CHAPTER-4

Fourier-Analysis

4.1 Fourier-Series

4.1.1 Periodic Functions

A function f is said to be periodic if there exists anon zero constant p such that $f(x+p)=f(x)$ for all x in the domain of f.

The smallest positive constant p such that $f(x+p)=f(x)$ for all x in the domain of f is said to be primitive or fundamental period.

Examples: Verify that the functions are periodic with period as indicated.

a) $f(x)=\sin x, p=2\pi$ *b)* $f(x)=\tan x, p=\pi$ c) $f(x)=\sin(4x), p=\dfrac{\pi}{2}$

d) $f(x)=\cos x, p=2\pi$ e) $f(x)=\sec x, p=2\pi$ *f)* $f(x)=\csc x, p=2\pi$

Solution:

a) Here, for f to be periodic, $f(x+p)=f(x)$ for x in the domain of f.

But $f(x+2\pi)=\sin(x+2\pi)=\sin x\cos 2\pi+\cos x\sin 2\pi=\sin x=f(x)$.

Therefore, $f(x)=\sin x$ is periodic with period $p=2\pi$.

b) Here, $f(x+\pi)=\tan(x+\pi)=\dfrac{\tan x-\tan\pi}{1-\tan x\tan\pi}=\dfrac{\tan x-0}{1-0}=\tan x=f(x)$.

Therefore, $f(x)=\tan x$ is periodic with period $p=\pi$.

c) Here $f(x+\dfrac{\pi}{2})=\sin(4(x+\dfrac{\pi}{2}))=\sin(4x+2\pi)$

$$=\sin 4x\cos 2\pi+\cos 4x\sin 2\pi=\sin 4x=f(x)$$

Therefore, $f(x)=\sin(4x)$ is periodic with period $p=\dfrac{\pi}{2}$.

In general, we can summarize the periodic trigonometric functions and their fundamental periods using table as follow.

Functions	Periods (p)	Fundamental period
$f(x)=\sin x,\ g(x)=\cos x$	$p=2\pi,4\pi,6\pi,8\pi...$	$p=2\pi$
$f(x)=\sin ax,\ g(x)=\cos ax,$ $f(x)=\sec ax,\ g(x)=\csc ax,\ a\neq 0$	$p=\left(\frac{2\pi}{\lvert a\rvert}\right)k,\ k\in N$	$p=\frac{2\pi}{\lvert a\rvert}$
$f(x)=\tan x, g(x)=\cot x$	$p=k\pi,\ k\in N$	$p=\pi$
$f(x)=\tan ax, g(x)=\cot ax,\ a\neq 0$	$p=(\frac{\pi}{\lvert a\rvert})k,\ k\in N$	$p=\frac{\pi}{\lvert a\rvert}$

Periods for Sum and Difference of Functions

The sum and difference of a number of periodic functions with *commensurable period* is a periodic function with period the least common multiple of their periods. That is if the period of f is p_1, the period of g is p_2 and the period of h is p_3, such that the periods are commensurable, then the fundamental period of their combination $F=af+bg+ch$ is $p=\text{LCM}(p_1,p_2,p_3)$.

(Here, functions with *commensurable period* means functions that have common periods).

Examples: Identify the fundamental period of the following functions.

a) $f(x)=3\sin\frac{x}{2}+6\cos\frac{x}{3}$ *b)* $f(x)=3\cos(\frac{x}{3})-4\sin(\frac{x}{2})+9\tan(\frac{x}{5})$

c) $f(x)=\tan(\frac{\pi}{3}x)+\sin(\frac{\pi}{2}x)$ *d)* $f(x)=8\tan(\frac{x}{3})-\cos(\frac{3x}{2})$

e) $f(x)=\sin(\pi x)+\cos(2x)$ *f)* $f(x)=4\sin(\frac{2x-7}{3})-5\csc(\frac{x}{5})+\cot\frac{x}{4}$

Solution: First observe that coefficients have no impact on periods.

For instance in $f(x)=3\sin\frac{x}{2}+6\cos\frac{x}{3}$, the coefficients 3 and 6 have no impact.

Now, let's find the fundamental period of the function.

a) From the table, the primitive period of $\sin ax$ is $p=\frac{2\pi}{|a|}$. Thus, the primitive period of $\sin\frac{x}{2}$ is. $p_1=\frac{2\pi}{1/2}=4\pi$. Again, since the primitive period of $\cos ax$ is $p=\frac{2\pi}{|a|}$, the primitive period of $\cos\frac{x}{3}$ is $p_2=6\pi$. Hence, the primitive period of f is the least common integral multiple given by $p=\text{LCM}(4\pi,6\pi)=12\pi$.

b) Since the primitive period of $\cos x$ and $\sin x$ is 2π, the primitive period of $\cos(\frac{x}{3})$ is $p_1=6\pi$ and $\sin(\frac{x}{2})$ is $p_2=4\pi$. Again, since the primitive period of $\tan x$ is π, the primitive period of $\tan(\frac{x}{5})$ is $p_3=\frac{\pi}{1/5}=5\pi$.

Hence, the primitive period of f is the least common integral multiple of $p_1=6\pi, p_2=4\pi$ and $p_3=5\pi$. That is $p=\text{LCM}(6\pi,4\pi,5\pi)=60\pi$.

c) As the primitive period of $\tan x$ is π, the primitive period of $\tan(\frac{\pi}{3}x)$ is $p_1=\frac{\pi}{\pi/3}=3$. Again, since the primitive period of $\sin x$ is 2π, the primitive period of $\sin(\frac{\pi}{2}x)$ is $p_2=\frac{2\pi}{\pi/2}=4$. Hence, the primitive period of f is the least common integral multiple of 3 and 4. That is $p=\text{LCM}(3,4)=12$.

d) The primitive period of $\sin x$ is 2π and that of $\tan\frac{x}{3}$ is as $p_1=\frac{\pi}{1/3}=3\pi$. Again, since the primitive period of $\cos x$ is 2π, the primitive period of $\cos(\frac{3x}{2})$ is $p_2=\frac{2\pi}{3/2}=\frac{4\pi}{3}$. Hence, the primitive period of f is the least common integral multiple given by $p=\text{LCM}\left(2\pi,3\pi,\frac{4\pi}{3}\right)=12\pi$.

e) The function has no primitive period. (Do you see why?)

f) The primitive period is $p=\text{LCM}(p_1,p_2,p_3)=\text{LCM}(3\pi,10\pi,4\pi)=60\pi$.

4.1.2 Fourier Series and Euler's Formula

Definition: The representation or expansion of a function f of period 2π in an open interval $(a,a+2\pi)$ in the form $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ is known as *Fourier Series representation*. Simply, it is called Fourier series. The coefficients a_0,a_n,b_n are called *Fourier coefficients* and they are given by

$$\begin{cases} a_0=\frac{1}{\pi}\int_a^{a+2\pi}f(x)dx \\ a_n=\frac{1}{\pi}\int_a^{a+2\pi}f(x)\cos nx dx \qquad \text{(Euler's formula)} \\ b_n=\frac{1}{\pi}\int_a^{a+2\pi}f(x)\sin nx dx \end{cases}$$

The most commonly used interval is determine by letting $a=-\pi$.
In this case, the Euler's formula becomes as follow:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx$$

Remark: Commonly encountered important integrals. (Please bear in mind!)

1) $\int x\sin nx dx=\frac{1}{n^2}(\sin nx-nx\cos nx)$

2) $\int x\cos nx dx=\frac{1}{n^2}(nx\sin nx+\cos nx)$

3) $\int x^2\sin nx dx=\frac{-x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}$

4) $\int x^2\cos nx dx=\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}$

5) $\int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$

6) $\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)dx$$

$$= \frac{e^{-x}}{\pi(n^2+1)}(-n\cos nx - \sin nx)\Big|_0^{2\pi} = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}$$

Therefore, $f(x) = \frac{1-e^{-2\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx\right)$

d) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\sin xdx = \frac{1}{\pi}(\sin x - x\cos x)\Big|_0^{2\pi} = -2$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nxdx = \frac{1}{2\pi}\left(\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]dx\right)$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{2}{n^2-1}, \text{for } n \neq 1$$

Here, we calculated above $a_n = \frac{2}{n^2-1}$ for $n \neq 1$. So, we have to compute a_1 separately. But for $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos xdx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos xdx = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}(2\sin x\cos x)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2xdx = \frac{1}{2\pi}\left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{2\pi} = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nxdx = \frac{1}{2\pi}\left(\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]dx\right)$$

$$= \frac{1}{2\pi}\left[\left(\frac{x\sin(n-1)x}{n-1} - \frac{x\sin(n+1)x}{n+1}\right) + \left(\frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= 0, \text{for } n \neq 1$$

Here, $b_n = 0$ for $n \neq 1$. So, we have to compute b_1 separately. But for $n = 1$, Examples:

1. Find the Fourier series expansion on the given intervals.

a) $f(x)=4x, -\pi<x<\pi$

b) $f(x)=3-2x, -\pi<x<\pi$

c) $f(x)=e^{-x}, 0<x<2\pi$

d) $f(x)=x\sin x, 0\le x\le 2\pi$

e) $f(x)=\dfrac{\pi-x}{2}, 0\le x\le 2\pi$

f) $f(x)=x+\dfrac{x^2}{4}, -\pi\le x\le \pi$

Solution:

$\int_{-\pi}^{\pi}=2$

a) $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)dx=\dfrac{1}{\pi}\int_{-\pi}^{\pi}4xdx=\dfrac{2x^2}{\pi}\Big|_{x=-\pi}^{x=\pi}=0$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}4x\cos nxdx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}4x\sin nxdx=\frac{-8\cos n\pi}{n}=\frac{8(-1)^{n+1}}{n}$$

Therefore, $f(x)=8\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx=8\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\dots\right)$

b) $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)dx=\dfrac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)dx=\dfrac{1}{\pi}\left(3x-x^2\right]_{-\pi}^{\pi}=6$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\cos nxdx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\sin nxdx=\frac{4(-1)^n}{n}$$

Therefore, $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=3+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

c) $a_0=\dfrac{1}{\pi}\displaystyle\int_{0}^{2\pi}f(x)dx=\int_{0}^{2\pi}e^{-x}dx=e^{-x}\Big|_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi}$

$$a_n=\frac{1}{\pi}\int_{0}^{2\pi}e^{-x}\cos(nx)dx=\frac{e^{-x}}{\pi(n^2+1)}(n\sin nx-\cos nx)\Big|_0^{2\pi}=\frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$b_1=\frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx=\frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx=\frac{1}{2\pi}(2\pi^2)=\pi$$

Therefore, $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

$$=\frac{a_0}{2}+a_1\cos x+b_1\sin x+\sum_{n=2}^{\infty}a_n\cos nx+\sum_{n=2}^{\infty}b_n\sin nx$$

$$=-1-\frac{1}{2}\cos x+\pi\sin x+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx$$

e) $a_0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx=\dfrac{1}{\pi}\int_0^{2\pi}\left(\dfrac{\pi-x}{2}\right)dx=\dfrac{1}{\pi}\left(\dfrac{\pi x}{2}-\dfrac{x^2}{4}\right)\Bigg|_0^{2\pi}=0$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi\cos nx-x\cos nx)\,dx=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)\sin nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi\sin nx-x\sin x)\,dx$$

$$=\frac{1}{2\pi}\left(\frac{-\pi\cos nx}{n}-\frac{1}{n^2}(\sin nx-nx\cos nx)\right)\Bigg|_0^{2\pi}=\frac{1}{n}$$

Therefore, $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=\dfrac{\pi-x}{2}=\sum_{n=1}^{\infty}\dfrac{1}{n}\sin nx$

f) $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\left(x+\dfrac{x^2}{4}\right)dx=\dfrac{1}{\pi}\left(\dfrac{x^2}{2}+\dfrac{x^3}{12}\right)\Bigg|_{-\pi}^{\pi}=\dfrac{\pi^2}{6}$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+\frac{x^2}{4}\right)\cos nx\,dx=\frac{(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+\frac{x^2}{4}\right)\sin nx\,dx=\frac{-2(-1)^n}{n}$$

Therefore, $f(x)=\dfrac{\pi^2}{12}+\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

2. Find the Fourier series of $f(x)=x^2, -\pi \le x \le \pi$ and deduce the sums

$i) \sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$ $\quad ii) \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$

Solution:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{1}{\pi}\left(\frac{x^3}{3}\right)\Bigg|_{-\pi}^{\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}x^2\cos nx dx\right)=\frac{1}{\pi}\left(\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right)\Bigg|_{-\pi}^{\pi}=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}x^2\sin nx dx\right)=\frac{1}{\pi}\left(\frac{-x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}\right)\Bigg|_{-\pi}^{\pi}=0$$

Therefore, $f(x)=x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$.

i) The series $\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ is obtained when $x=-\pi$.

So, equating $f(-\pi)$ and its Fourier series representation at $x=-\pi$ gives us

$$f(-\pi)=\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi)$$

$$\Rightarrow \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n=\pi^2, \quad \left(\text{Note:}\cos(n\pi)=(-1)^n \ \& \ (-1)^n(-1)^n=1\right)$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{2\pi^2}{3}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

ii) The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is obtained when $x=0$.

$$f(0)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(0)\Rightarrow \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=0\Rightarrow 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{-\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$$

4.1.3 Fourier Series of Discontinuous Functions

Suppose f is discontinuous at $x=c$ where $a<c<a+2\pi$. In this case, f is defined piece wisely as follow: $f(x)=\begin{cases} g(x), a<x<c \\ h(x), c<x<a+2\pi \end{cases}$ where g and h are continuous on the indicated intervals. Then, using properties of definite integral, the Fourier coefficients are obtained as follow:

$$a_0=\frac{1}{\pi}\int_a^{a+2\pi} f(x)dx=\frac{1}{\pi}\left(\int_a^c f(x)dx+\int_c^{a+2\pi} f(x)dx\right)=\frac{1}{\pi}\left(\int_a^c g(x)dx+\int_c^{a+2\pi} h(x)dx\right)$$

$$a_n=\frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos nxdx=\frac{1}{\pi}\left(\int_a^c g(x)\cos nxdx+\int_c^{a+2\pi} h(x)\cos nxdx\right)$$

$$b_n=\frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin nxdx=\frac{1}{\pi}\left(\int_a^c g(x)\sin nxdx+\int_c^{a+2\pi} h(x)\sin nxdx\right)$$

Remark: Dirichlet's Conditions

Since f is discontinuous at the point $x=c$, it is impossible to guess the value of the function at $x=c$ directly from the expansion. In such case, its value at $x=c$ is given by the average of the left and right limits of f at $x=c$.

Remember that as f has a finite jump at $x=c$, the left and right limits of f exists at $x=c$ but they may not be the same. Hence, the value of the function at $x=c$ is given by $f(c)=\dfrac{\lim\limits_{x\to c^-} f(x)+\lim\limits_{x\to c^+} f(x)}{2}=\dfrac{\lim\limits_{x\to c^-} g(x)+\lim\limits_{x\to c^+} h(x)}{2}$.

Examples:

1. Find the Fourier series expansion of the following piecewise functions.

a) $f(x)=\begin{cases}-2, -\pi<x<0\\ 2, 0<x<\pi\end{cases}$ and deduce $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$

b) $f(x)=\begin{cases}-\pi, -\pi<x<0\\ 3x, 0<x<\pi\end{cases}$ and deduce that $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$

Solution:

a) Observe that f is discontinuous at $x=0$.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-2dx+\int_{0}^{\pi}kdx\right)=\frac{1}{\pi}\left(-2x\Big|_{-\pi}^{0}+2x\Big|_{0}^{\pi}\right)=2\pi-2\pi=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}-2\cos nxdx+\int_{0}^{\pi}2\cos nxdx\right)$$

$$=\frac{1}{\pi}\left(\frac{-2}{n}\sin nx\Big|_{-\pi}^{0}+\frac{2}{n}\sin nx\Big|_{0}^{\pi}\right)=0$$

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{0}-2\sin nxdx+\int_{0}^{\pi}2\sin nxdx\right)=\frac{4}{n\pi}\left(1-(-1)^n\right)$$

Therefore, $f(x)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\left(1-(-1)^n\right)\sin nx=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)$.

Besides, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ is obtained when $x=\frac{\pi}{2}$. But when $x=\frac{\pi}{2}$, we have $f(\pi/2)=2$ (Because $f(x)=2$ for $0<x<\pi$).

So, $\frac{8}{\pi}\left(\sin(\pi/2)+\frac{\sin(3\pi/2)}{3}+\frac{\sin(5\pi/2)}{5}+\ldots\right)=f(\pi/2)=2$

$$\Rightarrow\frac{8}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)=2\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right) = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} 3x dx\right)$$

$$= \frac{1}{\pi}(-\pi x)\Big|_{-\pi}^{0} + \frac{1}{\pi}\frac{3x^2}{2}\Big|_{0}^{\pi} = -\pi + \frac{3\pi}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\cos nx dx + \int_{0}^{\pi} 3x\cos nx dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi\sin nx}{n}\right)\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{3}{n^2}(nx\sin nx + \cos nx)\right)\Big|_{0}^{\pi} = \frac{3}{\pi n^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin nx dx + \int_{0}^{\pi} f(x)\sin nx dx\right) = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\sin nx dx + \int_{0}^{\pi} 3x\sin nx dx\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi\cos nx}{n}\right)\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{3}{n^2}(\sin nx - nx\cos nx)\right)\Big|_{0}^{\pi} = \frac{1}{n}[1 - 4(-1)^n]$$

Therefore, $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{3}{\pi n^2}[(-1)^n - 1]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}[1 - 4(-1)^n]\sin nx$

Now let's deduce the sum $1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$. Since f is discontinuous at $x = 0$, it converges at $x = 0$ to

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{3}{\pi n^2}[(-1)^n - 1]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}[1 - 4(-1)^n]\sin nx = f(x)$$

$$\Rightarrow \frac{\pi}{4} - \frac{6}{\pi}(\cos 0 + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + ...) = \frac{\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)}{2} = -\frac{\pi}{2}$$

$$\Rightarrow -\frac{6}{\pi}(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...) = -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4} \Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{3\pi^2}{24} = \frac{\pi^2}{8}$$

2. Find the Fourier series expansion of the following piecewise functions.

a) $f(x)=\begin{cases}0, -\pi<x<0\\ x, 0<x<\pi\end{cases}$ b) $f(x)=\begin{cases}0, -\pi<x<0\\ x^2, 0<x<\pi\end{cases}$

c) $f(x)=\begin{cases}x+x^2, -\pi<x<\pi\\ \pi^2, x=\pm\pi\end{cases}$

Solution:

a) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx=\frac{1}{\pi}\int_{0}^{\pi}xdx=\frac{\pi}{2}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right)=\frac{1}{\pi}\int_{0}^{\pi}x\cos nxdx$$

$$=\frac{1}{\pi}\left(\frac{x\sin(nx)}{n}+\frac{\cos(nx)}{n^2}\right)\Bigg|_0^{\pi}=\frac{\cos(n\pi)-1}{n^2\pi}=\frac{(-1)^n-1}{n^2\pi}$$

$$b_n=\frac{1}{\pi}\int_{0}^{\pi}x\sin nxdx=\frac{1}{\pi}\left(\frac{-x\cos(nx)}{n}+\frac{\sin(nx)}{n^2}\right)\Bigg|_0^{\pi}=\frac{-\cos(n\pi)}{n}=\frac{(-1)^{n+1}}{n}$$

Therefore, $f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi}\cos nx+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

b) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx\right)=\frac{1}{\pi}\int_{0}^{\pi}x^2dx=\frac{x^3}{3\pi}\Bigg|_0^{\pi}=\frac{\pi^2}{3}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{0}^{\pi}x^2\cos nxdx=\frac{2(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{0}^{\pi}x^2\sin nxdx=\frac{\pi(-1)^{n+1}}{n}+\frac{2[(-1)^n-1]}{n^3\pi}$$

Therefore, $f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\left(\frac{\pi(-1)^{n+1}}{n}+\frac{2[(-1)^n-1]}{n^3\pi}\right)\sin nx$

c) $f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{n^2}\cos(nx)-\frac{2}{n}\sin(nx)\right)$

4.1.4 Fourier Series of Functions with Arbitrary Periods

The Fourier series of the periodic function f of period $2L$ defined on $(a, a+2L)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

where $a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx$, $a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}dx$, $b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}dx$

Remarks: Recall the following important integrals to save your time.

i) $a_n = \int_a^b x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{L^2}{(n\pi)^2}\left(\frac{n\pi}{L}x.\sin(\frac{n\pi x}{L}) + \cos(\frac{n\pi x}{L})\right)\Bigg|_a^b$

ii) $b_n = \int_a^b x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L^2}{(n\pi)^2}\left(\sin(\frac{n\pi x}{L}) - \frac{n\pi}{L}x\cos(\frac{n\pi x}{L})\right)\Bigg|_a^b$

Examples:

1. Find the Fourier series of

a) $f(x) = x,\ 0 < x < 2$ and $f(x+2) = f(x)$

b) $f(x) = \begin{cases} 2, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ and $f(x+4) = f(x)$

c) $f(x) = \begin{cases} 0, -2 < x < 0 \\ 1, 0 < x < 2 \end{cases}$ $f(x+2) = f(x)$

d) $f(x) = \begin{cases} 1, & 0 < x < 2 \\ x, & 2 < x < 4 \end{cases}$, $f(x+4) = f(x)$

e) $f(x) = \begin{cases} 1, -5 < x < 0 \\ x+1, 0 < x < 5 \end{cases}$, $f(x+10) = f(x)$

f) $f(x) = \begin{cases} x, -2 < x < 0 \\ 4, 0 < x < 2 \end{cases}$, $f(x+2) = f(x)$

g) $f(x) = \begin{cases} 0, -5 < x < 0 \\ 3, 0 < x < 5 \end{cases}$, $f(x+10) = f(x)$

Solution:

a) Here, a = 0, a + 2L = 2 ⇒ L = 1. $a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx = \int_0^2 xdx = \left.\frac{x^2}{2}\right|_0^2 = 2$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos(\frac{n\pi x}{L})dx = \int_0^2 x\cos(n\pi x)dx = \left.\frac{x\sin(n\pi x)}{n\pi}\right|_0^2 - \frac{1}{n\pi}\int_0^2 \sin(n\pi x)dx$$

$$= \frac{2}{n\pi}\sin(2n\pi) - \frac{1}{n\pi}\left.\left(\frac{-\cos(n\pi x}{n\pi}\right)\right|_0^2 = 0$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin(\frac{n\pi x}{L})dx = \int_0^2 x\sin(n\pi x)dx = \left.\frac{-x\cos(n\pi x)}{n\pi}\right|_0^2 + \frac{1}{n\pi}\int_0^2 \cos(n\pi x)dx$$

$$= \frac{-1}{n\pi}[2\cos(2n\pi) - 0] + \frac{1}{(n\pi)^2}[\sin(2n\pi) - 0] = \frac{-2}{n\pi}$$

Therefore, $f(x) = 1 - \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)$

b) Here, a = -2, a + 2L = 2 ⇒ L = 2.

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} f(x)dx + \frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_{-2}^{0} 2dx + \frac{1}{2}\int_0^2 xdx = 3$$

$$a_n = \frac{1}{2}\int_{-2}^{0} f(x)\cos(\frac{n\pi x}{2})dx + \frac{1}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})dx$$

$$= \frac{1}{2}\int_{-2}^{0} 2\cos(\frac{n\pi x}{2})dx + \frac{1}{2}\int_0^2 x\cos(\frac{n\pi x}{2})dx$$

$$= \left.\frac{2}{n\pi}\sin(\frac{n\pi x}{2})\right|_{-2}^{0} + \frac{4}{(n\pi)^2}\left.\left(\frac{n\pi}{2}x.\sin(\frac{n\pi x}{2}) + \cos(\frac{n\pi x}{2})\right)\right|_0^2 = \frac{4}{(n\pi)^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{0} 2\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2}{n\pi}\cos\left.\left(\frac{n\pi x}{2}\right)\right|_{-2}^{0} + \frac{4}{(n\pi)^2}\left.\left(\sin\frac{n\pi x}{2} - \frac{n\pi}{2}x\cos\frac{n\pi x}{2}\right)\right|_0^2 = \frac{-2[1 + (-1)^n]}{n\pi}$$

Therefore, $f(x)=\sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}[(-1)^n-1]\cos(\frac{n\pi x}{2})+\sum_{n=1}^{\infty}\frac{-2[1+(-1)^n]}{n\pi}\sin(\frac{n\pi x}{2})$

c) Here, a = -2, a + 2L = 2 ⇒ L = 2.

$$a_0=\frac{1}{L}\int_a^{a+2L}f(x)dx=\frac{1}{2}\int_{-2}^{2}f(x)dx=\frac{1}{2}\int_{-2}^{0}f(x)dx\frac{1}{2}\int_0^2 f(x)dx=\frac{1}{2}\int_0^2 1dx=1$$

$$a_n=\frac{1}{L}\int_{-L}^{a+2L}f(x)\cos\left(\frac{n\pi}{L}\right)dx=\frac{1}{2}\int_0^2\cos\left(\frac{n\pi}{2}x\right)dx=\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\Big|_0^2=0$$

$$b_n=\frac{1}{L}\int_{-L}^{a+2L}f(x)\sin\left(\frac{n\pi}{L}\right)dx=\frac{1}{2}\int_0^2\sin\left(\frac{n\pi}{2}x\right)dx=\frac{-1}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\Big|_0^2=\frac{1}{n\pi}[1-(-1)^n]$$

Therefore, $f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi}{2}x\right)$

d) Here, a = 0, a + 2L = 4 ⇒ 2L = 4 ⇒ L = 2.

$$a_0=\frac{1}{L}\int_a^{a+2L}f(x)dx=\frac{1}{2}\int_0^4 f(x)dx=\frac{1}{2}\int_0^2 f(x)dx+\frac{1}{2}\int_2^4 f(x)dx$$

$$=\frac{1}{2}\int_0^2 1dx+\frac{1}{2}\int_2^4 xdx=\frac{x}{2}\Big|_0^2+\frac{x^2}{4}\Big|_2^4=1+3=4$$

$$a_n=\frac{1}{L}\int_a^{a+2L}f(x)\cos(\frac{n\pi x}{L})dx=\frac{1}{2}\int_0^4 f(x)\cos(\frac{n\pi x}{2})dx$$

$$=\frac{1}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})dx+\frac{1}{2}\int_2^4 f(x)\cos(\frac{n\pi x}{2})dx$$

$$=\frac{1}{2}\int_0^2\cos(\frac{n\pi x}{2})dx+\frac{1}{2}\int_2^4 x\cos(\frac{n\pi x}{2})dx$$

$$=\frac{1}{n\pi}\sin(\frac{n\pi x}{2})\Big|_0^2+\frac{2}{(n\pi)^2}\left(\frac{n\pi}{2}x.\sin(\frac{n\pi x}{2})+\cos(\frac{n\pi x}{2})\right)\Big|_2^4$$

$$=\frac{2}{(n\pi)^2}(1-\cos n\pi)=\frac{2}{(n\pi)^2}[1-(-1)^n]$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^4 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_2^4 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_2^4 x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 + \frac{2}{(n\pi)^2}\left(\sin\frac{n\pi x}{2} - \frac{n\pi}{2}x\cos\frac{n\pi x}{2}\right)\Bigg|_2^4$$

$$= -\frac{1}{n\pi}[\cos n\pi - 1] + \frac{2}{(n\pi)^2}[-2n\pi + n\pi\cos n\pi]$$

$$= -\frac{1}{n\pi}[(-1)^n - 1] + \frac{2}{n\pi}[(-1)^n - 2] = \frac{1}{n\pi}[(-1)^n - 3]$$

Therefore,

$$f(x) = 2 + \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}[1-(-1)^n]\cos(\frac{n\pi x}{2}) + \sum_{n=1}^{\infty}\frac{1}{n\pi}[(-1)^n - 3]\sin(\frac{n\pi x}{2})$$

e) Here, $a = -5, a + 2L = 5 \Rightarrow L = 5.$

$$a_0 = \frac{1}{L}\int_{-L}^{a+2L} f(x)dx = \frac{1}{5}\int_{-5}^{5} f(x)dx = \frac{1}{5}\int_{-5}^{0} 1dx + \frac{1}{5}\int_0^5 (x+1)dx = \frac{9}{2}$$

$$a_n = \frac{1}{5}\int_{-5}^{0}\cos\left(\frac{n\pi}{5}x\right)dx + \frac{1}{5}\int_0^5 (x+1)\cos\left(\frac{n\pi}{5}x\right)dx = \frac{5}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{5}\int_{-5}^{0}\sin\left(\frac{n\pi x}{5}\right)dx + \frac{1}{5}\int_0^5 x\sin\left(\frac{n\pi x}{5}\right)dx + \frac{1}{5}\int_0^5 \sin\left(\frac{n\pi x}{5}\right)dx = \frac{5}{n\pi}(-1)^{n+1}$$

Therefore, $f(x) = \dfrac{9}{4} + \displaystyle\sum_{n=1}^{\infty}\frac{5}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi}{5}x\right) + \sum_{n=1}^{\infty}\frac{5}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi}{5}x\right)$

2. Find the Fourier series expansion of $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 2x, & 0 < x < 1 \end{cases}$, $f(x+2) = f(x)$ and

show that $f(0) = -\dfrac{1}{2}$.

4.1.5 Fourier Series of Even and Odd Functions

Revision about Even and Odd Functions:

Definition of even and odd functions:

i) A function f is *even* if and only if $f(-x)=f(x)$ for all x.

ii) A function f is odd if and only if $f(-x)=-f(x)$ for all x.

Examples:

a) $f(x)=x^2$ is even function because $f(-x)=(-x)^2=x^2=f(x)$.

b) $f(x)=e^{x^2}$ is an even function because $f(-x)=e^{(-x)^2}=e^{x^2}=f(x)$.

c) $f(x)=e^{2x}-e^{-2x}$ is odd because $f(-x)=e^{-2x}-e^{2x}=-(e^{2x}-e^{-2x})=-f(x)$

d) $f(x)=2x^3$ is an odd function because $f(-x)=2(-x)^3=-2x^3=-f(x)$.

Properties of even and odd functions:

i) The product of any two or more even functions is even.

ii) The product of an even and an odd function is odd.

iii) The product of any two odd functions is even.

Frequently use even and odd functions:

i) **The cosine** function: $g(x)=\cos ax$ is an even function.

ii) **The sine** function: $h(x)=\sin ax$ is an odd function.

iii) The product function $f(x)=x\cos ax$ is an odd function.

iv) The product function $f(x)=x\sin ax$ is an even function.

Integral Properties of even and odd functions:

i) $\int_{-L}^{L} f(x)dx=2\int_{0}^{L} f(x)dx$ if f is even

ii) $\int_{-L}^{L} f(x)dx=0$ if f is odd

Now let's discuss the Fourier Series of even and odd periodic functions

Suppose f is a periodic function on $(-L,L)$ with period $p=2L$.

Case-1: Suppose f is even. That is $f(-x)=f(x)$. The Fourier series of f is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L} \text{ where}$$

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx.$$

Since f is even, the product $f(x)\cos\frac{n\pi x}{L}$ is even. Then, by the above integral properties of even and odd functions, we have

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx=\frac{2}{L}\int_{0}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

Besides, since the product of an odd and an even function is odd, we have that $f(x)\sin\frac{n\pi x}{L}$ is odd. So, we have $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx=0$.

Therefore, the Fourier series of an even function f of period $2L$ is reduced to

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L} \text{ where } a_0=\frac{2}{L}\int_{0}^{L}f(x)dx,\ a_n=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

This series involves only the cosine terms and thus it is said to be cosine Fourier series expansion of f.

Case-2: When f is odd. That is $f(-x)=-f(x)$. Then, since f is odd

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx=0,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}=0,\ b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

Therefore the Fourier series of an odd function f of period $2L$ is reduced to

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L} \text{ where } b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{n\pi x}{L}dx.$$

This series involves only sine terms and thus it is said to be sine Fourier series expansion of f.

Examples

1. Find the Fourier series of the following functions:

a) $f(x) = x, -2 < x < 2$ and deduce $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \frac{\pi}{4}$

b) $f(x) = |x|, -\pi \le x \le \pi$ and deduce $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$

Solution:

a) Since f is odd, $a_0 = 0,\ a_n = 0$.

$$b_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^2 f(x)\sin nx dx = \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{4}{(n\pi)^2}\left(\sin\frac{n\pi x}{2} - \frac{n\pi}{2}x\cos\frac{n\pi x}{2}\right)\Bigg|_0^2 = \frac{-4n\pi\cos n\pi}{(n\pi)^2} = \frac{4(-1)^{n+1}}{n\pi}$$

Therefore, $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2} = \frac{4}{\pi}(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + ...)$

Besides, since the function is continuous at $x = 1$, we have

$$f(1) = \frac{4}{\pi}(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + ...)$$

$$\Rightarrow 1 = \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...) \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$$

b) Since $f(-x) = |-x| = |x| = f(x), f$ *is even* $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \frac{2}{n^2\pi}[(-1)^n - 1] = \frac{-4}{(2n-1)^2\pi}$$

Therefore, $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]\cos nx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}$

The series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$ is obtained when $x = 0$.

Thus, putting $x=0$ in $|x|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}$ gives

$$\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(0)}{(2n-1)^2}=0\Rightarrow\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi}{2}\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots=\frac{\pi^2}{8}$$

2. Find the Fourier series expansions of

a) $f(x)=x,\ -\pi\le x\le\pi$ b) $f(x)=x\cos,-\pi\le x\le\pi$

c) $f(x)=\pi^2-x^2,-\pi<x<\pi$ d) $f(x)=x|x|,-1\le x\le 1$

Solution:

a) Since $f(x)=x$ is an odd function, we have $a_0=0,a_n=0$. Besides,

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin xdx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nxdx=\frac{1}{\pi}\left(\frac{1}{n^2}(\sin nx-nx\cos nx)\right)\Big|_{-\pi}^{\pi}=\frac{2(-1)^{n+1}}{n}$$

Therefore, $x=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx=2(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\dots)$

b) Since $f(x)=x\cos x$ is the product of an odd and an even function ($h(x)=x$ is odd and $g(x)=\cos x$ is even), f is an odd function and thus $a_n=0$. Besides,

$$b_n=\frac{2}{\pi}\int_0^{\pi}x\cos x\sin nxdx=\frac{1}{\pi}\left(\int_0^{\pi}x[\sin(n+1)x+\sin(n-1)x]dx\right)$$

$$=\left(\frac{(-1)^n}{n+1}+\frac{(-1)^n}{n-1}\right)=(-1)^n\frac{2n}{n^2-1},\text{for } n\ne 1$$

Here, we calculated above $(-1)^n\frac{2n}{n^2-1}$, for $n\ne1$. So, we have to compute b_1 separately. But for $n=1$,

$$b_1=\frac{2}{\pi}\int_0^{\pi}x\cos x\sin xdx=\frac{1}{\pi}\int_0^{\pi}x\sin 2xdx=\frac{1}{4\pi}(\sin 2x-2x\cos 2x)\Big|_0^{\pi}=-\frac{1}{2}$$

Therefore, $f(x)=\cos x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

$$=b_1\sin x+\sum_{n=2}^{\infty}b_n\sin nx=-\frac{1}{2}\sin x+2\sum_{n=2}^{\infty}(-1)^n\frac{n}{n^2-1}\sin nx$$

4.1.6 Half-Range Expansions

A half range Fourier sine series is a series in which only sine terms are present and a half range Fourier cosine series is a series in which only cosine terms are present. When we want to determine a half range series to a given function, the function is generally defined in the interval $(0, L)$. This is half of the interval $(-L, L)$. That is why the expansion is referred as half range expansion.

i) Half-range sine Expansion:

It is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}$ where $a_n = 0, b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

ii) Half-range cosine Expansion:

It is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}$ where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, b_n = 0$$

Examples: Find the half range sine and cosine series expansion of

a) $f(x) = x, 0 < x < 2$ b) $f(x) = 2x, 0 < x < 1$

c) $f(x) = \sin x, 0 < x < \pi$ d) $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$

e) $f(x) = x(\pi - x), 0 \le x \le \pi$ f) $f(x) = \sin\left(\frac{\pi x}{3}\right), f(x+3) = f(x), 0 < x < 3$

Solution:

a) To find the half range sine expansion, $0 < x < L \Rightarrow 0 < x < 2 \Rightarrow L = 2$.
Since f is odd, $a_0 = a_n = 0$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^2 x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \left(\frac{-2x\cos(\frac{n\pi x}{2})}{n\pi} + \frac{4\sin(\frac{n\pi x}{2})}{n^2\pi^2}\right)\Bigg|_0^2 = \frac{-4\cos n\pi}{n\pi} = \frac{4(-1)^{n+1}}{n\pi}$$

Therefore, the half range sine series expansion of f(x) is given by

$$f(x)=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin(\tfrac{n\pi x}{2})=\frac{4}{\pi}\left(\sin(\tfrac{\pi x}{2})-\frac{\sin(\pi x)}{2}+\frac{1}{3}\sin(\tfrac{3\pi x}{2})-\ldots\right)$$

ii) Half range cosine series expansion:

Since f is even, $b_n=0,\ a_0=\frac{2}{L}\int_0^L f(x)dx=\int_0^2 xdx=2$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos(\tfrac{n\pi x}{L})dx=\int_0^2 x\cos(\tfrac{n\pi x}{2})dx$$

$$=\left(\frac{2x\sin(\frac{n\pi x}{2})}{n\pi}+\frac{4\cos(\frac{n\pi x}{2}))}{n^2\pi^2}\right)\Bigg|_0^2=\frac{4[\cos(n\pi)-1]}{n^2\pi^2}=\frac{4[(-1)^n-1]}{n^2\pi^2}$$

Therefore, the half range cosine series is $f(x)=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}\cos(\frac{n\pi x}{2})$

$$b)\ b_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx=2\int_0^1 2x\sin(n\pi x)dx$$

$$=4\left(\frac{1}{(n\pi)^2}(\sin(n\pi x)-n\pi x\cos(n\pi x))\right)\Bigg|_0^1=\frac{-4\cos n\pi}{n\pi}=\frac{4(-1)^{n+1}}{n\pi}$$

Therefore, the half range sine series expansion of $f(x)$ is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{L}x=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin(n\pi x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$$

For the half range cosine expansion, $b_n=0$.

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\int_0^1 2xdx=1$$

$$a_n=2\int_0^1 2x\cos(n\pi x)dx=4\left(\frac{1}{(n\pi)^2}(n\pi x.\sin(n\pi x)+\cos(n\pi x))\right)\Bigg|_0^1=\frac{4[(-1)^n-1]}{n^2\pi^2}$$

Therefore, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}=1+\sum_{n=1}^{\infty}\frac{4[(-1)^n-1]}{n^2\pi^2}\cos(n\pi x)$

c) $a_0 = \frac{2}{\pi}\int_0^L f(x)dx = \frac{2}{\pi}\int_0^\pi \sin x dx = \frac{2}{\pi}(-\cos x)\Big|_0^\pi = \frac{4}{\pi}$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{\pi}\int_0^\pi \sin x\cos nx dx$$

$$= \frac{2}{\pi}\int_0^\pi [\tfrac{1}{2}\sin[(1-n)x + \tfrac{1}{2}\sin[(1+n)x]dx = \frac{1}{\pi}\int_0^\pi [\sin[(1-n)x + \sin[(1+n)x]dx$$

$$= \frac{-\cos[(1-n)x]}{(1-n)\pi}\Bigg|_0^\pi - \frac{\cos[(1+n)x]}{(1+n)\pi}\Bigg|_0^\pi = \frac{1-\cos[(1-n)\pi]}{(1-n)\pi} + \frac{1-\cos[(1+n)\pi]}{(1+n)\pi}$$

$$= \frac{1+\cos n\pi}{(1+n)\pi} + \frac{1+\cos(n\pi)}{(1-n)\pi} = \frac{-2(1+\cos n\pi)}{(n^2-1)\pi}, \text{if } n \neq 1$$

If $n=1, a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Big|_0^\pi = 0.$

Therefore, $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right).$

d) Since we are interested in the half range cosine series expansion, $b_n = 0$.

$$a_0 = \int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 2xdx + \int_1^2 2(2-x)dx = 2$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^1 2x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 2(2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 2x\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 4\cos\left(\frac{n\pi x}{2}\right)dx - \int_1^2 2x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{16}{(n\pi)^2}\cos\frac{n\pi}{2} - \frac{8}{(n\pi)^2}(1+\cos n\pi)$$

Therefore, $f(x) = 1 - \frac{32}{\pi^2}\left(\frac{\cos\pi x}{2^2} + \frac{\cos 3\pi x}{6^2} + \frac{\cos 5\pi x}{10^2} + \ldots\right)$

e) Here, $a_0 = \frac{\pi^2}{3}$, $a_n = -\frac{1}{n^2}$ and thus $f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{n^2}\cos(2nx)$

4.1.7 Parseval's Identity (PI)

Let f be a periodic function with period $p = 2L$ such that its Fourier series expansion on $(a, a+2L)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$.

Then, $\int_{-L}^{L} [f(x)]^2 dx = L\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right)$ where $a_0, a_n,$ and b_n are the Fourier coefficients. This formula is known as Parseval's Identity.

Note: Parserval's Identity also works for half range expansions.

i) If the half-range cosine series of f in $(0, L)$, is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \text{ then } \int_0^L [f(x)]^2 dx = \frac{L}{2}\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right).$$

ii) If the half-range sine series of f in $(0, L)$, is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ then } \int_0^L [f(x)]^2 dx = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2.$$

Examples:

1. Find the Fourier series expansion of

a) $f(x) = x^2$ in $-\pi < x < \pi$ and use Parseval's identity to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{90}.$$

b) $f(x) = |\sin x|, -\pi < x < \pi$ and deduce using Parserval's identity that

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

Solution:

We have found the expansion earlier $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$.

From this expansion, we get $a_0 = \frac{2\pi^2}{3}, a_n = \frac{4}{n^2}(-1)^n, b_n = 0.$

Then, by Parseval's formula,

$$\int_{-\pi}^{\pi}[f(x)]^2\,dx=\pi\left[\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\,2}+b_n^{\,2}\right)\right]$$

$$\Rightarrow\int_{-\pi}^{\pi}[f(x)]^2\,dx=\pi\left[\frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\right)^2\right]$$

$$\Rightarrow\pi\left[\frac{4\pi^4}{18}+\sum_{n=1}^{\infty}\frac{16}{n^4}\right]=\frac{2\pi^5}{5}\Rightarrow\frac{4\pi^4}{18}+16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2\pi^4}{5}$$

$$\Rightarrow16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2\pi^4}{5}-\frac{4\pi^4}{18}\Rightarrow16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{16\pi^4}{90}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

$$\text{i.e } 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{90}$$

b) Since $f(-x)=|\sin(-x)|=|-\sin x|=|\sin x|=f(x)$, f is even. So, $b_n=0$.

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\left.\frac{-2\cos x}{\pi}\right|_0^{\pi}=\frac{4}{\pi}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}[\tfrac{1}{2}\sin[(1-n)x+\tfrac{1}{2}\sin[(1+n)x]dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin[(1-n)x+\sin[(1+n)x]dx$$

$$=\left.\frac{-\cos[(1-n)x]}{(1-n)\pi}\right|_0^{\pi}-\left.\frac{\cos[(1+n)x]}{(1+n)\pi}\right|_0^{\pi}=\frac{1-\cos[(1-n)\pi]}{(1-n)\pi}+\frac{1-\cos[(1+n)\pi]}{(1+n)\pi}$$

$$=\frac{2}{(1-2n)\pi}+\frac{2}{(1+2n)\pi}=\frac{-4}{(4n^2-1)\pi}$$

Therefore, $f(x)=\dfrac{2}{\pi}+\displaystyle\sum_{n=1}^{\infty}\frac{-4}{(4n^2-1)\pi}\cos(2nx)=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\ldots\right)$

Now from Parseval's Identity, sung, we have

$$\int_{-\pi}^{\pi}[f(x)]^2dx=\pi\left[\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\,2}+b_n^{\,2}\right)\right]$$

$$\Rightarrow\int_{-\pi}^{\pi}\sin^2 x dx=\pi\left[\frac{1}{2}\left(\frac{4}{\pi}\right)^2+\sum_{n=1}^{\infty}\left(\frac{-4}{(4n^2-1)\pi}\right)^2\right]$$

$$\Rightarrow\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2x)dx=\pi\left[\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\right]$$

$$\Rightarrow\pi\left[\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\right]=\left(\frac{x}{2}-\frac{1}{4}\sin 2x\right)\Big|_{-\pi}^{\pi}$$

$$\Rightarrow\pi=\pi\left[\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\right]\Rightarrow 1=\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}$$

$$\Rightarrow\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=1-\frac{8}{\pi^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=\frac{\pi^2}{16}-\frac{1}{2}$$

2. Find the half range cosine expansion of $f(x)=x,\ 0<x<4$. Using the result, show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$.

Solution: As we did earlier , the expansion is From the expansion, observe that is given by $x=2-\frac{16}{\pi^2}\left(\cos(\pi x/4)+\frac{\cos(3\pi x/4)}{3^2}+\frac{\cos(5\pi x/4)}{5^2}+\ldots\right)$.

Then, $a_0=4, a_n=\frac{16}{(2n-1)^2\pi^2}, b_n=0$. So, by Parseval's formula,

$$\int_0^L[f(x)]^2dx=\frac{L}{2}\left(\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}a_n^{\,2}\right)\Rightarrow\int_0^4 x^2dx=2\left[\frac{16}{2}+\sum_{n=1}^{\infty}\left(\frac{16}{(2n-1)^2\pi^2}\right)^2\right]$$

$$\Rightarrow 2\left[8+\sum_{n=1}^{\infty}\frac{256}{(2n-1)^4\pi^4}\right]=\frac{64}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{256}{(2n-1)^4\pi^4}=\frac{8}{3}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}\Rightarrow\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$$

3*. Using Parserval's Identity, show that $\int_{-\pi}^{\pi} \cos^4 x dx = \frac{3\pi}{4}$

Solution: Here, we infer. But $f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

So, we have $a_0 = 1,\ a_1 = \frac{1}{2}, a_n = 0, b_n = 0, \forall n \geq 2$

Therefore, $\int_{-\pi}^{\pi} [f(x)]^2 dx = \pi\left[\frac{a_0^{\ 2}}{2} + \sum_{n=1}^{\infty}\left(a_n^{\ 2} + b_n^{\ 2}\right)\right] \Rightarrow \int_{-\pi}^{\pi} \cos^4 x dx = \pi[\frac{1}{2} + \frac{1}{4}] = \frac{3\pi}{4}$

4*. Using Parserval's Identity, evaluate $\int_{-\pi}^{\pi} [4\cos^2(3x) - 2\cos(6x) + \frac{1}{2}\sin x]^2 dx$

Solution: Here, infer $f(x) = 4\cos^2(3x) - 2\cos(6x) + \frac{1}{2}\sin x$. But using half angle formula, $f(x) = 4\cos^2(3x) - 2\cos(6x) + \frac{1}{2}\sin x = 2 + \frac{1}{2}\sin x$.

So, we have $a_0 = 2,\ a_n = 0,\ b_1 = \frac{1}{2}, b_n = 0, \forall n \geq 2$

Therefore, $\int_{-\pi}^{\pi} [f(x)]^2 dx = \pi\left[\frac{a_0^{\ 2}}{2} + \sum_{n=1}^{\infty}\left(a_n^{\ 2} + b_n^{\ 2}\right)\right] = \pi[\frac{4}{2} + \frac{1}{4}] = \frac{9\pi}{4}$

4.2 Fourier Integrals and Transforms

4.2.1 Fourier Integrals

The Fourier Integral representation (simply the Fourier Integral) of a function f is an integral given by $f(x)=\int_{0}^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$

In this integral, $A(\omega)$ and $B(\omega)$ are called Fourier Integral coefficients.

They are given by $\begin{cases} A(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(\omega)\cos\omega x dx \\ B(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(\omega)\sin\omega x dx \end{cases}$

Examples: Given the function $f(x)=\begin{cases}1, \text{ if } |x|<1\\ 0, \text{ if } |x|>1\end{cases}$

a) Find the Fourier Integral of f

b) Show that $\displaystyle\int_{0}^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\begin{cases}\frac{\pi}{2}, |x|<1\\ \frac{\pi}{4}, |x|=1 \\ 0, |x|>1\end{cases}$ and deduce $\displaystyle\int_{0}^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$.

Solution: By definition, $f(x)=\int_{0}^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$.

a) So, the basic task is to determine the coefficients $A(\omega)$ and $B(\omega)$.

First, redefine f as $f(x)=\begin{cases}1, \text{ if } |x|<1\\ 0, \text{ if } |x|>1\end{cases} \Rightarrow f(x)=\begin{cases}1, \text{ if } -1<x<1\\ 0, \text{ if } x<-1 \text{ or } x>1\end{cases}$

Then, using the properties of definite integrals, we have

i) $$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\cos\omega x\,dx$$

$$=\frac{1}{\pi}\left(\int_{-\infty}^{-1} f(\omega)\cos\omega x\,dx+\int_{-1}^{1} f(\omega)\cos\omega x\,dx+\int_{1}^{\infty} f(\omega)\cos\omega x\,dx\right)$$

$$=\frac{1}{\pi}\left(\int_{-\infty}^{-1} 0.\cos\omega x\,dx+\int_{-1}^{1} 1.\cos\omega x\,dx+\int_{1}^{\infty} 0.\cos\omega x\,dx\right)$$

$$=\frac{1}{\pi}\int_{-1}^{1}\cos\omega x\,dx=\frac{\sin\omega x}{\pi\omega}\bigg|_{x=-1}^{x=1}=\frac{\sin\omega-\sin(-\omega)}{\pi\omega}=\frac{2\sin\omega}{\pi\omega}$$

ii) $$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x\,dx$$

$$=\frac{1}{\pi}\left(\int_{-\infty}^{-1} f(\omega)\sin\omega x\,dx+\int_{-1}^{1} f(\omega)\sin\omega x\,dx+\int_{1}^{\infty} f(\omega)\sin\omega x\,dx\right)$$

$$=\frac{1}{\pi}\int_{-1}^{1}\sin\omega x\,dx=-\frac{\cos\omega x}{\pi\omega}\bigg|_{x=-1}^{x=1}=-\frac{\cos\omega-\cos\omega}{\pi\omega}=0$$

$$[\text{Notice}:\sin(-\omega)=-\sin\omega,\ \cos(-\omega)=\cos\omega]$$

Therefore, the Fourier Integral of f becomes;

$$f(x)=\int_{0}^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\omega\cos\omega x}{\omega}d\omega$$

b) Equate the Fourier Integral and the function on the given interval.

For $|x|<1$, we have $f(x)=1$. So, using the result of part (a), we have

$$f(x)=1\Rightarrow\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=1\Rightarrow\int_{0}^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\frac{\pi}{2}.$$

For $|x|>1$, $f(x)=0 \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x \sin\omega}{\omega}d\omega = 0 \Rightarrow \int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = 0$.

For $|x|=1$, the function is discontinuous. So, the Fourier Integral is equal to the average of the limits as stated in the Dirichlet's conditions. That is

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = \frac{\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)}{2} = \frac{1}{2} \Rightarrow \int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}d\omega = \frac{\pi}{4}.$$

Finally, at $x=0$, $f(0)=1 \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\cos(0)\sin\omega}{\omega}d\omega = 1 \Rightarrow \int_0^\infty \frac{\sin\omega}{\omega}d\omega = \frac{\pi}{2}$.

4.2.2 Sine and cosine Fourier Integrals

Revision about Even and Odd Functions:

i) A function f is *even* if and only if $f(-x)=f(x)$ for all x .

ii) A function f is odd if and only if $f(-x)=-f(x)$ for all x .

Properties of even and odd functions:

i) The of any two or more even functions is even.

ii) The product of an even and an odd function is odd.

iii) The product of any two odd functions is even.

Integral Properties of even and odd functions:

i) $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$ if f is even

ii) $\int_{-L}^{L} f(x)dx = 0$ if f is odd

Fourier sine and Fourier cosine Integrals:

As we have discussed above, the Fourier Integral representation of a function f is given by $f(x)=\int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$.

Case-1: Suppose f is an even function. Since $\cos(\omega x)$ is even and $\sin(\omega x)$ is odd, the product $f(\omega)\cos(\omega x)$ is even while the product $f(\omega)\sin(\omega x)$ is odd. So, using the integral property of even and odd functions,

i) $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\cos\omega x dx=\frac{2}{\pi}\int_{0}^{\infty} f(\omega)\cos\omega x dx$

ii) $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx=0$

Then, the Fourier Integral representation of a function f becomes

$$f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x d\omega \text{ where } A(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(\omega)\cos\omega x dx.$$

The representation $f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x d\omega$ is known as Fourier cosine Integral.

Case-2: Suppose f is an odd function.
In this case, the product $f(\omega)\cos(\omega x)$ is odd and $f(\omega)\sin(\omega x)$ is even.
So, using the integral property of even and odd functions,

i) $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\cos\omega x dx=0$

ii) $B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx=\frac{2}{\pi}\int_{-\infty}^{\infty} f(\omega)\sin\omega x dx$

Then, the Fourier Integral representation of a function f becomes

$$f(x)=\int_{0}^{\infty} B(\omega)\sin\omega x d\omega \text{ where } B(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(\omega)\sin\omega x dx.$$

The representation $f(x)=\int_{0}^{\infty} B(\omega)\sin\omega x d\omega$ is known as Fourier sine Integral.

Examples:

1. Find the Fourier sine and cosine Integrals of $f(x)=e^{-ax}, x>0, a>0$.

Solution:

i) Fourier cosine integral: Using case-1; $f(x)=\int_{0}^{\infty} A(\omega)\cos\omega x d\omega$.

Here, using the important integral given in (4), we have

$$A(\omega)=\frac{2}{\pi}\int_0^{\infty} f(\omega)\cos\omega x dx=\frac{2}{\pi}\int_0^{\infty} e^{-ax}\cos\omega x dx=(\frac{2}{\pi})\frac{a}{a^2+\omega^2}=\frac{2a}{\pi(a^2+\omega^2)}$$

Therefore, the Fourier cosine integral becomes

$$f(x)=\int_0^{\infty} A(\omega)\cos\omega x d\omega=\int_0^{\infty}\frac{2a}{\pi(a^2+\omega^2)}\cos\omega x d\omega=\frac{2a}{\pi}\int_0^{\infty}\frac{\cos\omega x}{a^2+\omega^2}d\omega$$

ii) Fourier sine integral: Using case-2; $f(x)=\int_0^{\infty} B(\omega)\sin\omega x d\omega$.

Here, using the important integral given in (3), we have

$$B(\omega)=\frac{2}{\pi}\int_0^{\infty} f(\omega)\sin\omega x dx=\frac{2}{\pi}\int_0^{\infty} e^{-ax}\cos\omega x dx$$

$$=(\frac{2}{\pi})\frac{e^{-ax}}{a^2+\omega^2}\left[a\sin\omega x-\omega\cos\omega x\right]_{x=0}^{\infty}=(\frac{2}{\pi})\frac{\omega}{a^2+\omega^2}=\frac{2\omega}{\pi(a^2+\omega^2)}$$

Therefore, the Fourier sine integral becomes

$$f(x)=\int_0^{\infty} B(\omega)\sin\omega x d\omega=\int_0^{\infty}\frac{2\omega}{\pi(a^2+\omega^2)}\sin\omega x d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin\omega x}{a^2+\omega^2}d\omega$$

2. Given $\int_0^{\infty} f(t)\cos\omega t dt=\begin{cases}1-\omega, \text{ if } 0\le\omega\le 1\\ 0, \text{ if } \omega>1\end{cases}$. Find f if it is even.

Solution: From the Fourier cosine integral, $f(t)=\int_0^{\infty} A(\omega)\cos\omega t d\omega$.

Now, by using the given value of $\int_0^{\infty} f(t)\cos\omega t dt$ in the problem, we have

$$A(\omega)=\frac{2}{\pi}\int_0^{\infty} f(\omega)\cos\omega t dt=\frac{2}{\pi}\left(\int_0^{1} f(\omega)\cos\omega t dt+\int_1^{\infty} f(\omega)\cos\omega t dt\right)$$

$$=\frac{2}{\pi}\int_0^{1} f(\omega)\cos\omega t dt=\frac{2}{\pi}(1-\omega)$$

Then, $f(t)=\int_0^{\infty} A(\omega)\cos\omega x d\omega=\int_0^{\infty}\frac{2}{\pi}(1-\omega)\cos\omega t d\omega$.

Now, by using the important integral (2) revised above (integration by parts).

$$f(t)=\int_0^{\infty}\frac{2}{\pi}(1-\omega)\cos\omega t d\omega \frac{2\sin\omega t}{\pi t}-\frac{2}{\pi t^2}(\omega t\sin\omega t+\cos\omega t)\Big|_{\omega=0}^{\omega=1}$$

$$=(\frac{2\sin t}{\pi t}-\frac{2\sin t}{\pi t}-\frac{2\cos t}{\pi t^2})-(-\frac{2}{\pi t^2})=\frac{2}{\pi t^2}(1-\cos t)$$

3. Solve the equation $\int_0^{\infty} f(t)\sin\omega t dt=g(x)$ where $g(x)=\begin{cases}1, \text{if } 0\le x<\pi\\ 0, \text{ if } x>\pi\end{cases}$

Solution: Similarly as in problem 2, we get

$$f(t)=\int_0^{\infty}B(\omega)\sin\omega t d\omega=\int_0^{\pi}B(\omega)\sin\omega t d\omega+\int_{\pi}^{\infty}B(\omega)\sin\omega t d\omega$$

$$=\int_0^{\pi}\frac{2}{\pi}\sin\omega t d\omega=\frac{2}{\pi}(-\frac{\cos\omega t}{t})\Big|_{\omega=0}^{\omega=\pi}=\frac{2}{\pi t}(1-\cos\pi t)$$

4.2.3 Fourier Transforms

Suppose f is a function defined on $(-\infty,\infty)$. Then, the Fourier transform of f is given by $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$. Alternatively, using Euler's formula $e^{-iwx}=\cos wx-i\sin wx$, the Fourier transform of the function can be written as

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)[\cos wx-i\sin wx]dx.$$

Note:

i) The form $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$ is said to be *exponential form.*

ii) The form $F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)[\cos wx-i\sin wx]dx$ is *trigonometric form.*

iii) Which form is better? Since both forms give the same answer, we can use any of the forms. But we select one of the forms depending on the nature of the function whose Fourier transform is needed.

Parserval's Identity: $\int_{-\infty}^{\infty}[F(w)]^2\,dw = \int_{-\infty}^{\infty}[f(x)]^2\,dx$

Examples: 1. Find the Fourier transform of the function $f(x)=\begin{cases}1, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$ and deduce that $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi$ and thus $\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{\pi}{2}$.

Solution:

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-1}f(x)e^{-iwx}dx+\int_{-1}^{1}f(x)e^{-iwx}dx+\int_{1}^{\infty}f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-iwx}dx=-\frac{1}{\sqrt{2\pi}}\frac{e^{-iwx}}{iw}\Bigg|_{x=-1}^{x=1}=\frac{1}{\sqrt{2\pi}}.\frac{e^{iw}-e^{-iw}}{iw}=\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}$$

Now, apply Parserval's Identity.

$$\int_{-\infty}^{\infty}[F(w)]^2dw=\int_{-\infty}^{\infty}[f(x)]^2dx\Rightarrow\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}\right)^2dw=\int_{-1}^{1}[f(x)]^2dx$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sin^2 w}{w^2}dw=\int_{-1}^{1}1dx\Rightarrow\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 w}{w^2}dw=2\Rightarrow\int_{-\infty}^{\infty}\frac{\sin^2 w}{w^2}dw=\pi$$

Hence, replacing w by x in this result, we get $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi$.

Furthermore, since both $\sin^2 x$ and x^2 are even, the question $\frac{\sin^2 x}{x^2}$ is even.

Thus, by integral property of even function, $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=2\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx$.

So, $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi\Rightarrow 2\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi\Rightarrow\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{\pi}{2}$.

2. Find the Fourier transform of

$$a)\ f(x)=\begin{cases} e^{-ax}, \text{ if } x>0 \\ 0, \text{ if } x<0 \end{cases} \qquad b)\ f(x)=\begin{cases} e^{2x}, \text{ if } x>0 \\ e^{-2x}, \text{ if } x<0 \end{cases}.$$

Solution:

$$a)\ F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} f(x)e^{-iwx}dx+\int_{0}^{\infty} f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax}.e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+iw)x}dx=\frac{1}{\sqrt{2\pi}}.\frac{1}{(a+iw)}$$

$$b)\ F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} f(x)e^{-iwx}dx+\int_{0}^{\infty} f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-(2+iw)x}dx+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(2-iw)x}dx$$

$$=\frac{1}{\sqrt{2\pi}}.\frac{1}{(2+iw)}+\frac{1}{\sqrt{2\pi}}.\frac{1}{(2-iw)}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2+iw}+\frac{1}{2-iw}\right)=\frac{4}{\sqrt{2\pi}(4+w^2)}$$

3. Find the Fourier transform of $f(x)=\begin{cases} x^2e^{-x}, \text{ if } x>0 \\ 0, \text{ if } x<0 \end{cases}$.

Solution: Using integration by parts twice on $\int_{0}^{\infty} x^2e^{-(1+iw)x}dx$, we have

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} f(x)e^{-iwx}dx+\int_{0}^{\infty} f(x)e^{-iwx}dx\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-x}.e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-(1+iw)x}dx$$

$$=\frac{1}{\sqrt{2\pi}}.\frac{2}{(1+iw)^3}=\sqrt{\frac{2}{\pi}}.\frac{1}{(1+iw)^3}$$

4.2.4 Fourier Sine and Cosine Transforms

Fourier Transform of odd and even functions:

Consider the trigonometric form of the Fourier transform representation formula and expand it. That is

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)[\cos wx - i\sin wx]dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos wx dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin wx dx$$

Recall that is cosine function even and sine function is odd.

When f is **EVEN**, the product $f(x)\cos wx$ is even but $f(x)\sin wx$ is odd.

This gives $\int_{-\infty}^{\infty} f(x)\cos wx dx = 2\int_{0}^{\infty} f(x)\cos wx dx$ and $\int_{-\infty}^{\infty} f(x)\sin wx dx = 0$.

Therefore, the above transform formula is reduced as

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos wx dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin wx dx\pi$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)\cos wx dx = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos wx dx$$

When f is **ODD**, the product $f(x)\cos wx$ is odd but $f(x)\sin wx$ is even.

This gives $\int_{-\infty}^{\infty} f(x)\cos wx dx = 0$ and $\int_{-\infty}^{\infty} f(x)\sin wx dx = 2\int_{0}^{\infty} f(x)\sin wx dx$.

Again, the above transform formula is reduced to

$$F(w)=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)\sin wx dx = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin wx dx$$

Fourier Sine and Cosine Transforms:

From the discussion of Fourier Transforms of even and odd functions, we can summarize the results for Fourier sine and cosine transforms. But to talk about sine and cosine transforms from the transforms of even and odd functions, we have to restrict the domain of definition to be $(0,\infty)$ rather than $(-\infty,\infty)$. So, for a function on $(0,\infty)$, its Fourier sine and cosine transforms are as follow:

Fourier cosine transform: $F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx$.

Fourier sine transform: $F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx dx$.

Important relation between Fourier Integrals and Transforms:

It is crucially advisable to readers to understand the following important relations between Fourier sine and cosine Integrals with Fourier sine and cosine transforms. Because in many problematic situations, students are challenged to determine Fourier Integrals from the given Fourier transforms or Fourier transforms from the given Fourier integrals.

Fourier cosine transform: $F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx$.

Fourier cosine integral: $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(w)\cos \omega x d\omega$.

The two formula are said to be cosine transform of each other. If we are given $F_c(w)$, we can determine $f(x)$ from the cosine integral formula or vice-versa.

Fourier sine transform: $F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx dx$.

Fourier sine integral: $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(w)\sin \omega x d\omega$.

The two formula are said to be sine transform of each other. If we are given $F_s(w)$, we can determine $f(x)$ from the sine integral formula or vice-versa.

Examples:

1. Find the Fourier cosine and sine transforms of $f(x)=\begin{cases}1, \text{ if } 0<x<1\\ 0, \text{ if } x>1\end{cases}$

Using the result, show that $\int_0^\infty \frac{\sin^4 x}{x^2}dx=\frac{\pi}{2}$ & $\int_0^\infty \frac{(1-\cos x)^2}{x^2}dx=\frac{\pi}{2}$.

Solution:

Fourier cosine transform:

$$F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx=\sqrt{\frac{2}{\pi}}\int_0^1 f(x)\cos wx dx+\sqrt{\frac{2}{\pi}}\int_1^\infty f(x)\cos wx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 \cos wx dx=\sqrt{\frac{2}{\pi}}\frac{\sin \omega x}{\omega}\Big|_{x=0}^{x=1}=\sqrt{\frac{2}{\pi}}.\frac{\sin\omega}{\omega}$$

Fourier sine transform:

$$F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx dx=\sqrt{\frac{2}{\pi}}\int_0^1 f(x)\sin wx dx+\sqrt{\frac{2}{\pi}}\int_1^\infty f(x)\sin wx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 \sin wx dx=\sqrt{\frac{2}{\pi}}(-\frac{\cos \omega x}{\omega})\Big|_{x=0}^{x=1}=\sqrt{\frac{2}{\pi}}.\frac{(1-\cos\omega)}{\omega}$$

2. Find the Fourier sine and cosine transforms of $f(x)=e^{-x}, x\geq 0$.

i) Using the resulting transform, deduce that $\int_0^\infty \frac{x\sin ax}{x^2+1}dx=\frac{\pi}{2}e^{-a}, a>0$.

ii) At $x=0$, the representation $e^{-x}=\frac{2}{\pi}\int_0^\infty \frac{\omega \sin wx}{1+\omega^2}dw \Rightarrow 1=0$. Why it is wrong?

iii) Using the transform, deduce that $\int_0^\infty \frac{\cos \omega x}{\omega^2+1}d\omega=\frac{\pi}{2}e^{-x}, x\geq 0$.

Solution:

Fourier sine transform:

$$F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin wx dx$$

$$=(\sqrt{\frac{2}{\pi}})\frac{e^{-x}}{1+\omega^2}[\sin\omega x-\omega\cos\omega x]\Big|_{x=0}^{\infty}=\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{1+\omega^2}\right)$$

Fourier cosine transform:

$$F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx dx=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos wx dx$$

$$=(\sqrt{\frac{2}{\pi}})\frac{e^{-x}}{1+\omega^2}[-\cos \omega x+\sin \omega x\Big]_{x=0}^{\infty}=\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\omega^2}\right)$$

Now, apply the dual formula that relates integrals and transforms.

i) Use Fourier sine integral and sine transform.

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(w)\sin wx dw=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{1+\omega^2}\right)\sin wx dw=\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin wx}{1+\omega^2}dw$$

Hence, we get the representation $f(x)=e^{-x}=\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin wx}{1+\omega^2}dw$.

Then, using $x=a$ in $f(x)=e^{-x}$ and its Fourier integral representation, we have

$$f(x)=e^{-x}\Rightarrow\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin a\omega}{1+\omega^2}dw=e^{-a}\Rightarrow\int_0^{\infty}\frac{\omega\sin a\omega}{1+\omega^2}dw=\frac{\pi}{2}e^{-a}.$$

Fourier cosine integral: $f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f_c(w)\cos \omega x d\omega$.

iii) Use Fourier sine integral and sine transform.

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(w)\cos \omega x d\omega=\frac{2}{\pi}\int_0^{\infty}\left(\frac{1}{1+\omega^2}\right)\cos \omega x d\omega$$

Then, equating $f(x)=e^{-x}$ with its Fourier integral representation, we have

$$f(x)=e^{-x}\Rightarrow\frac{2}{\pi}\int_0^{\infty}\left(\frac{1}{1+\omega^2}\right)\cos \omega x d\omega=e^{-x}\Rightarrow\int_0^{\infty}\frac{\cos \omega x}{1+\omega^2}dw=\frac{\pi}{2}e^{-x},x\geq 0.$$

3. Find the Fourier transform of $f(x)=e^{-|x|}$.

Solution: From the trigonometric form of the definition, we have

$$F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos wx dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin wx dx.$$

Since $f(x)=e^{-|x|}$ is even function, its Fourier transform is the same as its cosine transform. That is $F(w)=F_c(w)$.

So, using the Fourier cosine transform formula, we have

$$F(w)=F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\cos wx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos wx\,dx \qquad (e^{-|x|}=e^{-x} \text{ for } x>0)$$

$$=\sqrt{\frac{2}{\pi}}\frac{e^{-x}}{1+\omega^2}\left[-\cos\omega x+\omega\sin\omega x\right]\Big|_{x=0}^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$$

4. Find the Fourier cosine and sine transforms of $f(x)=\begin{cases} x\text{, if } 0<x<1 \\ 0\text{, otherwise}\end{cases}$

Solution:

Fourier cosine transform:

$$F_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx=\sqrt{\frac{2}{\pi}}\int_0^1 f(x)\cos wx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 x\cos wx\,dx=\sqrt{\frac{2}{\pi}}(\omega x\sin\omega x+\cos\omega x)\Big|_{x=0}^{x=1}$$

$$=\sqrt{\frac{2}{\pi}}.\frac{\omega\sin\omega+\cos\omega-1}{\omega^2}$$