in many different ways and, therefore, may have many mathematical models, depending on one's perspective.

The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular system, for example, Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems. We must always keep in mind that deriving reasonable mathematical models is the most important part of the entire analysis of control systems.

We use mathematical models of physical systems to design and analyze control systems. Mathematical models are described by ordinary differential equations. If the coefficients of the describing differential equations are function of time, then the mathematical model is linear time-varying. On the other hand, if the coefficients describing differential equations are constants, the model is linear time-invariant.

The differential equations describing a LTI system can be reshaped into different forms for the convenience of analysis. For transient response or frequency response analysis of single-input-single-output linear systems, the transfer function representation is convenient. On the other hand, when the system has multiple inputs and outputs, the vector-matrix notation may be more convenient.

Powerful mathematical tools like Fourier and Laplace transforms are available for linear systems. Unfortunately no physical system in nature is perfectly linear. Certain assumptions must always be made to get a linear model. In the presence of strong nonlinearity or in presence of distributive effects it is not possible to obtain linear models.

A commonly adopted approach is to build a simplified linear model by ignoring certain nonlinearities and other physical properties that may be present in a system and thereby get an approximate idea of the dynamic response of the system. A more complete model is then built for more complete analysis.

Mathematical Models: Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models. For example, in optimal control problems, it is advantageous to use state-space representations. On the other hand, for the transient-response or frequency-response analysis of single-input-single-output, linear, time-invariant systems, the transfer function representation may be more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computer tools can be used for analysis and synthesis purposes.

Simplicity versus Accuracy: In obtaining a mathematical model, we must make a compromise between the simplicity of the model and the accuracy of the results of the analysis. In deriving a reasonably simplified mathematical model, we frequently find it necessary ***to ignore certain inherent physical properties*** of the system. In particular, if a linear lumped-parameter mathematical model (that is, one employing ordinary differential equations) is desired, it is always necessary to ignore certain nonlinearities and distributed parameters that may be present in the physical system. If the effects that these ignored properties have on the response are small, good agreements will be obtained between the results of the analysis of a mathematical model and the results of the experimental study of the physical system?

We must be well aware of the fact that a linear lumped-parameter model, which may be valid in low-frequency operations, may not be valid at sufficiently high frequencies since the neglected property of distributed parameters may become an important factor in the dynamic behavior of the system. For example, the mass of a spring may be neglected in low-frequency operations, but it becomes an important property of the system at high frequencies. (For the case where a mathematical model involves considerable errors, robust control theory may be applied.)

Differential and difference equations of systems

I. Modelling of Electrical System

Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors.

Table2.1summarises the components summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

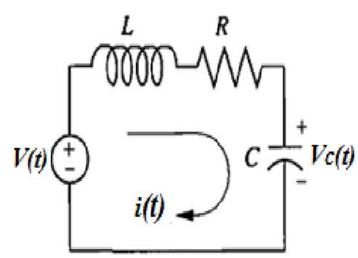
Our guiding principles are Kirchhoffs laws. We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and the equate the result to zero. From these relationships we can write the differential equations for the circuit.

Component	Voltage-current	Current-voltage	Voltage-charge
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$

Example 2.1: Find the deferential function relating the capacitor voltage, Vc(t),to the input voltage, V (t)?

Using mesh analysis: Summing the voltages around the loop, assuming zero initial conditions,

Yields the differential equation for this network as:



$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \text{ - - - -(2.1)}$$

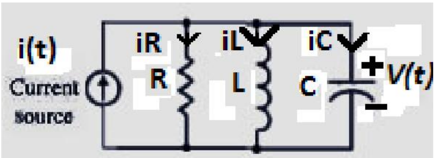
From the voltage-charge relationship for a capacitor in Table 2.1,

$$q(t) = Cv_C(t) \text{ - - - - - - - - -(2.2)}$$

Substituting Eq. (2.1) into Eq. (2.2) yields

$$\boxed{LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)}$$

Example 2.2:For parallel circuit



$$i = iR + iL + IC\ but \qquad \emptyset = \int v d(t) = v = \frac{d\emptyset}{dt}$$

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = i(t)$$

$$\boxed{v\frac{d^2\emptyset}{t^2} + \frac{1}{R}\frac{d\emptyset}{dt} + \frac{1}{L}\emptyset = i(t)}$$

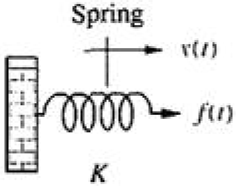
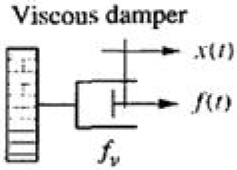
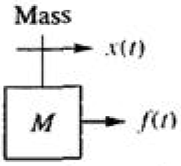
i. **Modelling of Mechanical System**

Mechanical system can be classified as:

1. Translational
2. Rotational

1 .Translational Mechanical system

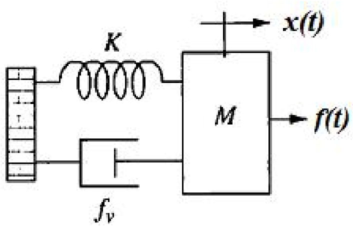
There are three characterizing elements in mechanical translational system: **spring, viscous damper and mass** and they can be modeled by differential equation as shown in the table below.

Component	Force-velocity	Force-displacement Differential Equation
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	**Hooks Law** $f(t) = Kx(t)$
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v\frac{dx(t)}{dt}$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	**Newtons Law** $f(t) = M\frac{d^2x(t)}{dt^2}$

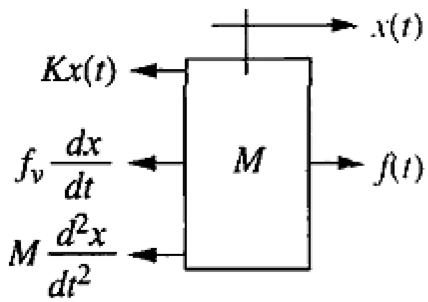
Note: The following set of symbols and units is used throughout this Note: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipates energy. The two energy-storage elements are analogous to the two electrical energy-storage elements, the inductor and capacitor. The energy dissipater is analogous to electrical resistance. Let us take a look at the mechanical elements, which are shown in table above. In the table above K, fv, and M are called spring constant, coefficient to fv is viscous friction, and mass, respectively.

Example 2.3: Find the differential equation f(t) for the mechanical system below?



The free body diagram is:

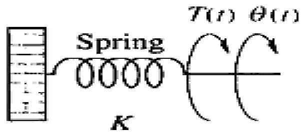
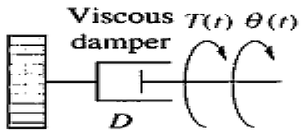
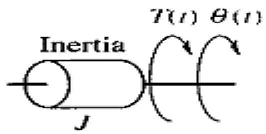


$$\boxed{M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)}$$

2. Rotational mechanical systems

Handled the same way as translational mechanical systems, except that torque replaces forceandangulardisplacementreplacestranslationaldisplacement.The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation.

Table2.5shows the components along with the relationships between torque and angular velocity, as well as angular displacement. Notice that the symbols for the

Component	Torque-angular velocity	Torque-angular displacement
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Also notice that the term associated with the mass is replaced by inertia. The Values of K,D, and J are called spring constant, coefficient of viscous friction, and moment of inertia ,respectively:

Laplace transform Review

A system represented by a differential equation is difficult to model as a block diagram. Thus, we now lay the ground work for the Laplace transform, with which we can represent the input, output, and system as separate entities.

$$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-) \qquad \text{Differentiation theorem}$$

$$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-) \qquad \text{Differentiation theorem}$$

$$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-) \qquad \text{Differentiation theorem}$$

$$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s} + \frac{1}{s}\int_{0-}^{t} f(\tau)d\tau \qquad \text{Integration theorem}$$

Transfer function of control systems

Is a function that algebraically relates a system's output to its input? The function will also allow us to algebraically combine mathematical representations of subsystems to yield a total system representation.

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

To derive the transfer function, we proceed according to the following steps.

1. Write the differential equation for the system.

2. Take the Laplace transform of the differential equation, assuming all initial conditions are zero.

3. Take the ratio of the output O(s) to the input I(s). This ratio is the transfer function

Example 2.4: Find the transfer function for the differential function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

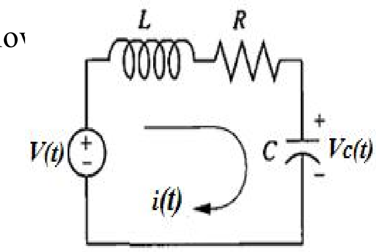
The transfer function, G(s),is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Using Matlab:

```
numf=[1]

denf=[1 2]

F=tf(numf,denf)
```

Example 2.5: Find the transfer function of the electrical circuit shown belo



$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

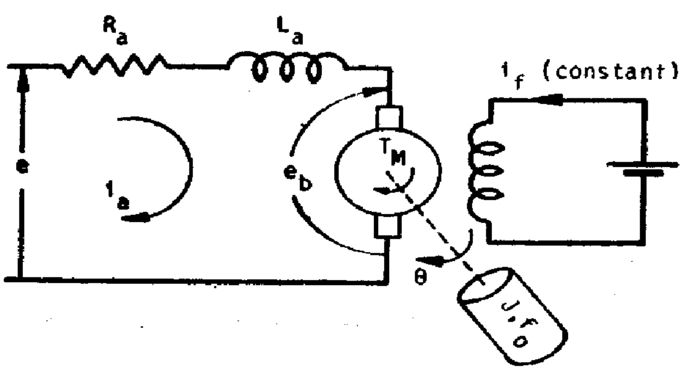
Taking the Laplace transform assuming zero initial conditions, rearranging terms,

And simplifying yields. $(LCs^2 + RCs + 1)V_C(s) = V(s)$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\boxed{\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}$$

A. Transfer Function of Armature Controlled DC Motor



$$\boxed{\phi = K_f i_f ; T_M = K_1 K_f i_f i_a = K_T i_a}$$

$$\boxed{e_b = K_b \frac{d\theta}{dt} ; L_a \frac{di_a}{dt} + R_a i_a + e_b = e}$$

$$\boxed{J\frac{d^2\theta}{dt^2} + f_0 \frac{d\theta}{dt} = T_M = K_T i_a}$$

In Laplace domain,

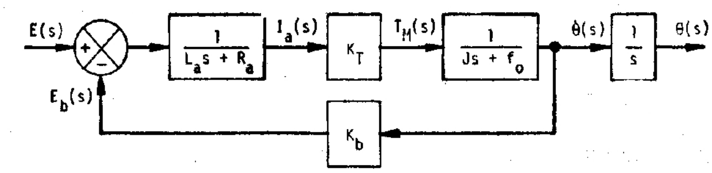
$$\begin{cases} E_b(s) = K_b s\theta(s) \\ (L_a s + R_a) I_a(s) = E(s) - E_b(s); \\ (Js^2 + f_0 s)\theta(s) = K_T I_a(s) \end{cases} \Rightarrow G(s) = \frac{\theta(s)}{E(s)} = \frac{K_T}{s\left[(R_a + sL_a)(Js + f_0) + K_T K_b\right]}$$

Neglecting L_a, $G(s) = \dfrac{K_T / R_a}{Js^2 + s(f_0 + K_T K_b / R_a)} = \dfrac{K_T / R_a}{s(Js + f)} = \dfrac{K_m}{s(s\tau_m + 1)}$;

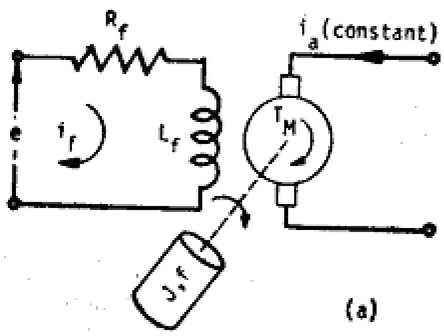
where, $f = f_0 + K_T K_b / R_a$ and $K_m = K_T / R_a f$; $\tau_m = J / f$.

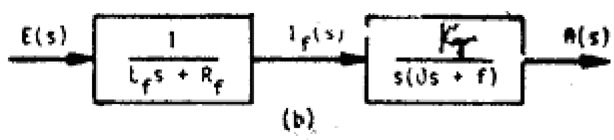
K_m and τ_m are called the motor gain and time constant respectively. These two parameters are usually supplied by the manufacturer.

The block diagram model is,



B. Transfer Function of a Field-controlled DC Motor





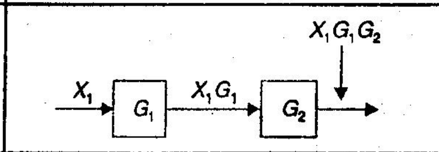
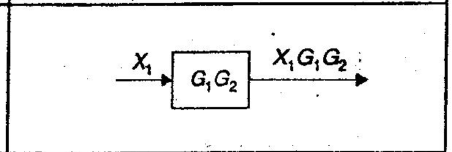
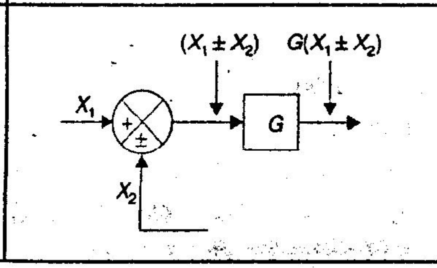
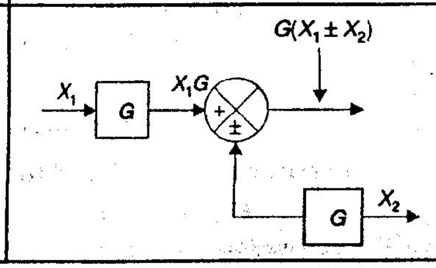
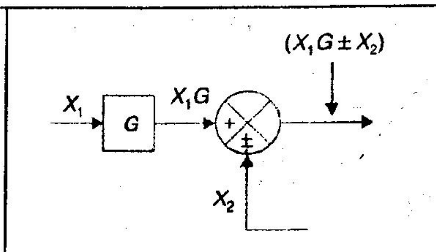
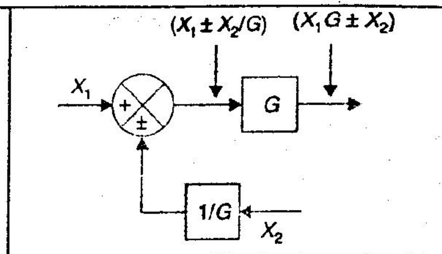
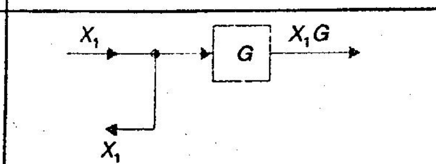
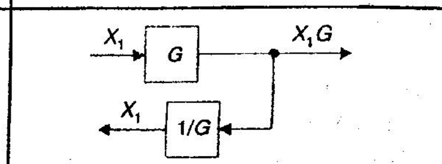
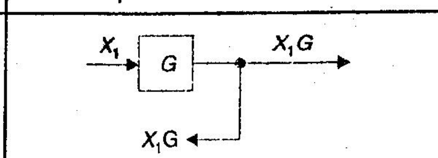
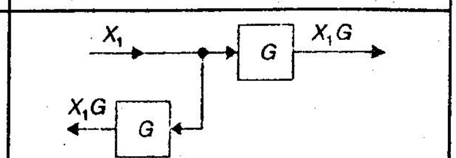
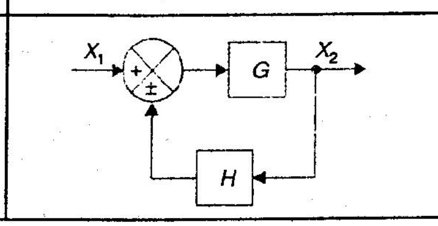
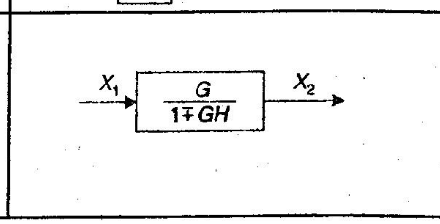
$$\begin{cases} T_M = K_1 \varphi i_a = K_1 K_f i_f i_a = K_T' i_f \\ L_f \dfrac{di_f}{dt} + R_f i_f = e \\ J \dfrac{d^2\theta}{dt^2} + f \dfrac{d\theta}{dt} = T_M = K_T' i_f \end{cases} ; \qquad \Rightarrow \qquad \begin{cases} (L_f s + R_f) I_f(s) = E(s) \\ (Js^2 + fs)\theta(s) = T_M(s) = K_T' I_f(s) \end{cases}$$

We obtain,

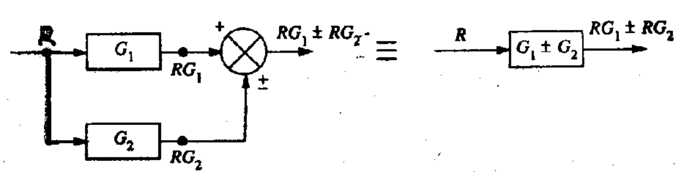
$$\boxed{G(s) = \frac{\theta(s)}{E(s)} = \frac{K_T'}{s(L_f s + R_f)(Js + f)}}$$

Block Diagram Algebra

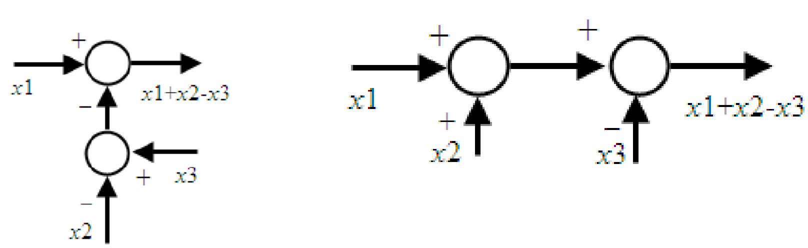
A complex system is represented by the interconnection of the blocks for individual elements. Evaluation of complex system requires simplification of block diagrams by block diagram rearrangement. Some of the important rules are given in figure below.

Rule	*Original diagram*	*Equivalent diagram*
1. Combining blocks in cascade		
2. Moving a summing point after a block		
3. Moving a summing point ahead of a block		
4. Moving a take off point after a block		
5. Moving a take off point ahead of a block		
6. Eliminating a feedback loop		

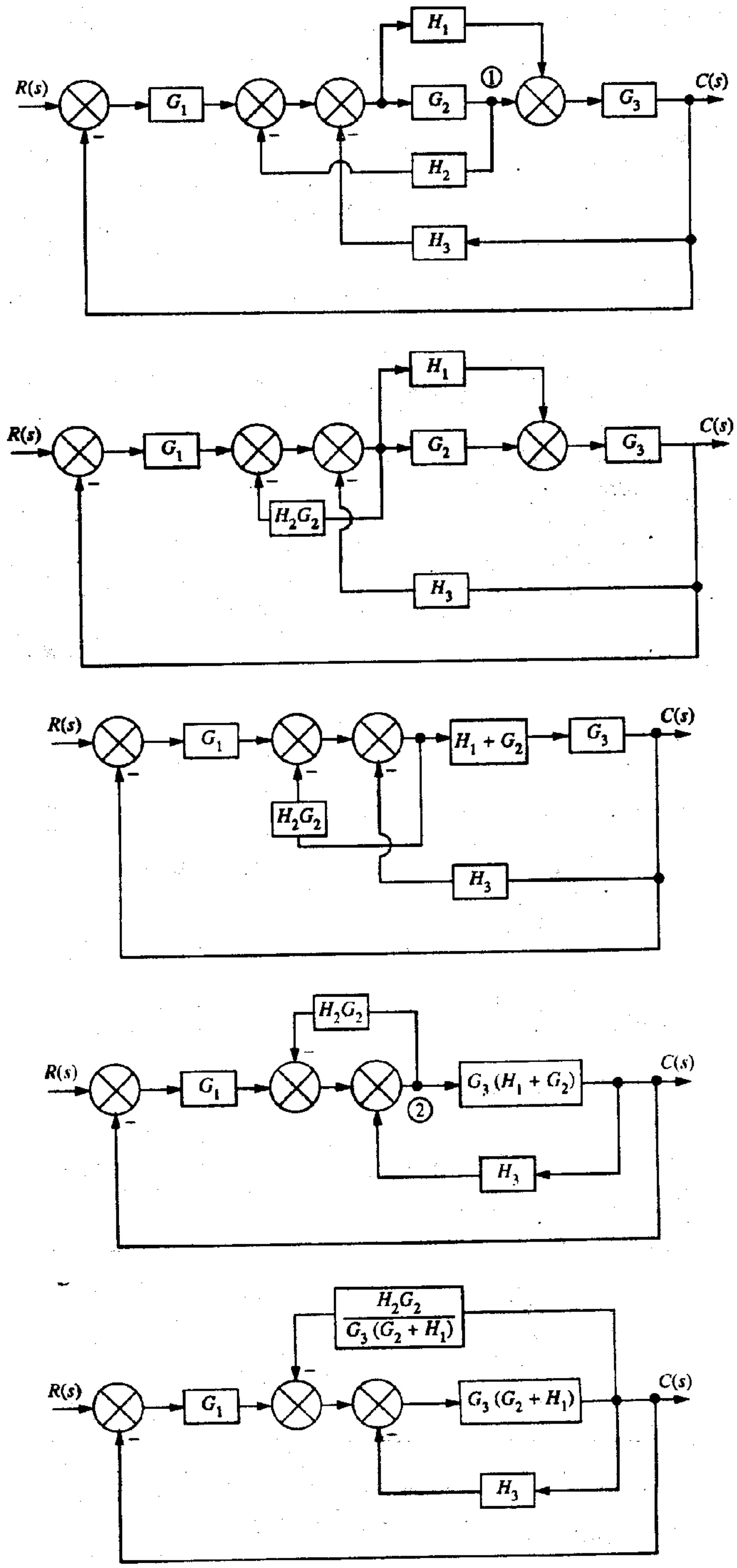
7. Combining Blocks in Parallel

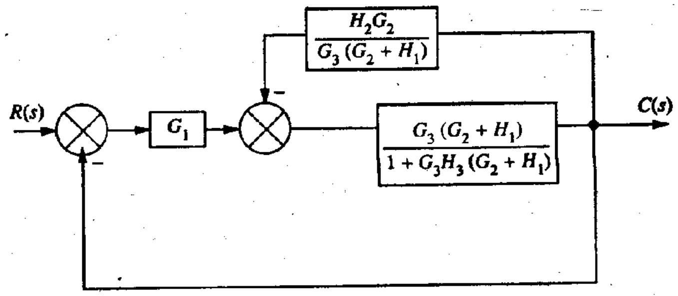


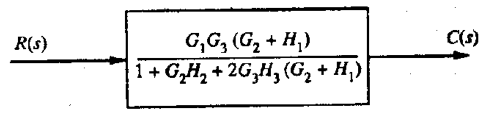
8. Moving summing point : ≡



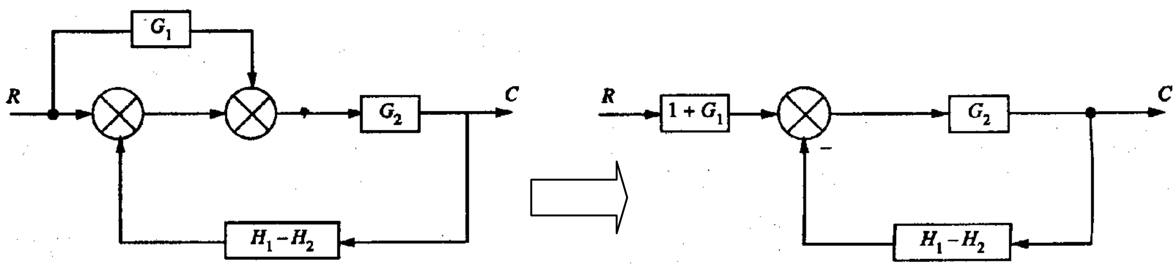
Example: Simplify the block diagram shown in Figure below.

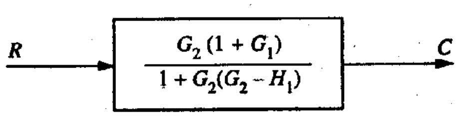






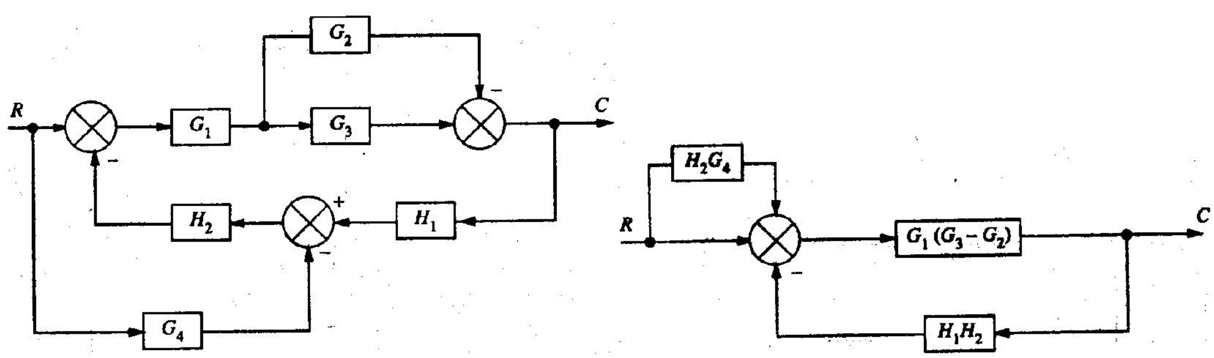
Example: Obtain the transfer function C/R of the block diagram shown in Figure below.



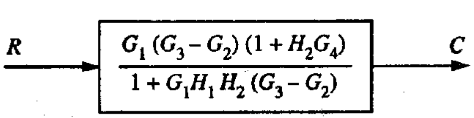


[Ans]

Example: Derive the transfer function of the system shown below.

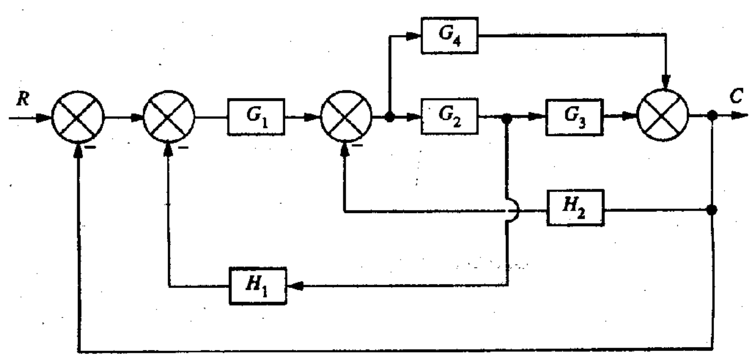


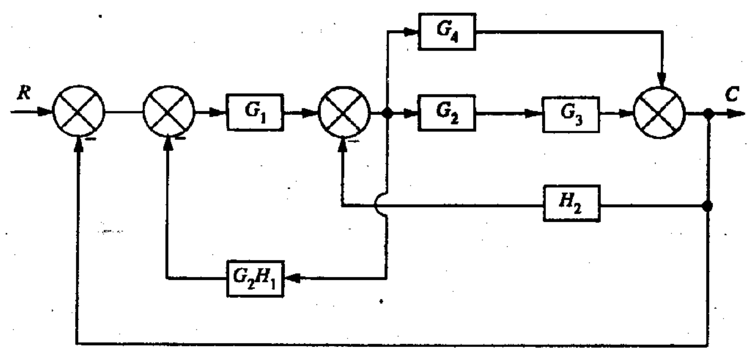
(a) (b)

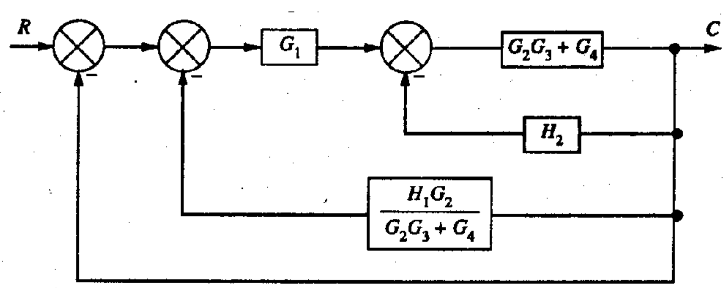


[Answer]

Example: Derive the transfer function of the system shown below.

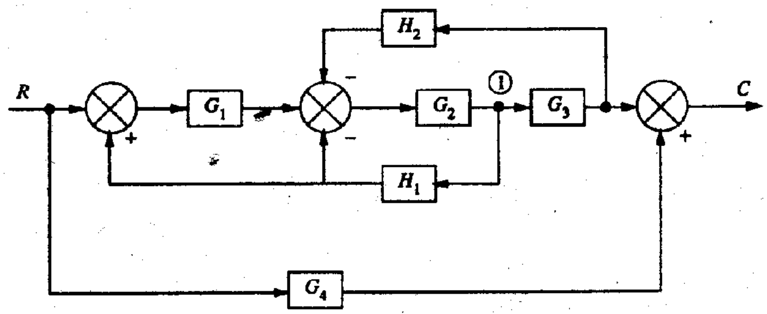


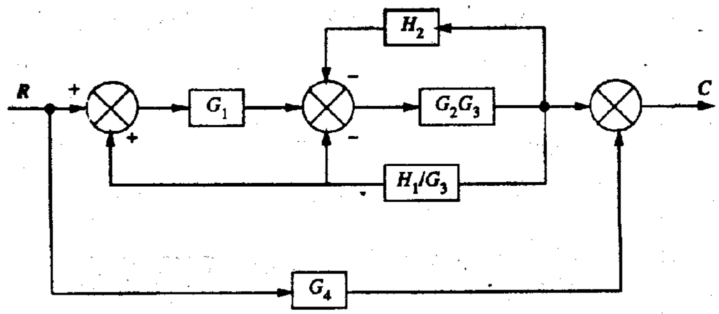




$$\frac{C}{R} = \frac{G_1\,[G_2\,G_3 + G_4)}{1 + H_2\,[G_2\,G_3 + G_4] + G_1\,G_2\,H_1 + G_1\,[G_2\,G_3 + G_4]}$$

Example: Find the transfer function of the following system.

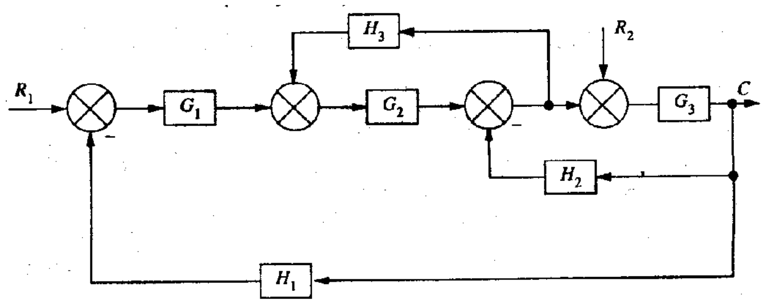




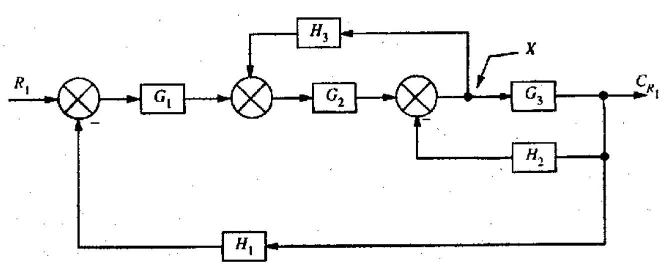
$$\frac{C}{R} = G_4 + \frac{G_1 G_2 G_3}{1 + H_2 G_2 G_3 + G_2 H_1 - H_1 G_1 G_2}$$

{Answer}

Example: Find the output of the system shown below.

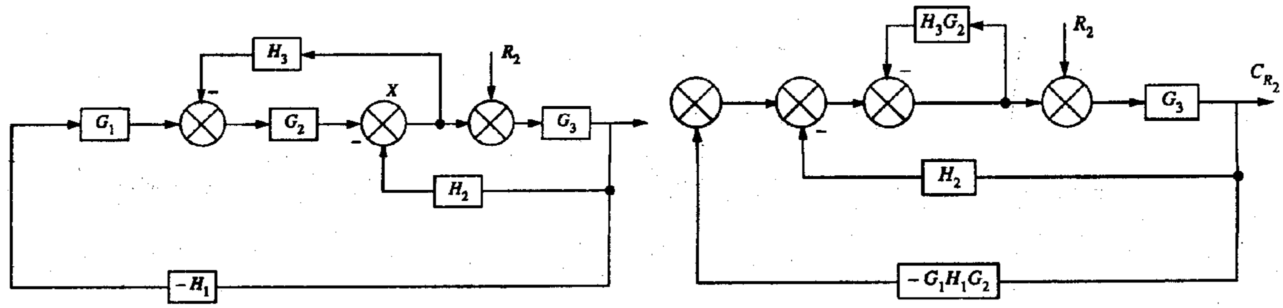


For Input R_1:



$$C_{R_1} = \left[\frac{G_1 G_2 G_3}{1 + G_3 H_2 + H_3 G_2 + G_1 G_2 G_3 H_1}\right] R_1 \quad \text{...} (1)$$

For input R_2:



$$C_{R_2} = \left[\frac{G_3\,[1 + G_2 H_3]}{1 + G_2 H_3 + G_3\,[G_1 G_2 H_1 + H_2]}\right] R_2 \qquad \text{...} (2)$$

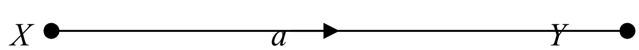
$$C = C_{R_1} + C_{R_2}$$

$$C = \left[\frac{G_1 G_2 G_3}{1 + G_3 H_2 + G_2 H_3 + G_1 G_2 G_3 H_1}\right] R_1 + \left[\frac{G_3\,[1 + G_2 H_3]}{1 + G_2 H_3 + G_3\,[G_1 G_2 H_1 + H_2]}\right] R_2$$

{Answer}

Signal Flow Graph

SFG is a diagram that represents a set of simultaneous linear algebraic equations which describe a system. Let us consider an equation, $Y = aX$. It may be represented graphically as,



where ‘a’ is called ***transmittance*** or transmission function.

Definitions in SFG

Node – A system variable, the value of which equals the sum of all incoming signals at the node.

Branch – A directed line segment joining two nodes.

Input/ Output node – node having only one outgoing/ incoming branch.

Path – A traversal of connected branches in the direction of branch arrows.

Forward path – A path from input to output node.

Loop – A closed path that originates and terminates on the same node.

Self-loop – A loop containing one branch.

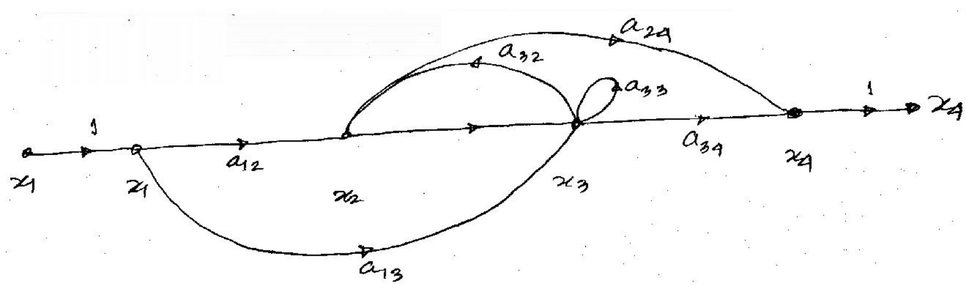
Non-touching loops – Loops which do not have a common node.

Gain – Transmittance of a branch.

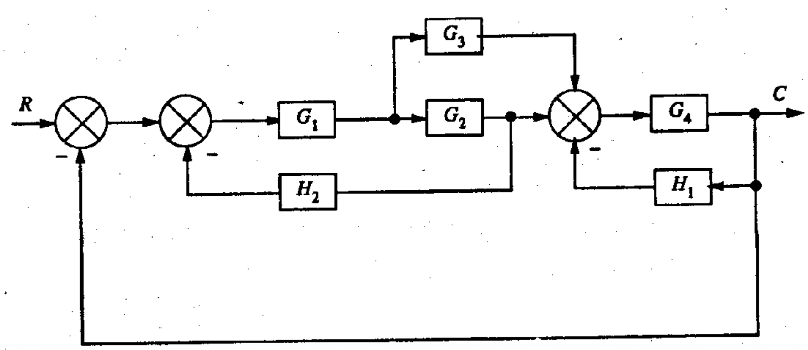
Construction of SFGs

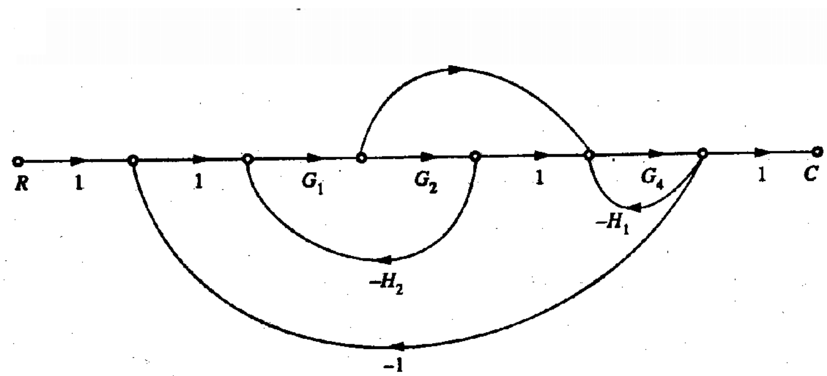
The SFG of a system can be constructed from the describing equations:

$$
\begin{aligned}
x_2 &= a_{12}x_1 + a_{32}x_3 \\
x_3 &= a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \\
x_4 &= a_{24}x_4 + a_{34}x_3
\end{aligned}
$$



SFG from Block Diagram





Each variable in the block diagram becomes a node, and each block becomes a branch.

Mason's Gain Formulae

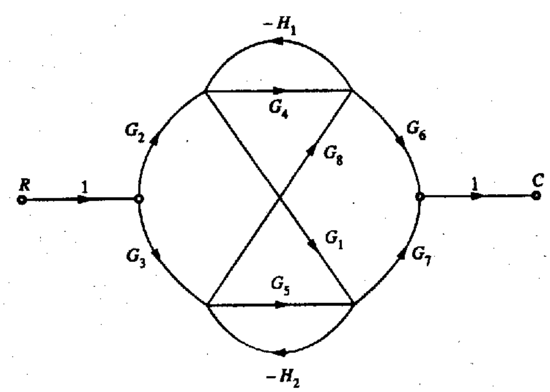
It is possible to write the overall transfer function of a system through inspection of SFG using Mason's gain formulae given by, $T = (\sum_i P_i \Delta_i)/\Delta$.

where T = overall gain of the system, P_i = path gain of *ith* forward path, $\Delta =$ determinant of SFG, $\Delta_i =$ value of Δ for that part of the graph not touching the *ith* forward path.

$\Delta = 1 - \sum_j P_{j1} + \sum_j P_{j2} - \sum_j P_{j3} + \cdots$ = 1 – [sum of loop gain of all individual loops] + [sum of all gain-products of two non-touching loops] – [sum of all gain-products of three non-touching loops] + ……;

$P_{jk} = j$th product of k non-touching loops.

Example



1. There are 6 forward paths with path gains

$$P_1 = G_2 G_4 G_6$$
$$P_2 = G_3 G_5 G_7$$
$$P_3 = G_2 G_1 \cdot G_7$$
$$P_4 = G_3 G_8 G_6$$
$$P_5 = -G_2 G_1 \cdot H_2 G_8 \cdot G_6$$
$$P_6 = -G_3 G_8 H_1 G_1 G_7$$

2. There are three $P_{11} = -H_1 G_4$, $P_{21} = -H_2 G_5$, $P_{31} = G_1 H_2 G_8 H_1$ individual loops with loop gains

3.There is only one combination of two non-touching loops

$$P_{12} = H_1 H_2 G_4 G_5$$

4. There are no combinations of more than two non-touching loops.

5. Hence,

$$\begin{aligned}\Delta &= 1 - [-H_1 G_4 - H_2 G_5 + G_1 H_2 G_8 H_1] + [H_1 H_2 G_4 G_5] \\ &= 1 - G_1 H_2 G_8 H_1 + H_2 G_5 - G_1 H_2 G_8 H_1 + H_1 H_2 G_4 G_5\end{aligned}$$

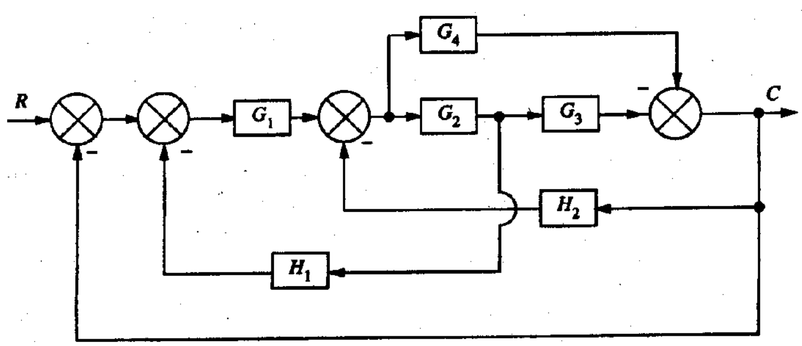
$\Delta_1 = 1 - (-H_2 G_5) = 1 + H_2 G_5$; $\Delta_2 = 1 - (-H_1 G_4) = 1 + H_1 G_4$;

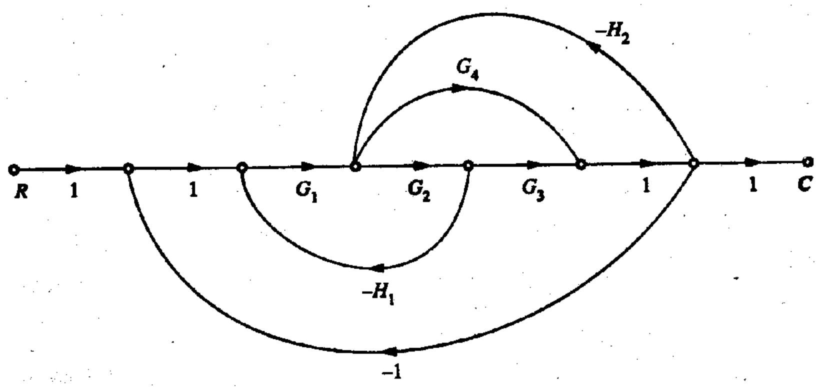
$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$

Thus, $T = \dfrac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{\Delta}$, where P_1, Δ_1, Δ etc. are derived before.

Example

Draw the SFG and determine C/ R for the block diagram shown in Figure below.





$$\frac{C}{R} = \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_4 + G_4 H_2}$$

{Answer}

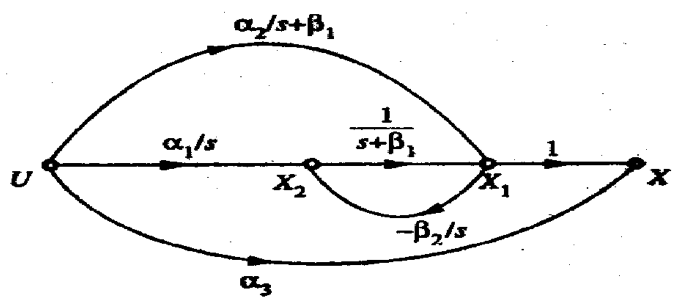
Example

For the system represented by the following equations, find the transfer function X(s)/U(s) by SFG technique.

$$\begin{aligned} x &= x_1 + \alpha_3 u \\ \dot{x}_1 &= -\beta_1 x_1 + x_2 + \alpha_2 u \\ \dot{x}_2 &= -\beta_2 x_1 + \alpha_1 u \end{aligned}$$

We need to Laplace transform the given sets of equations in order to represent differentiated variables.

$$\begin{aligned} X &= X_1 + \alpha_3 U \\ X_1 &= \frac{1}{s+\beta_1} X_2 + \frac{\alpha_2}{s+\beta_1} u \\ X_2 &= -\frac{\beta_2}{s} X_1 + \frac{\alpha_1}{s} u \end{aligned}$$

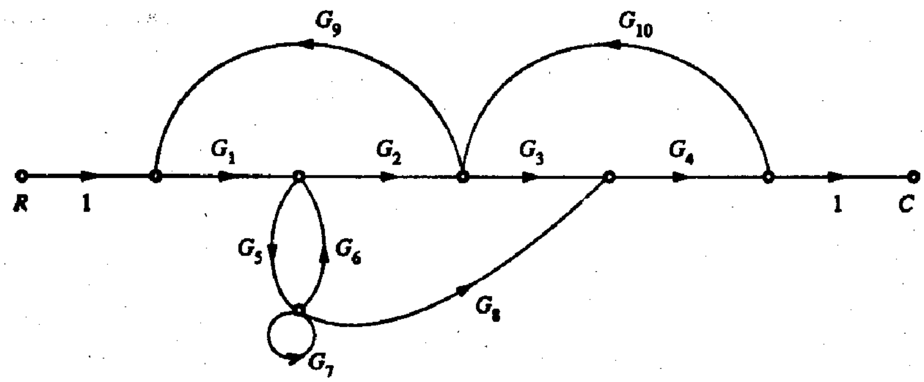


$$\frac{X(s)}{U(s)} = \frac{\alpha_1 + \alpha_2 s + \alpha_3 \cdot [s^2 + \beta_1 s + \beta_2]}{s^2 + \beta_1 s + \beta_2}$$

{Answer}

Example

Using Mason’s gain formulae find C/R of the SFG shown in Figure below.

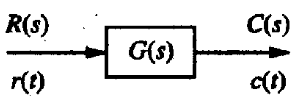


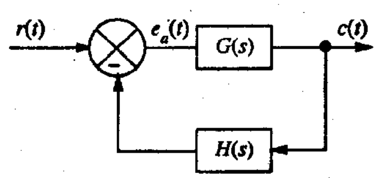
$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3G_4(1 - G_7) + G_1G_5G_8G_4}{1 - [G_1G_2G_9 + G_3G_4G_{10} + G_1G_5G_8G_4G_{10}G_9 + G_5G_6 + G_7] + [G_1G_2G_9G_7 + G_3G_4G_{10}G_5G_6 + G_3G_4G_{10}G_7]}$$

Feedback Characteristics of Control Systems

Consider the block diagram of the open-loop and the closed-loop system shown below.





For open-loop system, $C(s) = G(s)R(s)$

For closed-loop system, $C(s) = G(s)E_a(s) = G(s)[R(s) - H(s)C(s)]$

Hence, we have, $C(s) = \dfrac{G(s)}{1+G(s)H(s)}R(s)$ and, $E_a(s) = \dfrac{1}{1+G(s)H(s)}R(s)$

It is seen from the above equations that in order to reduce error, the loop-gain $G(s)H(s)$ should be made large over the range of frequencies of interest, i.e., $|G(s)H(s)| \gg 1$.

1. Reduction of parameter variations by use of feedback

One of the important properties of negative feedback systems is the reduction in the sensitivity to the variation in the parameters of the forward path. In the design of control systems, it is important that the transfer function of the closed-loop system be relatively insensitive to small changes in the values of the parameters of the components in the forward path of the system.

Let μ be a parameter of G(s). Then the sensitivity of G(s) with respect to the parameter μ is defined as,

$$S_\mu^G = \frac{\text{Fractional change in } G(s)}{\text{Fractional change in } \mu} = \frac{\Delta G / G}{\Delta \mu / \mu} = \frac{\mu}{G} \cdot \frac{\Delta G}{\Delta \mu} \ .$$

Now, $$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)};$$

$$S_\mu^T = \frac{\mu}{T} \cdot \frac{\Delta T}{\Delta \mu} = \frac{\mu}{G} \cdot \frac{G}{T} \cdot \frac{\Delta G}{\Delta \mu} \frac{\Delta T}{\Delta G} = S_\mu^G \cdot (1+GH) \cdot \frac{1 + \cancel{GH} - \cancel{GH}}{(1+GH)^2} = \frac{S_\mu^G}{1 + G(s)H(s)}$$

Thus feedback has reduced sensitivity in the variation in μ by the factor $\dfrac{1}{1+GH}$.

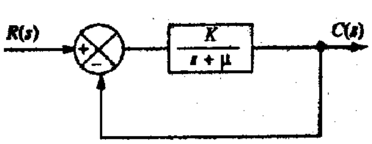
Again, $$S_\mu^T = \frac{\mu}{T} \cdot \frac{\Delta T}{\Delta \mu} = \frac{\mu}{H} \cdot \frac{H}{T} \cdot \frac{\Delta H}{\Delta \mu} \frac{\Delta T}{\Delta H} = S_\mu^H \cdot \frac{H(1+GH)}{\cancel{G}} \cdot \frac{-G.\cancel{G}}{(1+GH)^2} = -S_\mu^H \cdot \frac{GH}{1+GH} \cong -S_\mu^H \ .$$

It is seen that, the magnitude of two sensitivities are nearly equal for the variation of parameter in the feedback path. Thus, feedback does not reduce the sensitivity to variation in the parameter in feedback path.

Therefore, we can conclude that, G(s) in a closed-loop system may be less rigidly specified. On the other hand, we must be careful in accuracy of H(s) in the feedback loop.

2. Control over system dynamics by use of feedback

Let us consider the simple feedback system shown below.



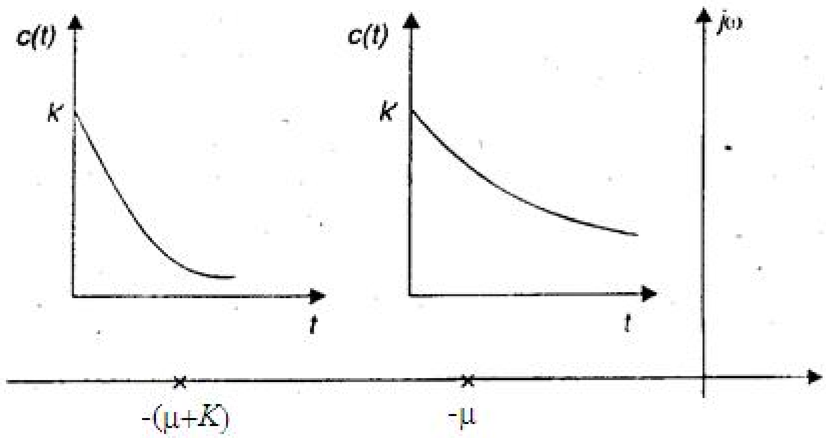
The open-loop transfer function is, $G(s) = \dfrac{K}{s+\mu}$.

The impulse response for the non-feedback system would be, $c(t) = Ke^{-\mu t}u(t) = Ke^{-t/\tau_1}u(t)$.

The closed-loop transfer function of the above system is, $T(s) = \dfrac{K}{s + \mu + K}$.

The impulse response of the closed-loop system is, $c(t) = Ke^{-(\mu+K)t}u(t) = Ke^{-t/\tau_2}u(t)$.

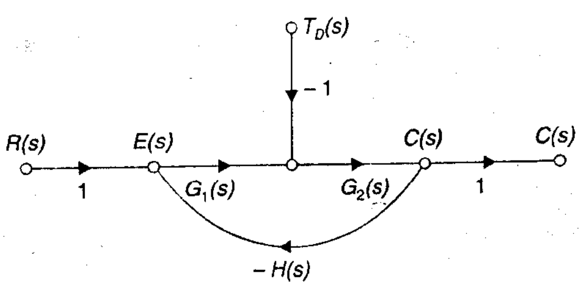
The location of the pole and the dynamic response of the non-feedback and feedback system are shown in Figure below.

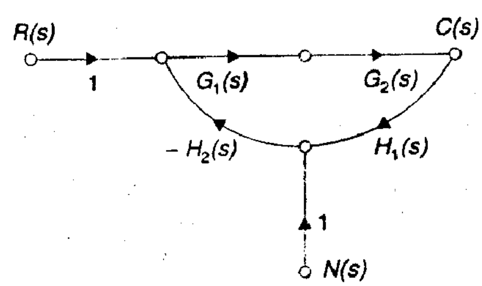


It is seen that the time-constant of open-loop system is $\tau_1 = 1/\mu$ and that of closed-loop system is $\tau_2 = 1/(\mu + K)$. As the time-constant of closed-loop system is less, its dynamic response is faster than the same of the open-loop system.

3. Control of the effect of disturbance signal by use of feedback

A. Disturbance in the forward path





$$\frac{C_d(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1(s)G_2(s)H(s)} \cong \frac{-1}{G_1(s)H(s)}; \quad \text{or,} \quad C_d(s) = \frac{-T_d(s)}{G_1(s)H(s)}$$

If $G_1(s)$ is made very large, the effect of disturbance on the output will be very small.

B. Disturbance in the feedback path

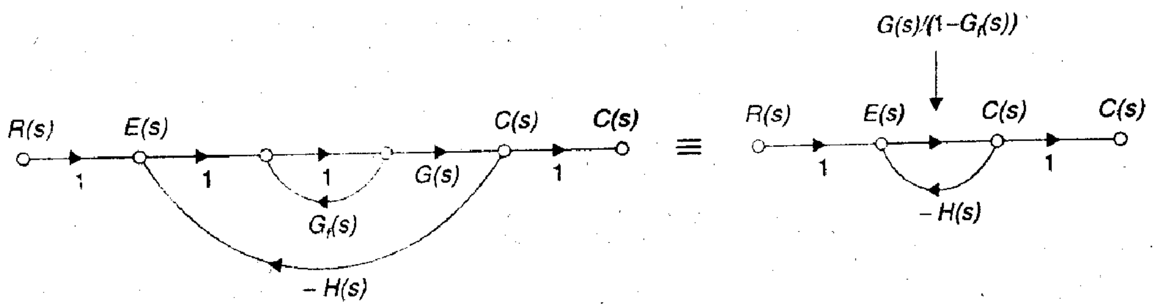
$$\frac{C_n(s)}{N(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} \cong \frac{-1}{H_1(s)}$$

Therefore, the effect of noise on output is, $C_n(s) \cong \dfrac{-1}{H_1(s)} \cdot N(s)$.

Thus, for the optimum performance of the system, the measurement sensor should be designed such that $H_1(s)$ is maximum. This is equivalent to maximizing the SNR of the sensor.

4. Regenerative Feedback

The regenerative feedback is sometimes used for increasing the loop gain of the feedback system. Figure in the following shows a feedback system where regenerative feedback occurs in the inner loop.



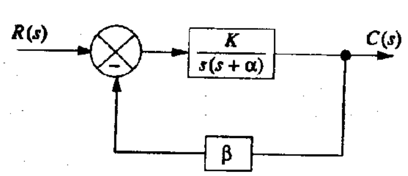
The open-loop gain is, $G_o(s) = \dfrac{G(s)}{1 - G_a(s)}$.

The system response is obtained as, $C(s) = \dfrac{R(s) \cdot G(s)/1 - G_a(s)}{1 + G_a(s)G(s)/1 - G_a(s)} = \dfrac{R(s) \cdot G(s)}{1 - G_a(s) + G(s)H(s)}$

When, $G_a(s) \gg 1$, $C(s) \cong \dfrac{R(s)}{H(s)}$. Due to high loop gain provided by the inner regenerative feedback loop, the closed-loop transfer function becomes insensitive to G(s).

Example

A position control system is shown below. Assume, *K*=10, $\alpha = 2$, $\beta = 1$. Evaluate: $S_K^T, S_\alpha^T, S_\beta^T$. For $r(t) = 2\cos 0.5t$ and a 5% change in K, evaluate the steady-state response and the change in steady-state response.



Here, $G(s) = \dfrac{K}{s(s+\alpha)}$, and $H(s) = \beta$

$$S_K^G = \frac{K}{G} \cdot \frac{dG}{dK} = s(s+\alpha) \cdot \frac{1}{s(s+\alpha)} = 1;$$

$$S_\alpha^G = \frac{\alpha}{G} \cdot \frac{dG}{d\alpha} = \frac{-\alpha}{s+\alpha} = \frac{-2}{s+2};\ S_\beta^H = \frac{\beta}{H} \cdot \frac{dH}{d\beta} = 1$$

$$S_K^T = \frac{S_K^G}{1+G(s)H(s)} = \frac{s(s+\alpha)}{s(s+\alpha)+K} = \frac{s^2+2s}{s^2+2s+10}$$

Therefore, $$S_\alpha^T = \frac{S_\alpha^G}{1+G(s)H(s)} = \frac{-\alpha}{s+\alpha}\cdot\frac{s(s+\alpha)}{s(s+\alpha)+K} = \frac{-2s}{s^2+2s+10}$$

$$S_\beta^T = \frac{-S_\beta^H \cdot G(s)H(s)}{1+G(s)H(s)} = \frac{-K}{s(s+\alpha)+K} = \frac{-10}{s^2+2s+10}$$

Now, $T(s) = \dfrac{K}{s^2+\alpha s+K\beta} = \dfrac{10}{s^2+2s+10}$; At $s = j0.5$, $T(j0.5) = 1.02e^{-j0.102}$

Thus, $c_{ss}(t) = 2.04\cos(0.5t - 0.102)$

Again, $S_K^T = \dfrac{K}{T}\cdot\dfrac{\Delta T}{\Delta K} \Rightarrow \dfrac{\Delta T}{T} = S_K^T \cdot \dfrac{\Delta K}{K} = \dfrac{s^2+2s}{s^2+2s+10}\cdot 0.05$

$$\Rightarrow \Delta T(s) = \frac{s^2+2s}{s^2+2s+10}\times 0.05\times\frac{10}{s^2+2s+10} = \frac{0.5s(s+2)}{(s^2+2s+10)^2}; \Rightarrow \Delta T(j0.5) = 0.005e^{-j4.672}$$

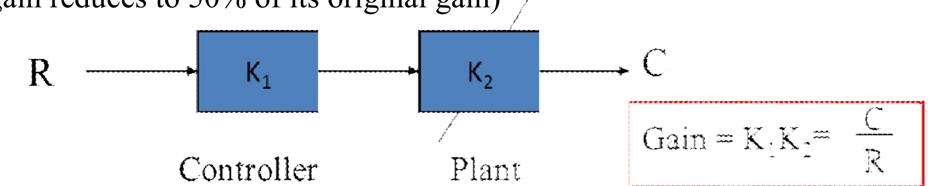
Thus, $\Delta c_{ss}(t) = \Delta T(j0.5)\times 2\cos 0.5t = 0.01\cos(0.5t - 4.672)$ {Answer}

Performance Characteristics of feedback control systems

The difference between the reference input $r(t)$ and controlled output $c(t)$ is known as error in control systems. To reduce error, feedback is used. Using feedback, not only this error is reduced but also the sensitivity of the system due to parameter variations and unwanted internal and external disturbances are also reduced.

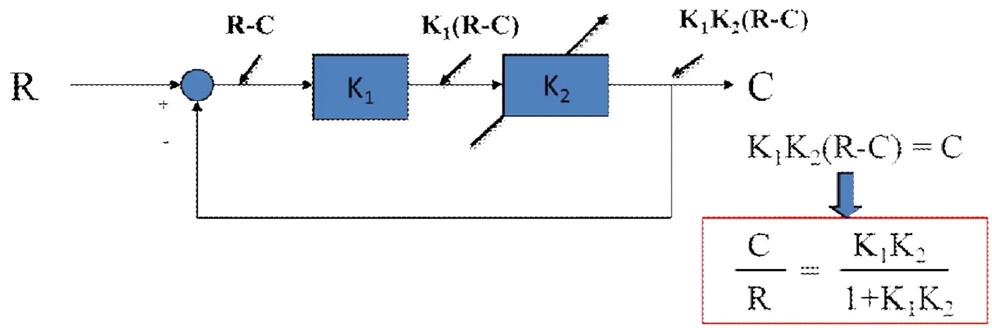
a. Effect of Feedback on sensitivity of parameter variation.

The open-loop system shown below, if K_2 is halved, then the system gain is also halved (i.e., the overall system gain reduces to 50% of its original gain)



‡ Note that for simplicity, we assume that K_1 and K_2 are constant. In general, they are frequency dependent.

Consider the closed-loop system shown below:



Assume that $K_1K_2 = 1$

If K_2 is halved, then the overall system gain reduces to 67% of the original gain.

Assume that $K_1K_2 = 9$

If K_2 is halved, then the overall system gain becomes 91% of the original gain.

The sensitivity is reduced as the loop gain (i.e. K_1K_2) is increased. Obvious advantage of using feedback.

b. Effect of Feedback on sensitivity of a control system

The parameters of a control system change due to environmental changes and other disturbances. This change has adverse effect on the system performance.

If the variable in a system be T due to the variation in parameter K of the system, the sensitivity of the system parameter T to the parameter K is given by

$$S = \frac{\%\ change\ in\ T}{\%\ change\ in\ K}$$

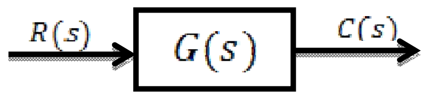
Mathematically, it can be written as

$$S_K^T = \frac{(\partial T/T)}{(\partial K/K)}$$

S_K^T Denotes the sensitivity of variable *T* with respect to the variations in parameter *K*. Let *T(s)* be the overall transfer function of a control system and its forward path gain *G(s)* be varying. The sensitivity of overall transfer function *T(s)* with respect to the variations in *G(s)* is given by

$$S_G^T = \frac{(\partial T(s)/T(s))}{(\partial G(s)/G(s))} \; or \; \frac{G(s)}{T(s)} * \frac{\partial G(s)}{\partial T(s)}$$

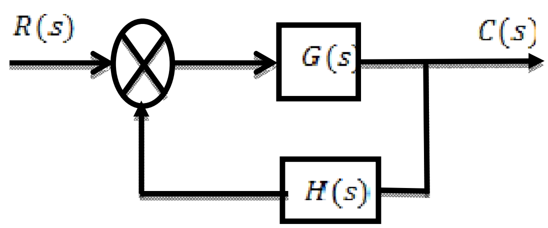
For open-loop system, *T(s) = G(s)*.



$$\frac{\partial G(s)}{\partial T(s)} = \frac{\partial G(s)}{\partial G(s)} = 1$$

$$S_G^T = \frac{G(s)}{T(s)} * \frac{\partial G(s)}{\partial T(s)} = \frac{G(s)}{G(s)} * \frac{\partial G(s)}{\partial G(s)} = 1 * \mathbf{1} = 1$$

For closed-loop system, $T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$



a) **Sensitivity *T(s)* with Respect to *G(s)***

$$\frac{\partial T(s)}{\partial G(s)} = \frac{[1 + G(s)H(s)] * 1 - [G(s)H(s)]}{[1 + G(s)H(s)]^2} = \frac{\mathbf{1}}{[1 + G(s)H(s)]^2}$$

The sensitivity of T(s) with respect to the variation of G(s) is given by

$$S_G^T = \frac{G(s)}{T(s)} * \frac{\partial T(s)}{\partial G(s)} = \frac{G(s)}{\left[\frac{G(s)}{1+G(s)H(s)}\right]} * \frac{1}{[1 + G(s)H(s)]^2}$$

$$S_G^T = \frac{1}{[1 + G(s)H(s)]^2} \qquad \text{.................................. (3.1)}$$

The sensitivity reduced by a factor $\frac{1}{1+G(s)H(s)}$ compared to open-loop sensitivity function due to the presence of feedback. In closed-loop systems, S_G^T is less sensitive to the variation of *G(s)*.

b) Sensitivity *T(s)* with Respect to *H(s)*

$$\frac{\partial T(s)}{\partial H(s)} = \frac{[1 + G(s)H(s)] * 0 - G(s)G(s)}{[1 + G(s)H(s)]^2} = -\frac{[G(s)]^2}{[1 + G(s)H(s)]^2}$$

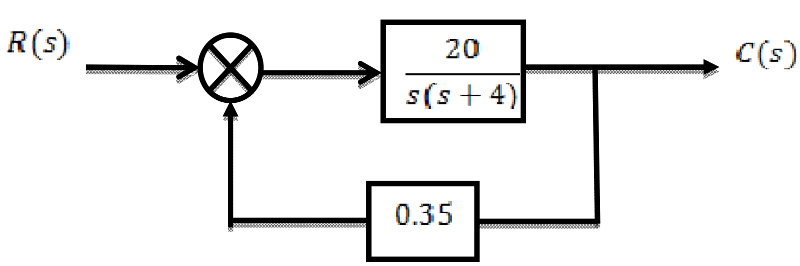
The sensitivity of the transfer function T(s) with respect the variation of H(s) is given by

$$S_G^T = \frac{H(s)}{T(s)} * \frac{\partial T(s)}{\partial H(s)} = \frac{H(s)}{\left[\frac{G(s)}{1+G(s)H(s)}\right]} * -\frac{[G(s)]^2}{[1 + G(s)H(s)]^2}$$

$$S_G^T = \frac{-G(s)H(s)}{1 + G(s)H(s)} \qquad \text{.................................. (3.2)}$$

Comparing eq. (3.1) and (3.2), we can conclude that a closed-loop system is more sensitive to the variations in the feedback path parameters compared to the variations in the forward path parameters.

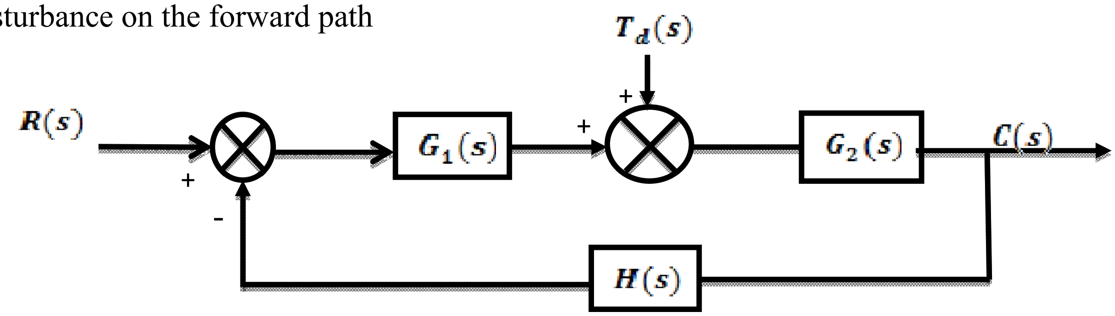
Exercise:- find the sensitivity of the overall transfer function of the system shown below with respect to (i) forward path transfer function and (b) feedback path transfer function. The value of $\omega = 1.2\, rad/sec.$



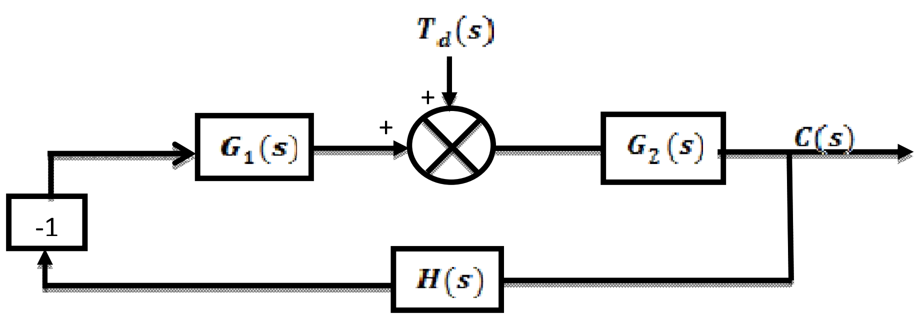
c. Effect of Feedback on external disturbance of a control system.

The output of control system becomes inaccurate due to some external disturbances such as high-frequency noise in electrical application, thermal noise in amplifier tubes, etc. The disturbance may be forward path feedback path or output of a system.

a) Disturbance on the forward path



To get the effect of disturbance on output, let us put *R(s) =0* as shown on the figure below



$$\frac{C(s)}{T_d(s)} = \frac{G_2(s)}{1 - G_2(s)[-G_1(s)H(s)]}$$

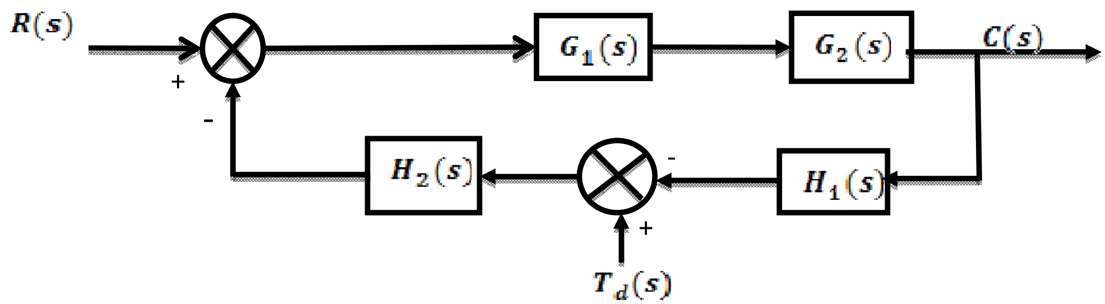
$$C(s) = \frac{T_d(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

Assuming $G_1(s)G_2(s)H(s) \gg 1$,

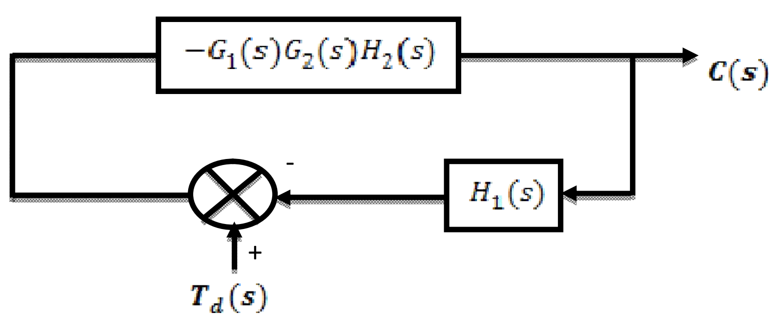
$$\frac{C(s)}{T_d(s)} = \frac{1}{G_1(s)H(s)}$$

From the above equation we can observe that $G_1(s)$ must be taken as large as possible to make the effect of disturbance on output as small as possible.

b) Disturbance on the feedback path



To study the effect of $T_d(s)$ on *C(s),* let input be *R(s) = 0.*

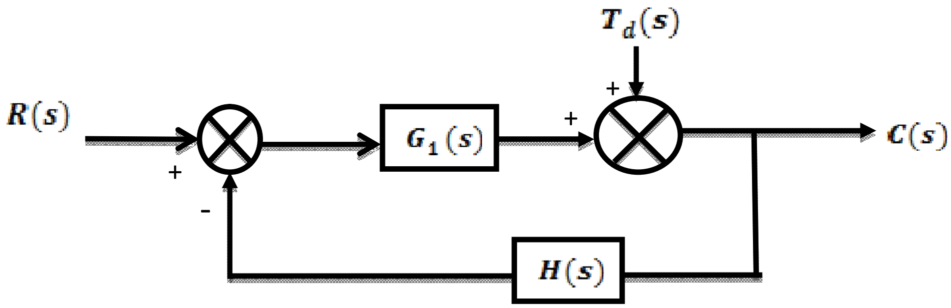


$$\frac{C(s)}{T_d(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1 + G_1(s)G_2(s)H_2(s)H_1(s)}$$

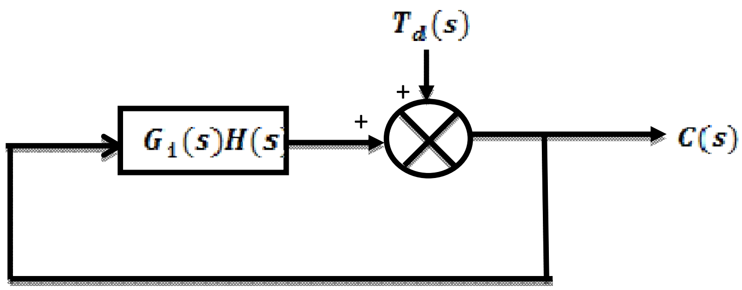
Assuming $G_1(s)G_2(s)H_2(s)H_1(s) \gg 1$

$$\frac{C(s)}{T_d(s)} = \frac{-1}{H_1(s)}$$

c) Disturbance on the output of a system



To study the effect of disturbance on output only, let us put *R(s) = 0*.



$$\frac{C(s)}{T_d(s)} = \frac{1}{1-[-G(s)H(s)]} = \frac{1}{1+G(s)H(s)}$$

Assuming $(s)H(s) \gg 1$,

$$\frac{C(s)}{T_d(s)} = \frac{1}{G(s)H(s)}$$

Therefore, changing the value of G(s) or H(s) or both, the effect of disturbance on output can be reduced.

CHAPTER THREE:

TIME DOMAIN ANALYSIS OF CONTROL SYSTEMS

INTRODUCTION:

In time-domain analysis the response of a dynamic system to an input is expressed as a function of time. It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

The time response of any system has two components: **transient response** and the **steady-state response**. $\lim_{t\to\infty} y_{tr}(t) = 0$

Transient response $y_{tr}(t)$is part of the response which goes to zero after interval of time.

The steady-state response $y_{ss}(t)$is part of the response which exists along time following any input signal (may be constant$y_{ss}(t) = C$, ramp$y_{ss}(t) = Ct$, sinusoidal $y_{ss}(t) = A sin(\omega t)$or parabolic$y_{ss}(t) = At^2$) thus, it depends on the input quantity therefore It is then examined using different test signals by final value theorem.

From the above two definitions the total response $y(t) = y_{tr}(t) + y_{ss}(t)$

Example: The total response of differential equation was determined as $y = \frac{1}{2} - \frac{1}{2}e^{-t}$. Clearly, the steady state response is given by $y_{ss} = \frac{1}{2}$. Since $\lim_{t\to\infty}\left\{-\frac{1}{2}e^{-t}\right\} = 0$,the transient response is $y_r = -\frac{1}{2}e^{-t}$

We have seen the time response of first order systems in the previous chapters. In this chapter we going to see second order time response of systems.

Time Response Analysis of Second order systems:

In the study of control systems, linear constant-coefficient second-order differential equations of the form:

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = \omega_n^2 u$$

Where y is output and u is input.

The standard form of a second order system can be shown below.

$$y(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Where the constants $\omega_n$and $\zeta$are called the natural undamped frequency and the damping ratio of the system respectively. The poles of G(s) are

$$s_{1,2} = -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right) = -\omega_n\zeta \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

where $\sigma = \omega_n\zeta$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ are called the *attenuation* or *damping* constant and the *damped natural frequency* of the system, respectively.

For a step input $R(s) = \frac{1}{s}$ the output response of the second order equation becomes

$$y(s) = \frac{{\omega_n}^2}{s(s^2 + 2\zeta\omega_n s + {\omega_n}^2)} \qquad \textbf{(4.3-4)}$$

Effect of ζ On Second-Order System

Case 1: Undamped ($\zeta = 0$)

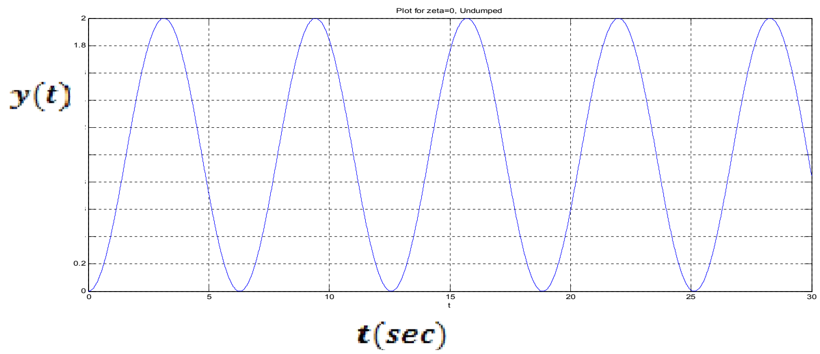
$$y(s) = \frac{{\omega_n}^2}{s(s^2 + {\omega_n}^2)}$$

If we expand $Y(s)$ in partial fractions, we have

$$Y(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}, \qquad \text{and thus } y(t) = 1 - \cos\omega_n t \qquad (4.3\text{-}5)$$

Thus, when $\zeta = 0$ we observe that the response $y(t)$ is a sustained oscillation with constant frequency to ω_n and constant amplitude equal to 1 (see Figure 4.4). In this case, we say that the system is *undamped*.

In this case the poles of $G(s)$ are imaginary since $s_{1,2} = \pm j\omega_n$ and relation (4.3-4) becomes

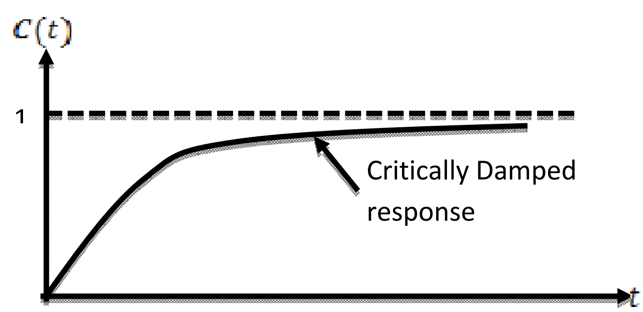


Case 2: Critically Damped ($\zeta = 1$)

In this case the poles of G(s) are the real and double pole $-\omega_n$. Substituting $\zeta = 1$ in e.q. (4.3-4)

$$y(s) = \frac{\omega_n{}^2}{s(s^2 + 2\omega_n s + \omega_n{}^2)} = \frac{\omega_n{}^2}{s(s + \omega_n)^2} = \frac{1}{s} + \frac{\omega_n}{(s + \omega_n)^2} - \frac{1}{s + \omega_n}$$

$$C(t) = 1 - \omega_n e^{-\omega_n t} t - e^{-\omega_n t} = 1 - (1 + \omega_n t) e^{-\omega_n t}$$

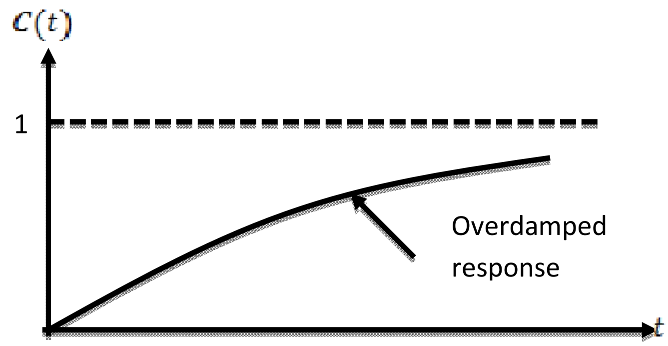


Case 3: Overdamped Case($\zeta > 1$)

In this case the poles of G(s) are both real and negative since $s_{1,2} = -\sigma \mp \omega_n \sqrt{\zeta^2 - 1}$.

$$y(s) = \frac{\omega_n{}^2}{s(s^2 + 2\omega_n s + \omega_n{}^2)}$$

$$y(t) = 1 - \left[A e^{-\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t} - B e^{-\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t} \right]$$

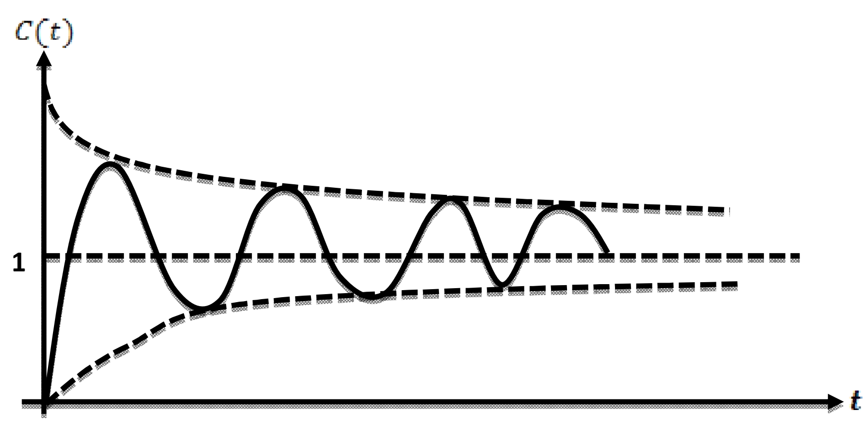


Case 4: Underdamped Case($0 < \zeta < 1$)

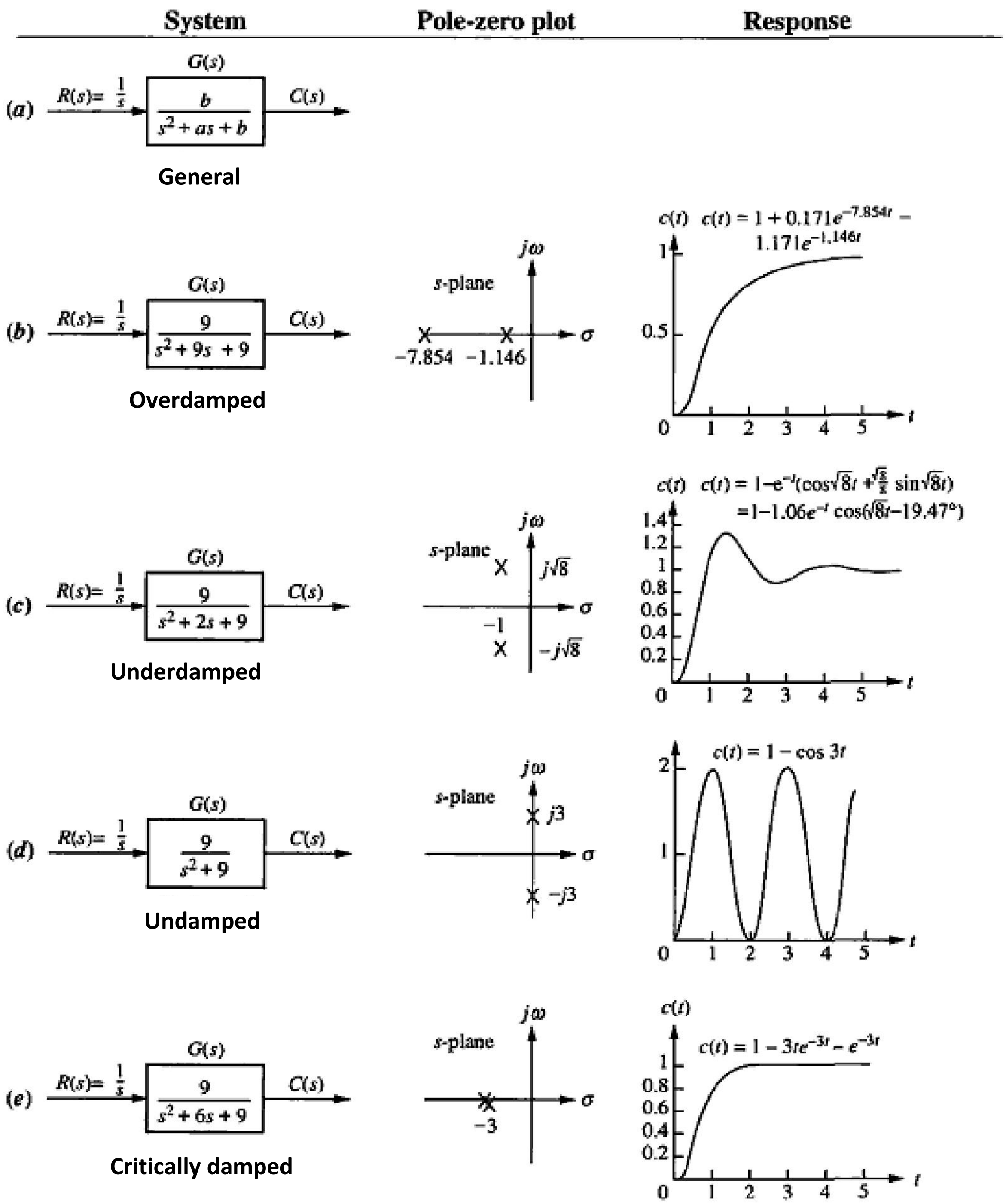
In this case the poles of $G(s)$ are a complex conjugate pair since $s_{1,2} = -\sigma \pm j\omega_d$ and relation (4.3-4) becomes

$$y(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)}$$

$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \alpha)$Where $\cos\alpha = \zeta$, $\sin\alpha$, $\alpha = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$



Pole Location Second order system for Different value of ζ


System
Pole-zero plot
Response
(a)
R(s)= 1/s
G(s)
b / (s^2 + as + b)
C(s)
General
(b)
G(s)
9 / (s^2 + 9s + 9)
Overdamped
s-plane
jω
σ
−7.854 −1.146
c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
(c)
G(s)
9 / (s^2 + 2s + 9)
Underdamped
j√8
−j√8
−1
c(t) = 1−e^{−t}(cos√8t + (√8/8) sin√8t) =1−1.06e^{−t} cos(√8t−19.47°)
(d)
G(s)
9 / (s^2 + 9)
Undamped
j3
−j3
c(t) = 1 − cos 3t
(e)
G(s)
9 / (s^2 + 6s + 9)
Critically damped
−3
c(t) = 1 − 3te^{−3t} − e^{−3t}

Characteristics of the graphical representation of time response:

The total system response (i.e both transient and steady state) of an asymptotically stable system with its time domain specifications is shown in the figure.4.2

Definitions of Transient Response Specifications:

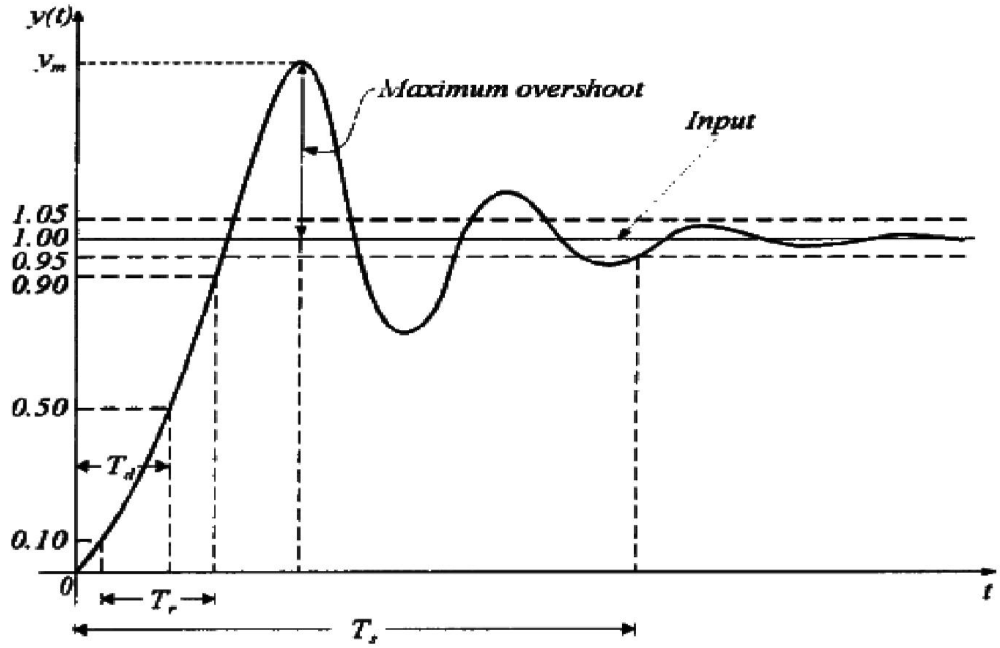


Figure 4.2 Typical unit step response of an asymptotically stable system.

1. Delay time, t_d: The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time, t_r: The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second-order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
3. Peak time, t_p: The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

   The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.
5. Settling time, t_s: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system. Which percentage error criterion to use may be determined from the objectives of the system design in question.

Derivation of Time Response Specifications

(a) Delay Time (T_d)

Is the time required to reach 50% of output is known as delay time.

$$y(t) = \frac{1}{2} \ at\ t = T_d$$

$$\frac{1}{2} = 1 - \frac{e^{-\zeta\omega_n T_d}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d T_d + \left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

The result of this equation is

$$\therefore \boldsymbol{T_d} = \frac{\boldsymbol{1 + 0.7\zeta}}{\boldsymbol{\omega_n}}$$

(b) Rise Time (T_r)

Since the system is underdamped, therefore, $c(t) = 1 \ at\ t = T_r$

$$1 = 1 - \frac{e^{-\zeta\omega_n T_r}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d T_r + \left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

$$T_r = \frac{-\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_d} \quad we\ can\ write \quad \tan(\pi - \alpha) = -\tan(\alpha)$$

$$\therefore \boldsymbol{T_r} = \frac{\boldsymbol{\pi - \alpha}}{\boldsymbol{\omega_d}} \qquad where \quad \alpha = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) radians$$

(c) Peak Time (T_p)

We may obtain peak time by differentiating y (t) with respect to time and letting the derivative equal to zero.

$$\frac{dy(t)}{dt} = 0 \ \ and\ t = T_p, we\ get$$

$$\therefore \boldsymbol{T_p} = \frac{\boldsymbol{n\pi}}{\boldsymbol{\omega_d}}$$

(d) Peak Overshoot (M_p)

The maximum overshoot occurs at the peak time or at $t = T_p = \pi/\omega_d$

$$M_p = y(t)\backslash_{t=T_p} - 1$$

$$y(t)\backslash_{t=T_p} = 1 - \frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d T_p + \alpha)$$

$$\therefore M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} r \qquad \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Settling Time (T_s)

For 2% criterion, $T_s = 4T = \frac{4}{\zeta\omega_n}$ $where\ T = \frac{1}{\zeta\omega_n} = time\ constant$ or $T_s = 3T = \frac{4}{\zeta\omega_n}$ for (5% criterion).

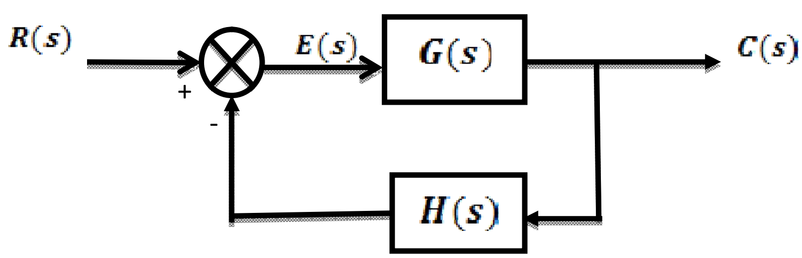
E.g.:-Given the transfer function of Eq. (4.23), find $\zeta\ and\ \omega_n,$

$$G(s) = \frac{36}{s^2 + 4.2s + 36} \tag{4.23}$$

$\omega_n{}^2 = 36 \Rightarrow \omega_n = 6 \quad 2\zeta\omega_n = 4.2 \Rightarrow \zeta = 0.35$

Analysis of Steady- State Error

A simple closed-loop system using negative feedback is shown in Figure below.



$E(s) = R(s) - C(s)H(s) \qquad and \qquad C(s) = E(s)G(s)$

From the two above equations $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Type of Input and Steady-State Error

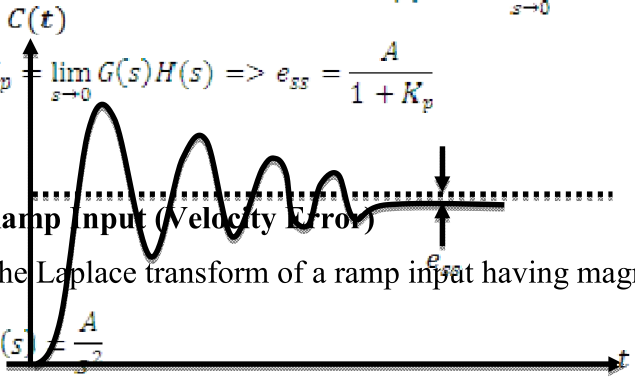
a) Step Input (Position Error)

Let the magnitude of the step be A. The Laplace transform of step input$[r(t)]$ is

$$R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s\left(\frac{A}{s}\right)}{1 + G(s)H(s)} = \frac{s\left(\frac{A}{s}\right)}{1 + \lim_{s\to 0} G(s)H(s)}$$

$$K_p = \lim_{s\to 0} G(s)H(s) \Rightarrow e_{ss} = \frac{A}{1 + K_p}$$



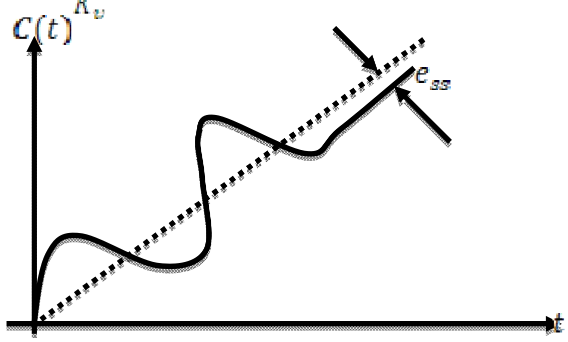
b) Ramp Input (Velocity Error)

The Laplace transform of a ramp input having magnitude A is given by

$$R(s) = \frac{A}{s^2}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s\left(\frac{A}{s^2}\right)}{1 + G(s)H(s)} = \frac{A}{s + \lim_{s\to 0} sG(s)H(s)} = \frac{A}{\lim_{s\to 0} sG(s)H(s)}$$

$$K_v = \lim_{s\to 0} sG(s)H(s) \quad and \quad e_{ss} = \frac{A}{K_v}$$



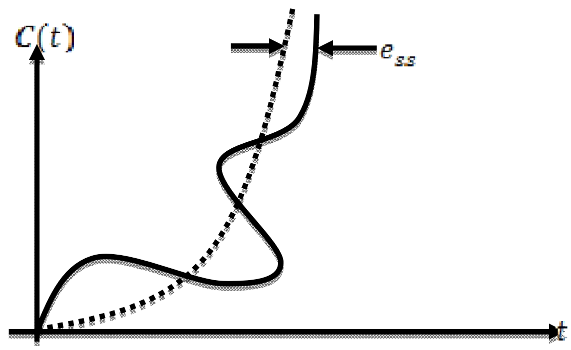
c) Parabolic Input (Acceleration Error)

The Laplace transform of a parabolic input having magnitude A is given by

$$R(s) = \frac{A}{s^3}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s\left(\frac{A}{s^3}\right)}{1 + G(s)H(s)} = \frac{A}{s^2 + \lim_{s\to 0} s^2G(s)H(s)} = \frac{A}{\lim_{s\to 0} s^2G(s)H(s)}$$

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) \ \ and \ \ e_{ss} = \frac{A}{K_a}$$



Error For different types of Input

Constant	Equation	Steady-state error(e_{ss})
Position Error (K_p)	$K_p = \lim_{s \to 0} G(s)H(s)$	$e_{ss} = \frac{A}{1 + K_p}$
Velocity Error (K_v)	$K_v = \lim_{s \to 0} sG(s)H(s)$	$e_{ss} = \frac{A}{K_v}$
Acceleration Error (K_a)	$K_a = \lim_{s \to 0} s^2 G(s)H(s)$	$e_{ss} = \frac{A}{K_a}$

Type	**Step Input**		**Ramp Input**		**Parabolic Input**	
	K_p	e_{ss}	K_v	e_{ss}	K_a	e_{ss}
Type 0	K	$\frac{A}{1 + K}$	0	∞	0	∞
Type 1	∞	0	K	$\frac{A}{k}$	0	∞
Type 2	∞	0	∞	0	K	$\frac{A}{k}$

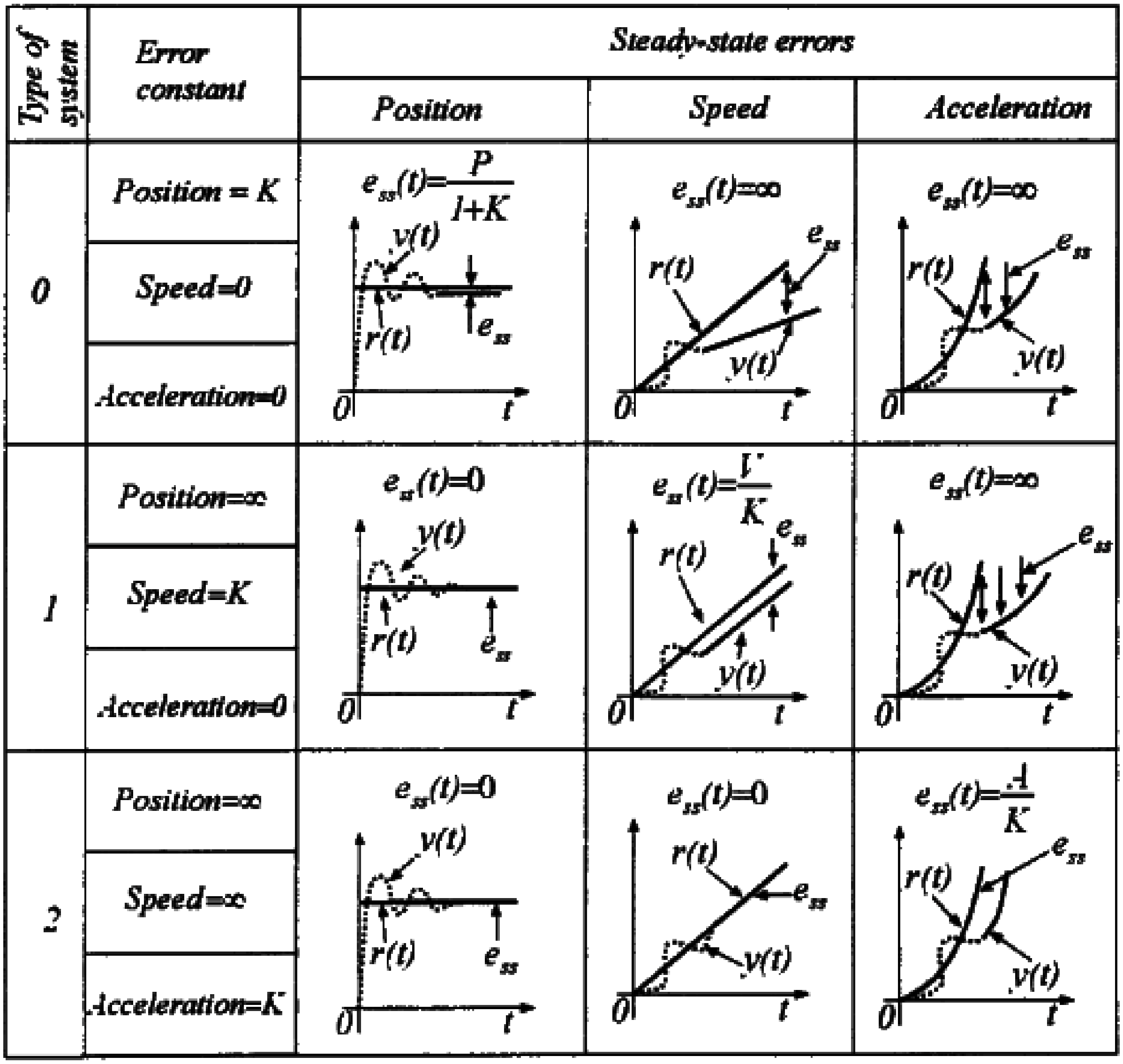
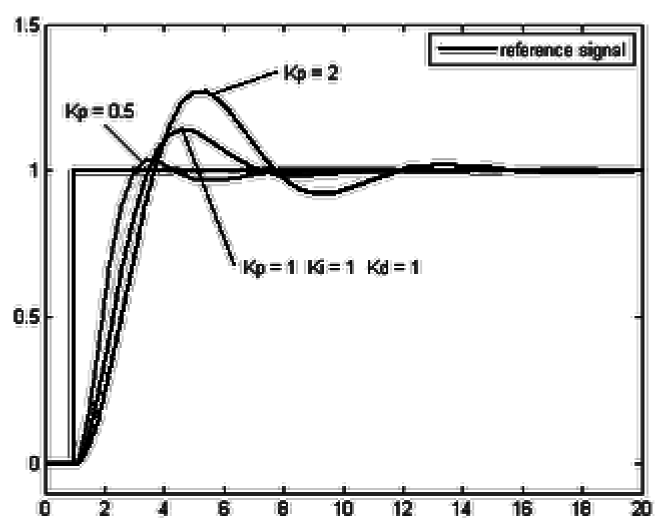


Figure 4.14 Position, speed, and acceleration errors.

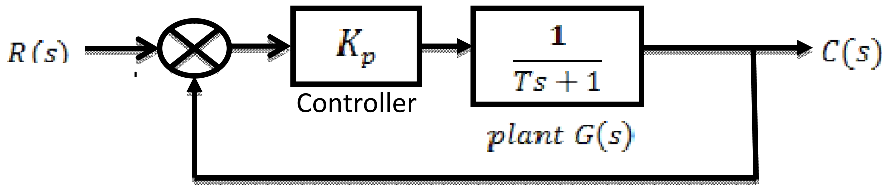
Effect of proportional derivative and integral terms in the transient response

a) Proportional Control Action:

The proportional term produces an output value that is proportional to the current error value. The proportional response can be adjusted by multiplying the error by a constant K_p, called the proportional gain constant.



A high proportional gain results in a large change in the output for a given change in the error. If the proportional gain is too high, the system can become unstable In contrast, a small gain results in a small output response to a large input error, and a less responsive or less sensitive controller. If the proportional gain is too low, the control action may be too small when responding to system disturbances.



Since

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$

the error $E(s)$ is given by

$$E(s) = \frac{1}{1 + G(s)} R(s) = \frac{1}{1 + \frac{K}{Ts + 1}} R(s)$$

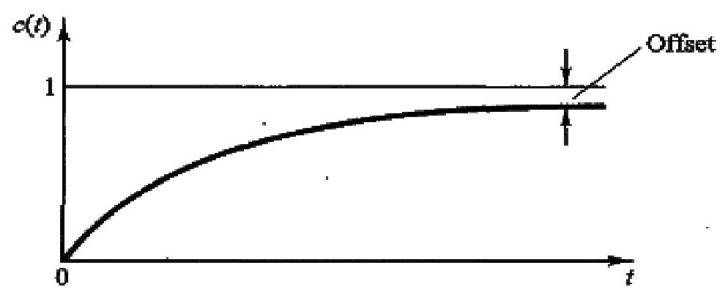
For the unit-step input $R(s) = 1/s$, we have

$$E(s) = \frac{Ts + 1}{Ts + 1 + K} \frac{1}{s}$$

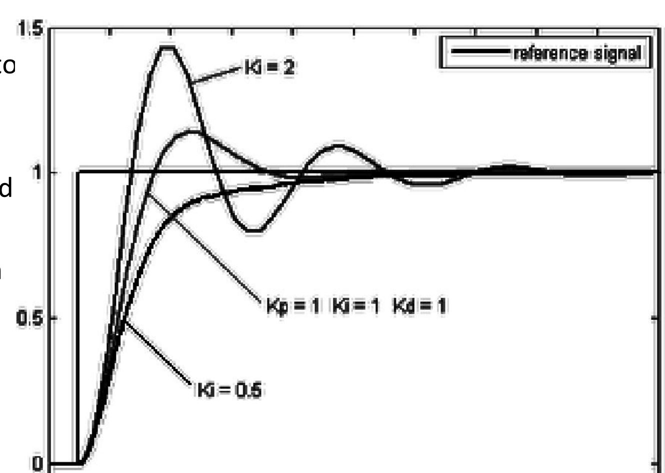
The steady-state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{Ts + 1}{Ts + 1 + K} = \frac{1}{K + 1}$$

Such a system without an integrator in the feedforward path always has a steady-state error in the step response. Such a steady-state error is called an offset. Figure 5–41 shows the unit-step response and the offset.



b) Integral Control Action:



The contribution from the integral term is proportional to both the magnitude of the error and the duration of the error. The integral in a PID controller is the sum of the instantaneous error over time and gives the accumulated offset that should have been corrected previously. The accumulated error is then multiplied by the integral gain (K_i) and added to the controller output.

The integral term accelerates the movement of the process towards set point and eliminates the residual steady-state error that occurs with a pure proportional

Integral Control of Systems. Consider the system shown in Figure 5–42. The controller is an integral controller. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K}{s(Ts + 1) + K}$$

Hence

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = \frac{s(Ts + 1)}{s(Ts + 1) + K}$$

Since the system is stable, the steady-state error for the unit-step response can be obtained by applying the final-value theorem, as follows:

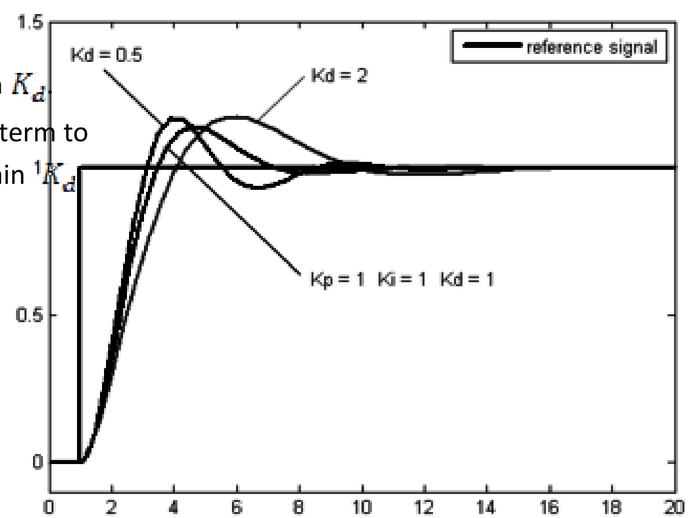
$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{s^2(Ts + 1)}{Ts^2 + s + K} \frac{1}{s} \\ &= 0 \end{aligned}$$

Integral control of the system thus eliminates the steady-state error in the response to the step input. This is an important improvement over the proportional control alone, which gives offset.

c) Derivative Control Action:

The derivative of the process error is calculated by determining the slope of the error over time and multiplying this rate of change by the derivative gain K_d. The magnitude of the contribution of the derivative term to the overall control action is termed the derivative gain [1] K_d

Derivative action predicts system behavior and thus improves settling time and stability of the system.



The derivative term is given by:

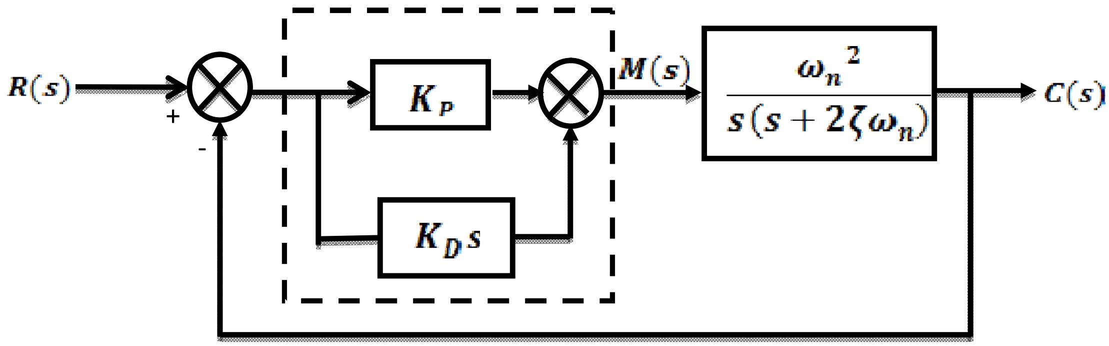
$$D_{out} = K_d \frac{d}{dt} e(t) \ or \ K_d sE(s)$$

Derivative Control Action. Derivative control action, when added to a proportional controller, provides a means of obtaining a controller with high sensitivity. An advantage of using derivative control action is that it responds to the rate of change of the actuating error and can produce a significant correction before the magnitude of the actuating error becomes too large. Derivative control thus anticipates the actuating error, initiates an early corrective action, and tends to increase the stability of the system.

Proportional plus Derivative Controller (PD)

In PD controller output in proportional to (i) error (ii) derivative of error. $M(s) = (K_p + sK_D)E(s)$

Taking inverse Laplace transform $m(t) = K_P e(t) + K_D + \frac{de(t)}{dt}$



$$G(s) = \frac{(K_P + sK_D)\omega_n^2}{s(s + 2\zeta\omega_n)} \qquad and \qquad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{(K_P + K_D s)\omega_n^2}{s^2 + s(2\zeta\omega_n + K_D\omega_n^2) + K_P\omega_n^2}$$

$\therefore$ Now $2\zeta\omega_n + K_{D\omega_n^2} = 2\omega_n\left(\zeta + \frac{K_D\omega_n}{2}\right) = 2\zeta'\omega_n$ where $\zeta' = (\zeta + \frac{K_D\omega_n}{2})$

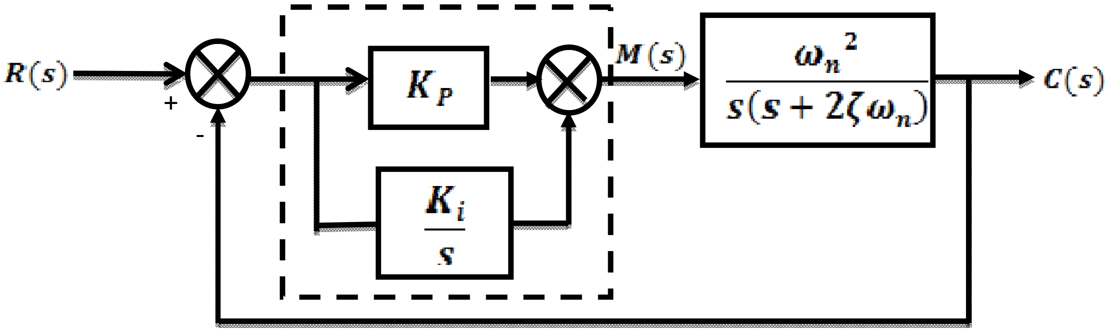
Therefore, the damping factor increases and $K_V = \lim_{s\to 0} sG(s)H(s) = \frac{K_P\omega_n}{2\zeta}$

In PD controllers, the following effects have been observed:

- Damping ratio improves and maximum overshoot reduces.
- Rise time and settling time are reduced.

Proportional Integral Controllers (PI)

The output is proportional to (i) error and (ii) integral of error



$$M(s) = \left(K_p + \frac{K_i}{s}\right)E(s) \quad or \quad m(t) = K_p e(t) + K_i \int_0^t e(t)dt \quad and \;\; H(s) = 1 \; G(s)H(s)$$

$$G(s)H(s) = \frac{(K_p + K_i)\omega_n^2}{s^2(s + 2\zeta\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{(K_p+K_i)\omega_n^2}{s^2(s+2\zeta\omega_n)}}{1 + \frac{(K_p+K_i)\omega_n^2}{s^2(s+2\zeta\omega_n)}} = \frac{(K_p + K_i)\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + K_i\omega_n^2}$$

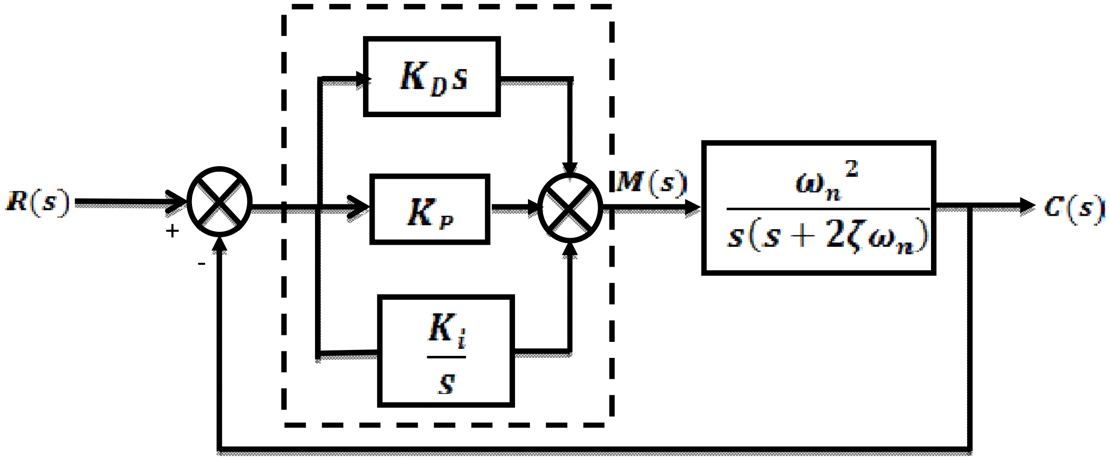
Since the order of the system increases, the system is susceptible from stability point of view. Therefore, K_i should be designed properly to make the system stable

$$e_{ss} = \frac{A}{K_V} \quad and \; K_V = \lim_{s\to 0} sG(s)\,H(s) = \infty \Rightarrow e_{ss} = 0$$

The main positive point is that system behavior is accurate due to reduction of steady-state error. The main features are as follows

- Order and type of the system increases by 1 (susceptible for instability)
- Improvement of damping and reduction of overshoot
- K_i must be designed properly

Proportional plus Integral Plus Derivative Controller (PID)



Since a PD controller improves the transient part and PI controller improves the steady-state part combination of PD and PI improves the overall system.

Effect of coefficients

Parameter	**Speed of Response**	**Stability**	**Accuracy**
Increasing K_P	Increases	Deteriorate	Improves
Increasing K_i	Decreases	Deteriorate	Improves
Increasing K_D	Increases	Improves	No impact

Effect of coefficients on the time response

Parameter	**Rise Time**	**Overshoot**	**Settling Time**	**S.S Error**	**Stability**
K_P	Decreases	Increases	Small change	Decreases	Worse
K_i	Decreases	Increases	Increases	Significant Decrease	Worse
K_D	Minor Dec.	Minor Dec.	Minor Dec.	No Change	If K_D small, better

THE CONCEPT OF STABILITY

We can say that a closed-loop feedback system is either stable or it is not stable. This type of stable/not stable characterization is referred to as **absolute stability**. A system possessing absolute stability is called a table system. Given that a closed-loop system is stable, we can further characterize the degree of stability. This is referred to as **relative stability**.

a) Stability using natural response

1. A linear, time-invariant system is **stable** if the natural response approaches zero as time approaches infinity.
2. A linear, time-invariant system is **unstable** if the natural response grows without bound as time approaches infinity.
3. A linear, time-invariant system is **marginally stable** if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

b) Stability using the total response (BIBO)

A system is stable if every bounded input yields a bounded output. We call this statement the bounded-input, bounded-output(BIBO) definition of stability.

c) Routh-Hurwitz Stability Criteria

Tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows.

Consider the characteristic equation of a system $a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \ldots \ldots \ldots (4.1)$

1. Check the necessary conditions
   a. All the coefficients of eqn. 4.1 have the same sign
   b. None of the coefficient vanish
2. If all coefficients are positive arrange the coefficients of the polynomial in rows and column according to the pattern:

$$
\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

The process of forming rows continues until we run out of elements. (The total number of rows is $n + 1$.) The coefficients b_1, b_2, b_3, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \qquad d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1} \qquad d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$\vdots$$

This process is continued until the nth row has been completed. The complete array of coefficients is triangular. Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Routh's stability criterion states that the number of roots of eqn. 4.1 with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed.

The necessary and sufficient conditions are that:

1. All roots of eqn. 4.1 lie in the left-half s plane
2. All the coefficients of eqn. 4.1 be positive
3. All terms in the first Colum of the array have positive signs.

Consider the following polynomial: $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array.)

s^4	1	3	5	s^4	1	3	5
s^3	2	4	0	s^3	2	$\cancel{4}$	$\cancel{0}$
					1	2	0
s^2	1	5		s^2	1	5	
s^1	-6			s^1	-3	0	
s^0	30			s^0	15		

- Number of sign changes in of coefficients in first column is 2, means there are two roots with +ve real parts. Therefore system is unstable.

Special Cases

Case 1: There is a zero in the first column, but some other elements of the row containing the zero in the first column are non-zero. If only one element in the array is zero, it may be replaced with a small positive number, ε that is allowed to approach zero after completing the array.

E.g. consider the following characteristics polynomial

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10.$$

The Routh array is then

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s^1 & d_1 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

Where

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon} \quad \text{and} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1} \to 6.$$

There are two sign changes due to the large negative number in the first column, $1 = -12/\varepsilon$. Therefore, the system is unstable, and two roots lie in the right half of the plane.

Case 2: All elements in any one row are zero, the situation with entire row of zero can be remedied by using the auxiliary eqn. $P(s) = 0$, which is formed from the coefficients of the row just above the row of zeros in the Routh's tabulation.

i. Form the auxiliary eqn. $A(s) = 0$ by use of the coefficients from the row just preceding the zero row.
ii. Take the derivative of the auxiliary eqn. w.r.t's this gives $dP(s)/ds$.
iii. Replace the row of zeros within the coefficient of $dp(s)/ds$.

Example: Consider the following characteristics equation.

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficients is

$$\begin{array}{cccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array} \quad \leftarrow \text{Auxiliary polynomial } P(s)$$

The terms in the s^3 row are all zero. (Note that such a case occurs only in an odd-numbered row.) The auxiliary polynomial is then formed from the coefficients of the s^4 row.

The auxiliary polynomial P(s) is $P(s) = 2s^4 + 48s^2 - 50$

which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex-conjugate roots on the imaginary axis).These pairs are obtained by solving the auxiliary polynomial equation P(s) = 0. The derivative of P(s) with respect to s is

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

The terms in the s^3 row are replaced by the coefficients of the last equation, that is, 8 and 96. The array of coefficients then becomes

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 8 & 96 & \leftarrow \text{Coefficients of } dP(s)/ds \\ s^2 & 24 & -50 & \\ s^1 & 112.7 & 0 & \\ s^0 & -50 & & \end{array}$$

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

we obtain

$$s^2 = 1, \qquad s^2 = -25$$

or

$$s = \pm 1, \qquad s = \pm j5$$

Since there is sign change the system in unstable.

Root locus

Introduction

Root locus, a graphical presentation of the closed loop poles as a system parameter is varied, is a powerful method of analysis and design for stability and transient response (Evans, 1948; 1950).

The root locus can be used to describe qualitatively the performance of a system as various parameters are changed. For example, the effect of varying gain upon percent overshoot, settling time, and peak time can be vividly displayed. The qualitative description can then be verified with quantitative analysis.

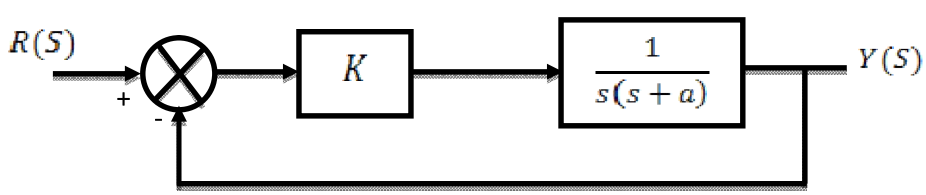
Besides transient response, the root locus also gives a graphical representation of a system's stability. We can clearly see ranges of stability, ranges of instability, and the conditions that cause a system to break in to oscillation.

It is frequently necessary to adjust one or more system parameter in order to obtain suitable root location. Therefore is worth to determine how the roots of characteristics equation of a given system migrate about the s-plane as the parameter are varied i.e. it is useful to determine the locus of roots in the s-plane as parameter varied.

This technique provides a graphical method of plotting the locus of the roots in the s-plane as a given system parameter varied over a complete range of values (may be zero to infinity). The root corresponding to a particular value of the system parameter can then be located on the locus or the value of the parameter for a desired root location can be determined from the locus.

Root Locus Concept

To understand the concept underlying the root locus technique, consider the simple second order below.



1. Closed loop system transfer function

$$T(s) = \frac{GH}{1+GH} = \frac{K}{s^2 + as + K}$$

2. The characteristic equation of the system

   $s^2 + as + K = 0$

The dynamic behavior of the system is controlled by the roots of the characteristic equation.

The roots is given by: $s_1, s_2,$ $\frac{a}{2} \mp \sqrt{\left(\frac{a}{2}\right)^2 + K}$

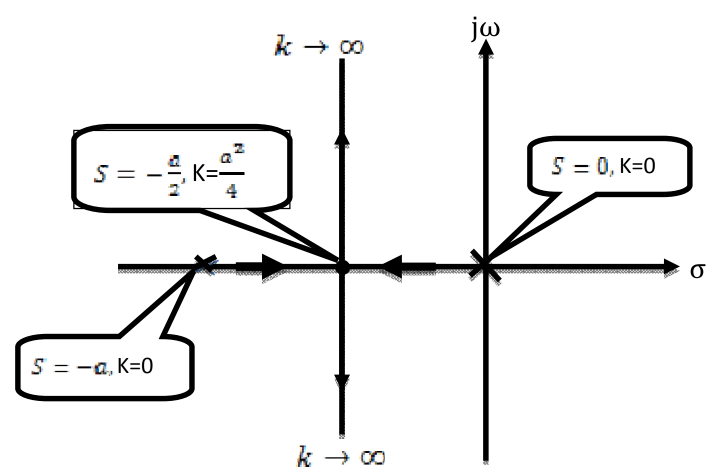
3. Taking a fixed ‘a’ as K varied from zero to infinity, the two roots ($s_1, s_2,$) describes loci in the s-Plane.

Root Location

i) $0 \le K < \frac{a^2}{4}$, the roots are $s_1 = 0, s_2 = -a$ for $k = 0$

ii) $K = \frac{a^2}{4}$, the roots are real and equal in values i.e $s_1 = s_2 = -\frac{a}{2}$

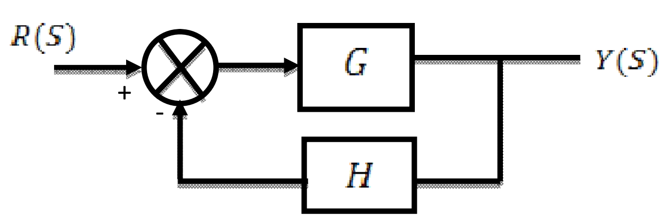
iii) $\frac{a^2}{4} \le K < \infty$ the roots are complex conjugate with real part $-\frac{a}{2}$



This shows the location of the roots depend on the variation of the gain and the root locus starts at the poles of open loop transfer function GH and ends at the zeros of GH.

Basic Properties of the Root-Loci

Consider the general closed-loop transfer function



$T(s) = \frac{Y(s)}{R(s)} = \frac{G}{1+GH}$ Characteristics equation $1 + G(s)H(s) = 0 \;=> G(s)H(s) = -1$

It is seen that the roots of the characteristics equation (i.e. the closed loop poles) of the system occurs only for those values of s where $G(s)H(s) = -1$.

Since s is a complex variable, the root loci satisfied the following conditions.

i) **Magnitude condition**

$$|G(s)H(s)| = 1$$

$$GH = \frac{k|s+Z_1| \ldots |s+Z_m|}{|s+P_1| \ldots |s+P_m|} = 1$$

$$|GH| = \frac{k\prod_{i=1}^{m}(s+Z_i)}{\prod_{i=1}^{n}(s+P_i)} = 1$$

ii) **Angle condition**

$$< GH = \pm 180 \mp 360k$$

$$< GH = \pm 180(1+2k) \qquad where \quad k = 0,1,2 \ldots$$

$$G(s)H(s) = \frac{k(s+Z_1)(s+Z_2) \ldots (s+Z_m)}{(s+P_1)(s+P_2) \ldots (s+P_n)} \quad m < n$$

$$< GH = \sum_{i=1}^{m} < S - Z_i \sum_{i=1}^{n} S + P_i = \pm 180(1+2k) \qquad where \quad k = 0,1,2 \ldots$$

Construction Rules of Root Loci of a System

Rule 1: The root locus starts at the poles of GH where k=0.

$$|GH| = \frac{\prod_{i=1}^{m}(s+Z_i)}{\prod_{i=1}^{n}(s+P_i)} = \frac{1}{k}$$

As ‘k’ approaches zero, GH approaches to infinity, so ‘S’ must approach the poles of GH. i.e. $s -> -P_i$ which are open-loop poles.

$$proof: K \rightarrow 0 \quad K\prod_{i=1}^{m}(s+Z_i)-\prod_{i=1}^{n}(s+P_i)=0 => s=-P_i$$

Therefore root locus branch therefore starts at the open loop-poles.

Rule 2: The root locus ends at the zeros of GH, where $k = \pm\infty$.

As 'k' approaches infinity, GH approaches zero this corresponds S must approach zero of GH.

$$proof: k \rightarrow \infty \prod_{i=1}^{m}(s+Z_i)-\frac{1}{|K|}\prod_{i=1}^{n}(s+P_i)=0 => s=-Z_i$$

The root locus of characteristics equation $1 + GH = 0$begins at the poles of GH and ends at the zero of GH as k increases from zero to infinity.

In case m<n, the GH has n-m zeros at infinity, therefore n-m branches of root loci terminates at infinity.

Rule 3. The number of separate loci (number of branch is equal to the number of poles since $m \leq n$ or the order of the characteristics equation).

Rule 4: The roots locus is symmetrical about the real axis (σ-axis). The roots of characteristics equation are either real or complex roots appears as pair of complex conjugate roots.

Note: the loci proceed to zeros at infinity along asymptotes centered at σ_A and with angle$\emptyset_A$, then the number of finite zeros of $GH\ (m)$ is less than the number of poles (n) by the number $N = n - m$, then N selection of loci must end at the zeros at infinity.

This section of loci proceeds to zeros at infinity along asymptotes as 'k' approaches infinity. These linear asymptotes are centered at a point on the axis given as:

$$\sigma_A = \frac{\sum Poles\ of\ GH - \sum Zeros\ of\ GH}{n-m}$$

$$\emptyset_A = \frac{(2K+1)*180°}{n-m} \quad K = 0,1,2, \ldots\ldots, (n-m-1)$$

Rule 5: The $(n-m)$ branches of the root which tends to infinity, do so along straight line asymptote wholes angle are given by

$$\emptyset_A = \frac{(2K+1)*180°}{n-m} \quad K = 0,1,2,\ldots\ldots,(n-m-1)$$

The asymptotes cross the real axis at a point known as centroid, determined by the relationship.

$$\sigma_A = \frac{(sum\ of\ real\ parts\ of\ poles) - (sum\ of\ the\ parts\ of\ zeros)}{number\ of\ poles\ - number\ of\ zeros}$$

Rule 6: The pint on the real axis lies on the locus if the number of open loop (GH) poles plus zeros on the real axis to the right of this point is odd.

E.g.:- Consider a feedback system which has characteristics equation.

$$1 + GH = 0 = 1 + \frac{k(s+2)}{s(s+3)(s^2+2s+2)}$$

$$GH = \frac{k(s+2)}{s(s+3)\big(s+(1+j)\big)(s+(1-j))}$$

Solution:

1. The root loci started at the poles of GH where $K = 0$.

   Poles of $GH = s = 0, s = -3, s = -1 - j, s = -1 + j$
2. The root loci ends at the zeros of GH where $k = \infty$, one finite zero at $s = -2$ and three zeros at infinity.
3. Number of separate loci are equal to the number of poles (order of the characteristics equation) equal to => 4 separate loci.
4. The root loci are symmetrical with respect to the axis.
5. Number of poles > number of zeros therefore $n - m$ branches of the root locus tends to infinity (rule 5,$4 - 1 = 3$ branches to infinity with asymptote angle.

   $$\emptyset_A = \frac{(2K+1)*180°}{n-m} \quad where \quad k = 0,1,2,\ldots.,(n-m-1)$$

   $n - m - 1 \Rightarrow k = 4 - 1 - 1 = 2\ three\ angles$

   - For $k = 0$

     $$\emptyset_A = \frac{(0+1)*180°}{4-1} = 60°$$

   - $For\ k = 1$

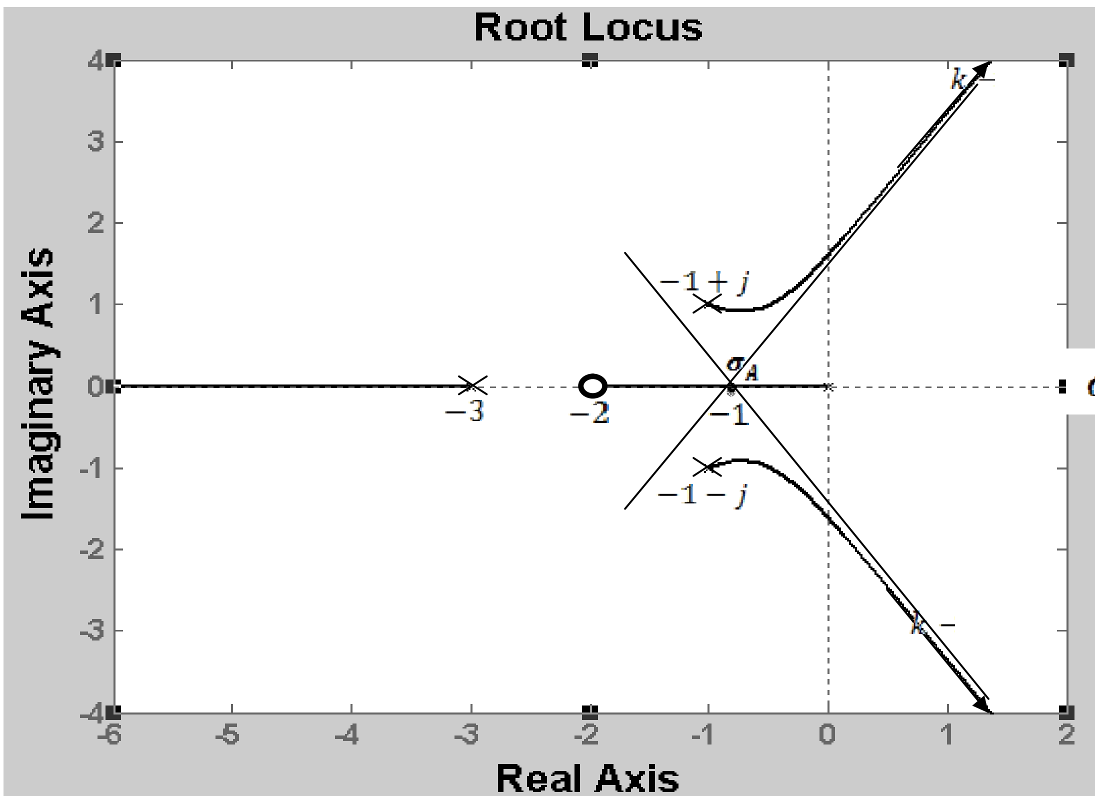
$$\phi_A = \frac{(2+1)*180°}{4-1} = 180°$$

- $For\ k = 2$

$$\phi_A = \frac{(4+1)*180°}{4-1} = 300°$$

Centroid

$$\sigma_A = \frac{\sum Poles\ of\ GH - \sum Zeros\ of\ GH}{n-m} = \frac{(0-3-1-j-1+j)-(-2)}{4-1} = -1$$



Breakaway points (saddle point)

The point where the root left the real axis. The locus leaves the real axis where there are a multiplicity of roots typically two.

Rule 7: Break away point (points at which multiple roots of characteristics equation occur) of the root locus are the solution of:

$$\frac{d}{ds}[G(s)H(s)] = 0$$

The point where two branches of the root loci meet at the break away point on the real axis and then depart from the axis in the opposite direction.

The breakaway point on the complete root locus of $1 + GH = 0$ must satisfy $\frac{d}{ds}[G(s)H(s)] = 0$

$$1 + G(s)H(s) = 0 \quad => \quad 1 + K\frac{B(s)}{A(s)} = 0$$

$$\frac{d}{ds}[G(s)H(s)] = k\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0$$

The breakaway point is roots of the solution of $\frac{d}{ds}[G(s)H(s)] = 0$

$$kB'(s)A(s) - B(s)A'(s) = 0$$

We can write

$$1 + k\frac{B(s)}{A(s)} = 0 => k\frac{B(s)}{A(s)} = -1 = k = -\frac{A(s)}{B(s)}$$

Differentiate k with respect to s:

$$\frac{d}{ds}k = \frac{d}{ds}\left(-\frac{A(s)}{B(s)}\right) = \frac{B'(s)A(s) - B(s)A'(s)}{B^2(s)} = 0$$

The breakaway point of the original equation are determined by $\frac{d}{ds}\boldsymbol{k} = \mathbf{0}$

E.g.:- The characteristics equation of the following feed-back loop system is given by:

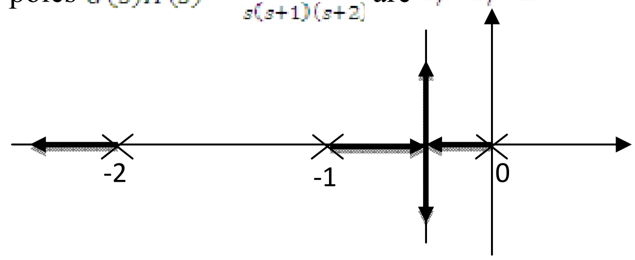
$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

Find the breakaway point?

Solution:

- Poles of open loop transfer function $G(s)H(s)$ poles $G(s)H(s) = \frac{K}{s(s+1)(s+2)}$ are $0, -1, -2$
- Zeros 3 at infinity

There is breakaway point on the real axis between 0 and -1



Breakaway point is the roots of $\frac{d}{ds}\boldsymbol{k} = \mathbf{0}$

$$K = -s(s+1)(s+2) \Rightarrow -\frac{dK}{ds}[(s^3 + 3s^2 + 2s)] = 0 = -3s^2 + 6s + 2 = 0$$

$s_1 = -0.423, s_2 = -1.577 \Rightarrow \mathbf{-0.423}$ *is a breakaway point.*

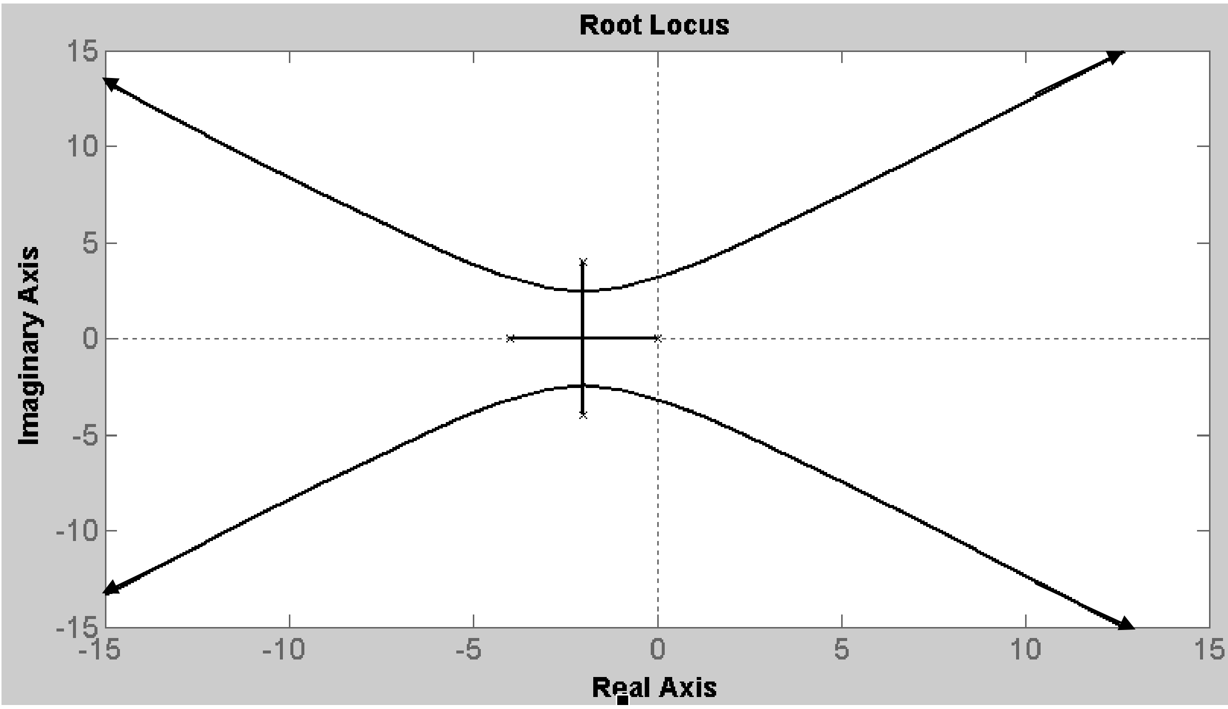
Breakaway direction of the root locus branches

The root locus branch must approach or leave the breakaway point on the real axis at an angle of $\pm \frac{180°}{r}$

Where r is the number of toots locus branches approaching or leaving the point.

E.g.:- The open loop transfer function of a feed-back system given as: $G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$

1. Poles of GH, $s_1 = 0, s_1 = -4, s_3 = -2 + j4, s_4 = -2 - j4$
2. Four (4 poles) =>4 branches originated at the poles of GH.
3. There are no open loop zeros in finite region, all the four $(n - m = 4 - 0 = 4)$ branches terminate at infinity along asymptotes whose angles with the real axis are



$$\emptyset_A = \frac{(2K+1) * 180°}{n-m} \quad where \quad k = 0,1,2, \ldots, (n-m-1)$$

$k = n - m - 1 = 4 - 0 - 1 = 3 => k = 0,1,2,3$ $(four\ degree)$

$= 45°, 135°, 225°, 315°$

4. Centroid

$$\sigma_A = \frac{(sum\ of\ real\ parts\ of\ poles) - (sum\ of\ the\ parts\ of\ zeros)}{number\ of\ poles\ - number\ of\ zeros}$$

$$\sigma_A = \frac{(0-4-2-2)-(0)}{4-0} = -2$$

5. The point between 0 and -4 on the real axis lie on the root locus.

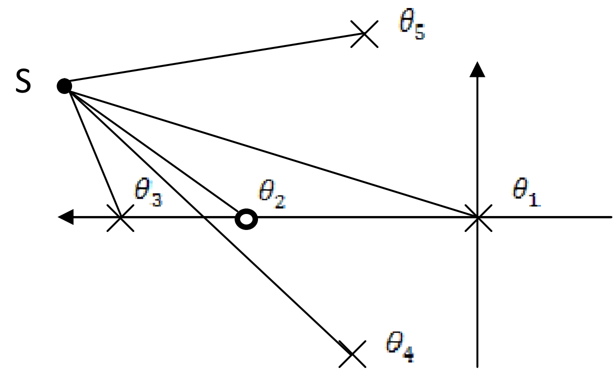
6. From the characteristics equation $1 + \frac{K}{s(s+4)(s^2+4s+20)} => k = -s(s+4)(s^2+4s+20)$

$$\frac{dK}{dS} = -(4s^3 + 24s^2 + 72s + 80) = 0 => s_1 = -2, s_2 = -2 \pm j2.45$$

Rule 8: Angle of departure and angle of arrival

The angle of departure or arrival of a root locus at a pole or zero of GH denotes the angle of tangent to the locus near the point.

Consider the given pole-zero configuration. The locus leaving the point placed close to the pole or zero at which the total angle contributions of the poles and zeros, of GH will have to satisfy the angle criterion.



CHAPTER FOUR

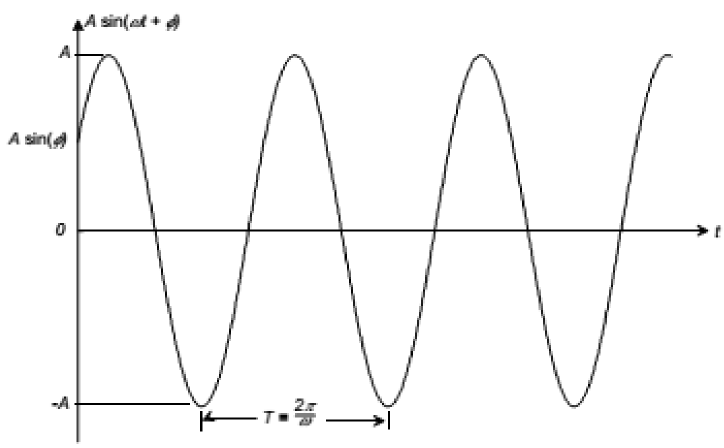
FREQUENCY DOMAIN ANALYSIS

4.1 Sinusoidal Frequency Response

4.1.1 Definitions

Consider a sinusoidal waveform

$f(t)= A \sin(\omega t + \varphi)$



where

A is the amplitude (in appropriate units);

ω is the angular frequency (rad/s) ;

φ is the phase (rad)

In addition we can define T the period $T = 2\pi/\omega$ (s); f the frequency, ($f = 1/T = \omega/2\pi$) (Hz).

The Euler Formulas: We will frequently need the Euler formulas

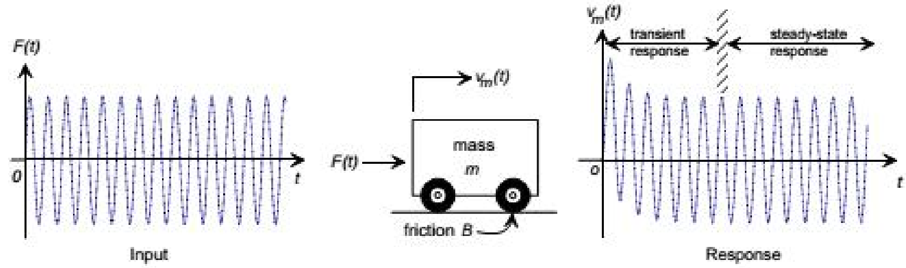
$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

or conversely $\cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})1/2$

$$\sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})1/2j$$

4.2 The Steady-State Sinusoidal Response



Assume a system, such as shown above, is excited by a sinusoidal input. The total response will have two components a) a transient component, and a steady-state component

$y(t)= y_h(t)+ y_p(t)$.

We define the steady-state component as the particular solution yp(t). Let the system differential equation be

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_1\frac{du}{dt} + b_0 u$$

with a complex exponential input

$u(t)= e^{j\omega t}$.

Assume a particular solution $y_p(t)$ to be of the same form as the input, that is

$y_p(t)= Ae^{j\omega t}$

and since

$$\frac{d^k y_p}{dt^k} = A(j\omega)^k e^{j\omega t}$$

Substitution into the differential equation gives

$$\begin{aligned}\left(a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0\right) Ae^{j\omega t} \\ = \left(b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \ldots + (b_1 j\omega) + b_0\right) e^{j\omega t}\end{aligned}$$

or

$$A = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \ldots + b_1(j\omega) + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0}$$

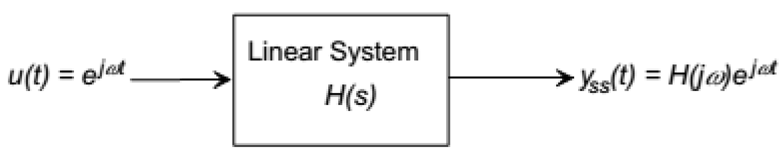
Examination of this equation shows its similarity to the transfer function H(s), in fact

$$A = H(s)\big|_{s=j\omega} = H(j\omega)$$

so that the steady-state response $y_{ss}(t)$ is

$y_{ss}(t)= y_p(t)= Ae^{j\omega t}= H(j\omega)e^{j\omega t}$………………………………(1)

or in other words, the steady-state response to a complex exponential input is defined bythe transfer function evaluated at s = jω, or along the imaginary axis of the s-plane. Note that H(jω) is in general complex.



We now extend this argument to a real sinusoidal input, for example u(t) = $\cos(\omega t)=(e^{j\omega t}+ e^{-j\omega t})1/2$.. The principle of superposition for linear systems allows us to express the response as the sum of the two responses to the complex exponentials:

$y_{ss}(t)= 1/2(H(j\omega)e^{j\omega t}+H(-j\omega))e^{-j\omega t}$

We show that H(−jω)=H*(−jω) where H*(jω) denotes the complex conjugate

$y_{ss}(t)= 1/2(H(j\omega)e^{j\omega t}+H^*(j\omega))e^{-j\omega t}$……………………………….(2)

We break up H(jω) into its real and imaginary parts,

$H(j\omega)\ \ =R\{H(j\omega)\}+jI\{H(j\omega)\}$

$H^*(j\omega)\ \ =R\{H(j\omega)\}-j\ I\{H(j\omega)\}$

and use the Euler formula to write

$e^{j\omega t}=\ \cos(\omega t)+ j\sin(\omega t)$

$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$

- We combine the real and imaginary parts of Eq. (2) to conclude

$$\boxed{y_{ss}(t) = \Re\{H(j\omega)\}\cos(\omega t) - \Im\{H(j\omega)\}\sin(\omega t)} \text{----------------(3)}$$

- We then use the trig. identity

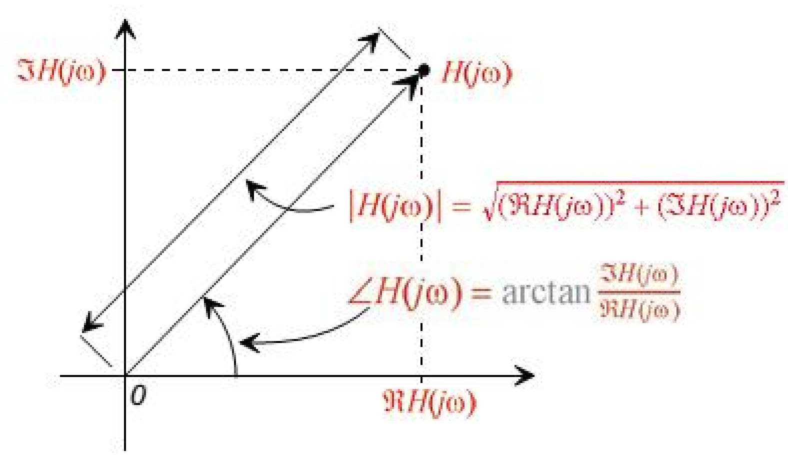
  $a\cos\theta - b\sin\theta = \sqrt{a^2+b^2}\,\cos(\theta + \emptyset)$

  to write Eq.(3) as

  $y(t)_{ss} = /H(j\omega)/\cos(\omega t + \angle H(j\omega))$..............(4)

  where

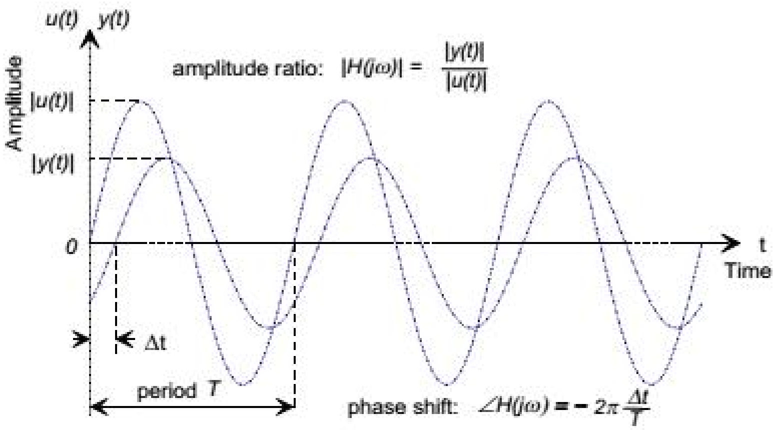
$$|H(j\omega)| = \sqrt{\Re^2\{H(j\omega)\} + \Im^2\{H(j\omega)\}}$$
$$\angle H(j\omega) = \arctan\left(\frac{\Im\{H(j\omega)\}}{\Re\{H(j\omega)\}}\right)$$



Equation (4) states the answer we seek. It shows that

> - The steady-state sinusoidal response is a sinusoid of the same angular frequency as the input,
> - The response differs from the input by (i) a change in amplitude as defined by |H(jω)|, and (ii) an added phase shift <H(jω).

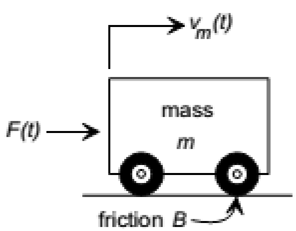
H(jω) is known as the frequency response function. | H(jω) | is the magnitude of the frequency response function, and <H(jω) is the phase.



Note that if |H(jω)|> 1 the sinusoidal input is amplified, while if |H(jω)| < 1 the input is attenuated by the system.

Example 1

The mechanical system



has a transfer function

$$H(s) = \frac{v_m(s)}{F(s)} = \frac{1}{ms + B}$$

where m = 1 kg, and B = 2 Ns/m. Find the steady-state response if F (t) =10sin(5t).

$$H(s) = \frac{1}{s + 2}$$

so that the frequency response function is

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{1}{j\omega + 2} = \frac{2 - j\omega}{\omega^2 + 4}$$

Then

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2 + 4}}, \qquad \angle H(j\omega) = \arctan\left(-\frac{\omega}{2}\right).$$

With ω = 5 rad/s,

$v_{ss}(t) = 10|H(j\omega)| \sin(5t + < H(j\omega))$

$$= \frac{10}{\sqrt{29}} \sin(5t - \arctan 2.5)$$
$$= 1.857 \sin(5t - 1.1903)$$

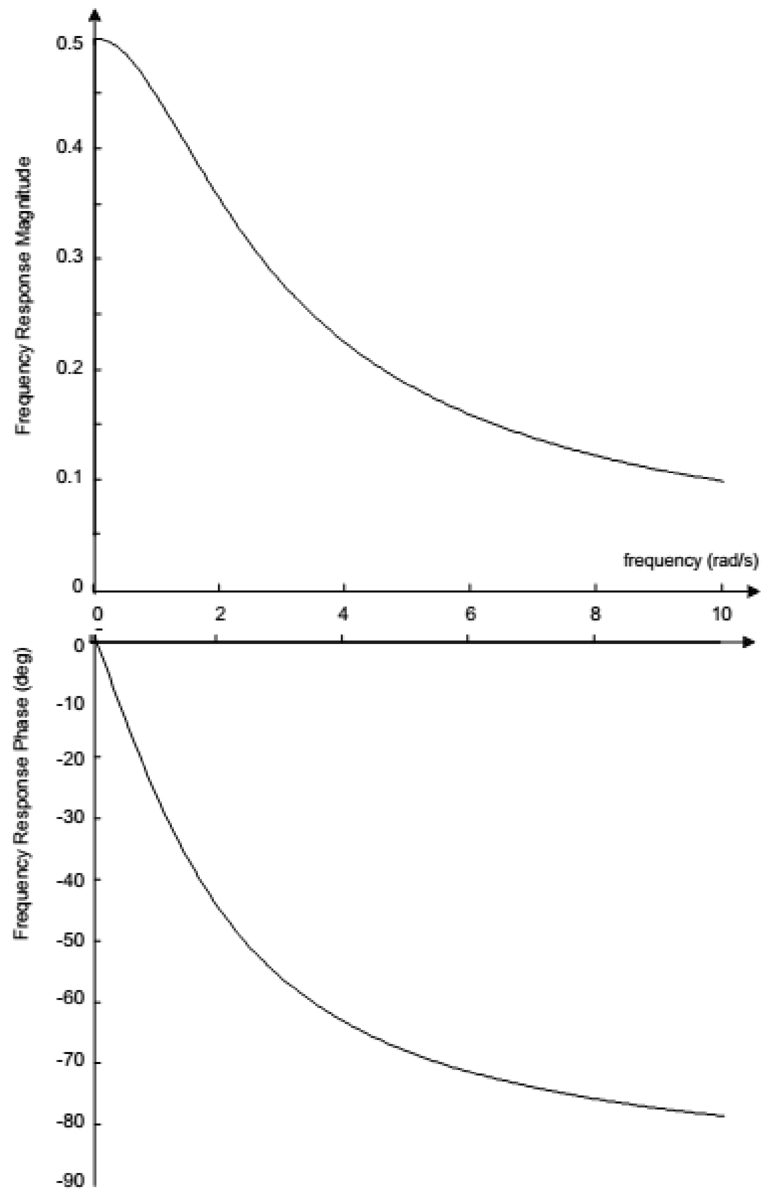
■ Example 2

Plot the variation of $|H(j\omega)|$ and $\angle H(j\omega)$ from $\omega = 0$ to 10 rad/s.

From above

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2 + 4}}, \quad \text{and} \quad \angle H(j\omega) = \arctan\left(-\frac{\omega}{2}\right).$$

These functions are plotted below:



Note that

- As the input frequency ω increases, the response *magnitude* decreases.
- At low frequencies the *phase* is a small negative number, but as the frequency increases the phase lag increases and apparently is tending toward $-90°$ at high frequencies.

4.2 The Frequency Response of Systems with Zeros

If a system has a transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

the frequency response function is

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)}.$$

For complex| a| and| b|, |a/b| = | a|/|b| and < (a/b)= <a-<b, so that

$$\boxed{\begin{aligned} |H(j\omega)| &= \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{\sqrt{\Re^2\{N(j\omega)\} + \Im^2\{N(j\omega)\}}}{\sqrt{\Re^2\{D(j\omega)\} + \Im^2\{D(j\omega)\}}} && (1) \\ \angle H(j\omega) &= \angle N(j\omega) - \angle D(j\omega) = \arctan\left(\frac{\Im\{N(j\omega)\}}{\Re\{N(j\omega)\}}\right) - \arctan\left(\frac{\Im\{D(j\omega)\}}{\Re\{D(j\omega)\}}\right) && (2) \end{aligned}}$$

Example 1

Find and plot the frequency response of

$$H(s) = \frac{s+5}{s+10}$$

$$H(j\omega) = \frac{j\omega+5}{j\omega+10}$$

$$\begin{aligned} |H(j\omega)| &= \frac{|N(j\omega)|}{|D(j\omega)|} = \frac{\sqrt{\omega^2+25}}{\sqrt{\omega^2+100}} \\ \angle H(j\omega) &= \angle N(j\omega) - \angle D(j\omega) = \arctan\left(\frac{\omega}{5}\right) - \arctan\left(\frac{\omega}{10}\right) \end{aligned}$$

The following MATLAB commands were used to plot the frequency response:

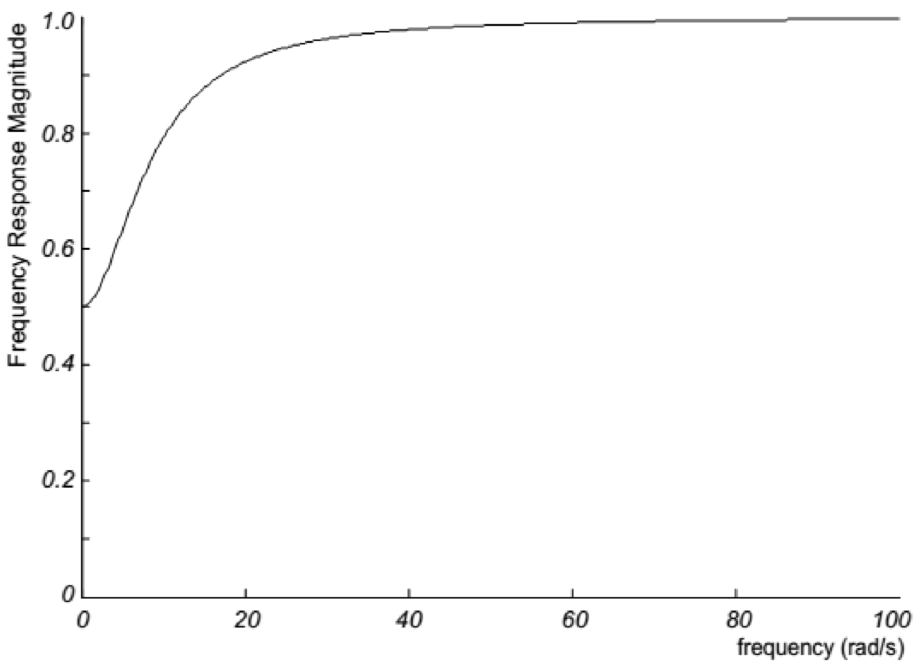
```
w=0:.2:100;

sys=zpk(-5,-10,1)
```

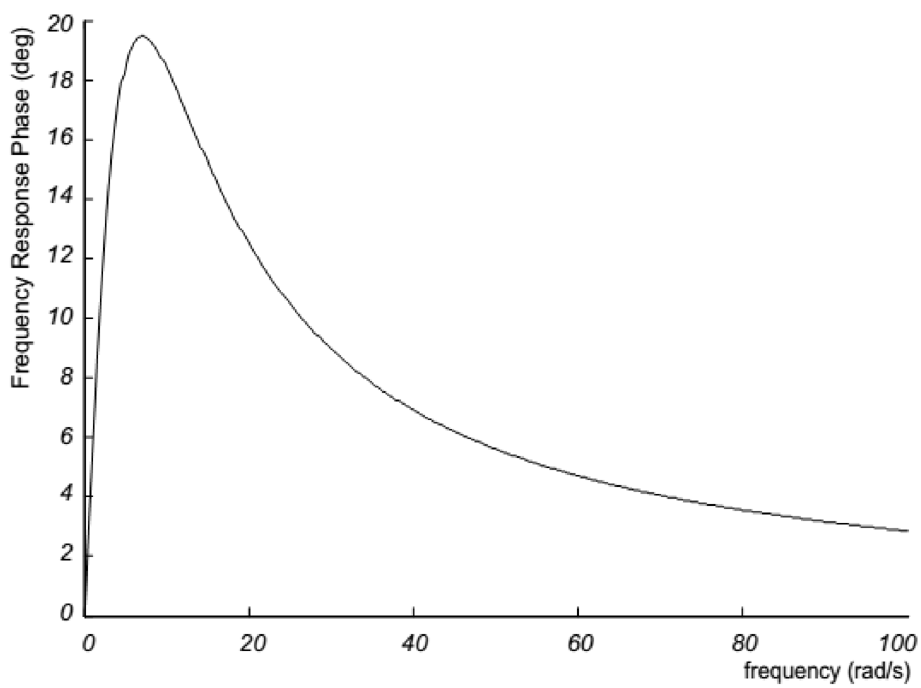
y=freqresp(sys,w);

plot(w,squeeze(abs(y)))

plot(w,squeeze(angle(y)))

which produced the following plots:





We note that

- When $\omega = 0$, $|H(j\omega)| = 0.5$ and $\angle H(j\omega) = 0°$.
- When $\omega \to \infty$, $|H(j\omega)| \to 1$ and $\angle H(j\omega) \to 0°$.

Example 2

Find the frequency response functions for (i) a differentiator, and (ii) an integrator.

(i)A differentiator. The transfer function is H(s) = s, so that H(jω) = jω.

Then|H(jω)| = ω, <H(jω)=π/2 (or 90^0)

(ii)An integrator. The transfer function is H(s)=1/s, so that H(jω)=1/jω.

Then|H(jω)| =1/ω<H(j ω)=- π/2 (0r -90^0)

4.2 .1 The Frequency Response of a Second-Order System

Consider the unity-gain second-order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The frequency response is

$$H(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

so that

$$\boxed{\begin{aligned} |H(j\omega)| &= \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \qquad (3) \\ \angle H(j\omega) &= -\tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \qquad (4) \end{aligned}}$$

Equations (3) and (4) show the following:

When $\omega = 0$, $|H(j\omega)| = 1$
$\angle H(j\omega) = 0$

When $\omega \to \infty$, $|H(j\omega)| \to 0$
$\angle H(j\omega) \to -\pi$ (or -180°)

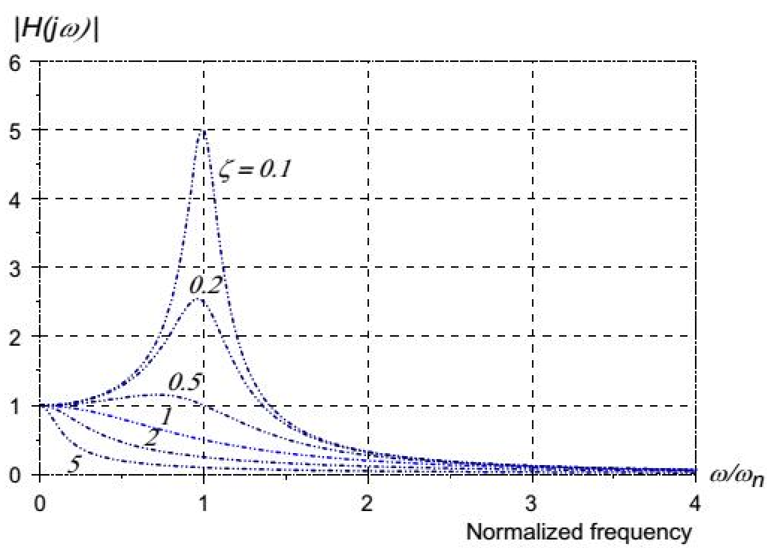
When $\omega = \omega_n$, $|H(j\omega)| = \dfrac{1}{2\zeta}$
$\angle H(j\omega) = -\dfrac{\pi}{2}$ (or -90°)

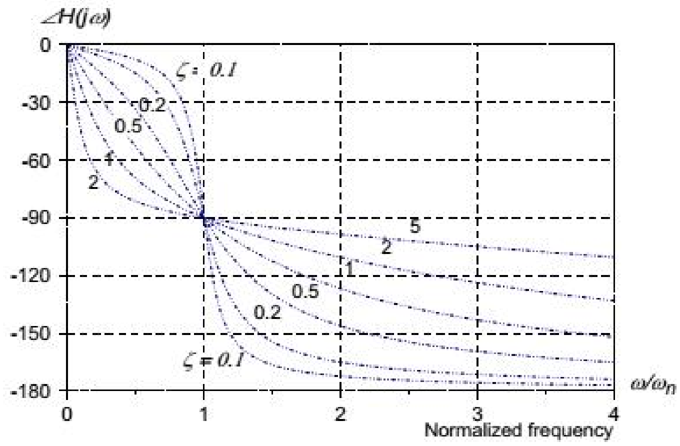
The response of the system to frequencies close to the undamped natural frequency clearly depends on the damping ratio ζ. For a lightly damped system $\zeta < 0.5$, $|H(j\omega_n)| > 1$ and the system demonstrates amplification due to resonance. Differentiation of Eq. (4) shows the ω_{peak}, the frequency of the peak response is not ω_n but is in fact

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$

$\zeta < \frac{1}{\sqrt{2}}$ and $|H(j\omega)| = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$

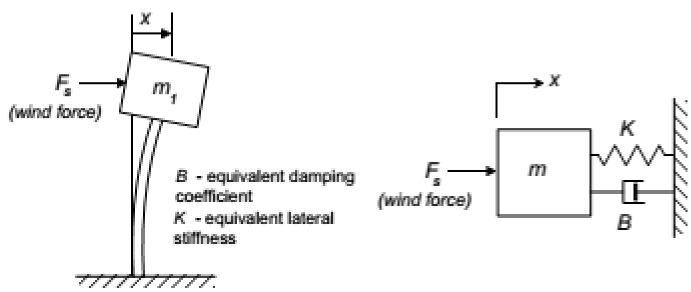
Frequency response plots for several values of ζ are shown below:





Example 3

A tall slender structure, excited by wind forces, is modeled as a mass-spring-damper system. Find the frequency response of the displacement x of the building to a sinusoidal wind loading.

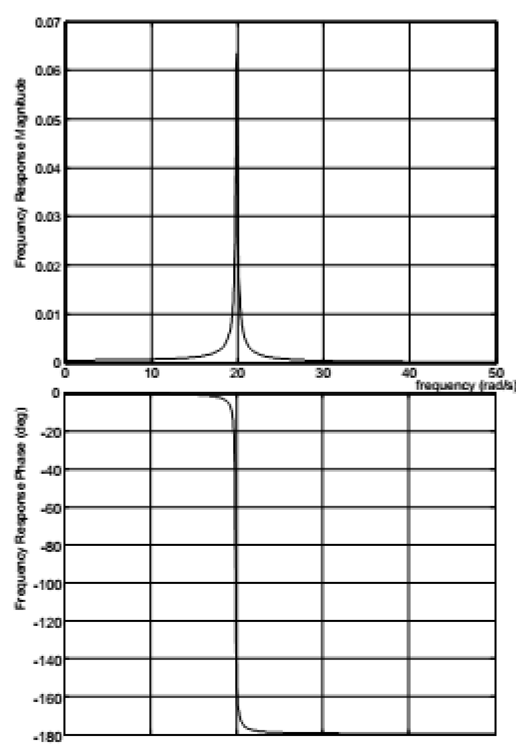


The transfer function is

$$H(s) = \frac{X(s)}{F_s(s)} = \frac{1}{ms^2 + Bs + K} \quad \rightarrow \quad H(j\omega) = \frac{1}{(K - m\omega^2) + jB}$$

$$|H(j\omega)| = \frac{1}{\sqrt{(K - m\omega^2)^2 + (B\omega)^2}}$$

$$\angle H(j\omega) = -\tan^{-1}\frac{B\omega}{K - m\omega^2}$$



With values m =5.11 kg, B =0.77 N-s/m, and K = 2020 N/m,

$$\omega_n = \sqrt{K/m} = 19.9 \text{ rad/s}$$
$$\zeta = B/(2m\omega_n) = 0.0038 \text{ N-s/m}$$

and the system is very lightly damped.

The frequency response is plotted:

Note the extremely sharp and high resonant peak in the magnitude plot, and the rapid phase transition about resonance in the phase plot.

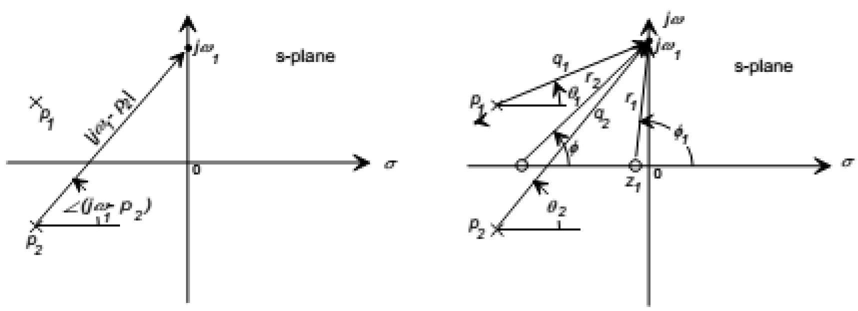
4.2.2 Frequency Response and the Pole-Zero Plot

The frequency response may be written in terms of the system poles and zeros by substituting directly into the factored form of the transfer function:

$$H(j\omega) = K\frac{(j\omega-z_1)(j\omega-z_2)\ldots\ldots(j\omega-z_{m-1})(j\omega-z_m)}{(j\omega-p_1)(j\omega-p_2)\ldots\ldots(j\omega-p_{n-1})(j\omega-p_m)} \qquad (5)$$

Because the frequency response is the transfer function evaluated on the imaginary axis of the s-plane, that is when $H(s)_{j\omega}$, the graphical method for evaluating the transfer function may be applied directly to the frequency response. Each of the vectors from the n system poles to a test point s = jω has a magnitude and an angle:

$$\begin{aligned} |j\omega - p_i| &= \sqrt{\sigma_i^2 + (\omega - \omega_i)^2}, \\ \angle(s - p_i) &= \tan^{-1}\left(\frac{\omega - \omega_i}{-\sigma_i}\right), \end{aligned}$$



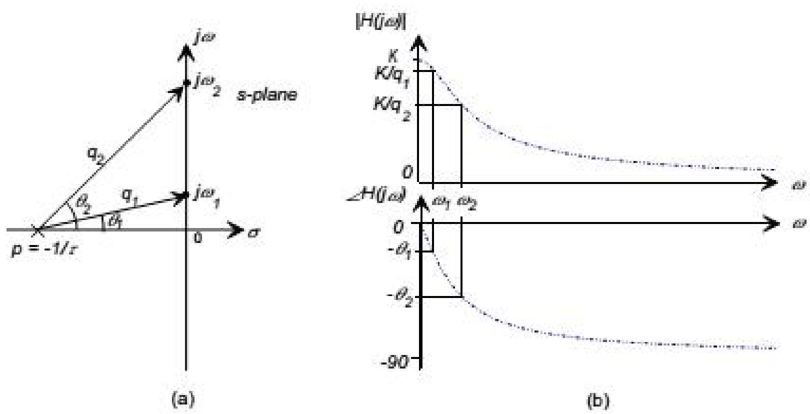
as shown above, with similar expressions for the vectors from the m zeros. The magnitude and phase angle of the complete frequency response may then be written in terms of the magnitudes and angles of these component vectors

$$\begin{aligned} |H(j\omega)| &= K\frac{\prod_{i=1}^{m}|(j\omega - z_i)|}{\prod_{i=1}^{n}|(j\omega - p_i)|} \\ \angle H(j\omega) &= \sum_{i=1}^{m}\angle(j\omega - z_i) - \sum_{i=1}^{n}\angle(j\omega - p_i). \end{aligned}$$

If the vector from the pole pi to the point s = jω has length q_i and an angle θ_i from the horizontal, and the vector from the zero z_i to the point jω has a length r_i and an angle φ_i, the value of the frequency response at the point jω is

$$
\begin{aligned}
|H(j\omega)| &= K\frac{r_1 \ldots r_m}{q_1 \ldots q_n} \\
\angle H(j\omega) &= (\phi_1 + \ldots + \phi_m) - (\theta_1 + \ldots + \theta_n)
\end{aligned}
$$

Example 4: Explain the nature of the sinusoidal response of a first-order system with a pole on the real axis at s = −1/τ as shown below, in terms of the pole-zero plot.



Even though the gain constant K cannot be determined from the pole-zero plot, the following observations may be made directly by noting the behavior of the magnitude and angle of the vector from the pole to the imaginary axis as the input frequency is varied:

- At low frequencies the gain approaches a finite value, and the phase angle has a small but finite lag.
- As the input frequency is increased the gain decreases (because the length of the vector increases), and the phase lag also increases (the angle of the vector becomes larger).
- At very high input frequencies the gain approaches zero, and the phase angle approaches π/2.

4.2.2 High Frequency Response

As ω →∞ we note the following

Magnitude Response: The magnitude response for the s-plane is

$$|H(j\omega)| = K\frac{r_1 \ldots r_m}{q_1 \ldots q_n}$$

and at high frequencies all vectors have approximately the same length, that is

$r_i \approx q_j \approx \omega$ for i =1 ...m, j =1 ...n

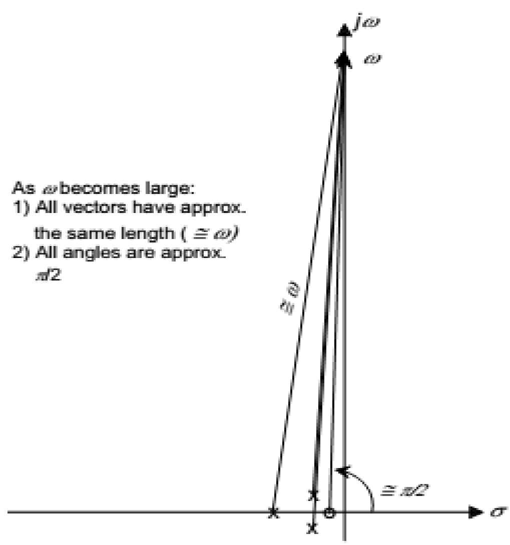
Then Phase Response: From the pole-zero plot $\angle H(j\omega)=(\varphi_1 + \ldots + \varphi_m) - (\theta_1 + \ldots + \theta_n)$ As ω becomes large all of the angles of the vectors approach π/2' $\varphi_i \approx \theta_j \approx \pi/2$ for i = 1 . .m, j = 1..n

And

$$\boxed{\lim_{\omega\to\infty} |H(j\omega)| = K\frac{1}{\omega^{n-m}}}$$

Phase response

$$\boxed{\lim_{\omega\to\infty} \angle H(j\omega) = -(n-m)\frac{\pi}{2}}$$



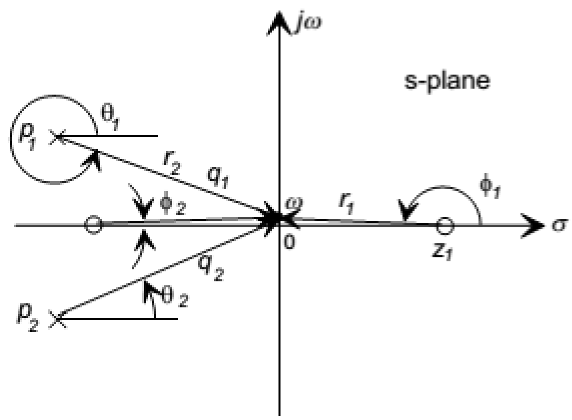
Then

	$n > m$	$n = m$	$n < m$
$\lim_{\omega\to\infty} \lvert H(j\omega)\rvert$	0	K	∞
$\lim_{\omega\to\infty} \angle H(j\omega)$	$-(n-m)\pi/2$	0	$(m-n)\pi/2$

If a system has an excess of poles over the number of zeros (n>m) the magnitude of the frequency response tends to zero as the frequency becomes large. Similarly, if a system has an excess of zeros the gain increases without bound as the frequency of the input increases.(This cannot happen in physical energetic systems because it implies an infinite power gain through the system.)

4.2.3Low Frequency Response

As $\omega \rightarrow 0$ we note the following



Magnitude Response: The magnitude response for the s-plane

$$|H(j\omega)| = K\frac{r_1 \dots r_m}{q_1 \dots q_n}$$

If any of the $r_i \rightarrow 0$, then $|H(j\omega)| \rightarrow 0$, and if any $q_i \rightarrow 0$, then $|H(j\omega)| \rightarrow \infty$ If a system has one or more zeros at the origin of the s-plane (corresponding to a pure differentiation), then the system will have zero gain at $\omega = 0$. Similarly, if the system has one or more poles at the origin (corresponding to a pure integration term in the transfer function), the system has infinite gain at zero frequency.

$$\boxed{\begin{array}{ll} \lim\limits_{\omega \to 0} |H(j\omega)| = 0 & \text{if there are zeros at the origin} \\ \lim\limits_{\omega \to 0} |H(j\omega)| = \infty & \text{if there are poles at the origin} \\ \lim\limits_{\omega \to 0} |H(j\omega)| = K\dfrac{r_1 \dots r_m}{q_1 \dots q_n} & \text{otherwise} \end{array}}$$

Phase Response: $\angle H(j\omega) = (\varphi_1 + \dots + \varphi_m) - (\theta_1 + \dots + \theta_n)$
As $\omega \rightarrow 0$:

- All real-axis LHP poles and zeros contribute 0 rad. to the phase response

- Each complex conjugate pole or zero pair contributes a total of 2π rad. to the phase response (effectively adding 0 rad. to the total response). A pole at the origin (s =0+ j0) contributes $-\pi/2$ rad. to the phase response.
- A zero at the origin (s =0+ j0) contributes $+\pi/2$ rad. to the phase response.
- Ar.h.p real zero contributes $+\pi$ rad. to the phase response.

The low frequency phase response is therefore

$$\boxed{\lim_{\omega \to 0} \angle H(j\omega) = -(N-M)\frac{\pi}{2} + L\pi \text{ rad.}}$$

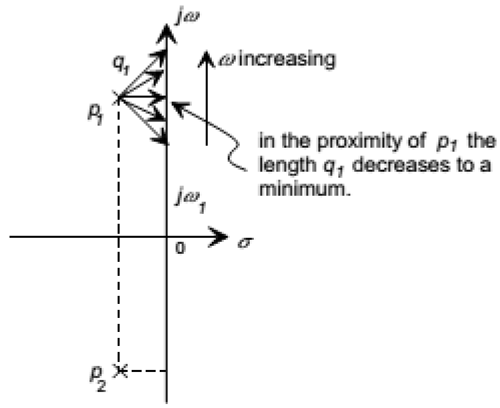
where N is the number of poles at the origin, M is the number of zeros at the origin, and L is the number of RHP real zeros.

4.2.4 Behavior in the Proximity of Poles and Zeros Close to the Imaginary Axis

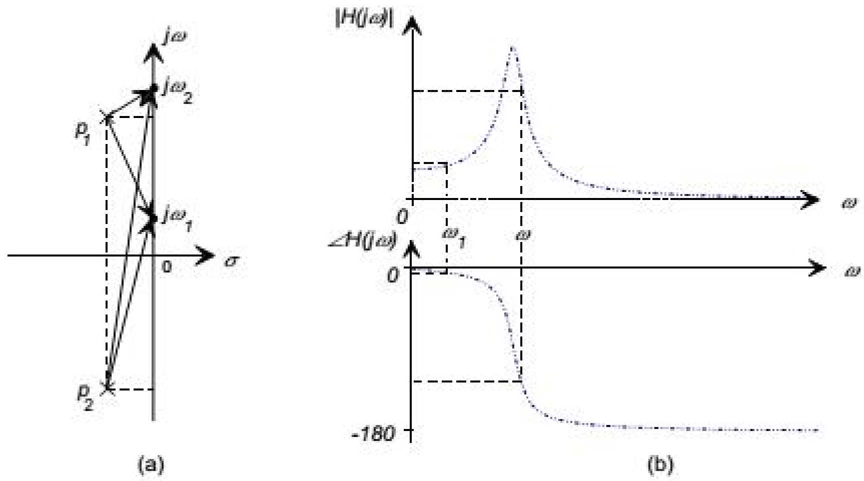
Consider a second-order system with a damping ratio $\zeta \ll 1$, so that the pair of complex conjugate poles are located close to the imaginary axis.

$$\begin{aligned} |H(j\omega)| &= \frac{K}{q_1 q_2} \\ \angle H(j\omega) &= -(\theta_1 + \theta_2) \end{aligned}$$

In this case there are a pair of vectors connecting the two poles to the imaginary axis, and the following conclusions may be drawn by noting how the lengths and angles of the vectors change as the test frequency moves up the imaginary axis: As the input frequency is increased and the test point on the imaginary axis approaches the pole, one of the vectors (associated with the pole in the second quadrant) decreases in length, and at some point reaches a minimum

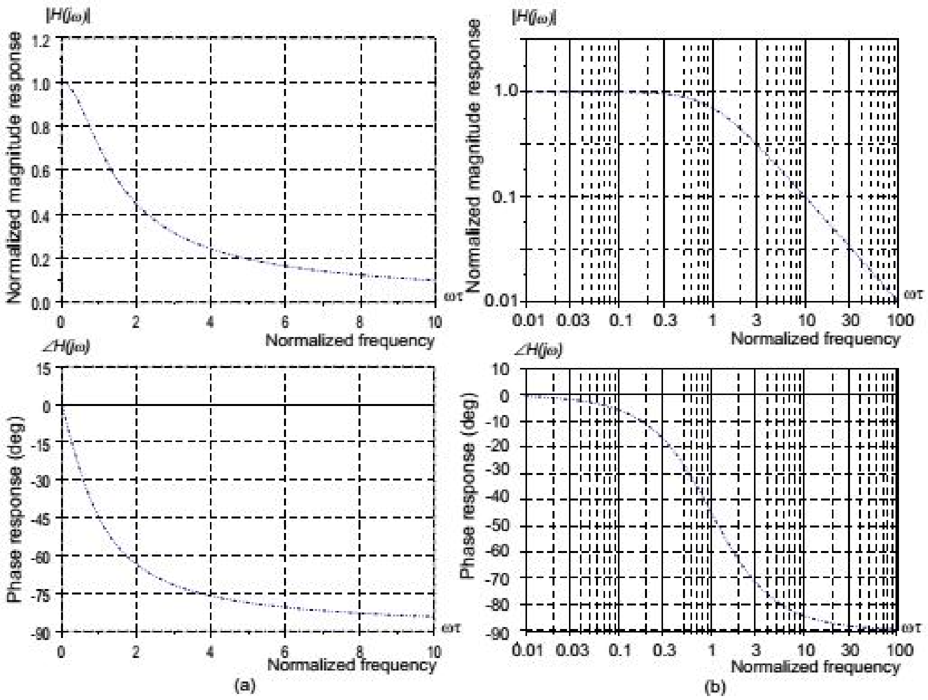


Because q_1 appears in the denominator of the magnitude function, over this range there is an increase in the value of $|H(j\omega)|$.

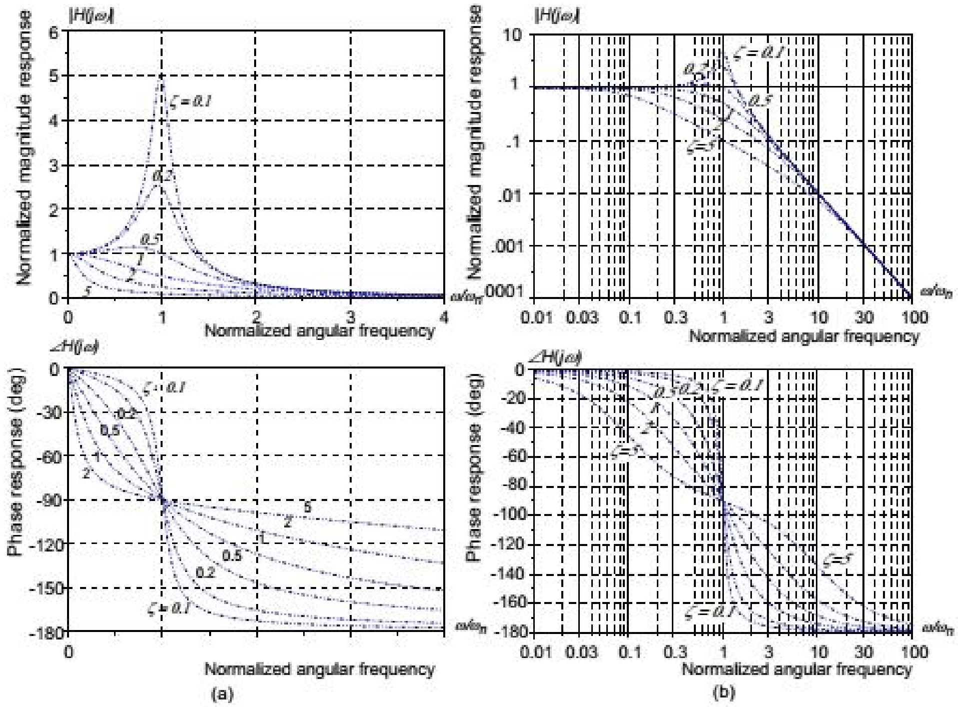


- If a system has a pair of complex conjugate poles close to the imaginary axis, the magnitude of the frequency response has a "peak", or resonance at frequencies in the proximity of the pole. If the pole pair lies directly upon the imaginary axis, the system exhibits an infinite gain at that frequency.
- Similarly, if a system has a pair of complex conjugate zeros close to the imaginary axis, the frequency response. Over this range has a "dip" or "notch" in its magnitude function at frequencies in the vicinity of the zero. Should the pair of zeros lie directly upon the imaginary axis, the response is identically zero at the frequency of the zero, and the system does not respond at all to sinusoidal excitation at that frequency. Similarly in the proximity of the pole there is a rapid change of the angle $\theta 1$ associated with the pole p_1

4.3 Logarithmic (Bode) Plots In system dynamic analyses, frequency response characteristics are almost always plotted using logarithmic scales. In particular, the magnitude function |H(jω)| is plotted against frequency on a log-log scale, and the phase < H(jω) is plotted on a linear-log scale. For example, the frequency response functions of a typical first-order system *τdy/dt + y = u(t)*is plotted below on (a) linear axes, and (b) logarithmically scaled axes.



Similarly the second-order frequency response is shown in linear and logarithmic forms below



It can be seen that while two sets of plots convey the same information, they have a different appearance. The logarithmic frequency scale has the effect of expanding the low frequency region of the plots while compressing the high frequencies. The logarithmic magnitude plot can be seen to exhibit straight line asymptotic behavior at high and low frequencies.

In the 1940's H. W. Bode introduced the logarithmic frequency response plots as a simplified method for sketching approximate frequency response characteristics of electronic feedback amplifiers. Bode plots, named after him, have subsequently been widely used in linear system design and analysis, and in feedback control system design and analysis. The Bode sketching method provides an effective means of approximating the frequency response of a complex system by combining of the responses of simple first and second-order systems.

4.3.1 Logarithmic Amplitude and Frequency Scales:

4.3.1.1 Logarithmic Amplitude Scale: The Decibel

Bode magnitude plots are frequently plotted using the decibel logarithmic scale to display the function $|H(j\omega)|$. The Bel, named after Alexander Graham Bell, is defined as the logarithm to base 10 of the ratio of two power levels. In practice the Bel is too large a unit, and the decibel (abbreviated dB), defined to be one tenth of a Bel, has become the standard unit of logarithmic power ratio. The power flow P into any element in a system, may be expressed in terms of a logarithmic ratio Q to a reference power level P_{ref}:

$$Q = \log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ Bel} \quad \text{or} \quad Q = 10\log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ dB.} \tag{2}$$

Because the power dissipated in a D–type element is proportional to the square of the amplitude of a system variable applied to it, when the ratio of across or through variables is computed the definition becomes

$$Q = 10\log_{10}\left(\frac{A}{A_{ref}}\right)^2 = 20\log_{10}\left(\frac{A}{A_{ref}}\right) \text{ dB.} \tag{3}$$

Where A and A_{ref} are amplitudes of variables.

Note: This definition is only strictly correct when the two amplitude quantities are measured across a common D–type (dissipative) element. Through common usage, however, the decibel has been effectively redefined to be simply a convenient logarithmic measure of amplitude ratio of any two variables. This practice is widespread in texts and references on system dynamics and control system theory

The table below expresses some commonly used decibel values in terms of the power and amplitude ratios.

Decibels	Power Ratio	Amplitude Ratio
-40	0.0001	0.01
-20	0.01	0.1
-10	0.1	0.3162
-6	0.25	0.5
-3	0.5	0.7071
0	1.0	1.0
3	2.0	1.414
6	4.0	2.0
10	10.0	3.162
20	100.0	10.0
40	10000.0	100.0

The magnitude of the frequency response function H (jω) is defined as the ratio of the | | amplitude of a sinusoidal output variable to the amplitude of a sinusoidal input variable. This ratio is expressed in decibels, that is

$$20\log_{10}|H(j\omega)| = 20\log_{10}\frac{|Y(j\omega)|}{|U(j\omega)|}\ \text{dB}.$$

As noted this usage is not strictly correct because the frequency response function does not define a power ratio, and the decibel is a dimensionless unit whereas $|H(j\omega)|$ may have physical units.

Example 1: An amplifier has a gain of 28. Express this gain in decibels.
We note that $28 = 10 \times 2 \times 1.4 \approx 10 \times 2\sqrt{2}$. The gain in dB is therefore
$20\log_{10} 10 + 20\log_{10} 2 + 20 \log10\sqrt{2}$, or Gain(dB)=20+6+3=29 dB.

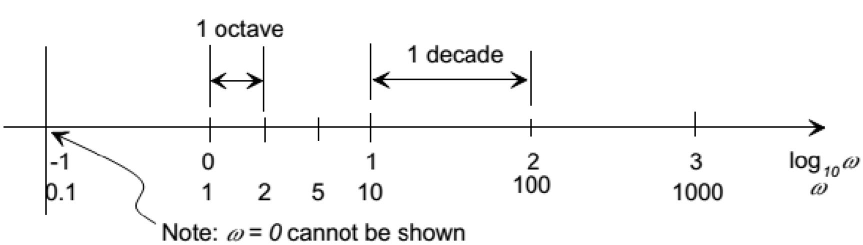
The advantages of a logarithmic amplitude scale include:

- Compression of a large dynamic range.
- Cascaded subsections may be handled by addition instead of multiplication, that is $\log(|H_1(j\omega)H_2(j\omega)H_3(j\omega)|) = \log(|H_1(j\omega)|) + \log(|H_2(j\omega)|) + \log(|H_3(j\omega)|)$ which is the basis for the sketching rules.
- High and low frequency asymptotes become straight lines when $\log(H(j\omega))$ is plotted against $\log(\omega)$.

4.3.1.2 Logarithmic Frequency Scales

In the Bode plots the frequency axis is plotted on a logarithmic scale. Two logarithmic units of

frequency



ratio are commonly used: the octave which is defined to be a frequency ratio of 2:1, and the decade which is a ratio of 10:1.

Given two frequencies $\omega 1$ and $\omega 2$ the frequency ratio $W = (\omega 1/\omega 2)$ between them may be expressed logarithmically in units of decades or octaves by the relationships

$W = \log_2(\omega 1/\omega 2)$ octaves $= \log_{10}(\omega 1/\omega 2)$ decades

The terms "above" and "below" are commonly used to express the positive and negative values of logarithmic values of W . A frequency of 100 rad/s is said to be two octaves (a factor of 2^2) above 25 rad/s, while it is three decades (a factor of 10^{-3}) below 100,000 rad/s.

4.3.1.3 Asymptotic Bode Plots of Low-Order Transfer Functions

The Bode plots consist of (1) a plot of the logarithmic magnitude (gain) function, and (2) a separate linear plot of the phase shift, both plotted on a logarithmic frequency scale. In this section we develop the plots for first and second-order terms in the transfer function. The approximate sketching methods described here are based on the fact that an approximate log–log magnitude plot can be derived from a set of simple straight line asymptotic plots that can be easily combined graphically. The system transfer function in terms of factored numerator and denominator polynomials is:

$$H(s) = K\frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-p_n)}, \qquad (3)$$

where the z_i, for $i = 1, \dots, m$, are the system zeros, and the p_i, for $i = 1, \dots, n$, are the system poles.

In general a system may have complex conjugate pole and zero pairs, real poles and zeros, and possibly poles or zeros at the origin of the s-plane. Bode plots are constructed from a rearranged form of Eq. (3), in which complex conjugate poles and zeros are combined into second-order terms with real coefficients. For example a pair of complex conjugate poles $s_i, s_{i+1} = \sigma_i \pm j\omega_i$ is written

$$\left.\frac{1}{(s-(\sigma_i+j\omega_i))(s-(\sigma_i-j\omega_i))}\right|_{s=j\omega} = \left(\frac{1}{\omega_n^2}\right)\frac{1}{(1-(\omega/\omega_n)^2)+j2\zeta\omega/\omega_n} \qquad (4)$$

and described by parameters ω_n and ζ. The constant terms $1/\omega_n^2$ is absorbed into a redefinition of the gain constant K. In the following sections Bode plots are developed for the first and second-order numerator and denominator terms:

1.**Constant Gain Term:** The simplest transfer function is a constant gain, that is $H(s)= K$

$$|H(j\omega)| = K \text{ and } < H(j\omega)=0,$$

and converting to the logarithmic decibel scale

$20\log_{10}|H(j\omega)| = 20\log_{10} K$ and $|H(j\omega)|=0$ dB

The Bode magnitude plot is a horizontal line at the appropriate gain and the phase plot is identically zero for all frequencies

2 A Pole at the Origin of the s-plane: A single pole at the origin of the s-plane, that is H(s)=1/s, has a frequency response

$|H(j\omega)| = 1/\omega$ *and* $\angle H(j\omega) = -\pi/2$.

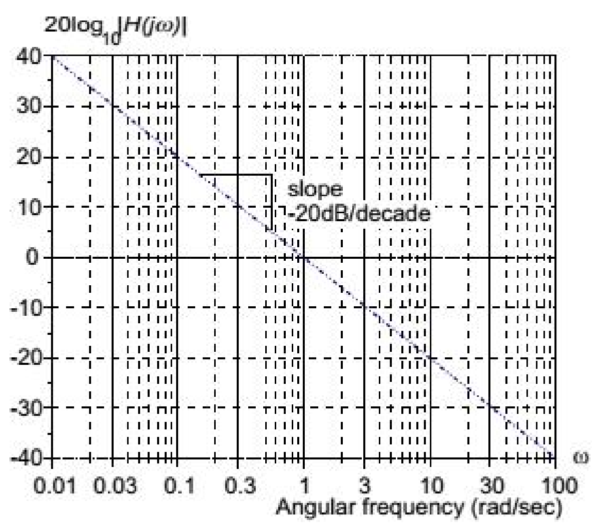
The value of the magnitude function in logarithmic units is

$\log|H(j\omega)| = -\log(\omega)$

or using the decibel scale

$20\log_{10}|H(j\omega)| = -20\log_{10}(\omega)$ dB

The decibel based Bode magnitude plot is therefore a straight line with a slope of -20 dB/decade and passing through the 0 dB line ($|H(j\omega)| = 1$) at a frequency of *1 rad/s*. The phase plot is a constant value of $-\pi/2$ *rad, or* -90°, at all frequencies. The magnitude Bode plot for this system is shown below.

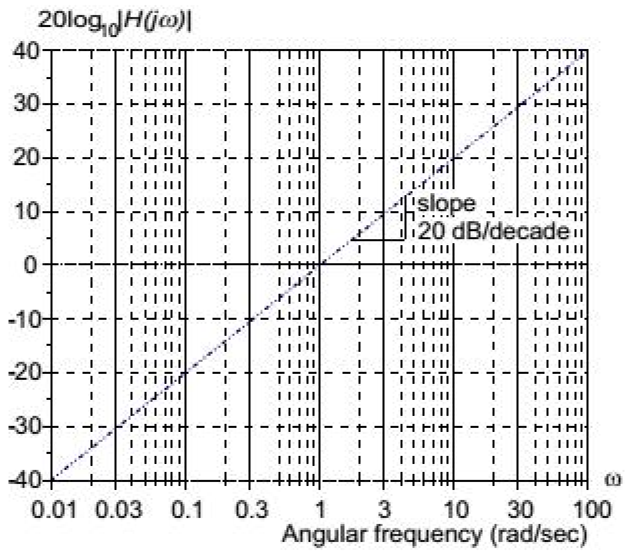


3 A Single Zero at the Origin: A single zero at the origin of the s-plane, that is $H(s) = s$, has a frequency response $H(j\omega)$ with magnitude and phase $|H(j\omega)| = \omega$ *and* $\angle H(j\omega) = \pi/2$. The logarithmic magnitude function is therefore

$Log|H(j\omega)| = |log(\omega)|$

or in decibels $20log_{10}\ |H(j\omega)| = 20\ log_{10}(\omega)\ dB$.

The Bode magnitude plot is a straight line with a slope of +20 dB/decade. This curve also has a gain of 0 dB (unity gain) at a frequency of 1 rad/s. The phase plot is a constant of π/2 radians, or +90◦ , at all frequencies. The magnitude plot is shown in below.



4.A Single Real Pole The frequency response of a unity-gain single real pole factor is

$$H(s) = \frac{1}{\tau s + 1}$$

and the frequency response is:

$$\boxed{|H(j\omega)| = \frac{1}{\sqrt{(\omega\tau)^2 + 1}} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(-\omega\tau).}$$

The logarithmic magnitude function is

$log\ |H(j\omega)| = -0.5log((\omega\tau)^2+1)$

or as a decibel function $\boldsymbol{log\ |H(j\omega)| = -10log_{10}((\omega\tau)^2+1)dB}$

- When $\omega\tau << 1$, the first term may be ignored and the magnitude may be approximated by a *low-frequency asymptote*

$$\lim_{\omega\tau \to 0} 20\log_{10}|H(j\omega)| = -10\log_{10}(1) = 0 \text{ dB}$$

- which is a horizontal line on the plot at 0dB (unity) gain

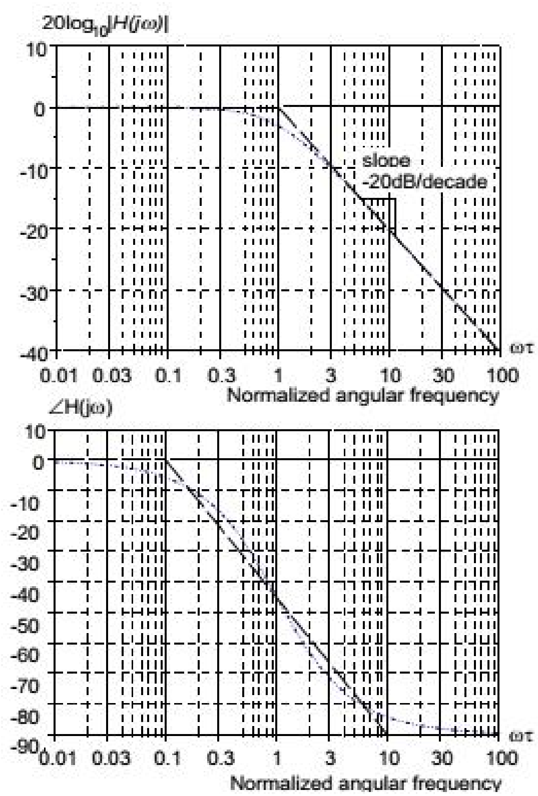
  At high frequencies, for which $\omega\tau >> 1$, the unity term in the magnitude expression may be ignored and the magnitude function is approximated by *a high-frequency asymptote*

  $20\log_{10}|H(j\omega)| \approx -10\log_{10}((\omega\tau)^2) = -20\log_{10}(\omega) - 20\log_{10}(\tau)$ dB

  which is a straight line when plotted against log(ω), with a slope of -20 dB/decade.

- The high and low frequency asymptotes intersect on the plot on the *0 dB* line at a corner or break frequency of $\omega = 1/\tau$. We note that when $\omega = 1/\tau$ the magnitude is $|H(j\omega)| = 1/\sqrt{2}$ *or -3 dB.*

The complete asymptotic Bode magnitude plot as defined by these two line segments is shown in (a) below using a normalized frequency axis. The exact response is also shown in the figure below;at the break frequency $\omega = 1/\tau$ the actual response is $20\log_{10}|H(j\omega)| = -10\log_{10}(2) = -3$dB.



The phase characteristic is also plotted against a normalized frequency scale in (a). At low frequencies the phase shift approaches 0 radians. It passes through a phase shift of $-\pi/4$ radians at the break frequency $\omega = 1/\tau$, and asymptotically approaches a maximum phase lag of $-\pi/2$ radians as the frequency becomes very large. A piece-wise linear approximation may be made by assuming that the curve has a phase shift of 0 radians at frequencies more than one decade below the break frequency, a phase shift of $-\pi/2$ radians at frequencies more than a decade above the break frequency, and a linear transition in phase between these two frequencies on the logarithmic frequency scale. This approximation is within 0.1 radians of the exact value at all frequencies.

5 A Single Real Zero

A numerator term, corresponding to a single real zero, written in the form $H(s)=\tau s+1$ (where τ is not strictly a time constant), is handled in a manner similar to a real pole. In this case

$H(j\omega)= j\omega\tau +1$

and the magnitude and phase responses are

$$\boxed{|H(j\omega)| = \sqrt{1+(\omega\tau)^2} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(\omega\tau)}$$

respectively. In decibels the magnitude expression is

$$\boxed{20\log_{10}|H(j\omega)| = 10\log_{10}(1+(\omega\tau)^2) \text{ dB.}}$$

- The low frequency asymptote is found by assuming that $\omega\tau \ll 1$ in which case

$$\boxed{\lim_{\omega\tau\to 0} 20\log_{10}|H(j\omega)| = 10\log10(1)=0 \text{ dB},}$$

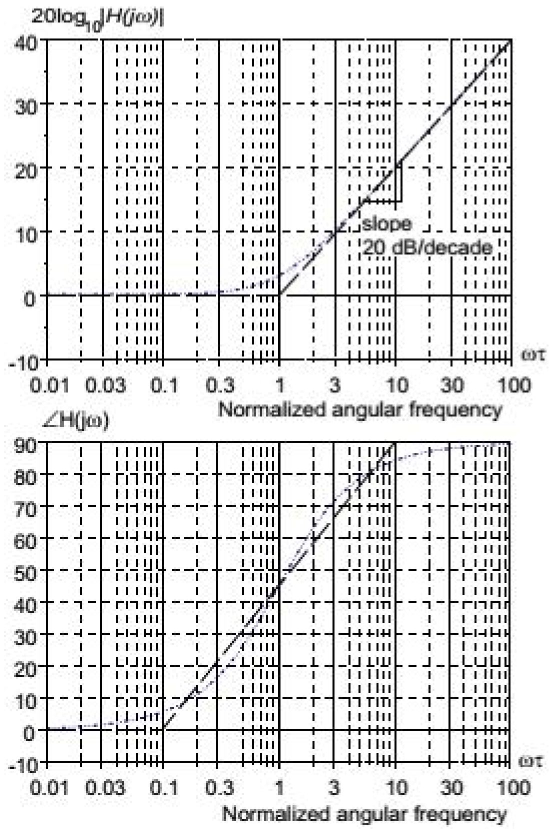
- The high frequency asymptote is found by assuming that $\omega\tau \gg 1$,

$$20\log10|H(j\omega)| \approx 20\log10(\omega\tau) = 20\log10(\omega) - 20\log10(\tau) \text{ dB} \quad \text{when } \omega \gg 1/\tau$$

which is a straight line on the log-log plot, with a slope of +20 dB/decade.

- The break frequency, defined by the intersection of these two asymptotes is at a fre-quency $\omega = 1/\tau$, and at this frequency the exact value of $|H(j\omega)|$ is$\sqrt{2}$ or +3 dB.

The complete asymptotic Bode magnitude plot using a normalized frequency scale is shown below.



The phase characteristic asymptotically approaches 0 radians at low frequencies and approaches a maximum phase lead of $\pi/2$ radians at frequencies much greater than the break frequency. At the break frequency the phase shift is $\pi/4$ radians. A piece-wise linear approximation, similar to that described for a real pole, is also shown below

6. Complex Conjugate Pole Pair:

The classical second-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\, s + \omega_n^2}$$

has a frequency response

$$|H(j\omega)| = \frac{1}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta(\omega/\omega_n))^2}}$$

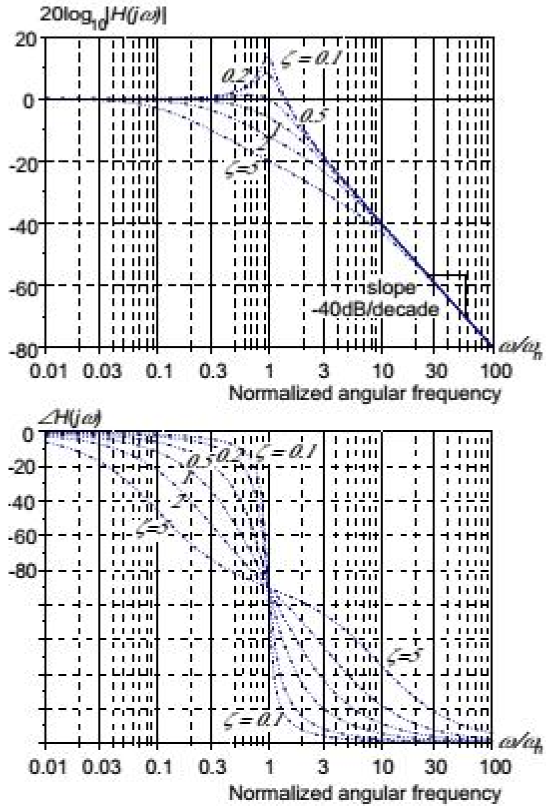
$$\text{and} \quad \angle H(j\omega) = \tan^{-1} \frac{-2\zeta\,(\omega/\omega_n)}{\left(1-(\omega/\omega_n)^2\right)}.$$

In logarithmic units the magnitude response is

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left[\left(1-(\omega/\omega_n)^2\right)^2 + (2\zeta(\omega/\omega_n))^2\right]$$

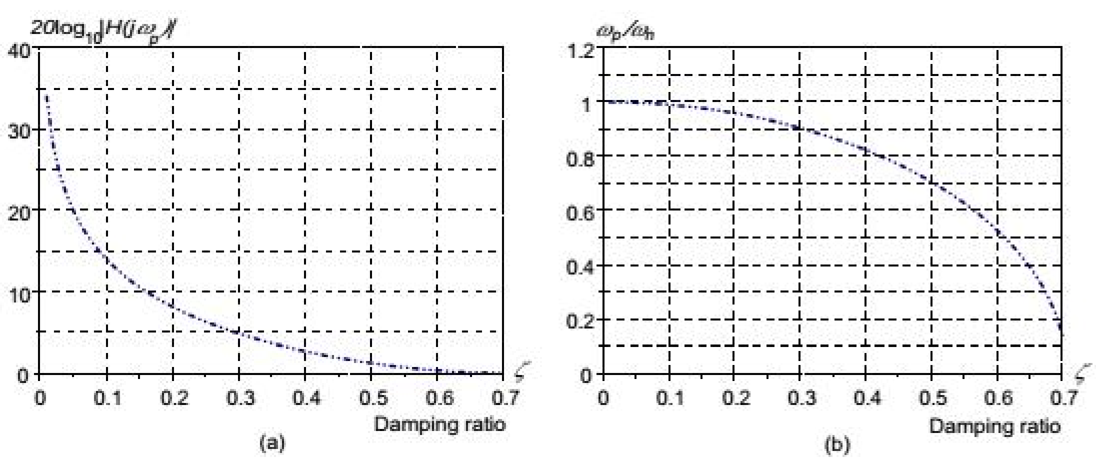
The Bode forms of the magnitude and phase responses are plotted in below, with the damping ratio ζ as a parameter.

- The low-frequency asymptote is found by assuming that $\omega/\omega_n \ll 1$ so that
  $$\lim_{(\omega/\omega_n)\to 0} (20\log_{10}|H(j\omega)|) = -10\log_{10}(1) = 0 \text{ dB}.$$
- The high frequency response can be found by retaining only the dominant term when
  $\omega/\omega_n \gg 1$:
  $20\log_{10}|H(j\omega)| \approx -10\log_{10}[(\omega/\omega_n)^4] = -40\log_{10}(\omega) + 40\log_{10}(\omega_n)$ *dB when* $\omega \gg \omega_n$,
- The two asymptotes intersect at a break frequency of $\omega = \omega_n$ as shown below. The straight line asymptotic form does not account in any way for the damping ratio.



The phase characteristic asymptotically approaches 0 radians at low frequencies, has a phase lag of $-\pi/2$ at the break frequency ω_n, and approaches $-\pi$ radians at high frequencies. The steepness of the transition is a function of the damping ratio ζ and so must be sketched using the information contained above.

The resonance peak (for values of $\zeta < 0.707$) must be sketched in after the asymptotes have been drawn. The figure below plots the logarithmic magnitude correction and frequency of the resonant peak as a function of ζ; it is a simple matter to sketch in the resonant peak from these values.



7 Complex Conjugate Zero Pair

Bode plots for a pair of complex conjugate zeros can be derived in a manner similar to the conjugate pole pair described above. In this case the block is assumed to have a transfer function

$$H(s) = \frac{1}{\omega_n^2}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$$

and a frequency response

$$\boxed{\begin{aligned} &|H(j\omega)| = \sqrt{\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2} \\ \text{and} \quad &\angle H(j\omega) = \tan^{-1}\frac{2\zeta\,(\omega/\omega_n)}{\left(1-(\omega/\omega_n)^2\right)}. \end{aligned}}$$

The logarithmic magnitude response is

$$20\log_{10}|H(j\omega)| = 10\log_{10}\left[\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2\right] \text{ dB}$$

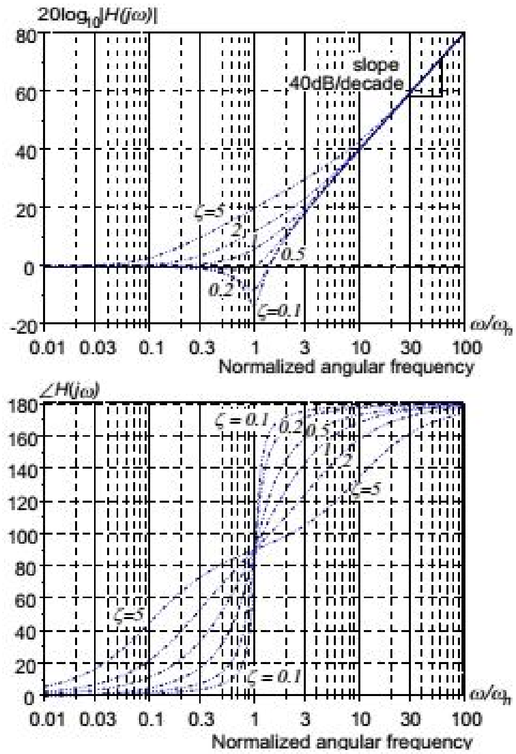
The asymptotic responses are derived in a similar manner to the complex pole pair; the low frequency asymptote is

$$\lim_{(\omega/\omega_n)\to 0}\left(20\log_{10}|H(j\omega)|\right) = 10\log_{10}(1) = 0 \text{ dB},$$

and the high frequency asymptote is

$$\begin{aligned} 20\log_{10}|H(j\omega)| &\approx 10\log_{10}\left[(\omega/\omega_n)^4\right] \\ &= 40\log_{10}(\omega) - 40\log_{10}(\omega_n) \text{ dB} \quad \text{for } \omega \gg \omega_n. \end{aligned}$$

The exact form of the magnitude response is plotted below. This is effectively an inverse of the characteristic of complex-conjugate pole pair described above. There is a “notch” in the response in the region of the frequency ω_n, and the depth is a function of the parameter ζ. The plot has a low frequency asymptote of 0 dB, a break frequency of $\omega = \omega_n$, and a high-frequency asymptote is a straight line with a slope of +40 dB/decade.



The phase characteristic is also a lipped version of that of a pair of complex conjugate poles; it approaches0 radians at low frequencies, passes through $-\pi/2$ at the break frequency, and shows a maximum phase lead of π radians at high frequencies. As above, the slope of the curve in the transition region is dependent on the value of ζ.

8 Summary

The essential features of the asymptotic forms of the seven components of the magnitude plot are summarized below

Description	Transfer Function	Break Frequency (radians/sec.)	High Frequency Slope (dB/decade)
Constant gain	K	-	0
Pole at the origin	$\frac{1}{s}$	-	-20
Zero at the origin	s	-	+20
Real pole	$\frac{1}{\tau s+1}$	$1/\tau$	-20
Real zero	$(\tau s+1)$	$1/\tau$	+20
Conjugate poles	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	ω_n	-40
Conjugate zeros	$\frac{1}{\omega_n^2}(s^2+2\zeta\omega_n s+\omega_n^2)$	ω_n	+40

4.4 Bode Plots of Higher Order Systems

If a system with transfer function H(s) = KN(s)/D(s) is expressed as a product of the terms in the table above, that is

$$
\begin{aligned}
H(s) &= K\frac{N(s)}{D(s)} \\
&= K'(N_1(s)\ldots N_m(s)) \times (\frac{1}{D_1(s)}\cdots\frac{1}{D_n(s)})
\end{aligned}
$$

where the factors Ni(s) are first-or second-order zero terms, and the $D_i(s)$ are pole terms, and K' is a modified constant factor. For example

$$
\begin{aligned}
H(s) &= \frac{10(s+3)}{(s+0.5)(s+5)} = K'N_1(s)\frac{1}{D_1(s)}\frac{1}{D_2(s)} \\
&= 12 \times (\frac{1}{3}s+1) \times \frac{1}{2s+1} \times \frac{1}{0.2s+1}.
\end{aligned}
$$

When complex numbers are represented in polar form, the magnitude of a product is the product of the component magnitudes, and the angle of a product is the sum of the component angles, the frequency response may be expressed in terms of its magnitude and phase functions:

$$
\begin{aligned}
|H(j\omega)| &= K' \times |N_1(j\omega)| \times \ldots \times |N_m(j\omega)| \times \left|\frac{1}{D_1(j\omega)}\right| \times \ldots \times \left|\frac{1}{D_n(j\omega)}\right| \\
\angle H(j\omega) &= \angle N_1(j\omega) + \ldots + \angle N_m(j\omega) + \angle\frac{1}{D_1(j\omega)} + \ldots + \angle\frac{1}{D_n(j\omega)}
\end{aligned}
$$

The logarithm of a product is the sum of the logarithms of its factors, so that

$$
\begin{aligned}
\log|H(j\omega)| &= \log K' + \log|N_1(j\omega)| + \ldots + \log|N_m(j\omega)| + \log\left|\frac{1}{D_1(j\omega)}\right| + \ldots + \log\left|\frac{1}{D_n(j\omega)}\right| \\
\angle H(j\omega) &= \angle N_1(j\omega) + \ldots + \angle N_m(j\omega) + \angle\frac{1}{D_1(j\omega)} + \ldots + \angle\frac{1}{D_n(j\omega)}
\end{aligned}
$$

which express the overall magnitude and phase responses as a sum of component responses of first and second-order elementary “building blocks”. In practice Bode plots are constructed by graphically adding the individual response components. Given the transfer function H(s)of a linear system, the following steps are used to construct the Bode magnitude plot:

1. Factor the numerator and denominator of the transfer function into the constant, first order and quadratic terms in the form described in the previous section.
2. Identify the break frequency associated with each factor.
3. Plot the asymptotic form of each of the factors on log–log axes.
4. Graphically add the component asymptotic plots to form the overall plot in straight line form.
5. "Round out" the corners in the straight line approximate curve by hand, using the known values of the responses at the break frequencies (±3dB for first-order sections, and dependent upon ζ for quadratic factors).

The phase plot is constructed by graphically by adding the component phase responses. The individual plots are drawn, either as the piece-wise linear approximation for the first-order poles, or in a smooth form from the exact plot, and then these are added to find the total phase shift at any frequency.

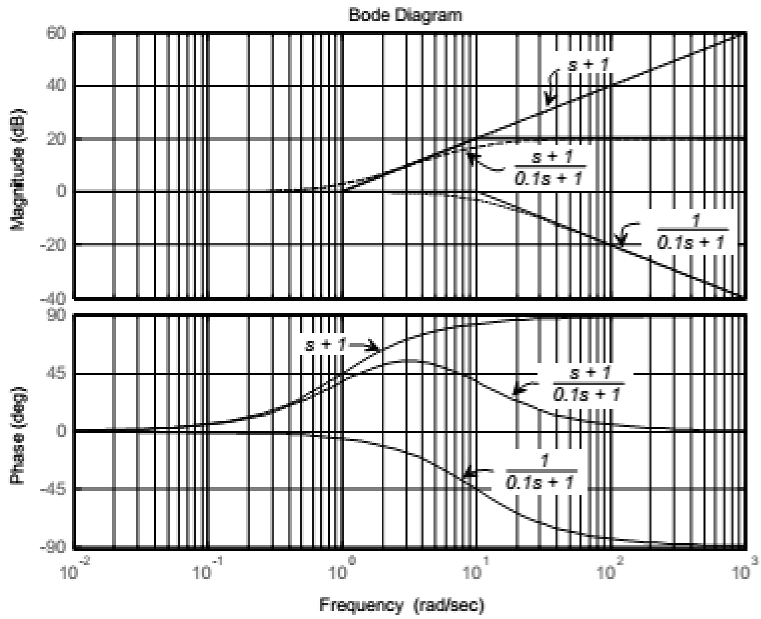
Example 1: Plot the Bode magnitude and phase plots of a first-order system described by the transfer function

$$H(S) = \frac{10(S+1)}{S+10}$$

$$H(s) = 10 \times (S+1) \times \frac{1}{10} \times \frac{1}{0.1S+1} = (S+1) \times \frac{1}{0.1S+1}$$

1. A single real zero term $H_1(s)=(s+1)$, with a break frequency of ω =1 radians/sec.
2. A single real pole term $H_2(S)=\frac{1}{0.1S+1}$, with a break frequency of ω = 10radians/sec.

The component terms are plotted and are added together to determine the total response for a frequency range of 0.01 to 1000 radians/sec. in the magnitude and phase plots below.



Example 2: Plot the Bode magnitude and phase plots of a third-order system described by the transfer function

$$H(s) = \frac{40s + 4}{s^3 + 2s^2 + 2s}$$

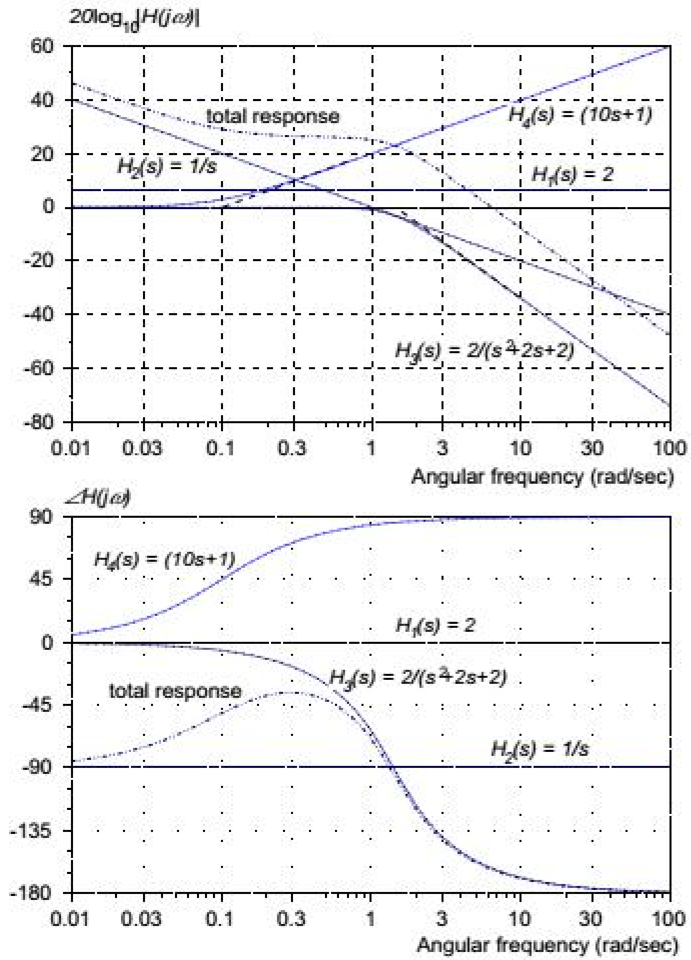
Solution: The transfer function is rewritten

$$H(s) = \frac{4(10s + 1)}{s\,(s^2 + 2s + 2)} = 2\,(10s + 1)\left(\frac{1}{s}\right)\left(\frac{2}{s^2 + 2s + 2}\right)$$

indicating four component terms:

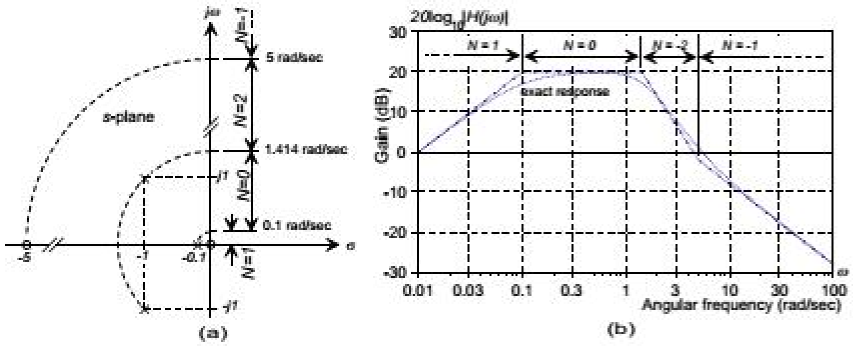
1. A constant gain term of H1(s) = 2,
2. A single real pole at the origin H2(s)=1/s,
3. A complex conjugate pole pair H(s) =2/(s^2 +2s + 2), characterized by ωn =√2 radians/sec. and a damping ratio of√2/2, and
4. A single real zero term H_3(s) = (10s +1), with a break frequency of ω =0.1 radians/sec.

The component terms are plotted and are added together to determine the total response for a frequency range of 0.01 to 100 radians/sec. in the magnitude and phase plots below.



A Simple Method for constructing the Magnitude Bode Plot directly from the Pole-Zero Plot The pole-zero plot of a system contains sufficient information to define the frequency response except for an arbitrary gain constant. It is often sufficient to know the shape of the magnitude Bode plot without knowing the absolute gain. The method described here allows the magnitude plot to be sketched by inspection, without drawing the individual component curves. The method is based on the fact that the overall magnitude curve undergoes a change in slope at each break frequency.

The first step is to identify the break frequencies, either by factoring the transfer function or directly from the pole-zero plot. Consider a typical pole-zero plot of a linear system as shown in (a) in the figure below. The break frequencies for the four first and second-order blocks are all at a frequency equal to the radial distance of the poles or zeros from the origin of the s-plane, that is $\omega_b = \sqrt{\omega^2 + \delta^2}$.Therefore all break frequencies may be found by taking a compass and drawing an arc from each pole or zero to the positive imaginary axis. These break frequencies may be transferred directly to the logarithmic frequency axis of the Bode plot



Because all low frequency asymptotes are horizontal lines with a gain of 0dB, a pole or zero does not contribute to the magnitude Bode plot below its break frequency. Each pole or zero contributes a change in the slope of the asymptotic plot of ±20 dB/decade above its break frequency. A complex conjugate pole or zero pair defines two coincident breaks of ±20 dB/decade (one from each member of the pair), giving a total change in the slope of ±40dB/decade. Therefore, at any frequency ω, the slope of the asymptotic magnitude function depends only on the number of break points at frequencies less than ω, or to the left on the Bode plot. If there are Z breakpoints due to zeros to the left, and P breakpoints due to poles, the slope of the curve at that frequency is $20 \times (Z - P)$ dB/decade. Any poles or zeros at the origin cannot be plotted on the Bode plot, because they are effectively to the left of all finite break frequencies. However, they define the initial slope. If an arbitrary starting frequency and an assumed gain (for example 0dB) at that frequency are chosen, the shape of the magnitude plot may be easily constructed by noting the initial slope, and constructing the curve from straight line segments that change in slope by units of ±20 dB/decade at the breakpoints. The arbitrary choice of the reference gain results in a vertical displacement of the curve.

In (b) the straight line magnitude plot for the system is shown, constructed using this method. A frequency range of 0.01 to 100 radians/sec was arbitrarily selected, and a gain of 0dB at 0.01 radians/sec was assigned as the reference level. The break frequencies at 0, 0.1, 1.414, and 5 radians/sec were transferred to the frequency axis from the pole-zero plot. The value of N at any frequency is $Z - P$, where Z is the number of zeros to the left, and P is the number of poles to the left. The curve was simply drawn by assigning the value of the slope in each of the frequency intervals and drawing connected lines.

CHAPTER 5:
CONTROLLER DESIGN

5.1 Introduction

Several common dynamic controllers appear very often in practice. They are known as PD, PI, PID, phase-lag, phase-lead, and phase-lag-lead controllers. In this section we introduce their structures and indicate their main properties. In the follow-up sections procedures for designing these controllers by using the root locus and bode plot techniques such that the given systems have the desired specifications are presented. In the most cases these controllers are placed in the forward path at the front of the plant (system) as presented in figure 1

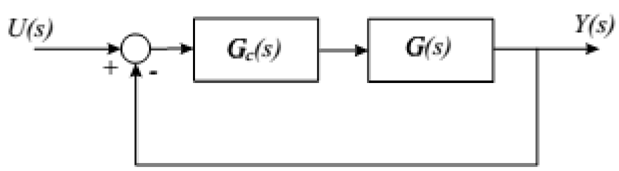


Figure 1 Simple unity feedback system

PD Controller: PD stands for a proportional and derivative controller. The output signal of this controller is equal to the sum of two signals: the signal obtained by multiplying the input signal by a constant gain K_p and the signal obtained by differentiating and multiplying the input signal by K_D , i.e. its transfer function is given by

$$Gc(s)=K_p+K_DS \tag{1a}$$

This controller is used to improve the system transient response.

PI controller : Similarly to the PD controller, the PI controller produces as its output a weighted sum of the input signal and its integral. Its transfer function is

$$G_c(s) = K_p + \frac{K_I}{S} \tag{1b}$$

In practical applications the PI controller zero is placed very close to its pole located at the origin so that the angular contribution of this "dipole" to the root locus is almost zero. *A PI controller is used to improve the system response steady state errors since it increases the control system type by one.*

PID controller: One of the most common controllers available commercially is the PID controller. Different processes are suited to different combinations of proportional, integral, and derivative control. The control engineer's task is to adjust the three gain factors to arrive at an acceptable degree of error reduction simultaneously with acceptable dynamic response. The compensator transfer function is

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s \qquad (1c)$$

The PID controller can be used to improve both the system transient response and steady state errors. This controller is very popular for industrial applications.

Phase Lead -Lag Controller: Other compensators, are lead, lag, and lead-lag compensators. A first-order compensator having a single zero and pole in its transfer function is

$$G_c(s) = \frac{s + Z_0}{s + P_0} \qquad (2)$$

The pole and zero are located in the left half s-plane as shown in Figure 1.

(a) Phase-lead (b) Phase-lag

Figure 1 Compensator phase angle contribution

For a given $s_1 = \sigma_1 + j\omega_1$, the transfer function angle given by $\theta_c = (\theta_{z_0} - \theta_{p_0})$ is positive if $z_0 < p_0$ as shown in Figure 1 (a), and the compensator is known as the *phase-lead controller*. On the other hand if $z_0 > p_0$ as shown in Figure 1 (b), the compensator angle $\theta_c = (\theta_{z_0} - \theta_{p_0})$ is negative, and the compensator is known as the *phase-lag controller*

5.2 Controller design Using Root locus

In general, the open-loop transfer function is given by

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where m is the number of finite zeros and n is the number of finite poles of the loop transfer function. If $n > m$, there are $(n-m)$ zeros at infinity. The characteristic equation of the closed-loop transfer function is

$$1 + KG(s)H(s) = 0$$

Therefore

$$\frac{(s+p_1)(s+p_2)\cdots(s+p_n)}{(s+z_1)(s+z_2)\cdots(s+z_m)} = -K$$

From the above expression, it follows that for a point in the s-plane to be on the root-locus, when $0 < K < \infty$, it must satisfy the following two conditions.

$$\begin{aligned} K &= \frac{|s+p_1||s+p_2|\cdots|s+p_n|}{|s+z_1||s+z_2|\cdots|s+z_m|} \quad \text{or} \\ K &= \frac{\text{product of vector lengths from finite poles}}{\text{product of vector lengths from finite zeros}} \end{aligned} \tag{3}$$

and

$$\sum \text{of zeros of } G(s)H(s) - \sum \text{angle of poles of } G(s)H(s) = r(180), \qquad r = \pm 1, \pm 3, \cdots$$

or

$$\sum_{i=1}^{m}\theta_{zi}-\sum_{i=1}^{n}\theta_{pi}=180r, \qquad r=\pm1,\pm3,\cdots \tag{4}$$

The magnitude and angle criteria given by (3) and (4) are used in the graphical root-locus design.

In addition to the MATLAB control system toolbox **rlocus(num, den)** for root locus plot, MATALB control system toolbox contain the following functions which are useful for interactively finding the gain at certain pole locations and intersect with constant ω_n circles. These are:

sgrid generates a grid over an existing continuous s-plane root locus or pole-zero map. Lines of constant damping ratio ζ and natural frequency ω_n are drawn. sgrid('new') clears the current axes first and sets hold on.

sgrid(Z, Wn) plots constant damping and frequency lines for the damping ratios in the vector Z and the natural frequencies in the vector Wn.

[K, poles] = rlocfind(num, den) puts up a crosshair cursor in the graphics window which is used to select a pole location on an existing root locus. The root locus gain associated with this point is returned in **K** and all the system poles for this gain are returned in **poles**.

rltool or **sistool** opens the SISO Design Tool. This Graphical User Interface allows you to design single-input/single-output (SISO) compensators by interacting with the root locus, Bode, and Nichols plots of the open-loop system.

1. Gain Factor Compensation or P-Controller Design

The proportional controller is a pure gain controller. The design is accomplished by choosing a value K_0, which results in a satisfactory transient response. The specification may be either the step response damping ratio or the step response time constant or the steady-state error. The procedure for finding K_0 is as follows:

- Construct an accurate root-locus plot
- For a given ζ draw a line from origin at angle $\theta = \cos^{-1}\zeta$ measured from negative real axis.
- The desired closed-loop pole s_1 is at the intersection of this line and the root-locus.
- Estimate the vector lengths from s_1 to poles and zeros and apply the magnitude criterion as given by (3) to find K_0.

Example 1

The open-loop transfer function of a control system is given by

$$KGH(s) = \frac{K}{s(s+1)(s+4)}$$

(a) Obtain the gain K_0 of a proportional controller such that the damping ratio of the closed-loop poles will be equal 0.6. Obtain root-locus, step response and the time-domain specifications for the compensated system.

The root-locus plot is shown in Figure 2. For $\zeta = 0.6$,

$$\theta = \cos^{-1} 0.6 = 53.13^\circ$$

The line drawn at this angle intersects the root-locus at approximately, $s_1 \approx -0.41 + j0.56$. The vector lengths from s_1 to the poles are marked on the diagram

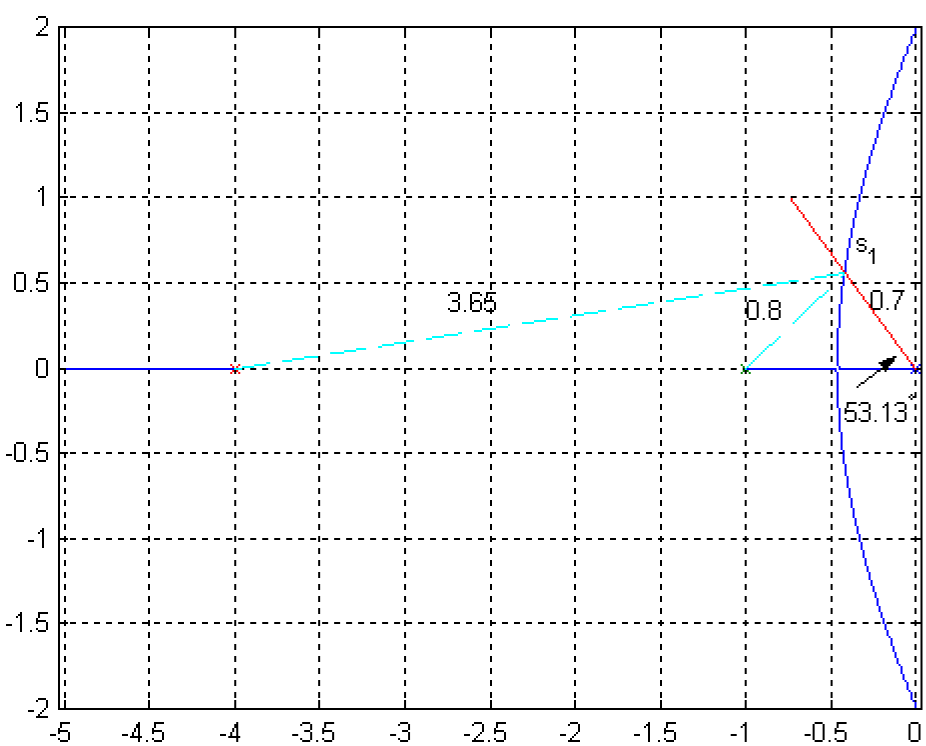


Figure 2 P-Controller Design

From (1), we have

$$K = (0.7)(0.8)(3.65) = 2.04$$

This gain will result in the velocity error constant of $K_v = \frac{2.04}{4} = 0.51$. Thus, the steady-state error due to a ramp input is $e_{ss} = \frac{1}{K_v} = \frac{1}{0.51} = 1.96$.

The compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2.05}{s^3 + 5s^2 + 4s + 2.05}$$

(b) Use the MATLAB Control System Toolbox functions **rlocus** and **sgrid(zeta, wn)** to obtain the root-locus and the gain K_0 for $\zeta = 0.6$. Also use the **ltiview** function to obtain the system step response and the time-domain specifications.

The following commands

```
num=1;
den=[1 5 4 0];
rlocus(num, den);
hold on
```

sgrid(0.6, 1) % plots constant line zeta=0.6 & constant line wn=1

result in

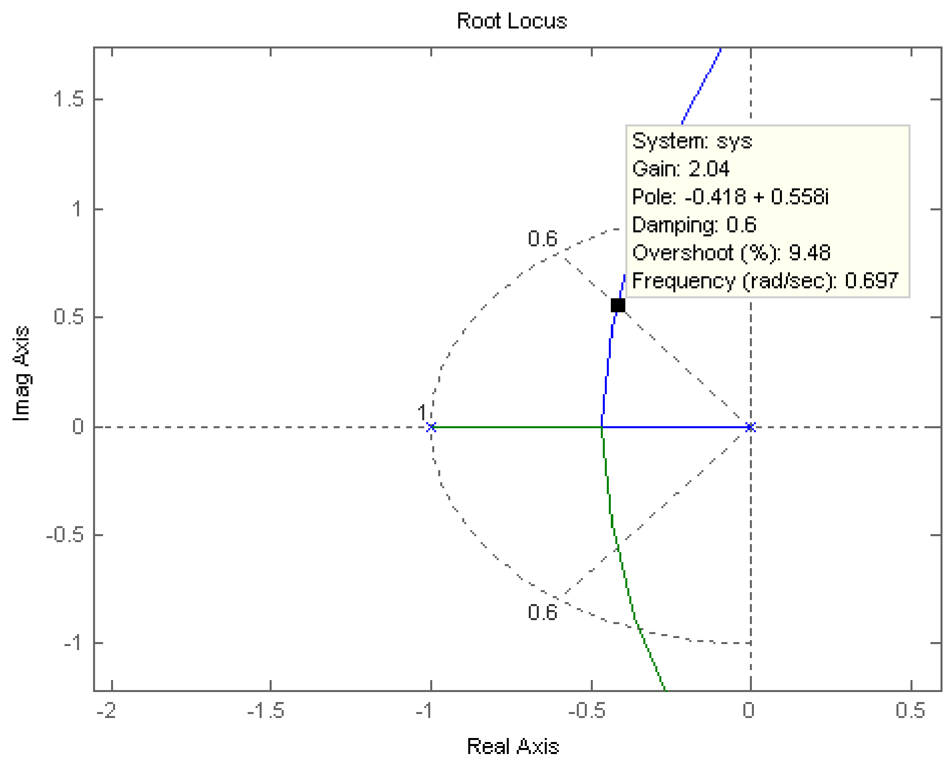


Figure 3

Zoom in at the area of intersection, click at the intersection, hold and move the mouse at intersection and adjust for Damping: 0.6. The gain is found to be 2.04. In addition, the percentage overshoot and natural frequency are obtained, i.e., $PO = 9.48\%$ and $\omega_n = 0.697$.

To obtain the step response and time-domain specifications, we use the following commands.

```
numc=2.04;

denc=[1 5 4 2.04];

T=tf(numc, denc)

ltiview('step', T)
```

The result is shown in Figure 4. Right-click on the LTI Viewer, use Chracteristics to mark peak response, peak time, settling time, and rise time. From File Menu use Print to Figure to obtain a Figure plot.

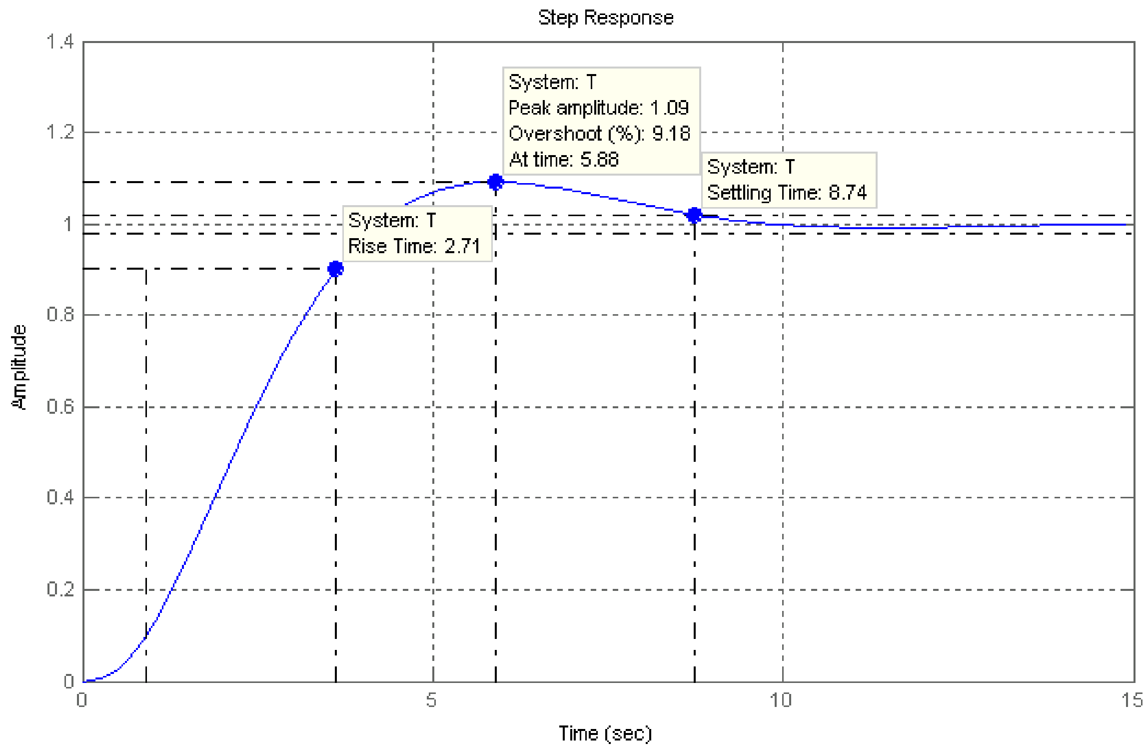


Figure4

2. *PD* Compensator Design

Here both the error and its derivative are used for control

$$G_c(s) = K_P + K_D s \tag{5}$$

or

$$G_c(s) = K_D\left(s + \frac{K_P}{K_D}\right) = K_D(s + z_0)$$

where

$$Z_0 = \frac{K_P}{K_D} \tag{6}$$

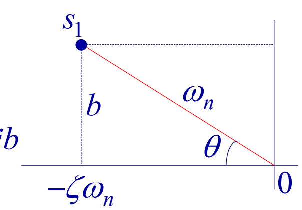
From above, it can be seen that the PD controller is equivalent to the addition of a simple zero at $Z_0 = K_P / K_D$ to the open-loop transfer function, which improves the transient response. From a different point of view, the PD controller may also be used to improve the steady-state error, because it anticipates large errors and attempts corrective action before they occur.

The procedure for the graphical root-locus PD compensator design is as follows:

- Construct an accurate root-locus plot
- From the design specifications; the desired damping ratio and time constant of the dominant closed-loop poles, obtain the desired location of the dominant closed-loop poles.

$$\zeta\omega_n = \frac{1}{\tau} \quad \text{and} \quad \theta = \cos^{-1}\zeta$$

$$b = \zeta\omega_n \tan\theta \quad \text{and} \quad s_1 = -\zeta\omega_n + jb$$



- Mark the poles and zeros of the open-loop plant transfer function. Find the location of the compensator zero Z_0 such that the angle criterion as given by (4) is satisfied.

  $$\theta_{z0} + (\theta_{z1} + \theta_{z2} + \cdots) - (\theta_{p1} + \theta_{p2} + \cdots) = -180$$

- Estimate the vector lengths from s_1 to all poles and zeros and apply the magnitude criterion as given by (3) to find K_D. Find K_P from (6)

Example 2

Consider the control system shown in Figure 5.

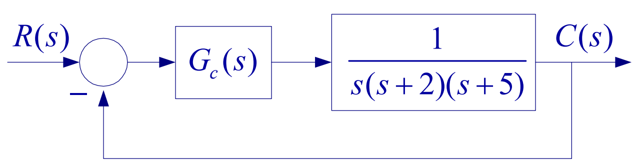


Figure 5

(a) Assume the compensator is a simple proportional controller K, obtain all pertinent pints for root locus and draw the root-locus. Determine the location of the dominant poles to have critically damped response, and find the time constant corresponding to this location. Also determine the value of K and the corresponding time constant for dominant poles damping ratio of 0.707. Obtain the compensated system step response.

(b) $G_c(s)$ is a PD compensator. Design the compensator for the following time-domain specifications.

- Dominant poles damping ratio $\zeta = 0.707$
- Dominant poles time constant $\tau = 0.5$ second

(a) First we construct the root locus

- The root-loci on the real axis are to the left of an odd number of finite poles and zeros.
- $n - m = 3$, i.e., there are three zeros at infinity.
- Three asymptotes with angles $\theta = 180^\circ$, and $\pm 60^\circ$.
- The asymptotes intersect on the real axis at

$$\sigma_a = \frac{\sum \text{finite poles of } GH(s) - \sum \text{finite zeros of } GH(s)}{n-m} = \frac{-(2+5)}{3} - 2.33$$

- Breakaway point on the real axis is given by

$$\frac{dK}{ds} = \frac{d}{ds}(s^3 + 7s^2 + 10s) = 0 \qquad \Rightarrow \qquad 3s^2 + 14s + 10 = 0$$

The roots of this equation are $s = -3.7863$, and $s = -0.8804$. But $s = -3.7863$ is not part of the root-locus for $K > 0$, therefore the breakaway point is at $s = -0.8804$. The Routh array gives the location of the $j\omega$-axis crossing.

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 7 & K \\ s^1 & 70-K & 0 \\ s^0 & K & 0 \end{array} \quad \Rightarrow \quad \text{for stability } 0 < K < 70 \text{ and } s = \pm j3.16$$

The root-locus is shown in Figure 6.

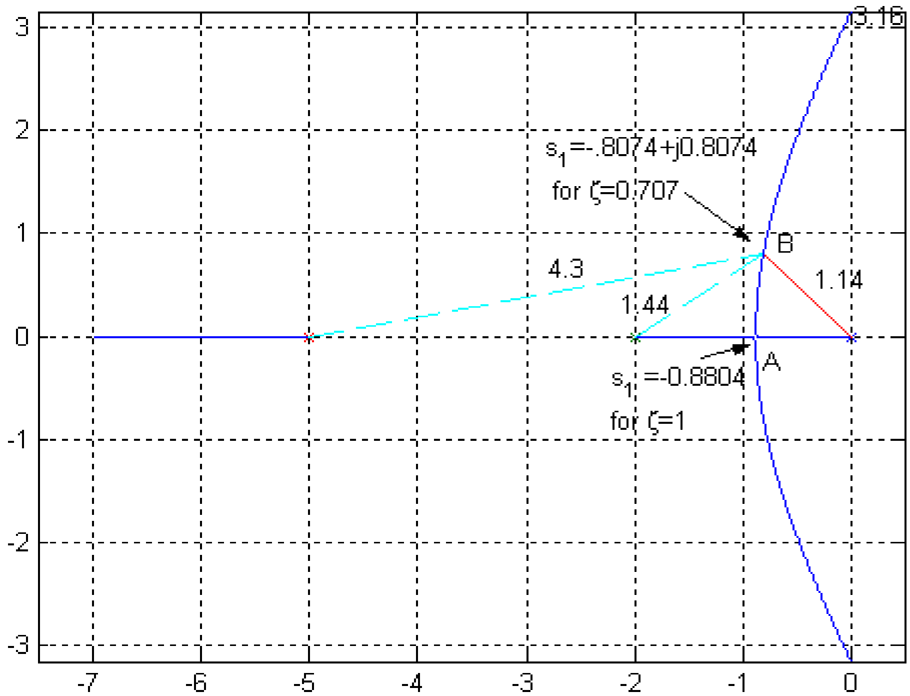


Figure 6

For the dominant poles to have critically damped response, the dominant poles are at the breakaway position A, i.e., $s_1 = s_2 = -0.8804$. The time constant and the gain K are

$$\tau = \frac{1}{0.8804} = 1.136 \text{ second}$$

$$K = (0.8804)(1.1296)(4.1196) = 4.06$$

For dominant poles damping ratio of 0.707, s_1 is at position B. The time constant and the gain K are

$$\tau = \frac{1}{0.8074} = 1.24 \text{ second}$$

$K = (1.14)(1.44)(4.3) = 7.06$

(b) The PD controller design

$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.5} = 2$, and $\theta = coc^{-1}(0.707) = 45^\circ$

Therefore

$$s_1 = -2 + j2$$

The desired location of s_1 requires the root-locus to be shifted towards the left half s-plane, which requires the addition of zero by the PD controller as shown in Figure 7.

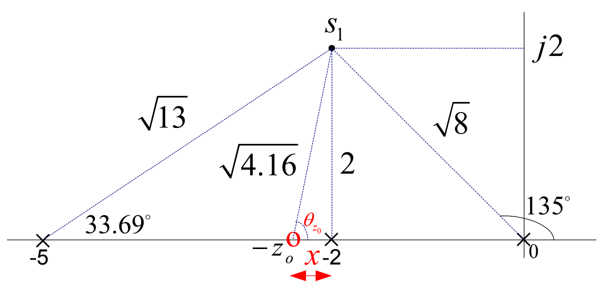


Figure 7

The position of z_0 is found by applying the angle criterion given by (4)

$$\theta_{z0} - (135 + 90 + 33.69) = -180 \qquad \Rightarrow \qquad \theta_{z0} = 78.69^\circ$$

$$\tan 78.69 = \frac{2}{x} \qquad \Rightarrow \qquad x = 0.4, \text{ and } z_0 = 2.4 = \frac{K_P}{K_D}$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_D(s + 2.4)}{s(s+2)(s+5)}$$

The vector lengths from s_1 are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_D = \frac{(\sqrt{8})(2)(\sqrt{13})}{\sqrt{4.16}} = 10$$

$$\frac{K_P}{K_D} = \frac{K_P}{10} = 2.4 \quad \Rightarrow \quad K_P = 24$$

Therefore, the controller transfer function is

$$G_c(s) = 24 + 10s$$

We use the following commands to obtain the closed-loop transfer function and the step response.

```
Gp = tf([0 0 1],[1  9  14])          % Plant transfer function
Gc = tf([10  24],[0  1])             % PD compensator
GpGc = series(Gp, Gc)                % Open-loop transfer function
T = feedback(GpGc, 1)                % closed-loop transfer function
ltiview('step', T)                   % obtains the step response
```

The result is shown in Figure 8.

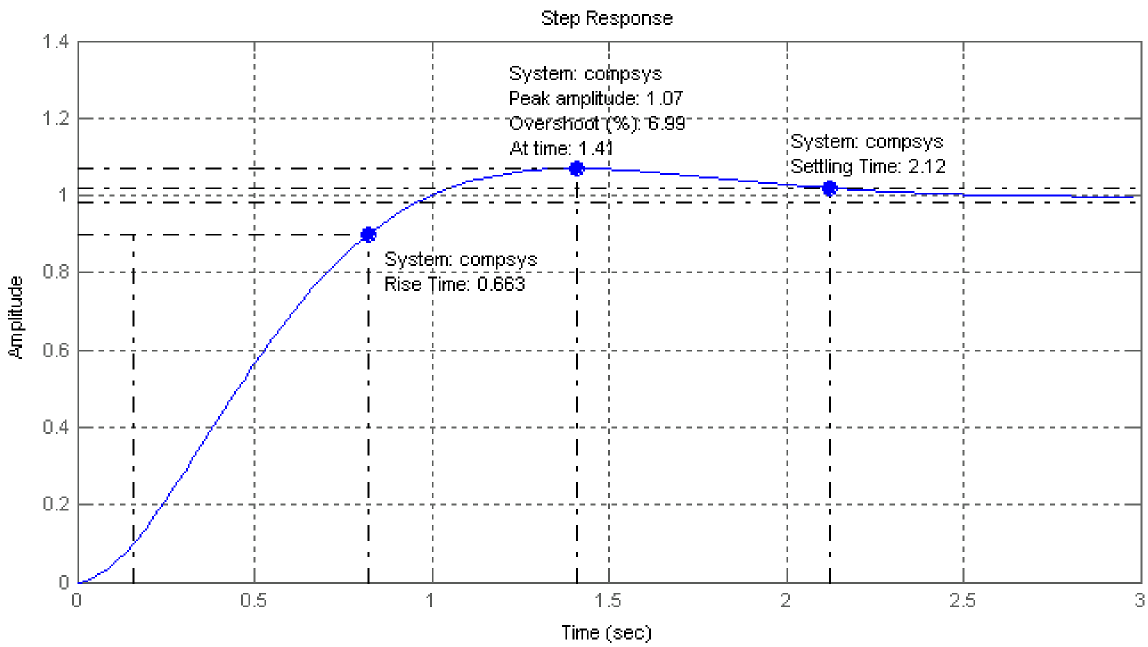


Figure 8 Step response for the system of Example 2

3. PI Compensator Design

The integral of the error as well as the error itself is used for control, and the compensator transfer function is

$$G_c(s) = K_P + \frac{K_I}{s} \tag{7}$$

or

$$G_c(s) = \frac{K_P s + K_I}{s} = \frac{K_P(s + \frac{K_I}{K_P})}{s} = \frac{K_P(s + z_0)}{s}$$

where

$$Z_0 = \frac{K_I}{K_P}$$

Integral control bases its corrective action on the cumulative error integrated over time. The controller increases the type of system by 1 and is used to eliminate the steady-state errors.

Example 3

For the control system shown in Figure 9 design a PI compensator for the following specifications:

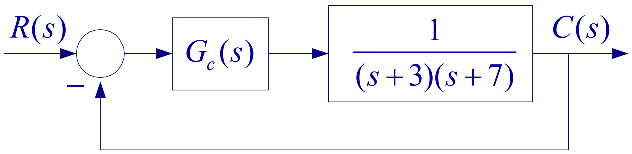


Figure 9

- Zero steady-state error due to a step input
- A pair of dominant closed-loop poles with a time constant of 0.25 seconds and a damping ratio of 0.8.

Obtain the compensated system step response.

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.25} = 4, \text{ and } \theta = \tan^{-1}(0.8) = 36.87^\circ$$

Therefore

$$s_1 = -4 + j4 * \tan 36.87^\circ \quad \Rightarrow \quad s_1 = -4 + j3$$

The poles of the open-loop transfer function and the controller pole at origin are marked in Figure 10.

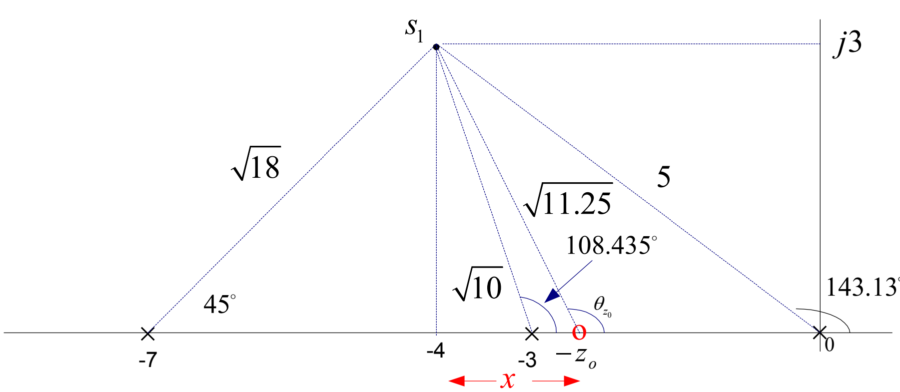


Figure 10

The position of controller zero for the desired location of s_1 is obtained by applying the angle criterion given by (4)

$$\theta_{z0} - (143.13 + 108.435 + 45) = -180 \quad \Rightarrow \quad \theta_{z0} = 116.565^\circ$$

$$\tan(180 - 116.56) = \frac{3}{x} \quad \Rightarrow \quad x = 1.5, \text{ and } z_0 = 4 - 1.5 = 2.5$$

Therefore

$$\frac{K_I}{K_P} = 2.5$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_P(s+2.5)}{s(s+3)(s+7)}$$

The vector lengths from s_1 are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_P = \frac{(5)(\sqrt{10})(\sqrt{18})}{\sqrt{11.25}} = 20$$

$$\frac{K_I}{K_P} = \frac{K_I}{20} = 2.5 \quad \Rightarrow \quad K_I = 50$$

Therefore, the controller transfer function is

$$G_c(s) = 20 + \frac{50}{s}$$

The PI controller increases the system type from zero to 1. That is, we have a type 1 system and the steady-state error due to a step input is zero. We use the following commands to obtain the closed-loop transfer function and the step response.

```
Gp = tf([0 0 1],[1  10  21])     % Plant transfer function

Gc = tf([20  50],[1  0])                  % PI compensator

GpGc = series(Gp, Gc)                     % Open-loop transfer function

T = feedback(GpGc, 1)                     % closed-loop transfer function

ltiview('step', T)               % obtains the step response
```

The result is shown in Figure 11.

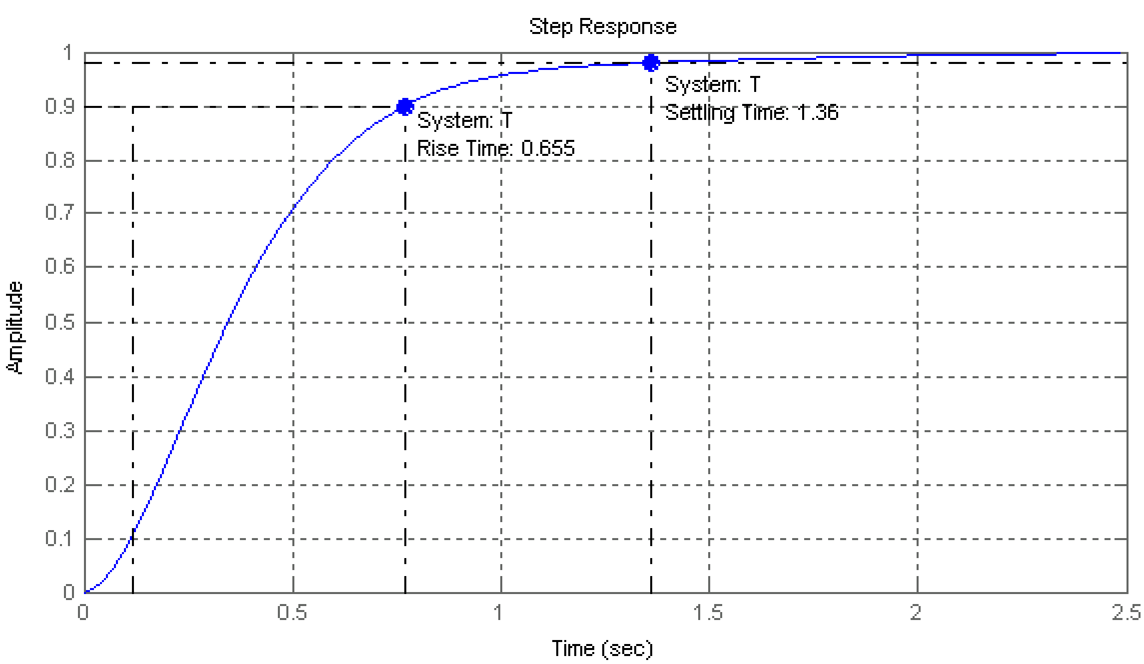


Figure 11 Step response for the system of Example 2

4. PID Compensator

The PID controller is used to improve the dynamic response as well as to reduce or eliminate the steady-state error. With a proportional controller increasing the controller gain will reduce the rise time and the steady-state error. However, in systems of third order or higher, large gain will make the system unstable. Derivative action contributes phase-lead and will improve the transient response, reducing the overshoot and settling time. The integral action increases the system type by 1 and eliminates the steady-state error, but it may make the transient response worse. When you are designing a PID controller, first set K_P to a large value to produce a fast response without loosing stability. Then add derivative gain K_D and adjust its value to meet the transient response specifications. If required introduce the Integral gain K_I to eliminate the steady-state error. Repeat the design and fine-tune the gains to obtain the desired response.

5. Phase-Lead Design

In the phase-lead controller $|z_0| < |p_0|$, thus the controller contributes a positive angle to the root-locus angle criterion and tends to shift the root-locus of the plant toward the left in the s-plane.Since $|z_0| < |p_0|$, the compensator is a high-pass filter. The phase-lead compensator has the same purpose as the PD compensator. It is utilized to improve the transient response, to raise bandwidth and to increase the speed of response. A lead compensator approximates derivative control and reduces the high-frequency noise present in the PD compensator. The procedure or the graphical root-locus design is as follows:

- From the time-domain specifications obtain the desired location of the closed-loop dominant poles.
- Select the controller zero. Place the zero to the left of the smallest plant's pole (or on the pole for *pole-zero cancellation*)
- Locate the compensator pole so that the angle criterion (3) is satisfied.
- Determine the compensator gain K_c such that the magnitude criterion (4) is satisfied.
- If the overall response rise time, overshoot and settling time is not satisfactory, place the controller zero at a different location and repeat the design

Moving the controller zero to the left away from the origin in the s-plane results in a faster response with increase in overshoot. Moving the controller zero to the right towards the origin will result in a slow response and reduces or eliminate the overshoot. The compensator angle $\theta_{z0} - \theta_{p0}$ must be positive Therefore, there is a limit on how far to the left along the real axis the compensator zero may be moved and still be able to satisfy the angle criterion.

Example 4

For the control system of Example 2 design a phase lead compensator to meet the following time-domain specifications:

- Dominant poles damping ratio $\zeta = 0.707$
- Dominant poles time constant $\tau = 0.5$ second

$\zeta\omega_n = \dfrac{1}{\tau} = \dfrac{1}{0.5} = 2$, and $\theta = coc^{-1}(0.707) = 45^\circ$

Therefore

$$s_1 = -2 + j2$$

The desired location of s_1 requires the root-locus to be shifted towards the left half s-plane, which requires the addition of phase lead controller as shown in Figure 12.

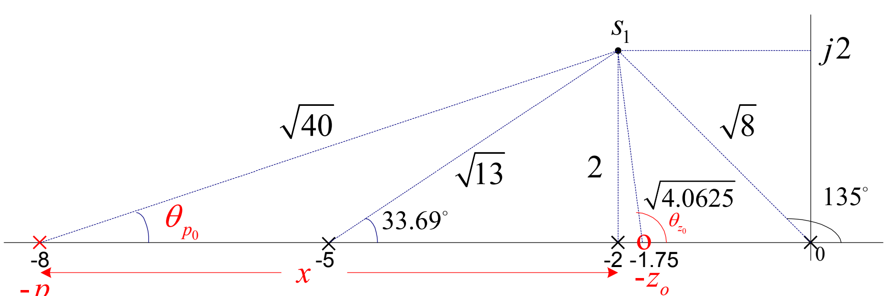


Figure 12

Let the controller zero be located at $z_0 = 1.75$. The position of the controller p_0 is found by applying the angle criterion given by (4)

$\theta_0 = 180 - \tan^{-1}(2/0.25) = 97.125^\circ$

$$97.125 - (135 + 90 + 33.69 + \theta_{p0}) = -180 \quad \Rightarrow \quad \theta_{p0} = 18.435^\circ$$

$$\tan 18.435 = \frac{2}{x} \quad \Rightarrow \quad x = 6, \text{ and } z_0 = 2 + 6 = 8$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_c(s+1.75)}{s(s+2)(s+5)(s+8)}$$

The vector lengths from s_1 are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_c = \frac{(\sqrt{8})(2)(\sqrt{13})(\sqrt{40})}{\sqrt{4.0625}} = 64$$

Therefore, the controller transfer function is

$$G_c(s) = \frac{64(s+1.75)}{(s+8)}$$

The controller dc gain is

$$a_0 = G_c(0) = \frac{(64)(1.75)}{8} = 14$$

We use the following commands to obtain the closed-loop transfer function and the step response.

```
Gp = tf([0 0  0 1],[1  7  10 0])          % Plant transfer function
Gc = tf(64*[1  1.75],[1  8])              % PI compensator
GpGc = series(Gp, Gc)                     % Open-loop transfer function
T = feedback(GpGc, 1)                     % closed-loop transfer function
ltiview('step', T)                        % obtains the step response
```

The result is shown in Figure 13.

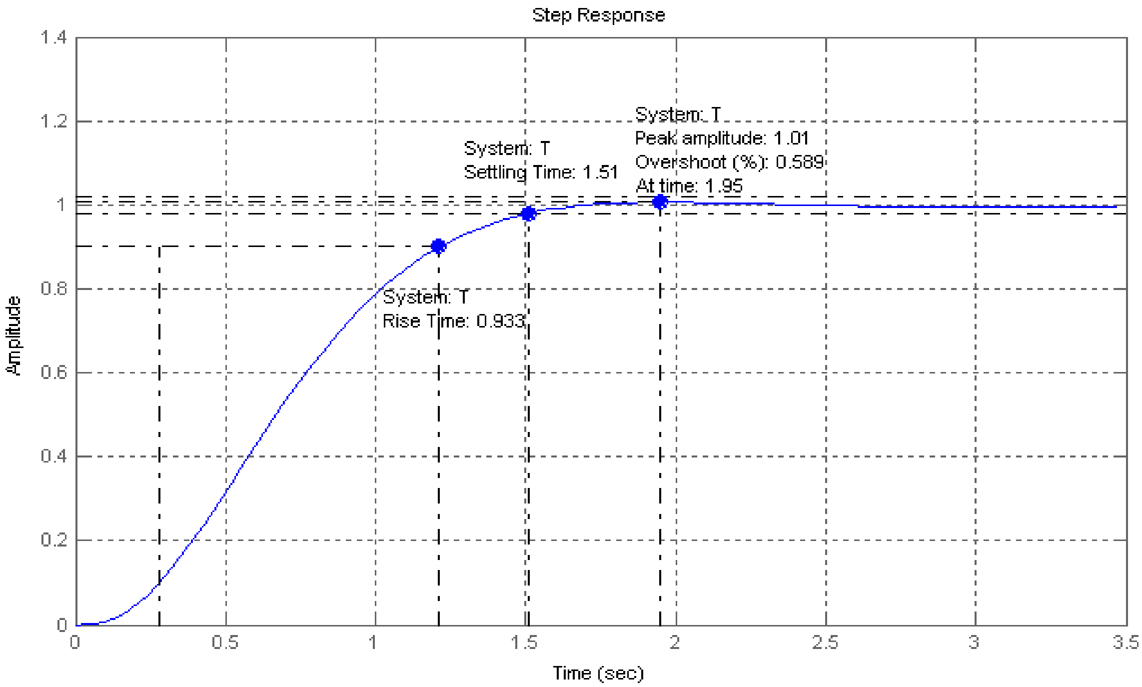


Figure 13

Step response for the system of Example 4.

6. Phase-lag compensator approximate design

The lag compensator is an approximate integral control. The phase-lag compensator is used when the system transient response is satisfactory but requires a reduction in the steady-state error. Since $|p_0| < |z_0|$, the compensator is a low-pass filter. It adds a negative angle to the angle criterion and tends to shift the root-locus to the right in the s-plane.

In the phase-lag control, the controller poles and zeros are placed very close together, and the combination is located relatively close to the origin of the s-plane. Thus, the root-loci in the compensated system are shifted only slightly from their original locations. The compensator contributes a magnitude of

$$|G_c(s)| = \frac{K_c |s_1 + z_0|}{|s_1 + p_0|} \approx K_c$$

The gain to satisfy the desired damping ratio is given by

$$K_0 \,|\, GH(s_1) \,|= 1 \qquad \Rightarrow \qquad |\, GH(s_1) \,|= \frac{1}{K_0}$$

For the compensated system, the magnitude criterion requires that

$$K \,|\, GH(s_1) \,||\, G_c(s_1) \,|= 1 \qquad \Rightarrow \qquad K \frac{1}{K_0} K_c = 1$$

or

$$K_c = \frac{K_0}{K} = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} \tag{7}$$

For a given desired location of a closed-loop pole s_1, the design can be accomplished by trial and error. The procedure for approximate phase-lag design is as follows:

- Obtain the root-locus and determine the gain K_0 to satisfy the desired damping ratio.
- Determine the gain K to satisfy the desired stead-state error.
- Evaluate the controller gain $K_c = \dfrac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} = \dfrac{K_0}{K}$
- Select the controller zero z_0 close to origin.
- Based on the compensator DC gain of unity, $\dfrac{K_c z_0}{p_0} = 1$, find the controller pole $p_0 = K_c z_0$

Example 5
Consider the control system shown in Figure 14.

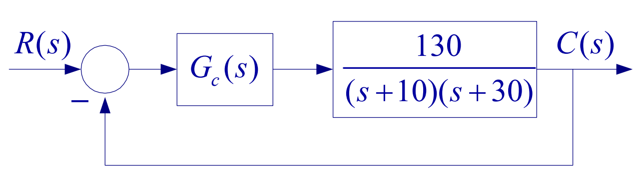


Figure 14

(a) Assume the compensator is a simple proportional controller K, obtain all pertinent pints for root locus and draw the root-locus. Determine the gain K_0 for the step response damping ratio of 0.8. Obtain the steady-state error and the system step response.

- The root-loci on the real axis are to the left of an odd number of finite poles and zeros.
- $n - m = 2$, i.e., there are two zeros at infinity.
- Two asymptotes with angles $\theta = \pm 90^\circ$.
- The asymptotes intersect on the real axis at

$$\sigma_a = \frac{\sum \text{finite poles of } GH(s) - \sum \text{finite zeros of } GH(s)}{n-m} = \frac{-(10+30)}{2} - 20$$

- Breakaway point on the real axis is given by

$$\frac{dK}{ds} = \frac{d}{ds}(s^2 + 40s + 300) = 0 \qquad \Rightarrow \qquad 2s + 40 = 0$$

Therefore the breakaway point is at $s = -20$.

The root-locus is shown in Figure 15

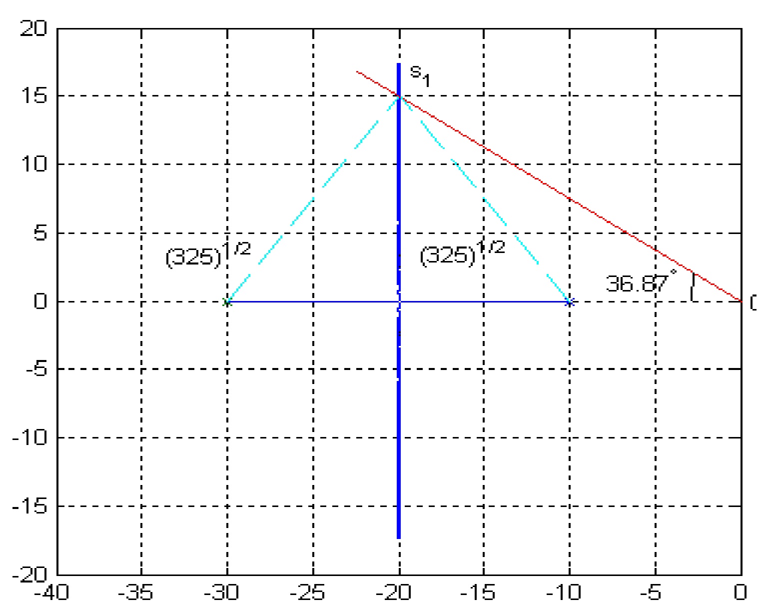


Figure 15

For $\zeta = 0.8 \quad \Rightarrow \quad \theta = \cos^{-1}(0.8) = 36.87^\circ$

The intersection of the line drawn from origin at this angle with root locus gives the desired complex pole $s_1 = -20 + j15$. Applying the magnitude criterion (3), the gain K_0 is found

$$130K_0 = \sqrt{325}\sqrt{325} \quad \Rightarrow \quad K_0 = 2.5$$

The position error constant is

$$K_p = \frac{(130)(2.5)}{(10)(30)} = 1.08333$$

The steady-state error is

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+1.08333} = 0.48$$

The step response is shown in Figure 16.

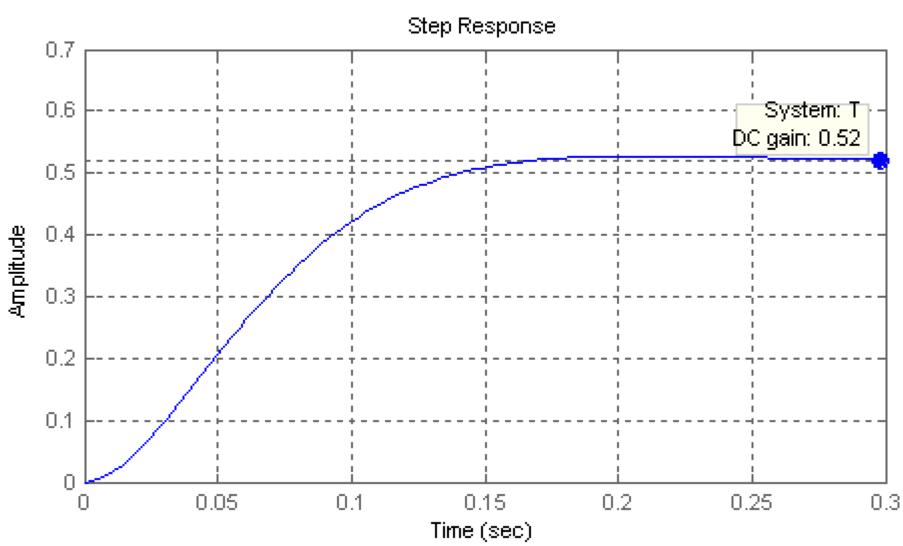


Figure 16 The step response for Example 5 (a).

(b) It is required to have approximately the same dominant closed-loop pole locations and the same damping ratio ($\zeta = 0.8$) as in part (a). Design a phase-lag compensator such that the steady-state error due to a unit step input e_{ss} will be equal to 0.0845. Obtain the step response, and the time-domain specifications for the compensated system.

The gain K, which results in $e_{ss} = 0.0845$ is given by

$$e_{ss} = 0.0845 = \frac{1}{1+K_p} \qquad \Rightarrow \qquad K_p = 10.8343 = \frac{130K}{(10)(30)}$$

Thus the gain to realize the steady-state error specification is $K = 25$

Using the approximate method, the controller gain is given by

$$K_c = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} = \frac{2.5}{25} = 0.1$$

Next choose a small value for the compensator zero, e.g., $z = 1.5$

Based on the controller dc gain of unity $K_c z_0 / p_0 = 1$, the controller pole is found

$p_0 = K_c z_0 = (0.1)(1.5) = 0.15$. Thus the controller transfer function is

$$G_c(s) = \frac{0.1(s+1.5)}{(s+0.15)}$$

and the compensated open-loop transfer function is

$$G_c(s)KG(s) = \frac{(0.1)(s+1.5)(130)(25)}{(s+0.15)(s+10)(s+30)} = \frac{325s+478.5}{s^3+40.15s^2+306s+45}$$

The compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{325s+478.5}{s^3+40.15s^2+631s+532.5}$$

The compensated characteristic equation roots are $-19.63 \pm j14.5$, and –0.894. The compensated step response is shown in Figure 17. The complex poles are shifted slightly to the left from the specified value of $-20 \pm j15$.

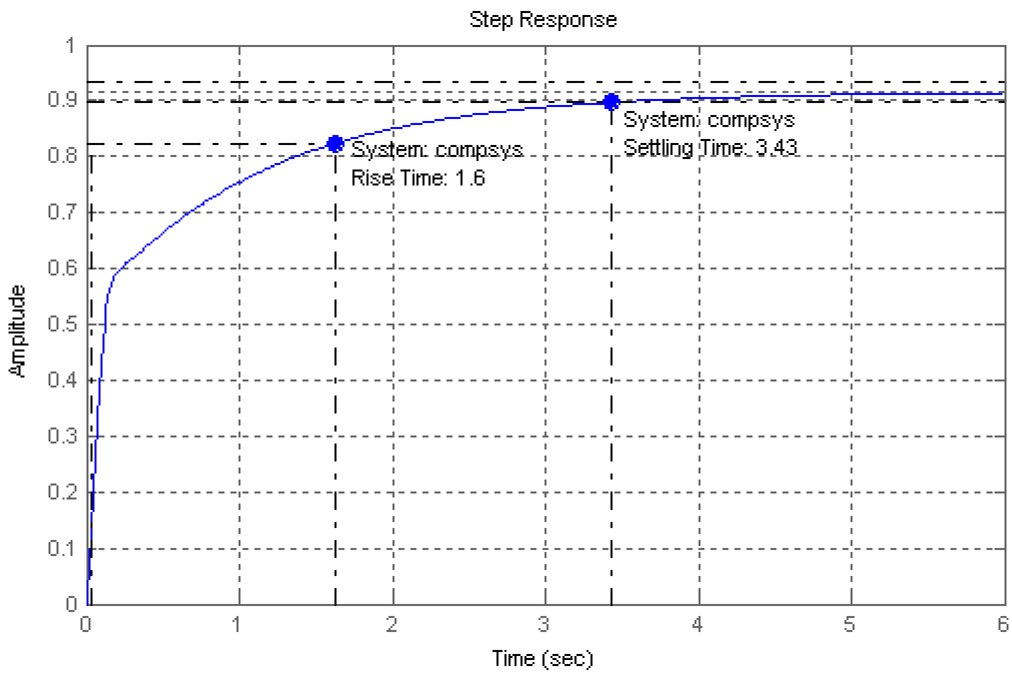


Figure 17 The compensated step response for Example 5 (b).

Note that that the complex poles are located approximately in the same location as in part (a). The steady-state error is greatly reduced, but because of the addition of the root at –0.894, the step response rise time and settling time are increased. If a faster response is desired, select the controller zero further to the left away from the origin. This would move the complex pole s_1 to the right further away from the specified value.

7. Phase-lead Compensator Analytical Design

The DC gain of the compensator $G_c(s) = \dfrac{K_c(s+z_0)}{(s+p_0)}$ is

$$a_0 = G_c(0) = \frac{K_c z_0}{p_0} \tag{8}$$

In the analytical design the controller dc gain a_0 is specified, usually in accordance to the steady-state error specification. Then, for a given location of the closed-loop pole

$$s_1 = |s_1| \angle \beta ,$$

z_0, and p_0 are obtained such that the equation

$$1 + G_c(s_1)GH(s_1) = 0$$

is satisfied. It can be shown that the above parameters are found from the following equations

$$z_0 = \frac{a_0}{a_1}, \quad p_0 = \frac{1}{b_1}, \quad \text{and} \quad K_c = \frac{a_0 p_0}{z_0} \tag{9}$$

where

$$\begin{aligned} a_1 &= \frac{\sin\beta + a_0 M \sin(\beta - \psi)}{|s_1| M \sin\psi} \\ b_1 &= -\frac{\sin(\beta + \psi) + a_0 M \sin\beta}{|s_1| \sin\psi} \end{aligned} \tag{10}$$

where M and ψ are the magnitude and phase angle of the open-loop plant transfer function evaluated at s_1, i.e.,

$$GH(s_1) = M \angle \psi \tag{11}$$

For the case that ψ is either 0° or 180°, (10) is given by

$$a_1 |s_1| \cos\beta \pm \frac{b_1 |s_1|}{M} \cos\beta \pm \frac{1}{M} + a_0 = 0 \tag{12}$$

where the plus sign applies for $\psi = 0^\circ$ and the minus sign applies for $\psi = 180^\circ$. For this case the zero of the compensator must also be assigned.

5.2.8. PID Compensator Analytical Design

For a desired location of the closed-loop pole s_1, as given by (3), the following equations are obtained to satisfy

$$\begin{aligned} K_P &= \frac{-\sin(\beta+\psi)}{M\sin\beta} - \frac{2K_I\cos\beta}{|s_1|} \\ K_D &= \frac{\sin\psi}{|s_1|\,M\sin\beta} + \frac{K_I}{|s_1|^2} \end{aligned} \tag{13}$$

For PD or PI controllers, the appropriate gain is set to zero. The above equations can be used only for the complex pole s_1. For the case that s_1 is real, the zero of the PD controller $(z_0 = K_P / KD)$ and the zero of the PI controller $(z_0 = K_I / K_P)$ are specified and the corresponding gains to satisfy angle and magnitude criteria are obtained accordingly. For the PID design, the value of K_I to achieve a desired steady state error is specified. Again, (13) is applied only for the complex pole s_1.

5.3 Controller Design Using Frequency Response Criteria

Frequency response concepts and techniques play an important role in control system design and analysis.

5.3.1Closed-Loop Behavior

In general, a feedback control system should satisfy the following design objectives:

1. Closed-loop stability
2. Good disturbance rejection (without excessive control action)
3. Fast set-point tracking (without excessive control action)
4. A satisfactory degree of robustness to process variations and model uncertainty
5. Low sensitivity to measurement noise

The Bode-Diagram is one of the most commonly used methods for the analysis and synthesis of linear feedback control systems.

5.3.2 Bode Stability Criterion

The Bode stability criterion has two important advantages in comparison with the Routh stability criterion of Chapter3:

1. .It provides exact results for processes with time delays, while the Routh stability criterion provides only approximate results due to the polynomial approximation that must be substituted for the time delay.
2. The Bode stability criterion provides a measure of the relative stability rather than merely a yes or no answer to the question.

Before stating the Bode stability criterion, we need to introduce two important definitions:

1. Again crossover frequency ω_c is defined to be a value of ω for which $(\varphi_\omega)_{OL}$=-180 .This frequency is also referred to as a phase crossover frequency.

2. A gain crossover frequency ω_{cg} is defined to be a value of ω for which $|G(j\omega)H(j\omega)|=M(j\omega)=-1$

For many control problems, there is only a single ωcp and a single ωg . But multiple values can occur, as shown in Fig. for .ωcp

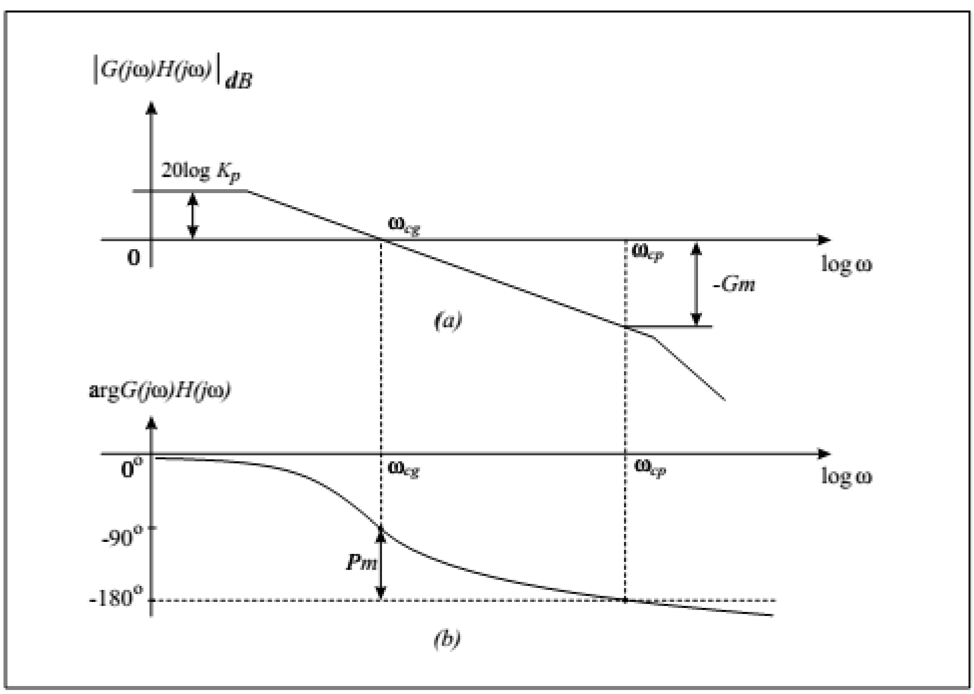


Figure 18

Gain Crossover point: This is the point at which the $\left|G(j\omega)\mathrm{H}(\mathrm{j}\omega)\right|_{dB} = 0$. The frequency at this point is called the gain crossover frequency, wc_g.

Phase crossover point: This is the point at which the phase angle of *G(jw),*

$\angle G(j\omega)\mathrm{H}(\mathrm{j}\omega)$ = -180^0. The frequency at this point is called the phase crossover frequency,ωc_p

Gain and Phase Margins:

Gain Margin: GM is one of the most frequently used criteria for measuring the relative stability of control systems. GM is defined as the amount of gain in dB that can be added to the system's loop before the closed loop system becomes unstable.

Phase Margin: PM is defined as the amount of angle in degrees that can be added to the loop before the closed loop system becomes unstable.

$$\text{GM} = 20\log_{10}\frac{1}{\left|L(jw_p)\right|}$$

and PM = $\angle L(jw_g) - 180^o$

Example 9.1: Consider the open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{(j\omega+1)}{j\omega(j\omega+2)\left[(j\omega)^2+2(j\omega)+2\right]}$$

The phase and gain stability margins and the crossover frequencies are

$$Gm = 8.9443\,\text{dB},\ Pm = 82.2462^\circ,\ \omega_{cp} = 1.7989\,\text{rad/s},\ \omega_{cg} = 0.2558\,\text{rad/s}$$

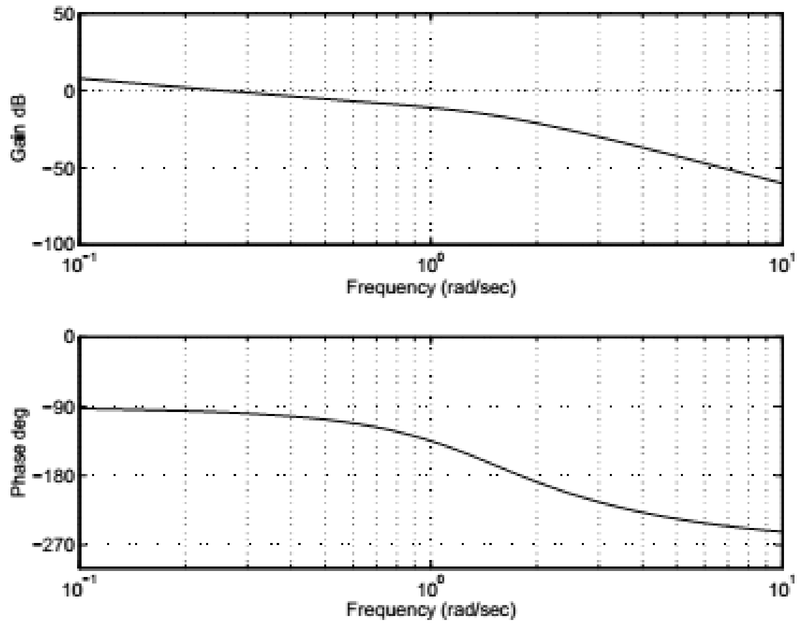


Figure 19: Bode diagrams

Stability via Bode plots – gain and phase margin

For a stable system (no closed-loop poles in the right half plane), the open-loop system will have these properties:

1. When $\varphi = -180°$, M should be < 1
2. When M = 1, φ should be $>-180°$

These facts are needed to understand gain and phase margin. The gain margin and phase margin are useful for determining relative stability, i.e. not just whether or not a system is stable but also how stable the system is.

5.2.3 Steady State Errors and Bode Diagrams

Steady state errors can be indirectly determined from Bode diagrams by reading the values for K_p, K_v,&K_a constants. The steady state errors and corresponding constants K_p, K_v,&K_a are first of all determined by the system type, which represents the multiplicity of the pole at the origin of the open-loop feedback transfer function, in general, represented by

$$G(j\omega)H(j\omega) = \frac{K(j\omega + z_1)(j\omega + z_2)\ \cdots}{(j\omega)^r(j\omega + p_1)(j\omega + p_2)\ \cdots} \qquad 15$$

This can be rewritten as

$$G(j\omega)H(j\omega) = \frac{Kz_1z_2\cdots\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots}{p_1p_2\cdots(j\omega)^r\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\ \cdots}$$

$$= \frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\ \cdots}{(j\omega)^r\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\ \cdots} \qquad (16)$$

Where

$$K_B = \frac{Kz_1z_2\ \cdots}{p_1p_2\ \cdots} \qquad (17)$$

is known as Bode's gain, and is the type of feedback control system.

For control systems of type r=0 , the position constant according to formula obtained from (17) as.

$$K_p = \frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\ \cdots}{(j\omega)^0\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\ \cdots}\Big|_{j\omega=0} = K_B \qquad (18)$$

It follows from (17)–(19) that the corresponding magnitude Bode diagram of type zero control systems for small values ofω is flat (has a slope of0dB) and the value of $20\log K_B = 20\log K_P$.

This is graphically represented in Figure 20

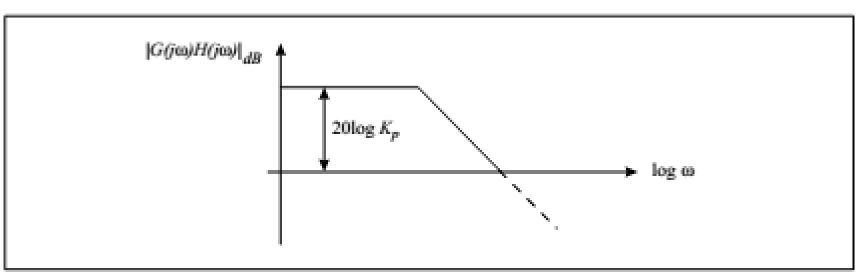


Figure 20: Magnitude Bode diagram of type zero control systems at small frequencies

For control systems of type r=1, the open-loop frequency transfer function is approximated at low frequencies by

$$\frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots}{(j\omega)^1\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\cdots} \approx \frac{K_B}{(j\omega)^1} \quad (19)$$

It follows that the corresponding magnitude Bode diagram of type one control

systems for small values ωhas a slope of -20dB/decades and the values of

$$20\log\left|\frac{K_B}{jw}\right| = 20\log|K_B| - 20\log|\omega| \quad (20)$$

From (20) it is easy to conclude that for type one control systems the velocity constant $K_v = K_B$.

Using this fact and the frequency plot of (21), we conclude that $K_v$is equal to the frequency ω^* at which the line (21) intersects the frequency axis, that is

$$0 = 20\log|K_B| - 20\log|\omega^*| \;\; \Rightarrow \;\; K_B = \omega^* = K_v \quad (21)$$

This is graphically represented as:

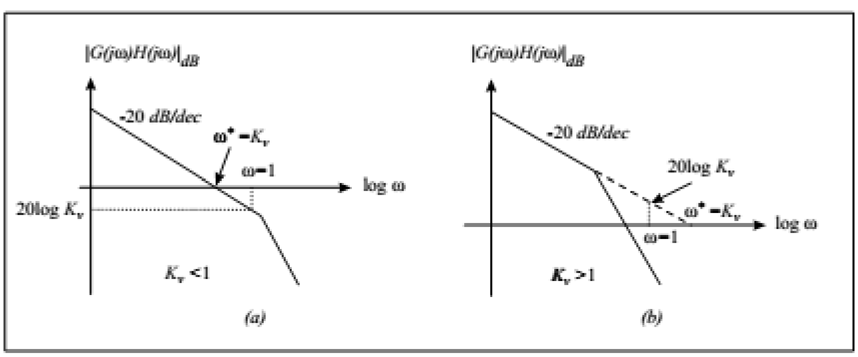


Figure 21:Magnitude Bode diagram of type one control systems at small frequencies

Note that if $K_v=\omega^*>1$, the corresponding frequency ω^* is obtained at the point where the extended initial curve, which has a slope of -20dB/decade , intersects the frequency axis (see Figure 21b).

Similarly, for type two control systems r=2, we have at low frequencies

$$\frac{K_B\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\ \cdots}{(j\omega)^2\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\ \cdots} \approx \frac{K_B}{(j\omega)^2} \qquad (22)$$

which indicates an initial slope of -40dB/decade and a frequency approximation of

$$20\log\left|\frac{K_B}{(jw)^2}\right| = 20\log|K_B| - 20\log\left|\omega^2\right| = 20\log|K_B| - 40\log\omega \qquad (23)$$

From (23) it is easy to conclude that for type two control systems the acceleration constant is $Ka=K_B$. From the frequency plot of the straight line (24), it follows that $K_B=(\omega^{**})^2$.Where ω^{**}

represents the intersection of the initial magnitude Bode plot with the frequency axis as represented in Figure 22

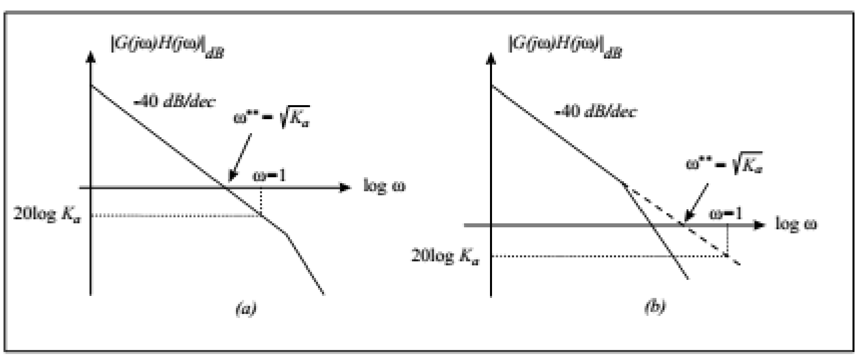


Figure 22: Magnitude Bode diagram of type two control systems at small frequencies

It can be seen from Figures 20–22 that by increasing the values for the magnitude Bode diagrams at low frequencies (i.e. by increasing K_B) the constants *Kp,Kv and Ka* are increased. According to the formulas for steady state errors, given in time domain analysis as:

$$e_{ss_{step}} = \frac{1}{1+K_p}, \quad e_{ss_{ramp}} = \frac{1}{K_v}, \quad e_{ss_{parabolic}} = \frac{2}{K_a}$$

we conclude that in this case the steady state errors are decreased. Thus, the bigger K_B, the smaller the steady state errors.

5.3.4 Compensator Design Using Bode Diagrams

Controller design techniques in the frequency domain will be governed by the

following facts:

a. Steady state errors are improved by increasing Bode's gain ➧.
b. System stability is improved by increasing phase and gain margins.
c. Overshoot is reduced by increasing the phase stability margin.
d. Rise time is reduced by increasing the system's bandwidth.

The first two items, (a) and (b), have been already clarified. In order to justify item (c), we consider the open-loop transfer function of a second-order system

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{(j\omega)(jw + 2\zeta\omega_n)} \qquad (24)$$

whose gain crossover frequency can be easily found from

$$|G(j\omega_{cg})H(j\omega_{cg})| = \frac{\omega_n^2}{\omega\sqrt{\omega^2 + 4\zeta^2\omega_n^2}} = 1 \qquad (25)$$

leading to

$$\omega_{cg} = \omega_n\sqrt{\sqrt{1+4\zeta^2} - 2\zeta^2} \qquad (26)$$

The phase of eq.(25) at the gain crossover frequency is

$$\arg\{G(j\omega_{cg})H(j\omega_{cg})\} = -90^\circ - \tan^{-1}\frac{\omega_{cg}}{2\zeta\omega_n} \tag{27}$$

so that the corresponding phase margin becomes

$$Pm = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^2}-2\zeta^2}} = Pm(\zeta) \tag{28}$$

Plotting the function$m(\xi)$, it can be shown that it is a monotonically increasing function with respect toξ ; we therefore conclude that the higher phase margin, the larger the damping ratio, which implies the smaller the overshoot.

Item (d) cannot be analytically justified since we do not have an analytical expression for the response rise time. However, it is very well known from undergraduate courses on linear systems and signals that rapidly changing signals have a wide bandwidth. Thus, *systems that are able to accommodate fast signals must have a wide bandwidth.*

1 Phase-Lag Controller

The transfer function of a phase-lag controller is given by

$$G_{lag}(j\omega) = \left(\frac{p_1}{z_1}\right)\frac{z_1 + j\omega}{p_1 + j\omega} = \frac{1 + j\frac{\omega}{z_1}}{1 + j\frac{\omega}{p_1}}, \quad z_1 > p_1 \tag{29}$$

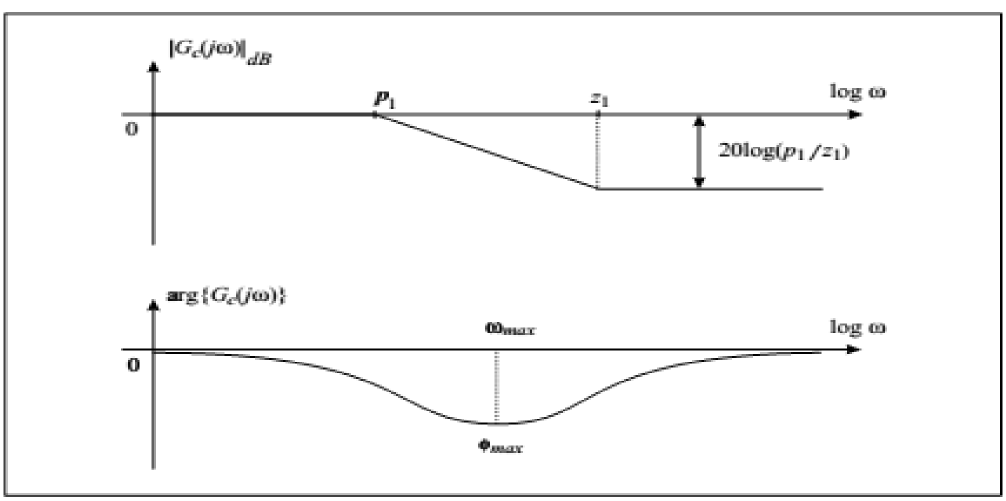


Figure 23: Magnitude approximation and exact phase of a phase-lag controller

Due to attenuation of the phase-lag controller at high frequencies, the frequency bandwidth of the compensated system (controller and system in series) is reduced. Thus, the phase-lag controllers are used in order to decrease the system bandwidth (to slow down the system response). In addition, they can be used to improve the stability margins (phase and gain) while keeping the steady state errors constant.

Expressions for ω_{max} and φ_{max} of a phase-lag controller will be derived in the next subsection in the context of the study of a phase-lead controller.

2 Phase-Lead Controller

The transfer function of a phase-lead controller is

$$G_{lead}(j\omega) = \left(\frac{p_2}{z_2}\right)\frac{z_2 + j\omega}{p_2 + j\omega} = \frac{1 + j\frac{\omega}{z_2}}{1 + j\frac{\omega}{p_2}}, \qquad p_2 > z_2 \tag{30}$$

Due to phase-lead controller (compensator) amplification at higher frequencies, it increases the bandwidth of the compensated system. *The phase-lead controllers are used to improve the gain and phase stability margins and to increase the system bandwidth (decrease the system response rise time).*

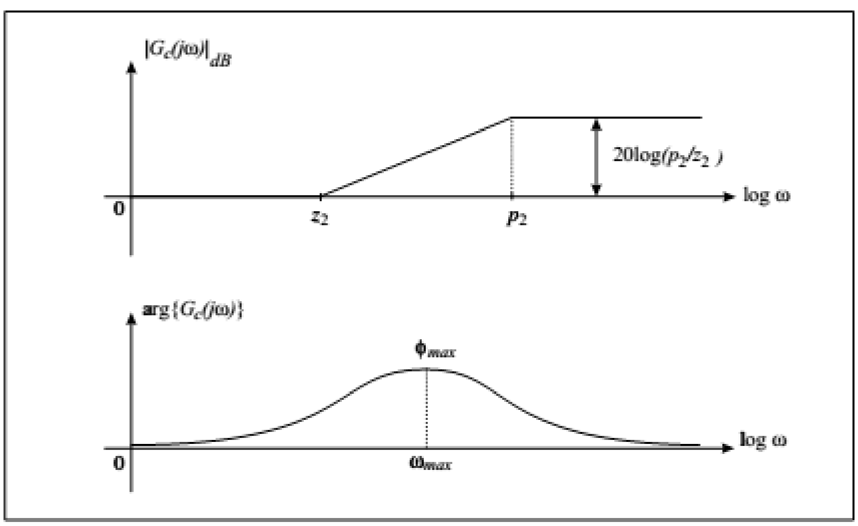


Figure 24: Magnitude approximation and exact phase of a phase-lead controller.

It follows from (31) that the phase of a phase-lead controller is given by

$$\arg\{G_{lead}(j\omega)\} = \tan^{-1}\left(\frac{\omega}{z_2}\right) - \tan^{-1}\left(\frac{\omega}{p_2}\right) \qquad (31)$$

so that

$$\frac{d}{d\omega}\arg\{G_{lead}(j\omega)\} = 0 \;\Rightarrow\; \omega_{max} = \sqrt{z_2 p_2} \qquad (32)$$

Assume that

$$p_2 = az_2, \quad a > 1 \quad \Rightarrow \quad \omega_{max} = \frac{p_2}{\sqrt{a}} \qquad (33)$$

Substitutingω_{max}in (9.32) implies

$$\tan\phi_{max} = \frac{a-1}{2\sqrt{a}} \qquad (34)$$

The parameter *a* can be found from (35) in terms ofϕ_{max}

$$a = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}} \qquad (35)$$

Note that the same formulas forω_{max} (33), and the parameter, (36), hold for a phase-lag controller with$p_1 z_1$ replacing p2$_{=}z_2$ and with$p_1 = az_1$, $a<1$.

3 Phase-Lag-Lead Controller

The phase-lag-lead controller has the features of both phase-lag and phase-lead controllers and can be used to improve both the transient response and steady state errors. The frequency transfer function of the phase-lag-lead controller is given by

$$G_c(j\omega) = \frac{(j\omega+z_1)(j\omega+z_2)}{(j\omega+p_1)(j\omega+p_2)} = \frac{z_1 z_2}{p_1 p_2}\frac{\left(1+j\frac{\omega}{z_1}\right)\left(1+j\frac{\omega}{z_2}\right)}{\left(1+j\frac{\omega}{p_1}\right)\left(1+j\frac{\omega}{p_2}\right)} \tag{36}$$

$$= \frac{\left(1+j\frac{\omega}{z_1}\right)\left(1+j\frac{\omega}{z_2}\right)}{\left(1+j\frac{\omega}{p_1}\right)\left(1+j\frac{\omega}{p_2}\right)}, \qquad z_1 z_2 = p_1 p_2, \qquad p_2 > z_2 > z_1 > p_1$$

The Bode diagrams of this controller are shown

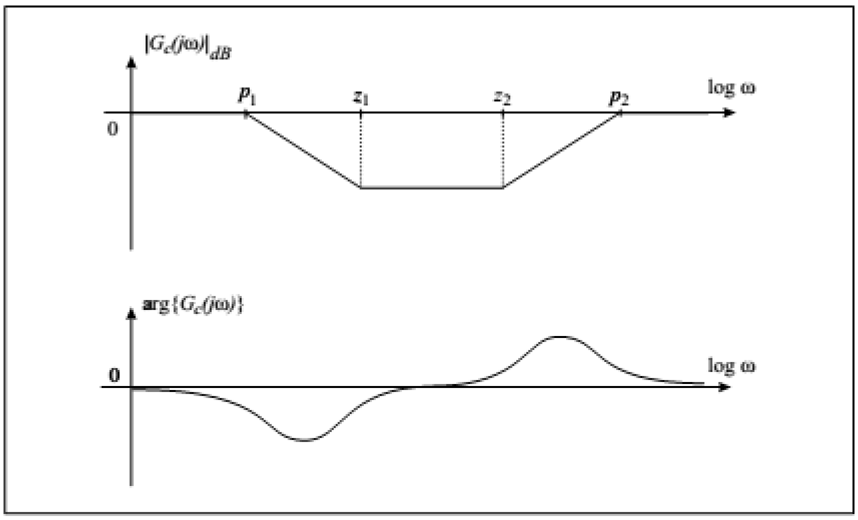


Figure25: Bode diagrams of a phase-lag-lead controller

Compensator Design with Phase-Lead Controller

The following algorithm can be used to design a controller (compensator) with a phase-lead network.

Algorithm 1:

1. Determine the value of the Bode gain K_B given by (18) as

$$K_B = \frac{K z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

such that the steady state error requirement is satisfied.

2. Find the phase and gain margins of the original system with K_B determined

in step 1.

3. Find the phase difference $\Delta\phi$, between the actual and desired phase margins and take ϕ_{max} to be 5^0-10^0 greater than this difference. Only in rare cases should this be greater than 10^0. This is due to the fact that we have to give an estimate of a new gain crossover frequency, which cannot be determined very accurately (see step 5).
4. Calculate the value for parameter from formula (36), i.e. by using

$$a = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}} > 1$$

5. Estimate a value for a compensator's pole such that ω_{max} is roughly located at the new gain crossover frequency $\omega_{max} = \omega_{cgnew}$. As a rule of thumb, add the gain of $\Delta G = 20\log(a)\ dB$. at high frequencies to the uncompensated system and estimate the intersection of the magnitude diagram with the frequency axis, say ω_1. The new gain crossover frequency is somewhere in between the old ω_{cg} and ω_1. Using the value for parameter a obtained in step 4 find the value for the compensator pole from (34) as
$-p_c = -\omega_{max}\sqrt{a}$ and the value for compensator's zero as $-z_c = -p_c/a$.
Note that one can also guess a value for p_c and then evaluate z_c and ω_{max}. The phase-lead compensator now can be represented by

$$G_c(s) = \frac{as+p_c}{s+p_c}$$

6. Draw the Bode diagram of the given system with controller and check the values for the gain and phase margins. If they are satisfactory, the controller design is done, otherwise repeat steps 1–5.

Example : Consider the following open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{K(j\omega+6)}{(j\omega+1)(j\omega+2)(j\omega+3)}$$

Step 1. Let the design requirements be set such that the steady state error due to a unit step is less than 2% and the phase margin is at least 45^0. Since

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K_B}, \quad K_B = \frac{K \times 6}{1 \times 2 \times 3} = K$$

we conclude that K.>=50 will satisfy the steady state error requirement of being less than 2%. . We know from the root locus technique that high static gains can damage system stability, and so for the rest of this design problem we take K=50 .

Step 2. We draw Bode diagrams of the uncompensated system with the Bode gain obtained in step 1 and determine the phase and gain margins and the crossover frequencies

The corresponding Bode diagrams are presented in Figure 18a. The phase and gain margins are $G_m = \infty$, $P_m=5.59^0$obtained as and the crossover frequencies are ω_{cg}=7.5423. Add 10^0 for the reason in Step 3. Since the desired phase is well above the actual one, the phase-lead controller

Step 3.ϕ_{max}=49.41.

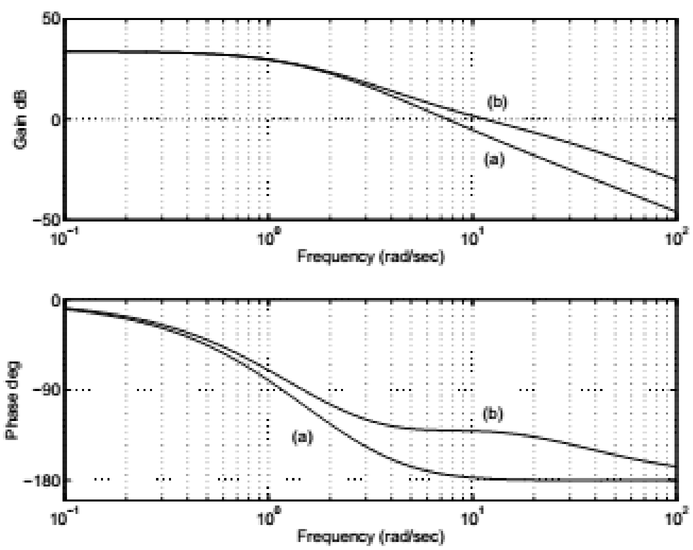


Figure26:Bode diagrams for the original system(a) and compensated system (b) of Example

Step 4. Here we evaluate the parameter *a* according to the formula (36) and get *a=7.3144.*

Step 5. In order to obtain an estimate for the new gain crossover frequency we first find the controller amplification at high frequencies, which is equal to$20\log(a) = 17.2836\,\mathrm{dB} = \Delta G_{dB}$.The magnitude Bode diagram increased byΔG_{dB}at high frequencies intersects the frequency axis atω_{max}=10.5 rad/sec.

We guess (estimate) the value for *pc, pc=25*which is roughly equal to$\omega_{max}\sqrt{a}$. .By using pc=25 and forming the corresponding compensator, we get $Pmc=48.2891^o$ *at* $\omega_{gcnew}=13.8519$.Which is satisfactory.

The phase-lead compensator obtained is given by

$$G_c(s) = \frac{7.3144s + 25}{s + 25} = \frac{as + p_2}{s + p_2}$$

Step 6. The Bode diagrams of the compensated control system are presented in Figure 26b. Both requirements are satisfied, and therefore the controller design procedure is successfully completed.

5 Compensator Design with Phase-Lag Controller

Compensator design using phase-lag controllers is based on the compensator's attenuation at high frequencies, which causes a shift of the gain crossover frequency to the lower frequency region where the phase margin is high. The phase-lag compensator can be designed by the following algorithm

Algorithm 2

1. Determine the value of the Bode gain K_B that satisfies the steady state error requirement.
2. Find on the phase Bode plot the frequency which has the phase margin equal to the desired phase margin increased by5^0-10^0.This frequency represents the new gain crossover frequencyω_{gcnew}.
3. Read the required attenuation at the new gain crossover frequency, i.e. $|\Delta G(j\omega_{cgnew})|_{dB}$, and find the parameter *a* from

   $$-20\log\left(\frac{p_1}{z_1}\right) = -20\log(a) = |\Delta G(j\omega_{cgnew})|_{dB}$$

   Which implies

   $$a = 10^{-\frac{1}{20}|\Delta G(j\omega_{cgnew})|_{dB}} = \frac{1}{|\Delta G(j\omega_{cgnew})|}$$

Note that

$$|\Delta G(j\omega_{cgnew})| = \frac{|K||j\omega_{cgnew} + z_1||j\omega_{cgnew} + z_2|\cdots}{|j\omega_{cgnew} + p_1||j\omega_{cgnew} + p_2|\cdots}$$

4. Place the controller zero one decade to the left of the new gain crossover frequency, that is

$$z_c = \frac{\omega_{cgnew}}{10}$$

Find the pole location from $p_c = az_c = a\omega_{cgnew}/10$. The required compensator has the form

$$G_c(s) = \frac{as + p_c}{s + p_c}$$

5. Redraw the Bode diagram of the given system with the controller and check the values for the gain and phase margins. If they are satisfactory, the controller design is done, otherwise repeat steps 1–5.

Example 9.5: Consider a control system represented by

$$G(s) = \frac{K}{s(s+2)(s+30)}$$

Design a phase-lag compensator such that the following specifications are met: $e_{ss_{ramp}} \le 0.05$, $Pm \ge 45°$. The minimum value for the static gain that produces the required steady state error is equal to $K = 1200$. The original system with this static gain has phase and gain margins given by $Pm = 6.6449°$, $Gm = 4.0824\,\mathrm{dB}$ and crossover frequencies of $\omega_{cg} = 6.1031\,\mathrm{rad/s}$, $\omega_{cp} = 7.746\,\mathrm{rad/s}$.

The new gain crossover frequency can be estimated as $\omega_{cgnew} = 1.4\,\mathrm{rad/s}$ since for that frequency the phase margin of the original system is approximately $50°$. At

$$\omega_{cgnew} = 1.4\,\mathrm{rad/s}\,.$$

which produces $|\Delta G(j1.4)| = 11.6906$ and $a = 1/|\Delta G(j1.4)| = 0.0855$. The compensator's pole and zero are obtained as $-z_c = -\omega_{cgnew}/10 = -0.14$ and $-p_c = -a\omega_{cgnew}/10 = -0.0120$ (see step 4 of Algorithm 9.2). The transfer function of the phase-lag compensator is

$$G_c(s) = \frac{0.0855s + 0.0120}{s + 0.0120}$$

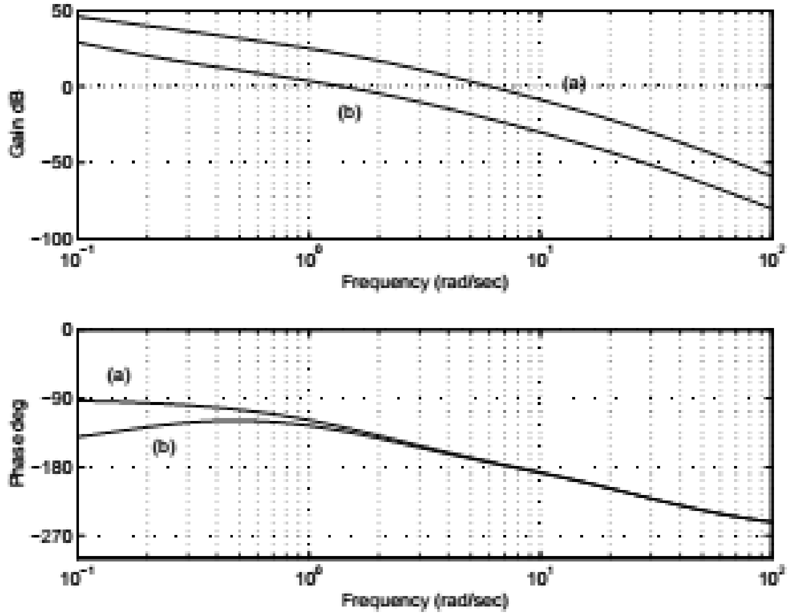


Figure 27: Bode diagrams for the original system (a) and compensated system (b) of the above Example

The new phase and gain margins and the actual crossover frequencies are $Pmc = 47.03°$, $Gmc = 24.82\,\text{dB}$, $\omega_{cgnew} = 1.405\,\text{rad/s}$, $\omega_{cpnew} = 7.477\,\text{rad/s}$ and so the design requirements are satisfied. The step responses of the original and compensated systems are presented in Figure 28.

It can be seen from this figure that the overshoot is reduced from roughly 0.83 to 0.3. In addition, it can be observed that the settling time is also reduced. Note that the phase-lag controller reduces the system bandwidth ($\omega_{cgnew} < \omega_{cg}$) so that the rise time of the compensated system is increased.

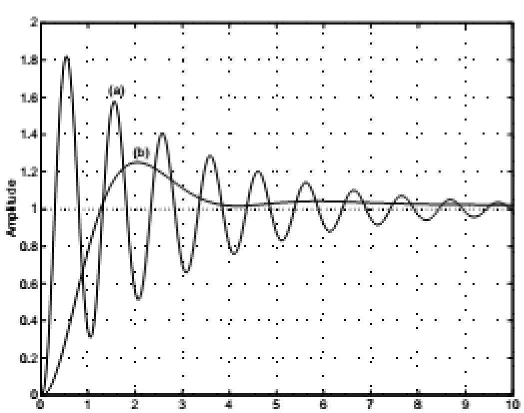


Figure 28:Step responses for the original system(a) and compensated system (b)

6 Compensator Design with Phase-Lag-Lead Controller

Compensator design using a phase-lag-lead controller can be performed according to the algorithm given below, in which we first form a phase-lead compensator and then a phase-lag compensator. Finally, we connect them together in series.

Algorithm 3:

1. Set a value for the static gain K_B such that the steady state error requirement is satisfied.
2. Draw Bode diagrams with K_B obtained in step 1 and find the corresponding phase and gain margins.
3. Find the difference between the actual and desired phase margins, $\Delta\phi = P_{md} - P_m$ and ϕ_{max} to be a little greater than $\Delta\phi$. Calculate the parameter a_2 of a phase-lead controller by using formula (36), that is

   $$a_2 = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}}$$

4. Locate the new gain crossover frequency at the point where

   $$20\log|G(j\omega_{cgnew})| = -10\log a_2 \tag{37}$$

5. Compute the values for the phase-lead compensator's pole and zero from

   $$p_{c2} = \omega_{cgnew}\sqrt{a_2}, \qquad z_{c2} = p_{c2}/a_2 \tag{38}$$

6. Select the phase-lag compensator's zero and pole according to

   $$z_{c1} = 0.1 z_{c2}, \qquad p_{c1} = z_{c1}/a_2 \tag{39}$$

7. Form the transfer function of the phase-lag-lead compensator as

   $$G_c(s) = G_{lag}(s) \times G_{lead}(s) = \frac{s+z_{c1}}{s+p_{c1}} \times \frac{s+z_{c2}}{s+p_{c2}} \tag{40}$$

8. Plot Bode diagrams of the compensated system and check whether the design specifications are met. If not, repeat some of the steps of the proposed algorithm in most cases go back to steps 3 or 4.

The phase-lead part of this compensator helps to increase the phase margin (increases the damping ratio, which reduces the maximum percent overshoot and settling time) and broaden the system's bandwidth (reduces the rise time). The phase-lag part, on the other hand, helps to improve the steady state errors

Example : Consider a control system that has the open-loop transfer function

$$G(s) = \frac{K(s+10)}{(s^2+2s+2)(s+20)}$$

For this system we design a phase-lag-lead controller by following Algorithm 3 such that the compensated system has a steady state error of less than 4% and a phase margin greater than $50°$. In the first step, we choose a value for the static gain K that produces the desired steady state error. It is easy to check that $K = 100 \Rightarrow e_{ss} = 3.85\%$, and therefore in the following we stick with this value for the static gain. Bode diagrams of the original system with $K = 100$ are presented in Figure 29 .

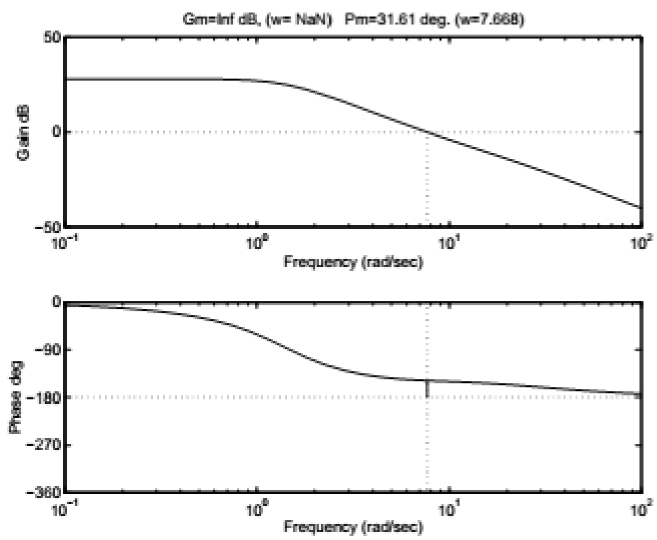


Figure 29: Bode diagrams of the original system

It can be seen from these diagrams that the phase and gain margins and the corresponding crossover frequencies are given by $Pm = 31.61°, Gm = \infty$, and $\omega_{cg} = 7.668\,\text{rad/s}$, $\omega_{cp} = \infty$.

According to step 3 of Algorithm 3, a controller has to introduce a phase lead of 18.39^o .we take

ϕ_{max}=25^{o}.and find the required parametera_2=2.4639.Takingω_{cgnew}=20rad/sec in step 4 and completing the design steps 5–8 we find that P_m=39.94^{o}which is not satisfactory. We go back to step 3 and takeϕ_{max}=30^{o}=0.5236 rad which implies a_2=3.

Step 4: since -10log 3=-10.9861 dB we search the magnitude diagram for frequency where the attenuation is approximately equal to -11 dB. We start search at ω=20 rad/sec since at that point, according to Figure 29, the attenuation is smaller than -11dB.

Then finding ω_{cgnew}=10 rad/sec.

In steps 5 and 6 the phase-lag-lead controller zeros and poles are obtained as:

$-p_{c2}=-17.3205$, $-zc2=-5.7735$the phase lead part and –pc1=-0.1925,-zc1=-0.5774 for the phase-lag-part ; hence the phase –lag-lead controller has the form

$$G_c(s) = \frac{s+0.5774}{s+0.1925} \times \frac{s+17.3205}{s+5.7735}$$

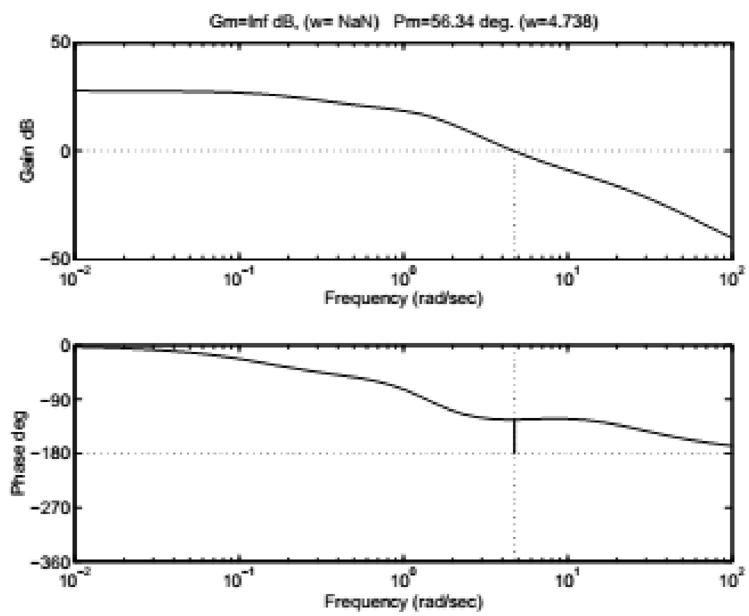


Figure 30:Bode diagrams of the compensated system

It can be seen that the phase margin obtained of 56.34^{o} meets the design requirement and that the actual gain crossover frequency 4.738 rad/sec. is considerably smaller than the one predicted. This contributes to the generally accepted inaccuracy of frequency methods for controller design based on Bode diagrams.

The step responses of the original and compensated systems are compared in

Figure 30 . The transient response of the compensated system is improved since the maximum percent overshoot is considerably reduced. However, the system rise time is increased due to the fact that the system bandwidth is shortened ($\omega_{cgnew} = 4.738 rad, \sec < \omega_{cg} = 7.668 rad/sec$).

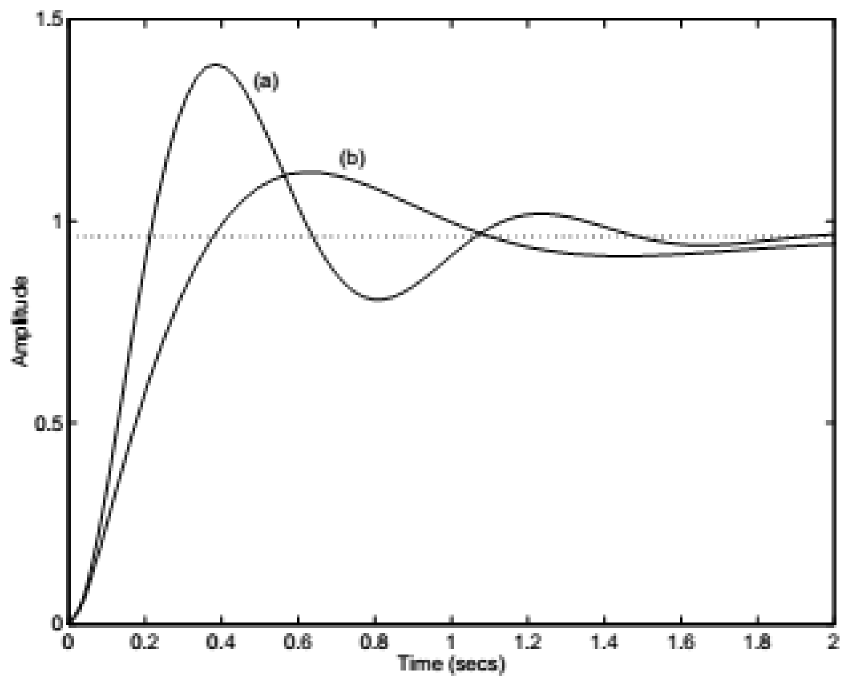


Figure 30:Step responses of the original (a) and compensated (b) systems.