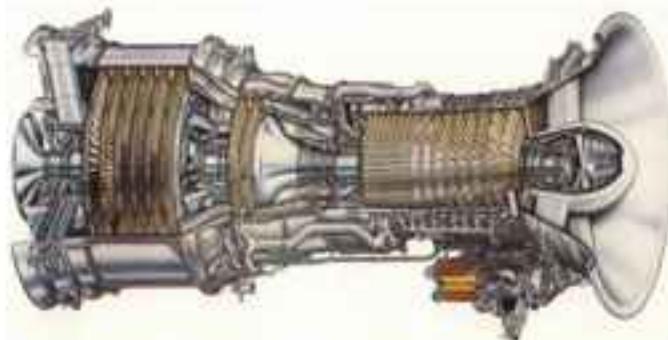


REVISED NINTH EDITION

A Textbook of

**FLUID MECHANICS  
AND  
HYDRAULIC MACHINES**

S.I. Units



Dr. R.K. Bansal

**A TEXTBOOK OF FLUID MECHANICS  
AND  
HYDRAULIC MACHINES**

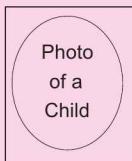
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A TEXTBOOK OF  
**FLUID MECHANICS**  
AND  
**HYDRAULIC MACHINES**

(In S.I. Units)

[For Degree, U.P.S.C. (Engg. Services), A.M.I.E. (India)]

*By*  
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*Dedicated  
to  
The loving memory  
of  
my daughter, Babli*



## PREFACE TO THE NINTH EDITION

---

The popularity of the eighth edition and reprints of the book **A Textbook of Fluid Mechanics and Hydraulic Machines** amongst the students and the teachers of the various Universities of the country, has prompted the bringing out of the ninth edition of the book so soon. The ninth edition has been thoroughly revised and brought up-to-date. A large number of problems from different B.E. degree examinations of Indian Universities and other examining bodies such as Institution of Engineers and U.P.S.C. upto Summer 2002 examinations have been selected and have been solved at proper places on this edition. Most of these problems have been worked out in S.I. units. All of the text along with existing problems have been converted into S.I. Units.

In the ninth edition, a new chapter entitled Ideal Flow (or Potential Flow) has been added. Potential flow has been included in most of Indian Universities. This chapter has been written in a simple and easy-to-follow language so that even an average student can grasp the subject matter by self-study. Also a few new topics such as "Liquids in Relative Equilibrium" and "Pipe Network" have been added in this edition. The topic of Pipe Network has been included in the chapter of Flow Through Pipes. The pipe network is mostly used in city water supply system, Laboratory supply system or house hold supply of water and gas.

The objective type multiple-choice questions are often asked in the various competitive examinations. Hence a large number of objective type questions with answers have been added in the end of the book.

With these additions, it is hoped that the book will be quite useful for the students of different branches of Engineering at various Engineering Institutions.

I express my sincere thanks to my colleagues, friends, students and the teachers of different Indian Universities for their valuable suggestions and recommending the book of their students.

Suggestions for the improvement of this book are most welcome and would be incorporated in the next edition with a view to make the book more useful.

- Author

## PREFACE TO THE FIRST EDITION

---

I am glad to present the book entitled, **A Textbook of Fluid Mechanics and Hydraulic Machines** to the engineering students of mechanical, civil, electrical, aeronautical and chemical and also to the students preparing for the new scheme of Section B of A.M.I.E. Examination of Institution of Engineers (India). The course contents have been planned in such a way that the general requirements of all engineering students are fulfilled.

During my long experience of teaching this subject to undergraduate and post-graduate engineering students for the past 16 years, I have observed that the students face difficulty in understanding clearly the basic principles, fundamental concepts and theory without adequate solved problems along with the text. To meet this very basic requirement to the students, a large number of the questions taken from the examinations of the various Universities of India and from other professional and competitive examinations (such as Institution of Engineers and U.P.S.C. Engineering Service Examination) have been solved along with the text in M.K.S. and S.I. units.

The book is written in a simple and easy-to-follow language, so that even an average students can grasp the subject by self-study. At the end of each chapter highlights, theoretical questions and many unsolved numerical problems with answer are given for the students to solve them.

I am thankful to my colleagues, friends and students who encouraged me to write this book. I am grateful to Institution of Engineers (India), various Universities of India and those authorities whose work have been consulted and gave me a great help in preparing the book.

I express my appreciation and gratefulness to my publisher, Shri R.K. Gupta (as Mechanical Engineer) for his most co-operative, painstaking attitude and untiring efforts for bringing out the book in a short period.

Mrs. Nirmal Bansal deserves special credit as she not only provided an ideal atmosphere at home for book writing but also gave inspiration and valuable suggestions.

Though every care has been taken in checking the manuscripts and proof reading, yet claiming perfection is very difficult. I shall be very grateful to the readers and users of this book for pointing any mistakes that might have crept in. Suggestions for improvement are most welcome and would be incorporated in the next edition with a view to make the book more useful.

- Author

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# 1

## CHAPTER

# PROPERTIES OF FLUIDS



### ► 1.1 INTRODUCTION

Fluid mechanics is that branch of science which deals with the behaviour of the fluids (liquids or gases) at rest as well as in motion. Thus this branch of science deals with the static, kinematics and dynamic aspects of fluids. The study of fluids at rest is called fluid statics. The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics and if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics.

### ► 1.2 PROPERTIES OF FLUIDS

**1.2.1 Density or Mass Density.** Density or mass density of a fluid is defined as the ratio of the mass of a fluid to its volume. Thus mass per unit volume of a fluid is called density. It is denoted by the symbol  $\rho$  (rho). The unit of mass density in SI unit is kg per cubic metre, i.e.,  $\text{kg/m}^3$ . The density of liquids may be considered as constant while that of gases changes with the variation of pressure and temperature.

Mathematically, mass density is written as

$$\rho = \frac{\text{Mass of fluid}}{\text{Volume of fluid}}.$$

The value of density of water is  $1 \text{ gm/cm}^3$  or  $1000 \text{ kg/m}^3$ .

**1.2.2 Specific Weight or Weight Density.** Specific weight or weight density of a fluid is the ratio between the weight of a fluid to its volume. Thus weight per unit volume of a fluid is called weight density and it is denoted by the symbol  $w$ .

$$\begin{aligned}\text{Thus mathematically, } w &= \frac{\text{Weight of fluid}}{\text{Volume of fluid}} = \frac{(\text{Mass of fluid}) \times \text{Acceleration due to gravity}}{\text{Volume of fluid}} \\ &= \frac{\text{Mass of fluid} \times g}{\text{Volume of fluid}} \\ &= \rho \times g \qquad \left\{ \because \frac{\text{Mass of fluid}}{\text{Volume of fluid}} = \rho \right\} \\ \therefore w &= \rho g \end{aligned} \quad \dots(1.1)$$

## 2 Fluid Mechanics

The value of specific weight or weight density ( $w$ ) for water is  $9.81 \times 1000$  Newton/m<sup>3</sup> in SI units.

**1.2.3 Specific Volume.** Specific volume of a fluid is defined as the volume of a fluid occupied by a unit mass or volume per unit mass of a fluid is called specific volume. Mathematically, it is expressed as

$$\text{Specific volume} = \frac{\text{Volume of fluid}}{\text{Mass of fluid}} = \frac{1}{\frac{\text{Mass of fluid}}{\text{Volume of fluid}}} = \frac{1}{\rho}$$

Thus specific volume is the reciprocal of mass density. It is expressed as m<sup>3</sup>/kg. It is commonly applied to gases.

**1.2.4 Specific Gravity.** Specific gravity is defined as the ratio of the weight density (or density) of a fluid to the weight density (or density) of a standard fluid. For liquids, the standard fluid is taken water and for gases, the standard fluid is taken air. Specific gravity is also called relative density. It is dimensionless quantity and is denoted by the symbol  $S$ .

$$\text{Mathematically, } S(\text{for liquids}) = \frac{\text{Weight density (density) of liquid}}{\text{Weight density (density) of water}}$$

$$S(\text{for gases}) = \frac{\text{Weight density (density) of gas}}{\text{Weight density (density) of air}}$$

$$\begin{aligned}\text{Thus weight density of a liquid} &= S \times \text{Weight density of water} \\ &= S \times 1000 \times 9.81 \text{ N/m}^3\end{aligned}$$

$$\begin{aligned}\text{The density of a liquid} &= S \times \text{Density of water} \\ &= S \times 1000 \text{ kg/m}^3.\end{aligned}\quad \dots(1.1A)$$

If the specific gravity of a fluid is known, then the density of the fluid will be equal to specific gravity of fluid multiplied by the density of water. For example, the specific gravity of mercury is 13.6, hence density of mercury =  $13.6 \times 1000 = 13600$  kg/m<sup>3</sup>.

**Problem 1.1** Calculate the specific weight, density and specific gravity of one litre of a liquid which weighs 7 N.

**Solution.** Given :

$$\text{Volume} = 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \quad \left( \because 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \text{ or } 1 \text{ litre} = 1000 \text{ cm}^3 \right)$$

$$\text{Weight} = 7 \text{ N}$$

$$(i) \text{ Specific weight } (w) = \frac{\text{Weight}}{\text{Volume}} = \frac{7 \text{ N}}{\left(\frac{1}{1000}\right) \text{ m}^3} = 7000 \text{ N/m}^3. \text{ Ans.}$$

$$(ii) \text{ Density } (\rho) = \frac{w}{g} = \frac{7000}{9.81} \text{ kg/m}^3 = 713.5 \text{ kg/m}^3. \text{ Ans.}$$

$$(iii) \text{ Specific gravity} = \frac{\text{Density of liquid}}{\text{Density of water}} = \frac{713.5}{1000} \quad \{ \because \text{Density of water} = 1000 \text{ kg/m}^3 \}$$
$$= 0.7135. \text{ Ans.}$$

**Problem 1.2** Calculate the density, specific weight and weight of one litre of petrol of specific gravity = 0.7

**Solution.** Given : Volume = 1 litre =  $1 \times 1000 \text{ cm}^3 = \frac{1000}{10^6} \text{ m}^3 = 0.001 \text{ m}^3$

$$\text{Sp. gravity} \quad S = 0.7$$

(i) Density ( $\rho$ )

Using equation (1.1A),

$$\text{Density } (\rho) = S \times 1000 \text{ kg/m}^3 = 0.7 \times 1000 = 700 \text{ kg/m}^3. \text{ Ans.}$$

(ii) Specific weight ( $w$ )

$$\text{Using equation (1.1), } w = \rho \times g = 700 \times 9.81 \text{ N/m}^3 = 6867 \text{ N/m}^3. \text{ Ans.}$$

(iii) Weight ( $W$ )

$$\text{We know that specific weight} = \frac{\text{Weight}}{\text{Volume}}$$

$$\text{or } w = \frac{W}{0.001} \text{ or } 6867 = \frac{W}{0.001}$$

$$\therefore W = 6867 \times 0.001 = 6.867 \text{ N. Ans.}$$

### ► 1.3 VISCOSITY

Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. When two layers of a fluid, a distance ' $dy$ ' apart, move one over the other at different velocities, say  $u$  and  $u + du$  as shown in Fig. 1.1, the viscosity together with relative velocity causes a shear stress acting between the fluid layers.

The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer. This shear stress is proportional to the rate of change of velocity with respect to  $y$ . It is denoted by symbol  $\tau$  (Tau).

$$\text{Mathematically, } \tau \propto \frac{du}{dy}$$

$$\text{or } \tau = \mu \frac{du}{dy}$$

where  $\mu$  (called  $\mu u$ ) is the constant of proportionality and is known as the co-efficient of dynamic viscosity or only viscosity.  $\frac{du}{dy}$  represents the rate of shear strain or rate of shear deformation or velocity gradient.

$$\text{From equation (1.2), we have } \mu = \frac{\tau}{\left( \frac{du}{dy} \right)} \quad \dots(1.3)$$

Thus viscosity is also defined as the shear stress required to produce unit rate of shear strain.

**1.3.1 Units of Viscosity.** The units of viscosity is obtained by putting the dimensions of the quantities in equation (1.3)

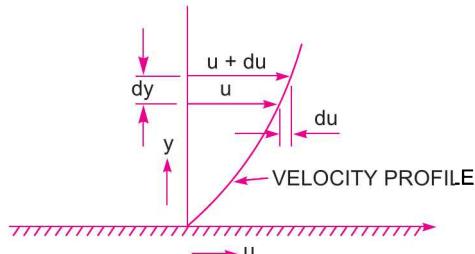


Fig. 1.1 Velocity variation near a solid boundary. ...(1.2)

## 4 Fluid Mechanics

$$\begin{aligned}\mu &= \frac{\text{Shear stress}}{\frac{\text{Change of velocity}}{\text{Change of distance}}} = \frac{\text{Force/Area}}{\left(\frac{\text{Length}}{\text{Time}}\right) \times \frac{1}{\text{Length}}} \\ &= \frac{\text{Force}/(\text{Length})^2}{\frac{1}{\text{Time}}} = \frac{\text{Force} \times \text{Time}}{(\text{Length})^2}\end{aligned}$$

In MKS system, force is represented by kgf and length by metre (m), in CGS system, force is represented by dyne and length by cm and in SI system force is represented by Newton (N) and length by metre (m).

$$\therefore \text{MKS unit of viscosity} = \frac{\text{kgf}\cdot\text{sec}}{\text{m}^2}$$

$$\text{CGS unit of viscosity} = \frac{\text{dyne}\cdot\text{sec}}{\text{cm}^2}$$

In the above expression  $\text{N/m}^2$  is also known as Pascal which is represented by Pa. Hence  $\text{N/m}^2 = \text{Pa}$   
= Pascal

$$\therefore \text{SI unit of viscosity} = \text{Ns/m}^2 = \text{Pa s.}$$

$$\text{SI unit of viscosity} = \frac{\text{Newton}\cdot\text{sec}}{\text{m}^2} = \frac{\text{Ns}}{\text{m}^2}$$

The unit of viscosity in CGS is also called Poise which is equal to  $\frac{\text{dyne}\cdot\text{sec}}{\text{cm}^2}$ .

The numerical conversion of the unit of viscosity from MKS unit to CGS unit is given below :

$$\frac{\text{one kgf}\cdot\text{sec}}{\text{m}^2} = \frac{9.81 \text{ N}\cdot\text{sec}}{\text{m}^2} \quad \{ \because 1 \text{ kgf} = 9.81 \text{ Newton} \}$$

$$\text{But one Newton} = \text{one kg (mass)} \times \text{one} \left( \frac{\text{m}}{\text{sec}^2} \right) \text{ (acceleration)}$$

$$\begin{aligned}&= \frac{(1000 \text{ gm}) \times (100 \text{ cm})}{\text{sec}^2} = 1000 \times 100 \frac{\text{gm}\cdot\text{cm}}{\text{sec}^2} \\ &= 1000 \times 100 \text{ dyne} \quad \left\{ \because \text{dyne} = \text{gm} \times \frac{\text{cm}}{\text{sec}^2} \right\}\end{aligned}$$

$$\begin{aligned}\therefore \frac{\text{one kgf}\cdot\text{sec}}{\text{m}^2} &= 9.81 \times 100000 \frac{\text{dyne}\cdot\text{sec}}{\text{cm}^2} = 9.81 \times 100000 \frac{\text{dyne}\cdot\text{sec}}{100 \times 100 \times \text{cm}^2} \\ &= 98.1 \frac{\text{dyne}\cdot\text{sec}}{\text{cm}^2} = 98.1 \text{ poise} \quad \left\{ \because \frac{\text{dyne}\cdot\text{sec}}{\text{cm}^2} = \text{Poise} \right\}\end{aligned}$$

Thus for solving numerical problems, if viscosity is given in poise, it must be divided by 98.1 to get its equivalent numerical value in MKS.

$$\text{But} \quad \frac{\text{one kgf}\cdot\text{sec}}{\text{m}^2} = \frac{9.81 \text{ Ns}}{\text{m}^2} = 98.1 \text{ poise}$$

$$\therefore \frac{\text{one Ns}}{\text{m}^2} = \frac{98.1}{9.81} \text{ poise} = 10 \text{ poise} \quad \text{or} \quad \text{One poise} = \frac{1}{10} \frac{\text{Ns}}{\text{m}^2}.$$

$$\text{Alternate Method. One poise} = \frac{\text{dyne} \times \text{s}}{\text{cm}^2} = \left( \frac{1 \text{ gm} \times 1 \text{ cm}}{\text{s}^2} \right) \times \frac{\text{s}}{\text{cm}^2}$$

$$\text{But dyne} = 1 \text{ gm} \times \frac{1 \text{ cm}}{\text{s}^2}$$

$$\therefore \text{One poise} = \frac{1 \text{ gm}}{\text{s cm}} = \frac{1}{1000} \frac{\text{kg}}{\text{s} \frac{1}{100} \text{ m}}$$

$$= \frac{1}{1000} \times 100 \frac{\text{kg}}{\text{sm}} = \frac{1}{10} \frac{\text{kg}}{\text{sm}} \quad \text{or} \quad 1 \frac{\text{kg}}{\text{sm}} = 10 \text{ poise.}$$

**Note.** (i) In SI units second is represented by 's' and not by 'sec'.

(ii) If viscosity is given in poise, it must be divided by 10 to get its equivalent numerical value in SI units.

Sometimes a unit of viscosity as centipoise is used where

$$1 \text{ centipoise} = \frac{1}{100} \text{ poise} \quad \text{or} \quad 1 \text{ cP} = \frac{1}{100} \text{ P} \quad [\text{cP} = \text{Centipoise}, \text{P} = \text{Poise}]$$

The viscosity of water at 20°C is 0.01 poise or 1.0 centipoise.

**1.3.2 Kinematic Viscosity.** It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by the Greek symbol ( $\nu$ ) called 'nu'. Thus, mathematically,

$$\nu = \frac{\text{Viscosity}}{\text{Density}} = \frac{\mu}{\rho} \quad \dots(1.4)$$

The units of kinematic viscosity is obtained as

$$\begin{aligned} \nu &= \frac{\text{Units of } \mu}{\text{Units of } \rho} = \frac{\text{Force} \times \text{Time}}{(\text{Length})^2 \times \frac{\text{Mass}}{(\text{Length})^3}} = \frac{\text{Force} \times \text{Time}}{\frac{\text{Mass}}{\text{Length}}} \\ &= \frac{\text{Mass} \times \frac{\text{Length}}{(\text{Time})^2} \times \text{Time}}{\left( \frac{\text{Mass}}{\text{Length}} \right)} \quad \left\{ \begin{array}{l} \text{Force} = \text{Mass} \times \text{Acc.} \\ = \text{Mass} \times \frac{\text{Length}}{\text{Time}^2} \end{array} \right. \\ &= \frac{(\text{Length})^2}{\text{Time}}. \end{aligned}$$

In MKS and SI, the unit of kinematic viscosity is metre<sup>2</sup>/sec or m<sup>2</sup>/sec while in CGS units it is written as cm<sup>2</sup>/s. In CGS units, kinematic viscosity is also known as stoke.

$$\text{Thus, one stoke} = \text{cm}^2/\text{s} = \left( \frac{1}{100} \right)^2 \text{ m}^2/\text{s} = 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Centistoke means} = \frac{1}{100} \text{ stoke.}$$

**1.3.3 Newton's Law of Viscosity.** It states that the shear stress ( $\tau$ ) on a fluid element layer is directly proportional to the rate of shear strain. The constant of proportionality is called the coefficient of viscosity. Mathematically, it is expressed as given by equation (1.2) or as

$$\tau = \mu \frac{du}{dy}.$$

## 6 Fluid Mechanics

Fluids which obey the above relation are known as **Newtonian** fluids and the fluids which do not obey the above relation are called **Non-Newtonian** fluids.

**1.3.4 Variation of Viscosity with Temperature.** Temperature affects the viscosity. The viscosity of liquids decreases with the increase of temperature while the viscosity of gases increases with the increase of temperature. This is due to reason that the viscous forces in a fluid are due to cohesive forces and molecular momentum transfer. In liquids, the cohesive forces predominates the molecular momentum transfer, due to closely packed molecules and with the increase in temperature, the cohesive forces decreases with the result of decreasing viscosity. But in case of gases the cohesive forces are small and molecular momentum transfer predominates. With the increase in temperature, molecular momentum transfer increases and hence viscosity increases. The relation between viscosity and temperature for liquids and gases are:

$$(i) \text{ For liquids, } \mu = \mu_0 \left( \frac{1}{1 + \alpha t + \beta t^2} \right) \quad \dots(1.4A)$$

where  $\mu$  = Viscosity of liquid at  $t^\circ\text{C}$ , in poise

$\mu_0$  = Viscosity of liquid at  $0^\circ\text{C}$ , in poise

$\alpha, \beta$  = Constants for the liquid

For water,  $\mu_0 = 1.79 \times 10^{-3}$  poise,  $\alpha = 0.03368$  and  $\beta = 0.000221$ .

Equation (1.4A) shows that with the increase of temperature, the viscosity decreases.

$$(ii) \text{ For a gas, } \mu = \mu_0 + \alpha t - \beta t^2 \quad \dots(1.4B)$$

where for air  $\mu_0 = 0.000017$ ,  $\alpha = 0.000000056$ ,  $\beta = 0.1189 \times 10^{-9}$ .

Equation (1.4B) shows that with the increase of temperature, the viscosity increases.

**1.3.5 Types of Fluids.** The fluids may be classified into the following five types :

- |                         |                             |
|-------------------------|-----------------------------|
| 1. Ideal fluid,         | 2. Real fluid,              |
| 3. Newtonian fluid,     | 4. Non-Newtonian fluid, and |
| 5. Ideal plastic fluid. |                             |

**1. Ideal Fluid.** A fluid, which is incompressible and is having no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity.

**2. Real Fluid.** A fluid, which possesses viscosity, is known as real fluid. All the fluids, in actual practice, are real fluids.

**3. Newtonian Fluid.** A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid.

**4. Non-Newtonian Fluid.** A real fluid, in which the shear stress is not proportional to the rate of shear strain (or velocity gradient), known as a Non-Newtonian fluid.

**5. Ideal Plastic Fluid.** A fluid, in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or velocity gradient), is known as ideal plastic fluid.

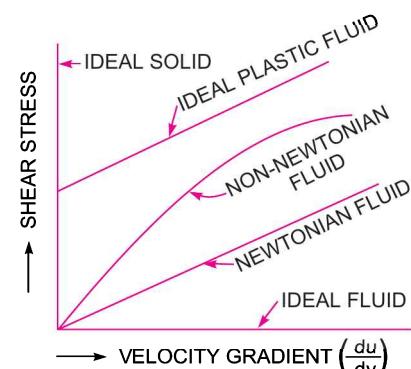


Fig. 1.2 Types of fluids.

**Problem 1.3** If the velocity distribution over a plate is given by  $u = \frac{2}{3} y - y^2$  in which  $u$  is the velocity in metre per second at a distance  $y$  metre above the plate, determine the shear stress at  $y = 0$  and  $y = 0.15$  m. Take dynamic viscosity of fluid as 8.63 poises.

**Solution.** Given :  $u = \frac{2}{3}y - y^2$   $\therefore \frac{du}{dy} = \frac{2}{3} - 2y$

$$\left( \frac{du}{dy} \right)_{\text{at } y=0} \text{ or } \left( \frac{du}{dy} \right)_{y=0} = \frac{2}{3} - 2(0) = \frac{2}{3} = 0.667$$

Also  $\left( \frac{du}{dy} \right)_{\text{at } y=0.15} \text{ or } \left( \frac{du}{dy} \right)_{y=0.15} = \frac{2}{3} - 2 \times .15 = .667 - .30 = 0.367$

Value of  $\mu = 8.63$  poise  $= \frac{8.63}{10}$  SI units  $= 0.863 \text{ N s/m}^2$

Now shear stress is given by equation (1.2) as  $\tau = \mu \frac{du}{dy}$ .

(i) Shear stress at  $y = 0$  is given by

$$\tau_0 = \mu \left( \frac{du}{dy} \right)_{y=0} = 0.863 \times 0.667 = 0.5756 \text{ N/m}^2. \text{ Ans.}$$

(ii) Shear stress at  $y = 0.15$  m is given by

$$(\tau)_{y=0.15} = \mu \left( \frac{du}{dy} \right)_{y=0.15} = 0.863 \times 0.367 = 0.3167 \text{ N/m}^2. \text{ Ans.}$$

**Problem 1.4** A plate 0.025 mm distant from a fixed plate, moves at 60 cm/s and requires a force of 2 N per unit area i.e.,  $2 \text{ N/m}^2$  to maintain this speed. Determine the fluid viscosity between the plates.

**Solution.** Given :

Distance between plates,  $dy = .025 \text{ mm}$   
 $= .025 \times 10^{-3} \text{ m}$

Velocity of upper plate,  $u = 60 \text{ cm/s} = 0.6 \text{ m/s}$

Force on upper plate,  $F = 2.0 \frac{\text{N}}{\text{m}^2}$ .

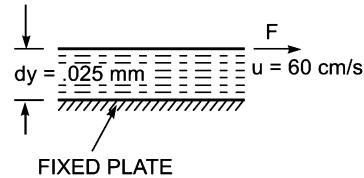


Fig. 1.3

This is the value of shear stress i.e.,  $\tau$

Let the fluid viscosity between the plates is  $\mu$ .

Using the equation (1.2), we have  $\tau = \mu \frac{du}{dy}$ .

where  $du = \text{Change of velocity} = u - 0 = u = 0.60 \text{ m/s}$

$dy = \text{Change of distance} = .025 \times 10^{-3} \text{ m}$

$$\tau = \text{Force per unit area} = 2.0 \frac{\text{N}}{\text{m}^2}$$

$$\therefore 2.0 = \mu \frac{0.60}{.025 \times 10^{-3}} \quad \therefore \mu = \frac{2.0 \times .025 \times 10^{-3}}{0.60} = 8.33 \times 10^{-5} \frac{\text{Ns}}{\text{m}^2}$$

$$= 8.33 \times 10^{-5} \times 10 \text{ poise} = 8.33 \times 10^{-4} \text{ poise. Ans.}$$

**Problem 1.5** A flat plate of area  $1.5 \times 10^6 \text{ mm}^2$  is pulled with a speed of 0.4 m/s relative to another plate located at a distance of 0.15 mm from it. Find the force and power required to maintain this speed, if the fluid separating them is having viscosity as 1 poise.

## 8 Fluid Mechanics

**Solution.** Given :

$$\text{Area of the plate, } A = 1.5 \times 10^6 \text{ mm}^2 = 1.5 \text{ m}^2$$

Speed of plate relative to another plate,  $du = 0.4 \text{ m/s}$

Distance between the plates,  $dy = 0.15 \text{ mm} = 0.15 \times 10^{-3} \text{ m}$

$$\text{Viscosity } \mu = 1 \text{ poise} = \frac{1}{10} \frac{\text{Ns}}{\text{m}^2}.$$

$$\text{Using equation (1.2) we have } \tau = \mu \frac{du}{dy} = \frac{1}{10} \times \frac{0.4}{0.15 \times 10^{-3}} = 266.66 \frac{\text{N}}{\text{m}^2}$$

$$(i) \therefore \text{Shear force, } F = \tau \times \text{area} = 266.66 \times 1.5 = 400 \text{ N. Ans.}$$

$$(ii) \text{Power* required to move the plate at the speed } 0.4 \text{ m/sec}$$

$$= F \times u = 400 \times 0.4 = 160 \text{ W. Ans.}$$

**Problem 1.6** Determine the intensity of shear of an oil having viscosity = 1 poise. The oil is used for lubricating the clearance between a shaft of diameter 10 cm and its journal bearing. The clearance is 1.5 mm and the shaft rotates at 150 r.p.m.

$$\text{Solution. Given : } \mu = 1 \text{ poise} = \frac{1}{10} \frac{\text{Ns}}{\text{m}^2}$$

$$\text{Dia. of shaft, } D = 10 \text{ cm} = 0.1 \text{ m}$$

Distance between shaft and journal bearing,

$$dy = 1.5 \text{ mm} = 1.5 \times 10^{-3} \text{ m}$$

$$\text{Speed of shaft, } N = 150 \text{ r.p.m.}$$

Tangential speed of shaft is given by

$$u = \frac{\pi DN}{60} = \frac{\pi \times 0.1 \times 150}{60} = 0.785 \text{ m/s}$$

$$\text{Using equation (1.2), } \tau = \mu \frac{du}{dy},$$

where  $du = \text{change of velocity between shaft and bearing} = u - 0 = u$

$$= \frac{1}{10} \times \frac{0.785}{1.5 \times 10^{-3}} = 52.33 \text{ N/m}^2. \text{ Ans.}$$

**Problem 1.7** Calculate the dynamic viscosity of an oil, which is used for lubrication between a square plate of size  $0.8 \text{ m} \times 0.8 \text{ m}$  and an inclined plane with angle of inclination  $30^\circ$  as shown in Fig. 1.4. The weight of the square plate is 300 N and it slides down the inclined plane with a uniform velocity of  $0.3 \text{ m/s}$ . The thickness of oil film is 1.5 mm.

**Solution.** Given :

$$\text{Area of plate, } A = 0.8 \times 0.8 = 0.64 \text{ m}^2$$

$$\text{Angle of plane, } \theta = 30^\circ$$

$$\text{Weight of plate, } W = 300 \text{ N}$$

$$\text{Velocity of plate, } u = 0.3 \text{ m/s}$$

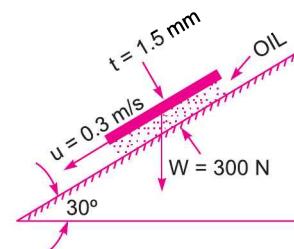


Fig. 1.4

\* Power =  $F \times u \text{ N m/s} = F \times u \text{ W} (\because \text{Nm/s} = \text{Watt})$

Thickness of oil film,  $t = dy = 1.5 \text{ mm} = 1.5 \times 10^{-3} \text{ m}$

Let the viscosity of fluid between plate and inclined plane is  $\mu$ .

Component of weight  $W$ , along the plane  $= W \cos 60^\circ = 300 \cos 60^\circ = 150 \text{ N}$

Thus the shear force,  $F$ , on the bottom surface of the plate  $= 150 \text{ N}$

$$\text{and shear stress, } \tau = \frac{F}{\text{Area}} = \frac{150}{0.64} \text{ N/m}^2$$

Now using equation (1.2), we have

$$\tau = \mu \frac{du}{dy}$$

where  $du = \text{change of velocity} = u - 0 = u = 0.3 \text{ m/s}$

$$dy = t = 1.5 \times 10^{-3} \text{ m}$$

$$\therefore \frac{150}{0.64} = \mu \frac{0.3}{1.5 \times 10^{-3}}$$

$$\therefore \mu = \frac{150 \times 1.5 \times 10^{-3}}{0.64 \times 0.3} = 1.17 \text{ N s/m}^2 = 1.17 \times 10 = 11.7 \text{ poise. Ans.}$$

**Problem 1.8** Two horizontal plates are placed 1.25 cm apart, the space between them being filled with oil of viscosity 14 poises. Calculate the shear stress in oil if upper plate is moved with a velocity of 2.5 m/s.

**Solution.** Given :

Distance between plates,  $dy = 1.25 \text{ cm} = 0.0125 \text{ m}$

$$\text{Viscosity, } \mu = 14 \text{ poise} = \frac{14}{10} \text{ N s/m}^2$$

Velocity of upper plate,  $u = 2.5 \text{ m/sec.}$

$$\text{Shear stress is given by equation (1.2) as, } \tau = \mu \frac{du}{dy}$$

where  $du = \text{Change of velocity between plates} = u - 0 = u = 2.5 \text{ m/sec.}$

$$dy = 0.0125 \text{ m.}$$

$$\therefore \tau = \frac{14}{10} \times \frac{2.5}{0.0125} = 280 \text{ N/m}^2. \text{ Ans.}$$

**Problem 1.9** The space between two square flat parallel plates is filled with oil. Each side of the plate is 60 cm. The thickness of the oil film is 12.5 mm. The upper plate, which moves at 2.5 metre per sec requires a force of 98.1 N to maintain the speed. Determine :

- (i) the dynamic viscosity of the oil in poise, and
- (ii) the kinematic viscosity of the oil in stokes if the specific gravity of the oil is 0.95.

**Solution.** Given :

Each side of a square plate  $= 60 \text{ cm} = 0.60 \text{ m}$

$$\therefore \text{Area, } A = 0.6 \times 0.6 = 0.36 \text{ m}^2$$

Thickness of oil film,  $dy = 12.5 \text{ mm} = 12.5 \times 10^{-3} \text{ m}$

Velocity of upper plate,  $u = 2.5 \text{ m/sec}$

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∴ Change of velocity between plates,  $du = 2.5 \text{ m/sec}$

Force required on upper plate,  $F = 98.1 \text{ N}$

$$\therefore \text{Shear stress, } \tau = \frac{\text{Force}}{\text{Area}} = \frac{F}{A} = \frac{98.1 \text{ N}}{0.36 \text{ m}^2}$$

(i) Let  $\mu$  = Dynamic viscosity of oil

$$\text{Using equation (1.2), } \tau = \mu \frac{du}{dy} \text{ or } \frac{98.1}{0.36} = \mu \times \frac{2.5}{12.5 \times 10^{-3}}$$

$$\therefore \mu = \frac{98.1}{0.36} \times \frac{12.5 \times 10^{-3}}{2.5} = 1.3635 \frac{\text{Ns}}{\text{m}^2} \quad \left( \because \frac{1 \text{ Ns}}{\text{m}^2} = 10 \text{ poise} \right)$$

$$= 1.3635 \times 10 = \mathbf{13.635 \text{ poise. Ans.}}$$

(ii) Sp. gr. of oil,  $S = 0.95$

Let  $\nu$  = kinematic viscosity of oil

Using equation (1.1A),

$$\text{Mass density of oil, } \rho = S \times 1000 = 0.95 \times 1000 = 950 \text{ kg/m}^3$$

$$\text{Using the relation, } \nu = \frac{\mu}{\rho}, \text{ we get } \nu = \frac{1.3635 \left( \frac{\text{Ns}}{\text{m}^2} \right)}{950} = .001435 \text{ m}^2/\text{sec} = .001435 \times 10^4 \text{ cm}^2/\text{s}$$

$$= \mathbf{14.35 \text{ stokes. Ans.}} \quad (\because \text{cm}^2/\text{s} = \text{stoke})$$

**Problem 1.10** Find the kinematic viscosity of an oil having density  $981 \text{ kg/m}^3$ . The shear stress at a point in oil is  $0.2452 \text{ N/m}^2$  and velocity gradient at that point is  $0.2 \text{ per second}$ .

**Solution.** Given :

$$\text{Mass density, } \rho = 981 \text{ kg/m}^3$$

$$\text{Shear stress, } \tau = 0.2452 \text{ N/m}^2$$

$$\text{Velocity gradient, } \frac{du}{dy} = 0.2 \text{ s}$$

$$\text{Using the equation (1.2), } \tau = \mu \frac{du}{dy} \text{ or } 0.2452 = \mu \times 0.2$$

$$\therefore \mu = \frac{0.2452}{0.200} = 1.226 \text{ Ns/m}^2$$

Kinematic viscosity  $\nu$  is given by

$$\therefore \nu = \frac{\mu}{\rho} = \frac{1.226}{981} = .125 \times 10^{-2} \text{ m}^2/\text{sec}$$

$$= 0.125 \times 10^{-2} \times 10^4 \text{ cm}^2/\text{s} = 0.125 \times 10^2 \text{ cm}^2/\text{s}$$

$$= 12.5 \text{ cm}^2/\text{s} = \mathbf{12.5 \text{ stoke. Ans.}} \quad (\because \text{cm}^2/\text{s} = \text{stoke})$$

**Problem 1.11** Determine the specific gravity of a fluid having viscosity  $0.05 \text{ poise}$  and kinematic viscosity  $0.035 \text{ stokes}$ .

**Solution.** Given :

$$\text{Viscosity, } \mu = 0.05 \text{ poise} = \frac{0.05}{10} \text{ N s/m}^2$$

$$\begin{aligned} \text{Kinematic viscosity, } v &= 0.035 \text{ stokes} \\ &= 0.035 \text{ cm}^2/\text{s} \\ &= 0.035 \times 10^{-4} \text{ m}^2/\text{s} \end{aligned} \quad \{ \because \text{ Stoke} = \text{cm}^2/\text{s} \}$$

Using the relation  $v = \frac{\mu}{\rho}$ , we get  $0.035 \times 10^{-4} = \frac{0.05}{10} \times \frac{1}{\rho}$

$$\therefore \rho = \frac{0.05}{10} \times \frac{1}{0.035 \times 10^{-4}} = 1428.5 \text{ kg/m}^3$$

$$\therefore \text{Sp. gr. of liquid} = \frac{\text{Density of liquid}}{\text{Density of water}} = \frac{1428.5}{1000} = 1.4285 \approx 1.43. \text{ Ans.}$$

**Problem 1.12** Determine the viscosity of a liquid having kinematic viscosity 6 stokes and specific gravity 1.9.

**Solution.** Given :

$$\text{Kinematic viscosity } v = 6 \text{ stokes} = 6 \text{ cm}^2/\text{s} = 6 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Sp. gr. of liquid} = 1.9$$

$$\text{Let the viscosity of liquid} = \mu$$

$$\text{Now sp. gr. of a liquid} = \frac{\text{Density of the liquid}}{\text{Density of water}}$$

$$\text{or } 1.9 = \frac{\text{Density of liquid}}{1000}$$

$$\therefore \text{Density of liquid} = 1000 \times 1.9 = 1900 \frac{\text{kg}}{\text{m}^3}$$

$$\therefore \text{Using the relation } v = \frac{\mu}{\rho}, \text{ we get}$$

$$6 \times 10^{-4} = \frac{\mu}{1900}$$

$$\text{or } \mu = 6 \times 10^{-4} \times 1900 = 1.14 \text{ Ns/m}^2 \\ = 1.14 \times 10 = 11.40 \text{ poise. Ans.}$$

**Problem 1.13** The velocity distribution for flow over a flat plate is given by  $u = \frac{3}{4} y - y^2$  in which  $u$  is the velocity in metre per second at a distance  $y$  metre above the plate. Determine the shear stress at  $y = 0.15 \text{ m}$ . Take dynamic viscosity of fluid as 8.6 poise.

$$\text{Solution. Given : } u = \frac{3}{4} y - y^2$$

$$\therefore \frac{du}{dy} = \frac{3}{4} - 2y$$

$$\text{At } y = 0.15, \frac{du}{dy} = \frac{3}{4} - 2 \times 0.15 = 0.75 - 0.30 = 0.45$$

$$\text{Viscosity, } \mu = 8.6 \text{ poise} = \frac{8.6}{10} \frac{\text{Ns}}{\text{m}^2} \quad \left( \because 10 \text{ poise} = 1 \frac{\text{Ns}}{\text{m}^2} \right)$$

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Using equation (1.2),  $\tau = \mu \frac{du}{dy} = \frac{8.5}{10} \times 0.45 \frac{\text{N}}{\text{m}^2} = 0.3825 \frac{\text{N}}{\text{m}^2}$ . Ans.

**Problem 1.14** The dynamic viscosity of an oil, used for lubrication between a shaft and sleeve is 6 poise. The shaft is of diameter 0.4 m and rotates at 190 r.p.m. Calculate the power lost in the bearing for a sleeve length of 90 mm. The thickness of the oil film is 1.5 mm.

**Solution.** Given :

Viscosity

$$\mu = 6 \text{ poise}$$

$$= \frac{6}{10} \frac{\text{Ns}}{\text{m}^2} = 0.6 \frac{\text{Ns}}{\text{m}^2}$$

Dia. of shaft,

$$D = 0.4 \text{ m}$$

Speed of shaft,

$$N = 190 \text{ r.p.m}$$

Sleeve length,

$$L = 90 \text{ mm} = 90 \times 10^{-3} \text{ m}$$

Thickness of oil film,

$$t = 1.5 \text{ mm} = 1.5 \times 10^{-3} \text{ m}$$

$$\text{Tangential velocity of shaft, } u = \frac{\pi D N}{60} = \frac{\pi \times 0.4 \times 190}{60} = 3.98 \text{ m/s}$$

Using the relation

$$\tau = \mu \frac{du}{dy}$$

where  $du = \text{Change of velocity} = u - 0 = u = 3.98 \text{ m/s}$

$dy = \text{Change of distance} = t = 1.5 \times 10^{-3} \text{ m}$

$$\tau = 10 \times \frac{3.98}{1.5 \times 10^{-3}} = 1592 \text{ N/m}^2$$

This is shear stress on shaft

$\therefore$  Shear force on the shaft,  $F = \text{Shear stress} \times \text{Area}$

$$= 1592 \times \pi D \times L = 1592 \times \pi \times 0.4 \times 90 \times 10^{-3} = 180.05 \text{ N}$$

Torque on the shaft,

$$T = \text{Force} \times \frac{D}{2} = 180.05 \times \frac{0.4}{2} = 36.01 \text{ Nm}$$

$\therefore$  \*Power lost

$$= \frac{2\pi NT}{60} = \frac{2\pi \times 190 \times 36.01}{60} = 716.48 \text{ W. Ans.}$$

**Problem 1.15** If the velocity profile of a fluid over a plate is parabolic with the vertex 20 cm from the plate, where the velocity is 120 cm/sec. Calculate the velocity gradients and shear stresses at a distance of 0, 10 and 20 cm from the plate, if the viscosity of the fluid is 8.5 poise.

**Solution.** Given :

Distance of vertex from plate = 20 cm

Velocity at vertex,

$$u = 120 \text{ cm/sec}$$

Viscosity,

$$\mu = 8.5 \text{ poise} = \frac{8.5}{10} \frac{\text{Ns}}{\text{m}^2} = 0.85.$$

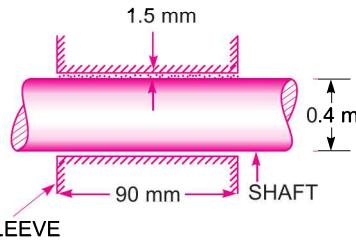


Fig. 1.5

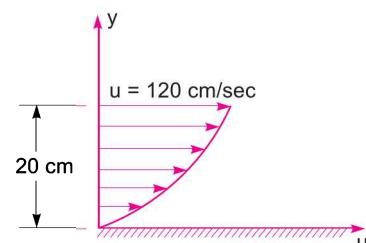


Fig. 1.6

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\* Power in S.I. unit =  $T \cdot \omega = T \times \frac{2\pi N}{60}$  Watt =  $\frac{2\pi NT}{60}$  Watt

The velocity profile is given parabolic and equation of velocity profile is

$$u = ay^2 + by + c \quad \dots(i)$$

where  $a$ ,  $b$  and  $c$  are constants. Their values are determined from boundary conditions as :

- (a) at  $y = 0$ ,  $u = 0$
- (b) at  $y = 20$  cm,  $u = 120$  cm/sec
- (c) at  $y = 20$  cm,  $\frac{du}{dy} = 0$ .

Substituting boundary condition (a) in equation (i), we get

$$c = 0.$$

Boundary condition (b) on substitution in (i) gives

$$120 = a(20)^2 + b(20) = 400a + 20b \quad \dots(ii)$$

Boundary condition (c) on substitution in equation (i) gives

$$\frac{du}{dy} = 2ay + b \quad \dots(iii)$$

or

$$0 = 2 \times a \times 20 + b = 40a + b$$

Solving equations (ii) and (iii) for  $a$  and  $b$

$$\text{From equation (iii), } b = -40a$$

Substituting this value in equation (ii), we get

$$120 = 400a + 20 \times (-40a) = 400a - 800a = -400a$$

$$\therefore a = \frac{120}{-400} = -\frac{3}{10} = -0.3$$

$$\therefore b = -40 \times (-0.3) = 12.0$$

Substituting the values of  $a$ ,  $b$  and  $c$  in equation (i),

$$u = -0.3y^2 + 12y.$$

## Velocity Gradient

$$\frac{du}{dy} = -0.3 \times 2y + 12 = -0.6y + 12$$

$$\text{at } y = 0, \text{ Velocity gradient, } \left( \frac{du}{dy} \right)_{y=0} = -0.6 \times 0 + 12 = 12/\text{s. Ans.}$$

$$\text{at } y = 10 \text{ cm, } \left( \frac{du}{dy} \right)_{y=10} = -0.6 \times 10 + 12 = -6 + 12 = 6/\text{s. Ans.}$$

$$\text{at } y = 20 \text{ cm, } \left( \frac{du}{dy} \right)_{y=20} = -0.6 \times 20 + 12 = -12 + 12 = 0. \text{ Ans.}$$

## Shear Stresses

$$\text{Shear stress is given by, } \tau = \mu \frac{du}{dy}$$

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(i) Shear stress at  $y = 0$ ,  $\tau = \mu \left( \frac{du}{dy} \right)_{y=0} = 0.85 \times 12.0 = 10.2 \text{ N/m}^2$ .

(ii) Shear stress at  $y = 10$ ,  $\tau = \mu \left( \frac{du}{dy} \right)_{y=10} = 0.85 \times 6.0 = 5.1 \text{ N/m}^2$ .

(iii) Shear stress at  $y = 20$ ,  $\tau = \mu \left( \frac{du}{dy} \right)_{y=20} = 0.85 \times 0 = 0. \text{ Ans.}$

**Problem 1.16** A Newtonian fluid is filled in the clearance between a shaft and a concentric sleeve. The sleeve attains a speed of 50 cm/s, when a force of 40 N is applied to the sleeve parallel to the shaft. Determine the speed if a force of 200 N is applied.

**Solution.** Given : Speed of sleeve,  $u_1 = 50 \text{ cm/s}$

when force,  $F_1 = 40 \text{ N}$ .

Let speed of sleeve is  $u_2$  when force,  $F_2 = 200 \text{ N}$ .

Using relation  $\tau = \mu \frac{du}{dy}$

where  $\tau = \text{Shear stress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$

$du = \text{Change of velocity} = u - 0 = u$

$dy = \text{Clearance} = y$

$\therefore \frac{F}{A} = \mu \frac{u}{y}$

$\therefore F = \frac{A\mu u}{y} \propto u \quad \{ \because A, \mu \text{ and } y \text{ are constant} \}$

$\therefore \frac{F_1}{u_1} = \frac{F_2}{u_2}$

Substituting values, we get  $\frac{40}{50} = \frac{200}{u_2}$

$\therefore u_2 = \frac{50 \times 200}{40} = 50 \times 5 = 250 \text{ cm/s. Ans.}$

**Problem 1.17** A 15 cm diameter vertical cylinder rotates concentrically inside another cylinder of diameter 15.10 cm. Both cylinders are 25 cm high. The space between the cylinders is filled with a liquid whose viscosity is unknown. If a torque of 12.0 Nm is required to rotate the inner cylinder at 100 r.p.m., determine the viscosity of the fluid.

**Solution.** Given :

Diameter of cylinder  $= 15 \text{ cm} = 0.15 \text{ m}$

Dia. of outer cylinder  $= 15.10 \text{ cm} = 0.151 \text{ m}$

Length of cylinders,  $L = 25 \text{ cm} = 0.25 \text{ m}$

Torque,  $T = 12.0 \text{ Nm}$

Speed,  $N = 100$  r.p.m.

Let the viscosity  $= \mu$

$$\text{Tangential velocity of cylinder, } u = \frac{\pi DN}{60} = \frac{\pi \times 0.15 \times 100}{60} = 0.7854 \text{ m/s}$$

$$\text{Surface area of cylinder, } A = \pi D \times L = \pi \times 0.15 \times 0.25 = .1178 \text{ m}^2$$

$$\text{Now using relation } \tau = \mu \frac{du}{dy}$$

where  $du = u - 0 = u = .7854 \text{ m/s}$

$$dy = \frac{0.151 - 0.150}{2} \text{ m} = .0005 \text{ m}$$

$$\tau = \frac{\mu \times .7854}{.0005}$$

$$\therefore \text{Shear force, } F = \text{Shear stress} \times \text{Area} = \frac{\mu \times .7854}{.0005} \times .1178$$

$$\therefore \text{Torque, } T = F \times \frac{D}{2}$$

$$12.0 = \frac{\mu \times .7854}{.0005} \times .1178 \times \frac{.15}{2}$$

$$\therefore \mu = \frac{12.0 \times .0005 \times 2}{.7854 \times .1178 \times .15} = 0.864 \text{ N s/m}^2$$

$$= 0.864 \times 10 = \mathbf{8.64 \text{ poise. Ans.}}$$

**Problem 1.18** Two large plane surfaces are 2.4 cm apart. The space between the surfaces is filled with glycerine. What force is required to drag a very thin plate of surface area 0.5 square metre between the two large plane surfaces at a speed of 0.6 m/s, if:

- (i) the thin plate is in the middle of the two plane surfaces, and
- (ii) the thin plate is at a distance of 0.8 cm from one of the plane surfaces ? Take the dynamic viscosity of glycerine  $= 8.10 \times 10^{-1} \text{ N s/m}^2$ .

**Solution.** Given :

Distance between two large surfaces = 2.4 cm

Area of thin plate,  $A = 0.5 \text{ m}^2$

Velocity of thin plate,  $u = 0.6 \text{ m/s}$

Viscosity of glycerine,  $\mu = 8.10 \times 10^{-1} \text{ N s/m}^2$

**Case I.** When the thin plate is in the middle of the two plane surfaces [Refer to Fig. 1.7 (a)]

Let  $F_1$  = Shear force on the upper side of the thin plate

$F_2$  = Shear force on the lower side of the thin plate

$F$  = Total force required to drag the plate

Then  $F = F_1 + F_2$

The shear stress ( $\tau_1$ ) on the upper side of the thin plate is given by equation,

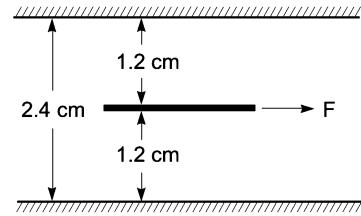


Fig. 1.7 (a)

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$$\tau_1 = \mu \left( \frac{du}{dy} \right)_1$$

where  $du$  = Relative velocity between thin plate and upper large plane surface  
 $= 0.6 \text{ m/sec}$

$dy$  = Distance between thin plate and upper large plane surface  
 $= 1.2 \text{ cm} = 0.012 \text{ m}$  (plate is a thin one and hence thickness of plate is neglected)

$$\therefore \tau_1 = 8.10 \times 10^{-1} \times \left( \frac{0.6}{0.012} \right) = 40.5 \text{ N/m}^2$$

Now shear force,  $F_1 = \text{Shear stress} \times \text{Area}$

$$= \tau_1 \times A = 40.5 \times 0.5 = 20.25 \text{ N}$$

Similarly shear stress ( $\tau_2$ ) on the lower side of the thin plate is given by

$$\tau_2 = \mu \left( \frac{du}{dy} \right)_2 = 8.10 \times 10^{-1} \times \left( \frac{0.6}{0.012} \right) = 40.5 \text{ N/m}^2$$

$\therefore$  Shear force,  $F_2 = \tau_2 \times A = 40.5 \times 0.5 = 20.25 \text{ N}$

$\therefore$  Total force,  $F = F_1 + F_2 = 20.25 + 20.25 = 40.5 \text{ N. Ans.}$

**Case II.** When the thin plate is at a distance of 0.8 cm from one of the plane surfaces [Refer to Fig. 1.7 (b)].

Let the thin plate is at a distance 0.8 cm from the lower plane surface.

Then distance of the plate from the upper plane surface  
 $= 2.4 - 0.8 = 1.6 \text{ cm} = .016 \text{ m}$

(Neglecting thickness of the plate)

The shear force on the upper side of the thin plate,

$$F_1 = \text{Shear stress} \times \text{Area} = \tau_1 \times A \\ = \mu \left( \frac{du}{dy} \right)_1 \times A = 8.10 \times 10^{-1} \times \left( \frac{0.6}{0.016} \right) \times 0.5 = 15.18 \text{ N}$$

The shear force on the lower side of the thin plate,

$$F_2 = \tau_2 \times A = \mu \left( \frac{du}{dy} \right)_2 \times A \\ = 8.10 \times 10^{-1} \times \left( \frac{0.6}{0.016} \right) \times 0.5 = 30.36 \text{ N}$$

$\therefore$  Total force required  $= F_1 + F_2 = 15.18 + 30.36 = 45.54 \text{ N. Ans.}$

**Problem 1.19** A vertical gap 2.2 cm wide of infinite extent contains a fluid of viscosity  $2.0 \text{ N s/m}^2$  and specific gravity 0.9. A metallic plate  $1.2 \text{ m} \times 1.2 \text{ m} \times 0.2 \text{ cm}$  is to be lifted up with a constant velocity of  $0.15 \text{ m/sec}$ , through the gap. If the plate is in the middle of the gap, find the force required. The weight of the plate is 40 N.

**Solution.** Given :

Width of gap  $= 2.2 \text{ cm}$ , viscosity,  $\mu = 2.0 \text{ N s/m}^2$

Sq. gr. of fluid  $= 0.9$

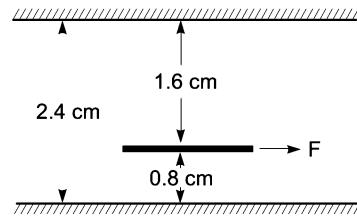


Fig. 1.7 (b)

∴ Weight density of fluid

$$= 0.9 \times 1000 = 900 \text{ kgf/m}^3 = 900 \times 9.81 \text{ N/m}^3 \\ (\because 1 \text{ kgf} = 9.81 \text{ N})$$

$$\begin{aligned} \text{Volume of plate} &= 1.2 \text{ m} \times 1.2 \text{ m} \times 0.2 \text{ cm} \\ &= 1.2 \times 1.2 \times .002 \text{ m}^3 = .00288 \text{ m}^3 \end{aligned}$$

$$\text{Thickness of plate} = 0.2 \text{ cm}$$

$$\text{Velocity of plate} = 0.15 \text{ m/sec}$$

$$\text{Weight of plate} = 40 \text{ N.}$$

When plate is in the middle of the gap, the distance of the plate from vertical surface of the gap

$$\begin{aligned} &= \left( \frac{\text{Width of gap} - \text{Thickness of plate}}{2} \right) \\ &= \frac{(2.2 - 0.2)}{2} = 1 \text{ cm} = .01 \text{ m.} \end{aligned}$$

Now the shear force on the left side of the metallic plate,

$$\begin{aligned} F_1 &= \text{Shear stress} \times \text{Area} \\ &= \mu \left( \frac{du}{dy} \right)_1 \times \text{Area} = 2.0 \times \left( \frac{0.15}{.01} \right) \times 1.2 \times 1.2 \text{ N} \\ &\quad (\because \text{Area} = 1.2 \times 1.2 \text{ m}^2) \\ &= 43.2 \text{ N.} \end{aligned}$$

Similarly, the shear force on the right side of the metallic plate,

$$F_2 = \text{Shear stress} \times \text{Area} = 2.0 \times \left( \frac{0.15}{.01} \right) \times 1.2 \times 1.2 = 43.2 \text{ N}$$

$$\therefore \text{Total shear force} = F_1 + F_2 = 43.2 + 43.2 = 86.4 \text{ N.}$$

In this case the weight of plate (which is acting vertically downward) and upward thrust is also to be taken into account.

The upward thrust = Weight of fluid displaced

$$\begin{aligned} &= (\text{Weight density of fluid}) \times \text{Volume of fluid displaced} \\ &= 9.81 \times 900 \times .00288 \text{ N} \\ &\quad (\because \text{Volume of fluid displaced} = \text{Volume of plate} = .00288) \\ &= 25.43 \text{ N.} \end{aligned}$$

The net force acting in the downward direction due to weight of the plate and upward thrust

$$= \text{Weight of plate} - \text{Upward thrust} = 40 - 25.43 = 14.57 \text{ N}$$

∴ Total force required to lift the plate up

$$= \text{Total shear force} + 14.57 = 86.4 + 14.57 = \mathbf{100.97 \text{ N. Ans.}}$$

#### ► 1.4 THERMODYNAMIC PROPERTIES

Fluids consist of liquids or gases. But gases are compressible fluids and hence thermodynamic properties play an important role. With the change of pressure and temperature, the gases undergo

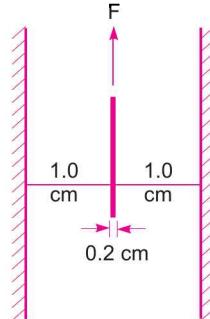


Fig. 1.8

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large variation in density. The relationship between pressure (absolute), specific volume and temperature (absolute) of a gas is given by the equation of state as

$$p \forall = RT \text{ or } \frac{p}{\rho} = RT \quad \dots(1.5)$$

where  $p$  = Absolute pressure of a gas in N/m<sup>2</sup>

$$\forall = \text{Specific volume} = \frac{1}{\rho}$$

$R$  = Gas constant

$T$  = Absolute temperature in °K

$\rho$  = Density of a gas.

**1.4.1 Dimension of R.** The gas constant,  $R$ , depends upon the particular gas. The dimension of  $R$  is obtained from equation (1.5) as

$$R = \frac{p}{\rho T}$$

(i) In MKS units  $R = \frac{\text{kgf}/\text{m}^2}{\left(\frac{\text{kg}}{\text{m}^3}\right)\text{°K}} = \frac{\text{kgf}\cdot\text{m}}{\text{kg}\text{°K}}$

(ii) In SI units,  $p$  is expressed in Newton/m<sup>2</sup> or N/m<sup>2</sup>.

$$\begin{aligned} \therefore R &= \frac{\text{N}/\text{m}^2}{\frac{\text{kg}}{\text{m}^3} \times \text{K}} = \frac{\text{Nm}}{\text{kg}\cdot\text{K}} = \frac{\text{Joule}}{\text{kg}\cdot\text{K}} \quad [\text{Joule} = \text{Nm}] \\ &= \frac{\text{J}}{\text{kg}\cdot\text{K}} \end{aligned}$$

For air,  $R$  in MKS = 29.3  $\frac{\text{kgf}\cdot\text{m}}{\text{kg}\text{°K}}$

$$R \text{ in SI} = 29.3 \times 9.81 \frac{\text{Nm}}{\text{kg}\text{°K}} = 287 \frac{\text{J}}{\text{kg}\cdot\text{K}}$$

**1.4.2 Isothermal Process.** If the change in density occurs at constant temperature, then the process is called isothermal and relationship between pressure ( $p$ ) and density ( $\rho$ ) is given by

$$\frac{p}{\rho} = \text{Constant} \quad \dots(1.6)$$

**1.4.3 Adiabatic Process.** If the change in density occurs with no heat exchange to and from the gas, the process is called adiabatic. And if no heat is generated within the gas due to friction, the relationship between pressure and density is given by

$$\frac{p}{\rho^k} = \text{Constant} \quad \dots(1.7)$$

where  $k$  = Ratio of specific heat of a gas at constant pressure and constant volume.  
= 1.4 for air.

#### 1.4.4 Universal Gas Constant

Let

$m$  = Mass of a gas in kg

$\forall$  = Volume of gas of mass  $m$

$p$  = Absolute pressure

$T$  = Absolute temperature

Then, we have

$$p\forall = mRT \quad \dots(1.8)$$

where  $R$  = Gas constant.

Equation (1.8) can be made universal, i.e., applicable to all gases if it is expressed in **mole-basis**.

Let

$n$  = Number of moles in volume of a gas

$\forall$  = Volume of the gas

$$M = \frac{\text{Mass of the gas molecules}}{\text{Mass of a hydrogen atom}}$$

$m$  = Mass of a gas in kg

Then, we have

$$n \times M = m.$$

Substituting the value of  $m$  in equation (1.8), we get

$$p\forall = n \times M \times RT \quad \dots(1.9)$$

The product  $M \times R$  is called universal gas constant and is equal to  $848 \frac{\text{kgt-m}}{\text{kg-mole } ^\circ\text{K}}$  in MKS units

and  $8314 \text{ J/kg-mole K}$  in SI units.

One kilogram mole is defined as the product of one kilogram mass of the gas and its molecular weight.

**Problem 1.20** A gas weighs  $16 \text{ N/m}^3$  at  $25^\circ\text{C}$  and at an absolute pressure of  $0.25 \text{ N/mm}^2$ . Determine the gas constant and density of the gas.

**Solution.** Given :

Weight density,  $w = 16 \text{ N/m}^3$

Temperature,  $t = 25^\circ\text{C}$

$$\therefore T = 273 + t = 273 + 25 = 288^\circ\text{K}$$

$$p = 0.25 \text{ N/mm}^2 (\text{abs.}) = 0.25 \times 10^6 \text{ N/m}^2 = 25 \times 10^4 \text{ N/m}^2$$

(i) Using relation  $w = \rho g$ , density is obtained as

$$\rho = \frac{w}{g} = \frac{16}{9.81} = 1.63 \text{ kg/m}^3. \text{ Ans.}$$

(ii) Using equation (1.5),  $\frac{p}{\rho} = RT$

$$\therefore R = \frac{p}{\rho T} = \frac{25 \times 10^4}{1.63 \times 288} = 532.55 \frac{\text{Nm}}{\text{kg K}}. \text{ Ans.}$$

**Problem 1.21** A cylinder of  $0.6 \text{ m}^3$  in volume contains air at  $50^\circ\text{C}$  and  $0.3 \text{ N/mm}^2$  absolute pressure. The air is compressed to  $0.3 \text{ m}^3$ . Find (i) pressure inside the cylinder assuming isothermal process and (ii) pressure and temperature assuming adiabatic process. Take  $k = 1.4$ .

**Solution.** Given :

Initial volume,  $\forall_1 = 0.6 \text{ m}^3$

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$$\begin{aligned}
 \text{Temperature} & \quad t_1 = 50^\circ\text{C} \\
 \therefore & \quad T_1 = 273 + 50 = 323^\circ\text{K} \\
 \text{Pressure} & \quad p_1 = 0.3 \text{ N/mm}^2 = 0.3 \times 10^6 \text{ N/m}^2 = 30 \times 10^4 \text{ N/m}^2 \\
 \text{Final volume} & \quad V_2 = 0.3 \text{ m}^3 \\
 & \quad k = 1.4
 \end{aligned}$$

(i) Isothermal process :

$$\begin{aligned}
 \text{Using equation (1.6),} \quad \frac{p}{V} &= \text{Constant or } pV = \text{Constant.} \\
 \therefore & \quad p_1 V_1 = p_2 V_2 \\
 \therefore & \quad p_2 = \frac{p_1 V_1}{V_2} = \frac{30 \times 10^4 \times 0.6}{0.3} = 0.6 \times 10^6 \text{ N/m}^2 = 0.6 \text{ N/mm}^2. \text{ Ans.}
 \end{aligned}$$

(ii) Adiabatic process :

$$\begin{aligned}
 \text{Using equation (1.7),} \quad \frac{p}{V^k} &= \text{Constant or } pV^k = \text{Constant} \\
 \therefore & \quad p_1 V_1^k = p_2 V_2^k \\
 \therefore & \quad p_2 = p_1 \left( \frac{V_1}{V_2} \right)^k = 30 \times 10^4 \times \left( \frac{0.6}{0.3} \right)^{1.4} = 30 \times 10^4 \times 2^{1.4} \\
 & \quad = 0.791 \times 10^6 \text{ N/m}^2 = 0.791 \text{ N/mm}^2. \text{ Ans.}
 \end{aligned}$$

For temperature, using equation (1.5), we get

$$\begin{aligned}
 pV &= RT \text{ and also } pV^k = \text{Constant} \\
 \therefore & \quad p = \frac{RT}{V} \text{ and } \frac{RT}{V} \times V^k = \text{Constant} \\
 \text{or} & \quad RTV^{k-1} = \text{Constant} \\
 \text{or} & \quad T V^{k-1} = \text{Constant} \quad \{ \because R \text{ is also constant} \} \\
 \therefore & \quad T_1 V_1^{k-1} = T_2 V_2^{k-1} \\
 \therefore & \quad T_2 = T_1 \left( \frac{V_1}{V_2} \right)^{k-1} = 323 \left( \frac{0.6}{0.3} \right)^{1.4 - 1.0} = 323 \times 2^{0.4} = 426.2^\circ\text{K} \\
 \therefore & \quad t_2 = 426.2 - 273 = 153.2^\circ\text{C. Ans.}
 \end{aligned}$$

**Problem 1.22** Calculate the pressure exerted by 5 kg of nitrogen gas at a temperature of  $10^\circ\text{C}$  if the volume is  $0.4 \text{ m}^3$ . Molecular weight of nitrogen is 28. Assume, ideal gas laws are applicable.

**Solution.** Given :

$$\begin{aligned}
 \text{Mass of nitrogen} & \quad = 5 \text{ kg} \\
 \text{Temperature,} & \quad t = 10^\circ\text{C} \\
 \therefore & \quad T = 273 + 10 = 283^\circ\text{K} \\
 \text{Volume of nitrogen,} & \quad V = 0.4 \text{ m}^3 \\
 \text{Molecular weight} & \quad = 28
 \end{aligned}$$

Using equation (1.9), we have  $pV = n \times M \times RT$

where  $M \times R = \text{Universal gas constant} = 8314 \frac{\text{Nm}}{\text{kg-mole } ^\circ\text{K}}$

and one kg-mole = (kg-mass)  $\times$  Molecular weight = (kg-mass)  $\times$  28

$$\therefore R \text{ for nitrogen} = \frac{8314}{28} = 296.9 \frac{\text{Nm}}{\text{kg } ^\circ\text{K}}$$

The gas laws for nitrogen is  $pV = mRT$ , where  $R = \text{Characteristic gas constant}$   
or  $p \times 0.4 = 5 \times 296.9 \times 283$

$$\therefore p = \frac{5 \times 296.9 \times 283}{0.4} = 1050283.7 \text{ N/m}^2 = 1.05 \text{ N/mm}^2. \text{ Ans.}$$

## ► 1.5 COMPRESSIBILITY AND BULK MODULUS

Compressibility is the reciprocal of the bulk modulus of elasticity,  $K$  which is defined as the ratio of compressive stress to volumetric strain.

Consider a cylinder fitted with a piston as shown in Fig. 1.9.

Let  $V$  = Volume of a gas enclosed in the cylinder

$p$  = Pressure of gas when volume is  $V$

Let the pressure is increased to  $p + dp$ , the volume of gas decreases from  $V$  to  $V - dV$ .

Then increase in pressure  $= dp \text{ kgf/m}^2$

Decrease in volume  $= dV$

$\therefore$  Volumetric strain  $= -\frac{dV}{V}$

- ve sign means the volume decreases with increase of pressure.

$$\begin{aligned} \therefore \text{ Bulk modulus} &= \frac{\text{Increase of pressure}}{\text{Volumetric strain}} \\ &= \frac{dp}{-\frac{dV}{V}} = \frac{-dp}{dV} V \end{aligned} \quad \dots(1.10)$$

$$\text{Compressibility} = \frac{1}{K} \quad \dots(1.11)$$

## Relationship between Bulk Modulus (K) and Pressure (p) for a Gas

The relationship between bulk modulus of elasticity ( $K$ ) and pressure for a gas for two different processes of compression are as :

(i) **For Isothermal Process.** Equation (1.6) gives the relationship between pressure ( $p$ ) and density ( $\rho$ ) of a gas as

$$\frac{p}{\rho} = \text{Constant}$$

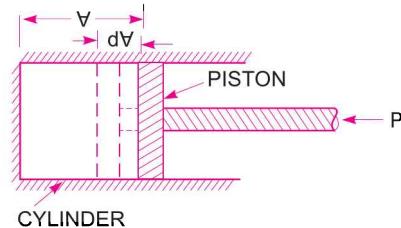


Fig. 1.9

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or

$$p \nabla = \text{Constant}$$

$$\left\{ \because \nabla = \frac{1}{\rho} \right\}$$

Differentiating this equation, we get ( $p$  and  $\nabla$  both are variables)

$$pd\nabla + \nabla dp = 0 \quad \text{or} \quad pd\nabla = -\nabla dp \quad \text{or} \quad p = \frac{-\nabla dp}{d\nabla}$$

Substituting this value in equation (1.10), we get

$$K = p \quad \dots(1.12)$$

(ii) **For Adiabatic Process.** Using equation (1.7) for adiabatic process

$$\frac{p}{\rho^k} = \text{Constant} \quad \text{or} \quad p \nabla^k = \text{Constant}$$

Differentiating, we get  $pd(\nabla^k) + \nabla^k(dp) = 0$

$$\text{or} \quad p \times k \times \nabla^{k-1} d\nabla + \nabla^k dp = 0$$

$$\text{or} \quad p k d\nabla + \nabla dp = 0$$

[Cancelling  $\nabla^{k-1}$  to both sides]

$$\text{or} \quad p k d\nabla = -\nabla dp \quad \text{or} \quad p k = -\frac{\nabla dp}{d\nabla}$$

Hence from equation (1.10), we have

$$K = p k \quad \dots(1.13)$$

where  $K$  = Bulk modulus and  $k$  = Ratio of specific heats.

**Problem 1.23** Determine the bulk modulus of elasticity of a liquid, if the pressure of the liquid is increased from  $70 \text{ N/cm}^2$  to  $130 \text{ N/cm}^2$ . The volume of the liquid decreases by 0.15 per cent.

**Solution.** Given :

$$\text{Initial pressure} = 70 \text{ N/cm}^2$$

$$\text{Final pressure} = 130 \text{ N/cm}^2$$

$$\therefore dp = \text{Increase in pressure} = 130 - 70 = 60 \text{ N/cm}^2$$

$$\text{Decrease in volume} = 0.15\%$$

$$\therefore -\frac{d\nabla}{\nabla} = +\frac{0.15}{100}$$

Bulk modulus,  $K$  is given by equation (1.10) as

$$K = \frac{dp}{-\frac{d\nabla}{\nabla}} = \frac{60 \text{ N/cm}^2}{\frac{.15}{100}} = \frac{60 \times 100}{.15} = 4 \times 10^4 \text{ N/cm}^2. \text{ Ans.}$$

**Problem 1.24** What is the bulk modulus of elasticity of a liquid which is compressed in a cylinder from a volume of  $0.0125 \text{ m}^3$  at  $80 \text{ N/cm}^2$  pressure to a volume of  $0.0124 \text{ m}^3$  at  $150 \text{ N/cm}^2$  pressure ?

**Solution.** Given :

$$\text{Initial volume, } \nabla = 0.0125 \text{ m}^3$$

$$\text{Final volume} = 0.0124 \text{ m}^3$$

$$\therefore \text{Decrease in volume, } d\nabla = .0125 - .0124 = .0001 \text{ m}^3$$

$$\therefore -\frac{dV}{V} = \frac{0.001}{0.0125}$$

Initial pressure = 80 N/cm<sup>2</sup>  
Final pressure = 150 N/cm<sup>2</sup>

$$\therefore \text{Increase in pressure, } dp = (150 - 80) = 70 \text{ N/cm}^2$$

Bulk modulus is given by equation (1.10) as

$$K = \frac{dp}{dV} = \frac{70}{0.001} = 70 \times 125 \text{ N/cm}^2$$

$$= \frac{70}{0.0125}$$

$$= 8.75 \times 10^3 \text{ N/cm}^2. \text{ Ans.}$$

## ► 1.6 SURFACE TENSION AND CAPILLARITY

Surface tension is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension. The magnitude of this force per unit length of the free surface will have the same value as the surface energy per unit area. It is denoted by Greek letter  $\sigma$  (called sigma). In MKS units, it is expressed as kgf/m while in SI units as N/m.

The phenomenon of surface tension is explained by Fig. 1.10. Consider three molecules A, B, C of a liquid in a mass of liquid. The molecule A is attracted in all directions equally by the surrounding molecules of the liquid. Thus the resultant force acting on the molecule A is zero. But the molecule B, which is situated near the free surface, is acted upon by upward and downward forces which are unbalanced. Thus a net resultant force on molecule B is acting in the downward direction. The molecule C, situated on the free surface of liquid, does experience a resultant downward force. All the molecules on the free surface experience a downward force. Thus the free surface of the liquid acts like a very thin film under tension of the surface of the liquid act as though it is an elastic membrane under tension.

**1.6.1 Surface Tension on Liquid Droplet.** Consider a small spherical droplet of a liquid of radius ' $r$ '. On the entire surface of the droplet, the tensile force due to surface tension will be acting.

Let  $\sigma$  = Surface tension of the liquid

$p$  = Pressure intensity inside the droplet (in excess of the outside pressure intensity)

$d$  = Dia. of droplet.

Let the droplet is cut into two halves. The forces acting on one half (say left half) will be

(i) tensile force due to surface tension acting around the circumference of the cut portion as shown in Fig. 1.11 (b) and this is equal to

$$= \sigma \times \text{Circumference}$$

$$= \sigma \times \pi d$$

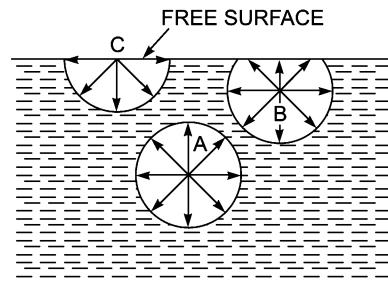


Fig. 1.10 Surface tension.

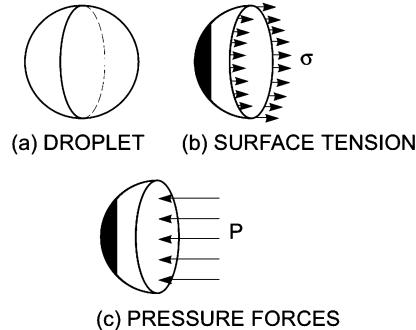
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(ii) pressure force on the area  $\frac{\pi}{4} d^2 = p \times \frac{\pi}{4} d^2$  as shown in

Fig. 1.11 (c). These two forces will be equal and opposite under equilibrium conditions, i.e.,

$$p \times \frac{\pi}{4} d^2 = \sigma \times \pi d$$

or  $p = \frac{\sigma \times \pi d}{\frac{\pi}{4} \times d^2} = \frac{4\sigma}{d}$  ... (1.14)



Equation (1.14) shows that with the decrease of diameter of the droplet, pressure intensity inside the droplet increases.

Fig. 1.11 Forces on droplet.

**1.6.2 Surface Tension on a Hollow Bubble.** A hollow bubble like a soap bubble in air has two surfaces in contact with air, one inside and other outside. Thus two surfaces are subjected to surface tension. In such case, we have

$$p \times \frac{\pi}{4} d^2 = 2 \times (\sigma \times \pi d)$$

$$\therefore p = \frac{2\sigma\pi d}{\frac{\pi}{4} d^2} = \frac{8\sigma}{d}$$
 ... (1.15)

**1.6.3 Surface Tension on a Liquid Jet.** Consider a liquid jet of diameter 'd' and length 'L' as shown in Fig. 1.12.

Let  $p$  = Pressure intensity inside the liquid jet above the outside pressure

$\sigma$  = Surface tension of the liquid.

Consider the equilibrium of the semi jet, we have

$$\begin{aligned} \text{Force due to pressure} &= p \times \text{area of semi jet} \\ &= p \times L \times d \end{aligned}$$

$$\text{Force due to surface tension} = \sigma \times 2L.$$

Equating the forces, we have

$$\begin{aligned} p \times L \times d &= \sigma \times 2L \\ \therefore p &= \frac{\sigma \times 2L}{L \times d} \end{aligned}$$
 ... (1.16)

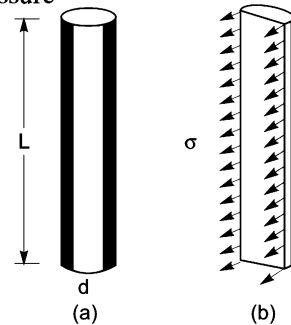


Fig. 1.12 Forces on liquid jet.

**Problem 1.25** The surface tension of water in contact with air at  $20^\circ C$  is  $0.0725 \text{ N/m}$ . The pressure inside a droplet of water is to be  $0.02 \text{ N/cm}^2$  greater than the outside pressure. Calculate the diameter of the droplet of water.

**Solution.** Given :

$$\text{Surface tension, } \sigma = 0.0725 \text{ N/m}$$

Pressure intensity,  $p$  in excess of outside pressure is

$$p = 0.02 \text{ N/cm}^2 = 0.02 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

Let

$d$  = dia. of the droplet

Using equation (1.14), we get  $p = \frac{4\sigma}{d}$  or  $0.02 \times 10^4 = \frac{4 \times 0.0725}{d}$

$$\therefore d = \frac{4 \times 0.0725}{0.02 \times (10)^4} = .00145 \text{ m} = .00145 \times 1000 = 1.45 \text{ mm. Ans.}$$

**Problem 1.26** Find the surface tension in a soap bubble of 40 mm diameter when the inside pressure is  $2.5 \text{ N/m}^2$  above atmospheric pressure.

**Solution.** Given :

$$\text{Dia. of bubble, } d = 40 \text{ mm} = 40 \times 10^{-3} \text{ m}$$

$$\text{Pressure in excess of outside, } p = 2.5 \text{ N/m}^2$$

For a soap bubble, using equation (1.15), we get

$$p = \frac{8\sigma}{d} \quad \text{or} \quad 2.5 = \frac{8 \times \sigma}{40 \times 10^{-3}}$$

$$\sigma = \frac{2.5 \times 40 \times 10^{-3}}{8} \text{ N/m} = 0.0125 \text{ N/m. Ans.}$$

**Problem 1.27** The pressure outside the droplet of water of diameter 0.04 mm is  $10.32 \text{ N/cm}^2$  (atmospheric pressure). Calculate the pressure within the droplet if surface tension is given as  $0.0725 \text{ N/m}$  of water.

**Solution.** Given :

$$\text{Dia. of droplet, } d = 0.04 \text{ mm} = .04 \times 10^{-3} \text{ m}$$

$$\text{Pressure outside the droplet} = 10.32 \text{ N/cm}^2 = 10.32 \times 10^4 \text{ N/m}^2$$

$$\text{Surface tension, } \sigma = 0.0725 \text{ N/m}$$

The pressure inside the droplet, in excess of outside pressure is given by equation (1.14)

$$\text{or } p = \frac{4\sigma}{d} = \frac{4 \times 0.0725}{.04 \times 10^{-3}} = 7250 \text{ N/m}^2 = \frac{7250 \text{ N}}{10^4 \text{ cm}^2} = 0.725 \text{ N/cm}^2$$

$$\therefore \text{Pressure inside the droplet} = p + \text{Pressure outside the droplet} \\ = 0.725 + 10.32 = 11.045 \text{ N/cm}^2. \text{ Ans.}$$

**1.6.4 Capillarity.** Capillarity is defined as a phenomenon of rise or fall of a liquid surface in a small tube relative to the adjacent general level of liquid when the tube is held vertically in the liquid. The rise of liquid surface is known as capillary rise while the fall of the liquid surface is known as capillary depression. It is expressed in terms of cm or mm of liquid. Its value depends upon the specific weight of the liquid, diameter of the tube and surface tension of the liquid.

**Expression for Capillary Rise.** Consider a glass tube of small diameter ' $d$ ' opened at both ends and is inserted in a liquid, say water. The liquid will rise in the tube above the level of the liquid.

Let  $h$  = height of the liquid in the tube. Under a state of equilibrium, the weight of liquid of height  $h$  is balanced by the force at the surface of the liquid in the tube. But the force at the surface of the liquid in the tube is due to surface tension.

Let  $\sigma$  = Surface tension of liquid

$\theta$  = Angle of contact between liquid and glass tube.

The weight of liquid of height  $h$  in the tube = (Area of tube  $\times h$ )  $\times \rho \times g$

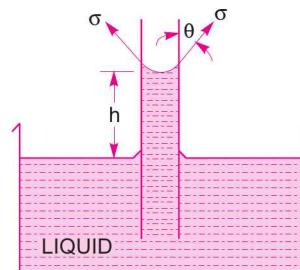


Fig. 1.13 Capillary rise.

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$$= \frac{\pi}{4} d^2 \times h \times \rho \times g \quad \dots(1.17)$$

where  $\rho$  = Density of liquid

Vertical component of the surface tensile force

$$\begin{aligned} &= (\sigma \times \text{Circumference}) \times \cos \theta \\ &= \sigma \times \pi d \times \cos \theta \end{aligned} \quad \dots(1.18)$$

For equilibrium, equating (1.17) and (1.18), we get

$$\frac{\pi}{4} d^2 \times h \times \rho \times g = \sigma \times \pi d \times \cos \theta$$

or

$$h = \frac{\sigma \times \pi d \times \cos \theta}{\frac{\pi}{4} d^2 \times \rho \times g} = \frac{4 \sigma \cos \theta}{\rho \times g \times d} \quad \dots(1.19)$$

The value of  $\theta$  between water and clean glass tube is approximately equal to zero and hence  $\cos \theta$  is equal to unity. Then rise of water is given by

$$h = \frac{4\sigma}{\rho \times g \times d} \quad \dots(1.20)$$

**Expression for Capillary Fall.** If the glass tube is dipped in mercury, the level of mercury in the tube will be lower than the general level of the outside liquid as shown in Fig. 1.14.

Let  $h$  = Height of depression in tube.

Then in equilibrium, two forces are acting on the mercury inside the tube. First one is due to surface tension acting in the downward direction and is equal to  $\sigma \times \pi d \times \cos \theta$ .

Second force is due to hydrostatic force acting upward and is equal to intensity of pressure at a depth ' $h$ '  $\times$  Area

$$= p \times \frac{\pi}{4} d^2 = \rho g \times h \times \frac{\pi}{4} d^2 \quad \{ \because p = \rho gh \}$$

Equating the two, we get

$$\begin{aligned} \sigma \times \pi d \times \cos \theta &= \rho g h \times \frac{\pi}{4} d^2 \\ \therefore h &= \frac{4 \sigma \cos \theta}{\rho g d} \end{aligned} \quad \dots(1.21)$$

Value of  $\theta$  for mercury and glass tube is  $128^\circ$ .

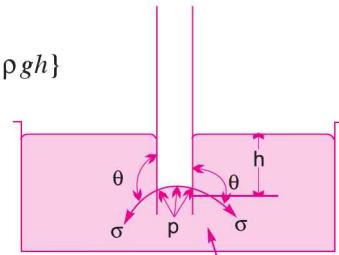


Fig. 1.14

**Problem 1.28** Calculate the capillary rise in a glass tube of 2.5 mm diameter when immersed vertically in (a) water and (b) mercury. Take surface tensions  $\sigma = 0.0725 \text{ N/m}$  for water and  $\sigma = 0.52 \text{ N/m}$  for mercury in contact with air. The specific gravity for mercury is given as 13.6 and angle of contact  $= 130^\circ$ .

**Solution.** Given :

Dia. of tube,	$d = 2.5 \text{ mm} = 2.5 \times 10^{-3} \text{ m}$
Surface tension, $\sigma$ for water	$= 0.0725 \text{ N/m}$
$\sigma$ for mercury	$= 0.52 \text{ N/m}$
Sp. gr. of mercury	$= 13.6$

$$\therefore \text{Density} = 13.6 \times 1000 \text{ kg/m}^3.$$

**(a) Capillary rise for water ( $\theta = 0^\circ$ )**

$$\begin{aligned}\text{Using equation (1.20), we get } h &= \frac{4\sigma}{\rho \times g \times d} = \frac{4 \times 0.0725}{1000 \times 9.81 \times 2.5 \times 10^{-3}} \\ &= .0118 \text{ m} = \mathbf{1.18 \text{ cm. Ans.}}\end{aligned}$$

**(b) For mercury**

Angle of contact between mercury and glass tube,  $\theta = 130^\circ$

$$\begin{aligned}\text{Using equation (1.21), we get } h &= \frac{4\sigma \cos\theta}{\rho \times g \times d} = \frac{4 \times 0.52 \times \cos 130^\circ}{13.6 \times 1000 \times 9.81 \times 2.5 \times 10^{-3}} \\ &= -.004 \text{ m} = \mathbf{-0.4 \text{ cm. Ans.}}\end{aligned}$$

The negative sign indicates the capillary depression.

**Problem 1.29** Calculate the capillary effect in millimetres in a glass tube of 4 mm diameter, when immersed in (i) water, and (ii) mercury. The temperature of the liquid is  $20^\circ\text{C}$  and the values of the surface tension of water and mercury at  $20^\circ\text{C}$  in contact with air are  $0.073575 \text{ N/m}$  and  $0.51 \text{ N/m}$  respectively. The angle of contact for water is zero and that for mercury is  $130^\circ$ . Take density of water at  $20^\circ\text{C}$  as equal to  $998 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Dia. of tube, } d = 4 \text{ mm} = 4 \times 10^{-3} \text{ m}$$

The capillary effect (i.e., capillary rise or depression) is given by equation (1.20) as

$$h = \frac{4\sigma \cos\theta}{\rho \times g \times d}$$

where  $\sigma$  = surface tension in  $\text{N/m}$

$\theta$  = angle of contact, and  $\rho$  = density

**(i) Capillary effect for water**

$$\sigma = 0.073575 \text{ N/m}, \theta = 0^\circ$$

$$\rho = 998 \text{ kg/m}^3 \text{ at } 20^\circ\text{C}$$

$$\therefore h = \frac{4 \times 0.073575 \times \cos 0^\circ}{998 \times 9.81 \times 4 \times 10^{-3}} = 7.51 \times 10^{-3} \text{ m} = \mathbf{7.51 \text{ mm. Ans.}}$$

**(ii) Capillary effect for mercury**

$$\sigma = 0.51 \text{ N/m}, \theta = 130^\circ \text{ and}$$

$$\rho = \text{sp. gr.} \times 1000 = 13.6 \times 1000 = 13600 \text{ kg/m}^2$$

$$\therefore h = \frac{4 \times 0.51 \times \cos 130^\circ}{13600 \times 9.81 \times 4 \times 10^{-3}} = -2.46 \times 10^{-3} \text{ m} = \mathbf{-2.46 \text{ mm. Ans.}}$$

The negative sign indicates the capillary depression.

**Problem 1.30** The capillary rise in the glass tube is not to exceed 0.2 mm of water. Determine its minimum size, given that surface tension for water in contact with air =  $0.0725 \text{ N/m}$ .

**Solution.** Given :

$$\text{Capillary rise, } h = 0.2 \text{ mm} = 0.2 \times 10^{-3} \text{ m}$$

$$\text{Surface tension, } \sigma = 0.0725 \text{ N/m}$$

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Let dia. of tube	$= d$
The angle $\theta$ for water	$= 0^\circ$
Density ( $\rho$ ) for water	$= 1000 \text{ kg/m}^3$

Using equation (1.20), we get

$$h = \frac{4\sigma}{\rho \times g \times d} \text{ or } 0.2 \times 10^{-3} = \frac{4 \times 0.0725}{1000 \times 9.81 \times d}$$

$$\therefore d = \frac{4 \times 0.0725}{1000 \times 9.81 \times 2 \times 10^{-3}} = 0.148 \text{ m} = 14.8 \text{ cm. Ans.}$$

Thus minimum diameter of the tube should be 14.8 cm.

**Problem 1.31** Find out the minimum size of glass tube that can be used to measure water level if the capillary rise in the tube is to be restricted to 2 mm. Consider surface tension of water in contact with air as 0.073575 N/m.

**Solution.** Given :

Capillary rise,	$h = 2.0 \text{ mm} = 2.0 \times 10^{-3} \text{ m}$
Surface tension,	$\sigma = 0.073575 \text{ N/m}$
Let dia. of tube	$= d$
The angle $\theta$ for water	$= 0^\circ$
The density for water,	$\rho = 1000 \text{ kg/m}^3$

Using equation (1.20), we get

$$h = \frac{4\sigma}{\rho \times g \times d} \text{ or } 2.0 \times 10^{-3} = \frac{4 \times 0.073575}{1000 \times 9.81 \times d}$$

$$\therefore d = \frac{4 \times 0.073575}{1000 \times 9.81 \times 2 \times 10^{-3}} = 0.015 \text{ m} = 1.5 \text{ cm. Ans.}$$

Thus minimum diameter of the tube should be 1.5 cm.

**Problem 1.32** An oil of viscosity 5 poise is used for lubrication between a shaft and sleeve. The diameter of the shaft is 0.5 m and it rotates at 200 r.p.m. Calculate the power lost in oil for a sleeve length of 100 mm. The thickness of oil film is 1.0 mm.

**Solution.** Given :

Viscosity,	$\mu = 5 \text{ poise}$
	$= \frac{5}{10} = 0.5 \text{ N s/m}^2$
Dia. of shaft,	$D = 0.5 \text{ m}$
Speed of shaft,	$N = 200 \text{ r.p.m.}$
Sleeve length,	$L = 100 \text{ mm} = 100 \times 10^{-3} \text{ m} = 0.1 \text{ m}$
Thickness of oil film,	$t = 1.0 \text{ mm} = 1 \times 10^{-3} \text{ m}$

$$\text{Tangential velocity of shaft, } u = \frac{\pi D N}{60} = \frac{\pi \times 0.5 \times 200}{60} = 5.235 \text{ m/s}$$

$$\text{Using the relation, } \tau = \mu \frac{du}{dy}$$

where,  $du$  = Change of velocity =  $u - 0 = u = 5.235 \text{ m/s}$

$dy$  = Change of distance =  $t = 1 \times 10^{-3} \text{ m}$

$$\therefore \tau = \frac{0.5 \times 5.235}{1 \times 10^{-3}} = 2617.5 \text{ N/m}^2$$

This is the shear stress on the shaft

$$\therefore \text{Shear force on the shaft, } F = \text{Shear stress} \times \text{Area} = 2617.5 \times \pi D \times L \quad (\because \text{Area} = \pi D \times L) \\ = 2617.5 \times \pi \times 0.5 \times 0.1 = 410.95 \text{ N}$$

$$\text{Torque on the shaft, } T = \text{Force} \times \frac{D}{2} = 410.95 \times \frac{0.5}{2} = 102.74 \text{ Nm}$$

$$\therefore \text{Power* lost} = T \times \omega \text{ Watts} = T \times \frac{2\pi N}{60} \text{ W} \\ = 102.74 \times \frac{2\pi \times 200}{60} = 2150 \text{ W} = 2.15 \text{ kW. Ans.}$$

## ► 1.7 VAPOUR PRESSURE AND CAVITATION

A change from the liquid state to the gaseous state is known as vaporization. The vaporization (which depends upon the prevailing pressure and temperature condition) occurs because of continuous escaping of the molecules through the free liquid surface.

Consider a liquid (say water) which is confined in a closed vessel. Let the temperature of liquid is  $20^\circ\text{C}$  and pressure is atmospheric. This liquid will vaporise at  $100^\circ\text{C}$ . When vaporization takes place, the molecules escapes from the free surface of the liquid. These vapour molecules get accumulated in the space between the free liquid surface and top of the vessel. These accumulated vapours exert a pressure on the liquid surface. This pressure is known as **vapour pressure** of the liquid or this is the pressure at which the liquid is converted into vapours.

Again consider the same liquid at  $20^\circ\text{C}$  at atmospheric pressure in the closed vessel. If the pressure above the liquid surface is reduced by some means, the boiling temperature will also reduce. If the pressure is reduced to such an extent that it becomes equal to or less than the vapour pressure, the boiling of the liquid will start, though the temperature of the liquid is  $20^\circ\text{C}$ . Thus a liquid may boil even at ordinary temperature, if the pressure above the liquid surface is reduced so as to be equal or less than the vapour pressure of the liquid at that temperature.

Now consider a flowing liquid in a system. If the pressure at any point in this flowing liquid becomes equal to or less than the vapour pressure, the vaporization of the liquid starts. The bubbles of these vapours are carried by the flowing liquid into the region of high pressure where they collapse, giving rise to high impact pressure. The pressure developed by the collapsing bubbles is so high that the material from the adjoining boundaries gets eroded and cavities are formed on them. This phenomenon is known as **cavitation**.

Hence the cavitation is the phenomenon of formation of vapour bubbles of a flowing liquid in a region where the pressure of the liquid falls below the vapour pressure and sudden collapsing of these vapour bubbles in a region of higher pressure. When the vapour bubbles collapse, a very high pressure is created. The metallic surfaces, above which the liquid is flowing, is subjected to these high pressures, which cause pitting action on the surface. Thus cavities are formed on the metallic surface and hence the name is cavitation.

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\* Power in case of S.I. Unit =  $T \times \omega$  or  $\frac{2\pi NT}{60}$  Watts or  $\frac{2\pi NT}{60,000}$  kW. The angular velocity  $\omega = \frac{2\pi N}{60}$ .

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### HIGHLIGHTS

1. The weight density or specific weight of a fluid is equal to weight per unit volume. It is also equal to,  
 $w = \rho \times g.$
2. Specific volume is the reciprocal of mass density.
3. The shear stress is proportional to the velocity gradient  $\frac{du}{dy}$ . Mathematically,  $\tau = \mu \frac{du}{dy}.$
4. Kinematic viscosity  $\nu$  is given by  $\nu = \frac{\mu}{\rho}.$
5. Poise and stokes are the units of viscosity and kinematic viscosity respectively.
6. To convert the unit of viscosity from poise to MKS units, poise should be divided by 98.1 and to convert poise into SI units, the poise should be divided by 10. SI unit of viscosity is  $\text{Ns/m}^2$  or  $\text{Pa s}$ , where  $\text{N/m}^2 = \text{Pa} = \text{Pascal}.$
7. For a perfect gas, the equation of state is  $\frac{P}{\rho} = RT$   
where R = gas constant and for air =  $29.3 \frac{\text{kgt-m}}{\text{kg}^\circ\text{K}} = 287 \text{ J/kg}^\circ\text{K}.$
8. For isothermal process,  $\frac{P}{\rho} = \text{Constant}$  whereas for adiabatic process,  $\frac{P}{\rho^k} = \text{constant}.$
9. Bulk modulus of elasticity is given as  $K = \frac{-dp}{\left(\frac{dV}{V}\right)}.$
10. Compressibility is the reciprocal of bulk modulus of elasticity or  $= \frac{1}{K}.$
11. Surface tension is expressed in N/m or dyne/cm. The relation between surface tension ( $\sigma$ ) and difference of pressure ( $p$ ) between the inside and outside of a liquid drop is given as  $p = \frac{4\sigma}{d}$   
For a soap bubble,  $p = \frac{8\sigma}{d}.$   
For a liquid jet,  $p = \frac{2\sigma}{d}.$
12. Capillary rise or fall of a liquid is given by  $h = \frac{4\sigma \cos \theta}{wd}.$   
The value of  $\theta$  for water is taken equal to zero and for mercury equal to  $128^\circ.$

### EXERCISE

#### (A) THEORETICAL PROBLEMS

1. Define the following fluid properties :  
Density, weight density, specific volume and specific gravity of a fluid.
2. Differentiate between : (i) Liquids and gases, (ii) Real fluids and ideal fluids, (iii) Specific weight and specific volume of a fluid.
3. What is the difference between dynamic viscosity and kinematic viscosity ? State their units of measurements.

4. Explain the terms : (i) Dynamic viscosity, and (ii) Kinematic viscosity. Give their dimensions.
5. State the Newton's law of viscosity and give examples of its application.
6. Enunciate Newton's law of viscosity. Explain the importance of viscosity in fluid motion. What is the effect of temperature on viscosity of water and that of air?
7. Define Newtonian and Non-Newtonian fluids.
8. What do you understand by terms : (i) Isothermal process, (ii) Adiabatic process, and (iii) Universal-gas constant.
9. Define compressibility. Prove that compressibility for a perfect gas undergoing isothermal compression is  $\frac{1}{p}$  while for a perfect gas undergoing isentropic compression is  $\frac{1}{wp}$ .
10. Define surface tension. Prove that the relationship between surface tension and pressure inside a droplet of liquid in excess of outside pressure is given by  $p = \frac{4\sigma}{d}$ .
11. Explain the phenomenon of capillarity. Obtain an expression for capillary rise of a liquid.
12. (a) Distinguish between ideal fluids and real fluids. Explain the importance of compressibility in fluid flow.  
(b) Define the terms : density, specific volume, specific gravity, vacuum pressure, compressible and incompressible fluids. *(R.G.P. Vishwavidyalaya, Bhopal S 2002)*
13. Define and explain Newton's law of viscosity.
14. Convert 1 kg/s-m dynamic viscosity in poise.
15. Why does the viscosity of a gas increases with the increase in temperature while that of a liquid decreases with increase in temperature ?
16. (a) How does viscosity of a fluid vary with temperature ?  
(b) Cite examples where surface tension effects play a prominent role. *(J.N.T.U., Hyderabad S 2002)*
17. (i) Develop the expression for the relation between gauge pressure  $P$  inside a droplet of liquid and the surface tension.  
(ii) Explain the following :  
Newtonian and Non-Newtonian fluids, vapour pressure, and compressibility. *(R.G.P.V., Bhopal S 2001)*

### **(B) NUMERICAL PROBLEMS**

1. One litre of crude oil weighs 9.6 N. Calculate its specific weight, density and specific gravity. **[Ans.** 9600 N/m<sup>3</sup>, 978.6 kg/m<sup>3</sup>, 0.978]
2. The velocity distribution for flow over a flat plate is given by  $u = \frac{3}{2} y - y^{3/2}$ , where  $u$  is the point velocity in metre per second at a distance  $y$  metre above the plate. Determine the shear stress at  $y = 9$  cm. Assume dynamic viscosity as 8 poise. *(Nagpur University)* **[Ans.** 0.839 N/m<sup>2</sup>]
3. A plate 0.025 mm distant from a fixed plate, moves at 50 cm/s and requires a force of 1.471 N/m<sup>2</sup> to maintain this speed. Determine the fluid viscosity between the plates in the poise. **[Ans.**  $7.357 \times 10^{-4}$ ]
4. Determine the intensity of shear of an oil having viscosity = 1.2 poise and is used for lubrication in the clearance between a 10 cm diameter shaft and its journal bearing. The clearance is 1.0 mm and shaft rotates at 200 r.p.m. **[Ans.** 125.56 N/m<sup>2</sup>]
5. Two plates are placed at a distance of 0.15 mm apart. The lower plate is fixed while the upper plate having surface area 1.0 m<sup>2</sup> is pulled at 0.3 m/s. Find the force and power required to maintain this speed, if the fluid separating them is having viscosity 1.5 poise. **[Ans.** 300 N, 89.8 W]
6. An oil film of thickness 1.5 mm is used for lubrication between a square plate of size 0.9 m × 0.9 m and an inclined plane having an angle of inclination 20°. The weight of the square is 392.4 N and it slides down the plane with a uniform velocity of 0.2 m/s. Find the dynamic viscosity of the oil. **[Ans.** 12.42 poise]

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7. In a stream of glycerine in motion, at a certain point the velocity gradient is 0.25 metre per sec per metre. The mass density of fluid is 1268.4 kg per cubic metre and kinematic viscosity is  $6.30 \times 10^{-4}$  square metre per second. Calculate the shear stress at the point. [Ans. 0.2 N/m<sup>2</sup>]
8. Find the kinematic viscosity of an oil having density 980 kg/m<sup>3</sup> when at a certain point in the oil, the shear stress is 0.25 N/m<sup>2</sup> and velocity gradient is 0.3/s. [Ans. 0.000849  $\frac{\text{m}^2}{\text{sec}}$  or 8.49 stokes]
9. Determine the specific gravity of a fluid having viscosity 0.07 poise and kinematic viscosity 0.042 stokes. [Ans. 1.667]
10. Determine the viscosity of a liquid having kinematic viscosity 6 stokes and specific gravity 2.0. [Ans. 11.99 poise]
11. If the velocity distribution of a fluid over a plate is given by  $u = (3/4)y - y^2$ , where  $u$  is the velocity in metre per second at a distance of  $y$  metres above the plate, determine the shear stress at  $y = 0.15$  metre. Take dynamic viscosity of the fluid as  $8.5 \times 10^{-5}$  kgf·sec/m<sup>2</sup>. [Ans.  $3.825 \times 10^{-5}$  kgf/m<sup>2</sup>]
12. An oil of viscosity 5 poise is used for lubrication between a shaft and sleeve. The diameter of shaft is 0.5 m and it rotates at 200 r.p.m. Calculate the power lost in the oil for a sleeve length of 100 mm. The thickness of the oil film is 1.0 mm. [Ans. 2.15 kW]
13. The velocity distribution over a plate is given by  $u = \frac{2}{3}y - y^2$  in which  $u$  is the velocity in m/sec at a distance of  $y$  m above the plate. Determine the shear stress at  $y = 0, 0.1$  and  $0.2$  m. Take  $\mu = 6$  poise. [Ans. 0.4, 0.028 and 0.159 N/m<sup>2</sup>]
14. In question 13, find the distance in metres above the plate, at which the shear stress is zero. [Ans. 0.333 m]
15. The velocity profile of a viscous fluid over a plate is parabolic with vertex 20 cm from the plate, where the velocity is 120 cm/s. Calculate the velocity gradient and shear stress at distances of 0, 5 and 15 cm from the plate, given the viscosity of the fluid = 6 poise. [Ans. 12/s, 7.18 N/m<sup>2</sup>; 9/s, 5.385 N/m<sup>2</sup>; 3/s, 1.795 N/m<sup>2</sup>]
16. The weight of a gas is given as 17.658 N/m<sup>3</sup> at 30°C and at an absolute pressure of 29.43 N/cm<sup>2</sup>. Determine the gas constant and also the density of the gas. [Ans.  $\frac{1.8 \text{ kg}}{\text{m}^3}, \frac{539.55 \text{ N m}}{\text{kg}^\circ\text{K}}$ ]
17. A cylinder of 0.9 m<sup>3</sup> in volume contains air at 0°C and 39.24 N/cm<sup>2</sup> absolute pressure. The air is compressed to 0.45 m<sup>3</sup>. Find (i) the pressure inside the cylinder assuming isothermal process, (ii) pressure and temperature assuming adiabatic process. Take  $k = 1.4$  for air. [Ans. (i) 78.48 N/cm<sup>2</sup>, (ii) 103.5 N/m<sup>2</sup>, 140°C]
18. Calculate the pressure exerted by 4 kg mass of nitrogen gas at a temperature of 15°C if the volume is 0.35 m<sup>3</sup>. Molecular weight of nitrogen is 28. [Ans. 97.8 N/cm<sup>2</sup>]
19. The pressure of a liquid is increased from 60 N/cm<sup>2</sup> to 100 N/cm<sup>2</sup> and volume decreases by 0.2 per cent. Determine the bulk modulus of elasticity. [Ans.  $2 \times 10^4$  N/cm<sup>2</sup>]
20. Determine the bulk modulus of elasticity of a fluid which is compressed in a cylinder from a volume of 0.009 m<sup>3</sup> at 70 N/cm<sup>2</sup> pressure to a volume of 0.0085 m<sup>3</sup> at 270 N/cm<sup>2</sup> pressure. [Ans.  $3.6 \times 10^3$  N/cm<sup>2</sup>]
21. The surface tension of water in contact with air at 20°C is given as 0.0716 N/m. The pressure inside a droplet of water is to be 0.0147 N/cm<sup>2</sup> greater than the outside pressure, calculate the diameter of the droplet of water. [Ans. 1.94 mm]
22. Find the surface tension in a soap bubble of 30 mm diameter when the inside pressure is 1.962 N/m<sup>2</sup> above atmosphere. [Ans. 0.00735 N/m]
23. The surface tension of water in contact with air is given as 0.0725 N/m. The pressure outside the droplet of water of diameter 0.02 mm is atmospheric  $\left(10.32 \frac{\text{N}}{\text{cm}^2}\right)$ . Calculate the pressure within the droplet of water. [Ans. 11.77 N/cm<sup>2</sup>]

24. Calculate the capillary rise in a glass tube of 3.0 mm diameter when immersed vertically in (a) water, and (b) mercury. Take surface tensions for mercury and water as 0.0725 N/m and 0.52 N/m respectively in contact with air. Specific gravity for mercury is given as 13.6. [Ans. 0.966 cm, 0.3275 cm]
25. The capillary rise in the glass tube used for measuring water level is not to exceed 0.5 mm. Determine its minimum size, given that surface tension for water in contact with air = 0.07112 N/m. [Ans. 5.8 cm]
26. (SI Units). One litre of crude oil weighs 9.6 N. Calculate its specific weight, density and specific gravity. [Ans. 9600 N/m<sup>3</sup>; 979.6 kg/m<sup>3</sup>; 0.9786]
27. (SI Units). A piston 796 mm diameter and 200 mm long works in a cylinder of 800 mm diameter. If the annular space is filled with a lubricating oil of viscosity 5 cp (centi-poise), calculate the speed of descent of the piston in vertical position. The weight of the piston and axial load are 9.81 N. [Ans. 7.84 m/s]
28. (SI Units). Find the capillary rise of water in a tube 0.03 cm diameter. The surface tension of water is 0.0735 N/m. [Ans. 9.99 cm]
29. Calculate the specific weight, density and specific gravity of two litres of a liquid which weight 15 N. [Ans. 7500 N/m<sup>3</sup>, 764.5 kg/m<sup>3</sup>, 0.764]
30. A 150 mm diameter vertical cylinder rotates concentrically inside another cylinder of diameter 151 mm. Both the cylinders are of 250 mm height. The space between the cylinders is filled with a liquid of viscosity 10 poise. Determine the torque required to rotate the inner cylinder at 100 r.p.m. [Ans. 13.87 Nm]
31. A shaft of diameter 120 mm is rotating inside a journal bearing of diameter 122 mm at a speed of 360 r.p.m. The space between the shaft and the bearing is filled with a lubricating oil of viscosity 6 poise. Find the power absorbed in oil if the length of bearing is 100 mm. [Ans. 115.73 W]
32. A shaft of diameter 100 mm is rotating inside a journal bearing of diameter 102 mm at a speed of 360 r.p.m. The space between the shaft and bearing is filled with a lubricating oil of viscosity 5 poise. The length of the bearing is 200 mm. Find the power absorbed in the lubricating oil. [Ans. 111.58 W]
33. Assuming that the bulk modulus of elasticity of water is  $2.07 \times 10^6$  kN/m<sup>2</sup> at standard atmospheric conditions, determine the increase of pressure necessary to produce 1% reduction in volume at the same temperature.

$$[\text{Hint. } K = 2.07 \times 10^6 \text{ kN/m}^2; \frac{-dV}{V} = \frac{1}{100} = 0.01]$$

$$\text{Increase in pressure } (dp) = K \times \left( \frac{-dV}{V} \right) = 2.07 \times 10^6 \times 0.01 = 2.07 \times 10^4 \text{ kN/m}^2.$$

34. A square plate of size 1 m × 1 m and weighing 350 N slides down an inclined plane with a uniform velocity of 1.5 m/s. The inclined plane is laid on a slope of 5 vertical to 12 horizontal and has an oil film of 1 mm thickness. Calculate the dynamic viscosity of oil. [J.N.T.U., Hyderabad, S 2002]

$$[\text{Hint. } A = 1 \times 1 = 1 \text{ m}^2, W = 350 \text{ N}, u = 1.5 \text{ m/s}, \tan \theta = \frac{5}{12} = \frac{BC}{AB}]$$

Component of weight along the plane =  $W \times \sin \theta$

$$\text{where } \sin \theta = \frac{BC}{AC} = \frac{5}{13} \quad \left( \because AC = \sqrt{AB^2 + BC^2} = \sqrt{12^2 + 5^2} = 13 \right)$$

$$\therefore F = W \sin \theta = 350 \times \frac{5}{13} = 134.615$$

$$\text{Now } \tau = \mu \frac{du}{dy}, \text{ where } du = u - 0 = u = 1.5 \text{ m/s and } dy = 1 \text{ mm} = 1 \times 10^{-3} \text{ m}$$

$$\text{or } \frac{F}{A} = \mu \frac{du}{dy}, \therefore \mu = \frac{F}{A} \times \frac{dy}{du} = \frac{134.615}{1} \times \frac{1 \times 10^{-3}}{1.5} = 0.0897 \frac{\text{Ns}}{\text{m}^2} = 0.897 \text{ poise}]$$

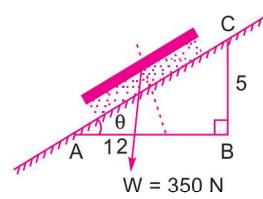


Fig. 1.15



# 2

## CHAPTER

# PRESSURE AND ITS MEASUREMENT



### ► 2.1 FLUID PRESSURE AT A POINT

Consider a small area  $dA$  in large mass of fluid. If the fluid is stationary, then the force exerted by the surrounding fluid on the area  $dA$  will always be perpendicular to the surface  $dA$ . Let  $dF$  is the force acting on the area  $dA$  in the normal direction. Then the ratio of  $\frac{dF}{dA}$  is known as the intensity of pressure or simply pressure and this ratio is represented by  $p$ . Hence mathematically the pressure at a point in a fluid at rest is

$$p = \frac{dF}{dA}.$$

If the force ( $F$ ) is uniformly distributed over the area ( $A$ ), then pressure at any point is given by

$$p = \frac{F}{A} = \frac{\text{Force}}{\text{Area}}.$$

$\therefore$  Force or pressure force,  $F = p \times A$ .

The units of pressure are : (i) kgf/m<sup>2</sup> and kgf/cm<sup>2</sup> in MKS units, (ii) Newton/m<sup>2</sup> or N/m<sup>2</sup> and N/mm<sup>2</sup> in SI units. N/m<sup>2</sup> is known as Pascal and is represented by Pa. Other commonly used units of pressure are :

$$\text{kPa} = \text{kilo pascal} = 1000 \text{ N/m}^2$$

$$\text{bar} = 100 \text{ kPa} = 10^5 \text{ N/m}^2.$$

### ► 2.2 PASCAL'S LAW

It states that the pressure or intensity of pressure at a point in a static fluid is equal in all directions. This is proved as :

The fluid element is of very small dimensions i.e.,  $dx$ ,  $dy$  and  $ds$ .

Consider an arbitrary fluid element of wedge shape in a fluid mass at rest as shown in Fig. 2.1. Let the width of the element perpendicular to the plane of paper is unity and  $p_x$ ,

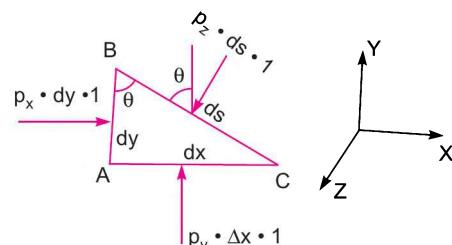


Fig. 2.1 Forces on a fluid element.

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$p_y$  and  $p_z$  are the pressures or intensity of pressure acting on the face  $AB$ ,  $AC$  and  $BC$  respectively. Let  $\angle ABC = \theta$ . Then the forces acting on the element are :

1. Pressure forces normal to the surfaces, and
2. Weight of element in the vertical direction.

The forces on the faces are :

$$\begin{aligned}\text{Force on the face } AB &= p_x \times \text{Area of face } AB \\ &= p_x \times dy \times 1\end{aligned}$$

Similarly force on the face  $AC = p_y \times dx \times 1$

$$\text{Force on the face } BC = p_z \times ds \times 1$$

$$\begin{aligned}\text{Weight of element} &= (\text{Mass of element}) \times g \\ &= (\text{Volume} \times \rho) \times g = \left( \frac{AB \times AC}{2} \times 1 \right) \times \rho \times g,\end{aligned}$$

where  $\rho$  = density of fluid.

Resolving the forces in  $x$ -direction, we have

$$\begin{aligned}p_x \times dy \times 1 - p_z (ds \times 1) \sin (90^\circ - \theta) &= 0 \\ \text{or} \quad p_x \times dy \times 1 - p_z ds \times 1 \cos \theta &= 0.\end{aligned}$$

$$\text{But from Fig. 2.1,} \quad ds \cos \theta = AB = dy$$

$$\therefore p_x \times dy \times 1 - p_z \times dy \times 1 = 0$$

$$\text{or} \quad p_x = p_z \quad \dots(2.1)$$

Similarly, resolving the forces in  $y$ -direction, we get

$$\begin{aligned}p_y \times dx \times 1 - p_z (ds \times 1) \cos (90^\circ - \theta) - \frac{dx \times dy}{2} \times 1 \times \rho \times g &= 0 \\ \text{or} \quad p_y \times dx - p_z ds \sin \theta - \frac{dxdy}{2} \times \rho \times g &= 0.\end{aligned}$$

But  $ds \sin \theta = dx$  and also the element is very small and hence weight is negligible.

$$\therefore p_y dx - p_z dx = 0$$

$$\text{or} \quad p_y = p_z \quad \dots(2.2)$$

From equations (2.1) and (2.2), we have

$$p_x = p_y = p_z \quad \dots(2.3)$$

The above equation shows that the pressure at any point in  $x$ ,  $y$  and  $z$  directions is equal.

Since the choice of fluid element was completely arbitrary, which means the pressure at any point is the same in all directions.

### ► 2.3 PRESSURE VARIATION IN A FLUID AT REST

The pressure at any point in a fluid at rest is obtained by the Hydrostatic Law which states that the rate of increase of pressure in a vertically downward direction must be equal to the specific weight of the fluid at that point. This is proved as :

Consider a small fluid element as shown in Fig. 2.2

Let  $\Delta A$  = Cross-sectional area of element

$\Delta Z$  = Height of fluid element

$p$  = Pressure on face  $AB$

$Z$  = Distance of fluid element from free surface.

The forces acting on the fluid element are :

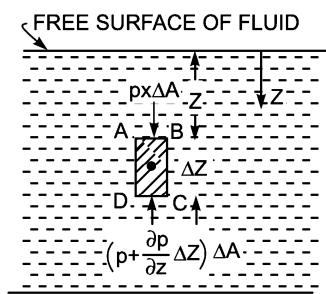


Fig. 2.2 Forces on a fluid element.

1. Pressure force on  $AB = p \times \Delta A$  and acting perpendicular to face  $AB$  in the downward direction.
2. Pressure force on  $CD = \left( p + \frac{\partial p}{\partial Z} \Delta Z \right) \times \Delta A$ , acting perpendicular to face  $CD$ , vertically upward direction.
3. Weight of fluid element = Density  $\times g \times$  Volume  $= \rho \times g \times (\Delta A \times \Delta Z)$ .
4. Pressure forces on surfaces  $BC$  and  $AD$  are equal and opposite. For equilibrium of fluid element, we have

$$p\Delta A - \left( p + \frac{\partial p}{\partial Z} \Delta Z \right) \Delta A + \rho \times g \times (\Delta A \times \Delta Z) = 0$$

or  $p\Delta A - p\Delta A - \frac{\partial p}{\partial Z} \Delta Z \Delta A + \rho \times g \times \Delta A \times Z = 0$

or  $- \frac{\partial p}{\partial Z} \Delta Z \Delta A + \rho \times g \times \Delta A \Delta Z = 0$

or  $\frac{\partial p}{\partial Z} \Delta Z \Delta A = \rho \times g \times \Delta A \Delta Z \quad \text{or} \quad \frac{\partial p}{\partial Z} = \rho \times g \quad [\text{cancelling } \Delta A \Delta Z \text{ on both sides}]$

$\therefore \frac{\partial p}{\partial Z} = \rho \times g = w \quad (\because \rho \times g = w) \quad \dots(2.4)$

where  $w$  = Weight density of fluid.

Equation (2.4) states that rate of increase of pressure in a vertical direction is equal to weight density of the fluid at that point. This is **Hydrostatic Law**.

By integrating the above equation (2.4) for liquids, we get

$$\int dp = \int \rho g dZ$$

or  $p = \rho g Z \quad \dots(2.5)$

where  $p$  is the pressure above atmospheric pressure and  $Z$  is the height of the point from free surfaces.

$$\text{From equation (2.5), we have } Z = \frac{p}{\rho \times g} \quad \dots(2.6)$$

Here  $Z$  is called **pressure head**.

**Problem 2.1** A hydraulic press has a ram of 30 cm diameter and a plunger of 4.5 cm diameter. Find the weight lifted by the hydraulic press when the force applied at the plunger is 500 N.

**Solution.** Given :

$$\text{Dia. of ram, } D = 30 \text{ cm} = 0.3 \text{ m}$$

$$\text{Dia. of plunger, } d = 4.5 \text{ cm} = 0.045 \text{ m}$$

$$\text{Force on plunger, } F = 500 \text{ N}$$

$$\text{Find weight lifted } = W$$

$$\text{Area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.3)^2 = 0.07068 \text{ m}^2$$

$$\text{Area of plunger, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.045)^2 = .00159 \text{ m}^2$$

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Pressure intensity due to plunger

$$= \frac{\text{Force on plunger}}{\text{Area of plunger}} = \frac{F}{a} = \frac{500}{.00159} \text{ N/m}^2.$$

Due to Pascal's law, the intensity of pressure will be equally transmitted in all directions. Hence the pressure intensity at the ram

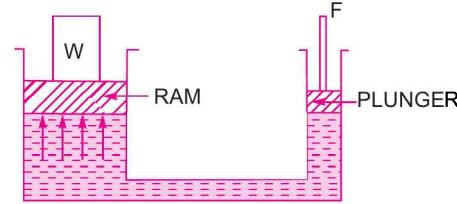


Fig. 2.3

$$= \frac{500}{.00159} = 314465.4 \text{ N/m}^2$$

$$\begin{aligned} \text{But pressure intensity at ram} &= \frac{\text{Weight}}{\text{Area of ram}} = \frac{W}{A} = \frac{W}{.07068} \text{ N/m}^2 \\ &= \frac{W}{.07068} = 314465.4 \end{aligned}$$

$$\therefore \text{Weight} = 314465.4 \times .07068 = 22222 \text{ N} = 22.222 \text{ kN. Ans.}$$

**Problem 2.2** A hydraulic press has a ram of 20 cm diameter and a plunger of 3 cm diameter. It is used for lifting a weight of 30 kN. Find the force required at the plunger.

**Solution.** Given :

$$\text{Dia. of ram, } D = 20 \text{ cm} = 0.2 \text{ m}$$

$$\therefore \text{Area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$$

$$\text{Dia. of plunger } d = 3 \text{ cm} = 0.03 \text{ m}$$

$$\therefore \text{Area of plunger, } a = \frac{\pi}{4} (0.03)^2 = 7.068 \times 10^{-4} \text{ m}^2$$

$$\text{Weight lifted, } W = 30 \text{ kN} = 30 \times 1000 \text{ N} = 30000 \text{ N.}$$

See Fig. 2.3.

$$\text{Pressure intensity developed due to plunger} = \frac{\text{Force}}{\text{Area}} = \frac{F}{a}$$

By Pascal's Law, this pressure is transmitted equally in all directions

$$\text{Hence pressure transmitted at the ram} = \frac{F}{a}$$

$$\therefore \text{Force acting on ram} = \text{Pressure intensity} \times \text{Area of ram}$$

$$= \frac{F}{a} \times A = \frac{F \times .0314}{7.068 \times 10^{-4}} \text{ N}$$

$$\text{But force acting on ram} = \text{Weight lifted} = 30000 \text{ N}$$

$$\therefore 30000 = \frac{F \times .0314}{7.068 \times 10^{-4}}$$

$$\therefore F = \frac{30000 \times 7.068 \times 10^{-4}}{.0314} = 675.2 \text{ N. Ans.}$$

**Problem 2.3** Calculate the pressure due to a column of 0.3 m of (a) water, (b) an oil of sp. gr. 0.8, and (c) mercury of sp. gr. 13.6. Take density of water,  $\rho = 1000 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Height of liquid column, } Z = 0.3 \text{ m.}$$

The pressure at any point in a liquid is given by equation (2.5) as

$$p = \rho g Z$$

(a) For water,

$$\rho = 1000 \text{ kg/m}^3$$

$\therefore$

$$\begin{aligned} p &= \rho g Z = 1000 \times 9.81 \times 0.3 = 2943 \text{ N/m}^2 \\ &= \frac{2943}{10^4} \text{ N/cm}^2 = 0.2943 \text{ N/cm}^2. \text{ Ans.} \end{aligned}$$

(b) For oil of sp. gr. 0.8,

From equation (1.1A), we know that the density of a fluid is equal to specific gravity of fluid multiplied by density of water.

$\therefore$  Density of oil,

$$\begin{aligned} \rho_0 &= \text{Sp. gr. of oil} \times \text{Density of water} & (\rho_0 = \text{Density of oil}) \\ &= 0.8 \times \rho = 0.8 \times 1000 = 800 \text{ kg/m}^3 \end{aligned}$$

Now pressure,

$$\begin{aligned} p &= \rho_0 \times g \times Z \\ &= 800 \times 9.81 \times 0.3 = 2354.4 \frac{\text{N}}{\text{m}^2} = \frac{2354.4}{10^4} \frac{\text{N}}{\text{cm}^2} \\ &= 0.2354 \frac{\text{N}}{\text{cm}^2}. \text{ Ans.} \end{aligned}$$

(c) For mercury, sp. gr.

$$= 13.6$$

From equation (1.1A) we know that the density of a fluid is equal to specific gravity of fluid multiplied by density of water

$\therefore$  Density of mercury,

$$\begin{aligned} \rho_s &= \text{Specific gravity of mercury} \times \text{Density of water} \\ &= 13.6 \times 1000 = 13600 \text{ kg/m}^3 \end{aligned}$$

$\therefore$

$$\begin{aligned} p &= \rho_s \times g \times Z \\ &= 13600 \times 9.81 \times 0.3 = 40025 \frac{\text{N}}{\text{m}^2} \\ &= \frac{40025}{10^4} = 4.002 \frac{\text{N}}{\text{cm}^2}. \text{ Ans.} \end{aligned}$$

**Problem 2.4** The pressure intensity at a point in a fluid is given  $3.924 \text{ N/cm}^2$ . Find the corresponding height of fluid when the fluid is : (a) water, and (b) oil of sp. gr. 0.9.

**Solution.** Given :

$$\text{Pressure intensity, } p = 3.924 \frac{\text{N}}{\text{cm}^2} = 3.924 \times 10^4 \frac{\text{N}}{\text{m}^2}.$$

The corresponding height,  $Z$ , of the fluid is given by equation (2.6) as

$$Z = \frac{p}{\rho \times g}$$

(a) For water,

$$\rho = 1000 \text{ kg/m}^3$$

$\therefore$

$$Z = \frac{p}{\rho \times g} = \frac{3.924 \times 10^4}{1000 \times 9.81} = 4 \text{ m of water. Ans.}$$

(b) For oil, sp. gr.

$$= 0.9$$

$\therefore$  Density of oil

$$\rho_0 = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$\therefore$

$$Z = \frac{p}{\rho_0 \times g} = \frac{3.924 \times 10^4}{900 \times 9.81} = 4.44 \text{ m of oil. Ans.}$$

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**Problem 2.5** An oil of sp. gr. 0.9 is contained in a vessel. At a point the height of oil is 40 m. Find the corresponding height of water at the point.

**Solution.** Given :

Sp. gr. of oil,

$$S_0 = 0.9$$

Height of oil,

$$Z_0 = 40 \text{ m}$$

Density of oil,

$$\rho_0 = \text{Sp. gr. of oil} \times \text{Density of water} = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

Intensity of pressure,

$$p = \rho_0 \times g \times Z_0 = 900 \times 9.81 \times 40 \frac{\text{N}}{\text{m}^2}$$

$$\therefore \text{Corresponding height of water} = \frac{p}{\text{Density of water} \times g}$$

$$= \frac{900 \times 9.81 \times 40}{1000 \times 9.81} = 0.9 \times 40 = 36 \text{ m of water. Ans.}$$

**Problem 2.6** An open tank contains water upto a depth of 2 m and above it an oil of sp. gr. 0.9 for a depth of 1 m. Find the pressure intensity (i) at the interface of the two liquids, and (ii) at the bottom of the tank.

**Solution.** Given :

Height of water,

$$Z_1 = 2 \text{ m}$$

Height of oil,

$$Z_2 = 1 \text{ m}$$

Sp. gr. of oil,

$$S_0 = 0.9$$

Density of water,

$$\rho_1 = 1000 \text{ kg/m}^3$$

Density of oil,

$$\rho_2 = \text{Sp. gr. of oil} \times \text{Density of water}$$

$$= 0.9 \times 1000 = 900 \text{ kg/m}^3$$

Pressure intensity at any point is given by

$$p = \rho \times g \times Z.$$

(i) At interface, i.e., at A

$$p = \rho_2 \times g \times 1.0$$

$$= 900 \times 9.81 \times 1.0$$

$$= 8829 \frac{\text{N}}{\text{m}^2} = \frac{8829}{10^4} = 0.8829 \text{ N/cm}^2. \text{ Ans.}$$

(ii) At the bottom, i.e., at B

$$p = \rho_2 \times gZ_2 + \rho_1 \times g \times Z_1 = 900 \times 9.81 \times 1.0 + 1000 \times 9.81 \times 2.0$$

$$= 8829 + 19620 = 28449 \text{ N/m}^2 = \frac{28449}{10^4} \text{ N/cm}^2 = 2.8449 \text{ N/cm}^2. \text{ Ans.}$$

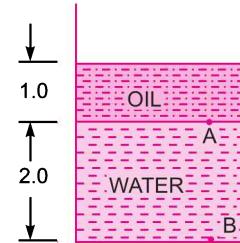


Fig. 2.4

**Problem 2.7** The diameters of a small piston and a large piston of a hydraulic jack are 3 cm and 10 cm respectively. A force of 80 N is applied on the small piston. Find the load lifted by the large piston when :

(a) the pistons are at the same level.

(b) small piston is 40 cm above the large piston.

The density of the liquid in the jack is given as 1000 kg/m<sup>3</sup>.

**Solution.** Given :

Dia. of small piston,

$$d = 3 \text{ cm}$$

∴ Area of small piston,

$$a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times (3)^2 = 7.068 \text{ cm}^2$$

Dia. of large piston,  $D = 10 \text{ cm}$

$$\therefore \text{Area of larger piston, } A = \frac{\pi}{4} \times (10)^2 = 78.54 \text{ cm}^2$$

Force on small piston,  $F = 80 \text{ N}$

Let the load lifted  $= W.$

**(a) When the pistons are at the same level**

Pressure intensity on small piston

$$\frac{F}{a} = \frac{80}{7.068} \text{ N/cm}^2$$

This is transmitted equally on the large piston.

$\therefore$  Pressure intensity on the large piston

$$= \frac{80}{7.068}$$

$\therefore$  Force on the large piston  $= \text{Pressure} \times \text{Area}$

$$= \frac{80}{7.068} \times 78.54 \text{ N} = 888.96 \text{ N. Ans.}$$

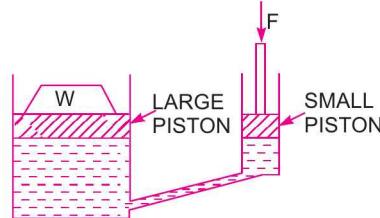


Fig. 2.5

**(b) When the small piston is 40 cm above the large piston**

Pressure intensity on the small piston

$$= \frac{F}{a} = \frac{80}{7.068} \frac{\text{N}}{\text{cm}^2}$$

$\therefore$  Pressure intensity at section A-A

$$= \frac{F}{a} + \text{Pressure intensity due to height of 40 cm of liquid.}$$

But pressure intensity due to 40 cm of liquid

$$= \rho \times g \times h = 1000 \times 9.81 \times 0.4 \text{ N/m}^2 \\ = \frac{1000 \times 9.81 \times 40}{10^4} \text{ N/cm}^2 = 0.3924 \text{ N/cm}^2$$

$\therefore$  Pressure intensity at section A-A

$$= \frac{80}{7.068} + 0.3924 \\ = 11.32 + 0.3924 = 11.71 \text{ N/cm}^2$$

$\therefore$  Pressure intensity transmitted to the large piston  $= 11.71 \text{ N/cm}^2$

$\therefore$  Force on the large piston  $= \text{Pressure} \times \text{Area of the large piston}$

$$= 11.71 \times A = 11.71 \times 78.54 = 919.7 \text{ N.}$$

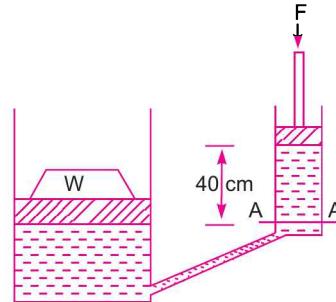


Fig. 2.6

## ► 2.4 ABSOLUTE, GAUGE, ATMOSPHERIC AND VACUUM PRESSURES

The pressure on a fluid is measured in two different systems. In one system, it is measured above the absolute zero or complete vacuum and it is called the absolute pressure and in other system, pressure is measured above the atmospheric pressure and it is called gauge pressure. Thus :

1. **Absolute pressure** is defined as the pressure which is measured with reference to absolute vacuum pressure.

2. **Gauge pressure** is defined as the pressure which is measured with the help of a pressure measuring instrument, in which the atmospheric pressure is taken as datum. The atmospheric pressure on the scale is marked as zero.

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**3. Vacuum pressure** is defined as the pressure below the atmospheric pressure.

The relationship between the absolute pressure, gauge pressure and vacuum pressure are shown in Fig. 2.7.

Mathematically :

(i) Absolute pressure

$$= \text{Atmospheric pressure} + \text{Gauge pressure}$$

or

$$P_{ab} = P_{atm} + P_{gauge}$$

(ii) Vacuum pressure

$$= \text{Atmospheric pressure} - \text{Absolute pressure}.$$

**Note.** (i) The atmospheric pressure at sea level at 15°C is 101.3 kN/m<sup>2</sup> or 10.13 N/cm<sup>2</sup> in SI unit. In case of MKS units, it is equal to 1.033 kgf/cm<sup>2</sup>.

(ii) The atmospheric pressure head is 760 mm of mercury or 10.33 m of water.

**Problem 2.8** What are the gauge pressure and absolute pressure at a point 3 m below the free surface of a liquid having a density of  $1.53 \times 10^3 \text{ kg/m}^3$  if the atmospheric pressure is equivalent to 750 mm of mercury ? The specific gravity of mercury is 13.6 and density of water =  $1000 \text{ kg/m}^3$ .

**Solution.** Given :

Depth of liquid,

$$Z_1 = 3 \text{ m}$$

Density of liquid,

$$\rho_1 = 1.53 \times 10^3 \text{ kg/m}^3$$

Atmospheric pressure head,

$$Z_0 = 750 \text{ mm of Hg}$$

$$= \frac{750}{1000} = 0.75 \text{ m of Hg}$$

$$\therefore \text{Atmospheric pressure, } P_{atm} = \rho_0 \times g \times Z_0$$

where  $\rho_0$  = Density of Hg = Sp. gr. of mercury × Density of water =  $13.6 \times 1000 \text{ kg/m}^3$

and  $Z_0$  = Pressure head in terms of mercury.

$$\therefore P_{atm} = (13.6 \times 1000) \times 9.81 \times 0.75 \text{ N/m}^2 \quad (\because Z_0 = 0.75) \\ = 100062 \text{ N/m}^2$$

Pressure at a point, which is at a depth of 3 m from the free surface of the liquid is given by,

$$P = \rho_1 \times g \times Z_1 \\ = (1.53 \times 1000) \times 9.81 \times 3 = 45028 \text{ N/m}^2$$

$\therefore$  Gauge pressure,

$$P = 45028 \text{ N/m}^2. \text{ Ans.}$$

Now absolute pressure

$$= \text{Gauge pressure} + \text{Atmospheric pressure} \\ = 45028 + 100062 = 145090 \text{ N/m}^2. \text{ Ans.}$$

### ► 2.5 MEASUREMENT OF PRESSURE

The pressure of a fluid is measured by the following devices :

- 1. Manometers
- 2. Mechanical Gauges.

**2.5.1 Manometers.** Manometers are defined as the devices used for measuring the pressure at a point in a fluid by balancing the column of fluid by the same or another column of the fluid. They are classified as :

(a) Simple Manometers,

(b) Differential Manometers.

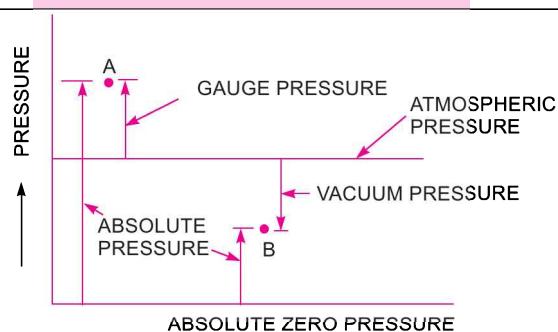


Fig. 2.7 Relationship between pressures.

**2.5.2 Mechanical Gauges.** Mechanical gauges are defined as the devices used for measuring the pressure by balancing the fluid column by the spring or dead weight. The commonly used mechanical pressure gauges are :

- (a) Diaphragm pressure gauge,
- (b) Bourdon tube pressure gauge,
- (c) Dead-weight pressure gauge, and
- (d) Bellows pressure gauge.

## ► 2.6 SIMPLE MANOMETERS

A simple manometer consists of a glass tube having one of its ends connected to a point where pressure is to be measured and other end remains open to atmosphere. Common types of simple manometers are :

1. Piezometer,
2. U-tube Manometer, and
3. Single Column Manometer.

**2.6.1 Piezometer.** It is the simplest form of manometer used for measuring gauge pressures. One end of this manometer is connected to the point where pressure is to be measured and other end is open to the atmosphere as shown in Fig. 2.8. The rise of liquid gives the pressure head at that point. If at a point A, the height of liquid say water is  $h$  in piezometer tube, then pressure at A

$$= \rho \times g \times h \frac{\text{N}}{\text{m}^2}.$$

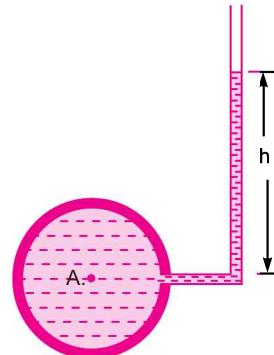


Fig. 2.8 Piezometer.

**2.6.2 U-tube Manometer.** It consists of glass tube bent in U-shape, one end of which is connected to a point at which pressure is to be measured and other end remains open to the atmosphere as shown in Fig. 2.9. The tube generally contains mercury or any other liquid whose specific gravity is greater than the specific gravity of the liquid whose pressure is to be measured.

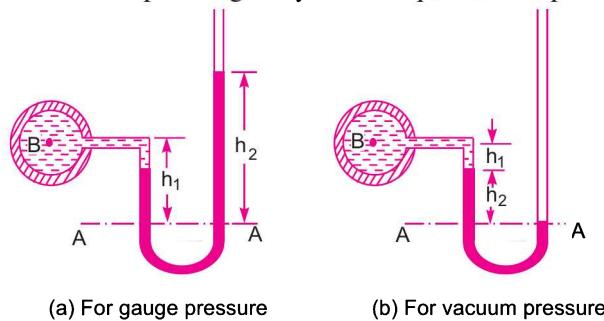


Fig. 2.9 U-tube Manometer.

**(a) For Gauge Pressure.** Let B is the point at which pressure is to be measured, whose value is  $p$ . The datum line is A-A.

- Let       $h_1$  = Height of light liquid above the datum line  
 $h_2$  = Height of heavy liquid above the datum line  
 $S_1$  = Sp. gr. of light liquid  
 $\rho_1$  = Density of light liquid =  $1000 \times S_1$   
 $S_2$  = Sp. gr. of heavy liquid  
 $\rho_2$  = Density of heavy liquid =  $1000 \times S_2$

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As the pressure is the same for the horizontal surface. Hence pressure above the horizontal datum line A-A in the left column and in the right column of U-tube manometer should be same.

$$\begin{aligned} \text{Pressure above A-A in the left column} &= p + \rho_1 \times g \times h_1 \\ \text{Pressure above A-A in the right column} &= \rho_2 \times g \times h_2 \\ \text{Hence equating the two pressures} \quad p + \rho_1 gh_1 &= \rho_2 gh_2 \\ \therefore \quad p &= (\rho_2 h_2 - \rho_1 \times g \times h_1). \end{aligned} \quad \dots(2.7)$$

(b) **For Vacuum Pressure.** For measuring vacuum pressure, the level of the heavy liquid in the manometer will be as shown in Fig. 2.9 (b). Then

$$\begin{aligned} \text{Pressure above A-A in the left column} &= \rho_2 gh_2 + \rho_1 gh_1 + p \\ \text{Pressure head in the right column above A-A} &= 0 \\ \therefore \quad \rho_2 gh_2 + \rho_1 gh_1 + p &= 0 \\ \therefore \quad p &= -(\rho_2 h_2 + \rho_1 h_1). \end{aligned} \quad \dots(2.8)$$

**Problem 2.9** The right limb of a simple U-tube manometer containing mercury is open to the atmosphere while the left limb is connected to a pipe in which a fluid of sp. gr. 0.9 is flowing. The centre of the pipe is 12 cm below the level of mercury in the right limb. Find the pressure of fluid in the pipe if the difference of mercury level in the two limbs is 20 cm.

**Solution.** Given :

$$\begin{aligned} \text{Sp. gr. of fluid,} \quad S_1 &= 0.9 \\ \therefore \text{Density of fluid,} \quad \rho_1 &= S_1 \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3 \\ \text{Sp. gr. of mercury,} \quad S_2 &= 13.6 \\ \therefore \text{Density of mercury,} \quad \rho_2 &= 13.6 \times 1000 \text{ kg/m}^3 \\ \text{Difference of mercury level,} \quad h_2 &= 20 \text{ cm} = 0.2 \text{ m} \\ \text{Height of fluid from A-A,} \quad h_1 &= 20 - 12 = 8 \text{ cm} = 0.08 \text{ m} \end{aligned}$$

Let  $p$  = Pressure of fluid in pipe

Equating the pressure above A-A, we get

$$\begin{aligned} p + \rho_1 gh_1 &= \rho_2 gh_2 \\ \text{or} \quad p + 900 \times 9.81 \times 0.08 &= 13.6 \times 1000 \times 9.81 \times 0.2 \\ p &= 13.6 \times 1000 \times 9.81 \times 0.2 - 900 \times 9.81 \times 0.08 \\ &= 26683 - 706 = 25977 \text{ N/m}^2 = 2.597 \text{ N/cm}^2. \text{ Ans.} \end{aligned}$$

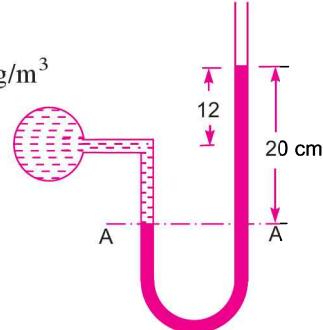


Fig. 2.10

**Problem 2.10** A simple U-tube manometer containing mercury is connected to a pipe in which a fluid of sp. gr. 0.8 and having vacuum pressure is flowing. The other end of the manometer is open to atmosphere. Find the vacuum pressure in pipe, if the difference of mercury level in the two limbs is 40 cm and the height of fluid in the left from the centre of pipe is 15 cm below.

**Solution.** Given :

$$\begin{aligned} \text{Sp. gr. of fluid,} \quad S_1 &= 0.8 \\ \text{Sp. gr. of mercury,} \quad S_2 &= 13.6 \\ \text{Density of fluid,} \quad \rho_1 &= 800 \\ \text{Density of mercury,} \quad \rho_2 &= 13.6 \times 1000 \end{aligned}$$

Difference of mercury level,  $h_2 = 40 \text{ cm} = 0.4 \text{ m}$ . Height of liquid in left limb,  $h_1 = 15 \text{ cm} = 0.15 \text{ m}$ . Let the pressure in pipe =  $p$ . Equating pressure above datum line A-A, we get

$$\rho_2 gh_2 + \rho_1 gh_1 + p = 0$$

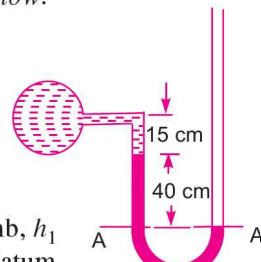


Fig. 2.11

$$\begin{aligned}\therefore p &= -[\rho_2 gh_2 + \rho_1 gh_1] \\ &= -[13.6 \times 1000 \times 9.81 \times 0.4 + 800 \times 9.81 \times 0.15] \\ &= -[53366.4 + 1177.2] = -54543.6 \text{ N/m}^2 = -5.454 \text{ N/cm}^2. \text{ Ans.}\end{aligned}$$

**Problem 2.11** A U-Tube manometer is used to measure the pressure of water in a pipe line, which is in excess of atmospheric pressure. The right limb of the manometer contains mercury and is open to atmosphere. The contact between water and mercury is in the left limb. Determine the pressure of water in the main line, if the difference in level of mercury in the limbs of U-tube is 10 cm and the free surface of mercury is in level with the centre of the pipe. If the pressure of water in pipe line is reduced to  $9810 \text{ N/m}^2$ , calculate the new difference in the level of mercury. Sketch the arrangements in both cases.

**Solution.** Given :

Difference of mercury = 10 cm = 0.1 m

The arrangement is shown in Fig. 2.11 (a)

### Ist Part

Let  $p_A$  = (pressure of water in pipe line (i.e., at point A)

The points B and C lie on the same horizontal line. Hence pressure at B should be equal to pressure at C. But pressure at B

$$\begin{aligned}&= \text{Pressure at } A + \text{Pressure due to } 10 \text{ cm (or } 0.1 \text{ m)} \\ &\quad \text{of water}\end{aligned}$$

$$= p_A + \rho \times g \times h$$

where  $\rho = 1000 \text{ kg/m}^3$  and  $h = 0.1 \text{ m}$

$$= p_A + 1000 \times 9.81 \times 0.1$$

$$= p_A + 981 \text{ N/m}^2 \quad \dots(i)$$

Pressure at C = Pressure at D + Pressure due to 10 cm of mercury

$$= 0 + \rho_0 \times g \times h_0$$

where  $\rho_0$  for mercury =  $13.6 \times 1000 \text{ kg/m}^3$

and  $h_0 = 10 \text{ cm} = 0.1 \text{ m}$

$$\therefore \text{Pressure at } C = 0 + (13.6 \times 1000) \times 9.81 \times 0.1$$

$$= 13341.6 \text{ N} \quad \dots(ii)$$

But pressure at B is equal to pressure at C. Hence equating the equations (i) and (ii), we get

$$p_A + 981 = 13341.6$$

$$\therefore p_A = 13341.6 - 981$$

$$= 12360.6 \frac{\text{N}}{\text{m}^2} \cdot \text{Ans.}$$

### IIInd Part

Given,  $p_A = 9810 \text{ N/m}^2$

Find new difference of mercury level. The arrangement is shown in Fig. 2.11 (b). In this case the pressure at A is  $9810 \text{ N/m}^2$  which is less than the  $12360.6 \text{ N/m}^2$ . Hence mercury in left limb will rise. The rise of mercury in left limb will be equal to the fall of mercury in right limb as the total volume of mercury remains same.

Let  $x$  = Rise of mercury in left limb in cm

Then fall of mercury in right limb =  $x$  cm

The points B, C and D show the initial conditions whereas points  $B^*$ ,  $C^*$  and  $D^*$  show the final conditions.

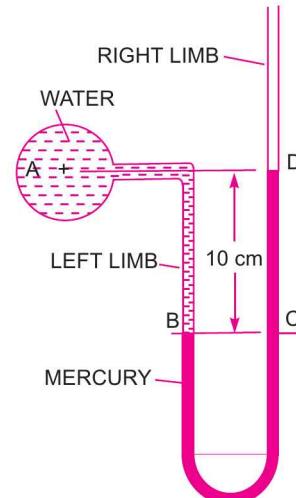


Fig. 2.11 (a)

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The pressure at  $B^*$  = Pressure at  $C^*$

or Pressure at  $A$  + Pressure due to  $(10 - x)$  cm of water  
 = Pressure at  $D^*$  + Pressure due to  
 $(10 - 2x)$  cm of mercury

or  $p_A + \rho_1 \times g \times h_1 = p_{D^*} + \rho_2 \times g \times h_2$

or  $1910 + 1000 \times 9.81 \times \left(\frac{10 - x}{100}\right)$   
 $= 0 + (13.6 \times 1000) \times 9.81 \times \left(\frac{10 - 2x}{100}\right)$

Dividing by 9.81, we get

or  $1000 + 100 - 10x = 1360 - 272x$

or  $272x - 10x = 1360 - 1100$

or  $262x = 260$

∴  $x = \frac{260}{262} = 0.992 \text{ cm}$

∴ New difference of mercury =  $10 - 2x \text{ cm} = 10 - 2 \times 0.992$   
 $= 8.016 \text{ cm. Ans.}$

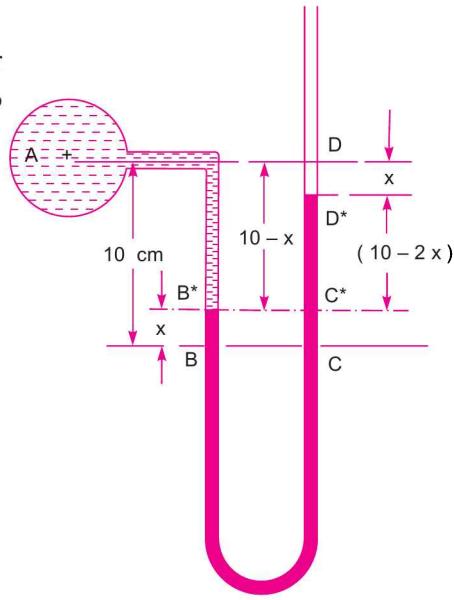


Fig. 2.11 (b)

**Problem 2.12** Fig. 2.12 shows a conical vessel having its outlet at  $A$  to which a U-tube manometer is connected. The reading of the manometer given in the figure shows when the vessel is empty. Find the reading of the manometer when the vessel is completely filled with water.

**Solution.** Vessel is empty. Given :

Difference of mercury level  $h_2 = 20 \text{ cm}$

Let  $h_1$  = Height of water above  $X-X$

Sp. gr. of mercury,  $S_2 = 13.6$

Sp. gr. of water,  $S_1 = 1.0$

Density of mercury,  $\rho_2 = 13.6 \times 1000$

Density of water,  $\rho_1 = 1000$

Equating the pressure above datum line  $X-X$ , we have

$$\rho_2 \times g \times h_2 = \rho_1 \times g \times h_1$$

or  $13.6 \times 1000 \times 9.81 \times 0.2 = 1000 \times 9.81 \times h_1$

$$h_1 = 2.72 \text{ m of water.}$$

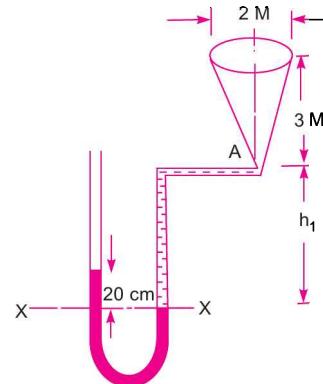


Fig. 2.12

**Vessel is full of water.** When vessel is full of water, the

pressure in the right limb will increase and mercury level in the right limb will go down. Let the distance through which mercury goes down in the right limb be,  $y \text{ cm}$  as shown in Fig. 2.13. The mercury will rise in the left by a distance of  $y \text{ cm}$ . Now the datum line is  $Z-Z$ . Equating the pressure above the datum line  $Z-Z$ .

Pressure in left limb = Pressure in right limb

$$13.6 \times 1000 \times 9.81 \times (0.2 + 2y/100) = 1000 \times 9.81 \times (3 + h_1 + y/100)$$

or  $13.6 \times (0.2 + 2y/100) = (3 + 2.72 + y/100)$  ( $\because h_1 = 2.72 \text{ cm}$ )

or  $2.72 + 27.2y/100 = 3 + 2.72 + y/100$

or  $(27.2y - y)/100 = 3.0$

or  $26.2y = 3 \times 100 = 300$

$$\therefore y = \frac{300}{26.2} = 11.45 \text{ cm}$$

The difference of mercury level in two limbs

$$= (20 + 2y) \text{ cm of mercury}$$

$$= 20 + 2 \times 11.45 = 20 + 22.90$$

$$= 42.90 \text{ cm of mercury}$$

$\therefore$  Reading of manometer = 42.90 cm. Ans.

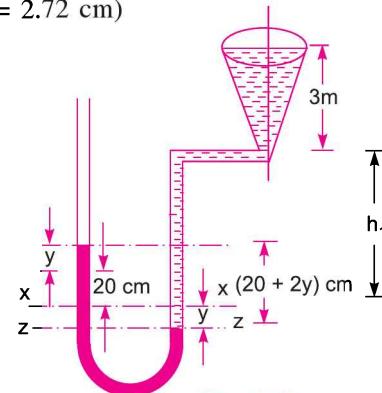


Fig. 2.13

**Problem 2.13** A pressure gauge consists of two cylindrical bulbs B and C each of 10 sq. cm cross-sectional area, which are connected by a U-tube with vertical limbs each of 0.25 sq. cm cross-sectional area. A red liquid of specific gravity 0.9 is filled into C and clear water is filled into B, the surface of separation being in the limb attached to C. Find the displacement of the surface of separation when the pressure on the surface in C is greater than that in B by an amount equal to 1 cm head of water.

**Solution.** Given :

Area of each bulb B and C,  $A = 10 \text{ cm}^2$

Area of each vertical limb,  $a = 0.25 \text{ cm}^2$

Sp. gr. of red liquid  $= 0.9 \quad \therefore \text{Its density} = 900 \text{ kg/m}^3$

Let  $X-X$  = Initial separation level

$h_C$  = Height of red liquid above  $X-X$

$h_B$  = Height of water above  $X-X$

Pressure above  $X-X$  in the left limb  $= 1000 \times 9.81 \times h_B$

Pressure above  $X-X$  in the right limb  $= 900 \times 9.81 \times h_C$

Equating the two pressure, we get

$$1000 \times 9.81 \times h_B = 900 \times 9.81 \times h_C$$

$$\therefore h_B = 0.9 h_C \quad \dots(i)$$

When the pressure head over the surface in C is increased by 1 cm of water, let the separation level falls by an amount equal to Z. Then Y-Y becomes the final separation level.

Now fall in surface level of C multiplied by cross-sectional area of bulb C must be equal to the fall in separation level multiplied by cross-sectional area of limb.

$\therefore$  Fall in surface level of C

$$= \frac{\text{Fall in separation level} \times a}{A}$$

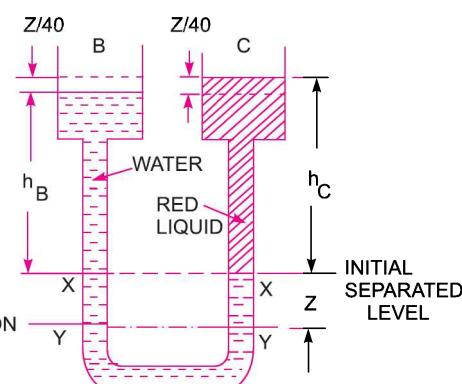


Fig. 2.14

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$$= \frac{Z \times a}{A} = \frac{Z \times 0.25}{10} = \frac{Z}{40}.$$

Also fall in surface level of C

= Rise in surface level of B

$$= \frac{Z}{40}$$

The pressure of 1 cm (or 0.01 m) of water =  $\rho gh = 1000 \times 9.81 \times 0.01 = 98.1 \text{ N/m}^2$

Consider final separation level Y-Y

$$\text{Pressure above } Y-Y \text{ in the left limb} = 1000 \times 9.81 \left( Z + h_B + \frac{Z}{40} \right)$$

$$\text{Pressure above } Y-Y \text{ in the right limb} = 900 \times 9.81 \left( Z + h_C - \frac{Z}{40} \right) + 98.1$$

Equating the two pressure, we get

$$1000 \times 9.81 \left( Z + h_B + \frac{Z}{40} \right) = \left( Z + h_C - \frac{Z}{40} \right) 900 \times 9.81 + 98.1$$

Dividing by 9.81, we get

$$1000 \left( Z + h_B + \frac{Z}{40} \right) = 900 \left( Z + h_C - \frac{Z}{40} \right) + 10$$

$$\text{Dividing by 1000, we get } Z + h_B + \frac{Z}{40} = 0.9 \left( Z + h_C - \frac{Z}{40} \right) + 0.01$$

But from equation (i),  $h_B = 0.9 h_C$

$$\therefore Z + 0.9 h_C + \frac{Z}{40} = \frac{39Z}{40} \times 0.9 + 0.9 h_C + 0.01$$

$$\text{or } \frac{41Z}{40} = \frac{39}{40} \times .9Z + .01$$

$$\text{or } Z \left( \frac{41}{40} - \frac{39 \times .9}{40} \right) = .01 \quad \text{or} \quad Z \left( \frac{41 - 35.1}{40} \right) = .01$$

$$\therefore Z = \frac{40 \times 0.01}{5.9} = \mathbf{0.0678 \text{ m} = 6.78 \text{ cm. Ans.}}$$

**2.6.3 Single Column Manometer.** Single column manometer is a modified form of a U-tube manometer in which a reservoir, having a large cross-sectional area (about 100 times) as compared to the area of the tube is connected to one of the limbs (say left limb) of the manometer as shown in Fig. 2.15. Due to large cross-sectional area of the reservoir, for any variation in pressure, the change in the liquid level in the reservoir will be very small which may be neglected and hence the pressure is given by the height of liquid in the other limb. The other limb may be vertical or inclined. Thus there are two types of single column manometer as :

1. Vertical Single Column Manometer.
2. Inclined Single Column Manometer.

### I. Vertical Single Column Manometer

Fig. 2.15 shows the vertical single column manometer. Let X-X be the datum line in the reservoir and in the right limb of the manometer, when it is not connected to the pipe. When the manometer is

connected to the pipe, due to high pressure at A, the heavy liquid in the reservoir will be pushed downward and will rise in the right limb.

Let  $\Delta h$  = Fall of heavy liquid in reservoir

$h_2$  = Rise of heavy liquid in right limb

$h_1$  = Height of centre of pipe above X-X

$p_A$  = Pressure at A, which is to be measured

A = Cross-sectional area of the reservoir

a = Cross-sectional area of the right limb

$S_1$  = Sp. gr. of liquid in pipe

$S_2$  = Sp. gr. of heavy liquid in reservoir and right limb

$\rho_1$  = Density of liquid in pipe

$\rho_2$  = Density of liquid in reservoir

Fall of heavy liquid in reservoir will cause a rise of heavy liquid level in the right limb.

$$\therefore A \times \Delta h = a \times h_2$$

$$\therefore \Delta h = \frac{a \times h_2}{A} \quad \dots(i)$$

Now consider the datum line Y-Y as shown in Fig. 2.15. Then pressure in the right limb above Y-Y.

$$= \rho_2 \times g \times (\Delta h + h_2)$$

$$\text{Pressure in the left limb above Y-Y} = \rho_1 \times g \times (\Delta h + h_1) + p_A$$

Equating these pressures, we have

$$\rho_2 \times g \times (\Delta h + h_2) = \rho_1 \times g \times (\Delta h + h_1) + p_A$$

or

$$\begin{aligned} p_A &= \rho_2 g (\Delta h + h_2) - \rho_1 g (\Delta h + h_1) \\ &= \Delta h [\rho_2 g - \rho_1 g] + h_2 \rho_2 g - h_1 \rho_1 g \end{aligned}$$

But from equation (i),

$$\Delta h = \frac{a \times h_2}{A}$$

$$\therefore p_A = \frac{a \times h_2}{A} [\rho_2 g - \rho_1 g] + h_2 \rho_2 g - h_1 \rho_1 g \quad \dots(2.9)$$

As the area A is very large as compared to a, hence ratio  $\frac{a}{A}$  becomes very small and can be neglected.

$$\text{Then } p_A = h_2 \rho_2 g - h_1 \rho_1 g \quad \dots(2.10)$$

From equation (2.10), it is clear that as  $h_1$  is known and hence by knowing  $h_2$  or rise of heavy liquid in the right limb, the pressure at A can be calculated.

## 2. Inclined Single Column Manometer

Fig. 2.16 shows the inclined single column manometer. This manometer is more sensitive. Due to inclination the distance moved by the heavy liquid in the right limb will be more.

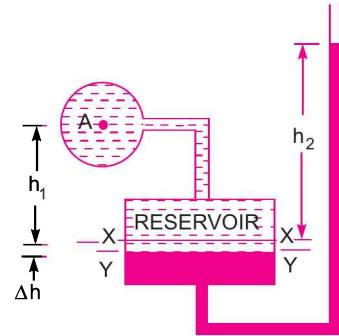


Fig. 2.15 Vertical single column manometer.

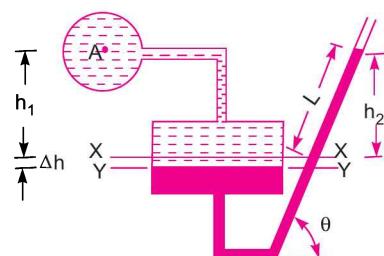


Fig. 2.16 Inclined single column manometer.

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Let  $L$  = Length of heavy liquid moved in right limb from  $X-X$

$\theta$  = Inclination of right limb with horizontal

$h_2$  = Vertical rise of heavy liquid in right limb from  $X-X = L \times \sin \theta$

From equation (2.10), the pressure at  $A$  is

$$p_A = h_2 \rho_2 g - h_1 \rho_1 g.$$

Substituting the value of  $h_2$ , we get

$$p_A = \sin \theta \times \rho_2 g - h_1 \rho_1 g. \quad \dots(2.11)$$

**Problem 2.14** A single column manometer is connected to a pipe containing a liquid of sp. gr. 0.9 as shown in Fig. 2.17. Find the pressure in the pipe if the area of the reservoir is 100 times the area of the tube for the manometer reading shown in Fig. 2.17. The specific gravity of mercury is 13.6.

**Solution.** Given :

Sp. gr. of liquid in pipe,  $S_1 = 0.9$

$\therefore$  Density  $\rho_1 = 900 \text{ kg/m}^3$

Sp. gr. of heavy liquid,  $S_2 = 13.6$

Density,  $\rho_2 = 13.6 \times 1000$

$$\frac{\text{Area of reservoir}}{\text{Area of right limb}} = \frac{A}{a} = 100$$

Height of liquid,  $h_1 = 20 \text{ cm} = 0.2 \text{ m}$

Rise of mercury in right limb,

$$h_2 = 40 \text{ cm} = 0.4 \text{ m}$$

Let

$p_A$  = Pressure in pipe

Using equation (2.9), we get

$$\begin{aligned} p_A &= \frac{a}{A} h_2 [\rho_2 g - \rho_1 g] + h_2 \rho_2 g - h_1 \rho_1 g \\ &= \frac{1}{100} \times 0.4 [13.6 \times 1000 \times 9.81 - 900 \times 9.81] + 0.4 \times 13.6 \times 1000 \times 9.81 - 0.2 \times 900 \times 9.81 \\ &= \frac{0.4}{100} [133416 - 8829] + 53366.4 - 1765.8 \\ &= 533.664 + 53366.4 - 1765.8 \text{ N/m}^2 = 52134 \text{ N/m}^2 = 5.21 \text{ N/cm}^2. \text{ Ans.} \end{aligned}$$

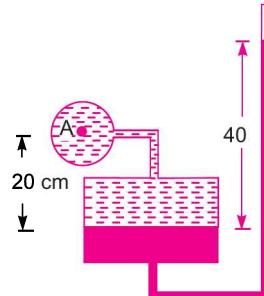


Fig. 2.17

## ► 2.7 DIFFERENTIAL MANOMETERS

Differential manometers are the devices used for measuring the difference of pressures between two points in a pipe or in two different pipes. A differential manometer consists of a U-tube, containing a heavy liquid, whose two ends are connected to the points, whose difference of pressure is to be measured. Most commonly types of differential manometers are :

1. U-tube differential manometer and
2. Inverted U-tube differential manometer.

**2.7.1 U-tube Differential Manometer.** Fig. 2.18 shows the differential manometers of U-tube type.

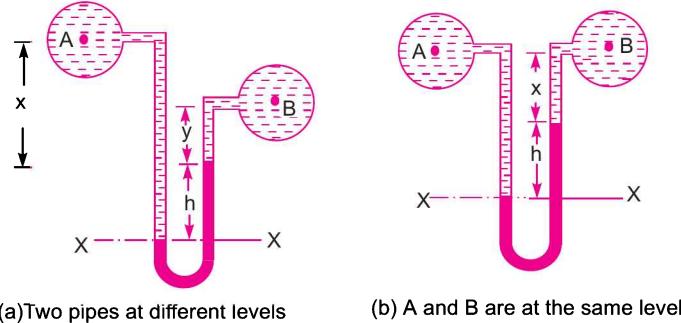


Fig. 2.18 U-tube differential manometers.

In Fig. 2.18 (a), the two points A and B are at different level and also contains liquids of different sp. gr. These points are connected to the U-tube differential manometer. Let the pressure at A and B are  $p_A$  and  $p_B$ .

Let  $h$  = Difference of mercury level in the U-tube.

$y$  = Distance of the centre of B, from the mercury level in the right limb.

$x$  = Distance of the centre of A, from the mercury level in the right limb.

$\rho_1$  = Density of liquid at A.

$\rho_2$  = Density of liquid at B.

$\rho_g$  = Density of heavy liquid or mercury.

Taking datum line at X-X.

Pressure above X-X in the left limb =  $\rho_1 g(h + x) + p_A$

where  $p_A$  = pressure at A.

Pressure above X-X in the right limb =  $\rho_g \times g \times h + \rho_2 \times g \times y + p_B$

where  $p_B$  = Pressure at B.

Equating the two pressure, we have

$$\begin{aligned} \rho_1 g(h + x) + p_A &= \rho_g \times g \times h + \rho_2 g y + p_B \\ \therefore p_A - p_B &= \rho_g \times g \times h + \rho_2 g y - \rho_1 g(h + x) \\ &= h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x \end{aligned} \quad \dots(2.12)$$

$\therefore$  Difference of pressure at A and B =  $h \times g(\rho_g - \rho_1) + \rho_2 g y - \rho_1 g x$

In Fig. 2.18 (b), the two points A and B are at the same level and contains the same liquid of density  $\rho_1$ . Then

Pressure above X-X in right limb =  $\rho_g \times g \times h + \rho_1 \times g \times x + p_B$

Pressure above X-X in left limb =  $\rho_1 \times g \times (h + x) + p_A$

Equating the two pressure

$$\begin{aligned} \rho_g \times g \times h + \rho_1 g x + p_B &= \rho_1 \times g \times (h + x) + p_A \\ \therefore p_A - p_B &= \rho_g \times g \times h + \rho_1 g x - \rho_1 g(h + x) \\ &= g \times h(\rho_g - \rho_1). \end{aligned} \quad \dots(2.13)$$

**Problem 2.15** A pipe contains an oil of sp. gr. 0.9. A differential manometer connected at the two points A and B shows a difference in mercury level as 15 cm. Find the difference of pressure at the two points.

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**Solution.** Given :

$$\text{Sp. gr. of oil, } S_1 = 0.9 \quad \therefore \text{ Density, } \rho_1 = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$\text{Difference in mercury level, } h = 15 \text{ cm} = 0.15 \text{ m}$$

$$\text{Sp. gr. of mercury, } S_g = 13.6 \quad \therefore \text{ Density, } \rho_g = 13.6 \times 1000 \text{ kg/m}^3$$

The difference of pressure is given by equation (2.13)

or

$$p_A - p_B = g \times h(\rho_g - \rho_1) \\ = 9.81 \times 0.15 (13600 - 900) = 18688 \text{ N/m}^2. \text{ Ans.}$$

**Problem 2.16** A differential manometer is connected at the two points A and B of two pipes as shown in Fig. 2.19. The pipe A contains a liquid of sp. gr. = 1.5 while pipe B contains a liquid of sp. gr. = 0.9. The pressures at A and B are 1 kgf/cm<sup>2</sup> and 1.80 kgf/cm<sup>2</sup> respectively. Find the difference in mercury level in the differential manometer.

**Solution.** Given :

$$\text{Sp. gr. of liquid at A, } S_1 = 1.5 \quad \therefore \rho_1 = 1500$$

$$\text{Sp. gr. of liquid at B, } S_2 = 0.9 \quad \therefore \rho_2 = 900$$

$$\text{Pressure at A, } p_A = 1 \text{ kgf/cm}^2 = 1 \times 10^4 \text{ kgf/m}^2 \\ = 10^4 \times 9.81 \text{ N/m}^2 (\because 1 \text{ kgf} = 9.81 \text{ N})$$

$$\text{Pressure at B, } p_B = 1.8 \text{ kgf/cm}^2 \\ = 1.8 \times 10^4 \text{ kgf/m}^2 \\ = 1.8 \times 10^4 \times 9.81 \text{ N/m}^2 (\because 1 \text{ kgf} = 9.81 \text{ N})$$

$$\text{Density of mercury} = 13.6 \times 1000 \text{ kg/m}^3$$

Taking X-X as datum line.

Pressure above X-X in the left limb

$$= 13.6 \times 1000 \times 9.81 \times h + 1500 \times 9.81 \times (2 + 3) + p_A \\ = 13.6 \times 1000 \times 9.81 \times h + 7500 \times 9.81 + 9.81 \times 10^4$$

Pressure above X-X in the right limb =  $900 \times 9.81 \times (h + 2) + p_B$

$$= 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Equating the two pressure, we get

$$13.6 \times 1000 \times 9.81h + 7500 \times 9.81 + 9.81 \times 10^4 \\ = 900 \times 9.81 \times (h + 2) + 1.8 \times 10^4 \times 9.81$$

Dividing by  $1000 \times 9.81$ , we get

$$13.6h + 7.5 + 10 = (h + 2.0) \times 9 + 18$$

$$\text{or } 13.6h + 17.5 = 0.9h + 1.8 + 18 = 0.9h + 19.8$$

$$\text{or } (13.6 - 0.9)h = 19.8 - 17.5 \text{ or } 12.7h = 2.3$$

$$\therefore h = \frac{2.3}{12.7} = 0.181 \text{ m} = 18.1 \text{ cm. Ans.}$$

**Problem 2.17** A differential manometer is connected at the two points A and B as shown in Fig. 2.20. At B air pressure is  $9.81 \text{ N/cm}^2$  (abs), find the absolute pressure at A.

**Solution.** Given :

$$\text{Air pressure at } B = 9.81 \text{ N/cm}^2$$

$$\text{or } p_B = 9.81 \times 10^4 \text{ N/m}^2$$

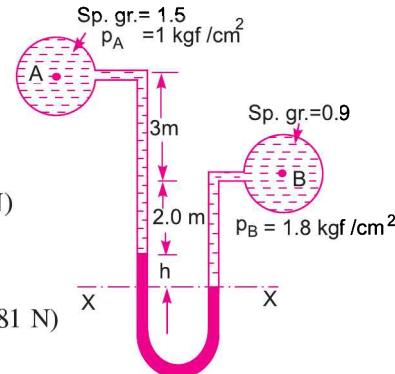


Fig. 2.19

$$\begin{aligned}\text{Density of oil} &= 0.9 \times 1000 = 900 \text{ kg/m}^3 \\ \text{Density of mercury} &= 13.6 \times 1000 \text{ kg/m}^3\end{aligned}$$

Let the pressure at A is  $p_A$

Taking datum line at X-X

Pressure above X-X in the right limb

$$\begin{aligned}&= 1000 \times 9.81 \times 0.6 + p_B \\ &= 5886 + 98100 = 103986\end{aligned}$$

Pressure above X-X in the left limb

$$\begin{aligned}&= 13.6 \times 1000 \times 9.81 \times 0.1 + 900 \\ &\quad \times 9.81 \times 0.2 + p_A \\ &= 13341.6 + 1765.8 + p_A\end{aligned}$$

Equating the two pressure heads

$$\begin{aligned}103986 &= 13341.6 + 1765.8 + p_A \\ \therefore p_A &= 103986 - 15107.4 = 88876.8 \\ \therefore p_A &= 88876.8 \text{ N/m}^2 = \frac{88876.8 \text{ N}}{10000 \text{ cm}^2} = 8.887 \frac{\text{N}}{\text{cm}^2}.\end{aligned}$$

$\therefore$  Absolute pressure at A =  $8.887 \text{ N/cm}^2$ . Ans.

**2.7.2 Inverted U-tube Differential Manometer.** It consists of an inverted U-tube, containing a light liquid. The two ends of the tube are connected to the points whose difference of pressure is to be measured. It is used for measuring difference of low pressures. Fig. 2.21 shows an inverted U-tube differential manometer connected to the two points A and B. Let the pressure at A is more than the pressure at B.

Let  $h_1$  = Height of liquid in left limb below the datum line X-X

$h_2$  = Height of liquid in right limb

$h$  = Difference of liquid height

$\rho_1$  = Density of liquid at A

$\rho_2$  = Density of liquid at B

$\rho_s$  = Density of light liquid

$p_A$  = Pressure at A

$p_B$  = Pressure at B.

Taking X-X as datum line. Then pressure in the left limb below X-X

$$= p_A - \rho_1 \times g \times h_1.$$

Pressure in the right limb below X-X

$$= p_B - \rho_2 \times g \times h_2 - \rho_s \times g \times h$$

Equating the two pressure

$$p_A - \rho_1 \times g \times h_1 = p_B - \rho_2 \times g \times h_2 - \rho_s \times g \times h$$

$$\text{or } p_A - p_B = \rho_1 \times g \times h_1 - \rho_2 \times g \times h_2 - \rho_s \times g \times h. \quad \dots(2.14)$$

**Problem 2.18** Water is flowing through two different pipes to which an inverted differential manometer having an oil of sp. gr. 0.8 is connected. The pressure head in the pipe A is 2 m of water, find the pressure in the pipe B for the manometer readings as shown in Fig. 2.22.

**Solution.** Given :

$$\text{Pressure head at } A = \frac{p_A}{\rho g} = 2 \text{ m of water}$$

$$\therefore p_A = \rho \times g \times 2 = 1000 \times 9.81 \times 2 = 19620 \text{ N/m}^2$$

Fig. 2.22 shows the arrangement. Taking X-X as datum line.

$$\text{Pressure below X-X in the left limb} = p_A - \rho_1 \times g \times h_1$$

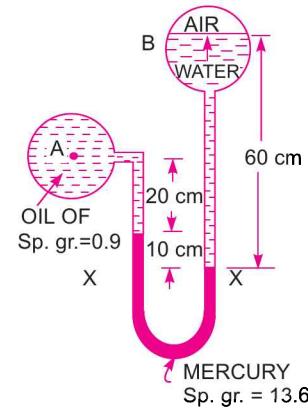


Fig. 2.20

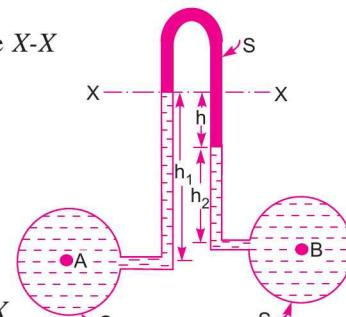


Fig. 2.21

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$$= 19620 - 1000 \times 9.81 \times 0.3 = 16677 \text{ N/m}^2.$$

Pressure below X-X in the right limb

$$\begin{aligned} &= p_B - 1000 \times 9.81 \times 0.1 - 800 \times 9.81 \times 0.12 \\ &= p_B - 981 - 941.76 = p_B - 1922.76 \end{aligned}$$

Equating the two pressure, we get

$$16677 = p_B - 1922.76$$

or

$$p_B = 16677 + 1922.76 = 18599.76 \text{ N/m}^2$$

or

$$p_B = 1.8599 \text{ N/cm}^2. \text{ Ans.}$$

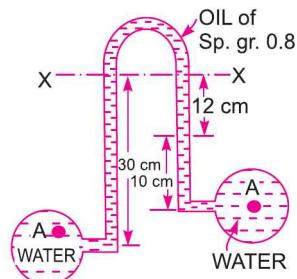


Fig. 2.22

**Problem 2.19** In Fig. 2.23, an inverted differential manometer is connected to two pipes A and B which convey water. The fluid in manometer is oil of sp. gr. 0.8. For the manometer readings shown in the figure, find the pressure difference between A and B.

**Solution.** Given :

$$\text{Sp. gr. of oil} = 0.8 \quad \therefore \rho_s = 800 \text{ kg/m}^3$$

Difference of oil in the two limbs

$$= (30 + 20) - 30 = 20 \text{ cm}$$

Taking datum line at X-X

Pressure in the left limb below X-X

$$= p_A - 1000 \times 9.81 \times 0$$

$$= p_A - 2943$$

Pressure in the right limb below X-X

$$= p_B - 1000 \times 9.81 \times 0.3 - 800 \times 9.81 \times 0.2$$

$$= p_B - 2943 - 1569.6 = p_B - 4512.6$$

Equating the two pressure  $p_A - 2943 = p_B - 4512.6$

$$\therefore p_B - p_A = 4512.6 - 2943 = 1569.6 \text{ N/m}^2. \text{ Ans.}$$

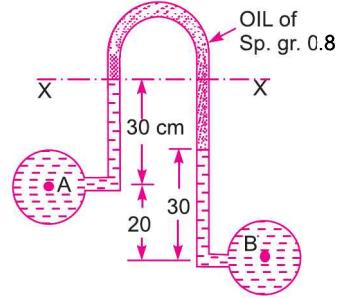


Fig. 2.23

**Problem 2.20** Find out the differential reading 'h' of an inverted U-tube manometer containing oil of specific gravity 0.7 as the manometric fluid when connected across pipes A and B as shown in Fig. 2.24 below, conveying liquids of specific gravities 1.2 and 1.0 and immiscible with manometric fluid. Pipes A and B are located at the same level and assume the pressures at A and B to be equal.

**Solution.** Given :

Fig. 2.24 shows the arrangement. Taking X-X as datum line.

Let

$$p_A = \text{Pressure at } A$$

$$p_B = \text{Pressure at } B$$

Density of liquid in pipe A

$$= \text{Sp. gr.} \times 1000$$

$$= 1.2 \times 1000$$

$$= 1200 \text{ kg/m}^3$$

Density of liquid in pipe B

$$= 1 \times 1000 = 1000 \text{ kg/m}^3$$

Density of oil

$$= 0.7 \times 1000 = 700 \text{ kg/m}^3$$

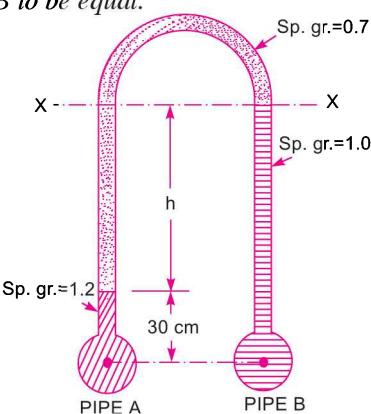


Fig. 2.24

Now pressure below X-X in the left limb

$$= p_A - 1200 \times 9.81 \times 0.3 - 700 \times 9.81 \times h$$

Pressure below X-X in the right limb

$$= p_B - 1000 \times 9.81 \times (h + 0.3)$$

Equating the two pressure, we get

$$p_A - 1200 \times 9.81 \times 0.3 - 700 \times 9.81 \times h = p_B - 1000 \times 9.81 (h + 0.3)$$

But

$$p_A = p_B \text{ (given)}$$

$$\therefore - 1200 \times 9.81 \times 0.3 - 700 \times 9.81 \times h = - 1000 \times 9.81 (h + 0.3)$$

Dividing by  $1000 \times 9.81$

$$- 1.2 \times 0.3 - 0.7h = - (h + 0.3)$$

or

$$0.3 \times 1.2 + 0.7h = h + 0.3 \text{ or } 0.36 - 0.3 = h - 0.7h = 0.3h$$

$$\therefore h = \frac{0.36 - 0.30}{0.30} = \frac{0.06}{0.30} \text{ m}$$

$$= \frac{1}{5} \text{ m} = \frac{1}{5} \times 100 = 20 \text{ cm. Ans.}$$

**Problem 2.21** An inverted U-tube manometer is connected to two horizontal pipes A and B through which water is flowing. The vertical distance between the axes of these pipes is 30 cm. When an oil of specific gravity 0.8 is used as a gauge fluid, the vertical heights of water columns in the two limbs of the inverted manometer (when measured from the respective centre lines of the pipes) are found to be same and equal to 35 cm. Determine the difference of pressure between the pipes.

**Solution.** Given :

Specific gravity of measuring liquid = 0.8

The arrangement is shown in Fig. 2.24 (a).

Let  $p_A$  = pressure at A

$p_B$  = pressure at B.

The points C and D lie on the same horizontal line.

Hence pressure at C should be equal to pressure at D.

$$\begin{aligned} \text{But pressure at } C &= p_A - \rho g h \\ &= p_A - 1000 \times 9.81 \times (0.35) \end{aligned}$$

$$\begin{aligned} \text{And pressure at } D &= p_B - \rho_1 g h_1 - \rho_2 g h_2 \\ &= p_B - 1000 \times 9.81 \times (0.35) - 800 \times 9.81 \times 0.3 \end{aligned}$$

But pressure at C = pressure at D

$$\begin{aligned} \therefore p_A - 1000 \times 9.81 \times 0.35 &= p_B - 1000 \times 9.81 \times 0.35 - 800 \times 9.81 \times 0.3 \\ &= p_B - 1000 \times 9.81 \times 0.35 - 800 \times 9.81 \times 0.3 \end{aligned}$$

$$\text{or } 800 \times 9.81 \times 0.3 = p_B - p_A$$

$$\text{or } p_B - p_A = 800 \times 9.81 \times 0.3 = 2354.4 \frac{\text{N}}{\text{m}^2}. \text{ Ans.}$$

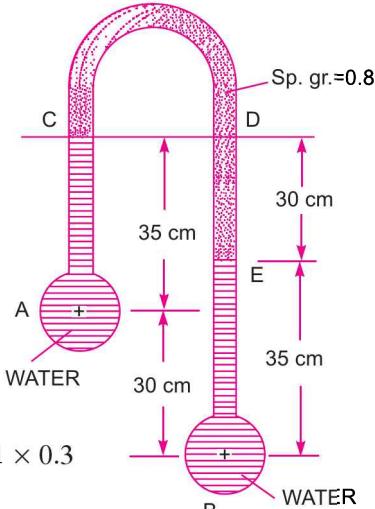


Fig. 2.24 (a)

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### ► 2.8 PRESSURE AT A POINT IN COMPRESSIBLE FLUID

For compressible fluids, density ( $\rho$ ) changes with the change of pressure and temperature. Such problems are encountered in aeronautics, oceanography and meteorology where we are concerned with atmospheric\* air where density, pressure and temperature changes with elevation. Thus for fluids with variable density, equation (2.4) cannot be integrated, unless the relationship between  $p$  and  $\rho$  is known. For gases the equation of state is

$$\frac{p}{\rho} = RT \quad \dots(2.15)$$

or

$$\rho = \frac{p}{RT}$$

Now equation (2.4) is  $\frac{dp}{dz} = w = \rho g = \frac{p}{RT} \times g$

$$\therefore \frac{dp}{p} = \frac{g}{RT} dz \quad \dots(2.16)$$

In equation (2.4),  $Z$  is measured vertically downward. But if  $Z$  is measured vertically up, then  $\frac{dp}{dz} = -\rho g$  and hence equation (2.16) becomes

$$\frac{dp}{p} = \frac{-g}{RT} dz \quad \dots(2.17)$$

**2.8.1 Isothermal Process.** **Case I.** If temperature  $T$  is constant which is true for **isothermal process**, equation (2.17) can be integrated as

$$\int_{p_0}^p \frac{dp}{p} = - \int_{Z_0}^Z \frac{g}{RT} dz = - \frac{g}{RT} \int_{Z_0}^Z dz$$

or  $\log \frac{p}{p_0} = \frac{-g}{RT} [Z - Z_0]$

where  $p_0$  is the pressure where height is  $Z_0$ . If the datum line is taken at  $Z_0$ , then  $Z_0 = 0$  and  $p_0$  becomes the pressure at datum line.

$$\therefore \log \frac{p}{p_0} = \frac{-g}{RT} Z$$

$$\frac{p}{p_0} = e^{-gZ/RT}$$

or pressure at a height  $Z$  is given by  $p = p_0 e^{-gZ/RT}$   $\dots(2.18)$

**2.8.2 Adiabatic Process.** If temperature  $T$  is not constant but the process follows adiabatic law then the relation between pressure and density is given by

$$\frac{p}{\rho^k} = \text{Constant} = C \quad \dots(i)$$

---

\* The standard atmospheric pressure, temperature and density referred to STP at the sea-level are :  
Pressure = 101.325 kN/m<sup>2</sup>; Temperature = 15°C and Density = 1.225 kg/m<sup>3</sup>.

where  $k$  is ratio of specific constant.

$$\therefore \rho^k = \frac{p}{C}$$

$$\text{or } \rho = \left( \frac{p}{C} \right)^{1/k} \quad \dots(ii)$$

Then equation (2.4) for  $Z$  measured vertically up becomes,

$$\frac{dp}{dZ} = -\rho g = -\left( \frac{p}{C} \right)^{1/k} g$$

$$\text{or } \frac{dp}{\left( \frac{p}{C} \right)^{1/k}} = -gdZ \text{ or } C^{1/k} \frac{dp}{p^k} = -gdZ$$

Integrating, we get  $\int_{p_0}^p C^{1/k} p^{-1/k} dp = \int_{Z_0}^Z -gdZ$

$$\text{or } C^{1/k} \left[ \frac{p^{-1/k+1}}{-\frac{1}{k} + 1} \right]_{p_0}^p = -g [Z]_{Z_0}^Z$$

$$\text{or } \left[ \frac{C^{1/k} p^{-1/k+1}}{-\frac{1}{k} + 1} \right]_{p_0}^p = -g [Z]_{Z_0}^Z \quad [C \text{ is a constant, can be taken inside}]$$

$$\text{But from equation (i), } C^{1/k} = \left( \frac{p}{\rho^k} \right)^{1/k} = \frac{p^{1/k}}{\rho}$$

Substituting this value of  $C^{1/k}$  above, we get

$$\left[ \frac{p^{1/k}}{\rho} \frac{p^{-1/k+1}}{-\frac{1}{k} + 1} \right]_{p_0}^p = -g [Z - Z_0]$$

$$\text{or } \left[ \frac{p^{\frac{1-1+k}{k}}}{\rho^{\frac{k-1}{k}}} \right]_{p_0}^p = -g [Z - Z_0] \text{ or } \left[ \frac{k}{k-1} \frac{p}{\rho} \right]_{p_0}^p = -g [Z - Z_0]$$

$$\text{or } \frac{k}{k-1} \left[ \frac{p}{\rho} - \frac{p_0}{\rho_0} \right] = -g [Z - Z_0]$$

If datum line is taken at  $Z_0$ , where pressure, temperature and density are  $p_0$ ,  $T_0$  and  $\rho_0$ , then  $Z_0 = 0$ .

$$\therefore \frac{k}{k-1} \left[ \frac{p}{\rho} - \frac{p_0}{\rho_0} \right] = -gZ \quad \text{or} \quad \frac{p}{\rho} - \frac{p_0}{\rho_0} = -gZ \frac{(k-1)}{k}$$

$$\text{or } \frac{p}{\rho} = \frac{p_0}{\rho_0} - gZ \frac{(k-1)}{k} = \frac{p_0}{\rho_0} \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]$$

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or  $\frac{p}{\rho} \times \frac{\rho_0}{p_0} = \left[ 1 + \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]^{1/k}$  ... (iii)

But from equation (i),  $\frac{p}{\rho^k} = \frac{p_0}{\rho_0^k}$  or  $\left( \frac{\rho_0}{\rho} \right)^k = \frac{p_0}{p}$  or  $\frac{\rho_0}{\rho} = \left( \frac{p_0}{p} \right)^{1/k}$

Substituting the value of  $\frac{\rho_0}{\rho}$  in equation (iii), we get

$$\frac{p}{p_0} \times \left( \frac{p_0}{p} \right)^{1/k} = \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]$$

or  $\frac{p}{p_0} \times \left( \frac{p}{p_0} \right)^{-1/k} = \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]$

or  $\left( \frac{p}{p_0} \right)^{1-\frac{1}{k}} = \left( \frac{p}{p_0} \right)^{\frac{k-1}{k}} = \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]$

$$\therefore \frac{p}{p_0} = \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]^{\frac{k}{k-1}}$$

$\therefore$  Pressure at a height  $Z$  from ground level is given by

$$p = p_0 \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]^{\frac{k}{k-1}} \quad \dots (2.19)$$

In equation (2.19),  $p_0$  = pressure at ground level, where  $Z_0 = 0$

$\rho_0$  = density of air at ground level

Equation of state is  $\frac{p_0}{\rho_0} = RT_0$  or  $\frac{\rho_0}{p_0} = \frac{1}{RT_0}$

Substituting the values of  $\frac{\rho_0}{p_0}$  in equation (2.19), we get

$$p = p_0 \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\frac{k}{k-1}} \quad \dots (2.20)$$

**2.8.3 Temperature at any Point in Compressible Fluid.** For the adiabatic process, the temperature at any height in air is calculated as :

Equation of state at ground level and at a height  $Z$  from ground level is written as

$$\frac{p_0}{\rho_0} = RT_0 \text{ and } \frac{p}{\rho} = RT$$

Dividing these equations, we get

$$\left( \frac{p_0}{\rho_0} \right) \div \frac{p}{\rho} = \frac{RT_0}{RT} = \frac{T_0}{T} \quad \text{or} \quad \frac{p_0}{\rho_0} \times \frac{\rho}{p} = \frac{T_0}{T}$$

or  $\frac{T}{T_0} = \frac{\rho_0}{p_0} \times \frac{p}{\rho} = \frac{p}{p_0} \times \frac{\rho_0}{\rho}$  ... (i)

But  $\frac{p}{p_0}$  from equation (2.20) is given by

$$\frac{p}{p_0} = \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\frac{1}{k-1}}$$

Also for adiabatic process  $\frac{p}{\rho^k} = \frac{p_0}{\rho_0^k}$  or  $\left( \frac{\rho_0}{\rho} \right)^k = \frac{p_0}{p}$

$$\text{or } \frac{\rho_0}{\rho} = \left( \frac{p_0}{p} \right)^{\frac{1}{k}} = \left( \frac{p}{p_0} \right)^{-\frac{1}{k}}$$

$$= \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\left( \frac{k}{k-1} \right) \times \left( -\frac{1}{k} \right)} = \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{-\frac{1}{k-1}}$$

Substituting the values of  $\frac{p}{p_0}$  and  $\frac{\rho_0}{\rho}$  in equation (i), we get

$$\begin{aligned} \frac{T}{T_0} &= \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\frac{1}{k-1}} \times \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{-\frac{1}{k-1}} \\ &= \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\frac{1}{k-1} - \frac{1}{k-1}} = \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right] \\ \therefore T &= T_0 \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right] \end{aligned} \quad \dots(2.21)$$

**2.8.4 Temperature Lapse-Rate (L).** It is defined as the rate at which the temperature changes with elevation. To obtain an expression for the temperature lapse-rate, the temperature given by equation (2.21) is differentiated with respect to Z as

$$\frac{dT}{dZ} = \frac{d}{dZ} \left[ T_0 \left( 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right) \right]$$

where  $T_0$ ,  $K$ ,  $g$  and  $R$  are constant

$$\therefore \frac{dT}{dZ} = -\frac{k-1}{k} \times \frac{g}{RT_0} \times T_0 = \frac{-g}{R} \left( \frac{k-1}{k} \right)$$

The temperature lapse-rate is denoted by  $L$  and hence

$$L = \frac{dT}{dZ} = \frac{-g}{R} \left( \frac{k-1}{k} \right) \quad \dots(2.22)$$

In equation (2.22), if (i)  $k = 1$  which means isothermal process,  $\frac{dT}{dZ} = 0$ , which means temperature is constant with height.

(ii) If  $k > 1$ , the lapse-rate is negative which means temperature decreases with the increase in height.

In atmosphere, the value of  $k$  varies with height and hence the value of temperature lapse-rate also varies. From the sea-level upto an elevation of about 11000 m (or 11 km), the temperature of air decreases uniformly at the rate of  $0.0065^\circ\text{C}/\text{m}$ . From 11000 m to 32000 m, the temperature remains constant at  $-56.5^\circ\text{C}$  and hence in this range lapse-rate is zero. Temperature rises again after 32000 m in air.

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**Problem 2.22 (SI Units)** If the atmosphere pressure at sea level is  $10.143 \text{ N/cm}^2$ , determine the pressure at a height of  $2500 \text{ m}$  assuming the pressure variation follows (i) Hydrostatic law, and (ii) isothermal law. The density of air is given as  $1.208 \text{ kg/m}^3$ .

**Solution.** Given :

Pressure at sea-level,

$$p_0 = 10.143 \text{ N/cm}^2 = 10.143 \times 10^4 \text{ N/m}^2$$

Height,

$$Z = 2500 \text{ m}$$

Density of air,

$$\rho_0 = 1.208 \text{ kg/m}^3$$

(i) **Pressure by hydrostatic law.** For hydrostatic law,  $\rho$  is assumed constant and hence  $p$  is given by equation  $\frac{dp}{dZ} = -\rho g$

Integrating, we get

$$\int_{p_0}^p dp = \int -\rho g dZ = -\rho g \int_{Z_0}^Z dZ$$

or

$$p - p_0 = -\rho g [Z - Z_0]$$

For datum line at sea-level,

$$Z_0 = 0$$

∴

$$\begin{aligned} p - p_0 &= -\rho g Z \quad \text{or} \quad p = p_0 - \rho g Z \\ &= 10.143 \times 10^4 - 1.208 \times 9.81 \times 2500 [\because \rho = \rho_0 = 1.208] \\ &= 101430 - 29626 = 71804 \frac{\text{N}}{\text{m}^2} \quad \text{or} \quad \frac{71804}{10^4} \text{ N/cm}^2 \\ &= 7.18 \text{ N/cm}^2. \text{ Ans.} \end{aligned}$$

(ii) **Pressure by Isothermal Law.** Pressure at any height  $Z$  by isothermal law is given by equation (2.18) as

$$\begin{aligned} p &= p_0 e^{-gZ/RT} \\ &= 10.143 \times 10^4 e^{-\frac{gZ \times \rho_0}{\rho_0}} \quad \left[ \because \frac{p_0}{\rho_0} = RT \text{ and } \rho_0 g = w_0 \right] \\ &= 10.143 \times 10^4 e^{-\frac{Z\rho_0 \times g}{\rho_0}} \\ &= 10.143 \times 10^4 e^{(-2500 \times 1.208 \times 9.81)/10.143 \times 10^4} \\ &= 101430 \times e^{-292} = 101430 \times \frac{1}{1.3391} = 75743 \text{ N/m}^2 \\ &= \frac{75743}{10^4} \text{ N/cm}^2 = 7.574 \text{ N/cm}^2. \text{ Ans.} \end{aligned}$$

**Problem 2.23** The barometric pressure at sea level is  $760 \text{ mm of mercury}$  while that on a mountain top is  $735 \text{ mm}$ . If the density of air is assumed constant at  $1.2 \text{ kg/m}^3$ , what is the elevation of the mountain top?

**Solution.** Given :

Pressure\* at sea,

$$p_0 = 760 \text{ mm of Hg}$$

$$= \frac{760}{1000} \times 13.6 \times 1000 \times 9.81 \text{ N/m}^2 = 101396 \text{ N/m}^2$$

\* Here pressure head ( $Z$ ) is given as  $760 \text{ mm of Hg}$ . Hence  $(p/\rho g) = 760 \text{ mm of Hg}$ . The density ( $\rho$ ) for mercury  $= 13.6 \times 1000 \text{ kg/m}^3$ . Hence pressure ( $p$ ) will be equal to  $\rho \times g \times Z$  i.e.,  $13.6 \times 1000 \times 9.81 \times \frac{760}{1000} \text{ N/m}^2$ .

Pressure at mountain,

$$p = 735 \text{ mm of Hg}$$

$$= \frac{735}{1000} \times 13.6 \times 1000 \times 9.81 = 98060 \text{ N/m}^2$$

Density of air,

$$\rho = 1.2 \text{ kg/m}^3$$

Let  $h$  = Height of the mountain from sea-level.

We know that as the elevation above the sea-level increases, the atmospheric pressure decreases. Here the density of air is given constant, hence the pressure at any height ' $h$ ' above the sea-level is given by the equation,

$$p = p_0 - \rho \times g \times h$$

$$\text{or } h = \frac{p_0 - p}{\rho \times g} = \frac{101396 - 98060}{1.2 \times 9.81} = 283.33 \text{ m. Ans.}$$

**Problem 2.24** Calculate the pressure at a height of 7500 m above sea level if the atmospheric pressure is 10.143 N/cm<sup>2</sup> and temperature is 15°C at the sea-level, assuming (i) air is incompressible, (ii) pressure variation follows isothermal law, and (iii) pressure variation follows adiabatic law. Take the density of air at the sea-level as equal to 1.285 kg/m<sup>3</sup>. Neglect variation of  $g$  with altitude.

**Solution.** Given :

Height above sea-level,  $Z = 7500 \text{ m}$

Pressure at sea-level,  $p_0 = 10.143 \text{ N/cm}^2 = 10.143 \times 10^4 \text{ N/m}^2$

Temperature at sea-level,  $t_0 = 15^\circ\text{C}$

$$\therefore T_0 = 273 + 15 = 288^\circ\text{K}$$

Density of air,  $\rho = \rho_0 = 1.285 \text{ kg/m}^3$

(i) Pressure when air is incompressible :

$$\frac{dp}{dZ} = -\rho g$$

$$\therefore \int_{p_0}^p dp = - \int_{Z_0}^Z \rho g dz \quad \text{or} \quad p - p_0 = -\rho g [Z - Z_0]$$

$$\begin{aligned} \text{or } p &= p_0 - \rho g Z && \{ \because Z_0 = \text{datum line} = 0 \} \\ &= 10.143 \times 10^4 - 1.285 \times 9.81 \times 7500 \\ &= 101430 - 94543 = 6887 \text{ N/m}^2 = \mathbf{0.688 \frac{N}{cm^2}}. \text{ Ans.} \end{aligned}$$

(ii) Pressure variation follows isothermal law :

Using equation (2.18), we have  $p = p_0 e^{-gZ/RT}$

$$\begin{aligned} &= p_0 e^{-gZ\rho_0/p_0} && \left\{ \because \frac{p_0}{\rho_0} = RT \therefore \frac{\rho_0}{p_0} = \frac{1}{RT} \right\} \\ &= 101430 e^{-gZ\rho_0/p_0} = 101430 e^{-7500 \times 1.285 \times 9.81/101430} \\ &= 101430 e^{-.9320} = 101430 \times .39376 \\ &= \mathbf{39939 \text{ N/m}^2 \text{ or } 3.993 \text{ N/cm}^2}. \text{ Ans.} \end{aligned}$$

(iii) Pressure variation follows adiabatic law : [ $k = 1.4$ ]

Using equation (2.19), we have  $p = p_0 \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]^{k/(k-1)}$ , where  $\rho_0 = 1.285 \text{ kg/m}^3$

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$$\therefore p = 101430 \left[ 1 - \frac{(1.4 - 1.0)}{1.4} \times 9.81 \times \frac{(7500 \times 1.285)}{101430} \right]^{\frac{1.4}{1.4 - 1.0}} \\ = 101430 [1 - .2662]^{1.4/1.4} = 101430 \times (.7337)^{3.5} \\ = 34310 \text{ N/m}^2 \text{ or } 3.431 \frac{\text{N}}{\text{cm}^2}. \text{ Ans.}$$

**Problem 2.25** Calculate the pressure and density of air at a height of 4000 m from sea-level where pressure and temperature of the air are  $10.143 \text{ N/cm}^2$  and  $15^\circ\text{C}$  respectively. The temperature lapse rate is given as  $0.0065^\circ\text{C}/\text{m}$ . Take density of air at sea-level equal to  $1.285 \text{ kg/m}^3$ .

**Solution.** Given :

Height,

$$Z = 4000 \text{ m}$$

Pressure at sea-level,

$$p_0 = 10.143 \text{ N/cm}^2 = 10.143 \times 10^4 = 101430 \frac{\text{N}}{\text{m}^2}$$

Temperature at sea-level,

$$t_0 = 15^\circ\text{C}$$

∴

$$T_0 = 273 + 15 = 288^\circ\text{K}$$

Temperature lapse-rate,

$$L = \frac{dT}{dZ} = -0.0065^\circ\text{K}/\text{m}$$

$$\rho_0 = 1.285 \text{ kg/m}^3$$

Using equation (2.22), we have  $L = \frac{dT}{dZ} = -\frac{g}{R} \left( \frac{k-1}{k} \right)$

$$\text{or } -0.0065 = -\frac{9.81}{R} \left( \frac{k-1}{k} \right), \text{ where } R = \frac{p_0}{\rho_0 T_0} = \frac{101430}{1.285 \times 288} = 274.09$$

$$\therefore -0.0065 = \frac{-9.81}{274.09} \times \left( \frac{k-1}{k} \right)$$

$$\therefore \frac{k-1}{k} = \frac{0.0065 \times 274.09}{9.81} = 0.1815$$

$$\therefore k[1 - .1815] = 1$$

$$\therefore k = \frac{1}{1 - .1815} = \frac{1}{.8184} = 1.222$$

This means that the value of power index  $k = 1.222$ .

(i) Pressure at 4000 m height is given by equation (2.19) as

$$p = p_0 \left[ 1 - \frac{k-1}{k} g Z \frac{\rho_0}{p_0} \right]^{\frac{k}{k-1}}, \text{ where } k = 1.222 \text{ and } \rho_0 = 1.285$$

$$\therefore p = 101430 \left[ 1 - \left( \frac{1.222 - 1.0}{1.222} \right) \times 9.81 \times \frac{4000 \times 1.285}{101430} \right]^{\frac{1.222}{1.222 - 1.0}} \\ = 101430 [1 - 0.09]^{5.50} = 101430 \times .595 \\ = 60350 \text{ N/m}^2 = 6.035 \frac{\text{N}}{\text{cm}^2}. \text{ Ans.}$$

(ii) **Density.** Using equation of state, we get

$$\frac{p}{\rho} = RT$$

where  $p$  = Pressure at 4000 m height

$\rho$  = Density at 4000 m height

$T$  = Temperature at 4000 m height

Now  $T$  is calculated from temperature lapse-rate as

$$t \text{ at } 4000 \text{ m} = t_0 + \frac{dT}{dZ} \times 4000 = 15 - .0065 \times 4000 = 15 - 26 = -11^\circ\text{C}$$

∴

$$T = 273 + t = 273 - 11 = 262^\circ\text{K}$$

∴ Density is given by

$$\rho = \frac{p}{RT} = \frac{60350}{274.09 \times 262} \text{ kg/m}^3 = 0.84 \text{ kg/m}^3. \text{ Ans.}$$

**Problem 2.26** An aeroplane is flying at an altitude of 5000 m. Calculate the pressure around the aeroplane, given the lapse-rate in the atmosphere as  $0.0065^\circ\text{K}/\text{m}$ . Neglect variation of  $g$  with altitude. Take pressure and temperature at ground level as  $10.143 \text{ N/cm}^2$  and  $15^\circ\text{C}$  and density of air as  $1.285 \text{ kg/cm}^3$ .

**Solution.** Given :

Height,

$$Z = 5000 \text{ m}$$

Lapse-rate,

$$L = \frac{dT}{dZ} = -.0065^\circ\text{K}/\text{m}$$

Pressure at ground level,

$$p_0 = 10.143 \times 10^4 \text{ N/m}^2$$

$$t_0 = 15^\circ\text{C}$$

∴

$$T_0 = 273 + 15 = 288^\circ\text{K}$$

Density,

$$\rho_0 = 1.285 \text{ kg/m}^3$$

$$\begin{aligned} \therefore \text{Temperature at } 5000 \text{ m height} &= T_0 + \frac{dT}{dZ} \times \text{Height} = 288 - .0065 \times 5000 \\ &= 288 - 32.5 = 255.5^\circ\text{K} \end{aligned}$$

First find the value of power index  $k$  as

$$\text{From equation (2.22), we have } L = \frac{dT}{dZ} = -\frac{g}{R} \left( \frac{k-1}{k} \right)$$

or

$$-.0065 = -\frac{9.81}{R} \left( \frac{k-1}{k} \right)$$

$$\text{where } R = \frac{p_0}{\rho_0 T_0} = \frac{101430}{1.285 \times 288} = 274.09$$

∴

$$-.0065 = -\frac{9.81}{274.09} \left( \frac{k-1}{k} \right)$$

∴

$$k = 1.222$$

The pressure is given by equation (2.19) as

$$p = p_0 \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]^{\left( \frac{k}{k-1} \right)}$$

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$$\begin{aligned}
 &= 101430 \left[ 1 - \left( \frac{1.222 - 1.0}{1.222} \right) \times 9.81 \times \frac{5000 \times 1.285}{101430} \right]^{\frac{1.222}{1.222 - 1.0}} \\
 &= 101430 \left[ 1 - \frac{.222}{1.222} \times 9.81 \times \frac{5000 \times 1.285}{101430} \right]^{\frac{1.222}{.222}} \\
 &= 101430 [1 - 0.11288]^{5.50} = 101430 \times 0.5175 = 52490 \text{ N/m}^3 \\
 &= \mathbf{5.249 \text{ N/cm}^2. \text{ Ans.}}
 \end{aligned}$$

### HIGHLIGHTS

1. The pressure at any point in a fluid is defined as the force per unit area.
  2. The Pascal's law states that intensity of pressure for a fluid at rest is equal in all directions.
  3. Pressure variation at a point in a fluid at rest is given by the hydrostatic law which states that the rate of increase of pressure in the vertically downward direction is equal to the specific weight of the fluid,  

$$\frac{dp}{dz} = w = \rho \times g.$$
  4. The pressure at any point in an incompressible fluid (liquid) is equal to the product of density of fluid at that point, acceleration due to gravity and vertical height from free surface of fluid,  

$$p = \rho \times g \times Z.$$
  5. Absolute pressure is the pressure in which absolute vacuum pressure is taken as datum while gauge pressure is the pressure in which the atmospheric pressure is taken as datum,
- $p_{\text{abs.}} = p_{\text{atm}} + p_{\text{gauge.}}$
6. Manometer is a device used for measuring pressure at a point in a fluid.
  7. Manometers are classified as (a) Simple manometers and (b) Differential manometers.
  8. Simple manometers are used for measuring pressure at a point while differential manometers are used for measuring the difference of pressures between the two points in a pipe, or two different pipes.
  9. A single column manometer (or micrometer) is used for measuring small pressures, where accuracy is required.
  10. The pressure at a point in static compressible fluid is obtained by combining two equations, i.e., equation of state for a gas and equation given by hydrostatic law.
  11. The pressure at a height  $Z$  in a static compressible fluid (gas) undergoing isothermal compression

$$\left( \frac{p}{\rho} = \text{const.} \right)$$

$$p = p_0 e^{-gZ/RT}$$

where  $p_0$  = Absolute pressure at sea-level or at ground level

$Z$  = Height from sea or ground level

$R$  = Gas constant

$T$  = Absolute temperature.

12. The pressure and temperature at a height  $Z$  in a static compressible fluid (gas) undergoing adiabatic compression ( $p/\rho^k = \text{const.}$ )

$$p = p_0 \left[ 1 - \frac{k-1}{k} gZ \frac{\rho_0}{p_0} \right]^{\frac{k}{k-1}} = p_0 \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\frac{k}{k-1}}$$

and temperature,

$$T = T_0 \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]$$

where  $p_0, T_0$  are pressure and temperature at sea-level  $k = 1.4$  for air.

13. The rate at which the temperature changes with elevation is known as Temperature Lapse-Rate. It is given by

$$L = \frac{-g}{R} \left( \frac{k-1}{k} \right)$$

if (i)  $k = 1$ , temperature is zero.  
(ii)  $k > 1$ , temperature decreases with the increase of height.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define pressure. Obtain an expression for the pressure intensity at a point in a fluid.
2. State and prove the Pascal's law.
3. What do you understand by Hydrostatic Law ?
4. Differentiate between : (i) Absolute and gauge pressure, (ii) Simple manometer and differential manometer, and (iii) Piezometer and pressure gauges.
5. What do you mean by vacuum pressure ?
6. What is a manometer ? How are they classified ?
7. What do you mean by single column manometers ? How are they used for the measurement of pressure ?
8. What is the difference between U-tube differential manometers and inverted U-tube differential manometers ? Where are they used ?
9. Distinguish between manometers and mechanical gauges. What are the different types of mechanical pressure gauges ?
10. Derive an expression for the pressure at a height  $Z$  from sea-level for a static air when the compression of the air is assumed isothermal. The pressure and temperature at sea-levels are  $p_0$  and  $T_0$  respectively.
11. Prove that the pressure and temperature for an adiabatic process at a height  $Z$  from sea-level for a static air are :

$$p_0 = p_0 \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]^{\frac{k}{k-1}} \text{ and } T = T_0 \left[ 1 - \frac{k-1}{k} \frac{gZ}{RT_0} \right]$$

where  $p_0$  and  $T_0$  are the pressure and temperature at sea-level.

12. What do you understand by the term, 'Temperature Lapse-Rate'? Obtain an expression for the temperature Lapse-Rate.
13. What is hydrostatic pressure distribution? Give one example where pressure distribution is non-hydrostatic.
14. Explain briefly the working principle of Bourdon Pressure Gauge with a neat sketch.

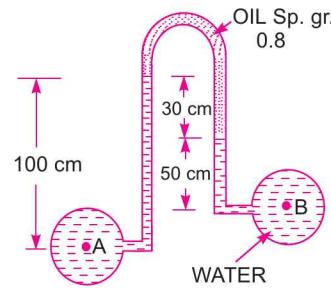
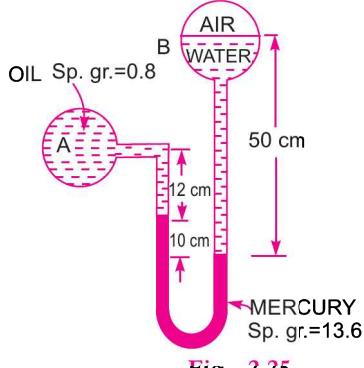
(J.N.T.U., Hyderabad, S 2002)

### (B) NUMERICAL PROBLEMS

1. A hydraulic press has a ram of 30 cm diameter and a plunger of 5 cm diameter. Find the weight lifted by the hydraulic press when the force applied at the plunger is 400 N. [Ans. 14.4 kN]
2. A hydraulic press has a ram of 20 cm diameter and a plunger of 4 cm diameter. It is used for lifting a weight of 20 kN. Find the force required at the plunger. [Ans. 800 N]

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3. Calculate the pressure due to a column of 0.4 m of (a) water, (b) an oil of sp. gr. 0.9, and (c) mercury of sp. gr. 13.6. Take density of water,  $\rho = 1000 \frac{\text{kg}}{\text{m}^3}$ . [Ans. (a)  $0.3924 \text{ N/cm}^2$ , (b)  $0.353 \text{ N/cm}^2$ , (c)  $5.33 \text{ N/cm}^2$ ]
4. The pressure intensity at a point in a fluid is given  $4.9 \text{ N/cm}^2$ . Find the corresponding height of fluid when it is : (a) water, and (b) an oil of sp. gr. 0.8. [Ans. (a) 5 m of water, (b) 6.25 m of oil]
5. An oil of sp. gr. 0.8 is contained in a vessel. At a point the height of oil is 20 m. Find the corresponding height of water at that point. [Ans. 16 m]
6. An open tank contains water upto a depth of 1.5 m and above it an oil of sp. gr. 0.8 for a depth of 2 m. Find the pressure intensity : (i) at the interface of the two liquids, and (ii) at the bottom of the tank. [Ans. (i)  $1.57 \text{ N/cm}^2$ , (ii)  $3.04 \text{ N/cm}^2$ ]
7. The diameters of a small piston and a large piston of a hydraulic jack are 2 cm and 10 cm respectively. A force of 60 N is applied on the small piston. Find the load lifted by the large piston, when : (a) the pistons are at the same level, and (b) small piston is 20 cm above the large piston. The density of the liquid in the jack is given as  $1000 \frac{\text{kg}}{\text{m}^3}$ . [Ans. (a) 1500 N, (b) 1520.5 N]
8. Determine the gauge and absolute pressure at a point which is 2.0 m below the free surface of water. Take atmospheric pressure as  $10.1043 \text{ N/cm}^2$ . [Ans.  $1.962 \text{ N/cm}^2$  (gauge),  $12.066 \text{ N/cm}^2$  (abs.)]
9. A simple manometer is used to measure the pressure of oil (sp. gr. = 0.8) flowing in a pipe line. Its right limb is open to the atmosphere and left limb is connected to the pipe. The centre of the pipe is 9 cm below the level of mercury (sp. gr. 13.6) in the right limb. If the difference of mercury level in the two limbs is 15 cm, determine the absolute pressure of the oil in the pipe in  $\text{N/cm}^2$ . [Ans.  $12.058 \text{ N/cm}^2$ ]
10. A simple manometer (U-tube) containing mercury is connected to a pipe in which an oil of sp. gr. 0.8 is flowing. The pressure in the pipe is vacuum. The other end of the manometer is open to the atmosphere. Find the vacuum, pressure in pipe, if the difference of mercury level in the two limbs is 20 cm and height of oil in the left limb from the centre of the pipe is 15 cm below. [Ans.  $-27.86 \text{ N/cm}^2$ ]
11. A single column vertical manometer (*i.e.*, micrometer) is connected to a pipe containing oil of sp. gr. 0.9. The area of the reservoir is 80 times the area of the manometer tube. The reservoir contains mercury of sp. gr. 13.6. The level of mercury in the reservoir is at a height of 30 cm below the centre of the pipe and difference of mercury levels in the reservoir and right limb is 50 cm. Find the pressure in the pipe. [Ans.  $6.474 \text{ N/cm}^2$ ]
12. A pipe contains an oil of sp. gr. 0.8. A differential manometer connected at the two points A and B of the pipe shows a difference in mercury level as 20 cm. Find the difference of pressure at the two points. [Ans.  $25113.6 \text{ N/m}^2$ ]
13. A U-tube differential manometer connects two pressure pipes A and B. Pipe A contains carbon tetrachloride having a specific gravity 1.594 under a pressure of  $11.772 \text{ N/cm}^2$  and pipe B contains oil of sp. gr. 0.8 under a pressure of  $11.772 \text{ N/cm}^2$ . The pipe A lies 2.5 m above pipe B. Find the difference of pressure measured by mercury as fluid filling U-tube. [Ans. 31.36 cm of mercury]
14. A differential manometer is connected at the two points A and B as shown in Fig. 2.25. At B air pressure is  $7.848 \text{ N/cm}^2$  (abs.), find the absolute pressure at A. [Ans.  $6.91 \text{ N/cm}^2$ ]



15. An inverted differential manometer containing an oil of sp. gr. 0.9 is connected to find the difference of pressures at two points of a pipe containing water. If the manometer reading is 40 cm, find the difference of pressures. [Ans. 392.4 N/m<sup>2</sup>]
16. In above Fig. 2.26 shows an inverted differential manometer connected to two pipes A and B containing water. The fluid in manometer is oil of sp. gr. 0.8. For the manometer readings shown in the figure, find the difference of pressure head between A and B. [Ans. 0.26 m of water]
17. If the atmospheric pressure at sea-level is 10.143 N/cm<sup>2</sup>, determine the pressure at a height of 2000 m assuming that the pressure variation follows : (i) Hydrostatic law, and (ii) Isothermal law. The density of air is given as 1.208 kg/m<sup>3</sup>. [Ans. (i) 7.77 N/cm<sup>2</sup>, (ii) 8.03 N/cm<sup>2</sup>]
18. Calculate the pressure at a height of 8000 m above sea-level if the atmospheric pressure is 101.3 kN/m<sup>2</sup> and temperature is 15°C at the sea-level assuming (i) air is incompressible, (ii) pressure variation follows adiabatic law, and (iii) pressure variation follows isothermal law. Take the density of air at the sea-level as equal to 1.285 kg/m<sup>3</sup>. Neglect variation of  $g$  with altitude. [Ans. (i) 607.5 N/m<sup>2</sup>, (ii) 31.5 kN/m<sup>2</sup> (iii) 37.45 kN/m<sup>2</sup>]
19. Calculate the pressure and density of air at a height of 3000 m above sea-level where pressure and temperature of the air are 10.143 N/cm<sup>2</sup> and 15°C respectively. The temperature lapse-rate is given as 0.0065°K/m. Take density of air at sea-level equal to 1.285 kg/m<sup>3</sup>. [Ans. 6.896 N/cm<sup>2</sup>, 0.937 kg/m<sup>3</sup>]
20. An aeroplane is flying at an altitude of 4000 m. Calculate the pressure around the aeroplane, given the lapse-rate in the atmosphere as 0.0065°K/m. Neglect variation of  $g$  with altitude. Take pressure and temperature at ground level as 10.143 N/cm<sup>2</sup> and 15°C respectively. The density of air at ground level is given as 1.285 kg/m<sup>3</sup>. [Ans. 6.038 N/cm<sup>2</sup>]
21. The atmospheric pressure at the sea-level is 101.3 kN/m<sup>2</sup> and the temperature is 15°C. Calculate the pressure 8000 m above sea-level, assuming (i) air is incompressible, (ii) isothermal variation of pressure and density, and (iii) adiabatic variation of pressure and density. Assume density of air at sea-level as 1.285 kg/m<sup>3</sup>. Neglect variation of ' $g$ ' with altitude. [Ans. (i) 501.3 N/m<sup>2</sup>, (ii) 37.45 kN/m<sup>2</sup>, (iii) 31.5 kN/m<sup>2</sup>]
22. An oil of sp. gr. is 0.8 under a pressure of 137.2 kN/m<sup>2</sup>
- (i) What is the pressure head expressed in metre of water ?
  - (ii) What is the pressure head expressed in metre of oil ? [Ans. (i) 14 m, (ii) 17.5 m]
23. The atmospheric pressure at the sea-level is 101.3 kN/m<sup>2</sup> and temperature is 15°C. Calculate the pressure 8000 m above sea-level, assuming : (i) isothermal variation of pressure and density, and (ii) adiabatic variation of pressure and density. Assume density of air at sea-level as 1.285 kg/m<sup>3</sup>. Neglect variation of ' $g$ ' with altitude.
- Derive the formula that you may use. [Ans. (i) 37.45 kN/m<sup>2</sup>, (ii) 31.5 kN/m<sup>2</sup>]
24. What are the gauge pressure and absolute pressure at a point 4 m below the free surface of a liquid of specific gravity 1.53, if atmospheric pressure is equivalent to 750 mm of mercury. [Ans. 60037 N/m<sup>2</sup> and 160099 N/m<sup>2</sup>]
25. Find the gauge pressure and absolute pressure in N/m<sup>2</sup> at a point 4 m below the free surface of a liquid of sp. gr. 1.2, if the atmospheric pressure is equivalent to 750 mm of mercury. [Ans. 47088 N/m<sup>2</sup> ; 147150 N/m<sup>2</sup>]
26. A tank contains a liquid of specific gravity 0.8. Find the absolute pressure and gauge pressure at a point, which is 2 m below the free surface of the liquid. The atmospheric pressure head is equivalent to 760 mm of mercury. [Ans. 117092 N/m<sup>2</sup> ; 15696 N/m<sup>2</sup>]



# 3

## CHAPTER

# HYDROSTATIC FORCES ON SURFACES



### ► 3.1 INTRODUCTION

This chapter deals with the fluids (*i.e.*, liquids and gases) at rest. This means that there will be no relative motion between adjacent or neighbouring fluid layers. The velocity gradient, which is equal to the change of velocity between two adjacent fluid layers divided by the distance between the layers, will be zero or  $\frac{du}{dy} = 0$ . The shear stress which is equal to  $\mu \frac{\partial u}{\partial y}$  will also be zero. Then the forces acting on the fluid particles will be :

1. due to pressure of fluid normal to the surface,
2. due to gravity (or self-weight of fluid particles).

### ► 3.2 TOTAL PRESSURE AND CENTRE OF PRESSURE

Total pressure is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface.

**Centre of pressure** is defined as the point of application of the total pressure on the surface. There are four cases of submerged surfaces on which the total pressure force and centre of pressure is to be determined. The submerged surfaces may be :

1. Vertical plane surface,
2. Horizontal plane surface,
3. Inclined plane surface, and
4. Curved surface.

### ► 3.3 VERTICAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane vertical surface of arbitrary shape immersed in a liquid as shown in Fig. 3.1.

Let  $A$  = Total area of the surface

$\bar{h}$  = Distance of C.G. of the area from free surface of liquid

$G$  = Centre of gravity of plane surface

$P$  = Centre of pressure

$h^*$  = Distance of centre of pressure from free surface of liquid.

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(a) **Total Pressure (F).** The total pressure on the surface may be determined by dividing the entire surface into a number of small parallel strips. The force on small strip is then calculated and the total pressure force on the whole area is calculated by integrating the force on small strip.

Consider a strip of thickness  $dh$  and width  $b$  at a depth of  $h$  from free surface of liquid as shown in Fig. 3.1

$$\text{Pressure intensity on the strip, } p = \rho gh$$

(See equation 2.5)

$$\text{Area of the strip, } dA = b \times dh$$

$$\begin{aligned} \text{Total pressure force on strip, } dF &= p \times \text{Area} \\ &= \rho gh \times b \times dh \end{aligned}$$

$\therefore$  Total pressure force on the whole surface,

$$F = \int dF = \int \rho gh \times b \times dh = \rho g \int b \times h \times dh$$

But

$$\int b \times h \times dh = \int h \times dA$$

$$\begin{aligned} &= \text{Moment of surface area about the free surface of liquid} \\ &= \text{Area of surface} \times \text{Distance of C.G. from free surface} \\ &= A \times \bar{h} \end{aligned}$$

$$\therefore F = \rho g A \bar{h} \quad \dots(3.1)$$

For water the value of  $\rho = 1000 \text{ kg/m}^3$  and  $g = 9.81 \text{ m/s}^2$ . The force will be in Newton.

(b) **Centre of Pressure ( $h^*$ ).** Centre of pressure is calculated by using the "Principle of Moments", which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis.

The resultant force  $F$  is acting at  $P$ , at a distance  $h^*$  from free surface of the liquid as shown in Fig. 3.1. Hence moment of the force  $F$  about free surface of the liquid  $= F \times h^*$   $\dots(3.2)$

Moment of force  $dF$ , acting on a strip about free surface of liquid

$$\begin{aligned} &= dF \times h && \{ \because dF = \rho gh \times b \times dh \} \\ &= \rho gh \times b \times dh \times h \end{aligned}$$

Sum of moments of all such forces about free surface of liquid

$$\begin{aligned} &= \int \rho gh \times b \times dh \times h = \rho g \int b \times h \times h dh \\ &= \rho g \int bh^2 dh = \rho g \int h^2 dA && (\because bdh = dA) \end{aligned}$$

But

$$\int h^2 dA = \int bh^2 dh$$

$$\begin{aligned} &= \text{Moment of Inertia of the surface about free surface of liquid} \\ &= I_0 \end{aligned}$$

$\therefore$  Sum of moments about free surface

$$= \rho g I_0 \quad \dots(3.3)$$

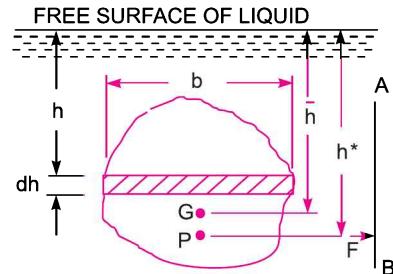


Fig. 3.1

Equating (3.2) and (3.3), we get

$$F \times h^* = \rho g I_0$$

But

$$F = \rho g A \bar{h}$$

∴

$$\rho g A \bar{h} \times h^* = \rho g I_0$$

or

$$h^* = \frac{\rho g I_0}{\rho g A \bar{h}} = \frac{I_0}{A \bar{h}} \quad \dots(3.4)$$

By the theorem of parallel axis, we have

$$I_0 = I_G + A \times \bar{h}^2$$

where  $I_G$  = Moment of Inertia of area about an axis passing through the C.G. of the area and parallel to the free surface of liquid.

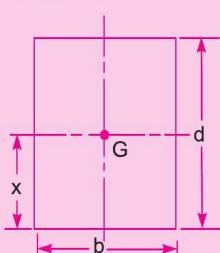
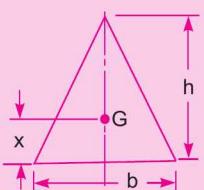
Substituting  $I_0$  in equation (3.4), we get

$$h^* = \frac{I_G + A \bar{h}^2}{A \bar{h}} = \frac{I_G}{A \bar{h}} + \bar{h} \quad \dots(3.5)$$

In equation (3.5),  $\bar{h}$  is the distance of C.G. of the area of the vertical surface from free surface of the liquid. Hence from equation (3.5), it is clear that :

- (i) Centre of pressure (*i.e.*,  $h^*$ ) lies below the centre of gravity of the vertical surface.
- (ii) The distance of centre of pressure from free surface of liquid is independent of the density of the liquid.

**Table 3.1** The moments of inertia and other geometric properties of some important plane surfaces

Plane surface	C.G. from the base	Area	Moment of inertia about an axis passing through C.G. and parallel to base ( $I_G$ )	Moment of inertia about base ( $I_0$ )
1. Rectangle		$x = \frac{d}{2}$	$bd$	$\frac{bd^3}{12}$
2. Triangle		$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$

Contd...

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Plane surface	C.G. from the base	Area	Moment of inertia about an axis passing through C.G. and parallel to base ( $I_G$ )	Moment of inertia about base ( $I_0$ )
3. Circle	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	—
4. Trapezium	$x = \left( \frac{2a+b}{a+b} \right) \frac{h}{3}$	$\frac{(a+b)}{2} \times h$	$\left( \frac{a^2 + 4ab + b^2}{36(a+b)} \right) \times h^3$	—

**Problem 3.1** A rectangular plane surface is 2 m wide and 3 m deep. It lies in vertical plane in water. Determine the total pressure and position of centre of pressure on the plane surface when its upper edge is horizontal and (a) coincides with water surface, (b) 2.5 m below the free water surface.

**Solution.** Given :

Width of plane surface,  $b = 2$  m

Depth of plane surface,  $d = 3$  m

(a) **Upper edge coincides with water surface (Fig. 3.2).** Total pressure is given by equation (3.1) as

$$F = \rho g A \bar{h}$$

where  $\rho = 1000 \text{ kg/m}^3$ ,  $g = 9.81 \text{ m/s}^2$

$$A = 3 \times 2 = 6 \text{ m}^2, \bar{h} = \frac{1}{2} (3) = 1.5 \text{ m.}$$

$$\therefore F = 1000 \times 9.81 \times 6 \times 1.5 \\ = 88290 \text{ N. Ans.}$$

Depth of centre of pressure is given by equation (3.5) as

$$h^* = \frac{I_G}{Ah} + \bar{h}$$

where  $I_G = \text{M.O.I. about C.G. of the area of surface}$

$$= \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

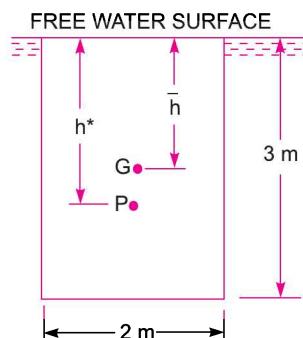


Fig. 3.2

$$\therefore h^* = \frac{4.5}{6 \times 1.5} + 1.5 = 0.5 + 1.5 = 2.0 \text{ m. Ans.}$$

(b) **Upper edge is 2.5 m below water surface (Fig. 3.3).** Total pressure ( $F$ ) is given by (3.1)

$$F = \rho g A \bar{h}$$

where  $\bar{h}$  = Distance of C.G. from free surface of water

$$= 2.5 + \frac{3}{2} = 4.0 \text{ m}$$

$$\therefore F = 1000 \times 9.81 \times 6 \times 4.0 \\ = 235440 \text{ N. Ans.}$$

Centre of pressure is given by  $h^* = \frac{I_G}{A \bar{h}} + \bar{h}$

where  $I_G = 4.5, A = 6.0, \bar{h} = 4.0$

$$\therefore h^* = \frac{4.5}{6.0 \times 4.0} + 4.0$$

$$= 0.1875 + 4.0 = 4.1875 = 4.1875 \text{ m. Ans.}$$

**Problem 3.2** Determine the total pressure on a circular plate of diameter 1.5 m which is placed vertically in water in such a way that the centre of the plate is 3 m below the free surface of water. Find the position of centre of pressure also.

**Solution.** Given : Dia. of plate,  $d = 1.5 \text{ m}$

$$\therefore \text{Area, } A = \frac{\pi}{4} (1.5)^2 = 1.767 \text{ m}^2$$

$$\bar{h} = 3.0 \text{ m}$$

Total pressure is given by equation (3.1),

$$F = \rho g A \bar{h} \\ = 1000 \times 9.81 \times 1.767 \times 3.0 \text{ N} \\ = 52002.81 \text{ N. Ans.}$$

Position of centre of pressure ( $h^*$ ) is given by equation (3.5),

$$h^* = \frac{I_G}{A \bar{h}} + \bar{h}$$

$$\text{where } I_G = \frac{\pi d^4}{64} = \frac{\pi \times 1.5^4}{64} = 0.2485 \text{ m}^4$$

$$\therefore h^* = \frac{0.2485}{1.767 \times 3.0} + 3.0 = 0.0468 + 3.0 \\ = 3.0468 \text{ m. Ans.}$$

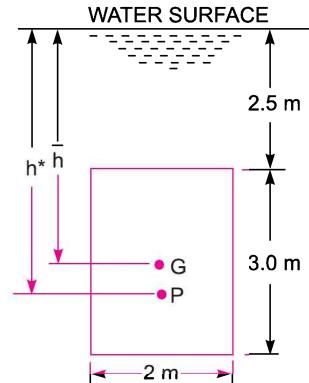


Fig. 3.3

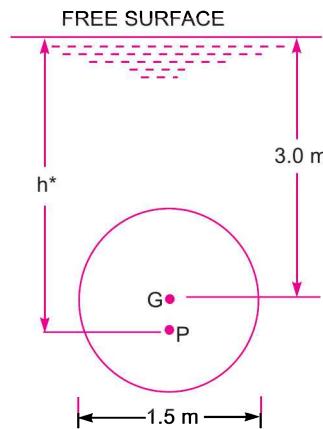


Fig. 3.4

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**Problem 3.3** A rectangular sluice gate is situated on the vertical wall of a lock. The vertical side of the sluice is 'd' metres in length and depth of centroid of the area is 'p' m below the water surface.

Prove that the depth of pressure is equal to  $\left( p + \frac{d^2}{12p} \right)$ .

**Solution.** Given :

$$\begin{aligned}\text{Depth of vertical gate} &= d \text{ m} \\ \text{Let the width of gate} &= b \text{ m} \\ \therefore \text{Area,} &A = b \times d \text{ m}^2\end{aligned}$$

Depth of C.G. from free surface

$$\bar{h} = p \text{ m.}$$

Let  $h^*$  is the depth of centre of pressure from free surface, which is given by equation (3.5) as

$$\begin{aligned}h^* &= \frac{I_G}{Ah} + \bar{h}, \text{ where } I_G = \frac{bd^3}{12} \\ \therefore h^* &= \left( \frac{bd^3}{12} / b \times d \times p \right) + p = \frac{d^2}{12p} + p \quad \text{or} \quad p + \frac{d^2}{12p} \cdot \text{Ans.}\end{aligned}$$

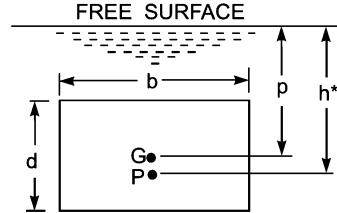


Fig. 3.5

**Problem 3.4** A circular opening, 3 m diameter, in a vertical side of a tank is closed by a disc of 3 m diameter which can rotate about a horizontal diameter. Calculate :

- (i) the force on the disc, and
- (ii) the torque required to maintain the disc in equilibrium in the vertical position when the head of water above the horizontal diameter is 4 m.

**Solution.** Given :

$$\text{Dia. of opening,} \quad d = 3 \text{ m}$$

$$\therefore \text{Area,} \quad A = \frac{\pi}{4} \times 3^2 = 7.0685 \text{ m}^2$$

$$\text{Depth of C.G.,} \quad \bar{h} = 4 \text{ m}$$

(i) Force on the disc is given by equation (3.1) as

$$\begin{aligned}F &= \rho g A \bar{h} = 1000 \times 9.81 \times 7.0685 \times 4.0 \\ &= 277368 \text{ N} = 277.368 \text{ kN. Ans.}\end{aligned}$$

(ii) To find the torque required to maintain the disc in equilibrium, first calculate the point of application of force acting on the disc, i.e., centre of pressure of the force  $F$ . The depth of centre of pressure ( $h^*$ ) is given by equation (3.5) as

$$\begin{aligned}h^* &= \frac{I_G}{Ah} + \bar{h} = \frac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2 \times 4.0} + 4.0 \quad \left\{ \because I_G = \frac{\pi}{64} d^4 \right\} \\ &= \frac{d^2}{16 \times 4.0} + 4.0 = \frac{3^2}{16 \times 4.0} + 4.0 = 0.14 + 4.0 = 4.14 \text{ m}\end{aligned}$$

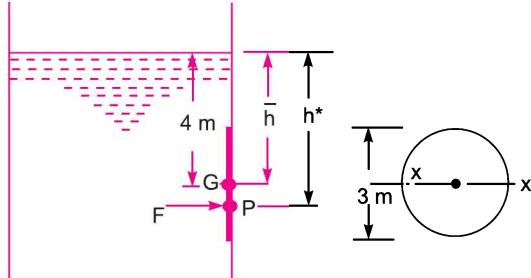


Fig. 3.6

The force  $F$  is acting at a distance of 4.14 m from free surface. Moment of this force about horizontal diameter  $X-X$

$$= F \times (h^* - \bar{h}) = 277368 (4.14 - 4.0) = 38831 \text{ Nm. Ans.}$$

Hence a torque of 38831 Nm must be applied on the disc in the clockwise direction.

**Problem 3.5** A pipe line which is 4 m in diameter contains a gate valve. The pressure at the centre of the pipe is 19.6 N/cm<sup>2</sup>. If the pipe is filled with oil of sp. gr. 0.87, find the force exerted by the oil upon the gate and position of centre of pressure.

**Solution.** Given :

Dia. of pipe,

$$d = 4 \text{ m}$$

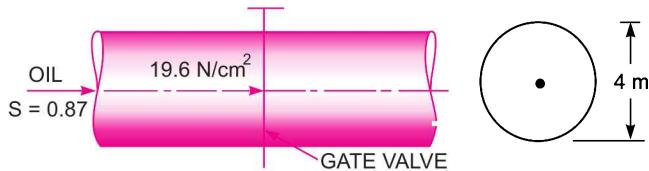


Fig. 3.7

∴ Area,

$$A = \frac{\pi}{4} \times 4^2 = 4\pi \text{ m}^2$$

Sp. gr. of oil,

$$S = 0.87$$

∴ Density of oil,

$$\rho_0 = 0.87 \times 1000 = 870 \text{ kg/m}^3$$

∴ Weight density of oil,  $w_0 = \rho_0 \times g = 870 \times 9.81 \text{ N/m}^3$

Pressure at the centre of pipe,  $p = 19.6 \text{ N/cm}^2 = 19.6 \times 10^4 \text{ N/m}^2$

$$\therefore \text{Pressure head at the centre} = \frac{p}{w_0} = \frac{19.6 \times 10^4}{870 \times 9.81} = 22.988 \text{ m}$$

∴ The height of equivalent free oil surface from the centre of pipe = 22.988 m.

The depth of C.G. of the gate valve from free oil surface  $\bar{h} = 22.988 \text{ m}$ .

(i) Now the force exerted by the oil on the gate is given by

$$F = \rho g A \bar{h}$$

where  $\rho$  = density of oil = 870 kg/m<sup>3</sup>

$$F = 870 \times 9.81 \times 4\pi \times 22.988 = 2465500 \text{ N} = 2.465 \text{ MN. Ans.}$$

(ii) Position of centre of pressure ( $h^*$ ) is given by (3.5) as

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$$h^* = \frac{I_G}{Ah} + \bar{h} = \frac{\frac{\pi}{4} d^4}{\frac{\pi}{4} d^2 \times \bar{h}} + \bar{h} = \frac{d^2}{16 \bar{h}} + \bar{h} = \frac{4^2}{16 \times 22.988} + 22.988 \\ = 0.043 + 22.988 = 23.031 \text{ m. Ans.}$$

Or centre of pressure is below the centre of the pipe by a distance of 0.043 m. Ans.

**Problem 3.6** Determine the total pressure and centre of pressure on an isosceles triangular plate of base 4 m and altitude 4 m when it is immersed vertically in an oil of sp. gr. 0.9. The base of the plate coincides with the free surface of oil.

**Solution.** Given :

$$\text{Base of plate, } b = 4 \text{ m}$$

$$\text{Height of plate, } h = 4 \text{ m}$$

$$\therefore \text{Area, } A = \frac{b \times h}{2} = \frac{4 \times 4}{2} = 8.0 \text{ m}^2$$

$$\text{Sp. gr. of oil, } S = 0.9$$

$$\therefore \text{Density of oil, } \rho = 900 \text{ kg/m}^3.$$

The distance of C.G. from free surface of oil,

$$\bar{h} = \frac{1}{3} \times h = \frac{1}{3} \times 4 = 1.33 \text{ m.}$$

Total pressure ( $F$ ) is given by  $F = \rho g A \bar{h}$

$$= 900 \times 9.81 \times 8.0 \times 1.33 \text{ N} = 9597.6 \text{ N. Ans.}$$

Centre of pressure ( $h^*$ ) from free surface of oil is given by

$$h^* = \frac{I_G}{Ah} + \bar{h}$$

where  $I_G$  = M.O.I. of triangular section about its C.G.

$$= \frac{bh^3}{36} = \frac{4 \times 4^3}{36} = 7.11 \text{ m}^4$$

$$\therefore h^* = \frac{7.11}{8.0 \times 1.33} + 1.33 = 0.6667 + 1.33 = 1.99 \text{ m. Ans.}$$

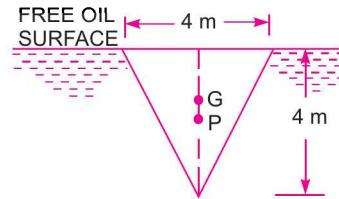


Fig. 3.8

**Problem 3.7** A vertical sluice gate is used to cover an opening in a dam. The opening is 2 m wide and 1.2 m high. On the upstream of the gate, the liquid of sp. gr. 1.45, lies upto a height of 1.5 m above the top of the gate, whereas on the downstream side the water is available upto a height touching the top of the gate. Find the resultant force acting on the gate and position of centre of pressure. Find also the force acting horizontally at the top of the gate which is capable of opening it. Assume that the gate is hinged at the bottom.

**Solution.** Given :

$$\text{Width of gate, } b = 2 \text{ m}$$

$$\text{Depth of gate, } d = 1.2 \text{ m}$$

$$\therefore \text{Area, } A = b \times d = 2 \times 1.2 = 2.4 \text{ m}^2$$

$$\text{Sp. gr. of liquid} = 1.45$$

∴ Density of liquid,  $\rho_1 = 1.45 \times 1000 = 1450 \text{ kg/m}^3$

Let  $F_1$  = Force exerted by the fluid of sp. gr. 1.45 on gate

$F_2$  = Force exerted by water on the gate.

The force  $F_1$  is given by  $F_1 = \rho_1 g \times A \times \bar{h}_1$

where  $\rho_1 = 1.45 \times 1000 = 1450 \text{ kg/m}^2$

$\bar{h}_1$  = Depth of C.G. of gate from free surface of liquid

$$= 1.5 + \frac{1.2}{2} = 2.1 \text{ m.}$$

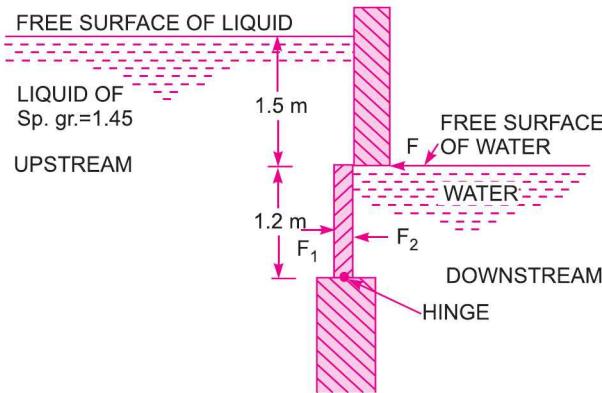


Fig. 3.9

$$\therefore F_1 = 1450 \times 9.81 \times 2.4 \times 2.1 = 71691 \text{ N}$$

$$\text{Similarly, } F_2 = \rho_2 g A \bar{h}_2$$

where  $\rho_2 = 1,000 \text{ kg/m}^3$

$\bar{h}_2$  = Depth of C.G. of gate from free surface of water

$$= \frac{1}{2} \times 1.2 = 0.6 \text{ m}$$

$$\therefore F_2 = 1000 \times 9.81 \times 2.4 \times 0.6 = 14126 \text{ N}$$

(i) Resultant force on the gate =  $F_1 - F_2 = 71691 - 14126 = 57565 \text{ N. Ans.}$

(ii) Position of centre of pressure of resultant force. The force  $F_1$  will be acting at a depth of  $h_1^*$  from free surface of liquid, given by the relation

$$h_1^* = \frac{I_G}{A \bar{h}_1} + \bar{h}_1$$

$$\text{where } I_G = \frac{bd^3}{12} = \frac{2 \times 1.2^3}{12} = 0.288 \text{ m}^4$$

$$\therefore h_1^* = \frac{0.288}{2.4 \times 2.1} + 2.1 = 0.0571 + 2.1 = 2.1571 \text{ m}$$

∴ Distance of  $F_1$  from hinge

$$= (1.5 + 1.2) - h_1^* = 2.7 - 2.1571 = 0.5429 \text{ m}$$

The force  $F_2$  will be acting at a depth of  $h_2^*$  from free surface of water and is given by

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$$h_2^* = \frac{I_G}{A h_2} + \bar{h}_2$$

where  $I_G = 0.288 \text{ m}^4$ ,  $\bar{h}_2 = 0.6 \text{ m}$ ,  $A = 2.4 \text{ m}^2$ ,

$$h_2^* = \frac{0.288}{2.4 \times 0.6} + 0.6 = 0.2 + 0.6 = 0.8 \text{ m}$$

Distance of  $F_2$  from hinge =  $1.2 - 0.8 = 0.4 \text{ m}$

The resultant force 57565 N will be acting at a distance given by

$$\begin{aligned} &= \frac{71691 \times .5429 - 14126 \times 0.4}{57565} \\ &= \frac{38921 - 5650.4}{57565} \text{ m above hinge} \\ &= \mathbf{0.578 \text{ m above the hinge. Ans.}} \end{aligned}$$

(iii) Force at the top of gate which is capable of opening the gate. Let  $F$  is the force required on the top of the gate to open it as shown in Fig. 3.9. Taking the moments of  $F$ ,  $F_1$  and  $F_2$  about the hinge, we get

$$F \times 1.2 + F_2 \times 0.4 = F_1 \times .5429$$

$$\begin{aligned} \text{or } F &= \frac{F_1 \times .5429 - F_2 \times 0.4}{1.2} \\ &= \frac{71691 \times .5429 - 14126 \times 0.4}{1.2} = \frac{38921 - 5650.4}{1.2} \\ &= \mathbf{27725.5 \text{ N. Ans.}} \end{aligned}$$

**Problem 3.8** A caisson for closing the entrance to a dry dock is of trapezoidal form 16 m wide at the top and 10 m wide at the bottom and 6 m deep. Find the total pressure and centre of pressure on the caisson if the water on the outside is just level with the top and dock is empty.

**Solution.** Given :

$$\text{Width at top} = 16 \text{ m}$$

$$\text{Width at bottom} = 10 \text{ m}$$

$$\text{Depth, } d = 6 \text{ m}$$

Area of trapezoidal  $ABCD$ ,

$$\begin{aligned} A &= \frac{(BC + AD)}{2} \times d \\ &= \frac{(10 + 16)}{2} \times 6 = 78 \text{ m}^2 \end{aligned}$$

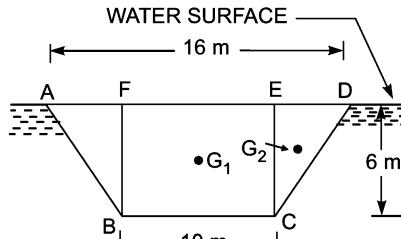


Fig. 3.10

Depth of C.G. of trapezoidal area  $ABCD$  from free surface of water,

$$\begin{aligned} \bar{h} &= \frac{10 \times 6 \times 3 + \frac{(16 - 10)}{2} \times 6 \times \frac{1}{3} \times 6}{78} \\ &= \frac{180 + 36}{78} = 2.769 \text{ m from water surface.} \end{aligned}$$

(i) **Total Pressure (F).** Total pressure,  $F$  is given by

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 78 \times 2.769 \text{ N} \\ = 2118783 \text{ N} = \mathbf{2.118783 \text{ MN. Ans.}}$$

(ii) **Centre of Pressure ( $h^*$ )**. Centre of pressure is given by equation (3.5) as

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where  $I_G$  = M.O.I. of trapezoidal  $ABCD$  about its C.G.

Let  $I_{G_1}$  = M.O.I. of rectangle  $FBCE$  about its C.G.

$I_{G_2}$  = M.O.I. of two  $\Delta$ s  $ABF$  and  $ECD$  about its C.G.

Then

$$I_{G_1} = \frac{bd^3}{12} = \frac{10 \times 6^3}{12} = 180 \text{ m}^4$$

$I_{G_1}$  is the M.O.I. of the rectangle about the axis passing through  $G_1$ .

$\therefore$  M.O.I. of the rectangle about the axis passing through the C.G. of the trapezoidal  $I_{G_1} + \text{Area of rectangle} \times x_1^2$

where  $x_1$  is distance between the C.G. of rectangle and C.G. of trapezoidal  
 $= (3.0 - 2.769) = 0.231 \text{ m}$

$$\therefore \text{M.O.I. of } FBCE \text{ passing through C.G. of trapezoidal} \\ = 180 + 10 \times 6 \times (0.231)^2 = 180 + 3.20 = 183.20 \text{ m}^4$$

Now

$$I_{G_2} = \text{M.O.I. of } \Delta ABD \text{ in Fig. 3.11 about } G_2 = \frac{bd^3}{36} \\ = \frac{(16 - 10) \times 6^3}{36} = 36 \text{ m}^4$$

The distance between the C.G. of triangle and C.G. of trapezoidal  
 $= (2.769 - 2.0) = 0.769$

$\therefore$  M.O.I. of the two  $\Delta$ s about an axis passing through C.G. of trapezoidal

$$= I_{G_2} + \text{Area of triangles} \times (.769)^2 \\ = 36.0 + \frac{6 \times 6}{2} \times (.769)^2 \\ = 36.0 + 10.64 = 46.64$$

$\therefore I_G = \text{M.O.I. of trapezoidal about its C.G.}$

$$= \text{M.O.I. of rectangle about the C.G. of trapezoidal} \\ + \text{M.O.I. of triangles about the C.G. of the trapezoidal} \\ = 183.20 + 46.64 = 229.84 \text{ m}^4$$

$$\therefore h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where  $A = 78, \bar{h} = 2.769$

$$h^* = \frac{229.84}{78 \times 2.769} + 2.769 = 1.064 + 2.769 = \mathbf{3.833 \text{ m. Ans.}}$$

#### Alternate Method

The distance of the C.G. of the trapezoidal channel from surface  $AD$  is given by (refer to Table 3.1 on page 71)

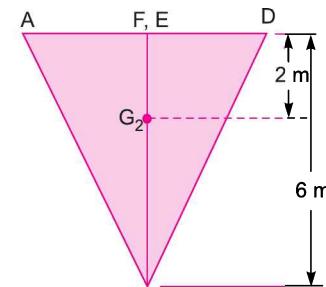


Fig. 3.11

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$$\begin{aligned}
 x &= \frac{(2a+b)}{(a+b)} \times \frac{h}{3} \\
 &= \frac{(2 \times 10 + 16)}{(10 + 16)} \times \frac{6}{3} \quad (\because a = 10, b = 16 \text{ and } h = 6) \\
 &= \frac{36}{26} \times 2 = 2.769 \text{ m}
 \end{aligned}$$

This is also equal to the distance of the C.G. of the trapezoidal from free surface of water.

$$\bar{h} = 2.769 \text{ m}$$

$$\begin{aligned}
 \therefore \text{ Total pressure, } F &= \rho g A \bar{h} \quad (\because A = 78) \\
 &= 1000 \times 9.81 \times 78 \times 2.769 \text{ N} = 2118783 \text{ N. Ans.}
 \end{aligned}$$

$$\text{Centre of Pressure, } (h^*) = \frac{I_G}{Ah} + \bar{h}$$

Now  $I_G$  from Table 3.1 is given by,

$$\begin{aligned}
 I_G &= \frac{(a^2 + 4ab + b^2)}{36(a+b)} \times h^3 = \frac{(10^2 + 4 \times 10 \times 16 + 16^2)}{36(10+16)} \times 6^3 \\
 &= \frac{(100 + 640 + 256)}{36 \times 26} \times 216 = 229.846 \text{ m}^4
 \end{aligned}$$

$$\begin{aligned}
 \therefore h^* &= \frac{229.846}{78 \times 2.769} + 2.769 \quad (\because A = 78 \text{ m}^2) \\
 &= 3.833 \text{ m. Ans.}
 \end{aligned}$$

**Problem 3.9** A trapezoidal channel 2 m wide at the bottom and 1 m deep has side slopes 1 : 1. Determine :

- (i) the total pressure, and
- (ii) the centre of pressure on the vertical gate closing the channel when it is full of water.

**Solution.** Given :

$$\begin{aligned}
 \text{Width at bottom} &= 2 \text{ m} \\
 \text{Depth, } d &= 1 \text{ m} \\
 \text{Side slopes} &= 1 : 1 \\
 \therefore \text{Top width, } AD &= 2 + 1 + 1 = 4 \text{ m} \\
 \text{Area of rectangle } FBEC, &A_1 = 2 \times 1 = 2 \text{ m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Area of two triangles } ABF \text{ and } ECD, &A_2 = \frac{(4 - 2)}{2} \times 1 = 1 \text{ m}^2 \\
 \therefore \text{Area of trapezoidal } ABCD, &A = A_1 + A_2 = 2 + 1 = 3 \text{ m}^2
 \end{aligned}$$

Depth of C.G. of rectangle  $FBEC$  from water surface,

$$\bar{h}_1 = \frac{1}{2} = 0.5 \text{ m}$$

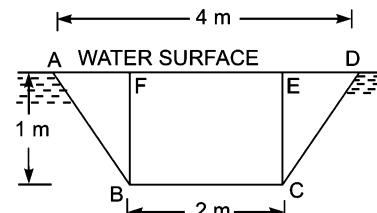


Fig. 3.12

Depth of C.G. of two triangles  $ABF$  and  $ECD$  from water surface,

$$\bar{h}_2 = \frac{1}{3} \times 1 = \frac{1}{3} \text{ m}$$

$\therefore$  Depth of C.G. of trapezoidal  $ABCD$  from free surface of water

$$\bar{h} = \frac{A_1 \times \bar{h}_1 + A_2 \times \bar{h}_2}{(A_1 + A_2)} = \frac{2 \times 0.5 + 1 \times 0.33333}{(2 + 1)} = .44444$$

(i) **Total Pressure (F).** Total pressure  $F$  is given by

$$\begin{aligned} F &= \rho g A \bar{h} \\ &= 1000 \times 9.81 \times 3.0 \times 0.44444 = \mathbf{13079.9 \text{ N. Ans.}} \end{aligned}$$

(ii) **Centre of Pressure ( $h^*$ ).** M.O.I. of rectangle  $FBCE$  about its C.G.,

$$I_{G_1} = \frac{bd^3}{12} = \frac{2 \times 1^3}{12} = \frac{1}{6} \text{ m}^4$$

M.O.I. of  $FBCE$  about an axis passing through the C.G. of trapezoidal

$$\begin{aligned} \text{or } I_{G_1}^* &= I_{G_1} + A_1 \times [\text{Distance between C.G. of rectangle and C.G.} \\ &\quad \text{of trapezoidal}]^2 \\ &= \frac{1}{6} + 2 \times [\bar{h}_1 - \bar{h}]^2 \\ &= \frac{1}{6} + 2 \times [0.5 - .4444]^2 = .1666 + .006182 = 0.1727 \end{aligned}$$

M.O.I. of the two triangles  $ABF$  and  $ECD$  about their C.G.,

$$I_{G_2} = \frac{bd^3}{36} = \frac{(1+1) \times 1^3}{36} = \frac{2}{36} = \frac{1}{18} \text{ m}^4.$$

M.O.I. of the two triangles about the C.G. of trapezoidal,

$$\begin{aligned} I_{G_2}^* &= I_{G_2} + A_2 \times [\text{Distance between C.G. of triangles and C.G.} \\ &\quad \text{of trapezoidal}]^2 \\ &= \frac{1}{18} + 1 \times [\bar{h} - \bar{h}_2]^2 = \frac{1}{18} + 1 \times \left[.4444 - \frac{1}{3}\right]^2 \\ &= \frac{1}{18} + (.1111)^2 = 0.0555 + (.1111)^2 \\ &= .0555 + 0.01234 = 0.06789 \text{ m}^4 \end{aligned}$$

$\therefore$  M.O.I. of the trapezoidal about its C.G.

$$I_G = I_{G_1}^* + I_{G_2}^* = .1727 + .06789 = 0.24059 \text{ m}^4$$

Then centre of pressure ( $h^*$ ) on the vertical trapezoidal,

$$\begin{aligned} h^* &= \frac{I_G}{Ah} + \bar{h} = \frac{0.24059}{3 \times .44444} + .4444 = 0.18046 + .4444 = 0.6248 \\ &\simeq \mathbf{0.625 \text{ m. Ans.}} \end{aligned}$$

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### Alternate Method

The distance of the C.G. of the trapezoidal channel from surface  $AD$  is given by (refer to Table 3.1 on page 71).

$$x = \frac{(2a+b)}{(a+b)} \times \frac{h}{3} = \frac{(2 \times 2 + 4)}{(2 + 4)} \times \frac{1}{3} \quad (\because a = 2, b = 4 \text{ and } h = 1)$$

$$= 0.444 \text{ m}$$

$$\therefore \bar{h} = x = 0.444 \text{ m}$$

$$\therefore \text{Total pressure, } F = \rho g A \bar{h} = 1000 \times 9.81 \times 3.0 \times .444 \quad (\because A = 3.0)$$

$$= 13079 \text{ N. Ans.}$$

$$\text{Centre of pressure, } h^* = \frac{I_G}{Ah} + \bar{h}$$

where  $I_G$  from Table 3.1 is given by

$$I_G = \frac{(a^2 + 4ab + b^2)}{36(a+b)} \times h^3 = \frac{(2^2 + 4 \times 2 \times 4 + 4^2)}{36(2+4)} \times 1^3 = \frac{52}{36 \times 6} = 0.2407 \text{ m}^4$$

$$\therefore h^* = \frac{0.2407}{3.0 \times .444} + .444 = 0.625 \text{ m. Ans.}$$

**Problem 3.10** A square aperture in the vertical side of a tank has one diagonal vertical and is completely covered by a plane plate hinged along one of the upper sides of the aperture. The diagonals of the aperture are 2 m long and the tank contains a liquid of specific gravity 1.15. The centre of aperture is 1.5 m below the free surface. Calculate the thrust exerted on the plate by the liquid and position of its centre of pressure.

**Solution.** Given : Diagonals of aperture,  $AC = BD = 2 \text{ m}$

$\therefore$  Area of square aperture,  $A = \text{Area of } \triangle ACB + \text{Area of } \triangle ACD$

$$= \frac{AC \times BO}{2} + \frac{AC \times OD}{2} = \frac{2 \times 1}{2} + \frac{2 \times 1}{2} = 1 + 1 = 2.0 \text{ m}^2$$

Sp. gr. of liquid  $= 1.15$

$\therefore$  Density of liquid,  $\rho = 1.15 \times 1000 = 1150 \text{ kg/m}^3$

Depth of centre of aperture from free surface,

$$\bar{h} = 1.5 \text{ m.}$$

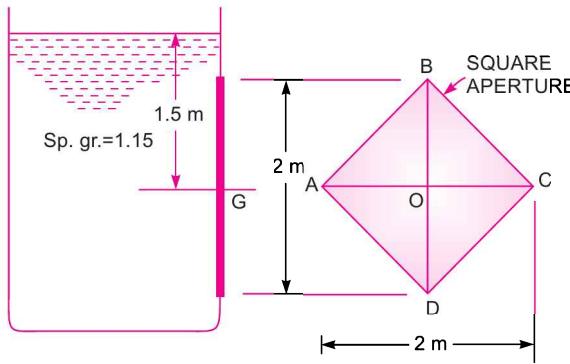


Fig. 3.13

(i) The thrust on the plate is given by

$$F = \rho g A \bar{h} = 1150 \times 9.81 \times 2 \times 1.5 = 33844.5. \text{ Ans.}$$

(ii) Centre of pressure ( $h^*$ ) is given by

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where  $I_G$  = M.O.I. of  $ABCD$  about diagonal  $AC$

= M.O.I. of triangle  $ABC$  about  $AC$  + M.O.I. of triangle  $ACD$  about  $AC$

$$= \frac{AC \times OB^3}{12} + \frac{AC \times OD^3}{12} \quad \left( \because \text{M.O.I. of a triangle about its base} = \frac{bh^3}{12} \right)$$

$$= \frac{2 \times 1^3}{12} + \frac{2 \times 1^3}{12} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \text{ m}^4$$

$$\therefore h^* = \frac{\frac{1}{3}}{2 \times 1.5} + 1.5 = \frac{1}{3 \times 2 \times 1.5} + 1.5 = 1.611 \text{ m. Ans.}$$

**Problem 3.11** A tank contains water upto a height of 0.5 m above the base. An immiscible liquid of sp. gr. 0.8 is filled on the top of water upto 1 m height. Calculate :

(i) total pressure on one side of the tank,

(ii) the position of centre of pressure for one side of the tank, which is 2 m wide.

**Solution.** Given :

Depth of water = 0.5 m

Depth of liquid = 1 m

Sp. gr. of liquid = 0.8

Density of liquid,  $\rho_1 = 0.8 \times 1000 = 800 \text{ kg/m}^3$

Density of water,  $\rho_2 = 1000 \text{ kg/m}^3$

Width of tank = 2 m

(i) Total pressure on one side is calculated by drawing pressure diagram, which is shown in Fig. 3.14.

Intensity of pressure on top,  $p_A = 0$

Intensity of pressure on  $D$  (or  $DE$ ),  $p_D = \rho_1 g h_1$   
 $= 800 \times 9.81 \times 1.0 = 7848 \text{ N/m}^2$

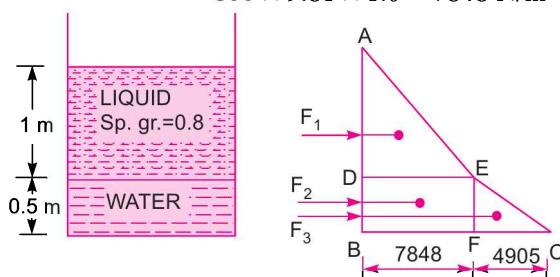


Fig. 3.14

Intensity of pressure on base (or  $BC$ ),  $p_B = \rho_1 g h_1 + \rho_2 g \times 0.5$

$$= 7848 + 1000 \times 9.81 \times 0.5 = 7848 + 4905 = 12753 \text{ N/m}^2$$

Now force

$F_1 = \text{Area of } \Delta ADE \times \text{Width of tank}$

$$= \frac{1}{2} \times AD \times DE \times 2.0 = \frac{1}{2} \times 1 \times 7848 \times 2.0 = 7848 \text{ N}$$

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Force

$$F_2 = \text{Area of rectangle } DBFE \times \text{Width of tank}$$

$$= 0.5 \times 7848 \times 2 = 7848 \text{ N}$$

$$F_3 = \text{Area of } \Delta EFC \times \text{Width of tank}$$

$$= \frac{1}{2} \times EF \times FC \times 2.0 = \frac{1}{2} \times 0.5 \times 4905 \times 2.0 = 2452.5 \text{ N}$$

$\therefore$  Total pressure,

$$F = F_1 + F_2 + F_3 \\ = 7848 + 7848 + 2452.5 = 18148.5 \text{ N. Ans.}$$

(ii) **Centre of Pressure ( $h^*$ )**. Taking the moments of all force about A, we get

$$F \times h^* = F_1 \times \frac{2}{3} AD + F_2 (AD + \frac{1}{2} BD) + F_3 [AD + \frac{2}{3} BD]$$

$$18148.5 \times h^* = 7848 \times \frac{2}{3} \times 1 + 7848 \left( 1.0 + \frac{0.5}{2} \right) + 2452.5 \left( 1.0 + \frac{2}{3} \times 0.5 \right) \\ = 5232 + 9810 + 3270 = 18312$$

$$\therefore h^* = \frac{18312}{18148.5} = 1.009 \text{ m from top. Ans.}$$

**Problem 3.12** A cubical tank has sides of 1.5 m. It contains water for the lower 0.6 m depth. The upper remaining part is filled with oil of specific gravity 0.9. Calculate for one vertical side of the tank:

- (a) total pressure, and
- (b) position of centre of pressure.

**Solution.** Given :

Cubical tank of sides 1.5 m means the dimensions of the tank are  $1.5 \text{ m} \times 1.5 \text{ m} \times 1.5 \text{ m}$ .

$$\text{Depth of water} = 0.6 \text{ m}$$

$$\text{Depth of liquid} = 1.5 - 0.6 = 0.9 \text{ m}$$

$$\text{Sp. gr. of liquid} = 0.9$$

$$\text{Density of liquid, } \rho_1 = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$\text{Density of water, } \rho_2 = 1000 \text{ kg/m}^3$$

(a) **Total pressure** on one vertical side is calculated by drawing pressure diagram, which is shown in Fig. 3.15.

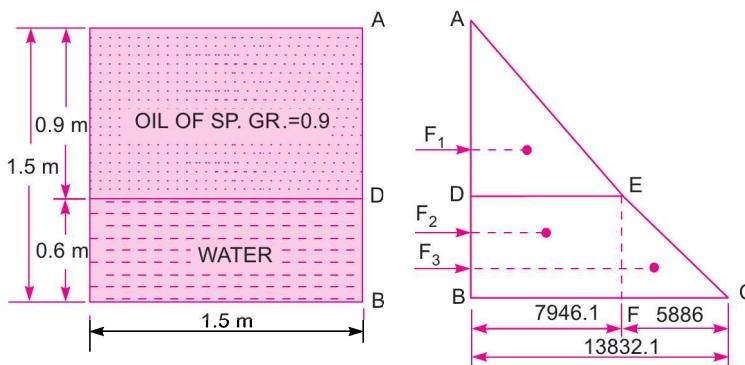


Fig. 3.15

Intensity of pressure at A,  $p_A = 0$

Intensity of pressure at D,  $p_D = \rho_1 g \times h = 900 \times 9.81 \times 0.9 = 7946.1 \text{ N/m}^2$

$$\begin{aligned}\text{Intensity of pressure at } B, p_B &= \rho_1 gh_1 + \rho_2 gh_2 = 900 \times 9.81 \times 0.9 + 1000 \times 9.81 \times 0.6 \\ &= 7946.1 + 5886 = 13832.1 \text{ N/m}^2\end{aligned}$$

Hence in pressure diagram :

$$DE = 7946.1 \text{ N/m}^2, BC = 13832.1 \text{ N/m}^2, FC = 5886 \text{ N/m}^2$$

The pressure diagram is split into triangle ADE, rectangle BDEF and triangle EFC. The total pressure force consists of the following components :

$$(i) \text{ Force } F_1 = \text{Area of triangle } ADE \times \text{Width of tank}$$

$$\begin{aligned}&= \left(\frac{1}{2} \times AD \times DE\right) \times 1.5 \quad (\because \text{Width} = 1.5 \text{ m}) \\ &= \left(\frac{1}{2} \times 0.9 \times 7946.1\right) \times 1.5 = 5363.6 \text{ N}\end{aligned}$$

This force will be acting at the C.G. of the triangle ADE, i.e., at a distance of  $\frac{2}{3} \times 0.9 = 0.6 \text{ m}$  below A

$$(ii) \text{ Force } F_2 = \text{Area of rectangle } BDEF \times \text{Width of tank}$$

$$= (BD \times DE) \times 1.5 = (0.6 \times 7946.1) \times 1.5 = 7151.5$$

This force will be acting at the C.G. of the rectangle BDEF i.e., at a distance of  $0.9 + \frac{0.6}{2} = 1.2 \text{ m}$

below A.

$$(iii) \text{ Force } F_3 = \text{Area of triangle } EFC \times \text{Width of tank}$$

$$= \left(\frac{1}{2} \times EF \times FC\right) \times 1.5 = \left(\frac{1}{2} \times 0.6 \times 5886\right) \times 1.5 = 2648.7 \text{ N}$$

This force will be acting at the C.G. of the triangle EFC, i.e., at a distance of  $0.9 + \frac{2}{3} \times 0.6 = 1.30 \text{ m}$

below A.

$\therefore$  Total pressure force on one vertical face of the tank,

$$\begin{aligned}F &= F_1 + F_2 + F_3 \\ &= 5363.6 + 7151.5 + 2648.7 = 15163.8 \text{ N. Ans.}\end{aligned}$$

#### (b) Position of centre of pressure

Let the total force  $F$  is acting at a depth of  $h^*$  from the free surface of liquid, i.e., from A.

Taking the moments of all forces about A, we get

$$F \times h^* = F_1 \times 0.6 + F_2 \times 1.2 + F_3 \times 1.3$$

or

$$\begin{aligned}h^* &= \frac{F_1 \times 0.6 + F_2 \times 1.2 + F_3 \times 1.3}{F} \\ &= \frac{5363.6 \times 0.6 + 7151.5 \times 1.2 + 2648.7 \times 1.3}{15163.8} \\ &= 1.005 \text{ m from A. Ans.}\end{aligned}$$

### ► 3.4 HORIZONTAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane horizontal surface immersed in a static fluid. As every point of the surface is at the same depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface and equal to,  $p = \rho gh$ , where  $h$  is depth of surface.

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Let

$$A = \text{Total area of surface}$$

Then total force,  $F$ , on the surface

$$= p \times \text{Area} = \rho g \times h \times A = \rho g A \bar{h}$$

where  $\bar{h}$  = Depth of C.G. from free surface of liquid =  $h$

also  $h^*$  = Depth of centre of pressure from free surface =  $h$ .

**Problem 3.13** Fig. 3.17 shows a tank full of water. Find :

(i) Total pressure on the bottom of tank.

(ii) Weight of water in the tank.

(iii) Hydrostatic paradox between the results of (i) and (ii). Width of tank is 2 m.

**Solution.** Given :

Depth of water on bottom of tank

$$h_1 = 3 + 0.6 = 3.6 \text{ m}$$

Width of tank

$$= 2 \text{ m}$$

Length of tank at bottom

$$= 4 \text{ m}$$

$$\therefore \text{Area at the bottom, } A = 4 \times 2 = 8 \text{ m}^2$$

(i) Total pressure  $F$ , on the bottom is

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 8 \times 3.6$$

$$= 282528 \text{ N. Ans.}$$

(ii) Weight of water in tank =  $\rho g \times \text{Volume of tank}$

$$= 1000 \times 9.81 \times [3 \times 0.4 \times 2 + 4 \times .6 \times 2]$$

$$= 1000 \times 9.81 [2.4 + 4.8] = 70632 \text{ N. Ans.}$$

(iii) From the results of (i) and (ii), it is observed that the total weight of water in the tank is much less than the total pressure at the bottom of the tank. This is known as Hydrostatic paradox.

### ► 3.5 INCLINED PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane surface of arbitrary shape immersed in a liquid in such a way that the plane of the surface makes an angle  $\theta$  with the free surface of the liquid as shown in Fig. 3.18.

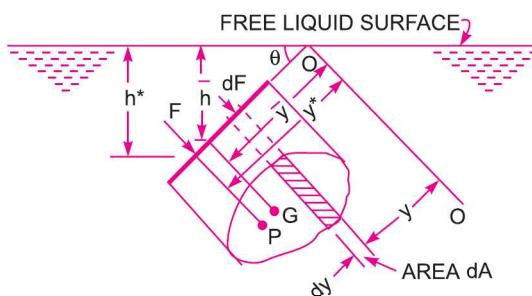


Fig. 3.18 Inclined immersed surface.

Let  $A$  = Total area of inclined surface

$\bar{h}$  = Depth of C.G. of inclined area from free surface

$h^*$  = Distance of centre of pressure from free surface of liquid

$\theta$  = Angle made by the plane of the surface with free liquid surface.

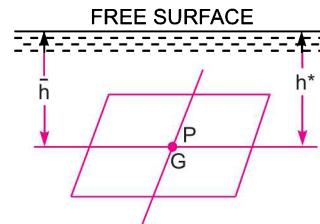


Fig. 3.16

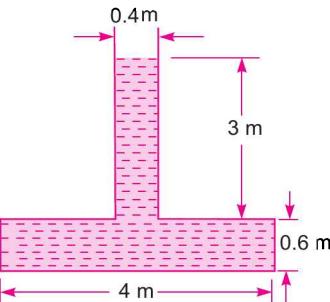


Fig. 3.17

Let the plane of the surface, if produced meet the free liquid surface at  $O$ . Then  $O-O$  is the axis perpendicular to the plane of the surface.

Let  $\bar{y}$  = distance of the C.G. of the inclined surface from  $O-O$   
 $y^*$  = distance of the centre of pressure from  $O-O$ .

Consider a small strip of area  $dA$  at a depth ' $h$ ' from free surface and at a distance  $y$  from the axis  $O-O$  as shown in Fig. 3.18.

Pressure intensity on the strip,  $p = \rho gh$   
 $\therefore$  Pressure force,  $dF$ , on the strip,  $dF = p \times \text{Area of strip} = \rho gh \times dA$

Total pressure force on the whole area,  $F = \int dF = \int \rho g h dA$

$$\begin{aligned} \text{But from Fig. 3.18, } \frac{h}{y} &= \frac{\bar{h}}{\bar{y}} = \frac{h^*}{y^*} = \sin \theta \\ \therefore h &= y \sin \theta \\ \therefore F &= \int \rho g \times y \times \sin \theta \times dA = \rho g \sin \theta \int y dA \end{aligned}$$

$$\text{But } \int y dA = A \bar{y}$$

where  $\bar{y}$  = Distance of C.G. from axis  $O-O$

$$\begin{aligned} \therefore F &= \rho g \sin \theta \bar{y} \times A \\ &= \rho g A \bar{h} \quad (\because \bar{h} = \bar{y} \sin \theta) \dots(3.6) \end{aligned}$$

### Centre of Pressure ( $h^*$ )

$$\begin{aligned} \text{Pressure force on the strip, } dF &= \rho g h dA \\ &= \rho g y \sin \theta dA \quad [h = y \sin \theta] \end{aligned}$$

$$\begin{aligned} \text{Moment of the force, } dF, \text{ about axis } O-O \\ &= dF \times y = \rho g y \sin \theta dA \times y = \rho g \sin \theta y^2 dA \end{aligned}$$

Sum of moments of all such forces about  $O-O$

$$= \int \rho g \sin \theta y^2 dA = \rho g \sin \theta \int y^2 dA$$

$$\text{But } \int y^2 dA = \text{M.O.I. of the surface about } O-O = I_0$$

$$\therefore \text{Sum of moments of all forces about } O-O = \rho g \sin \theta I_0 \dots(3.7)$$

$$\begin{aligned} \text{Moment of the total force, } F, \text{ about } O-O \text{ is also given by} \\ &= F \times y^* \dots(3.8) \end{aligned}$$

where  $y^*$  = Distance of centre of pressure from  $O-O$ .

Equating the two values given by equations (3.7) and (3.8)

$$F \times y^* = \rho g \sin \theta I_0$$

$$\text{or } y^* = \frac{\rho g \sin \theta I_0}{F} \dots(3.9)$$

$$\text{Now } y^* = \frac{h^*}{\sin \theta}, F = \rho g A \bar{h}$$

and  $I_0$  by the theorem of parallel axis =  $I_G + A \bar{y}^2$ .

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Substituting these values in equation (3.9), we get

$$\frac{h^*}{\sin \theta} = \frac{\rho g \sin \theta}{\rho g A \bar{h}} [I_G + A \bar{y}^2]$$

$$\therefore h^* = \frac{\sin^2 \theta}{A \bar{h}} [I_G + A \bar{y}^2]$$

But  $\frac{\bar{h}}{y} = \sin \theta$  or  $\bar{y} = \frac{\bar{h}}{\sin \theta}$

$$\therefore h^* = \frac{\sin^2 \theta}{A \bar{h}} \left[ I_G + A \times \frac{\bar{h}^2}{\sin^2 \theta} \right]$$

or  $h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$  ... (3.10)

If  $\theta = 90^\circ$ , equation (3.10) becomes same as equation (3.5) which is applicable to vertically plane submerged surfaces.

In equation (3.10),  $I_G$  = M.O.I. of inclined surfaces about an axis passing through G and parallel to O-O.

**Problem 3.14 (a)** A rectangular plane surface 2 m wide and 3 m deep lies in water in such a way that its plane makes an angle of  $30^\circ$  with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge is 1.5 m below the free water surface.

**Solution.** Given :

Width of plane surface,  $b = 2 \text{ m}$

Depth,  $d = 3 \text{ m}$

Angle,  $\theta = 30^\circ$

Distance of upper edge from free water surface = 1.5 m

(i) **Total pressure force** is given by equation (3.6) as

$$F = \rho g A \bar{h}$$

where  $\rho = 1000 \text{ kg/m}^3$

$$A = b \times d = 3 \times 2 = 6 \text{ m}^2$$

$\therefore \bar{h}$  = Depth of C.G. from free water surface

$$= 1.5 + 1.5 \sin 30^\circ$$

$$\{ \because \bar{h} = AE + EB = 1.5 + BC \sin 30^\circ = 1.5 + 1.5 \sin 30^\circ \}$$

$$= 1.5 + 1.5 \times \frac{1}{2} = 2.25 \text{ m}$$

$$\therefore F = 1000 \times 9.81 \times 6 \times 2.25 = 132435 \text{ N. Ans.}$$

(ii) **Centre of pressure ( $h^*$ )**

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}, \quad \text{where } I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5 \text{ m}^4$$

$$\begin{aligned} \therefore h^* &= \frac{4.5 \times \sin^2 30^\circ}{6 \times 2.25} + 2.25 = \frac{4.5 \times \frac{1}{4}}{6 \times 2.25} + 2.25 \\ &= 0.0833 + 2.25 = 2.3333 \text{ m. Ans.} \end{aligned}$$

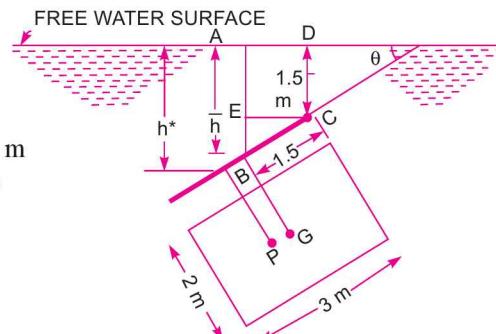


Fig. 3.19

**Problem 3.14 (b)** A rectangular plane surface 3 m wide and 4 m deep lies in water in such a way that its plane makes an angle of  $30^\circ$  with the free surface of water. Determine the total pressure force and position of centre of pressure, when the upper edge is 2 m below the free surface.

**Solution.** Given :

$$b = 3 \text{ m}, d = 4 \text{ m}, \theta = 30^\circ$$

Distance of upper edge from free surface of water = 2 m

(i) Total pressure force is given by equation (3.6) as

$$F = \rho g A h$$

where  $\rho = 1000 \text{ kg/m}^3$ ,

$$A = b \times d = 3 \times 4 = 12 \text{ m}^2$$

and  $\bar{h}$  = Depth of C.G. of plate from free water surface

$$= 2 + BE = 2 + BC \sin \theta$$

$$= 2 + 2 \sin 30^\circ = 2 + 2 \times \frac{1}{2} = 3 \text{ m}$$

$$\therefore F = 1000 \times 9.81 \times 12 \times 3 = 353167 \text{ N} = 353.167 \text{ kN. Ans.}$$

(ii) Centre of pressure ( $h^*$ )

$$\text{Using equation (3.10), we have } h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$$

$$\text{where } I_G = \frac{bd^3}{12} = \frac{3 \times 4^3}{12} = 16 \text{ m}^4$$

$$\therefore h^* = \frac{16 \times \sin^2 30^\circ}{12 \times 3} + 3 = \frac{16 \times \frac{1}{4}}{36} + 3 = 3.111 \text{ m. Ans.}$$

**Problem 3.15 (a)** A circular plate 3.0 m diameter is immersed in water in such a way that its greatest and least depth below the free surface are 4 m and 1.5 m respectively. Determine the total pressure on one face of the plate and position of the centre of pressure.

**Solution.** Given :

Dia. of plate,

$$d = 3.0 \text{ m}$$

$\therefore$  Area,

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (3.0)^2 = 7.0685 \text{ m}^2$$

Distance

$$DC = 1.5 \text{ m}, BE = 4 \text{ m}$$

Distance of C.G. from free surface

$$= \bar{h} = CD + GC \sin \theta = 1.5 + 1.5 \sin \theta$$

But

$$\begin{aligned} \sin \theta &= \frac{AB}{BC} = \frac{BE - AE}{BC} = \frac{4.0 - DC}{3.0} = \frac{4.0 - 1.5}{3.0} \\ &= \frac{2.5}{3.0} = 0.8333 \end{aligned}$$

$\therefore$

$$\bar{h} = 1.5 + 1.5 \times 0.8333 = 1.5 + 1.249 = 2.749 \text{ m}$$

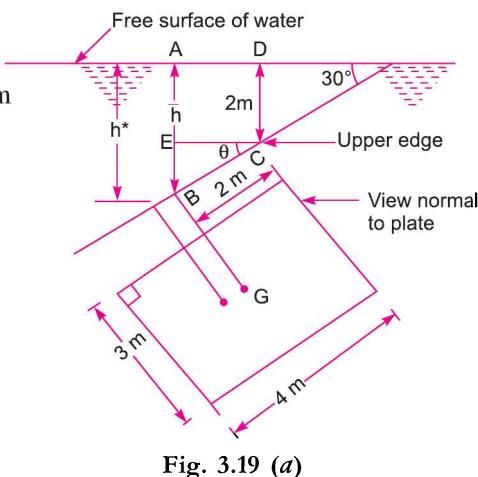


Fig. 3.19 (a)

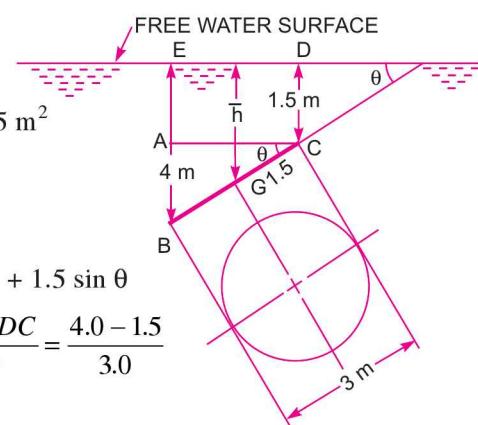


Fig. 3.20

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### (i) Total pressure (F)

$$F = \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times 7.0685 \times 2.749 = 190621 \text{ N. Ans.}$$

### (ii) Centre of pressure ( $h^*$ )

Using equation (3.10), we have  $h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$

$$\text{where } I_G = \frac{\pi}{64} d^4 = \frac{\pi}{64} (3)^4 = 3.976 \text{ m}^4$$

$$h^* = \frac{3.976 \times (.8333) \times .8333}{7.0685 \times 2.749} + 2.749 = 0.1420 + 2.749$$

$$= 2.891 \text{ m. Ans.}$$

**Problem 3.15 (b)** If in the above problem, the given circular plate is having a concentric circular hole of diameter 1.5 m, then calculate the total pressure and position of the centre of pressure on one face of the plate.

**Solution.** Given : [Refer to Fig. 3.20 (a)]

$$\text{Dia. of plate, } d = 3.0 \text{ m}$$

$$\therefore \text{Area of solid plate} = \frac{\pi}{4} d^2 = \frac{\pi}{4} (3)^2 = 7.0685 \text{ m}^2$$

$$\text{Dia. of hole in the plate, } d_0 = 1.5 \text{ m}$$

$$\therefore \text{Area of hole} = \frac{\pi}{4} d_0^2 = \frac{\pi}{4} (1.5)^2 = 1.7671 \text{ m}^2$$

$$\therefore \text{Area of the given plate, } A = \text{Area of solid plate} - \text{Area of hole}$$

$$= 7.0685 - 1.7671 = 5.3014 \text{ m}^2$$

$$\text{Distance } CD = 1.5, BE = 4 \text{ m}$$

$$\text{Distance of C.G. from the free surface,}$$

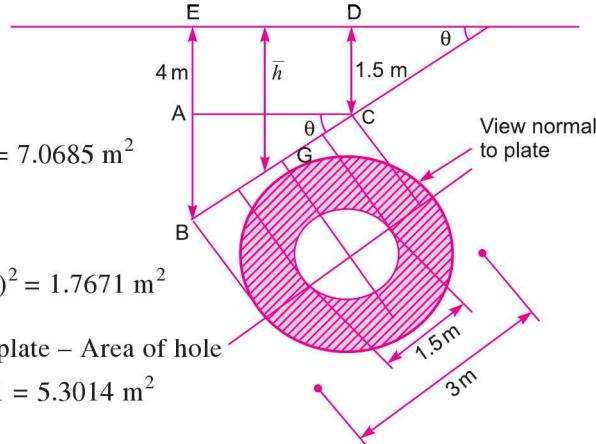


Fig. 3.20 (a)

$$\bar{h} = CD + GC \sin \theta$$

$$= 1.5 + 1.5 \sin \theta$$

$$\text{But } \sin \theta = \frac{AB}{BC} = \frac{BE - AE}{BC} = \frac{4 - 1.5}{3} = \frac{2.5}{3}$$

$$\therefore \bar{h} = 1.5 + 1.5 \times \frac{2.5}{3} = 1.5 + 1.25 = 2.75 \text{ m}$$

### (i) Total pressure force (F)

$$F = \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times 5.3014 \times 2.75$$

$$= 143018 \text{ N} = 143.018 \text{ kN. Ans.}$$

(ii) Position of centre of pressure ( $h^*$ )

Using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$$

$$\text{where } I_G = \frac{\pi}{64} [d^4 - d_0^4] = \frac{\pi}{64} [3^4 - 1.5^4] \text{ m}^4$$

$$A = \frac{\pi}{4} [d^2 - d_0^2] = \frac{\pi}{4} [3^2 - 1.5^2] \text{ m}^2$$

$$\sin \theta = \frac{2.5}{3} \text{ and } \bar{h} = 2.75$$

$$\begin{aligned} \therefore h^* &= \frac{\frac{\pi}{64} [3^4 - 1.5^4] \times \left(\frac{2.5}{3}\right)^2}{\frac{\pi}{4} [3^2 - 1.5^2] \times 2.75} + 2.75 \\ &= \frac{\frac{1}{16} [3^2 + 1.5^2] \times \left(\frac{2.5}{3}\right)^2}{2.75} + 2.75 = \frac{1 \times 11.25 \times 6.25}{16 \times 2.75 \times 9} + 2.75 \\ &= 0.177 + 2.75 = \mathbf{2.927 \text{ m. Ans.}} \end{aligned}$$

**Problem 3.16** A circular plate 3 metre diameter is submerged in water as shown in Fig. 3.21. Its greatest and least depths are below the surfaces being 2 metre and 1 metre respectively. Find : (i) the total pressure on front face of the plate, and (ii) the position of centre of pressure.

**Solution.** Given :

$$\text{Dia. of plate, } d = 3.0 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} (3.0)^2 = 7.0685 \text{ m}^2$$

$$\text{Distance, } DC = 1 \text{ m, } BE = 2 \text{ m}$$

$$\text{In } \triangle ABC, \quad \sin \theta = \frac{AB}{AC} = \frac{BE - AE}{BC} = \frac{BE - DC}{BC} = \frac{2.0 - 1.0}{3.0} = \frac{1}{3}$$

The centre of gravity of the plate is at the middle of  $BC$ , i.e., at a distance 1.5 m from  $C$ .

The distance of centre of gravity from the free surface of the water is given by

$$\bar{h} = CD + CG \sin \theta = 1.0 + 1.5 \times \frac{1}{3} = 1.5 \text{ m.} \quad (\because \sin \theta = \frac{1}{3})$$

(i) Total pressure on the front face of the plate is given by

$$\begin{aligned} F &= \rho g A \bar{h} \\ &= 1000 \times 9.81 \times 7.0685 \times 1.5 = \mathbf{104013 \text{ N. Ans.}} \end{aligned}$$

(ii) Let the distance of the centre of pressure from the free surface of the water be  $h^*$ . Then using equation (3.10), we have

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$$

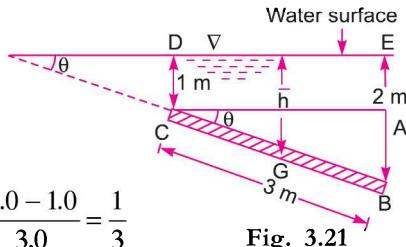


Fig. 3.21

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where  $I_G = \frac{\pi}{64} d^4 = \frac{\pi}{64} (3)^4, A = \frac{\pi}{4} d^2, \bar{h} = 1.5 \text{ m}$  and  $\sin \theta = \frac{1}{3}$

Substituting the values, we get

$$\begin{aligned} h^* &= \frac{\frac{\pi}{64} d^4 \times \left(\frac{1}{3}\right)^2}{\frac{\pi}{4} d^2 \times 1.5} + 1.5 = \frac{d^2}{16} \times \frac{1}{9 \times 1.5} + 1.5 \\ &= \frac{3^2}{16 \times 9 \times 1.5} + 1.5 = .0416 + 1.5 = \mathbf{1.5416 \text{ m. Ans.}} \end{aligned}$$

**Problem 3.17** A rectangular gate  $5 \text{ m} \times 2 \text{ m}$  is hinged at its base and inclined at  $60^\circ$  to the horizontal as shown in Fig. 3.22. To keep the gate in a stable position, a counter weight of 5000 kgf is attached at the upper end of the gate as shown in figure. Find the depth of water at which the gate begins to fall. Neglect the weight of the gate and friction at the hinge and pulley.

**Solution.** Given :

$$\text{Length of gate} = 5 \text{ m}$$

$$\text{Width of gate} = 2 \text{ m}$$

$$\theta = 60^\circ$$

$$\begin{aligned} \text{Weight, } W &= 5000 \text{ kgf} \\ &= 5000 \times 9.81 \text{ N} \\ &= 49050 \text{ N} \quad (\because 1 \text{ kgf} = 9.81 \text{ N}) \end{aligned}$$

As the pulley is frictionless, the force acting at  $B = 49050 \text{ N}$ . First find the total force  $F$  acting on the gate  $AB$  for a given depth of water.

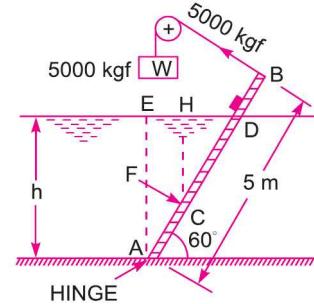


Fig. 3.22

$$\text{From figure, } AD = \frac{AE}{\sin \theta} = \frac{h}{\sin 60^\circ} = \frac{5}{\sqrt{3}/2} = \frac{2h}{\sqrt{3}}$$

$$\therefore \text{Area of gate immersed in water, } A = AD \times \text{Width} \times \frac{2h}{\sqrt{3}} \times 2 = \frac{4h}{\sqrt{3}} \text{ m}^2$$

$$\text{Also depth of the C.G. of the immersed area} = \bar{h} = \frac{h}{2} = 0.5 h$$

$$\therefore \text{Total force } F \text{ is given by } F = \rho g A \bar{h} = 1000 \times 9.81 \times \frac{4h}{\sqrt{3}} \times \frac{h}{\sqrt{2}} = \frac{19620}{\sqrt{3}} h^2 \text{ N}$$

The centre of pressure of the immersed surface,  $h^*$  is given by

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$$

where  $I_G = \text{M.O.I. of the immersed area}$

$$\begin{aligned} &= \frac{b \times (AD)^3}{12} = \frac{2}{12} \times \left(\frac{2h}{\sqrt{3}}\right)^3 \quad \left\{ \because AD = \frac{2h}{\sqrt{3}} \right\} \\ &= \frac{16h^3}{12 \times 3 \times \sqrt{3}} = \frac{4h^3}{9 \times \sqrt{3}} \text{ m}^4 \end{aligned}$$

$$\therefore h^* = \frac{4h^3}{9 \times \sqrt{3}} \times \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\frac{4h}{\sqrt{3}} \times \frac{h}{2}} + \frac{h}{2} = \frac{3h^3}{18h^2} + \frac{h}{2} = \frac{h}{6} + \frac{h}{2} = \frac{h+3h}{6} = \frac{2h}{3}$$

Now in the  $\Delta CHD$ ,  $CH = h^* = \frac{2h}{3}$ ,  $\angle CDH = 60^\circ$

$$\therefore \frac{CH}{CD} = \sin 60^\circ$$

$$\therefore CD = \frac{CH}{\sin 60^\circ} = \frac{h^*}{\sin 60^\circ} = \frac{2h}{3 \times \frac{\sqrt{3}}{2}} = \frac{4h}{3 \times \sqrt{3}}$$

$$\therefore AC = AD - CD = \frac{2h}{\sqrt{3}} - \frac{4h}{3\sqrt{3}} = \frac{6h - 4h}{3\sqrt{3}} = \frac{2h}{3\sqrt{3}} \text{ m}$$

Taking the moments about hinge, we get

$$49050 \times 5.0 = F \times AC = \frac{19620}{\sqrt{3}} h^2 \times \frac{2h}{3\sqrt{3}}$$

$$\text{or } .245250 = \frac{39240 h^3}{3 \times 3}$$

$$\therefore h^3 = \frac{9 \times 245250}{39240} = 56.25$$

$$\therefore h = (56.25)^{1/3} = 3.83 \text{ m. Ans.}$$

**Problem 3.18** An inclined rectangular sluice gate AB, 1.2 m by 5 m size as shown in Fig. 3.23 is installed to control the discharge of water. The end A is hinged. Determine the force normal to the gate applied at B to open it.

**Solution.** Given :

$$A = \text{Area of gate} = 1.2 \times 5.0 = 6.0 \text{ m}^2$$

Depth of C.G. of the gate from free surface of the water =  $\bar{h}$

$$\begin{aligned} &= DG = BC - BE \\ &= 5.0 - BG \sin 45^\circ \\ &= 5.0 - 0.6 \times \frac{1}{\sqrt{2}} = 4.576 \text{ m} \end{aligned}$$

The total pressure force ( $F$ ) acting on the gate,

$$\begin{aligned} F &= \rho g A \bar{h} \\ &= 1000 \times 9.81 \times 6.0 \times 4.576 \\ &= 269343 \text{ N} \end{aligned}$$

This force is acting at H, where the depth of H from free surface is given by

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$$

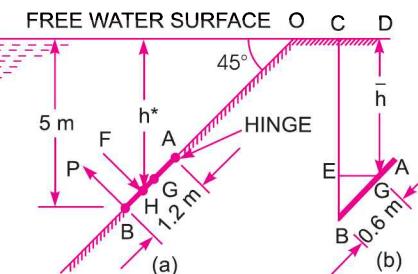


Fig. 3.23

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where  $I_G = \text{M.O.I. of gate} = \frac{bd^3}{12} = \frac{5.0 \times 1.2^3}{12} = 0.72 \text{ m}$

$$\therefore \text{Depth of centre of pressure } h^* = \frac{0.72 \times \sin^2 45^\circ}{6 \times 4.576} + 4.576 = .013 + 4.576 = 4.589 \text{ m}$$

But from Fig. 3.23 (a),  $\frac{h^*}{OH} = \sin 45^\circ$

$$\therefore \text{Distance, } OH = \frac{h^*}{\sin 45^\circ} = \frac{4.589}{\frac{1}{\sqrt{2}}} = 4.589 \times \sqrt{2} = 6.489 \text{ m}$$

Distance,  $BO = \frac{5}{\sin 45^\circ} = 5 \times \sqrt{2} = 7.071 \text{ m}$

Distance,  $BH = BO - OH = 7.071 - 6.489 = 0.582 \text{ m}$

$\therefore$  Distance  $AH = AB - BH = 1.2 - 0.582 = 0.618 \text{ m}$

Taking the moments about the hinge A

$$P \times AB = F \times (AH)$$

where P is the force normal to the gate applied at B

$$\therefore P \times 1.2 = 269343 \times 0.618$$

$$\therefore P = \frac{269343 \times 0.618}{1.2} = 138708 \text{ N. Ans.}$$

**Problem 3.19** A gate supporting water is shown in Fig. 3.24. Find the height h of the water so that the gate tips about the hinge. Take the width of the gate as unity.

**Solution.** Given :

$$\theta = 60^\circ$$

Distance,  $AC = \frac{h}{\sin 60^\circ} = \frac{2h}{\sqrt{3}}$

where  $h$  = Depth of water.

The gate will start tipping about hinge B if the resultant pressure force acts at B. If the resultant pressure force passes through a point which is lying from B to C anywhere on the gate, the gate will tip over the hinge. Hence limiting case is when the resultant force passes through B. But the resultant force passes through the centre of pressure. Hence for the given position, point B becomes the centre of pressure. Hence depth of centre of pressure,

$$h^* = (h - 3) \text{ m}$$

But  $h^*$  is also given by  $= \frac{I_G \sin^2 \theta}{Ah} + \bar{h}$

Taking width of gate unity. Then

Area,  $A = AC \times 1 = \frac{2h}{\sqrt{3}} \times 1 ; \bar{h} = \frac{h}{2}$

$$I_G = \frac{bd^3}{12} = \frac{1 \times AC^3}{12} = \frac{1 \times \left(\frac{2h}{\sqrt{3}}\right)^3}{12} = \frac{8h^3}{12 \times 3 \times \sqrt{3}} = \frac{2h^3}{9 \times \sqrt{3}}$$

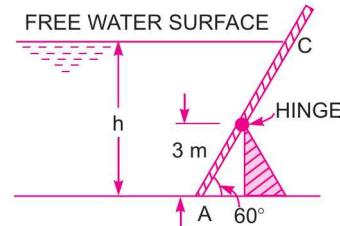


Fig. 3.24

$$\therefore h^* = \frac{2h^3}{9 \times \sqrt{3}} \times \frac{\sin^2 60^\circ}{\frac{2h}{\sqrt{3}} \times \frac{h}{2}} + \frac{h}{2} = \frac{2h^3 \times 3}{9h^2} + \frac{h}{2} = \frac{h}{6} + \frac{h}{2} = \frac{2h}{3}$$

Equating the two values of  $h^*$ ,

$$h - 3 = \frac{2h}{3} \quad \text{or} \quad h - \frac{2h}{3} = 3 \quad \text{or} \quad \frac{h}{3} = 3$$

$$\therefore h = 3 \times 3 = 9 \text{ m}$$

$\therefore$  Height of water for tipping the gate = 9 m. Ans.

**Problem 3.20** A rectangular sluice gate AB, 2 m wide and 3 m long is hinged at A as shown in Fig. 3.25. It is kept closed by a weight fixed to the gate. The total weight of the gate and weight fixed to the gate is 343350 N. Find the height of the water 'h' which will just cause the gate to open. The centre of gravity of the weight and gate is at G.

**Solution.** Given :

$$\text{Width of gate, } b = 2 \text{ m ; Length of gate } L = 3 \text{ m}$$

$$\therefore \text{Area, } A = 2 \times 3 = 6 \text{ m}^2$$

$$\text{Weight of gate and } W = 343350 \text{ N}$$

$$\text{Angle of inclination, } \theta = 45^\circ$$

Let  $h$  is the required height of water.

Depth of C.G. of the gate and weight =  $\bar{h}$

From Fig. 3.25 (a),

$$\begin{aligned} \bar{h} &= h - ED = h - (AD - AE) \\ &= h - (AB \sin \theta - EG \tan \theta) \quad \left\{ \because \tan \theta = \frac{AE}{EG} \therefore AE = EG \tan \theta \right\} \\ &= h - (3 \sin 45^\circ - 0.6 \tan 45^\circ) \\ &= h - (2.121 - 0.6) = (h - 1.521) \text{ m} \end{aligned}$$

The total pressure force,  $F$  is given by

$$\begin{aligned} F &= \rho g A \bar{h} = 1000 \times 9.81 \times 6 \times (h - 1.521) \\ &= 58860 (h - 1.521) \text{ N.} \end{aligned}$$

The total force  $F$  is acting at the centre of pressure as shown in Fig. 3.25 (b) at H. The depth of H from free surface is given by  $h^*$  which is equal to

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}, \text{ where } I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12} = \frac{54}{12} = 4.5 \text{ m}^4$$

$$\therefore h^* = \frac{4.5 \times \sin^2 45^\circ}{6 \times (h - 1.521)} + (h - 1.521) = \frac{0.375}{(h - 1.521)} + (h - 1.521) \text{ m}$$

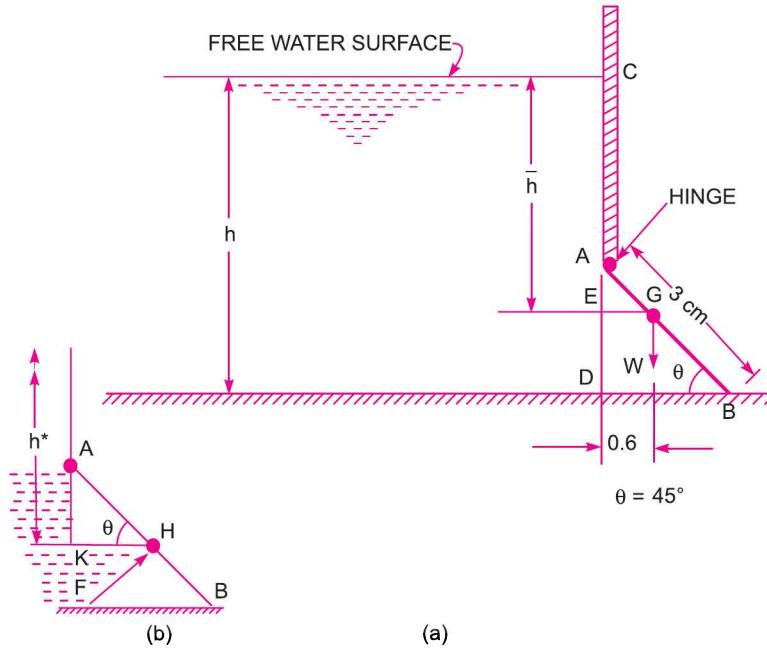


Fig. 3.25

Now taking moments about hinge A, we get

$$343350 \times EG = F \times AH$$

or

$$343350 \times 0.6 = F \times \frac{AK}{\sin 45^\circ}$$

$$\left[ \text{From } \Delta AKH, \text{ Fig. 3.25 (b)} \quad AK = AH \sin \theta = AH \sin 45^\circ \therefore AH = \frac{AK}{\sin 45^\circ} \right]$$

$$= \frac{58860(h - 1.521) \times AK}{\sin 45^\circ}$$

$$\therefore AK = \frac{343350 \times 0.6 \times \sin 45^\circ}{58860(h - 1.521)} = \frac{0.3535 \times 7}{(h - 1.521)} \quad \dots(i)$$

$$\text{But } AK = h^* - AC = \frac{375}{(h - 1.521)} + (h - 1.521) - AC \quad \dots(ii)$$

$$\text{But } AC = CD - AD = h - AB \sin 45^\circ = h - 3 \times \sin 45^\circ = h - 2.121$$

$\therefore$  Substituting this value in (ii), we get

$$AK = \frac{375}{h - 1.521} + (h - 1.521) - (h - 2.121)$$

$$= \frac{375}{h - 1.521} + 2.121 - 1.521 = \frac{375}{h - 1.521} + 0.6 \quad \dots(iii)$$

Equating the two values of AK from (i) and (iii)

$$\frac{0.3535 \times 7}{h - 1.521} = \frac{0.375}{h - 1.521} + 0.6$$

or

$$0.3535 \times 7 = 0.375 + 0.6(h - 1.521) = 0.375 + 0.6h - 0.6 \times 1.521$$

or

$$0.6h = 2.4745 - .375 + 0.6 \times 1.521 = 2.0995 + 0.9126 = 3.0121$$

$$\therefore h = \frac{3.0121}{0.6} = 5.02 \text{ m. Ans.}$$

**Problem 3.21** Find the total pressure and position of centre of pressure on a triangular plate of base 2 m and height 3 m which is immersed in water in such a way that the plane of the plate makes an angle of  $60^\circ$  with the free surface of the water. The base of the plate is parallel to water surface and at a depth of 2.5 m from water surface.

**Solution.** Given :

Base of plate,  $b = 2 \text{ m}$

Height of plate,  $h = 3 \text{ m}$

$$\therefore \text{Area, } A = \frac{b \times h}{2} = \frac{2 \times 3}{2} = 3 \text{ m}^2$$

Inclination,  $\theta = 60^\circ$

Depth of centre of gravity from free surface of water,

$$\begin{aligned} \bar{h} &= 2.5 + AG \sin 60^\circ \\ &= 2.5 + \frac{1}{3} \times 3 \times \frac{\sqrt{3}}{2} \\ &= 2.5 + .866 \text{ m} = 3.366 \text{ m} \end{aligned} \quad \left\{ \because AG = \frac{1}{3} \text{ of height of triangle} \right\}$$

(i) **Total pressure force (F)**

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times 3 \times 3.366 = 99061.38 \text{ N. Ans.}$$

(ii) **Centre of pressure ( $h^*$ )**. Depth of centre of pressure from free surface of water is given by

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h}$$

$$\text{where } I_G = \frac{bh^3}{36} = \frac{2 \times 3^3}{36} = \frac{3}{2} = 1.5 \text{ m}^4$$

$$\therefore h^* = \frac{1.5 \times \sin^2 60^\circ}{3 \times 3.366} + 3.366 = 0.111 + 3.366 = 3.477 \text{ m. Ans.}$$

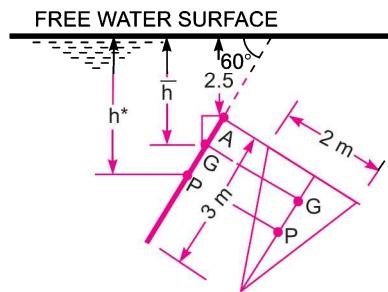


Fig. 3.26

### ► 3.6 CURVED SURFACE SUB-MERGED IN LIQUID

Consider a curved surface  $AB$ , sub-merged in a static fluid as shown in Fig. 3.27. Let  $dA$  is the area of a small strip at a depth of  $h$  from water surface.

Then pressure intensity on the area  $dA$  is  $= \rho gh$

and pressure force,  $dF = p \times \text{Area} = \rho gh \times dA$  ... (3.11)

This force  $dF$  acts normal to the surface.

Hence total pressure force on the curved surface should be

$$F = \int \rho g h dA \quad \dots(3.12)$$

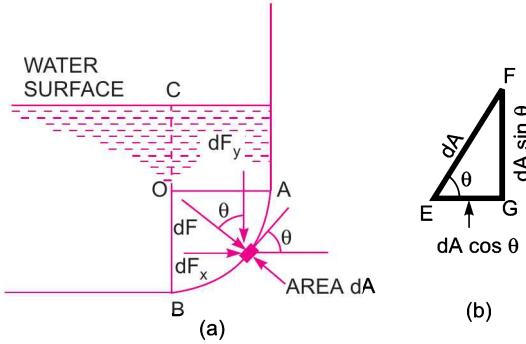


Fig. 3.27

But here as the direction of the forces on the small areas are not in the same direction, but varies from point to point. Hence integration of equation (3.11) for curved surface is impossible. The problem can, however, be solved by resolving the force  $dF$  in two components  $dF_x$  and  $dF_y$  in the  $x$  and  $y$  directions respectively. The total force in the  $x$  and  $y$  directions, i.e.,  $F_x$  and  $F_y$  are obtained by integrating  $dF_x$  and  $dF_y$ . Then total force on the curved surface is

$$F = \sqrt{F_x^2 + F_y^2} \quad \dots(3.13)$$

$$\text{and inclination of resultant with horizontal is } \tan \phi = \frac{F_y}{F_x} \quad \dots(3.14)$$

Resolving the force  $dF$  given by equation (3.11) in  $x$  and  $y$  directions :

$$dF_x = dF \sin \theta = \rho g h dA \sin \theta \quad \{ \because dF = \rho g h dA \}$$

$$\text{and} \quad dF_y = dF \cos \theta = \rho g h dA \cos \theta$$

Total forces in the  $x$  and  $y$  direction are :

$$F_x = \int dF_x = \int \rho g h dA \sin \theta = \rho g \int h dA \sin \theta \quad \dots(3.15)$$

$$\text{and} \quad F_y = \int dF_y = \int \rho g h dA \cos \theta = \rho g \int h dA \cos \theta \quad \dots(3.16)$$

Fig. 3.27 (b) shows the enlarged area  $dA$ . From this figure, i.e.,  $\Delta EFG$ ,

$$EF = dA$$

$$FG = dA \sin \theta$$

$$EG = dA \cos \theta$$

Thus in equation (3.15),  $dA \sin \theta = FG$  = Vertical projection of the area  $dA$  and hence the expression  $\rho g \int h dA \sin \theta$  represents the total pressure force on the projected area of the curved surface on the vertical plane. Thus

$$F_x = \text{Total pressure force on the projected area of the curved surface on vertical plane.} \quad \dots(3.17)$$

Also  $dA \cos \theta = EG$  = horizontal projection of  $dA$  and hence  $hdA \cos \theta$  is the volume of the liquid contained in the elementary area  $dA$  upto free surface of the liquid. Thus  $\int hdA \cos \theta$  is the total volume contained between the curved surface extended upto free surface.

Hence  $\rho g \int hdA \cos \theta$  is the total weight supported by the curved surface. Thus

$$F_y = \rho g \int hdA \cos \theta$$

$$= \text{weight of liquid supported by the curved surface upto free surface of liquid.} \quad \dots(3.18)$$

In Fig. 3.28, the curved surface  $AB$  is not supporting any fluid. In such cases,  $F_y$  is equal to the weight of the imaginary liquid supported by  $AB$  upto free surface of liquid. The direction of  $F_y$  will be taken in upward direction.

**Problem 3.22** Compute the horizontal and vertical components of the total force acting on a curved surface  $AB$ , which is in the form of a quadrant of a circle of radius 2 m as shown in Fig. 3.29. Take the width of the gate as unity.

**Solution.** Given :

$$\text{Width of gate} = 1.0 \text{ m}$$

$$\text{Radius of the gate} = 2.0 \text{ m}$$

$$\therefore \text{Distance } AO = OB = 2 \text{ m}$$

Horizontal force,  $F_x$  exerted by water on gate is given by equation (3.17) as

$F_x$  = Total pressure force on the projected area of curved surface  $AB$  on vertical plane

$$= \text{Total pressure force on } OB$$

{projected area of curved surface on vertical plane =  $OB \times 1$ }

$$= \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times 2 \times 1 \times \left( 1.5 + \frac{2}{2} \right)$$

$$\{ \because \text{Area of } OB = A = BO \times 1 = 2 \times 1 = 2,$$

$$\bar{h} = \text{Depth of C.G. of } OB \text{ from free surface} = 1.5 + \frac{2}{2} \}$$

$$F_x = 9.81 \times 2000 \times 2.5 = 49050 \text{ N. Ans.}$$

The point of application of  $F_x$  is given by  $h^* = \frac{I_G}{Ah} + \bar{h}$

$$\text{where } I_G = \text{M.O.I. of } OB \text{ about its C.G.} = \frac{bd^3}{12} = \frac{1 \times 2^3}{12} = \frac{2}{3} \text{ m}^4$$

$$\therefore h^* = \frac{\frac{2}{3}}{2 \times 2.5} + 2.5 = \frac{1}{7.5} + 2.5 \text{ m}$$

$$= 0.1333 + 2.5 = 2.633 \text{ m from free surface.}$$

Vertical force,  $F_y$ , exerted by water is given by equation (3.18)

$$F_y = \text{Weight of water supported by } AB \text{ upto free surface}$$

$$= \text{Weight of portion } DABOC$$

$$= \text{Weight of } DAOC + \text{Weight of water } AOB$$

$$= \rho g [\text{Volume of } DAOC + \text{Volume of } AOB]$$

$$= 1000 \times 9.81 \left[ AD \times AO \times 1 + \frac{\pi}{4} (AO)^2 \times 1 \right]$$

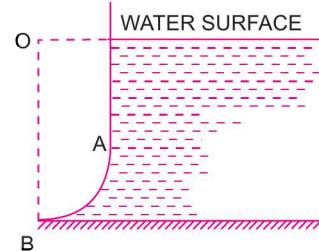


Fig. 3.28

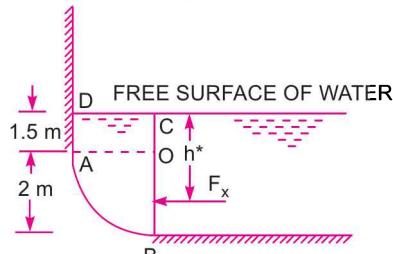


Fig. 3.29

$$= 1000 \times 9.81 \left[ 1.5 \times 2.0 \times 1 + \frac{\pi}{4} \times 2^2 \times 1 \right]$$

$$= 1000 \times 9.81 [3.0 + \pi] N = 60249.1 N. Ans.$$

**Problem 3.23** Fig. 3.30 shows a gate having a quadrant shape of radius 2 m. Find the resultant force due to water per metre length of the gate. Find also the angle at which the total force will act.

**Solution.** Given :

$$\text{Radius of gate} = 2 \text{ m}$$

$$\text{Width of gate} = 1 \text{ m}$$

**Horizontal Force**

$F_x$  = Force on the projected area of the curved surface on vertical plane

$$= \text{Force on } BO = \rho g A \bar{h}$$

$$\text{where } A = \text{Area of } BO = 2 \times 1 = 2 \text{ m}^2, \bar{h} = \frac{1}{2} \times 2 = 1 \text{ m};$$

$$F_x = 1000 \times 9.81 \times 2 \times 1 = 19620 \text{ N}$$

This will act at a depth of  $\frac{2}{3} \times 2 = \frac{4}{3} \text{ m}$  from free surface of liquid,

**Vertical Force,  $F_y$**

$F_y$  = Weight of water (imagined) supported by  $AB$   
 $= \rho g \times \text{Area of } AOB \times 1.0$

$$= 1000 \times 9.81 \times \frac{\pi}{4} (2)^4 \times 1.0 = 30819 \text{ N}$$

This will act at a distance of  $\frac{4R}{3\pi} = \frac{4 \times 2.0}{3\pi} = 0.848 \text{ m}$  from  $OB$ .

$\therefore$  Resultant force,  $F$  is given by

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{19620 + 30819} = \sqrt{384944400 + 949810761} \\ &= 36534.4 \text{ N. Ans.} \end{aligned}$$

The angle made by the resultant with horizontal is given by

$$\tan \theta = \frac{F_y}{F_x} = \frac{30819}{19620} = 1.5708$$

$$\therefore \theta = \tan^{-1} 1.5708 = 57^\circ 31'. Ans.$$

**Problem 3.24** Find the magnitude and direction of the resultant force due to water acting on a roller gate of cylindrical form of 4.0 m diameter, when the gate is placed on the dam in such a way that water is just going to spill. Take the length of the gate as 8 m.

**Solution.** Given :

$$\text{Dia. of gate} = 4 \text{ m}$$

$$\therefore \text{Radius, } R = 2 \text{ m}$$

$$\text{Length of gate, } l = 8 \text{ m}$$

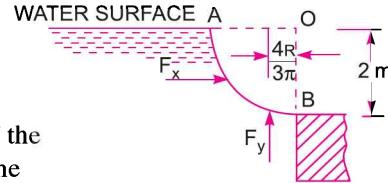


Fig. 3.30

**Horizontal force,  $F_x$**  acting on the gate is

$$F_x = \rho g A \bar{h} = \text{Force on projected area of curved surface } ACB \text{ on vertical plane}$$

= Force on vertical area  $AOB$

where  $A = \text{Area of } AOB = 4.0 \times 8.0 = 32.0 \text{ m}^2$

$\bar{h} = \text{Depth of C.G. of } AOB = 4/2 = 2.0 \text{ m}$

$$\therefore F_x = 1000 \times 9.81 \times 32.0 \times 2.0 \\ = 627840 \text{ N.}$$

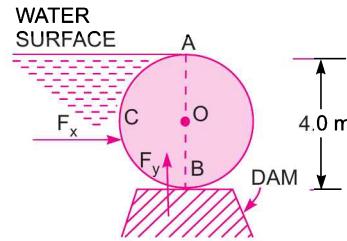


Fig. 3.31

**Vertical force,  $F_y$**  is given by

$$F_y = \text{Weight of water enclosed or supported (actually or imaginary) by the curved surface } ACB \\ = \rho g \times \text{Volume of portion } ACB \\ = \rho g \times \text{Area of } ACB \times l \\ = 1000 \times 9.81 \times \frac{\pi}{2}(R)^2 \times 8.0 = 9810 \times \frac{\pi}{2}(2)^2 \times 8.0 = 493104 \text{ N}$$

It will be acting in the upward direction.

$$\therefore \text{Resultant force, } F = \sqrt{F_x^2 + F_y^2} = \sqrt{627840 + 493104} = 798328 \text{ N. Ans.}$$

$$\text{Direction of resultant force is given by } \tan \theta = \frac{F_y}{F_x} = \frac{493104}{627840} = 0.7853$$

$$\therefore \theta = 31^\circ 8'. \text{ Ans.}$$

**Problem 3.25** Find the horizontal and vertical component of water pressure acting on the face of aainter gate of  $90^\circ$  sector of radius 4 m as shown in Fig. 3.32. Take width of gate unity.

**Solution.** Given :

Radius of gate,  $R = 4 \text{ m}$

Horizontal component of force acting on the gate is

$$F_x = \text{Force on area of gate} \\ \text{projected on vertical plane} \\ = \text{Force on area } ADB \\ = \rho g A \bar{h}$$

where  $A = AB \times \text{Width of gate}$

$$= 2 \times AD \times 1 \quad (\because AB = 2AD)$$

$$= 2 \times 4 \times \sin 45^\circ = 8 \times .707 = 5.656 \text{ m}^2 \quad \{\because AD = 4 \sin 45^\circ\}$$

$$\bar{h} = \frac{AB}{2} = \frac{5.656}{2} = 2.828 \text{ m}$$

$$\therefore F_x = 1000 \times 9.81 \times 5.656 \times 2.828 \text{ N} = 156911 \text{ N. Ans.}$$

**Vertical component**

$$F_y = \text{Weight of water supported or enclosed by the curved surface} \\ = \text{Weight of water in portion } ACBDA \\ = \rho g \times \text{Area of } ACBDA \times \text{Width of gate} \\ = 1000 \times 9.81 \times [\text{Area of sector } ACBOA - \text{Area of } \Delta ABO] \times 1$$

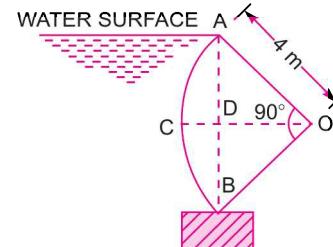


Fig. 3.32

$$\begin{aligned}
 &= 9810 \times \left[ \frac{\pi}{4} R^2 - \frac{AO \times BO}{2} \right] \quad [\because \Delta AOB \text{ is a right angled}] \\
 &= 9810 \times \left[ \frac{\pi}{4} 4^2 - \frac{4 \times 4}{2} \right] = 44796 \text{ N. Ans.}
 \end{aligned}$$

**Problem 3.26** Calculate the horizontal and vertical components of the water pressure exerted on a tainter gate of radius 8 m as shown in Fig. 3.33. Take width of gate unity.

**Solution.** The horizontal component of water pressure is given by

$$\begin{aligned}
 F_x &= \rho g A \bar{h} = \text{Force on the area projected on vertical plane} \\
 &= \text{Force on the vertical area of } BD \\
 \text{where } A &= BD \times \text{Width of gate} = 4.0 \times 1 = 4.0 \text{ m}
 \end{aligned}$$

$$\bar{h} = \frac{1}{2} \times 4 = 2 \text{ m}$$

$$F_x = 1000 \times 9.81 \times 4.0 \times 2.0 = 78480 \text{ N. Ans.}$$

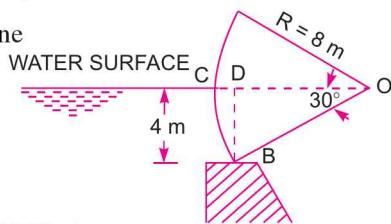


Fig. 3.33

Vertical component of the water pressure is given by

$$\begin{aligned}
 F_y &= \text{Weight of water supported or enclosed (imaginary) by curved} \\
 &\quad \text{surface } CB \\
 &= \text{Weight of water in the portion } CBDC \\
 &= \rho g \times [\text{Area of portion } CBDC] \times \text{Width of gate} \\
 &= \rho g \times [\text{Area of sector } CBO - \text{Area of the triangle } BOD] \times 1 \\
 &= 1000 \times 9.81 \times \left[ \frac{30}{360} \times \pi R^2 - \frac{BD \times DO}{2} \right] \\
 &= 9810 \times \left[ \frac{1}{12} \pi \times 8^2 - \frac{4.0 \times 8.8 \cos 30^\circ}{2} \right] \\
 &\quad \{ \because DO = BO \cos 30^\circ = 8 \times \cos 30^\circ \} \\
 &= 9810 \times [16.755 - 13.856] = 28439 \text{ N. Ans.}
 \end{aligned}$$

**Problem 3.27** A cylindrical gate of 4 m diameter 2 m long has water on its both sides as shown in Fig. 3.34. Determine the magnitude, location and direction of the resultant force exerted by the water on the gate. Find also the least weight of the cylinder so that it may not be lifted away from the floor.

**Solution.** Given :

$$\text{Dia. of gate} = 4 \text{ m}$$

$$\text{Radius} = 2 \text{ m}$$

(i) The forces acting on the left side of the cylinder are :

The horizontal component,  $F_{x_1}$

where  $F_{x_1} = \text{Force of water on area projected on vertical plane}$

= Force on area  $AOC$

$$= \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times 8 \times 2$$

$$= 156960 \text{ N}$$

$$\begin{aligned}
 \text{where } A &= AC \times \text{Width} = 4 \times 2 \\
 &= 8 \text{ m}^2
 \end{aligned}$$

$$= \bar{h} = \frac{1}{2} \times 4 = 2 \text{ m}$$

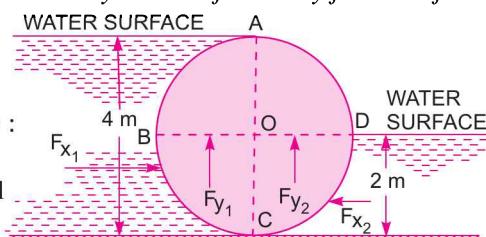


Fig. 3.34

$$\begin{aligned}
 F_{y_1} &= \text{weight of water enclosed by } ABCOA \\
 &= 1000 \times 9.81 \times \left[ \frac{\pi}{2} R^2 \right] \times 2.0 = 9810 \times \frac{\pi}{2} \times 2^2 \times 2.0 = 123276 \text{ N.}
 \end{aligned}$$

### Right Side of the Cylinder

$$\begin{aligned}
 F_{x_2} &= \rho g A_2 \bar{h}_2 = \text{Force on vertical area } CO \\
 &= 1000 \times 9.81 \times 2 \times 2 \times \frac{2}{2} \left\{ A_2 = CO \times 1 = 2 \times 1 = 2 \text{ m}^2, \bar{h}_2 = \frac{2}{2} = 1.0 \right\} \\
 &= 39240 \text{ N} \\
 F_{y_2} &= \text{Weight of water enclosed by } DOCD \\
 &= \rho g \times \left[ \frac{\pi}{4} R^2 \right] \times \text{Width of gate} \\
 &= 1000 \times 9.81 \times \frac{\pi}{4} \times 2^2 \times 2 = 61638 \text{ N}
 \end{aligned}$$

∴ Resultant force in the direction of  $x$ ,

$$F_x = F_{x_1} - F_{x_2} = 156960 - 39240 = 117720 \text{ N}$$

Resultant force in the direction of  $y$ ,

$$F_y = F_{y_1} + F_{y_2} = 123276 + 61638 = 184914 \text{ N}$$

(i) **Resultant force,  $F$**  is given as

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(117720)^2 + (184914)^2} = 219206 \text{ N. Ans.}$$

(ii) **Direction of resultant force** is given by

$$\tan \theta = \frac{F_y}{F_x} = \frac{184914}{117720} = 1.5707$$

$$\therefore \theta = 57^\circ 31'. \text{ Ans.}$$

(iii) **Location of the resultant force**

Force,  $F_{x_1}$  acts at a distance of  $\frac{2 \times 4}{3} = 2.67$  m from the top surface of water on left side, while  $F_{x_2}$  acts at a distance of  $\frac{2}{3} \times 2 = 1.33$  m from free surface on the right side of the cylinder. The resultant force  $F_x$  in the direction of  $x$  will act at a distance of  $y$  from the bottom as

$$F_x \times y = F_{x_1} [4 - 2.67] - F_{x_2} [2 - 1.33]$$

$$\text{or} \quad 117720 \times y = 156960 \times 1.33 - 39240 \times .67 = 208756.8 - 26290.8 = 182466$$

$$\therefore y = \frac{182466}{117720} = 1.55 \text{ m from the bottom.}$$

Force  $F_{y_1}$  acts at a distance  $\frac{4R}{3\pi}$  from  $AOC$  or at a distance  $\frac{4 \times 2.0}{3\pi} = 0.8488$  m from  $AOC$  towards left of  $AOC$ .

Also  $F_{y_2}$  acts at a distance  $\frac{4R}{3\pi} = 0.8488$  m from  $AOC$  towards the right of  $AOC$ . The resultant force  $F_y$  will act at a distance  $x$  from  $AOC$  which is given by

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$$F_y \times x = F_{y_1} \times .8488 - F_{y_2} \times .8488$$

or  $184914 \times x = 123276 \times .8488 - 61638 \times .8488 = .8488 [123276 - 61638] = 52318.4$

$$\therefore x = \frac{52318.4}{184914} = 0.2829 \text{ m from AOC.}$$

(iv) **Least weight of cylinder.** The resultant force in the upward direction is

$$F_y = 184914 \text{ N}$$

Thus the weight of cylinder should not be less than the upward force  $F_y$ . Hence least weight of cylinder should be at least.

$$= 184914 \text{ N. Ans.}$$

**Problem 3.28** Fig. 3.35 shows the cross-section of a tank full of water under pressure. The length of the tank is 2 m. An empty cylinder lies along the length of the tank on one of its corner as shown. Find the horizontal and vertical components of the force acting on the curved surface ABC of the cylinder.

**Solution.** Radius,  $R = 1 \text{ m}$

Length of tank,  $l = 2 \text{ m}$

$$\begin{aligned} \text{Pressure, } p &= 0.2 \text{ kgf/cm}^2 = 0.2 \times 9.81 \text{ N/cm}^2 \\ &= 1.962 \text{ N/cm}^2 = 1.962 \times 10^4 \text{ N/m}^2 \end{aligned}$$

$$\therefore \text{Pressure head, } h = \frac{p}{\rho g} = \frac{1.962 \times 10^4}{1000 \times 9.81} = 2 \text{ m}$$

$\therefore$  Free surface of water will be at a height of 2 m from the top of the tank.

$\therefore$  Fig. 3.36 shows the equivalent free surface of water.

### (i) Horizontal Component of Force

$$F_x = \rho g A \bar{h}$$

$$\begin{aligned} \text{where } A &= \text{Area projected on vertical plane} \\ &= 1.5 \times 2.0 = 3.0 \text{ m}^2 \end{aligned}$$

$$\bar{h} = 2 + \frac{1.5}{2} = 2.75$$

$$\therefore F_x = 1000 \times 9.81 \times 3.0 \times 2.75$$

$$= 80932.5 \text{ N. Ans.}$$

### (ii) Vertical Component of Force

$$\begin{aligned} F_y &= \text{Weight of water enclosed or supported} \\ &\quad \text{actually or imaginary by curved surface ABC} \\ &= \text{Weight of water in the portion CODE ABC} \\ &= \text{Weight of water in CODFBC} - \text{Weight of water in AEFB} \end{aligned}$$

But weight of water in CODFBC

$$= \text{Weight of water in } [COB + ODFBO]$$

$$\begin{aligned} &= \rho g \left[ \frac{\pi R^2}{4} + BO \times OD \right] \times 2 = 1000 \times 9.81 \left[ \frac{\pi}{4} \times 1^2 + 1.0 \times 2.5 \right] \times 2 \\ &= 64458.5 \text{ N} \end{aligned}$$

Weight of water in AEFB =  $\rho g [\text{Area of AEFB}] \times 2.0$

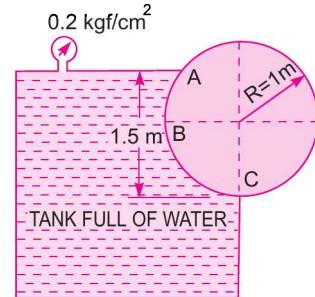


Fig. 3.35

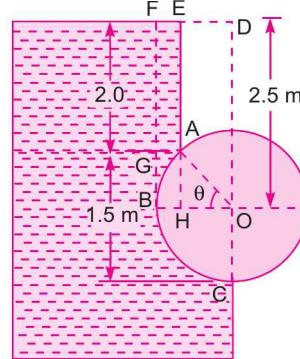


Fig. 3.36

$$= 1000 \times 9.81 [\text{Area of } (AEFG + AGBH - AHB)] \times 2.0$$

$$\text{In } \triangle AHO, \quad \sin \theta = \frac{AH}{AO} = \frac{0.5}{1.0} = 0.5 \quad \therefore \theta = 30^\circ$$

$$BH = BO - HO = 1.0 - AO \cos \theta = 1.0 - 1 \times \cos 30^\circ = 0.134$$

$$\text{Area, } ABH = \text{Area } ABO - \text{Area } AHO$$

$$= \pi R^2 \times \frac{30}{360} - \frac{AH \times HO}{2.0} = \frac{\pi R^2}{12} - \frac{0.5 \times .866}{2} = 0.0453$$

$\therefore$  Weight of water in  $AEFB$

$$= 9810 \times [AE \times AG + AG \times AH - 0.0453] \times 2.0$$

$$= 9810 \times [2.0 \times .134 + .134 \times .5 - .0453] \times 2.0$$

$$= 9810 \times [.268 + .067 - .0453] \times 2.0 = 5684 \text{ N}$$

$$\therefore F_y = 64458.5 - 5684 = 58774.5 \text{ N. Ans.}$$

**Problem 3.29** Find the magnitude and direction of the resultant water pressure acting on a curved face of a dam which is shaped according to the relation  $y = \frac{x^2}{9}$  as shown in Fig. 3.37. The height of the water retained by the dam is 10 m. Consider the width of the dam as unity.

**Solution.** Equation of curve  $AB$  is

$$y = \frac{x^2}{9} \quad \text{or} \quad x^2 = 9y$$

$$\therefore x = \sqrt{9y} = 3\sqrt{y}$$

$$\text{Height of water, } h = 10 \text{ m}$$

$$\text{Width, } b = 1 \text{ m}$$

The horizontal component,  $F_x$  is given by

$$\begin{aligned} F_x &= \text{Pressure due to water on the curved area projected on vertical plane} \\ &= \text{Pressure on area } BC \\ &= \rho g A \bar{h} \end{aligned}$$

$$\text{where } A = BC \times 1 = 10 \times 1 \text{ m}^2, \bar{h} = \frac{1}{2} \times 10 = 5 \text{ m}$$

$$F_x = 1000 \times 9.81 \times 10 \times 5 = 490500 \text{ N}$$

Vertical component,  $F_y$  is given by

$$F_y = \text{Weight of water supported by the curve } AB$$

$$= \text{Weight of water in the portion } ABC$$

$$= \rho g [\text{Area of } ABC] \times \text{Width of dam}$$

$$= \rho g \left[ \int_0^{10} x \times dy \right] \times 1.0 \quad \left\{ \text{Area of strip} = xdy \quad \therefore \text{Area } ABC = \int_0^{10} xdy \right\}$$

$$= 1000 \times 9.81 \times \int_0^{10} 3\sqrt{y} dy \quad \left\{ \because x = 3\sqrt{y} \right\}$$

$$= 29430 \left[ \frac{y^{3/2}}{3/2} \right]_0^{10} = 29430 \times \frac{2}{3} \left[ y^{3/2} \right]_0^{10} = 19620 [10^{3/2}]$$

$$= 19620 \times 31.622 = 620439 \text{ N}$$

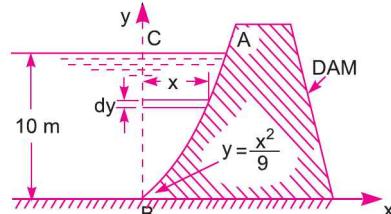


Fig. 3.37

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∴ Resultant water pressure on dam

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(490500)^2 + (620439)^2} \\ = 790907 \text{ N} = 790.907 \text{ kN. Ans.}$$

Direction of the resultant is given by

$$\tan \theta = \frac{F_y}{F_x} = \frac{620439}{490500} = 1.265 \\ \therefore \theta = 51^\circ 40'. \text{ Ans.}$$

**Problem 3.30** A dam has a parabolic shape  $y = y_0 \left( \frac{x}{x_0} \right)^2$  as shown in Fig. 3.38 below having  $x_0 = 6 \text{ m}$  and  $y_0 = 9 \text{ m}$ . The fluid is water with density =  $1000 \text{ kg/m}^3$ . Compute the horizontal, vertical and the resultant thrust exerted by water per metre length of the dam.

**Solution.** Given :

Equation of the curve  $OA$  is

$$y = y_0 \left( \frac{x}{x_0} \right)^2 = 9 \left( \frac{x}{6} \right)^2 = 9 \times \frac{x^2}{36} = \frac{x^2}{4}$$

or

$$x^2 = 4y$$

∴

$$x = \sqrt{4y} = 2y^{1/2}$$

Width of dam,

$$b = 1 \text{ m.}$$

(i) **Horizontal thrust exerted by water**

$F_x$  = Force exerted by water on vertical surface  $OB$ , i.e., the surface obtained by projecting the curved surface on vertical plane

$$= \rho g A \bar{h}$$

$$= 1000 \times 9.81 \times (9 \times 1) \times \frac{9}{2} = 397305 \text{ N. Ans.}$$

(ii) **Vertical thrust exerted by water**

$F_y$  = Weight of water supported by curved surface  $OA$  upto free surface of water

= Weight of water in the portion  $ABO$

=  $\rho g \times \text{Area of } OAB \times \text{Width of dam}$

$$= 1000 \times 9.81 \times \left[ \int_0^9 x \times dy \right] \times 1.0$$

$$= 1000 \times 9.81 \times \left[ \int_0^9 2y^{1/2} \times dy \right] \times 1.0 \quad (\because x = 2y^{1/2})$$

$$= 19620 \times \left[ \frac{y^{3/2}}{(3/2)} \right]_0^9 = 19620 \times \frac{2}{3} [9^{3/2}]$$

$$= 19620 \times \frac{2}{3} \times 27 = 353160 \text{ N. Ans.}$$

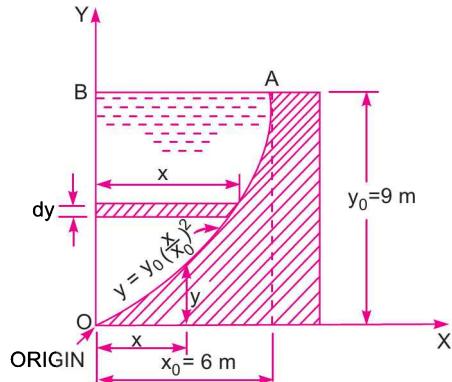


Fig. 3.38

## (iii) Resultant thrust exerted by water

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{397305 + 353160} = 531574 \text{ N. Ans.}$$

Direction of resultant is given by

$$\tan \theta = \frac{F_y}{F_x} = \frac{353160}{397305} = 0.888$$

$$\theta = \tan^{-1} 0.888 = 41.63^\circ. \text{ Ans.}$$

**Problem 3.31** A cylinder 3 m in diameter and 4 m long retains water on one side. The cylinder is supported as shown in Fig. 3.39. Determine the horizontal reaction at A and the vertical reaction at B. The cylinder weighs 196.2 kN. Ignore friction.

**Solution.** Given :

$$\text{Dia. of cylinder} = 3 \text{ m}$$

$$\text{Length of cylinder} = 4 \text{ m}$$

$$\text{Weight of cylinder, } W = 196.2 \text{ kN} = 196200 \text{ N}$$

Horizontal force exerted by water

$$\begin{aligned} F_x &= \text{Force on vertical area } BOC \\ &= \rho g A \bar{h} \end{aligned}$$

$$\text{where } A = BOC \times l = 3 \times 4 = 12 \text{ m}^2, \bar{h} = \frac{1}{2} \times 3 = 1.5 \text{ m}$$

$$F_x = 1000 \times 9.81 \times 12 \times 1.5 = 176580 \text{ N}$$

The vertical force exerted by water

$$\begin{aligned} F_y &= \text{Weight of water enclosed in } BDCOB \\ &= \rho g \times \left( \frac{\pi}{2} R^2 \right) \times l = 1000 \times 9.81 \times \frac{\pi}{2} \times (1.5)^2 \times 4 = 138684 \text{ N} \end{aligned}$$

Force  $F_y$  is acting in the upward direction.

**For the equilibrium of cylinder**

$$\text{Horizontal reaction at } A = F_x = 176580 \text{ N}$$

$$\begin{aligned} \text{Vertical reaction at } B &= \text{Weight of cylinder} - F_y \\ &= 196200 - 138684 = 57516 \text{ N. Ans.} \end{aligned}$$

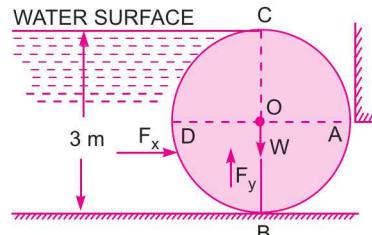


Fig. 3.39

### ► 3.7 TOTAL PRESSURE AND CENTRE OF PRESSURE ON LOCK GATES

Lock gates are the devices used for changing the water level in a canal or a river for navigation. Fig. 3.40 shows plan and elevation of a pair of lock gates. Let AB and BC be the two lock gates. Each gate is supported on two hinges fixed on their top and bottom at the ends A and C. In the closed position, the gates meet at B.

Let  $F$  = Resultant force due to water on the gate AB or BC acting are right angles to the gate

$R$  = Reaction at the lower and upper hinge

$P$  = Reaction at the common contact surface of the two gates and acting perpendicular to the contact surface.

Let the force  $P$  and  $F$  meet at O. Then the reaction  $R$  must pass through O as the gate AB is in the equilibrium under the action of three forces. Let  $\theta$  is the inclination of the lock gate with the normal to the side of the lock.

In  $\angle ABO$ ,  $\angle OAB = \angle ABO = \theta$ .

Resolving all forces along the gate  $AB$  and putting equal to zero, we get

$$R \cos \theta - P \cos \theta = 0 \text{ or } R = P \quad \dots(3.19)$$

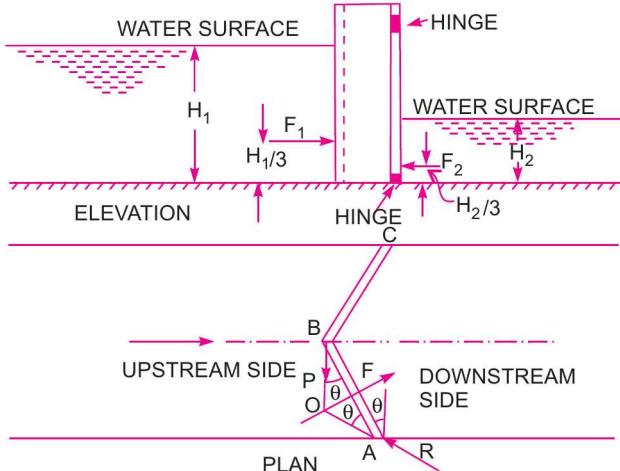


Fig. 3.40

Resolving forces normal to the gate  $AB$

$$R \sin \theta + P \sin \theta - F = 0$$

or

$$F = R \sin \theta + P \sin \theta = 2P \sin \theta \quad \{\because R = P\}$$

$\therefore$

$$P = \frac{F}{2 \sin \theta} \quad \dots(3.20)$$

#### To calculate $P$ and $R$

In equation (3.20),  $P$  can be calculated if  $F$  and  $\theta$  are known. The value of  $\theta$  is calculated from the angle between the lock gates. The angle between the two lock gate is equal to  $180^\circ - 2\theta$ . Hence  $\theta$  can be calculated. The value of  $F$  is calculated as :

Let

$$H_1 = \text{Height of water on the upstream side}$$

$$H_2 = \text{Height of water on the downstream side}$$

$$F_1 = \text{Water pressure on the gate on upstream side}$$

$$F_2 = \text{Water pressure on the gate on downstream side of the gate}$$

$$l = \text{Width of gate}$$

Now

$$F_1 = \rho g A_1 \bar{h}_1$$

$$= \rho g \times H_1 \times l \times \frac{H_1}{2}$$

$$= \rho g l \frac{H_1^2}{2} \quad \left[ \because A_1 = H_1 \times l, \bar{h}_1 = \frac{H_1}{2} \right]$$

Similarly,

$$F_2 = \rho g A_2 \bar{h}_2 = \rho g \times (H_2 \times l) \times \frac{H_2}{2} = \frac{\rho g l H_2^2}{2}$$

$$\therefore \text{Resultant force } F = F_1 - F_2 = \frac{\rho g l H_1^2}{2} - \frac{\rho g l H_2^2}{2}$$

Substituting the value of  $\theta$  and  $F$  in equation (3.20), the value of  $P$  and  $R$  can be calculated.

#### Reactions at the top and bottom hinges

Let

$$R_t = \text{Reaction of the top hinge}$$

$R_b$  = Reaction of the bottom hinge

Then

$$R = R_t + R_b$$

The resultant water pressure  $F$  acts normal to the gate. Half of the value of  $F$  is resisted by the hinges of one lock gates and other half will be resisted by the hinges of other lock gate. Also  $F_1$  acts at a distance of  $\frac{H_1}{3}$  from bottom while  $F_2$  acts at a distance of  $\frac{H_2}{3}$  from bottom.

Taking moments about the lower hinge

$$R_t \times \sin \theta \times H = \frac{F_1}{2} \times \frac{H_1}{3} - \frac{F_2}{2} \times \frac{H_2}{3} \quad \dots(i)$$

where  $H$  = Distance between two hinges

Resolving forces horizontally

$$R_t \sin \theta + R_b \sin \theta = \frac{F_1}{2} - \frac{F_2}{2} \quad \dots(ii)$$

From equations (i) and (ii), we can find  $R_t$  and  $R_b$ .

**Problem 3.32** Each gate of a lock is 6 m high and is supported by two hinges placed on the top and bottom of the gate. When the gates are closed, they make an angle of  $120^\circ$ . The width of lock is 5 m. If the water levels are 4 m and 2 m on the upstream and downstream sides respectively, determine the magnitude of the forces on the hinges due to water pressure.

**Solution.** Given :

$$\text{Height of lock} = 6 \text{ m}$$

$$\text{Width of lock} = 5 \text{ m}$$

$$\text{Width of each lock gate} = AB$$

$$\text{or } l = \frac{AD}{\cos 30^\circ} = \frac{2.5}{\cos 30^\circ} = 2.887 \text{ m}$$

$$\text{Angle between gates} = 120^\circ$$

$$\therefore \theta = \frac{180^\circ - 120^\circ}{2} = \frac{60^\circ}{2} = 30^\circ$$

$$\text{Height of water on upstream side}$$

$$H_1 = 4 \text{ m}$$

$$\text{and } H_2 = 2 \text{ m}$$

$$\therefore \text{Total water pressure on upstream side}$$

$$F_1 = \rho g A_1 \bar{h}_1, \text{ where } A_1 = H_1 \times l = 4.0 \times 2.887 \text{ m}^2$$

$$= 1000 \times 9.81 \times 4 \times 2.887 \times 2.0$$

$$\left\{ \bar{h}_1 = \frac{H_1}{2} = \frac{4}{2} = 2.0 \text{ m} \right\}$$

$$= 226571 \text{ N}$$

Force  $F_1$  will be acting at a distance of  $\frac{H_1}{3} = \frac{4}{3} = 1.33 \text{ m}$  from bottom.

Similarly, total water pressure on the downstream side

$$F_2 = \rho g A_2 \bar{h}_2, \text{ where } A_2 = H_2 \times l = 2 \times 2.887 \text{ m}^2$$

$$= 1000 \times 9.81 \times 2 \times 2.887 \times 1.0$$

$$\bar{h}_2 = \frac{H_2}{2} = \frac{2}{2} = 1.0 \text{ m}$$

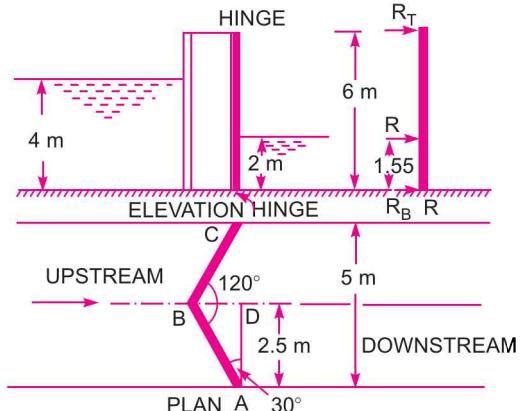


Fig. 3.41

$$= 56643 \text{ N}$$

$F_2$  will act at a distance of  $\frac{H_2}{3} = \frac{2}{3} = 0.67 \text{ m}$  from bottom,

Resultant water pressure on each gate

$$F = F_1 - F_2 = 226571 - 56643 = 169928 \text{ N.}$$

Let  $x$  is height of  $F$  from the bottom, then taking moments of  $F_1$ ,  $F_2$  and  $F$  about the bottom, we have

$$F \times x = F_1 \times 1.33 - F_2 \times 0.67$$

or

$$169928 \times x = 226571 \times 1.33 - 56643 \times 0.67$$

$$\therefore x = \frac{226571 \times 1.33 - 56643 \times 0.67}{169928} = \frac{301339 - 37950}{169928} = 1.55 \text{ m}$$

From equation (3.20),  $P = \frac{F}{2 \sin \theta} = \frac{169928}{2 \sin 30} = 169928 \text{ N.}$

From equation (3.19),  $R = P = 169928 \text{ N.}$

If  $R_T$  and  $R_B$  are the reactions at the top and bottom hinges, then  $R_T + R_B = R = 169928 \text{ N.}$

Taking movements of hinge reactions  $R_T$ ,  $R_B$  and  $R$  about the bottom hinges, we have

$$R_T \times 6.0 + R_B \times 0 = R \times 1.55$$

$$\therefore R_T = \frac{169928 \times 1.55}{6.0} = 43898 \text{ N}$$

$$\therefore R_B = R - R_T = 169928 - 43898 = \mathbf{126030 \text{ N. Ans.}}$$

**Problem 3.33** The end gates ABC of a lock are 9 m high and when closed include an angle of  $120^\circ$ . The width of the lock is 10 m. Each gate is supported by two hinges located at 1 m and 6 m above the bottom of the lock. The depths of water on the two sides are 8 m and 4 m respectively. Find:

- (i) Resultant water force on each gate,
- (ii) Reaction between the gates AB and BC, and
- (iii) Force on each hinge, considering the reaction of the gate acting in the same horizontal plane as resultant water pressure.

**Solution.** Given :

Height of gate  $= 9 \text{ m}$

Inclination of gate  $= 120^\circ$

$$\therefore \theta = \frac{180^\circ - 120^\circ}{2} = 30^\circ$$

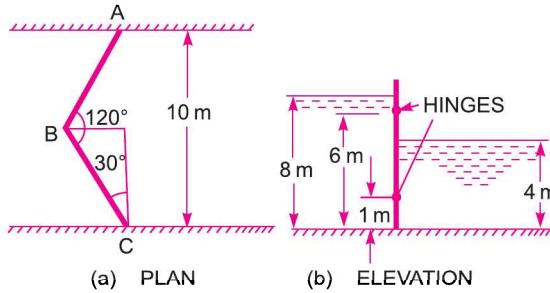


Fig. 3.42

$$\text{Width of lock} = 10 \text{ m}$$

$$\therefore \text{Width of each lock} = \frac{5}{\cos 30^\circ} \text{ or } l = 5.773 \text{ m}$$

$$\text{Depth of water on upstream side, } H_1 = 8 \text{ m}$$

$$\text{Depth of water on downstream side, } H_2 = 4 \text{ m}$$

**(i) Water pressure on upstream side**

$$F_1 = \rho g A_1 \bar{h}_1$$

$$\text{where } A_1 = l \times H_1 = 5.773 \times 8 = 46.184 \text{ m}, \bar{h}_1 = \frac{H_1}{2} = \frac{8}{2} = 4.0 \text{ m}$$

$$F_1 = 1000 \times 9.81 \times 46.184 \times 4.0 = 1812260 \text{ N} = 1812.26 \text{ kN}$$

Water pressure on downstream side,

$$F_2 = \rho g A_2 \bar{h}_2$$

$$\text{where } A_2 = l \times H_2 = 5.773 \times 4 = 23.092 \text{ m}, \bar{h}_2 = \frac{H_2}{2} = \frac{4}{2} = 2.0$$

$$F_2 = 1000 \times 9.81 \times 23.092 \times 2.0 = 453065 \text{ N} = 453.065 \text{ kN}$$

$\therefore$  Resultant water pressure

$$= F_1 - F_2 = 1812.26 - 453.065 = 1359.195 \text{ kN}$$

**(ii) Reaction between the gates AB and BC.** The reaction ( $P$ ) between the gates  $AB$  and  $BC$  is given by equation (3.20) as

$$F = \frac{F}{2 \sin \theta} = \frac{1359.195}{2 \times \sin 30^\circ} = 1359.195 \text{ kN. Ans.}$$

**(iii) Force on each hinge.** If  $R_T$  and  $R_B$  are the reactions at the top and bottom hinges then

$$R_T + R_B = R$$

But from equation (3.19),  $R = P = 1359.195$

$$\therefore R_T + R_B = 1359.195$$

The force  $F_1$  is acting at  $\frac{H_1}{3} = \frac{8}{3} = 2.67$  m from bottom and  $F_2$  at  $\frac{H_2}{3} = \frac{4}{3} = 1.33$  m from bottom.

The resultant force  $F$  will act at a distance  $x$  from bottom is given by

$$F \times x = F_1 \times 2.67 - F_2 \times 1.33$$

$$\begin{aligned} \text{or } x &= \frac{F_1 \times 2.67 - F_2 \times 1.33}{F} = \frac{1812.26 \times 2.67 - 453.065 \times 1.33}{1359.195} \\ &= \frac{4838.734 - 602.576}{1359.195} = 3.116 = 3.11 \text{ m} \end{aligned}$$

Hence  $R$  is also acting at a distance 3.11 m from bottom.

Taking moments of  $R_T$  and  $R$  about the bottom hinge

$$R_T \times [6.0 - 1.0] = R \times (x - 1.0)$$

$$\therefore R_T = \frac{R \times (x - 1.0)}{5.0} = \frac{1359.195 \times 2.11}{5.0} = 573.58 \text{ N}$$

$$\begin{aligned} \therefore R_B &= R - R_T = 1359.195 - 573.58 \\ &= 785.615 \text{ kN. Ans.} \end{aligned}$$

### ► 3.8 PRESSURE DISTRIBUTION IN A LIQUID SUBJECTED TO CONSTANT HORIZONTAL/VERTICAL ACCELERATION

In chapters 2 and 3, the containers which contains liquids, are assumed to be at rest. Hence the liquids are also at rest. They are in static equilibrium with respect to containers. But if the container containing a liquid is made to move with a constant acceleration, the liquid particles initially will move relative to each other and after some time, there will not be any relative motion between the liquid particles and boundaries of the container. The liquid will take up a new position under the effect of acceleration imparted to its container. The liquid will come to rest in this new position relative to the container. The entire fluid mass moves as a single unit. Since the liquid after attaining a new position is in static condition relative to the container, the laws of hydrostatic can be applied to determine the liquid pressure. As there is no relative motion between the liquid particles, hence the shear stresses and shear forces between liquid particles will be zero. The pressure will be normal to the surface in contact with the liquid.

The following are the important cases under consideration :

- (i) Liquid containers subject to constant horizontal acceleration.
- (ii) Liquid containers subject to constant vertical acceleration.

**3.8.1 Liquid Containers Subject to Constant Horizontal Acceleration.** Fig. 3.43 (a) shows a tank containing a liquid upto a certain depth. The tank is stationary and free surface of liquid is horizontal. Let this tank is moving with a constant acceleration ' $a$ ' in the horizontal direction towards right as shown in Fig. 3.43 (b). The initial free surface of liquid which was horizontal, now takes the shape as shown in Fig. 3.43 (b). Now  $AB$  represents the new free surface of the liquid. Thus the free surface of liquid due to horizontal acceleration will become a downward sloping inclined plane, with the liquid rising at the back end, the liquid falling at the front end. The equation for the free liquid surface can be derived by considering the equilibrium of a fluid element  $C$  lying on the free surface. The forces acting on the element  $C$  are :

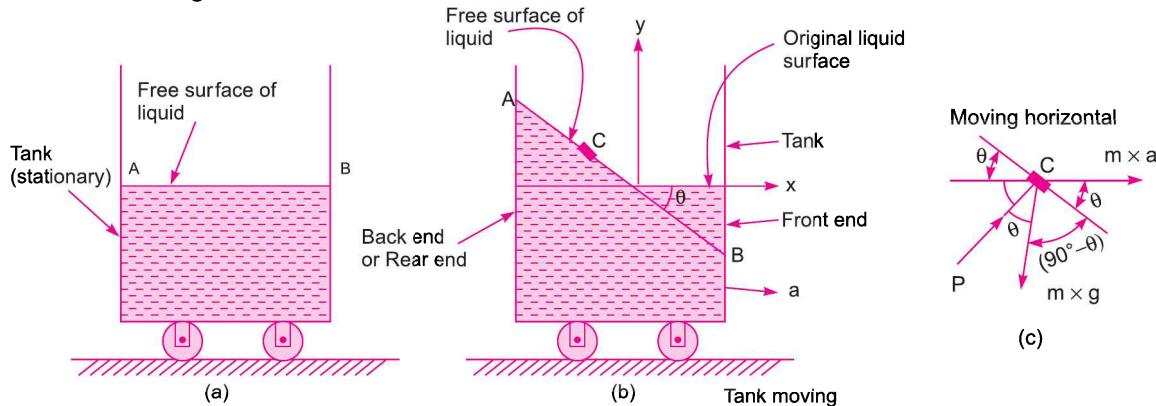


Fig. 3.43

- (i) the pressure force  $P$  exerted by the surrounding fluid on the element  $C$ . This force is normal to the free surface.
- (ii) the weight of the fluid element i.e.,  $m \times g$  acting vertically downward.
- (iii) accelerating force i.e.,  $m \times a$  acting in horizontal direction.

Resolving the forces horizontally, we get

$$P \sin \theta + m \times a = 0$$

or  $P \sin \theta = -ma$  ... (i)

Resolving the forces vertically, we get

$$P \cos \theta - mg = 0$$

or  $P \cos \theta = m \times g$  ... (ii)

Dividing (i) by (ii), we get

$$\tan \theta = -\frac{a}{g} \quad \left( \text{or } \frac{a}{g} \text{ Numerically} \right) \quad \dots (3.20A)$$

The above equation, gives the slope of the free surface of the liquid which is contained in a tank which is subjected to horizontal constant acceleration. The term  $(a/g)$  is a constant and hence  $\tan \theta$  will be constant. The -ve sign shows that the free surface of liquid is sloping downwards. Hence the free surface is a straight plane inclined down at an angle  $\theta$  along the direction of acceleration.

Now let us find the expression for the pressure at any point  $D$  in the liquid mass subjected to horizontal acceleration. Let the point  $D$  is at a depth of ' $h$ ' from the free surface. Consider an elementary prism  $DE$  of height ' $h$ ' and cross-sectional area  $dA$  as shown in Fig. 3.44.

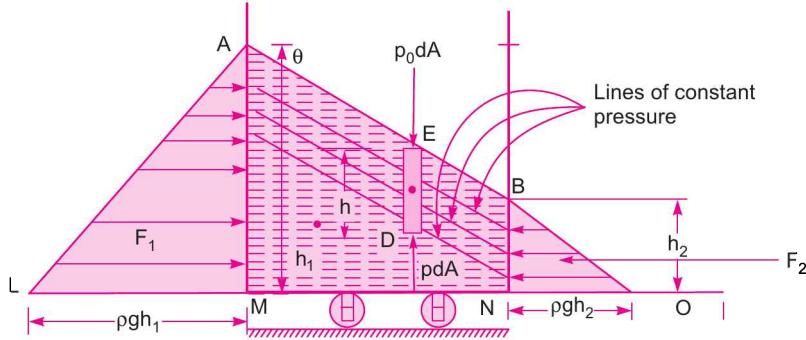


Fig. 3.44

Consider the equilibrium of the elementary prism  $DE$ .

The forces acting on this prism  $DE$  in the vertical direction are :

- (i) the atmospheric pressure force ( $p_0 \times dA$ ) at the top end of the prism acting downwards,
- (ii) the weight of the element ( $\rho \times g \times h \times dA$ ) at the C.G. of the element acting in the downward direction, and
- (iii) the pressure force ( $p \times dA$ ) at the bottom end of the prism acting upwards.

Since there is no vertical acceleration given to the tank, hence net force acting vertically should be zero.

$$\therefore p \times dA - p_0 \times dA - \rho gh dA = 0$$

or  $p - p_0 - \rho gh = 0 \quad \text{or} \quad p = p_0 + \rho gh$

or  $p - p_0 = \rho gh$

or gauge pressure at point  $D$  is given by

$$p = \rho gh$$

or pressure head at point  $D$ ,  $\frac{p}{\rho g} = h$ .

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From the above equation, it is clear that pressure head at any point in a liquid subjected to a constant horizontal acceleration is equal to the height of the liquid column above that point. Therefore the pressure distribution in a liquid subjected to a constant horizontal acceleration is same as hydrostatic pressure distribution. The planes of constant pressure are therefore, parallel to the inclined surface as shown in Fig. 3.44. This figure also shows the variation of pressure on the rear and front end of the tank.

If  $h_1$  = Depth of liquid at the rear end of the tank

$h_2$  = Depth of liquid at the front end of the tank

$F_1$  = Total pressure force exerted by liquid on the rear side of the tank

$F_2$  = Total pressure force exerted by liquid on the front side of the tank,

then  $F_1 = (\text{Area of triangle } AML) \times \text{Width}$

$$= \left( \frac{1}{2} \times LM \times AM \times b \right) = \frac{1}{2} \times \rho g h_1 \times h_1 \times b = \frac{\rho g \cdot b \cdot h_1^2}{2}$$

and  $F_2 = (\text{Area of triangle } BNO) \times \text{Width}$

$$= \left( \frac{1}{2} \times BN \times NO \right) = \frac{1}{2} \times h_2 \times \rho g h_2 \times b = \frac{\rho g \cdot b \cdot h_2^2}{2}$$

where  $b$  = Width of tank perpendicular to the plane of the paper.

The values of  $F_1$  and  $F_2$  can also be obtained as

[Refer to Fig. 3.44 (a)]

$$F_1 = \rho \times g \times A_1 \times \bar{h}_1, \text{ where } A_1 = h_1 \times b \text{ and } \bar{h}_1 = \frac{h_1}{2}$$

$$= \rho \times g \times (h_1 \times b) \times \frac{h_1}{2} = \frac{1}{2} \rho g \cdot b \cdot h_1^2$$

and  $F_2 = \rho \times g \times A_2 \times \bar{h}_2, \text{ where } A_2 = h_2 \times b \text{ and } \bar{h}_2 = \frac{h_2}{2}$

$$= \rho \times g \times (h_2 \times b) \times \frac{h_2}{2}$$

$$= \frac{1}{2} \rho g \cdot b \cdot h_2^2.$$

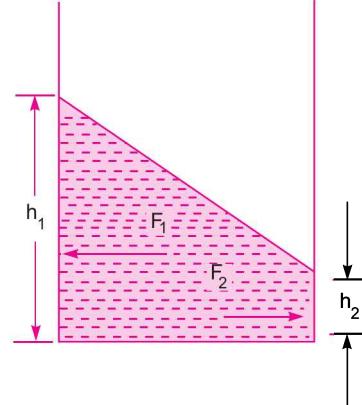


Fig. 3.44(a)

It can also be proved that the difference of these two forces (*i.e.*,  $F_1 - F_2$ ) is equal to the force required to accelerate the mass of the liquid contained in the tank *i.e.*,

$$F_1 - F_2 = M \times a$$

where  $M$  = Total mass of the liquid contained in the tank

$a$  = Horizontal constant acceleration.

Note : (i) If a tank completely filled with liquid and open at the top is subjected to a constant horizontal acceleration, then some of the liquid will spill out from the tank and new free surface with its slope given by equation  $\tan \theta = -\frac{a}{g}$  will be developed.

(ii) If a tank partly filled with liquid and open at the top is subjected to a constant horizontal acceleration, spilling of the liquid may take place depending upon the magnitude of the acceleration.

(iii) If a tank completely filled with liquid and closed at the top is subjected to a constant horizontal acceleration, then the liquid would not spill out from the tank and also there will be no adjustment in the surface elevation of the liquid. But the equation  $\tan \theta = -\frac{a}{g}$  is applicable for this case also.

(iv) The example for a tank with liquid subjected to a constant horizontal acceleration, is a fuel tank on an airplane during take off.

**Problem 3.34** A rectangular tank is moving horizontally in the direction of its length with a constant acceleration of  $2.4 \text{ m/s}^2$ . The length, width and depth of the tank are 6 m, 2.5 m and 2 m respectively. If the depth of water in the tank is 1 m and tank is open at the top then calculate :

- the angle of the water surface to the horizontal,
- the maximum and minimum pressure intensities at the bottom,
- the total force due to water acting on each end of the tank.

**Solution.** Given :

Constant acceleration,  $a = 2.4 \text{ m/s}^2$ .

Length = 6 m ; Width = 2.5 m and depth = 2 m.  $2 \text{ m}$

Depth of water in tank,  $h = 1 \text{ m}$

**(i) The angle of the water surface to the horizontal**

Let  $\theta$  = the angle of water surface to the horizontal

Using equation (3.20), we get

$$\tan \theta = -\frac{a}{g} = -\frac{2.4}{9.81} = -0.2446$$

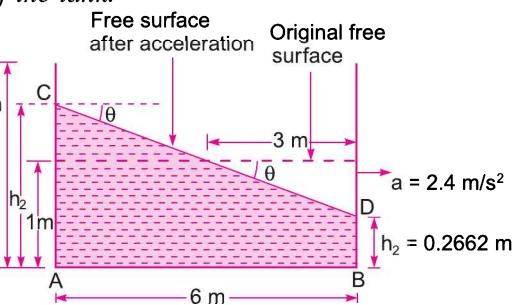


Fig. 3.45

(the -ve sign shows that the free surface of water is sloping downward as shown in Fig. 3.45)

$$\therefore \tan \theta = 0.2446 \text{ (slope downward)}$$

$$\therefore \theta = \tan^{-1} 0.2446 = 13.7446^\circ \text{ or } 13^\circ 44.6'. \text{ Ans.}$$

**(ii) The maximum and minimum pressure intensities at the bottom of the tank**

From the Fig. 3.45,

Depth of water at the front end,

$$h_1 = 1 - 3 \tan \theta = 1 - 3 \times 0.2446 = 0.2662 \text{ m}$$

Depth of water at the rear end,

$$h_2 = 1 + 3 \tan \theta = 1 + 3 \times 0.2446 = 1.7338 \text{ m}$$

The pressure intensity will be maximum at the bottom, where depth of water is maximum.

Now the maximum pressure intensity at the bottom will be at point A and it is given by,

$$\begin{aligned} p_{\max} &= \rho \times g \times h_2 \\ &= 1000 \times 9.81 \times 1.7338 \text{ N/m}^2 = 17008.5 \text{ N/m}^2. \text{ Ans.} \end{aligned}$$

The minimum pressure intensity at the bottom will be at point B and it is given by

$$\begin{aligned} p_{\min} &= \rho \times g \times h_1 \\ &= 1000 \times 9.81 \times 0.2662 = 2611.4 \text{ N/m}^2. \text{ Ans.} \end{aligned}$$

**(iii) The total force due to water acting on each end of the tank**

Let

$F_1$  = total force acting on the front side (i.e., on face BD)

$F_2$  = total force acting on the rear side (i.e., on face AC)

Then

$F_1 = \rho g A_1 \bar{h}_1$ , where  $A_1 = BD \times \text{width of tank} = h_1 \times 2.5 = 0.2662 \times 2.5$

$$\begin{aligned} \text{and } \bar{h}_1 &= \frac{BD}{2} = \frac{h_1}{2} = \frac{0.2662}{2} = 0.1331 \text{ m} \\ &= 1000 \times 9.81 \times (0.2662 \times 2.5) \times 0.1331 \\ &= 868.95 \text{ N. Ans.} \end{aligned}$$

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and

$$F_2 = \rho.g.A_2.\bar{h}_2, \text{ where } A_2 = AB \times \text{width of tank} = h_2 \times 2.5 = 1.7338 \times 2.5$$

$$\bar{h}_2 = \frac{AB}{2} = \frac{h_2}{2} = \frac{1.7338}{2} = 0.8669 \text{ m}$$

$$= 1000 \times 9.81 \times (1.7338 \times 2.5) \times 0.8669$$

$$= 36861.8 \text{ N. Ans.}$$

$$\therefore \text{Resultant force} = F_1 - F_2$$

$$= 36861.8 \text{ N} - 868.95$$

$$= 35992.88 \text{ N}$$

**Note.** The difference of the forces acting on the two ends of the tank is equal to the force necessary to accelerate the liquid mass. This can be proved as shown below :

Consider the control volume of the liquid *i.e.*, control volume is  $ACDBA$  as shown in Fig. 3.46. The net force acting on the control volume in the horizontal direction must be equal to the product of mass of the liquid in control volume and acceleration of the liquid.

$\therefore$

$$(F_1 - F_2) = M \times a$$

$$= (\rho \times \text{volume of control volume}) \times a$$

$$= (1000 \times \text{Area of } ABDCE \times \text{width}) \times 2.4$$

$$= \left[ 1000 \times \left( \frac{AC + BD}{2} \right) \times AB \times \text{width} \right] \times 2.4$$

$$\left[ \because \text{Area of trapezium} = \left( \frac{AC + BD}{2} \right) \times AB \right]$$

$$= 1000 \times \left( \frac{1.7338 + 0.2662}{2} \right) \times 6 \times 2.5 \times 2.4$$

$$= 36000 \text{ N}$$

$$(\because AC = h_2 = 1.7338 \text{ m}, BD = h_1 = 0.2662 \text{ m}, \text{ and } AB = 6 \text{ m, width} = 2.5 \text{ m})$$

The above force is nearly the same as the difference of the forces acting on the two ends of the tank. (*i.e.*,  $35992.88 \approx 36000$ ).

**Problem 3.35** The rectangular tank of the above problem contains water to a depth of 1.5 m. Find the horizontal acceleration which may be imparted to the tank in the direction of its length so that

- (i) the spilling of water from the tank is just on the verge of taking place,
- (ii) the front bottom corner of the tank is just exposed,
- (iii) the bottom of the tank is exposed upto its mid-point.

Also calculate the total forces exerted by the water on each end of the tank in each case. Also prove that the difference between these forces is equal to the force necessary to accelerate the mass of water tank.

**Solution.** Given :

Dimensions of the tank from previous problem,

$$L = 6 \text{ m, width } (b) = 2.5 \text{ m and depth} = 2 \text{ m}$$

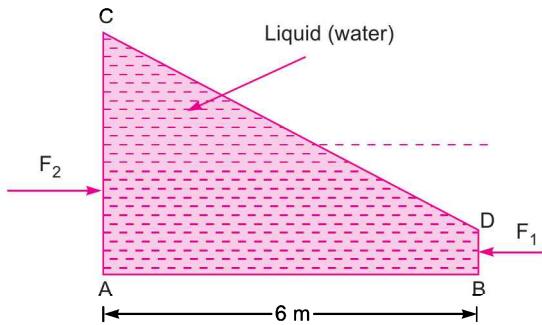


Fig. 3.46

Depth of water in tank,  $h = 1.5 \text{ m}$

### Horizontal acceleration imparted to the tank

(i) (a) When the spilling of water from the tank is just on the verge of taking place

Let  $a$  = required horizontal acceleration

When the spilling of water from the tank is just on the verge of taking place, the water would rise upto the rear top corner of the tank as shown in Fig. 3.47 (a)

$$\therefore \tan \theta = \frac{AC}{AO} = \frac{(2 - 1.5)}{3} = \frac{0.5}{3} = 0.1667$$

But from equation (3.20)  $\tan \theta = \frac{a}{g}$  (Numerically)

$$\therefore a = g \times \tan \theta = 9.81 \times 0.1667 = 1.635 \text{ m/s}^2. \text{ Ans.}$$

(b) Total forces exerted by water on each end of the tank

The force exerted by water on the end  $CE$  of the tank is

$$F_1 = \rho g A_1 \bar{h}_1, \text{ where } A_1 = CE \times \text{width of the tank} = 2 \times 2.5$$

$$\bar{h}_1 = \frac{CE}{2} = \frac{2}{2} = 1 \text{ m}$$

$$= 1000 \times 9.81 \times (2 \times 2.5) \times 1$$

$$= 49050 \text{ N. Ans.}$$

The force exerted by water on the end  $FD$  of the tank is

$$F_2 = \rho g A_2 \times \bar{h}_2, \text{ where } A_2 = FD \times \text{width} = 1 \times 2.5$$

$$(\because AC = BD = 0.5 \text{ m}, \therefore FD = BF - BD = 1.5 - 0.5 = 1)$$

$$= 1000 \times 9.81 \times (1 \times 2.5) \times 0.5 \quad \bar{h}_2 = \frac{FD}{2} = \frac{1}{2} = 0.5 \text{ m}$$

$$= 12262.5 \text{ N. Ans.}$$

(c) Difference of the forces is equal to the force necessary to accelerate the mass of water in the tank

Difference of the forces  $= F_1 - F_2$

$$= 49050 - 12262.5 = 36787.5 \text{ N}$$

Volume of water in the tank before acceleration is imparted to it  $= L \times b \times \text{depth of water}$

$$= 6 \times 2.5 \times 1.5 = 22.5 \text{ m}^3.$$

The force necessary to accelerate the mass of water in the tank

$$= \text{Mass of water in tank} \times \text{Acceleration}$$

$$= (\rho \times \text{volume of water}) \times 1.635 \quad (\because a = 1.635 \text{ m/s}^2)$$

$$= 1000 \times 22.5 \times 1.635 \quad [\text{There is no spilling of water and volume of water} = 22.5 \text{ m}^3]$$

$$= 36787.5 \text{ N}$$

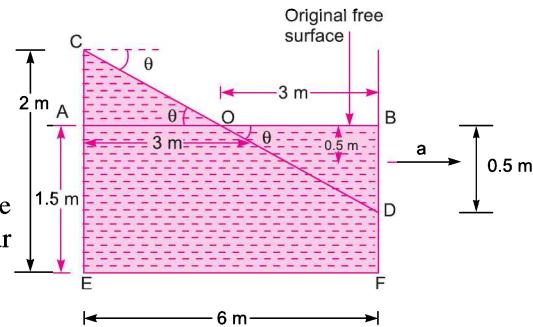


Fig. 3.47 (a) Spilling of water is just on the verge of taking place.

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Hence the difference between the forces on the two ends of the tank is equal to the force necessary to accelerate the mass of water in the tank.

Volume of water in the tank can also be calculated as volume =  $\left(\frac{CE + FD}{2}\right) \times EF \times \text{Width}$  [Refer to Fig. 3.47 (a)]

$$= \left(\frac{2+1}{2}\right) \times 6 \times 2.5 = 22.5 \text{ m}^3.$$

(ii) (a) Horizontal acceleration when the front bottom corner of the tank is just exposed

Refer to Fig. 3.47 (b). In this case the free surface of water in the tank will be along  $CD$ .

Let  $a$  = required horizontal acceleration.

$$\text{In this case, } \tan \theta = \frac{CE}{ED} = \frac{2}{6} = \frac{1}{3}$$

But from equation (3.17),

$$\tan \theta = \frac{a}{g} \text{ (Numerically)}$$

$$\therefore a = g \times \tan \theta = 9.81 \times \frac{1}{3} = 3.27 \text{ m/s}^2. \text{ Ans.}$$

(b) Total forces exerted by water on each end of the tank

The force exerted by water on the end  $CE$  of the tank is

$$F_1 = \rho g \times A_1 \times \bar{h}_1$$

$$\text{where } A_1 = CE \times \text{width} = 2 \times 2.5 = 5 \text{ m}^2$$

$$\begin{aligned} \bar{h}_1 &= \frac{CE}{2} = \frac{2}{2} = 1 \text{ m} \\ &= 1000 \times 9.81 \times 5 \times 1 \\ &= 49050 \text{ N. Ans.} \end{aligned}$$

The force exerted by water on the end  $BD$  of the tank is zero as there is no water against the face  $BD$

$$\therefore F_2 = 0$$

$$\therefore \text{Difference of forces} = 49050 - 0 = 49050 \text{ N}$$

(c) Difference of forces is equal to the force necessary to accelerate the mass of water in the tank.

Volume of water in the tank = Area of  $CED \times$  Width of tank

$$\begin{aligned} &= \left(\frac{CE \times ED}{2}\right) \times 2.5 \quad (\because \text{Width of tank} = 2.5 \text{ m}) \\ &= \frac{2 \times 6}{2} \times 2.5 = 15 \text{ m}^3 \end{aligned}$$

$\therefore$  Force necessary to accelerate the mass of water in the tank

$$\begin{aligned} &= \text{Mass of water in tank} \times \text{Acceleration} \\ &= (1000 \times \text{Volume of water}) \times 3.27 \\ &= 1000 \times 15 \times 3.27 = 49050 \text{ N} \end{aligned}$$

Difference of two forces is also = 49050 N

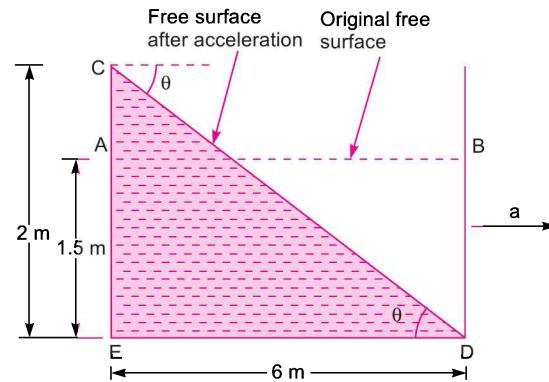


Fig. 3.47 (b)

Hence difference between the forces on the two ends of the tank is equal to the force necessary to accelerate the mass of water in the tank.

(iii) (a) *Horizontal acceleration when the bottom of the tank is exposed upto its mid-point*

Refer to Fig. 3.47 (c). In this case the free surface of water in the tank will be along  $CD^*$ , where  $D^*$  is the mid-point of  $ED$ .

Let  $a$  = required horizontal acceleration from

Fig. 3.47 (c), it is clear that

$$\tan \theta = \frac{CE}{ED^*} = \frac{2}{3}$$

But from equation (3.20) numerically

$$\tan \theta = \frac{a}{g}$$

$$\therefore a = g \times \tan \theta = 9.81 \times \frac{2}{3} = 6.54 \text{ m/s}^2. \text{ Ans.}$$

(b) *Total forces exerted by water on each end of the tank*

The force exerted by water on the end  $CE$  of the tank is

$$F_1 = \rho \times g \times A_1 \times \bar{h}_1$$

$$\text{where } A_1 = CE \times \text{Width} = 2 \times 2.5 = 5 \text{ m}^2$$

$$\begin{aligned} \bar{h}_1 &= \frac{CE}{2} = \frac{2}{2} = 1 \text{ m} \\ &= 1000 \times 9.81 \times 5 \times 1 \\ &= 49050 \text{ N. Ans.} \end{aligned}$$

The force exerted by water on the end  $BD$  is zero as there is no water against the face  $BD$ .

$$\therefore F_2 = 0$$

$$\therefore \text{Difference of the forces} = F_1 - F_2 = 49050 - 0 = 49050 \text{ N}$$

(c) *Difference of the two forces is equal to the force necessary to accelerate the mass of water remaining in the tank*

Volume of water in the tank = Area  $CED^*$  × Width of tank

$$= \frac{CE \times ED^*}{2} \times 2.5 = \frac{2 \times 3}{2} \times 2.5 = 7.5 \text{ m}^3$$

Force necessary to accelerate the mass of water in the tank

$$\begin{aligned} &= \text{Mass of water} \times \text{Acceleration} \\ &= \rho \times \text{Volume of water} \times 6.54 \quad (\because a = 6.54 \text{ m/s}^2) \\ &= 1000 \times 7.5 \times 6.54 \\ &= 49050 \text{ N} \end{aligned}$$

This is the same force as the difference of the two forces on the two ends of the tank.

**Problem 3.36** A rectangular tank of length 6 m, width 2.5 m and height 2 m is completely filled with water when at rest. The tank is open at the top. The tank is subjected to a horizontal constant linear acceleration of  $2.4 \text{ m/s}^2$  in the direction of its length. Find the volume of water spilled from the tank.

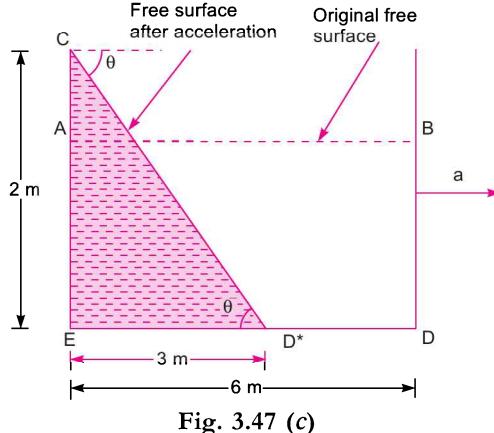


Fig. 3.47 (c)

**Solution.** Given :

$$L = 6 \text{ m}, b = 2.5 \text{ m} \text{ and height, } H = 2 \text{ m}$$

$$\text{Horizontal acceleration, } a = 2.4 \text{ m/s}^2.$$

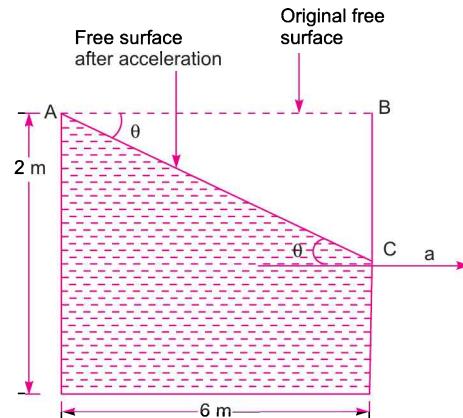
The slope of the free surface of water after the tank is subjected to linear constant acceleration is given by equation (3.20) as

$$\begin{aligned}\tan \theta &= \frac{a}{g} \text{ (Numerically)} \\ &= \frac{2.4}{9.81} = 0.2446\end{aligned}$$

From Fig. 3.48,

$$\tan \theta = \frac{BC}{AB}$$

$$\begin{aligned}\therefore BC &= AB \times \tan \theta \\ &= 6 \times 0.2446 \\ &= 1.4676 \text{ m}\end{aligned}$$



( $\because AB = \text{Length} = 6 \text{ m}; \tan \theta = 0.2446$ ) Fig. 3.48

$\therefore$  Volume of water spilled = Area of  $ABC \times$  Width of tank

$$\begin{aligned}&= \left( \frac{1}{2} \times AB \times BC \right) \times 2.5 \quad (\because \text{Width} = 2.5 \text{ m}) \\ &= \frac{1}{2} \times 6 \times 1.4676 \times 2.5 \quad (\because BC = 1.4676 \text{ m}) \\ &= 11.007 \text{ m}^3. \text{ Ans.}\end{aligned}$$

**3.8.2 Liquid Container Subjected to Constant Vertical Acceleration.** Fig. 3.49 shows a tank containing a liquid and the tank is moving vertically upward with a constant acceleration. The liquid in the tank will be subjected to the same vertical acceleration. To obtain the expression for the pressure at any point in the liquid mass subjected to vertical upward acceleration, consider a vertical elementary prism of liquid  $CDFE$ .

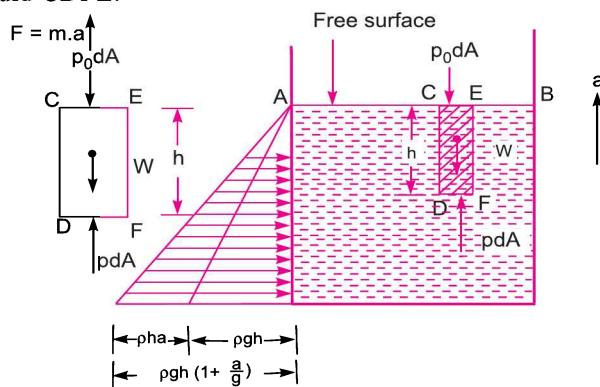


Fig. 3.49

Let  $dA$  = Cross-sectional area of prism

$h$  = Height of prism

$p_0$  = Atmospheric pressure acting on the face  $CE$

$p$  = Pressure at a depth  $h$  acting on the face  $DF$

The forces acting on the elementary prism are :

- Pressure force equal to  $p_0 \times dA$  acting on the face  $CE$  vertically downward
- Pressure force equal to  $p \times dA$  acting on the face  $DF$  vertically upward
- Weight of the prism equal to  $\rho \times g \times dA \times h$  acting through C.G. of the element vertically downward.

According to Newton's second law of motion, the net force acting on the element must be equal to mass multiplied by acceleration in the same direction.

$\therefore$  Net force in vertically upward direction = Mass  $\times$  acceleration

$$p \times dA - p_0 \times dA - \rho g dA \cdot h = (\rho \times dA \times h) \times a \quad (\because \text{Mass} = \rho \times dA \times h)$$

or 
$$p - p_0 - \rho gh = \rho h \times a \quad (\text{Cancelling } dA \text{ from both sides})$$

$$\begin{aligned} \text{or} \quad p - p_0 &= \rho gh + \rho ha \\ &= \rho gh \left[ 1 + \frac{a}{g} \right] \end{aligned} \quad \dots(3.21)$$

But  $(p - p_0)$  is the gauge pressure. Hence gauge pressure at any point in the liquid mass subjected to a constant vertical upward acceleration, is given by

$$p_g = \rho gh \left[ 1 + \frac{a}{g} \right] \quad \dots(3.22)$$

$$= \rho gh + \rho ha \quad \dots(3.22A)$$

where  $p_g = p - p_0 = \text{gauge pressure}$

In equation (3.22)  $\rho$ ,  $g$  and  $a$  are constant. Hence variation of gauge pressure is linear. Also when  $h = 0$ ,  $p_g = 0$ . This means  $p - p_0 = 0$  or  $p = p_0$ . Hence when  $h = 0$ , the pressure is equal to atmospheric pressure. Hence free surface of liquid subjected to constant vertical acceleration will be horizontal.

From equation (3.22A) it is also clear that the pressure at any point in the liquid mass is greater than the hydrostatic pressure (hydrostatic pressure is  $\rho gh$ ) by an amount of  $\rho \times h \times a$ .

Fig. 3.49 shows the variation of pressure for the liquid mass subjected to a constant vertical upward acceleration.

If the tank containing liquid is moving vertically downward with a constant acceleration, then the gauge pressure at any point in the liquid at a depth of  $h$  from the free surface will be given by

$$(p - p_0) = \rho gh \left[ 1 - \frac{a}{g} \right] = \rho gh - \rho ha \quad \dots(3.23)$$

The above equation shows that the pressure at any point in the liquid mass is less than the hydrostatic pressure by an amount of  $\rho ha$ . Fig. 3.50 shows the variation of pressure for the liquid mass subjected to a constant vertical downward acceleration.

If the tank containing liquid is moving downward with a constant acceleration equal to  $g$  (i.e., when  $a = g$ ), then equation reduces to  $p - p_0 = 0$  or  $p = p_0$ . This means the pressure at any point in the liquid is equal to surrounding atmospheric pressure. There will be no force on the walls or on the base of the tank.

**Note.** If a tank containing a liquid is subjected to a constant acceleration in the inclined direction, then the acceleration may be resolved along the horizontal direction and vertical direction. Then each of these cases may be separately analysed in accordance with the above procedure.

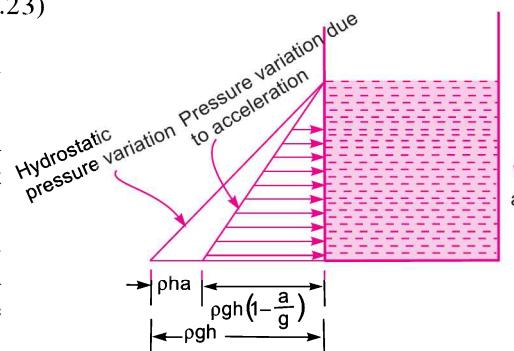


Fig. 3.50

**Problem 3.37** A tank containing water upto a depth of 500 mm is moving vertically upward with a constant acceleration of  $2.45 \text{ m/s}^2$ . Find the force exerted by water on the side of the tank. Also calculate the force on the side of the tank when the width of tank is 2 m and

- tank is moving vertically downward with a constant acceleration of  $2.45 \text{ m/s}^2$ , and
- the tank is not moving at all.

**Solution.** Given :

$$\text{Depth of water, } h = 500 \text{ mm} = 0.5 \text{ m}$$

$$\text{Vertical acceleration, } a = 2.45 \text{ m/s}^2$$

$$\text{Width of tank, } b = 2 \text{ m}$$

To find the force exerted by water on the side of the tank when moving vertically upward, let us first find the pressure at the bottom of the tank.

The gauge pressure at the bottom (i.e., at point B) for this case is given by equation as

$$\begin{aligned} p_B &= \rho gh \left(1 + \frac{a}{g}\right) \\ &= 1000 \times 9.81 \times 0.5 \left(1 + \frac{2.45}{9.81}\right) = 6131.25 \text{ N/m}^2 \end{aligned}$$

This pressure is represented by line BC.

Now the force on the side AB = Area of triangle ABC  $\times$  Width of tank

$$\begin{aligned} &= \left(\frac{1}{2} \times AB \times BC\right) \times b \\ &= \left(\frac{1}{2} \times 0.5 \times 6131.25\right) \times 2 \quad (\because BC = 6131.25 \text{ and } b = 2 \text{ m}) \\ &= \mathbf{3065.6 \text{ N. Ans.}} \end{aligned}$$

**(i) Force on the side of the tank, when tank is moving vertically downward.**

The pressure variation is shown in Fig. 3.52. For this case, the pressure at the bottom of the tank (i.e., at point B) is given by equation (3.23) as

$$\begin{aligned} p_B &= \rho gh \left(1 - \frac{a}{g}\right) \\ &= 1000 \times 9.81 \times 0.5 \left(1 - \frac{2.45}{9.81}\right) \\ &= 3678.75 \text{ N/m}^2 \end{aligned}$$

This pressure is represented by line BC.

Now the force on the side AB = Area of triangle ABC  $\times$  Width

$$\begin{aligned} &= \left(\frac{1}{2} \times AB \times BC\right) \times b \\ &= \left(\frac{1}{2} \times 0.5 \times 3678.75\right) \times 2 \quad (\because BC = 3678.75, b = 2) \\ &= \mathbf{1839.37 \text{ N. Ans.}} \end{aligned}$$

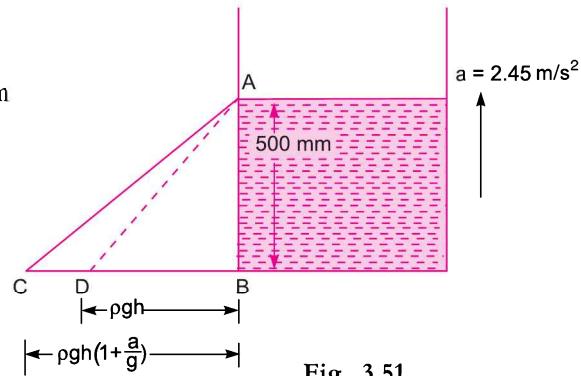


Fig. 3.51

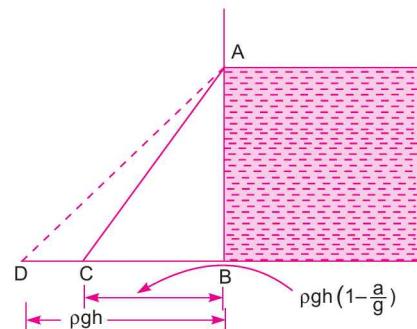


Fig. 3.52

**(ii) Force on the side of the tank, when tank is stationary.**

The pressure at point *B* is given by,

$$p_B = \rho gh = 1000 \times 9.81 \times 0.5 = 4905 \text{ N/m}^2$$

This pressure is represented by line *BD* in Fig. 3.52

$$\begin{aligned} \text{Force on the side } AB &= \text{Area of triangle } ABD \times \text{Width} \\ &= \left(\frac{1}{2} \times AB \times BD\right) \times b \\ &= \left(\frac{1}{2} \times 0.5 \times 4905\right) \times 2 && (\because BD = 4905) \\ &= 2452.5 \text{ N. Ans.} \end{aligned}$$

For this case, the force on *AB* can also be obtained as

$$F_{AB} = \rho g A \bar{h}$$

$$\text{where } A = AB \times \text{Width} = 0.5 \times 2 = 1 \text{ m}^2$$

$$\begin{aligned} \bar{h} &= \frac{AB}{2} = \frac{0.5}{2} = 0.25 \text{ m} = 1000 \times 9.81 \times 1 \times 0.25 \\ &= 2452.5 \text{ N. Ans.} \end{aligned}$$

**Problem 3.38** A tank contains water upto a depth of 1.5 m. The length and width of the tank are 4 m and 2 m respectively. The tank is moving up an inclined plane with a constant acceleration of 4 m/s<sup>2</sup>. The inclination of the plane with the horizontal is 30° as shown in Fig. 3.53. Find,

- (i) the angle made by the free surface of water with the horizontal.
- (ii) the pressure at the bottom of the tank at the front and rear ends.

**Solution.** Given :

Depth of water,  $h = 1.5 \text{ m}$ ; Length,  $L = 4 \text{ m}$  and Width,  $b = 2 \text{ m}$

Constant acceleration along the inclined plane,

$$a = 4 \text{ m/s}^2$$

Inclination of plane,  $\alpha = 30^\circ$

Let  $\theta$  = Angle made by the free surface of water after the acceleration is imparted to the tank

$p_A$  = Pressure at the bottom of the tank at the front end and  $p_D$  = Pressure at the bottom of the tank at the rear end.

This problem can be done by resolving the given acceleration along the horizontal direction and vertical direction. Then each of these cases may be separately analysed according to the set procedure.

Horizontal and vertical components of the acceleration are :

$$a_x = a \cos \alpha = 4 \cos 30^\circ = 3.464 \text{ m/s}^2$$

$$a_y = a \sin \alpha = 4 \sin 30^\circ = 2 \text{ m/s}^2$$

When the tank is stationary on the inclined plane, free surface of liquid will be along *EF* as shown in Fig. 3.53. But when the tank is moving upward along the inclined plane the free surface of liquid will be along *BC*. When the tank containing a liquid is moving up an inclined plane with a constant acceleration, the angle made by the free surface of the liquid with the horizontal is given by

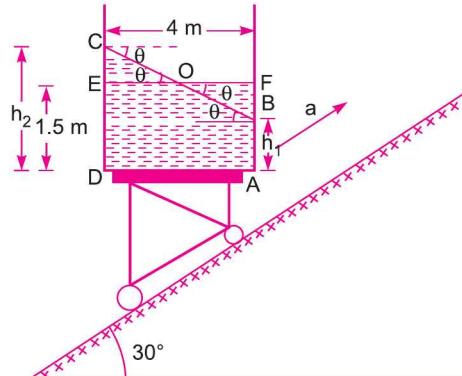


Fig. 3.53

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$$\tan \theta = \frac{a_x}{a_y + g} = \frac{3.464}{2 + 9.81} = 0.2933$$

$$\therefore \theta = \tan^{-1} 0.2933 = 16.346^\circ \text{ or } 16^\circ 20.8'. \text{ Ans.}$$

Now let us first find the depth of liquid at the front and rear end of the tank.

Depth of liquid at front end =  $h_1 = AB$

Depth of liquid at rear end =  $h_2 = CD$

From Fig. 3.53, in triangle  $COE$ ,  $\tan \theta = \frac{CE}{EO}$

or

$$CE = EO \tan \theta = 2 \times 0.2933 \quad (\because EO = 2 \text{ m}, \tan \theta = 0.2933)$$

$$= 0.5866 \text{ m}$$

$$\therefore CD = h_2 = ED + CE = 1.5 + 0.5866 = 2.0866 \text{ m}$$

Similarly

$$h_1 = AB = AF - BF$$

$$= 1.5 - 0.5866 \quad (\because AF = 1.5, BF = CE = 0.5866)$$

$$= 0.9134 \text{ m}$$

The pressure at the bottom of tank at the rear end is given by,

$$p_D = \rho gh_2 \left( 1 + \frac{a_y}{g} \right)$$

$$= 1000 \times 9.81 \times 2.0866 \left( 1 + \frac{2}{9.81} \right) = 24642.7 \text{ N/m}^2. \text{ Ans.}$$

The pressure at the bottom of tank at the front end is given by

$$p_A = \rho gh_1 \left( 1 + \frac{a_y}{g} \right)$$

$$= 1000 \times 9.81 \times 0.9134 \left( 1 + \frac{2}{9.81} \right) = 10787.2 \text{ N/m}^2. \text{ Ans.}$$

### HIGHLIGHTS

- When the fluid is at rest, the shear stress is zero.
- The force exerted by a static fluid on a vertical, horizontal or an inclined plane immersed surface,

$$F = \rho g A \bar{h}$$

where  $\rho$  = Density of the liquid,

$A$  = Area of the immersed surface, and

$\bar{h}$  = Depth of the centre of gravity of the immersed surface from free surface of the liquid.

- Centre of pressure is defined as the point of application of the resultant pressure.
- The depth of centre of pressure of an immersed surface from free surface of the liquid,

$$h^* = \frac{I_G}{A\bar{h}} + \bar{h} \quad \text{for vertically immersed surface.}$$

$$= \frac{I_G \sin^2 \theta}{A\bar{h}} + \bar{h} \quad \text{for inclined immersed surface.}$$

5. The centre of pressure for a plane vertical surface lies at a depth of two-third the height of the immersed surface.
6. The total force on a curved surface is given by  $F = \sqrt{F_x^2 + F_y^2}$   
 where  $F_x$  = Horizontal force on curved surface and is equal to total pressure force on the projected area of the curved surface on the vertical plane,  
 $= \rho g A \bar{h}$   
 and  $F_y$  = Vertical force on sub-merged curved surface and is equal to the weight of liquid actually or imaginary supported by the curved surface.
7. The inclination of the resultant force on curved surface with horizontal,  $\tan \theta = \frac{F_y}{F_x}$ .
8. The resultant force on a sluice gate,  $F = F_1 - F_2$   
 where  $F_1$  = Pressure force on the upstream side of the sluice gate and  
 $F_2$  = Pressure force on the downstream side of the sluice gate.
9. For a lock gate, the reaction between the two gates is equal to the reaction at the hinge,  $R = P$ .  
 Also the reaction between the two gates,  $P = \frac{F}{2 \sin \theta}$   
 where  $F$  = Resultant water pressure on the lock gate  $= F_1 - F_2$   
 and  $\theta$  = Inclination of the gate with the normal to the side of the lock.

## EXERCISE

### (A) THEORETICAL PROBLEMS

- What do you understand by 'Total Pressure' and 'Centre of Pressure' ?
- Derive an expression for the force exerted on a sub-merged vertical plane surface by the static liquid and locate the position of centre of pressure.
- Prove that the centre of pressure of a completely sub-merged plane surface is always below the centre of gravity of the sub-merged surface or at most coincide with the centre of gravity when the plane surface is horizontal.
- Prove that the total pressure exerted by a static liquid on an inclined plane sub-merged surface is the same as the force exerted on a vertical plane surface as long as the depth of the centre of gravity of the surface is unaltered.
- Derive an expression for the depth of centre of pressure from free surface of liquid of an inclined plane surface sub-merged in the liquid.
- (a) How would you determine the horizontal and vertical components of the resultant pressure on a sub-merged curved surface ?  
(b) Explain the procedure of finding hydrostatic forces on curved surfaces.

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- Explain how you would find the resultant pressure on a curved surface immersed in a liquid.
- Why the resultant pressure on a curved sub-merged surface is determined by first finding horizontal and vertical forces on the curved surface ? Why is the same method not adopted for a plane inclined surface sub-merged in a liquid ?

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9. Describe briefly with sketches the various methods used for measuring pressure exerted by fluids.
10. Prove that the vertical component of the resultant pressure on a sub-merged curved surface is equal to the weight of the liquid supported by the curved surface.
11. What is the difference between sluice gate and lock gate ?
12. Prove that the reaction between the gates of a lock is equal to the reaction at the hinge.
13. Derive an expression for the reaction between the gates as  $P = \frac{F}{2 \sin \theta}$   
where  $F$  = Resultant water pressure on lock gate,  $\theta$  = inclination of the gate with normal to the side of the lock.
14. When will centre of pressure and centre of gravity of an immersed plane surface coincide ?
15. Find an expression for the force exerted and centre of pressure for a completely sub-merged inclined plane surface. Can the same method be applied for finding the resultant force on a curved surface immersed in the liquid ? If not, why ?
16. What do you understand by the hydrostatic equation ? With the help of this equation derive the expressions for the total thrust on a sub-merged plane area and the buoyant force acting on a sub-merged body.

### (B) NUMERICAL PROBLEMS

1. Determine the total pressure and depth of centre of pressure on a plane rectangular surface of 1 m wide and 3 m deep when its upper edge is horizontal and (a) coincides with water surface (b) 2 m below the free water surface.  
[Ans. (a) 44145 N, 2.0 m, (b) 103005 N, 3.714 m]
2. Determine the total pressure on a circular plate of diameter 1.5 m which is placed vertically in water in such a way that centre of plate is 2 m below the free surface of water. Find the position of centre of pressure also.  
[Ans. 34668.54 N, 2.07 m]
3. A rectangular sluice gate is situated on the vertical wall of a lock. The vertical side of the sluice is 6 m in length and depth of centroid of area is 8 m below the water surface. Prove that the depth of centre of pressure is given by 8.475 m.
4. A circular opening, 3 m diameter, in a vertical side of a tank is closed by a disc of 3 m diameter which can rotate about a horizontal diameter. Calculate : (i) the force on the disc, and (ii) the torque required to maintain the disc in equilibrium in the vertical position when the head of water above the horizontal diameter is 6 m.  
[Ans. (i) 416.05 kN, (ii) 39005 Nm]
5. The pressure at the centre of a pipe of diameter 3 m is  $29.43 \text{ N/cm}^2$ . The pipe contains oil of sp. gr. 0.87 and is filled with a gate valve. Find the force exerted by the oil on the gate and position of centre of pressure.  
[Ans. 2.08 MN, .016 m below centre of pipe]
6. Determine the total pressure and centre of pressure on an isosceles triangular plate of base 5 m and altitude 5 m when the plate is immersed vertically in an oil of sp. gr. 0.8. The base of the plate is 1 m below the free surface of water.  
[Ans. 261927 N, 3.19 m]
7. The opening in a dam is 3 m wide and 2 m high. A vertical sluice gate is used to cover the opening. On the upstream of the gate, the liquid of sp. gr. 1.5, lies upto a height of 2.0 m above the top of the gate, whereas on the downstream side, the water is available upto a height of the top of the gate. Find the resultant force acting on the gate and position of centre of pressure. Assume that the gate is higher at the bottom.  
[Ans. 206010 N, 0.964 m above the hinge]

8. A caisson for closing the entrance to a dry dock is of trapezoidal form 16 m wide at the top and 12 m wide at the bottom and 8 m deep. Find the total pressure and centre of pressure on the caisson if the water on the outside is 1 m below the top level of the caisson and dock is empty.

[Ans. 3.164 MN, 4.56 m below water surface]

9. A sliding gate 2 m wide and 1.5 m high lies in a vertical plane and has a co-efficient of friction of 0.2 between itself and guides. If the gate weighs one tonne, find the vertical force required to raise the gate if its upper edge is at a depth of 4 m from free surface of water.

[Ans. 37768.5 N]

10. A tank contains water upto a height of 1 m above the base. An immiscible liquid of sp. gr. 0.8 is filled on the top of water upto 1.5 m height. Calculate : (i) total pressure on one side of the tank, (ii) the position of centre of pressure for one side of the tank, which is 3 m wide.

[Ans. 76518 N, 1.686 m from top]

11. A rectangular tank 4 m long, 1.5 m wide contains water upto a height of 2 m. Calculate the force due to water pressure on the base of the tank. Find also the depth of centre of pressure from free surface.

[Ans. 117720 N, 2 m from free surface]

12. A rectangular plane surface 1 m wide and 3 m deep lies in water in such a way that its plane makes an angle of  $30^\circ$  with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge of the plate is 2 m below the free water surface.

[Ans. 80932.5 N, 2.318 m]

13. A circular plate 3.0 m diameter is immersed in water in such a way that the plane of the plate makes an angle of  $60^\circ$  with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge of the plate is 2 m below the free water surface.

[Ans. 228.69 kN, 3.427 m from free surface]

14. A rectangular gate  $6 \text{ m} \times 2 \text{ m}$  is hinged at its base and inclined at  $60^\circ$  to the horizontal as shown in Fig. 3.54. To keep the gate in a stable position, a counter weight of 29430 N is attached at the upper end of the gate. Find the depth of water at which the gate begins to fall. Neglect the weight of the gate and also friction at the hinge and pulley.

[Ans. 3.43 m]

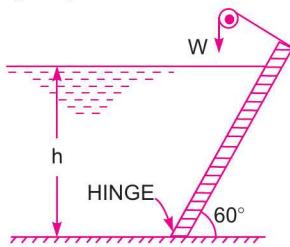


Fig. 3.54

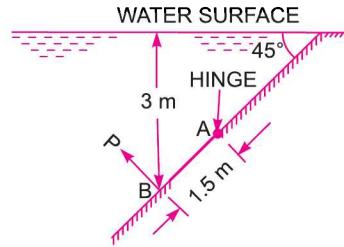


Fig. 3.55

15. An inclined rectangular gate of width 5 m and depth 1.5 m is installed to control the discharge of water as shown in Fig. 3.55. The end A is hinged. Determine the force normal to the gate applied at B to open it.

[Ans. 97435.8 N]

16. A gate supporting water is shown in Fig. 3.56. Find the height 'h' of the water so that the gate begins to tip about the hinge. Take the width of the gate as unity.

[Ans.  $3 \times \sqrt{3}$  m]

17. Find the total pressure and depth of centre of pressure on a triangular plate of base 3 m and height 3 m which is immersed in water in such a way that plane of the plate makes an angle of  $60^\circ$  with the free surface. The base of the plate is parallel to water surface and at a depth of 2 m from water surface.

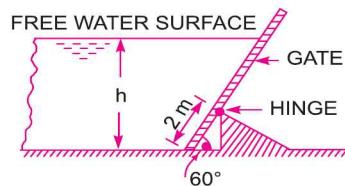


Fig. 3.56

[Ans. 126.52 kN, 2.996 m]

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18. Find the horizontal and vertical components of the total force acting on a curved surface AB, which is in the form of a quadrant of a circle of radius 2 m as shown in Fig. 3.57. Take the width of the gate 2 m.

[Ans.  $F_x = 117.72$  kN,  $F_y = 140.114$  kN]

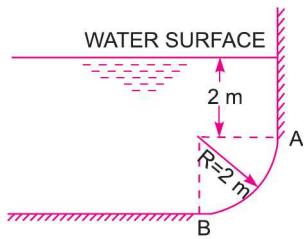


Fig. 3.57

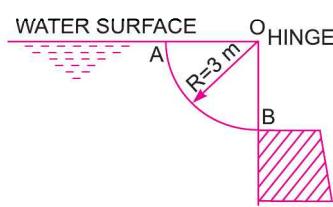


Fig. 3.58

19. Fig. 3.58 shows a gate having a quadrant shape of radius of 3 m. Find the resultant force due to water per metre length of the gate. Find also the angle at which the total force will act. [Ans. 82.201 kN,  $\theta = 57^\circ 31'$ ]

20. A roller gate is shown in Fig. 3.59. It is cylindrical form of 6.0 m diameter. It is placed on the dam. Find the magnitude and direction of the resultant force due to water acting on the gate when the water is just going to spill. The length of the gate is given 10 m. [Ans. 2.245 MN,  $\theta = 38^\circ 8'$ ]

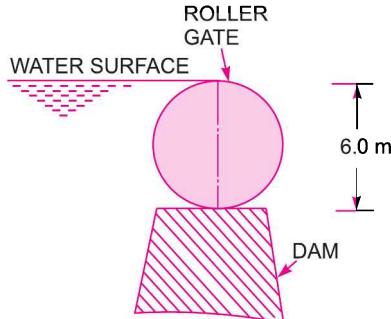


Fig. 3.59

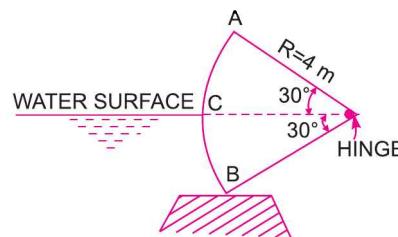


Fig. 3.60

21. Find the horizontal and vertical components of the water pressure exerted on a tainter gate of radius 4 m as shown in Fig. 3.60. Consider width of the gate unity. [Ans.  $F_x = 19.62$  kN,  $F_y = 7102.44$  N]

22. Find the magnitude and direction of the resultant water pressure acting on a curved face of a dam which is shaped

according to the relation  $y = \frac{x^2}{6}$  as shown in Fig. 3.61. The

height of water retained by the dam is 12 m. Take the width of dam as unity. [Ans. 970.74 kN,  $\theta = 43^\circ 19'$ ]

23. Each gate of a lock is 5 m high and is supported by two hinges placed on the top and bottom of the gate. When the gates are closed, they make an angle of 120°. The width of the lock is 4 m. If the depths of water on the two sides of the gates are 4 m and 3 m respectively, determine : (i) the magnitude of resultant pressure on each gate, and (ii) magnitude of the hinge reactions. [Ans. (i) 79.279 kN, (ii)  $R_T = 27.924$  kN,  $R_B = 51.355$  kN]
24. The end gates ABC of a lock are 8 m high and when closed make an angle of 120°. The width of lock is 10 m. Each gate is supported by two hinges located at 1 m and 5 m above the bottom of the lock. The depth of water on the upstream and downstream sides of the lock are 6 m and 4 m respectively. Find : (i) Resultant water force on each gate.

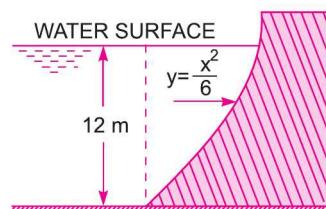


Fig. 3.61

- (ii) Reaction between the gates  $AB$  and  $BC$ , and  
 (iii) Force on each hinge, considering the reaction of the gate acting in the same horizontal plane as resultant water pressure. [Ans. 566.33 kN, (ii) 566.33 kN, and (iii)  $R_T = 173.64$  kN,  $R_B = 392.69$  kN]
25. A hollow circular plate of 2 m external and 1 m internal diameter is immersed vertically in water such that the centre of plate is 4 m deep from water surface. Find the total pressure and depth of centre of pressure.  
 [Ans. 92.508 kN, 4.078 m]
26. A rectangular opening 2 m wide and 1 m deep in the vertical side of a tank is closed by a sluice gate of the same size. The gate can turn about the horizontal centroidal axis. Determine : (i) the total pressure on the sluice gate and (ii) the torque on the sluice gate. The head of water above the upper edge of the gate is 1.5 m.  
 [Ans. (i) 39.24 kN, (ii) 1635 Nm]
27. Determine the total force and location of centre of pressure on one face of the plate shown in Fig. 3.62 immersed in a liquid of specific gravity 0.9.  
 [Ans. 62.4 kN, 3.04 m]
28. A circular opening, 3 m diameter, in the vertical side of water tank is closed by a disc of 3 m diameter which can rotate about a horizontal diameter ? Calculate: (i) the force on the disc, and (ii) the torque required to maintain the disc in equilibrium in the vertical position when the head of water above the horizontal diameter is 4 m. [Ans. (i) 270 kN, and (ii) 38 kN m]
29. A penstock made up by a pipe of 2 m diameter contains a circular disc of same diameter to act as a valve which controls the discharge passing through it. It can rotate about a horizontal diameter. If the head of water above its centre is 20 m, find the total force acting on the disc and the torque required to maintain it in the vertical position.
30. A circular drum 1.8 m diameter and 1.2 m height is submerged with its axis vertical and its upper end at a depth of 1.8 m below water level. Determine :  
 (i) total pressure on top, bottom and curved surfaces of the drum,  
 (ii) resultant pressure on the whole surface, and  
 (iii) depth of centre of pressure on curved surface.
31. A circular plate of diameter 3 m is immersed in water in such a way that its least and greatest depth from the free surface of water are 1 m and 3 m respectively. For the front side of the plate, find (i) total force exerted by water and (ii) the position of centre of pressure. [Ans. (i) 138684 N ; (ii) 2.125 m]
32. A tank contains water upto a height of 10 m. One of the sides of the tank is inclined. The angle between free surface of water and inclined side is  $60^\circ$ . The width of the tank is 5 m. Find : (i) the force exerted by water on inclined side and (ii) position of centre of pressure. [Ans. (i) 283.1901 kN, (ii) 6.67 m]
33. A circular plate of 3 m diameter is under water with its plane making an angle of  $30^\circ$  with the water surface. If the top edge of the plate is 1 m below the water surface, find the force on one side of the plate and its location. (J.N.T.U., Hyderabad S 2002)

[Hint.  $d = 3$  m,  $\theta = 30^\circ$ , height of top edge = 1 m,  $h = 1 + 1.5 \times \sin 30^\circ$

$$F = \rho g A \bar{h} = 1000 \times 9.81 \times \left( \frac{\pi}{4} \times 3^2 \right) \times 1.75 = 121.35 \text{ kN.}$$

$$h^* = \frac{I_G \sin^2 \theta}{A \bar{h}} + \bar{h} = \frac{\frac{\pi}{4} (3^4)^4 \times \frac{1}{4}}{\frac{\pi}{4} (3^2) \times 1.75} + 1.75 = 0.08 + 1.75 = 1.83 \text{ m.}$$

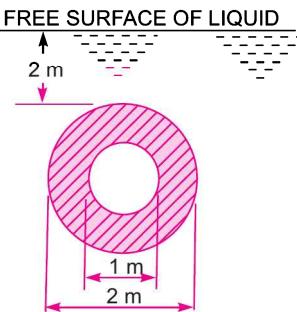


Fig. 3.62

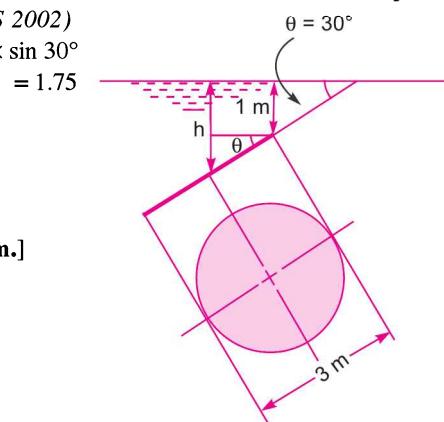


Fig. 3.63



# 4

## CHAPTER

# BUOYANCY AND FLOTATION



### ► 4.1 INTRODUCTION

In this chapter, the equilibrium of the floating and sub-merged bodies will be considered. Thus the chapter will include : 1. Buoyancy, 2. Centre of buoyancy, 3. Metacentre, 4. Metacentric height, 5. Analytical method for determining metacentric height, 6. Conditions of equilibrium of a floating and sub-merged body, and 7. Experimental method for metacentric height.

### ► 4.2 BUOYANCY

When a body is immersed in a fluid, an upward force is exerted by the fluid on the body. This upward force is equal to the weight of the fluid displaced by the body and is called the force of buoyancy or simply buoyancy.

### ► 4.3 CENTRE OF BUOYANCY

It is defined as the point, through which the force of buoyancy is supposed to act. As the force of buoyancy is a vertical force and is equal to the weight of the fluid displaced by the body, the centre of buoyancy will be the centre of gravity of the fluid displaced.

**Problem 4.1** Find the volume of the water displaced and position of centre of buoyancy for a wooden block of width 2.5 m and of depth 1.5 m, when it floats horizontally in water. The density of wooden block is  $650 \text{ kg/m}^3$  and its length 6.0 m.

**Solution.** Given :

Width	= 2.5 m
Depth	= 1.5 m
Length	= 6.0 m
Volume of the block	$= 2.5 \times 1.5 \times 6.0 = 22.50 \text{ m}^3$
Density of wood,	$\rho = 650 \text{ kg/m}^3$
$\therefore$ Weight of block	$= \rho \times g \times \text{Volume}$ $= 650 \times 9.81 \times 22.50 \text{ N} = 143471 \text{ N}$

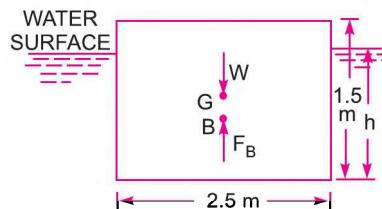


Fig. 4.1

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For equilibrium the weight of water displaced = Weight of wooden block  
 $= 143471 \text{ N}$

$\therefore$  Volume of water displaced

$$= \frac{\text{Weight of water displaced}}{\text{Weight density of water}} = \frac{143471}{1000 \times 9.81} = 14.625 \text{ m}^3. \text{ Ans.}$$

( $\because$  Weight density of water =  $1000 \times 9.81 \text{ N/m}^3$ )

**Position of Centre of Buoyancy.** Volume of wooden block in water

$$= \text{Volume of water displaced}$$

$$\text{or } 2.5 \times h \times 6.0 = 14.625 \text{ m}^3, \text{ where } h \text{ is depth of wooden block in water}$$

$$\therefore h = \frac{14.625}{2.5 \times 6.0} = 0.975 \text{ m}$$

$$\therefore \text{Centre of Buoyancy} = \frac{0.975}{2} = 0.4875 \text{ m from base. Ans.}$$

**Problem 4.2** A wooden log of  $0.6 \text{ m}$  diameter and  $5 \text{ m}$  length is floating in river water. Find the depth of the wooden log in water when the sp. gravity of the log is  $0.7$ .

**Solution.** Given :

$$\text{Dia. of log} = 0.6 \text{ m}$$

$$\text{Length, } L = 5 \text{ m}$$

$$\text{Sp. gr., } S = 0.7$$

$$\therefore \text{Density of log} = 0.7 \times 1000 = 700 \text{ kg/m}^3$$

$$\therefore \text{Weight density of log, } w = \rho \times g \\ = 700 \times 9.81 \text{ N/m}^3$$

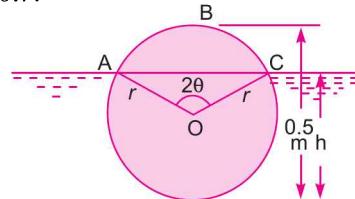


Fig. 4.2

Find depth of immersion or  $h$

$$\text{Weight of wooden log} = \text{Weight density} \times \text{Volume of log}$$

$$= 700 \times 9.81 \times \frac{\pi}{4} (D)^2 \times L$$

$$= 700 \times 9.81 \times \frac{\pi}{4} (.6)^2 \times 5 \text{ N} = 989.6 \times 9.81 \text{ N}$$

For equilibrium,

$$\begin{aligned} \text{Weight of wooden log} &= \text{Weight of water displaced} \\ &= \text{Weight density of water} \times \text{Volume of water displaced} \\ \therefore \text{Volume of water displaced} &= \frac{989.6 \times 9.81}{1000 \times 9.81} = 0.9896 \text{ m}^3 \\ &\quad (\because \text{Weight density of water} = 1000 \times 9.81 \text{ N/m}^3) \end{aligned}$$

Let  $h$  is the depth of immersion

$$\begin{aligned} \therefore \text{Volume of log inside water} &= \text{Area of } ADCA \times \text{Length} \\ &= \text{Area of } ADCA \times 5.0 \end{aligned}$$

$$\text{But volume of log inside water} = \text{Volume of water displaced} = 0.9896 \text{ m}^3$$

$$\therefore 0.9896 = \text{Area of } ADCA \times 5.0$$

$$\therefore \text{Area of } ADCA = \frac{0.9896}{5.0} = 0.1979 \text{ m}^2$$

$$\begin{aligned}\text{But area of } ADCA &= \text{Area of curved surface } ADCOA + \text{Area of } \Delta AOC \\ &= \pi r^2 \left[ \frac{360^\circ - 2\theta}{360^\circ} \right] + \frac{1}{2} r \cos \theta \times 2r \sin \theta \\ &= \pi r^2 \left[ 1 - \frac{\theta}{180^\circ} \right] + r^2 \cos \theta \sin \theta\end{aligned}$$

$$\therefore 0.1979 = \pi (.3)^2 \left[ 1 - \frac{\theta}{180^\circ} \right] + (.3)^2 \cos \theta \sin \theta$$

$$0.1979 = .2827 - .00157 \theta + 0.9 \cos \theta \sin \theta$$

$$\text{or } .00157 \theta - .09 \cos \theta \sin \theta = .2827 - .1979 = 0.0848$$

$$\theta - \frac{.09}{.00157} \cos \theta \sin \theta = \frac{.0848}{.00157}$$

$$\text{or } \theta - 57.32 \cos \theta \sin \theta = 54.01.$$

$$\text{or } \theta - 57.32 \cos \theta \sin \theta - 54.01 = 0$$

$$\text{For } \theta = 60^\circ, \quad 60 - 57.32 \times 0.5 \times .866 - 54.01 = 60 - 24.81 - 54.01 = - 18.82$$

$$\text{For } \theta = 70^\circ, \quad 70 - 57.32 \times .342 \times 0.9396 - 54.01 = 70 - 18.4 - 54.01 = - 2.41$$

$$\text{For } \theta = 72^\circ, \quad 72 - 57.32 \times .309 \times .951 - 54.01 = 72 - 16.84 - 54.01 = + 1.14$$

$$\text{For } \theta = 71^\circ, \quad 71 - 57.32 \times .325 \times .9455 - 54.01 = 71 - 17.61 - 54.01 = - 0.376$$

$$\therefore \theta = 71.5^\circ, \quad 71.5 - 57.32 \times .3173 \times .948 - 54.01 = 71.5 - 17.24 - 54.01 = + .248$$

$$\text{Then } h = r + r \cos 71.5^\circ$$

$$= 0.3 + 0.3 \times 0.3173 = \mathbf{0.395 \text{ m. Ans.}}$$

**Problem 4.3** A stone weighs 392.4 N in air and 196.2 N in water. Compute the volume of stone and its specific gravity.

**Solution.** Given :

$$\text{Weight of stone in air} = 392.4 \text{ N}$$

$$\text{Weight of stone in water} = 196.2 \text{ N}$$

For equilibrium,

$$\text{Weight in air} - \text{Weight of stone in water} = \text{Weight of water displaced}$$

$$\text{or } 392.4 - 196.2 = 196.2 = 1000 \times 9.81 \times \text{Volume of water displaced}$$

$\therefore$  Volume of water displaced

$$= \frac{196.2}{1000 \times 9.81} = \frac{1}{50} \text{ m}^3 = \frac{1}{50} \times 10^6 \text{ cm}^3 = \mathbf{2 \times 10^4 \text{ cm}^3. \text{ Ans.}}$$

= Volume of stone

$$\therefore \text{Volume of stone} = \mathbf{2 \times 10^4 \text{ cm}^3. \text{ Ans.}}$$

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### Specific Gravity of Stone

$$\begin{aligned}\text{Mass of stone} &= \frac{\text{Weight in air}}{g} = \frac{392.4}{9.81} = 40 \text{ kg} \\ \text{Density of stone} &= \frac{\text{Mass in air}}{\text{Volume}} = \frac{40.0 \text{ kg}}{\frac{1}{50} \text{ m}^3} = 40 \times 50 = 2000 \frac{\text{kg}}{\text{m}^3} \\ \therefore \text{Sp. gr. of stone} &= \frac{\text{Density of stone}}{\text{Density of water}} = \frac{2000}{1000} = 2.0. \text{ Ans.}\end{aligned}$$

**Problem 4.4** A body of dimensions  $1.5 \text{ m} \times 1.0 \text{ m} \times 2 \text{ m}$ , weighs  $1962 \text{ N}$  in water. Find its weight in air. What will be its specific gravity?

**Solution.** Given :

$$\begin{aligned}\text{Volume of body} &= 1.50 \times 1.0 \times 2.0 = 3.0 \text{ m}^3 \\ \text{Weight of body in water} &= 1962 \text{ N} \\ \text{Volume of the water displaced} &= \text{Volume of the body} = 3.0 \text{ m}^3 \\ \therefore \text{Weight of water displaced} &= 1000 \times 9.81 \times 3.0 = 29430 \text{ N}\end{aligned}$$

For the equilibrium of the body

$$\begin{aligned}\text{Weight of body in air} - \text{Weight of water displaced} &= \text{Weight in water} \\ \therefore W_{\text{air}} - 29430 &= 1962\end{aligned}$$

$$\begin{aligned}W_{\text{air}} &= 29430 + 1962 = 31392 \text{ N} \\ \text{Mass of body} &= \frac{\text{Weight in air}}{g} = \frac{31392}{9.81} = 3200 \text{ kg} \\ \text{Density of the body} &= \frac{\text{Mass}}{\text{Volume}} = \frac{3200}{3.0} = 1066.67 \\ \therefore \text{Sp. gravity of the body} &= \frac{1066.67}{1000} = 1.067. \text{ Ans.}\end{aligned}$$

**Problem 4.5** Find the density of a metallic body which floats at the interface of mercury of sp. gr. 13.6 and water such that 40% of its volume is sub-merged in mercury and 60% in water.

**Solution.** Let the volume of the body =  $V \text{ m}^3$

Then volume of body sub-merged in mercury

$$= \frac{40}{100} V = 0.4 V \text{ m}^3$$

Volume of body sub-merged in water

$$= \frac{60}{100} \times V = 0.6 V \text{ m}^3$$

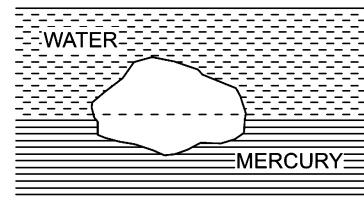


Fig. 4.3

For the equilibrium of the body

Total buoyant force (upward force) = Weight of the body

But total buoyant force = Force of buoyancy due to water + Force of buoyancy due to mercury

Force of buoyancy due to water = Weight of water displaced by body

$$\begin{aligned}&= \text{Density of water} \times g \times \text{Volume of water displaced} \\ &= 1000 \times g \times \text{Volume of body in water}\end{aligned}$$

$$= 1000 \times g \times 0.6 \times V \text{ N}$$

and Force of buoyancy due to mercury

$$= \text{Weight of mercury displaced by body}$$

$$= g \times \text{Density of mercury} \times \text{Volume of mercury displaced}$$

$$= g \times 13.6 \times 1000 \times \text{Volume of body in mercury}$$

$$= g \times 13.6 \times 1000 \times 0.4 \text{ V N}$$

$$= \text{Density} \times g \times \text{Volume of body} = \rho \times g \times V$$

Weight of the body

where  $\rho$  is the density of the body

$\therefore$  For equilibrium, we have

$$\text{Total buoyant force} = \text{Weight of the body}$$

$$1000 \times g \times 0.6 \times V + 13.6 \times 1000 \times g \times .4 V = \rho \times g \times V$$

or

$$\rho = 600 + 13600 \times .4 = 600 + 54400 = 6040.00 \text{ kg/m}^3$$

$\therefore$  Density of the body

$$= 6040.00 \text{ kg/m}^3. \text{ Ans.}$$

**Problem 4.6** A float valve regulates the flow of oil of sp. gr. 0.8 into a cistern. The spherical float is 15 cm in diameter.  $AOB$  is a weightless link carrying the float at one end, and a valve at the other end which closes the pipe through which oil flows into the cistern. The link is mounted in a frictionless hinge at  $O$  and the angle  $AOB$  is  $135^\circ$ . The length of  $OA$  is 20 cm, and the distance between the centre of the float and the hinge is 50 cm. When the flow is stopped  $AO$  will be vertical. The valve is to be pressed on to the seat with a force of 9.81 N to completely stop the flow of oil into the cistern. It was observed that the flow of oil is stopped when the free surface of oil in the cistern is 35 cm below the hinge. Determine the weight of the float.

**Solution.** Given :

$$\text{Sp. gr. of oil} = 0.8$$

$$\therefore \text{Density of oil, } \rho_0 = 0.8 \times 1000 \\ = 800 \text{ kg/m}^3$$

$$\text{Dia. of float, } D = 15 \text{ cm}$$

$$\angle AOB = 135^\circ$$

$$OA = 20 \text{ cm}$$

Force,

$$P = 9.81 \text{ N}$$

$$OB = 50 \text{ cm}$$

Find the weight of the float. Let it is equal to  $W$ .

When the flow of oil is stopped, the centre of float is shown in Fig. 4.4

The level of oil is also shown. The centre of float is below the level of oil, by a depth ' $h$ '.

$$\text{From } \Delta BOD, \quad \sin 45^\circ = \frac{OD}{OB} = \frac{OC + CD}{OB} = \frac{35 + h}{50}$$

$$\therefore 50 \times \sin 45^\circ = 35 + h$$

$$\text{or } h = 50 \times \frac{1}{\sqrt{2}} - 35 = 35.355 - 35 = 0.355 \text{ cm} = .00355 \text{ m.}$$

The weight of float is acting through  $B$ , but the upward buoyant force is acting through the centre of weight of oil displaced.

$$\text{Volume of oil displaced} = \frac{2}{3} \pi r^3 + h \times \pi r^2 \quad \left\{ r = \frac{D}{2} = \frac{15}{2} = 7.5 \text{ cm} \right\}$$

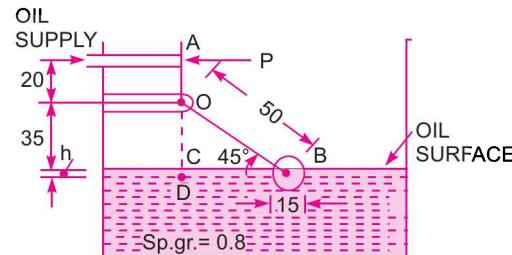


Fig. 4.4

$$= \frac{2}{3} \times \pi \times (.075)^3 + .00355 \times \pi \times (.075)^2 = 0.000945 \text{ m}^3$$

$$\begin{aligned}\therefore \text{Buoyant force} &= \text{Weight of oil displaced} \\ &= \rho_0 \times g \times \text{Volume of oil} \\ &= 800 \times 9.81 \times .000945 = 7.416 \text{ N}\end{aligned}$$

The buoyant force and weight of the float passes through the same vertical line, passing through  $B$ . Let the weight of float is  $W$ . Then net vertical force on float

$$= \text{Buoyant force} - \text{Weight of float} = (7.416 - W)$$

Taking moments about the hinge  $O$ , we get

$$P \times 20 = (7.416 - W) \times BD = (7.416 - W) \times 50 \times \cos 45^\circ$$

or

$$9.81 \times 20 = (7.416 - W) \times 35.355$$

$$\therefore W = 7.416 - \frac{20 \times 9.81}{35.355} = 7.416 - 5.55 = \mathbf{1.866 \text{ N. Ans.}}$$

#### ► 4.4 META-CENTRE

It is defined as the point about which a body starts oscillating when the body is tilted by a small angle. The meta-centre may also be defined as the point at which the line of action of the force of buoyancy will meet the normal axis of the body when the body is given a small angular displacement.

Consider a body floating in a liquid as shown in Fig. 4.5 (a). Let the body is in equilibrium and  $G$  is the centre of gravity and  $B$  the centre of buoyancy. For equilibrium, both the points lie on the normal axis, which is vertical.

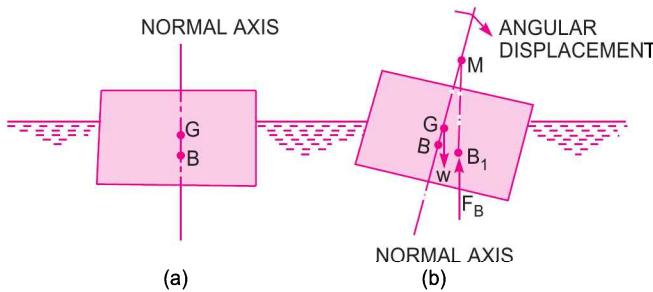


Fig. 4.5 Meta-centre

Let the body is given a small angular displacement in the clockwise direction as shown in Fig. 4.5 (b). The centre of buoyancy, which is the centre of gravity of the displaced liquid or centre of gravity of the portion of the body submersed in liquid, will now be shifted towards right from the normal axis. Let it is at  $B_1$  as shown in Fig. 4.5 (b). The line of action of the force of buoyancy in this new position, will intersect the normal axis of the body at some point say  $M$ . This point  $M$  is called **Meta-centre**.

#### ► 4.5 META-CENTRIC HEIGHT

The distance  $MG$ , i.e., the distance between the meta-centre of a floating body and the centre of gravity of the body is called meta-centric height.

## ► 4.6 ANALYTICAL METHOD FOR META-CENTRE HEIGHT

Fig. 4.6 (a) shows the position of a floating body in equilibrium. The location of centre of gravity and centre of buoyancy in this position is at  $G$  and  $B$ . The floating body is given a small angular displacement in the clockwise direction. This is shown in Fig. 4.6 (b). The new centre of buoyancy is at  $B_1$ . The vertical line through  $B_1$  cuts the normal axis at  $M$ . Hence  $M$  is the meta-centre and  $GM$  is meta-centric height.

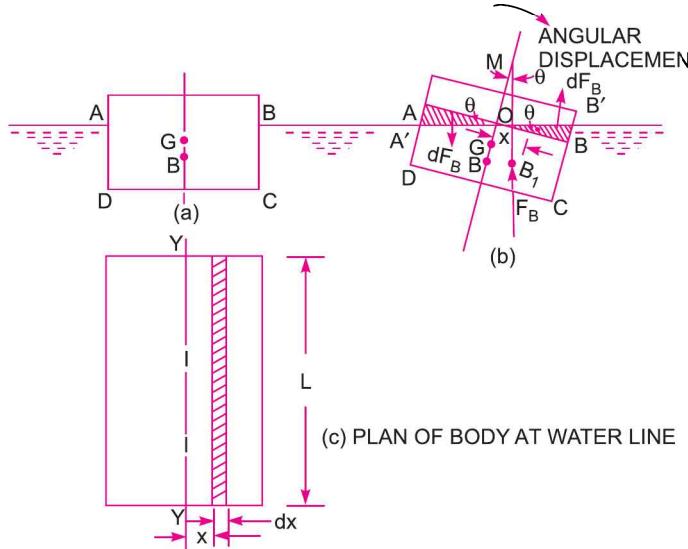


Fig. 4.6 Meta-centre height of floating body.

The angular displacement of the body in the clockwise direction causes the wedge-shaped prism  $BOB'$  on the right of the axis to go inside the water while the identical wedge-shaped prism represented by  $AOA'$  emerges out of the water on the left of the axis. These wedges represent a gain in buoyant force on the right side and a corresponding loss of buoyant force on the left side. The gain is represented by a vertical force  $dF_B$  acting through the C.G. of the prism  $BOB'$  while the loss is represented by an equal and opposite force  $dF_B$  acting vertically downward through the centroid of  $AOA'$ . The couple due to these buoyant forces  $dF_B$  tends to rotate the ship in the counterclockwise direction. Also the moment caused by the displacement of the centre of buoyancy from  $B$  to  $B_1$  is also in the counterclockwise direction. Thus these two couples must be equal.

**Couple Due to Wedges.** Consider towards the right of the axis a small strip of thickness  $dx$  at a distance  $x$  from  $O$  as shown in Fig. 4.5 (b). The height of strip  $x \times \angle BOB' = x \times \theta$ .

$$\{\because \angle BOB' = \angle AOA' = BMB_1' = \theta\}$$

$$\therefore \text{Area of strip} = \text{Height} \times \text{Thickness} = x \times \theta \times dx$$

If  $L$  is the length of the floating body, then

$$\begin{aligned}\text{Volume of strip} &= \text{Area} \times L \\ &= x \times \theta \times L \times dx\end{aligned}$$

$$\therefore \text{Weight of strip} = \rho g \times \text{Volume} = \rho g x \theta L dx$$

Similarly, if a small strip of thickness  $dx$  at a distance  $x$  from  $O$  towards the left of the axis is considered, the weight of strip will be  $\rho g x \theta L dx$ . The two weights are acting in the opposite direction and hence constitute a couple.

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$$\begin{aligned}
 \text{Moment of this couple} &= \text{Weight of each strip} \times \text{Distance between these two weights} \\
 &= \rho g x \theta L dx [x + x] \\
 &= \rho g x \theta L dx \times 2x = 2\rho g x^2 \theta L dx \\
 \therefore \text{Moment of the couple for the whole wedge} &= \int 2\rho g x^2 \theta L dx \quad \dots(4.1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Moment of couple due to shifting of centre of buoyancy from } B \text{ to } B_1 &= F_B \times BB_1 \\
 &= F_B \times BM \times \theta \quad \{ \because BB_1 = BM \times \theta \text{ if } \theta \text{ is very small} \} \\
 &= W \times BM \times \theta \quad \{ \because F_B = W \} \quad \dots(4.2)
 \end{aligned}$$

But these two couples are the same. Hence equating equations (4.1) and (4.2), we get

$$\begin{aligned}
 W \times BM \times \theta &= \int 2\rho g x^2 \theta L dx \\
 W \times BM \times \theta &= 2\rho g \theta \int x^2 L dx \\
 W \times BM &= 2\rho g \int x^2 L dx
 \end{aligned}$$

Now  $L dx$  = Elemental area on the water line shown in Fig. 4.6 (c) and  $= dA$

$$\therefore W \times BM = 2\rho g \int x^2 dA.$$

But from Fig. 4.5 (c) it is clear that  $2 \int x^2 dA$  is the second moment of area of the plan of the body at water surface about the axis Y-Y. Therefore

$$\begin{aligned}
 W \times BM &= \rho g I \quad \{ \text{where } I = 2 \int x^2 dA \} \\
 \therefore BM &= \frac{\rho g I}{W} \\
 \text{But } W &= \text{Weight of the body} \\
 &= \text{Weight of the fluid displaced by the body} \\
 &= \rho g \times \text{Volume of the fluid displaced by the body} \\
 &= \rho g \times \text{Volume of the body submersed in water} \\
 &= \rho g \times V \\
 \therefore BM &= \frac{\rho g \times I}{\rho g \times V} = \frac{I}{V} \quad \dots(4.3)
 \end{aligned}$$

$$GM = BM - BG = \frac{I}{V} - BG$$

$$\therefore \text{Meta-centric height} = GM = \frac{I}{V} - BG. \quad \dots(4.4)$$

**Problem 4.7** A rectangular pontoon is 5 m long, 3 m wide and 1.20 m high. The depth of immersion of the pontoon is 0.80 m in sea water. If the centre of gravity is 0.6 m above the bottom of the pontoon, determine the meta-centric height. The density for sea water = 1025 kg/m<sup>3</sup>.

**Solution.** Given :

$$\begin{aligned}
 \text{Dimension of pontoon} &= 5 \text{ m} \times 3 \text{ m} \times 1.20 \text{ m} \\
 \text{Depth of immersion} &= 0.8 \text{ m}
 \end{aligned}$$

Distance	$AG = 0.6 \text{ m}$
Distance	$AB = \frac{1}{2} \times \text{Depth of immersion}$ $= \frac{1}{2} \times 0.8 = 0.4 \text{ m}$
Density for sea water	$= 1025 \text{ kg/m}^3$

Meta-centre height  $GM$ , given by equation (4.4) is

$$GM = \frac{I}{\nabla} - BG$$

where  $I$  = M.O. Inertia of the plan of the pontoon about  $Y-Y$  axis

$$= \frac{1}{12} \times 5 \times 3^3 \text{ m}^4 = \frac{45}{4} \text{ m}^4$$

$$\nabla = \text{Volume of the body submersed in water}$$

$$= 3 \times 0.8 \times 5.0 = 12.0 \text{ m}^3$$

$$BG = AG - AB = 0.6 - 0.4 = 0.2 \text{ m}$$

$$\therefore GM = \frac{45}{4} \times \frac{1}{12.0} - 0.2 = \frac{45}{48} - 0.2 = 0.9375 - 0.2 = \mathbf{0.7375 \text{ m. Ans.}}$$

**Problem 4.8** A uniform body of size  $3 \text{ m long} \times 2 \text{ m wide} \times 1 \text{ m deep}$  floats in water. What is the weight of the body if depth of immersion is  $0.8 \text{ m}$ ? Determine the meta-centric height also.

**Solution.** Given :

$$\text{Dimension of body} = 3 \times 2 \times 1$$

$$\text{Depth of immersion} = 0.8 \text{ m}$$

Find (i) Weight of body,  $W$

(ii) Meta-centric height,  $GM$

(i) **Weight of Body,  $W$**

= Weight of water displaced

=  $\rho g \times \text{Volume of water displaced}$

=  $1000 \times 9.81 \times \text{Volume of body in water}$

=  $1000 \times 9.81 \times 3 \times 2 \times 0.8 \text{ N}$

= **47088 N. Ans.**

(ii) **Meta-centric Height,  $GM$**

Using equation (4.4), we get

$$GM = \frac{I}{\nabla} - BG$$

where  $I$  = M.O.I about  $Y-Y$  axis of the plan of the body

$$= \frac{1}{12} \times 3 \times 2^3 = \frac{3 \times 2^3}{12} = 2.0 \text{ m}^4$$

$\nabla$  = Volume of body in water

$$= 3 \times 2 \times 0.8 = 4.8 \text{ m}^3$$

$$BG = AG - AB = \frac{1.0}{2} - \frac{0.8}{2} = 0.5 - 0.4 = 0.1$$

$$\therefore GM = \frac{2.0}{4.8} - 0.1 = 0.4167 - 0.1 = \mathbf{0.3167 \text{ m. Ans.}}$$

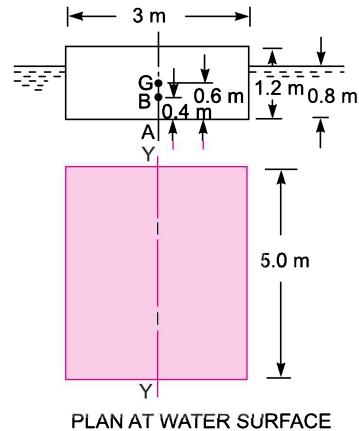


Fig. 4.7

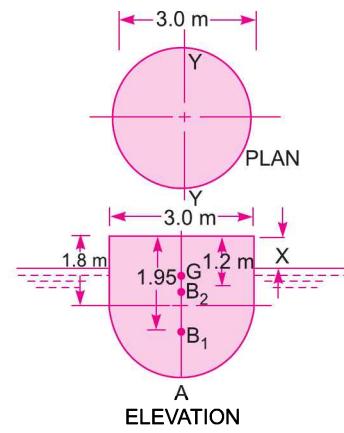


Fig. 4.8

**Problem 4.9** A block of wood of specific gravity 0.7 floats in water. Determine the meta-centric height of the block if its size is 2 m × 1 m × 0.8 m.

**Solution.** Given :

$$\text{Dimension of block} = 2 \times 1 \times 0.8$$

$$\text{Let depth of immersion} = h \text{ m}$$

$$\text{Sp. gr. of wood} = 0.7$$

$$\begin{aligned}\text{Weight of wooden piece} &= \text{Weight density of wood}^* \times \text{Volume} \\ &= 0.7 \times 1000 \times 9.81 \times 2 \times 1 \times 0.8 \text{ N}\end{aligned}$$

$$\begin{aligned}\text{Weight of water displaced} &= \text{Weight density of water} \\ &\times \text{Volume of the wood sub-merged in water} \\ &= 1000 \times 9.81 \times 2 \times 1 \times h \text{ N}\end{aligned}$$

For equilibrium,

$$\text{Weight of wooden piece} = \text{Weight of water displaced}$$

$$\therefore 700 \times 9.81 \times 2 \times 1 \times 0.8 = 1000 \times 9.81 \times 2 \times 1 \times h$$

$$\therefore h = \frac{700 \times 9.81 \times 2 \times 1 \times 0.8}{1000 \times 9.81 \times 2 \times 1} = 0.7 \times 0.8 = 0.56 \text{ m}$$

∴ Distance of centre of Buoyancy from bottom, i.e.,

$$AB = \frac{h}{2} = \frac{0.56}{2} = 0.28 \text{ m}$$

and

$$AG = 0.8/2.0 = 0.4 \text{ m}$$

$$\therefore BG = AG - AB = 0.4 - 0.28 = 0.12 \text{ m}$$

The meta-centric height is given by equation (4.4) or

$$GM = \frac{I}{\nabla} - BG$$

$$\text{where } I = \frac{1}{12} \times 2 \times 1.0^3 = \frac{1}{6} \text{ m}^4$$

$$\begin{aligned}\nabla &= \text{Volume of wood in water} \\ &= 2 \times 1 \times h = 2 \times 1 \times 0.56 = 1.12 \text{ m}^3\end{aligned}$$

$$\therefore GM = \frac{1}{6} \times \frac{1}{1.12} - 0.12 = 0.1488 - 0.12 = \mathbf{0.0288 \text{ m. Ans.}}$$

**Problem 4.10** A solid cylinder of diameter 4.0 m has a height of 3 metres. Find the meta-centric height of the cylinder when it is floating in water with its axis vertical. The sp. gr. of the cylinder = 0.6.

**Solution.** Given :

$$\text{Dia. of cylinder, } D = 4.0 \text{ m}$$

$$\text{Height of cylinder, } h = 3.0 \text{ m}$$

\* Weight density of wood  
=  $\rho \times g$ , where  $\rho$  = density of wood  
 $= 0.7 \times 1000 = 700 \text{ kg/m}^3$ . Hence  $w$  for wood =  $700 \times 9.81 \text{ N/m}^3$ .

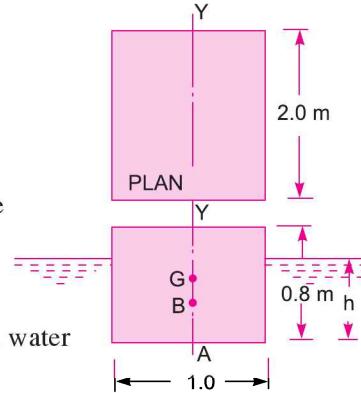


Fig. 4.9

$$\text{Sp. gr. of cylinder} = 0.6$$

$$\text{Depth of immersion of cylinder} = 0.6 \times 3.0 = 1.8 \text{ m}$$

$$\therefore AB = \frac{1.8}{2} = 0.9 \text{ m}$$

$$\text{and } AG = \frac{3}{2} = 1.5 \text{ m}$$

$$\therefore BG = AG - AB \\ = 1.5 - 0.9 = 0.6 \text{ m}$$

Now the meta-centric height  $GM$  is given by equation (4.4)

$$GM = \frac{I}{\nabla} - BG$$

But

$I = \text{M.O.I. about } Y-Y \text{ axis of the plan of the body}$

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64} \times (4.0)^4$$

and

$\nabla = \text{Volume of cylinder in water}$

$$= \frac{\pi}{4} D^2 \times \text{Depth of immersion}$$

$$= \frac{\pi}{4} (4)^2 \times 1.8 \text{ m}^3$$

$$\therefore GM = \frac{\frac{\pi}{64} \times (4.0)^4}{\frac{\pi}{4} \times (4.0)^2 \times 1.8} - 0.6 \\ = \frac{1}{16} \times \frac{4.0^2}{1.8} - 0.6 = \frac{1}{1.8} - 0.6 = 0.55 - 0.6 = - 0.05 \text{ m. Ans.}$$

– ve sign means that meta-centre, ( $M$ ) is below the centre of gravity ( $G$ ).

**Problem 4.11** A body has the cylindrical upper portion of 3 m diameter and 1.8 m deep. The lower portion is a curved one, which displaces a volume of  $0.6 \text{ m}^3$  of water. The centre of buoyancy of the curved portion is at a distance of 1.95 m below the top of the cylinder. The centre of gravity of the whole body is 1.20 m below the top of the cylinder. The total displacement of water is 3.9 tonnes. Find the meta-centric height of the body.

**Solution.** Given :

$$\text{Dia. of body} = 3.0 \text{ m}$$

$$\text{Depth of body} = 1.8 \text{ m}$$

Volume displaced by curved portion

$$= 0.6 \text{ m}^3 \text{ of water.}$$

Let  $B_1$  is the centre of buoyancy of the curved surface and  $G$  is the centre of gravity of the whole body.

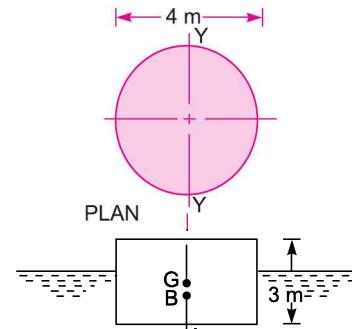


Fig. 4.10

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Then

$$CB_1 = 1.95 \text{ m}$$

$$CG = 1.20 \text{ m}$$

Total weight of water displaced by body = 3.9 tonnes

$$= 3.9 \times 1000 = 3900 \text{ kgf}$$

$$= 3900 \times 9.81 \text{ N} = 38259 \text{ N}$$

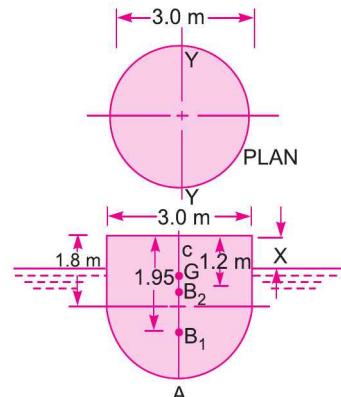
Find meta-centric height of the body.

Let the height of the body above the water surface  $x$  m. Total weight of water displaced by body

$$= \text{Weight density of water} \times [\text{Volume of water displaced}]$$

$$= 1000 \times 9.81 \times [\text{Volume of the body in water}]$$

$$= 9810 [\text{Volume of cylindrical part in water} + \text{Volume of curved portion}]$$



ELEVATION

Fig. 4.11

$$= 9810 \left[ \frac{\pi}{4} \times D^2 \times \text{Depth of cylindrical part in water} + \text{Volume displaced by curved portion} \right]$$

$$\text{or } 38259 = 9810 \left[ \frac{\pi}{4} (3)^2 \times (1.8 - x) + 0.6 \right]$$

$$\therefore \frac{\pi}{4} (3)^2 \times (1.8 - x) + 0.6 = \frac{38259}{9810} = 3.9$$

$$\therefore \frac{\pi}{4} \times 3^2 \times (1.8 - x) = 3.9 - 0.6 = 3.3$$

$$\text{or } 1.8 - x = \frac{3.3 \times 4}{\pi \times 3 \times 3} = 0.4668$$

$$\therefore x = 1.8 - 0.4668 = 1.33 \text{ m}$$

Let  $B_2$  is the centre of buoyancy of cylindrical part and  $B$  is the centre of buoyancy of the whole body.

Then depth of cylindrical part in water =  $1.8 - x = 0.467 \text{ m}$

$$\therefore CB_2 = x + \frac{.467}{2} = 1.33 + .2335 = 1.5635 \text{ m.}$$

The distance of the centre of buoyancy of the whole body from the top of the cylindrical part is given as

$$\begin{aligned} CB &= (\text{Volume of curved portion} \times CB_1 + \text{Volume of cylindrical part in water} \times CB_2) \\ &\quad \div (\text{Total volume of water displaced}) \\ &= \frac{0.6 \times 1.95 + 3.3 \times 1.5635}{(0.6 + 3.3)} = \frac{1.17 + 5.159}{3.9} = 1.623 \text{ m.} \end{aligned}$$

Then  $BG = CB - CG = 1.623 - 1.20 = .423 \text{ m.}$

Meta-centric height,  $GM$ , is given by

$$GM = \frac{I}{\nabla} - BG$$

where  $I = \text{M.O.I. of the plan of the body at water surface about } Y-Y \text{ axis}$

$$= \frac{\pi}{64} \times D^4 = \frac{\pi}{64} \times 3^4 \text{ m}^4$$

$\forall = \text{Volume of the body in water} = 3.9 \text{ m}^3$

$$\therefore GM = \frac{\pi}{64} \times \frac{3^4}{3.9} - .423 = 1.019 - .423 = \mathbf{0.596 \text{ m. Ans.}}$$

#### ► 4.7 CONDITIONS OF EQUILIBRIUM OF A FLOATING AND SUB-MERGED BODIES

A sub-merged or a floating body is said to be stable if it comes back to its original position after a slight disturbance. The relative position of the centre of gravity ( $G$ ) and centre of buoyancy ( $B_1$ ) of a body determines the stability of a sub-merged body.

**4.7.1 Stability of a Sub-merged Body.** The position of centre of gravity and centre of buoyancy in case of a completely sub-merged body are fixed. Consider a balloon, which is completely submerged in air. Let the lower portion of the balloon contains heavier material, so that its centre of gravity is lower than its centre of buoyancy as shown in Fig. 4.12 (a). Let the weight of the balloon is  $W$ . The weight  $W$  is acting through  $G$ , vertically in the downward direction, while the buoyant force  $F_B$  is acting vertically up, through  $B$ . For the equilibrium of the balloon  $W = F_B$ . If the balloon is given an angular displacement in the clockwise direction as shown in Fig. 4.12 (a), then  $W$  and  $F_B$  constitute a couple acting in the anti-clockwise direction and brings the balloon in the original position. Thus the balloon in the position, shown by Fig. 4.12 (a) is in stable equilibrium.

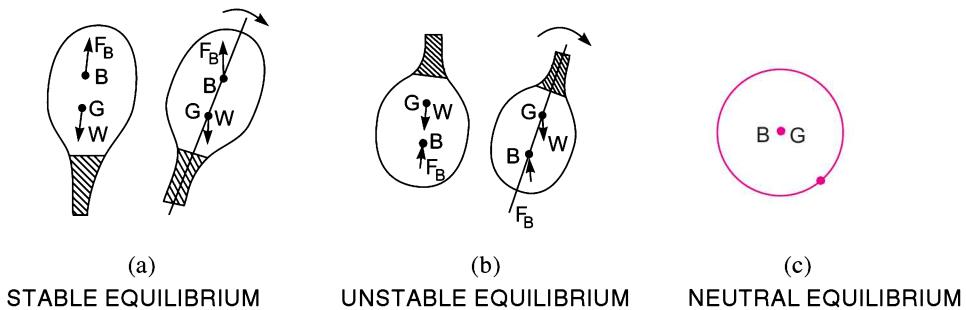


Fig. 4.12 *Stabilities of sub-merged bodies.*

(a) **Stable Equilibrium.** When  $W = F_B$  and point  $B$  is above  $G$ , the body is said to be in stable equilibrium.

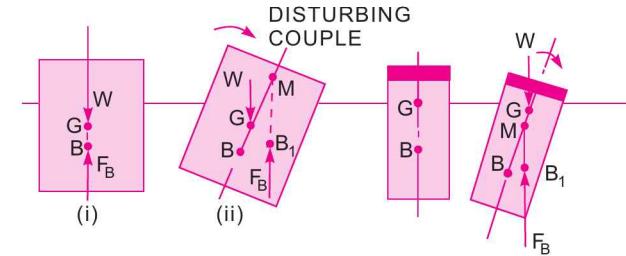
(b) **Unstable Equilibrium.** If  $W = F_B$ , but the centre of buoyancy ( $B$ ) is below centre of gravity ( $G$ ), the body is in unstable equilibrium as shown in Fig. 4.12 (b). A slight displacement to the body, in the clockwise direction, gives the couple due to  $W$  and  $F_B$  also in the clockwise direction. Thus the body does not return to its original position and hence the body is in unstable equilibrium.

(c) **Neutral Equilibrium.** If  $F_B = W$  and  $B$  and  $G$  are at the same point, as shown in Fig. 4.12 (c), the body is said to be in neutral equilibrium.

**4.7.2 Stability of Floating Body.** The stability of a floating body is determined from the position of Meta-centre ( $M$ ). In case of floating body, the weight of the body is equal to the weight of liquid displaced.

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(a) **Stable Equilibrium.** If the point  $M$  is above  $G$ , the floating body will be in stable equilibrium as shown in Fig. 4.13 (a). If a slight angular displacement is given to the floating body in the clockwise direction, the centre of buoyancy shifts from  $B$  to  $B_1$  such that the vertical line through  $B_1$  cuts at  $M$ . Then the buoyant force  $F_B$  through  $B_1$  and weight  $W$  through  $G$  constitute a couple acting in the anti-clockwise direction and thus bringing the floating body in the original position.



(a) Stable equilibrium  $M$  is above  $G$

(b) Unstable equilibrium  $M$  is below  $G$ .

Fig. 4.13 Stability of floating bodies.

(b) **Unstable Equilibrium.** If the point  $M$  is below  $G$ , the floating body will be in unstable equilibrium as shown in Fig. 4.13 (b). The disturbing couple is acting in the clockwise direction. The couple due to buoyant force  $F_B$  and  $W$  is also acting in the clockwise direction and thus overturning the floating body.

(c) **Neutral Equilibrium.** If the point  $M$  is at the centre of gravity of the body, the floating body will be in neutral equilibrium.

**Problem 4.12** A solid cylinder of diameter 4.0 m has a height of 4.0 m. Find the meta-centric height of the cylinder if the specific gravity of the material of cylinder = 0.6 and it is floating in water with its axis vertical. State whether the equilibrium is stable or unstable.

**Solution.** Given :

$$D = 4 \text{ m}$$

Height,

$$h = 4 \text{ m}$$

Sp. gr.

$$= 0.6$$

Depth of cylinder in water

$$= \text{Sp. gr.} \times h$$

$$= 0.6 \times 4.0 = 2.4 \text{ m}$$

$\therefore$  Distance of centre of buoyancy ( $B$ ) from  $A$

$$\text{or } AB = \frac{2.4}{2} = 1.2 \text{ m}$$

Distance of centre of gravity ( $G$ ) from  $A$

$$\text{or } AG = \frac{h}{2} = \frac{4.0}{2} = 2.0 \text{ m}$$

$$\therefore BG = AG - AB = 2.0 - 1.2 = 0.8 \text{ m}$$

Now the meta-centric height  $GM$  is given by

$$GM = \frac{I}{\nabla} - BG$$

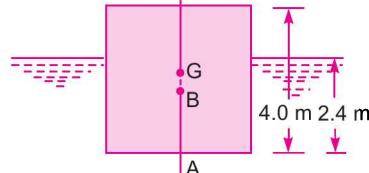
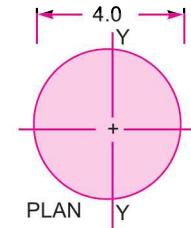


Fig. 4.14

where  $I = \text{M.O.I. of the plan of the body about } Y-Y \text{ axis}$

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64} \times (4.0)^4$$

$\nabla = \text{Volume of cylinder in water}$

$$= \frac{\pi}{4.0} \times D^2 \times \text{Depth of cylinder in water} = \frac{\pi}{4} \times 4^2 \times 2.4 \text{ m}^3$$

$$\therefore \frac{I}{\nabla} = \frac{\frac{\pi}{64} \times 4^4}{\frac{\pi}{4} \times 4^2 \times 2.4} = \frac{1}{16} \times \frac{4^2}{2.4} = \frac{1}{2.4} = 0.4167 \text{ m}$$

$$\therefore GM = \frac{I}{\nabla} - BG = 0.4167 - 0.8 = -0.3833 \text{ m. Ans.}$$

-ve sign means that the meta-centre ( $M$ ) is below the centre of gravity ( $G$ ). Thus the cylinder is in unstable equilibrium. Ans.

**Problem 4.13** A solid cylinder of 10 cm diameter and 40 cm long, consists of two parts made of different materials. The first part at the base is 1.0 cm long and of specific gravity = 6.0. The other part of the cylinder is made of the material having specific gravity 0.6. State, if it can float vertically in water.

**Solution.** Given :

$$D = 10 \text{ cm}$$

$$\text{Length, } L = 40 \text{ cm}$$

$$\text{Length of 1st part, } l_1 = 1.0 \text{ cm}$$

$$\text{Sp. gr., } S_1 = 6.0$$

$$\text{Density of 1st part, } \rho_1 = 6 \times 1000 = 6000 \text{ kg/m}^3$$

$$\text{Length of 2nd part, } l_2 = 40 - 1.0 = 39.0 \text{ cm}$$

$$\text{Sp. gr., } S_2 = 0.6$$

$$\text{Density of 2nd part, } \rho_2 = 0.6 \times 1000 = 600 \text{ kg/m}^3$$

The cylinder will float vertically in water if its meta-centric height  $GM$  is positive. To find meta-centric height, find the location of centre of gravity ( $G$ ) and centre of buoyancy ( $B$ ) of the combined solid cylinder. The distance of the centre of gravity of the solid cylinder from  $A$  is given as

$$AG = [(Weight \text{ of 1st part} \times \text{Distance of C.G. of 1st part from } A) + (\text{Weight of 2nd part of cylinder} \times \text{Distance of C.G. of 2nd part from } A)] \div [\text{Weight of 1st part} + \text{weight of 2nd part}]$$

$$= \frac{\left( \frac{\pi}{4} D^2 \times 1.0 \times 6.0 \times 0.5 \right) + \left( \frac{\pi}{4} D^2 \times 39.0 \times 0.6 \times (1.0 \times 39/2) \right)}{\left( \frac{\pi}{4} D^2 \times 1.0 \times 6.0 + \frac{\pi}{4} D^2 \times 39 \times 0.6 \right)}$$

$$= \frac{1.0 \times 6.0 \times 0.5 + 39.0 \times 0.6 \times (20.5)}{1.0 \times 6.0 + 39.0 \times 0.6}$$

$$\text{Cancel } \frac{\pi}{4} D^2 \text{ in the Numerator and Denominator} = \frac{3.0 + 479.7}{6.0 + 23.4} = \frac{482.7}{29.4} = 16.42.$$

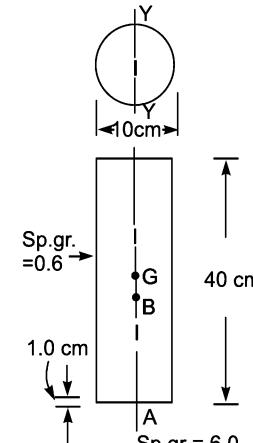


Fig. 4.15

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To find the centre of buoyancy of the combined two parts or of the cylinder, determine the depth of immersion of the cylinder. Let the depth of immersion of the cylinder is  $h$ . Then

Weight of the cylinder = Weight of water displaced

$$\frac{\pi}{4} \times (.1)^2 \times \frac{39.0}{100} \times 600 \times 9.81 + \frac{\pi}{4} (.1)^2 \times \frac{1.0}{100} \times 6000 \times 9.81 = \frac{\pi}{4} (.1)^2 \times \frac{h}{100} \times 1000 \times 9.81$$

[∴  $h$  is in cm]

or cancelling  $\frac{\pi}{4} (.1)^2 \times \frac{1000 \times 9.81}{100}$  throughout, we get

$$39.0 \times 0.6 + 1.0 \times 6.0 = h \quad \text{or} \quad h = 23.4 + 6.0 = 29.4$$

∴ The distance of the centre of the buoyancy  $B$ , of the cylinder from  $A$  is

$$AB = h/2 = \frac{29.4}{2} = 14.7$$

$$\therefore BG = AG - AB = 16.42 - 14.70 = 1.72 \text{ cm.}$$

Meta-centric height  $GM$  is given by

$$GM = \frac{I}{\nabla} - BG$$

where  $I$  = M.O.I. of plan of the body about  $Y-Y$  axis

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64} (10)^4 \text{ cm}^4$$

$\nabla$  = Volume of cylinder in water

$$= \frac{\pi}{4} D^2 \times h = \frac{\pi}{4} (10)^2 \times 29.4 \text{ m}^3$$

$$\therefore \frac{I}{\nabla} = \frac{\pi}{64} (10)^4 / \frac{\pi}{4} (10)^2 \times 29.4 = \frac{1}{16} \times \frac{10^2}{29.4} = \frac{100}{19 \times 29.4} = 0.212$$

$$\therefore GM = 0.212 - 1.72 = -1.508 \text{ cm}$$

As  $GM$  is - ve, it means that the Meta-centre  $M$  is below the centre of gravity ( $G$ ). Thus the cylinder is in unstable equilibrium and so it cannot float vertically in water. Ans.

**Problem 4.14** A rectangular pontoon 10.0 m long, 7 m broad and 2.5 m deep weighs 686.7 kN. It carries on its upper deck an empty boiler of 5.0 m diameter weighing 588.6 kN. The centre of gravity of the boiler and the pontoon are at their respective centres along a vertical line. Find the meta-centric height. Weight density of sea water is 10.104 kN/m<sup>3</sup>.

**Solution.** Given : Dimension of pontoon = 10 × 7 × 2.5

Weight of pontoon,  $W_1 = 686.7 \text{ kN}$

Dia. of boiler,  $D = 5.0 \text{ m}$

Weight of boiler,  $W_2 = 588.6 \text{ kN}$

$w$  for sea water  $= 10.104 \text{ kN/m}^3$

To find the meta-centric height, first determine the common centre of gravity  $G$  and common centre of buoyancy  $B$  of the boiler and pontoon. Let  $G_1$  and  $G_2$  are the centre of gravities of pontoon and boiler respectively. Then

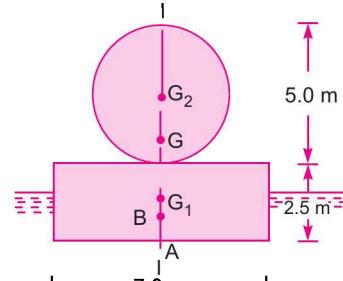


Fig. 4.16

$$AG_1 = \frac{2.5}{2} = 1.25 \text{ m}$$

$$AG_2 = 2.5 + \frac{5.0}{2} = 2.5 + 2.5 = 5.0 \text{ m}$$

The distance of common centre of gravity  $G$  from  $A$  is given as

$$\begin{aligned} AG &= \frac{W_1 \times AG_1 + W_2 \times AG_2}{W_1 + W_2} \\ &= \frac{686.7 \times 1.25 + 588.6 \times 5.0}{(686.7 + 588.6)} = 2.98 \text{ m.} \end{aligned}$$

Let  $h$  is the depth of immersion. Then

$$\begin{aligned} \text{Total weight of pontoon and boiler} &= \text{Weight of sea water displaced} \\ \text{or } (686.7 + 588.6) &= w \times \text{Volume of the pontoon in water} \\ &= 10.104 \times L \times b \times \text{Depth of immersion} \\ \therefore 1275.3 &= 10.104 \times 10 \times 7 \times h \end{aligned}$$

$$h = \frac{1275.3}{10 \times 7 \times 10.104} = 1.803 \text{ m}$$

$\therefore$  The distance of the common centre of buoyancy  $B$  from  $A$  is

$$AB = \frac{h}{2} = \frac{1.803}{2} = .9015 \text{ m}$$

$$\therefore BG = AG - AB = 2.98 - .9015 = 2.0785 \text{ m} \approx 2.078 \text{ m}$$

Meta-centric height is given by  $GM = \frac{I}{\nabla} - BG$

where  $I$  = M.O.I. of the plan of the body at the water level along  $Y-Y$

$$= \frac{1}{12} \times 10.0 \times 7^3 = \frac{10 \times 49 \times 7}{12} \text{ m}^4$$

$\nabla$  = Volume of the body in water

$$= L \times b \times h = 10.0 \times 7 \times 1.857$$

$$\therefore \frac{I}{\nabla} = \frac{10 \times 49 \times 7}{12 \times 10 \times 7 \times 1.857} = \frac{49}{12 \times 1.857} = 2.198 \text{ m}$$

$$\therefore GM = \frac{I}{\nabla} - BG = 2.198 - 2.078 = 0.12 \text{ m.}$$

$\therefore$  Meta-centric height of both the pontoon and boiler = **0.12 m. Ans.**

**Problem 4.15** A wooden cylinder of sp. gr. = 0.6 and circular in cross-section is required to float in oil (sp. gr. = 0.90). Find the L/D ratio for the cylinder to float with its longitudinal axis vertical in oil, where  $L$  is the height of cylinder and  $D$  is its diameter.

**Solution.** Given :

$$\text{Dia. of cylinder} = D$$

$$\text{Height of cylinder} = L$$

$$\text{Sp. gr. of cylinder, } S_1 = 0.6$$

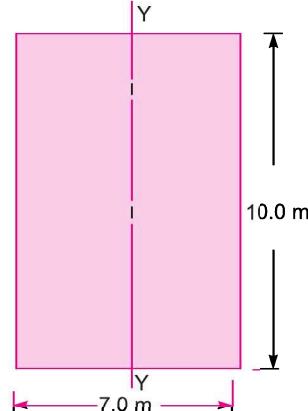


Fig. 4.17 Plan of the body at water-line

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Sp. gr. of oil  $S_2 = 0.9$

Let the depth of cylinder immersed in oil =  $h$

For the principle of buoyancy

Weight of cylinder = wt. of oil displaced

$$\frac{\pi}{4} D^2 \times L \times 0.6 \times 1000 \times 9.81 = \frac{\pi}{4} D^2 \times h \times 0.9 \times 1000 \times 9.81$$

or

$$L \times 0.6 = h \times 0.9$$

$$\therefore h = \frac{0.6 \times L}{0.9} = \frac{2}{3} L.$$

The distance of centre of gravity  $G$  from  $A$ ,  $AG = \frac{L}{2}$

The distance of centre of buoyancy  $B$  from  $A$ ,

$$AB = \frac{h}{2} = \frac{1}{2} \left[ \frac{2}{3} L \right] = \frac{L}{3}$$

$$\therefore BG = AG - AB = \frac{L}{2} - \frac{L}{3} = \frac{3L - 2L}{6} = \frac{L}{6}$$

The meta-centric height  $GM$  is given by

$$GM = \frac{I}{\nabla} - BG$$

where  $I = \frac{\pi}{64} D^4$  and  $\nabla = \text{Volume of cylinder in oil} = \frac{\pi}{4} D^2 \times h$

$$\therefore \frac{I}{\nabla} = \left( \frac{\pi}{64} D^4 \Big/ \frac{\pi}{4} D^2 h \right) = \frac{1}{16} \frac{D^2}{h} = \frac{D^2}{16 \times \frac{2}{3} L} = \frac{3D^2}{32L} \quad \left\{ \because h = \frac{2}{3} L \right\}$$

$$\therefore GM = \frac{3D^2}{32L} - \frac{L}{6}.$$

For stable equilibrium,  $GM$  should be +ve or

$$GM > 0 \quad \text{or} \quad \frac{3D^2}{32L} - \frac{L}{6} > 0$$

$$\text{or} \quad \frac{3D^2}{32L} > \frac{L}{6} \quad \text{or} \quad \frac{3 \times 6}{32} > \frac{L^2}{D^2}$$

$$\text{or} \quad \frac{L^2}{D^2} < \frac{18}{32} \quad \text{or} \quad \frac{9}{16}$$

$$\therefore \frac{L}{D} < \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\therefore L/D < 3/4. \text{ Ans.}$$

**Problem 4.16** Show that a cylindrical buoy of 1 m diameter and 2.0 m height weighing 7.848 kN will not float vertically in sea water of density  $1030 \text{ kg/m}^3$ . Find the force necessary in a vertical chain attached at the centre of base of the buoy that will keep it vertical.

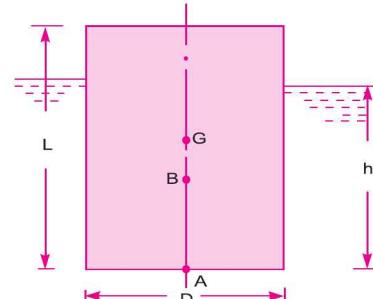


Fig. 4.18

**Solution.** Given : Dia. of buoy,  $D = 1 \text{ m}$

$$\text{Height, } H = 2.0 \text{ m}$$

$$\begin{aligned} \text{Weight, } W &= 7.848 \text{ kN} \\ &= 7.848 \times 1000 = 7848 \text{ N} \end{aligned}$$

$$\text{Density, } \rho = 1030 \text{ kg/m}^3$$

(i) Show the cylinder will not float vertically.

(ii) Find the force in the chain.

**Part I.** The cylinder will not float if meta-centric height is -ve.

Let the depth of immersion be  $h$

Then for equilibrium, Weight of cylinder

= Weight of water displaced

= Density  $\times g \times$  Volume of cylinder in water

$$\therefore 7848 = 1030 \times 9.81 \times \frac{\pi}{4} D^2 \times h$$

$$= 10104.3 \times \frac{\pi}{4} (1)^2 \times h$$

$$\therefore h = \frac{4 \times 7848}{10104.3 \times \pi} = 0.989 \text{ m.}$$

$\therefore$  The distance of centre of buoyancy  $B$  from  $A$ ,

$$AB = \frac{h}{2} = \frac{0.989}{2} = 0.494 \text{ m.}$$

And the distance of centre of gravity  $G$ , from  $A$  is  $AG = \frac{2.0}{2} = 1.0 \text{ m}$

$$\therefore BG = AG - AB = 1.0 - .494 = .506 \text{ m.}$$

Now meta-centric height  $GM$  is given by  $GM = \frac{I}{\nabla} - BG$

$$\text{where } I = \frac{\pi}{64} D^4 = \frac{\pi}{64} \times (1)^4 \text{ m}^4$$

$$\text{and } \nabla = \text{Volume of cylinder in water} = \frac{\pi}{4} D^2 \times h = \frac{\pi}{4} 1^2 \times .989$$

$$\begin{aligned} \therefore \frac{I}{\nabla} &= \frac{\frac{\pi}{64} \times 1^4}{\frac{\pi}{4} D^2 \times h} = \frac{\frac{\pi}{64} \times 1^4}{\frac{\pi}{4} \times 1^2 \times .989} \\ &= \frac{1}{16} \times 1^2 \times \frac{1}{.989} = \frac{1}{16 \times .989} = 0.063 \text{ m} \end{aligned}$$

$$\therefore GM = .063 - .506 = -0.443 \text{ m. Ans.}$$

As the meta-centric height is -ve, the point  $M$  lies below  $G$  and hence the cylinder will be in unstable equilibrium and hence cylinder will not float vertically.

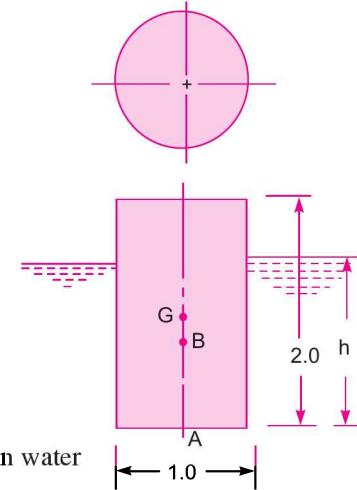


Fig. 4.19

**Part II.** Let the force applied in a vertical chain attached at the centre of the base of the buoy is  $T$  to keep the buoy vertical.

Now find the combined position of centre of gravity ( $G'$ ) and centre of buoyancy ( $B'$ ). For the combined centre of buoyancy, let  $h' = \text{depth of immersion when the force } T \text{ is applied}$ . Then

Total downward force = Weight of water displaced  
or  $(7848 + T) = \text{Density of water} \times g \times \text{Volume of cylinder in water}$

$$\begin{aligned} &= 1030 \times 9.81 \times \frac{\pi}{4} D' \times h' \quad [\text{where } h' = \text{depth of immersion}] \\ \therefore h' &= \frac{7848 + T}{10104.3 \times \frac{\pi}{4} \times D^2} = \frac{7848 + T}{10104.3 \times \frac{\pi}{4} \times 1^2} = \frac{10104.3 + T}{7935.9} \text{ m} \\ \therefore AB' &= \frac{h'}{2} = \frac{1}{2} \left[ \frac{7848 + T}{7935.9} \right] = \frac{7848 + T}{15871.8} \text{ m.} \end{aligned}$$

The combined centre of gravity ( $G'$ ) due to weight of cylinder and due to tension  $T$  in the chain from  $A$  is

$$\begin{aligned} AG' &= [\text{Wt. of cylinder} \times \text{Distance of C.G. of cylinder from } A \\ &\quad + T \times \text{Distance of C.G. of } T \text{ from } A] + [\text{Weight of cylinder} + T] \\ &= \left( 7848 \times \frac{2}{2} + T \times 0 \right) + [7848 + T] = \frac{7848}{7848 + T} \text{ m} \\ \therefore B'G' &= AG' - AB' = \frac{7848}{(7848 + T)} - \frac{(7848 + T)}{15871.8} \end{aligned}$$

The meta-centric height  $GM$  is given by  $GM = \frac{I}{\forall} - B'G'$

$$\text{where } I = \frac{\pi}{64} \times D^4 = \frac{\pi}{64} \times 1^4 = \frac{\pi}{64} \text{ m}^4$$

$$\text{and } \forall = \frac{\pi}{4} D^2 \times h' = \frac{\pi}{4} \times 1^2 \times \frac{(7848 + T)}{7935.9} = \frac{\pi}{4} \times \frac{7848 + T}{7935.9}$$

$$\begin{aligned} \therefore \frac{I}{\forall} &= \frac{\frac{\pi}{64}}{\frac{\pi}{4} \frac{(7848 + T)}{7935.9}} = \frac{1}{16} \times \frac{7935.9}{(7848 + T)} \\ \therefore GM &= \frac{7935.9}{16(7848 + T)} - \left[ \frac{7848}{(7848 + T)} - \frac{(7848 + T)}{15871.8} \right] \end{aligned}$$

For stable equilibrium  $GM$  should be positive

$$\text{or } GM > 0$$

$$\text{or } \frac{7935.9}{16(7848 + T)} - \left[ \frac{7848}{(7848 + T)} - \frac{(7848 + T)}{15871.8} \right] > 0$$

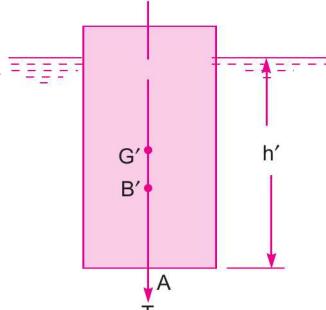


Fig. 4.20

$$\text{or } \frac{7935.9}{16(7848+T)} - \frac{7848}{(7848+T)} + \frac{7848+T}{15871.8} > 0$$

$$\text{or } \frac{7935.9 - 16 \times 7848}{16(7848+T)} + \frac{(7848+T)}{15871.8} > 0$$

$$\text{or } \frac{-117632}{16(7848+T)} + \frac{(7848+T)}{15871.8} > 0$$

$$\text{or } \frac{(7848+T)}{15871.8} > \frac{117632}{16(7848+T)}$$

$$\begin{aligned} \text{or } (7848+T)^2 &> \frac{117632}{16.0} \times 15871.8 \\ &> 116689473.5 \\ &> (10802.3)^2 \end{aligned}$$

$$\therefore 7848 + T > 10802.3$$

$$\therefore T > 10802.3 - 7848$$

**> 2954.3 N. Ans.**

∴ The force in the chain must be at least 2954.3 N so that the cylindrical buoy can be kept in vertical position. **Ans.**

**Problem 4.17** A solid cone floats in water with its apex downwards. Determine the least apex angle of cone for stable equilibrium. The specific gravity of the material of the cone is given 0.8.

**Solution.** Given :

$$\text{Sp. gr. of cone} = 0.8$$

$$\text{Density of cone, } \rho = 0.8 \times 1000 = 800 \text{ kg/m}^3$$

Let

$D$  = Dia. of the cone

$d$  = Dia. of cone at water level

$2\theta$  = Apex angle of cone

$H$  = Height of cone

$h$  = Depth of cone in water

$G$  = Centre of gravity of the cone

$B$  = Centre of buoyancy of the cone

For the cone, the distance of centre of gravity from the apex  $A$  is

$$AC = \frac{3}{4} \text{ height of cone} = \frac{3}{4} H$$

$$\text{also } AB = \frac{3}{4} \text{ depth of cone in water} = \frac{3}{4} h$$

$$\text{Volume of water displaced} = \frac{1}{3} \pi r^2 \times h$$

$$\text{Volume of cone} = \frac{1}{3} \times \pi R^2 \times h$$

$$\therefore \text{Weight of cone} = 800 \times g \times \frac{1}{3} \times \pi R^2 \times H$$

$$\text{Now from } \Delta AEF, \tan \theta = \frac{EF}{EA} = \frac{R}{H}$$

$$\therefore R = H \tan \theta$$

$$\text{Similarly, } r = h \tan \theta$$

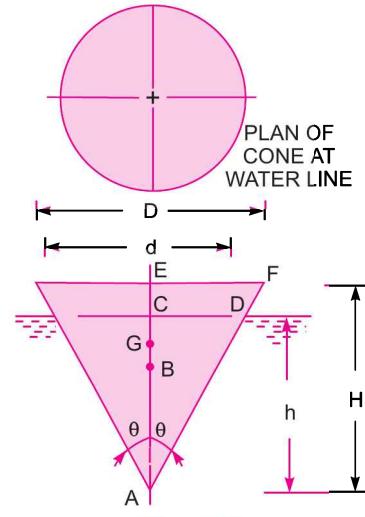


Fig. 4.21

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$$\therefore \text{Weight of cone} = 800 \times g \times \frac{1}{3} \times \pi \times (H \tan \theta)^2 \times H = \frac{800 \times g \times \pi \times H^3 \tan^2 \theta}{3}$$

$$\therefore \text{Weight of water displaced} = 1000 \times g \times \frac{1}{3} \times \pi r^2 \times h$$

$$= 1000 \times g \times \frac{1}{3} \times \pi (h \tan \theta)^2 \times h = \frac{1000 \times g \times \pi \times h^3 \tan^2 \theta}{3.0}$$

For equilibrium

$$\text{Weight of cone} = \text{Weight of water displaced}$$

$$\text{or } \frac{800 \times g \times \pi \times H^3 \tan^2 \theta}{3.0} = \frac{1000 \times 9.81 \times \pi \times h^3 \times \tan^2 \theta}{3.0}$$

$$\text{or } 800 \times H^3 = 1000 \times h^3$$

$$\therefore H^3 = \frac{1000}{800} \times h^3 \text{ or } \frac{H}{h} = \left( \frac{1000}{800} \right)^{1/3}$$

For stable equilibrium, Meta-centric height  $GM$  should be positive. But  $GM$  is given by

$$GM = \frac{I}{\nabla} - BG$$

$$\text{where } I = \text{M.O.I. of cone at water-line} = \frac{\pi}{64} d^4$$

$$\nabla = \text{Volume of cone in water} = \frac{1}{3} \frac{\pi}{4} d^2 \times h$$

$$\begin{aligned} \therefore \frac{I}{\nabla} &= \frac{\pi}{64} d^4 / \left( \frac{1}{3} \times \frac{\pi}{4} d^2 \times h \right) \\ &= \frac{1 \times 3}{16} \times \frac{d^2}{h} = \frac{3d^2}{16h} = \frac{3}{16h} \times (2r)^2 = \frac{3}{4} \frac{r^2}{h} \\ &= \frac{3}{4} \frac{(h \tan \theta)^2}{h} \\ &= \frac{3}{4} h \tan^2 \theta \end{aligned} \quad \{ \because r = h \tan \theta \}$$

and

$$BG = AG - AB = \frac{3}{4} H - \frac{3}{4} h = \frac{3}{4} (H - h)$$

$$\therefore GM = \frac{3}{4} h \tan^2 \theta - \frac{3}{4} (H - h)$$

For stable equilibrium  $GM$  should be positive or

$$\frac{3}{4} h \tan^2 \theta - \frac{3}{4} (H - h) > 0 \quad \text{or} \quad h \tan^2 \theta - (H - h) > 0$$

$$\text{or} \quad h \tan^2 \theta > (H - h) \quad \text{or} \quad h \tan^2 \theta + h > H$$

$$\text{or} \quad h[\tan^2 \theta + 1] > H \quad \text{or} \quad 1 + \tan^2 \theta > H/h \quad \text{or} \quad \sec^2 \theta > \frac{H}{h}$$

$$\text{But} \quad \frac{H}{h} = \left( \frac{1000}{800} \right)^{1/3} = 1.077$$

$$\therefore \sec^2 \theta > 1.077 \quad \text{or} \quad \cos^2 \theta > \frac{1}{1.077} = 0.9285$$

$$\therefore \cos \theta > 0.9635$$

$$\therefore \theta > 15^\circ 30' \quad \text{or} \quad 2\theta > 31^\circ$$

$\therefore$  Apex angle ( $2\theta$ ) should be at least  $31^\circ$ . Ans.

**Problem 4.18** A cone of specific gravity  $S$ , is floating in water with its apex downwards. It has a diameter  $D$  and vertical height  $H$ . Show that for stable equilibrium of the cone  $H < \frac{1}{2} \left[ \frac{D^2 \cdot S^{1/3}}{2 - S^{1/3}} \right]^{1/2}$ .

**Solution.** Given :

Dia. of cone =  $D$

Height of cone =  $H$

Sp. gr. of cone =  $S$

Let  $G$  = Centre of gravity of cone

$B$  = Centre of buoyancy

$2\theta$  = Apex angle

$A$  = Apex of the cone

$h$  = Depth of immersion

$d$  = Dia. of cone at water surface

Then

$$AG = \frac{3}{4} H$$

$$AB = \frac{3}{4} h$$

Also weight of cone = Weight of water displaced.

$$1000 S \times g \times \frac{1}{3} \pi R^2 \times H = 1000 \times g \times \frac{1}{3} \pi r^2 \times h \quad \text{or} \quad SR^2 H = r^2 h$$

$$\therefore h = \frac{SR^2 H}{r^2}$$

$$\text{But} \quad \tan \theta = \frac{R}{H} = \frac{r}{h}$$

$$\therefore R = H \tan \theta, r = h \tan \theta$$

$$\therefore h = \frac{S \times (H \tan \theta)^2 \times H}{(h \tan \theta)^2}$$

$$h = \frac{S \times H^2 \times \tan^2 \theta \times H}{h^2 \tan^2 \theta} = \frac{SH^3}{h^2} \quad \text{or} \quad h^3 = SH^3$$

$$\text{or} \quad h = (SH^3)^{1/3} = S^{1/3} H \quad \dots(1)$$

Distance,

$$BG = AG - AB$$

$$\begin{aligned} &= \frac{3}{4} H - \frac{3}{4} h = \frac{3}{4} (H - h) = \frac{3}{4} (H - S^{1/3} H) \quad \{\because h = S^{1/3} H\} \\ &= \frac{3}{4} H [1 - S^{1/3}] \end{aligned} \quad \dots(2)$$

Also

$I$  = M.O. Inertia of the plan of body at water surface

$$= \frac{\pi}{64} d^4$$

$$\forall = \text{Volume of cone in water} = \frac{1}{3} \times \frac{\pi}{4} \times d^2 \times h = \frac{1}{3} \frac{\pi}{4} d^2 [H \cdot S^{1/3}]$$

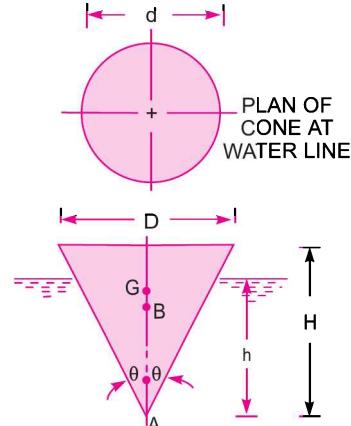


Fig. 4.22

$$\therefore \frac{I}{\nabla} = \frac{\frac{\pi}{64} d^4}{\frac{1}{3} \times \frac{\pi}{4} d^2 H \cdot S^{1/3}} = \frac{3d^2}{16 \cdot H \cdot S^{1/3}}$$

Now Meta-centric height  $GM$  is given as

$$GM = \frac{I}{\nabla} - BG = \frac{3d^2}{16 \cdot H \cdot S^{1/3}} - \frac{3H}{4} [1 - S^{1/3}]$$

$GM$  should be +ve for stable equilibrium or  $GM > 0$

$$\text{or } \frac{3d^2}{16 \cdot H \cdot S^{1/3}} - \frac{3H}{4} (1 - S^{1/3}) > 0$$

$$\text{or } \frac{3d^2}{16 \cdot H \cdot S^{1/3}} > \frac{3H}{4} (1 - S^{1/3}) \quad \dots(3)$$

Also we know  $R = H \tan \theta$  and  $r = h \tan \theta$

$$\therefore \frac{R}{r} = \frac{H}{h} = \frac{D}{d}$$

$$\therefore d = \frac{Dh}{H} = \frac{D}{H} \times HS^{1/3} = DS^{1/3}$$

Substituting the value of  $d$  in equation (3), we get

$$\frac{3(DS^{1/3})^2}{16 \cdot H \cdot S^{1/3}} > \frac{3H}{4} (1 - S^{1/3}) \quad \text{or} \quad \frac{D^2 \cdot S^{1/3}}{4 \cdot H} > H (1 - S^{1/3})$$

$$\text{or } \frac{D^2 \cdot S^{1/3}}{4(1 - S^{1/2})} > H^2 \quad \text{or} \quad H^2 < \frac{D^2 \cdot S^{1/3}}{4(1 - S^{1/3})}$$

$$\text{or } H < \frac{1}{2} \left[ \frac{D^2 \cdot S^{1/3}}{1 - S^{1/3}} \right]^{1/2}. \text{ Ans.}$$

#### ► 4.8 EXPERIMENTAL METHOD OF DETERMINATION OF META-CENTRIC HEIGHT

The meta-centric height of a floating vessel can be determined, provided we know the centre of gravity of the floating vessel. Let  $w_1$  is a known weight placed over the centre of the vessel as shown in Fig. 4.23 (a) and the vessel is floating.

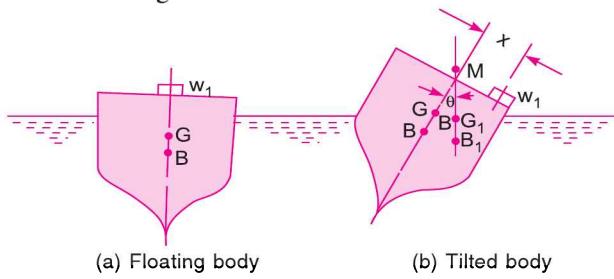


Fig. 4.23 Meta-centric height.

Let  $W$  = Weight of vessel including  $w_1$

$G$  = Centre of gravity of the vessel

$B$  = Centre of buoyancy of the vessel

The weight  $w_1$  is moved across the vessel towards right through a distance  $x$  as shown in Fig. 4.23 (b). The vessel will be tilted. The angle of heel  $\theta$  is measured by means of a plumbline and a protractor attached on the vessel. The new centre of gravity of the vessel will shift to  $G_1$  as the weight  $w_1$  has been moved towards the right. Also the centre of buoyancy will change to  $B_1$  as the vessel has tilted. Under equilibrium, the moment caused by the movement of the load  $w_1$  through a distance  $x$  must be equal to the moment caused by the shift of the centre of gravity from  $G$  to  $G_1$ . Thus

$$\text{The moment due to change of } G = GG_1 \times W = W \times GM \tan \theta$$

$$\text{The moment due to movement of } w_1 = w_1 \times x$$

$$\therefore w_1x = WGM \tan \theta$$

$$\text{Hence } GM = \frac{w_1x}{W \tan \theta} \quad \dots(4.5)$$

**Problem 4.19** A ship 70 m long and 10 m broad has a displacement of 19620 kN. A weight of 343.35 kN is moved across the deck through a distance of 6 m. The ship is tilted through  $6^\circ$ . The moment of inertia of the ship at water-line about its fore and aft axis is 75% of M.O.I. of the circumscribing rectangle. The centre of buoyancy is 2.25 m below water-line. Find the meta-centric height and position of centre of gravity of ship. Specific weight of sea water is  $10104 \text{ N/m}^3$ .

**Solution.** Given :

$$\text{Length of ship, } L = 70 \text{ m}$$

$$\text{Breadth of ship, } b = 10 \text{ m}$$

$$\text{Displacement, } W = 19620 \text{ kN}$$

$$\text{Angle of heel, } \theta = 6^\circ$$

$$\text{M.O.I. of ship at water-line} = 75\% \text{ of M.O.I. of circumscribing rectangle}$$

$$w \text{ for sea-water} = 10104 \text{ N/m}^3 = 10.104 \text{ kN/m}^3$$

$$\text{Movable weight, } w_1 = 343.35 \text{ kN}$$

$$\text{Distance moved by } w_1, x = 6 \text{ m}$$

$$\text{Centre of buoyancy} = 2.25 \text{ m below water surface}$$

Find (i) Meta-centric height,  $GM$

(ii) Position of centre of gravity,  $G$ .

(i) **Meta-centric height,  $GM$**  is given by equation (4.5)

$$\begin{aligned} \therefore GM &= \frac{w_1x}{W \tan \theta} = \frac{343.35 \text{ kN} \times 6.0}{19620 \text{ kN} \times \tan 6^\circ} \\ &= \frac{343.35 \text{ kN} \times 6.0}{19620 \text{ kN} \times .1051} = 0.999 \text{ m. Ans.} \end{aligned}$$

(ii) **Position of Centre of Gravity,  $G$**

$$GM = \frac{I}{\nabla} - BG$$

where  $I$  = M.O.I. of the ship at water-line about  $Y-Y$

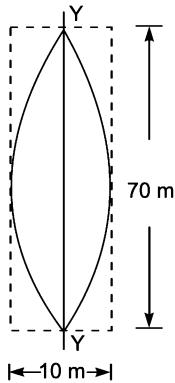


Fig. 4.24

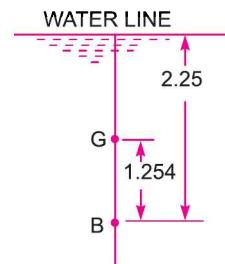


Fig. 4.25

$$= 75\% \text{ of } \frac{1}{12} \times 70 \times 10^3 = .75 \times \frac{1}{12} \times 70 \times 10^3 = 4375 \text{ m}^4$$

and  $\forall = \text{Volume of ship in water} = \frac{\text{Weight of ship}}{\text{Weight density of water}} = \frac{19620}{10.104} = 1941.74 \text{ m}^3$

$$\therefore \frac{I}{\forall} = \frac{4375}{1941.74} = 2.253 \text{ m}$$

$$\therefore GM = 2.253 - BG \text{ or } .999 = 2.253 - BG$$

$$\therefore BG = 2.253 - .999 = 1.254 \text{ m.}$$

From Fig. 4.25, it is clear that the distance of  $G$  from free surface of the water = distance of  $B$  from water surface -  $BG$

$$= 2.25 - 1.254 = 0.996 \text{ m. Ans.}$$

**Problem 4.20** A pontoon of 15696 kN displacement is floating in water. A weight of 245.25 kN is moved through a distance of 8 m across the deck of pontoon, which tilts the pontoon through an angle  $4^\circ$ . Find meta-centric height of the pontoon.

**Solution.** Given :

$$\text{Weight of pontoon} = \text{Displacement}$$

or  $W = 15696 \text{ kN}$

Movable weight,  $w_1 = 245.25 \text{ kN}$

Distance moved by weight  $w_1$ ,  $x = 8 \text{ m}$

Angle of heel,  $\theta = 4^\circ$

The meta-centric height,  $GM$  is given by equation (4.5)

$$\begin{aligned} \text{or } GM &= \frac{w_1 x}{W \tan \theta} = \frac{245.25 \text{ kN} \times 8}{15696 \text{ kN} \times \tan 4^\circ} \\ &= \frac{1962}{15696 \times 0.0699} = 1.788 \text{ m. Ans.} \end{aligned}$$

#### ► 4.9 OSCILLATION (ROLLING) OF A FLOATING BODY

Consider a floating body, which is tilted through an angle by an overturning couple as shown in Fig. 4.26. Let the overturning couple is suddenly removed. The body will start oscillating. Thus, the

body will be in a state of oscillation as if suspended at the meta-centre  $M$ . This is similar to the case of a pendulum. The only force acting on the body is due to the restoring couple due to the weight  $W$  of the body force of buoyancy  $F_B$ .

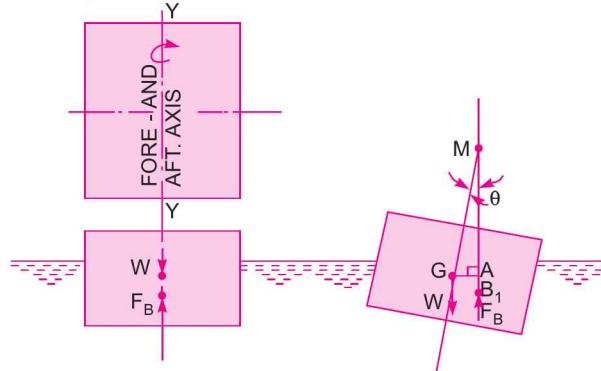


Fig. 4.26

$$\therefore \text{Restoring couple} = W \times \text{Distance } GA \\ = W \times GM \sin \theta \quad \dots(i)$$

This couple tries to decrease the angle

$$\text{Angular acceleration of the body, } \alpha = -\frac{d^2\theta}{dt^2}.$$

-ve sign has been introduced as the restoring couple tries to decrease the angle  $\theta$ .

$$\text{Torque due to inertia} = \text{Moment of Inertia about } Y-Y \times \text{Angular acceleration}$$

$$= I_{Y-Y} \times \left( -\frac{d^2\theta}{dt^2} \right)$$

$$\text{But} \quad I_{Y-Y} = \frac{W}{g} K^2$$

where  $W$  = Weight of body,  $K$  = Radius of gyration about  $Y-Y$

$$\therefore \text{Inertia torque} = \frac{W}{g} K^2 \left( -\frac{d^2\theta}{dt^2} \right) = -\frac{W}{g} K^2 \frac{d^2\theta}{dt^2} \quad \dots(ii)$$

Equating (i) and (ii), we get

$$W \times GM \sin \theta = -\frac{W}{g} K^2 \frac{d^2\theta}{dt^2} \quad \text{or} \quad GM \sin \theta = -\frac{K^2}{g} \frac{d^2\theta}{dt^2}$$

$$\text{For small angle } \theta, \quad \sin \theta \approx \theta$$

$$\therefore GM \times \theta = -\frac{K^2}{g} \frac{d^2\theta}{dt^2} \quad \text{or} \quad \frac{K^2}{g} \frac{d^2\theta}{dt^2} + GM \times \theta = 0$$

$$\text{Dividing by } \frac{K^2}{g}, \text{ we get } \frac{d^2\theta}{dt^2} + \frac{GM \times g \times \theta}{K^2} = 0$$

The above equation is a differential equation of second degree. The solution is

$$\theta = C_1 \sin \sqrt{\frac{GM \cdot g}{K^2}} \times t + C_2 \cos \sqrt{\frac{GM \cdot g}{K^2}} \times t \quad \dots(iii)$$

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where  $C_1$  and  $C_2$  are constants of integration.

The values of  $C_1$  and  $C_2$  are obtained from boundary conditions which are

(i) at  $t = 0, \theta = 0$

(ii) at  $t = \frac{T}{2}, \theta = 0$

where  $T$  is the time period of one complete oscillation.

Substituting the 1st boundary condition in (iii), we get

$$0 = C_1 \times 0 + C_2 \times 1 \quad \{ \because \sin \theta = 0, \cos \theta = 1 \}$$

$$\therefore C_2 = 0$$

Substituting 2nd boundary conditions in (iii), we get

$$0 = C_1 \sin \sqrt{\frac{GM.g}{K^2}} \times \frac{T}{2}$$

But  $C_1$  cannot be equal to zero and so the other alternative is

$$\sin \sqrt{\frac{GM.g}{K^2}} \times \frac{T}{2} = 0 = \sin \pi \quad \{ \because \sin \pi = 0 \}$$

$$\therefore \sqrt{\frac{GM.g}{K^2}} \times \frac{T}{2} = \pi \quad \text{or} \quad T = 2\pi \sqrt{\frac{K^2}{GM.g}} \quad \dots(4.6)$$

$\therefore$  Time period of oscillation is given by equation (4.6).

**Problem 4.21** The least radius of gyration of a ship is 8 m and meta-centric height 70 cm. Calculate the time period of oscillation of the ship.

**Solution.** Given :

Least radius of gyration,  $K = 8$  m

Meta-centric height,  $GM = 70$  cm = 0.70 m

The time period of oscillation is given by equation (4.6).

$$T = 2\pi \sqrt{\frac{K^2}{GM.g}} = 2\pi \sqrt{\frac{8 \times 8}{0.7 \times 9.81}} = 19.18 \text{ sec. Ans.}$$

**Problem 4.22** The time period of rolling of a ship of weight 29430 kN in sea water is 10 seconds. The centre of buoyancy of the ship is 1.5 m below the centre of gravity. Find the radius of gyration of the ship if the moment of inertia of the ship at the water line about fore and aft axis is  $1000 \text{ m}^4$ . Take specific weight of sea water as =  $10100 \text{ N/m}^3$ .

**Solution.** Given :

Time period,  $T = 10$  sec

Distance between centre of buoyancy and centre of gravity,  $BG = 1.5$  m

Moment of Inertia,  $I = 10000 \text{ m}^4$

Weight,  $W = 29430 \text{ kN} = 29430 \times 1000 \text{ N}$

Let the radius of gyration =  $K$

First calculate the meta-centric height  $GM$ , which is given as

$$GM = BM - BG = \frac{I}{V} - BG$$

where  $I$  = M.O. Inertia

and  $\nabla$  = Volume of water displaced

$$= \frac{\text{Weight of ship}}{\text{Sp. weight of sea water}} = \frac{29430 \times 1000}{10104} = 2912.6 \text{ m}^3$$

$$\therefore GM = \frac{10000}{2912.6} - 1.5 = 3.433 - 1.5 = 1.933 \text{ m.}$$

Using equation (4.6), we get  $T = 2\pi \sqrt{\frac{K^2}{GM \times g}}$

or  $10 = 2\pi \sqrt{\frac{K^2}{1.933 \times 9.81}} = \frac{2\pi K}{\sqrt{1.933 \times 9.81}}$

or  $K = \frac{10 \times \sqrt{1.933 \times 9.81}}{2\pi} = 6.93 \text{ m. Ans.}$

## HIGHLIGHTS

1. The upward force exerted by a liquid on a body when the body is immersed in the liquid is known as buoyancy or force of buoyancy.
2. The point through which force of buoyancy is supposed to act is called centre of buoyancy.
3. The point about which a body starts oscillating when the body is tilted is known as meta-centre.
4. The distance between the meta-centre and centre of gravity is known as meta-centric height.
5. The meta-centric height ( $GM$ ) is given by  $GM = \frac{I}{\nabla} - BG$

where  $I$  = Moment of Inertia of the floating body (in plan) at water surface about the axis  $Y-Y$

$\nabla$  = Volume of the body sub-merged in water

$BG$  = Distance between centre of gravity and centre of buoyancy.

6. Conditions of equilibrium of a floating and sub-merged body are :

Equilibrium	Floating Body	Sub-merged Body
(i) Stable Equilibrium	$M$ is above $G$	$B$ is above $G$
(ii) Unstable Equilibrium	$M$ is below $G$	$B$ is below $G$
(iii) Neutral Equilibrium	$M$ and $G$ coincide	$B$ and $G$ coincide

7. The value of meta-centric height  $GM$ , experimentally is given as  $GM = \frac{w_1 x}{W \tan \theta}$

where  $w_1$  = Movable weight

$x$  = Distance through which  $w_1$  is moved

$W$  = Weight of the ship or floating body including  $w_1$

$\theta$  = Angle through the ship or floating body is tilted due to the movement of  $w_1$ .

8. The time period of oscillation or rolling of a floating body is given by  $T = 2\pi \sqrt{\frac{K^2}{GM \times g}}$

where  $K$  = Radius of gyration,  $GM$  = Meta-centric height

$T$  = Time of one complete oscillation.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define the terms 'buoyancy' and 'centre of buoyancy'.
  2. Explain the terms 'meta-centre' and 'meta-centric height'.
  3. Derive an expression for the meta-centric height of a floating body.
  4. Show that the distance between the meta-centre and centre of buoyancy is given by  $BM = \frac{I}{\nabla}$   
where  $I$  = Moment of inertia of the plan of the floating body at water surface about longitudinal axis.  
 $\nabla$  = Volume of the body sub-merged in liquid.
  5. What are the conditions of equilibrium of a floating body and a sub-merged body ?
  6. How will you determine the meta-centric height of a floating body experimentally ? Explain with neat sketch.
  7. Select the correct statement :
    - (a) The buoyant force for a floating body passes through the
 

(i) centre of gravity of the body	(ii) centroid of volume of the body
(iii) meta-centre of the body	(iv) centre of gravity of the sub-merged part of the body
(v) centroid of the displaced volume.	
    - (b) A body sub-merged in liquid is in equilibrium when :
 

(i) its meta-centre is above the centre of gravity	(ii) its meta-centre is above the centre of buoyancy
(iii) its centre of gravity is above the centre of buoyancy	(iv) its centre of buoyancy is above the centre of gravity
(v) none of these.	
- [Ans. 7 (a) (v), (b) (iv)]
8. Derive an expression for the time period of the oscillation of a floating body in terms of radius of gyration and meta-centric height of the floating body.
  9. Define the terms : meta-centre, centre of buoyancy, meta-centric height, gauge pressure and absolute pressure.
  10. What do you understand by the hydrostatic equation ? With the help of this equation, derive the expression for the buoyant force acting on a sub-merged body.
  11. With neat sketches, explain the conditions of equilibrium for floating and sub-merged bodies.
  12. Differentiate between :
    - (i) Dynamic viscosity and kinematic viscosity, (ii) Absolute and gauge pressure (iii) Simple and differential manometers (iv) Centre of gravity and centre of buoyancy.

*(Delhi University, Dec. 2002)*

### (B) NUMERICAL PROBLEMS

1. A wooden block of width 2 m, depth 1.5 m and length 4 m floats horizontally in water. Find the volume of water displaced and position of centre of buoyancy. The specific gravity of the wooden block is 0.7.  
[Ans.  $8.4 \text{ m}^3$ , 0.525 m from the base]

2. A wooden log of 0.8 m diameter and 6 m length is floating in river water. Find the depth of wooden log in water when the sp. gr. of the wooden log is 0.7. [Ans. 0.54 m]
3. A stone weighs 490.5 N in air and 196.2 N in water. Determine the volume of stone and its specific gravity. [Ans.  $0.03 \text{ m}^3$  or  $3 \times 10^4 \text{ cm}^3$ , 1.67]
4. A body of dimensions  $2.0 \text{ m} \times 1.0 \text{ m} \times 3.0 \text{ m}$  weighs 3924 N in water. Find its weight in air. What will be its specific gravity ? [Ans. 62784 N, 1.0667]
5. A metallic body floats at the interface of mercury of sp. gr. 13.6 and water in such a way that 30% of its volume is sub-merged in mercury and 70% in water. Find the density of the metallic body. [Ans.  $4780 \text{ kg/m}^3$ ]
6. A body of dimensions  $0.5 \text{ m} \times 0.5 \text{ m} \times 1.0 \text{ m}$  and of sp. gr. 3.0 is immersed in water. Determine the least force required to lift the body. [Ans. 4905 N]
7. A rectangular pontoon is 4 m long, 3 m wide and 1.40 m high. The depth of immersion of the pontoon is 1.0 m in sea-water. If the centre of gravity is 0.70 m above the bottom of the pontoon, determine the meta-centric height. Take the density of sea-water as  $1030 \text{ kg/m}^3$ . [Ans. 0.45 m]
8. A uniform body of size  $4 \text{ m long} \times 2 \text{ m wide} \times 1 \text{ m deep}$  floats in water. What is the weight of the body if depth of immersion is 0.6 m ? Determine the meta-centric height also. [Ans. 47088 N, 0.355 m]
9. A block of wood of specific gravity 0.8 floats in water. Determine the meta-centric height of the block if its size is  $3 \text{ m} \times 2 \text{ m} \times 1 \text{ m}$ . [Ans. 0.316 m]
10. A solid cylinder of diameter 3.0 m has a height of 2 m. Find the meta-centric height of the cylinder when it is floating in water with its axis vertical. The sp. gr. of the cylinder is 0.7. [Ans. 0.1017 m]
11. A body has the cylindrical upper portion of 4 m diameter and 2 m deep. The lower portion is a curved one, which displaces a volume of  $0.9 \text{ m}^3$  of water. The centre of buoyancy of the curved portion is at a distance of 2.10 m below the top of the cylinder. The centre of gravity of the whole body is 1.50 m below the top of the cylinder. The total displacement of water is 4.5 tonnes. Find the meta-centric height of the body. [Ans. 2.387 m]
12. A solid cylinder of diameter 5.0 m has a height of 5.0 m. Find the meta-centric height of the cylinder if the specific gravity of the material of cylinder is 0.7 and it is floating in water with its axis vertical. State whether the equilibrium is stable or unstable. [Ans.  $-0.304 \text{ m}$ , Unstable Equilibrium]
13. A solid cylinder of 15 cm diameter and 60 cm long, consists of two parts made of different materials. The first part at the base is 1.20 cm long and of specific gravity = 5.0. The other parts of the cylinder is made of the material having specific gravity 0.6. State, if it can float vertically in water. [Ans.  $GM = -5.26$ , Unstable, Equilibrium]
14. A rectangular pontoon 8.0 m long, 7 m broad and 3.0 m deep weighs 588.6 kN. It carries on its upper deck an empty boiler of 4.0 m diameter weighing 392.4 kN. The centre of gravity of the boiler and the pontoon are at their respective centres along a vertical line. Find the meta-centric height. Weight density of sea-water is  $10104 \text{ N/m}^3$ . [Ans. 0.325 m]
15. A wooden cylinder of sp. gr. 0.6 and circular in cross-section is required to float in oil (sp. gr. 0.8). Find the  $L/D$  ratio for the cylinder to float with its longitudinal axis vertical in oil where  $L$  is the height of cylinder and  $D$  is its diameter. [Ans.  $(L/D) < 0.8164$ ]
16. Show that a cylindrical buoy of 1.5 m diameter and 3 m long weighing 2.5 tonnes will not float vertically in sea-water of density  $1030 \text{ kg/m}^3$ . Find the force necessary in a vertical chain attached at the centre of the base of the buoy that will keep it vertical. [Ans. 10609.5 N]
17. A solid cone floats in water its apex downwards. Determine the least apex angle of cone for stable equilibrium. The specific gravity of the material of the cone is given 0.7. [Ans.  $39^\circ 7'$ ]
18. A ship 60 m long and 12 m broad has a displacement of 19620 kN. A weight of 294.3 kN is moved across the deck through a distance of 6.5 m. The ship is tilted through  $5^\circ$ . The moment of inertia of the ship at

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water line about its fore and aft axis is 75% of moment of inertia the circumscribing rectangle. The centre of buoyancy is 2.75 m below water line. Find the meta-centric height and position of centre of gravity of ship. Take specific weight of sea water =  $10104 \text{ N/m}^3$ . [Ans. 1.1145 m, 0.53 m below water surface]

19. A pontoon of 1500 tonnes displacement is floating in water. A weight of 20 tonnes is moved through a distance of 6 m across the deck of pontoon, which tilts the pontoon through an angle of  $5^\circ$ . Find meta-centric height of the pontoon. [Ans. 0.9145 m]
20. Find the time period of rolling of a solid circular cylinder of radius 2.5 m and 5.0 m long. The specific gravity of the cylinder is 0.9 and is floating in water with its axis vertical. [Ans. 0.35 sec]

# 5

## CHAPTER



# KINEMATICS OF FLOW AND IDEAL FLOW

## A. KINEMATICS OF FLOW

### ► 5.1 INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

### ► 5.2 METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods. They are —(i) Lagrangian Method, and (ii) Eulerian Method. In the Lagrangian method, a **single fluid particle** is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density etc., are described **at a point** in flow field. The Eulerian method is commonly used in fluid mechanics.

### ► 5.3 TYPES OF FLUID FLOW

The fluid flow is classified as :

- (i) Steady and unsteady flows ;
- (ii) Uniform and non-uniform flows ;
- (iii) Laminar and turbulent flows ;
- (iv) Compressible and incompressible flows ;
- (v) Rotational and irrotational flows ; and
- (vi) One, two and three-dimensional flows.

**5.3.1 Steady and Unsteady Flows.** Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

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$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where  $(x_0, y_0, z_0)$  is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

**5.3.2 Uniform and Non-uniform Flows.** Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t=\text{constant}} = 0$$

where  $\partial V =$  Change of velocity

$\partial s =$  Length of flow in the direction  $S$ .

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t=\text{constant}} \neq 0.$$

**5.3.3 Laminar and Turbulent Flows.** Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a zig-zag way. Due to the movement of fluid particles in a zig-zag way, the eddies formation takes place which are responsible

for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number  $\frac{VD}{v}$

called the Reynold number,

where  $D =$  Diameter of pipe

$V =$  Mean velocity of flow in pipe

and  $v =$  Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

**5.3.4 Compressible and Incompressible Flows.** Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density ( $\rho$ ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant.}$$

**5.3.5 Rotational and Irrotational Flows.** Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

**5.3.6 One-, Two- and Three-Dimensional Flows.** **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only, say  $x$ . For a steady one-dimensional flow, the velocity is a function of one-space-co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence mathematically, for one-dimensional flow

$$u = f(x), v = 0 \text{ and } w = 0$$

where  $u$ ,  $v$  and  $w$  are velocity components in  $x$ ,  $y$  and  $z$  directions respectively.

**Two-dimensional flow** is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say  $x$  and  $y$ . For a steady two-dimensional flow the velocity is a function of two space co-ordinates only. The variation of velocity in the third direction is negligible. Thus, mathematically for two-dimensional flow

$$u = f_1(x, y), v = f_2(x, y) \text{ and } w = 0.$$

**Three-dimensional flow** is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are functions of three space co-ordinates ( $x$ ,  $y$  and  $z$ ) only. Thus, mathematically, for three-dimensional flow

$$u = f_1(x, y, z), v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

## ► 5.4 RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

- (i) For liquids the units of  $Q$  are  $\text{m}^3/\text{s}$  or litres/s
- (ii) For gases the units of  $Q$  is  $\text{kgf/s}$  or Newton/s

Consider a liquid flowing through a pipe in which

$A$  = Cross-sectional area of pipe

$V$  = Average velocity of fluid across the section

Then discharge  $Q = A \times V.$  ... (5.1)

## ► 5.5 CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig. 5.1.

Let  $V_1$  = Average velocity at cross-section 1-1

$\rho_1$  = Density at section 1-1

$A_1$  = Area of pipe at section 1-1

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and  $V_2, \rho_2, A_2$  are corresponding values at section, 2-2.

Then rate of flow at section 1-1 =  $\rho_1 A_1 V_1$

Rate of flow at section 2-2 =  $\rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

$$\text{or } \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \dots(5.2)$$

Equation (5.2) is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then  $\rho_1 = \rho_2$  and continuity equation (5.2) reduces to

$$A_1 V_1 = A_2 V_2 \quad \dots(5.3)$$

**Problem 5.1** The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.

**Solution.** Given :

At section 1,

$$D_1 = 10 \text{ cm} = 0.1 \text{ m}$$

$$A_1 = \frac{\pi}{4} (D_1)^2 = \frac{\pi}{4} (0.1)^2 = 0.007854 \text{ m}^2$$

$$V_1 = 5 \text{ m/s.}$$

At section 2,

$$D_2 = 15 \text{ cm} = 0.15 \text{ m}$$

$$A_2 = \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2$$

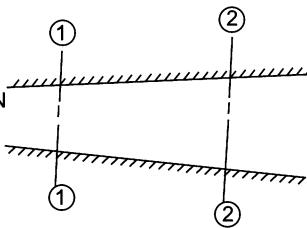


Fig. 5.1 Fluid flowing through a pipe.

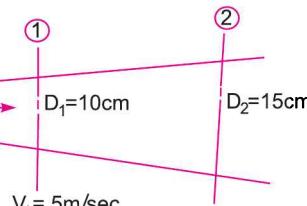


Fig. 5.2

(i) Discharge through pipe is given by equation (5.1)

or

$$Q = A_1 \times V_1 \\ = 0.007854 \times 5 = 0.03927 \text{ m}^3/\text{s. Ans.}$$

Using equation (5.3), we have  $A_1 V_1 = A_2 V_2$

$$(ii) \therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = 2.22 \text{ m/s. Ans.}$$

**Problem 5.2** A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.

**Solution.** Given :

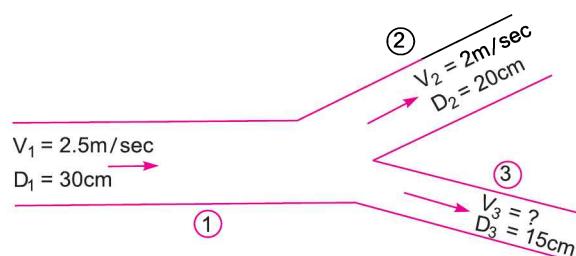


Fig. 5.3

$$D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

$$V_1 = 2.5 \text{ m/s}$$

$$D_2 = 20 \text{ cm} = 0.20 \text{ m}$$

$$\therefore A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2,$$

$$V_2 = 2 \text{ m/s}$$

$$D_3 = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$$

Find (i) Discharge in pipe 1 or  $Q_1$

(ii) Velocity in pipe of dia. 15 cm or  $V_3$

Let  $Q_1$ ,  $Q_2$  and  $Q_3$  are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad \dots(1)$$

(i) The discharge  $Q_1$  in pipe 1 is given by

$$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3/\text{s} = 0.1767 \text{ m}^3/\text{s. Ans.}$$

(ii) Value of  $V_3$

$$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3/\text{s}$$

Substituting the values of  $Q_1$  and  $Q_2$  in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$$\therefore Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3/\text{s}$$

$$\text{But } Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$$

$$\therefore V_3 = \frac{0.1139}{0.01767} = 6.44 \text{ m/s. Ans.}$$

**Problem 5.3** Water flows through a pipe AB 1.2 m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter. At C, the pipe branches. Branch CD is 0.8 m in diameter and carries one-third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.

**Solution.** Given :

$$\text{Diameter of pipe AB, } D_{AB} = 1.2 \text{ m}$$

$$\text{Velocity of flow through AB, } V_{AB} = 3.0 \text{ m/s}$$

$$\text{Dia. of pipe BC, } D_{BC} = 1.5 \text{ m}$$

$$\text{Dia. of branched pipe CD, } D_{CD} = 0.8 \text{ m}$$

$$\text{Velocity of flow in pipe CE, } V_{CE} = 2.5 \text{ m/s}$$

$$\text{Let the flow rate in pipe AB} = Q \text{ m}^3/\text{s}$$

$$\text{Velocity of flow in pipe BC} = V_{BC} \text{ m/s}$$

$$\text{Velocity of flow in pipe CD} = V_{CD} \text{ m/s}$$

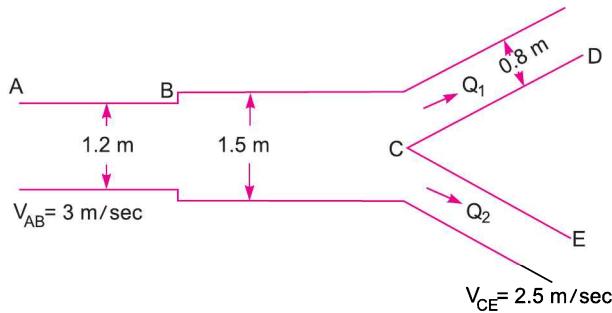


Fig. 5.4

Diameter of pipe

$$CE = D_{CE}$$

Then flow rate through

$$CD = Q/3$$

and flow rate through

$$CE = Q - Q/3 = \frac{2Q}{3}$$

(i) Now volume flow rate through AB =  $Q = V_{AB} \times \text{Area of } AB$

$$= 3.0 \times \frac{\pi}{4} (D_{AB})^2 = 3.0 \times \frac{\pi}{4} (1.2)^2 = 3.393 \text{ m}^3/\text{s. Ans.}$$

(ii) Applying continuity equation to pipe AB and pipe BC,

$$V_{AB} \times \text{Area of pipe } AB = V_{BC} \times \text{Area of pipe } BC$$

or

$$3.0 \times \frac{\pi}{4} (D_{AB})^2 = V_{BC} \times \frac{\pi}{4} (D_{BC})^2$$

or

$$3.0 \times (1.2)^2 = V_{BC} \times (1.5)^2$$

Divide by  $\frac{\pi}{4}$

or

$$V_{BC} = \frac{3 \times 1.2^2}{1.5^2} = 1.92 \text{ m/s. Ans.}$$

(iii) The flow rate through pipe

$$CD = Q_1 = \frac{Q}{3} = \frac{3.393}{3} = 1.131 \text{ m}^3/\text{s}$$

∴

$$Q_1 = V_{CD} \times \text{Area of pipe } CD \times \frac{\pi}{4} (D_{CD})^2$$

or

$$1.131 = V_{CD} \times \frac{\pi}{4} \times 0.8^2 = 0.5026 V_{CD}$$

∴

$$V_{CD} = \frac{1.131}{0.5026} = 2.25 \text{ m/s. Ans.}$$

(iv) Flow rate through CE,

$$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.262 \text{ m}^3/\text{s}$$

∴

$$Q_2 = V_{CE} \times \text{Area of pipe } CE = V_{CE} \frac{\pi}{4} (D_{CE})^2$$

or

$$2.263 = 2.5 \times \frac{\pi}{4} \times (D_{CE})^2$$

or

$$D_{CE} = \sqrt{\frac{2.263 \times 4}{2.5 \times \pi}} = \sqrt{1.152} = 1.0735 \text{ m}$$

∴

$$\text{Diameter of pipe } CE = 1.0735 \text{ m. Ans.}$$

**Problem 5.4** A 25 cm diameter pipe carries oil of sp. gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.

**Solution.** Given :

at section 1,

$$D_1 = 25 \text{ cm} = 0.25 \text{ m}$$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049 \text{ m}^3$$

$$V_1 = 3 \text{ m/s}$$

at section 2,

$$D_2 = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$$V_2 = ?$$

Mass rate of flow of oil = ?

Applying continuity equation at sections 1 and 2,

$$A_1 V_1 = A_2 V_2$$

or

$$0.049 \times 3.0 = 0.0314 \times V_2$$

∴

$$V_2 = \frac{0.049 \times 3.0}{0.0314} = 4.68 \text{ m/s. Ans.}$$

Mass rate of flow of oil = Mass density  $\times Q = \rho \times A_1 \times V_1$

Sp. gr. of oil =  $\frac{\text{Density of oil}}{\text{Density of water}}$

∴ Density of oil = Sp. gr. of oil  $\times$  Density of water  
 $= 0.9 \times 1000 \text{ kg/m}^3 = \frac{900 \text{ kg}}{\text{m}^3}$

∴ Mass rate of flow =  $900 \times 0.049 \times 3.0 \text{ kg/s} = 132.23 \text{ kg/s. Ans.}$

**Problem 5.5** A jet of water from a 25 mm diameter nozzle is directed vertically upwards. Assuming that the jet remains circular and neglecting any loss of energy, that will be the diameter at a point 4.5 m above the nozzle, if the velocity with which the jet leaves the nozzle is 12 m/s.

**Solution.** Given :

Dia. of nozzle,  $D_1 = 25 \text{ mm} = 0.025 \text{ m}$

Velocity of jet at nozzle,  $V_1 = 12 \text{ m/s}$

Height of point A,  $h = 4.5 \text{ m}$

Let the velocity of the jet at a height 4.5 m =  $V_2$

Consider the vertical motion of the jet from the outlet of the nozzle to the point A (neglecting any loss of energy).

Initial velocity,  $u = V_1 = 12 \text{ m/s}$

Final velocity,  $V = V_2$

Value of  $g = -9.81 \text{ m/s}^2$  and  $h = 4.5 \text{ m}$

Using,  $V^2 - u^2 = 2gh$ , we get

$$V_2^2 - 12^2 = 2 \times (-9.81) \times 4.5$$

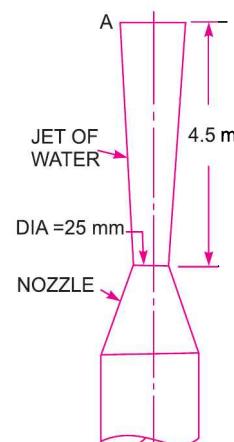


Fig. 5.5

$$\therefore V_2 = \sqrt{12^2 - 2 \times 9.81 \times 4.5} = \sqrt{144 - 88.29} = 7.46 \text{ m/s}$$

Now applying continuity equation to the outlet of nozzle and at point A,  
we get

$$A_1 V_1 = A_2 V_2$$

$$\text{or } A_2 = \frac{A_1 V_1}{V_2} = \frac{\frac{\pi}{4} D_1^2 \times V_1}{V_2} = \frac{\pi \times (0.025)^2 \times 12}{4 \times 7.46} = 0.0007896$$

Let  $D_2$  = Diameter of jet at point A.

$$\text{Then } A_2 = \frac{\pi}{4} D_2^2 \text{ or } 0.0007896 = \frac{\pi}{4} \times D_2^2$$

$$\therefore D_2 = \sqrt{\frac{0.0007896 \times 4}{\pi}} = 0.0317 \text{ m} = 31.7 \text{ mm. Ans.}$$

## ► 5.6 CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths  $dx$ ,  $dy$  and  $dz$  in the direction of  $x$ ,  $y$  and  $z$ . Let  $u$ ,  $v$  and  $w$  are the inlet velocity components in  $x$ ,  $y$  and  $z$  directions respectively. Mass of fluid entering the face  $ABCD$  per second

$$\begin{aligned} &= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD \\ &= \rho \times u \times (dy \times dz) \end{aligned}$$

$$\text{Then mass of fluid leaving the face } EFGH \text{ per second} = \rho u dy dz + \frac{\partial}{\partial x} (\rho u dy dz) dx$$

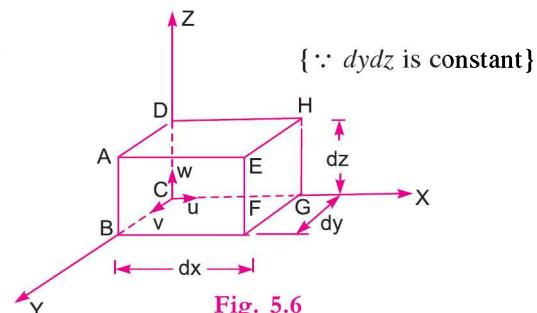
$\therefore$  Gain of mass in  $x$ -direction

$$\begin{aligned} &= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second} \\ &= \rho u dy dz - \rho u dy dz - \frac{\partial}{\partial x} (\rho u dy dz) dx \\ &= - \frac{\partial}{\partial x} (\rho u dy dz) dx \\ &= - \frac{\partial}{\partial x} (\rho u) dx dy dz \end{aligned}$$

Similarly, the net gain of mass in  $y$ -direction

$$= - \frac{\partial}{\partial y} (\rho v) dx dy dz$$

$$\text{and in } z\text{-direction} \quad = - \frac{\partial}{\partial z} (\rho w) dx dy dz$$



$$\therefore \text{Net gain of masses} = - \left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] dx dy dz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass

of fluid in the element is  $\rho \cdot dx \cdot dy \cdot dz$  and its rate of increase with time is  $\frac{\partial}{\partial t} (\rho \cdot dx \cdot dy \cdot dz)$  or

$$\frac{\partial \rho}{\partial t} \cdot dx \cdot dy \cdot dz.$$

Equating the two expressions,

$$\text{or } -\left[ \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] dx dy dz = \frac{\partial \rho}{\partial t} \cdot dx dy dz$$

$$\text{or } \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad [\text{Cancelling } dx \cdot dy \cdot dz \text{ from both sides}] \dots(5.3A)$$

Equation (5.3A) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

- (i) Steady and unsteady flow,
- (ii) Uniform and non-uniform flow, and
- (iii) Compressible and incompressible fluids.

For steady flow,  $\frac{\partial \rho}{\partial t} = 0$  and hence equation (5.3A) becomes as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \dots(5.3B)$$

If the fluid is incompressible, then  $\rho$  is constant and the above equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \dots(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component  $w = 0$  and hence continuity equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \dots(5.5)$$

**5.6.1 Continuity Equation in Cylindrical Polar Co-ordinates.** The continuity equation in cylindrical polar co-ordinates (*i.e.*,  $r$ ,  $\theta$ ,  $z$  co-ordinates) is derived by the procedure given below.

Consider a two-dimensional incompressible flow field. The two-dimensional polar co-ordinates are  $r$  and  $\theta$ . Consider a fluid element  $ABCD$  between the radii  $r$  and  $r + dr$  as shown in Fig. 5.7. The angle subtended by the element at the centre is  $d\theta$ . The components of the velocity  $V$  are  $u_r$  in the radial direction and  $u_\theta$  in the tangential direction. The sides of the element are having the lengths as

Side  $AB = rd\theta$ ,  $BC = dr$ ,  $DC = (r + dr)d\theta$ ,  $AD = dr$ .

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

*Consider the flow in radial direction*

Mass of fluid entering the face  $AB$  per unit time

$$= \rho \times \text{Velocity in } r\text{-direction} \times \text{Area}$$

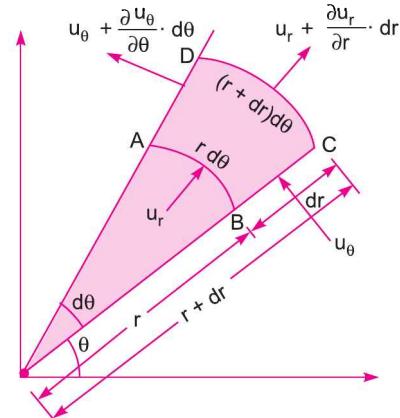


Fig. 5.7

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$$= \rho \times u_r \times (AB \times 1) \quad (\because \text{Area} = AB \times \text{Thickness} = rd\theta \times 1)$$

$$= \rho \times u_r \times (rd\theta \times 1) = \rho \cdot u_r \cdot rd\theta$$

Mass of fluid leaving the face  $CD$  per unit time

$$\begin{aligned} &= \rho \times \text{Velocity} \times \text{Area} \\ &= \rho \times \left( u_r + \frac{\partial u_r}{\partial r} \cdot dr \right) \times (CD \times 1) \quad (\because \text{Area} = CD \times 1) \\ &= \rho \times \left( u_r + \frac{\partial u_r}{\partial r} dr \right) \times (r + dr)d\theta \quad [\because CD = (r + dr)d\theta] \\ &= \rho \times \left[ u_r \times r + u_r dr + r \frac{\partial u_r}{\partial r} dr + \frac{\partial u_r}{\partial r} (dr)^2 \right] d\theta \\ &= \rho \left[ u_r \times r + u_r \times dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta \end{aligned}$$

[The term containing  $(dr)^2$  is very small and has been neglected]

$\therefore$  Gain of mass in  $r$ -direction per unit time

$$\begin{aligned} &= (\text{Mass through } AB - \text{Mass through } CD) \text{ per unit time} \\ &= \rho \cdot u_r \cdot rd\theta - \rho \left[ u_r \cdot r + u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta \\ &= \rho \cdot u_r \cdot rd\theta - \rho \cdot u_r \cdot r \cdot d\theta - \rho \left[ u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta \\ &= -\rho \left[ u_r \cdot dr + r \frac{\partial u_r}{\partial r} \cdot dr \right] d\theta \\ &= -\rho \left[ \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] r \cdot dr \cdot d\theta \quad [\text{This is written in this form because } (r \cdot d\theta \cdot dr \cdot 1) \text{ is equal to volume of element}] \end{aligned}$$

Now consider the flow in  $\theta$ -direction

Gain in mass in  $\theta$ -direction per unit time

$$\begin{aligned} &= (\text{Mass through } BC - \text{Mass through } AD) \text{ per unit time} \\ &= [\rho \times \text{Velocity through } BC \times \text{Area} - \rho \times \text{Velocity through } AD \times \text{Area}] \\ &= \left[ \rho \cdot u_\theta \cdot dr \times 1 - \rho \left( u_\theta + \frac{\partial u_\theta}{\partial \theta} \cdot d\theta \right) \times dr \times 1 \right] \\ &= -\rho \left( \frac{\partial u_\theta}{\partial \theta} \cdot d\theta \right) dr \times 1 \quad (\because \text{Area} = dr \times 1) \\ &= -\rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{r \cdot d\theta \cdot dr}{r} \quad [\text{Multiplying and dividing by } r] \end{aligned}$$

$\therefore$  Total gain in fluid mass per unit time

$$= -\rho \left[ \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] \cdot r \cdot dr \cdot d\theta - \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{r d\theta \cdot dr}{r} \quad \dots(5.5A)$$

$$\begin{aligned}\text{But mass of fluid element} &= \rho \times \text{Volume of fluid element} \\ &= \rho \times [rd\theta \times dr \times 1] \\ &= \rho \times rd\theta \cdot dr\end{aligned}$$

Rate of increase of fluid mass in the element with time

$$= \frac{\partial}{\partial t} [\rho \cdot rd\theta \cdot dr] = \frac{\partial \rho}{\partial t} \cdot rd\theta \cdot dr \quad \dots(5.5B)$$

( $\because rd\theta \cdot dr \cdot 1$  is the volume of element and is a constant quantity)

Since the mass is neither created nor destroyed in the fluid element, hence net gain of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Hence equating the two expressions given by equations (5.5 A) and (5.5 B), we get

$$\begin{aligned}-\rho \left[ \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] r \cdot dr \cdot d\theta - \rho \frac{\partial u_\theta}{\partial \theta} \frac{rd\theta \cdot dr}{r} &= \frac{\partial \rho}{\partial t} \cdot rd\theta \cdot dr \\ \text{or} \quad -\rho \left[ \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] - \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} &= \frac{\partial \rho}{\partial t} \quad [\text{Cancelling } rdr \cdot d\theta \text{ from both sides}] \\ \text{or} \quad \frac{\partial \rho}{\partial t} + \rho \left[ \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] + \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} &= 0 \quad \dots(5.5C)\end{aligned}$$

Equation (5.5 C) is the continuity equation in polar co-ordinates for two-dimensional flow.

For steady flow  $\frac{\partial \rho}{\partial t} = 0$  and hence equation (5.5 C) reduces to

$$\begin{aligned}\rho \left[ \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right] + \rho \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} &= 0 \\ \text{or} \quad \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} \cdot \frac{1}{r} &= 0 \\ \text{or} \quad u_r + r \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} &= 0 \\ \text{or} \quad \frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial \theta} (u_\theta) &= 0 \quad \left[ \because \frac{\partial}{\partial r} (r \cdot u_r) = r \cdot \frac{\partial u_r}{\partial r} + u_r \right] \quad \dots(5.5D)\end{aligned}$$

Equation (5.5 D) represents the continuity equation in polar co-ordinates for two-dimensional steady incompressible flow.

**Problem 5.5A** Examine whether the following velocity components represent a physically possible flow ?

$$u_r = r \sin \theta, u_\theta = 2r \cos \theta.$$

**Solution.** Given :  $u_r = r \sin \theta$  and  $u_\theta = 2r \cos \theta$

For physically possible flow, the continuity equation,

$$\frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial \theta} (u_\theta) = 0 \text{ should be satisfied.}$$

Now  $u_r = r \sin \theta$

Multiplying the above equation by  $r$ , we get

$$ru_r = r^2 \sin \theta$$

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Differentiating the preceding equation w.r.t.  $r$ , we get

$$\begin{aligned}\frac{\partial}{\partial r} (ru_r) &= \frac{\partial}{\partial r} (r^2 \sin \theta) \\ &= 2r \sin \theta \quad (\because \sin \theta \text{ is constant w.r.t. } r)\end{aligned}$$

Now

$$u_\theta = 2r \cos \theta$$

Differentiating the above equation w.r.t.  $\theta$ , we get

$$\begin{aligned}\frac{\partial}{\partial \theta} (u_\theta) &= \frac{\partial}{\partial \theta} (2r \cos \theta) \\ &= 2r (-\sin \theta) \quad (\because 2r \text{ is constant w.r.t. } \theta) \\ &= -2r \sin \theta\end{aligned}$$

$$\therefore \frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial \theta} (u_\theta) = 2r \sin \theta - 2r \sin \theta = 0$$

Hence the continuity equation is satisfied. Hence the given velocity components represent a physically possible flow.

### ► 5.7 VELOCITY AND ACCELERATION

Let  $V$  is the resultant velocity at any point in a fluid flow. Let  $u$ ,  $v$  and  $w$  are its component in  $x$ ,  $y$  and  $z$  directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$

$$v = f_2(x, y, z, t)$$

$$w = f_3(x, y, z, t)$$

and Resultant velocity,  $V = \sqrt{u^2 + v^2 + w^2}$

Let  $a_x$ ,  $a_y$  and  $a_z$  are the **total acceleration** in  $x$ ,  $y$  and  $z$  directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But  $\frac{dx}{dt} = u$ ,  $\frac{dy}{dt} = v$  and  $\frac{dz}{dt} = w$

$$\therefore a_x = \frac{du}{dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \quad [$$

Similarly,  $a_y = \frac{dv}{dt} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$

$$a_z = \frac{dw}{dt} = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \quad ] \quad \dots(5.6)$$

For steady flow,  $\frac{\partial V}{\partial t} = 0$ , where  $V$  is resultant velocity

or  $\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$

Hence acceleration in  $x, y$  and  $z$  directions becomes

$$\left. \begin{aligned} a_x &= \frac{du}{dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \end{aligned} \right] \quad \dots(5.7)$$

Acceleration vector  $A = a_x i + a_y j + a_z k$  \dots(5.8)  
 $= \sqrt{a_x^2 + a_y^2 + a_z^2}$ .

**5.7.1 Local Acceleration and Convective Acceleration.** Local acceleration is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the equation given

by (5.6), the expression  $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$  or  $\frac{\partial w}{\partial t}$  is known as local acceleration.

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expressions other than  $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$  and  $\frac{\partial w}{\partial t}$  in equation (5.6) are known as convective acceleration.

**Problem 5.6** The velocity vector in a fluid flow is given

$$V = 4x^3 i - 10x^2 y j + 2t k$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time  $t = 1$ .

**Solution.** The velocity components  $u, v$  and  $w$  are  $u = 4x^3, v = -10x^2 y, w = 2t$

For the point (2, 1, 3), we have  $x = 2, y = 1$  and  $z = 3$  at time  $t = 1$ .

Hence velocity components at (2, 1, 3) are

$$\begin{aligned} u &= 4 \times (2)^3 = 32 \text{ units} \\ v &= -10(2)^2(1) = -40 \text{ units} \\ w &= 2 \times 1 = 2 \text{ units} \end{aligned}$$

$\therefore$  Velocity vector  $V$  at (2, 1, 3) =  $32i - 40j + 2k$

or Resultant velocity =  $\sqrt{u^2 + v^2 + w^2}$   
 $= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = 51.26 \text{ units. Ans.}$

Acceleration is given by equation (5.6)

$$\begin{aligned} a_x &= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ a_y &= u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \end{aligned}$$

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$$a_z = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \quad \frac{\partial v}{\partial y} = -10x^2, \quad \frac{\partial v}{\partial z} = 0 \quad \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time  $t = 1$  are

$$\begin{aligned} a_x &= 4x^3(12x^2) + (-10x^2y)(0) + 2t \times (0) + 0 \\ &= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units} \\ a_y &= 4x^3(-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0 \\ &= -80x^4y + 100x^4y \\ &= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.} \\ a_z &= 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units} \end{aligned}$$

$\therefore$  Acceleration is  $A = a_x i + a_y j + a_z k = 1536i + 320j + 2k. \text{ Ans.}$

or Resultant  $A = \sqrt{(1536)^2 + (320)^2 + (2)^2} \text{ units}$

$$= \sqrt{2359296 + 102400 + 4} = 1568.9 \text{ units. Ans.}$$

**Problem 5.7** The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :

$$(i) \quad u = x^2 + y^2 + z^2 ; v = xy^2 - yz^2 + xy$$

$$(ii) \quad v = 2y^2, w = 2xyz.$$

**Solution.** The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

<b>Case I.</b>	$u = x^2 + y^2 + z^2 \quad \therefore \quad \frac{\partial u}{\partial x} = 2x$
	$v = xy^2 - yz^2 + xy \quad \therefore \quad \frac{\partial v}{\partial y} = 2xy - z^2 + x$

Substituting the values of  $\frac{\partial u}{\partial x}$  and  $\frac{\partial v}{\partial y}$  in continuity equation.

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or  $\frac{\partial w}{\partial z} = -3x - 2xy + z^2 \text{ or } \partial w = (-3x - 2xy + z^2) \partial z$

Integration of both sides gives  $\int dw = \int (-3x - 2xy + z^2) dz$

or  $w = \left( -3xz - 2xyz + \frac{z^3}{3} \right) + \text{Constant of integration,}$

where constant of integration cannot be a function of  $z$ . But it can be a function of  $x$  and  $y$  that is  $f(x, y)$ .

$$\therefore w = \left( -3xz - 2xyz + \frac{z^3}{3} \right) + f(x, y). \text{ Ans.}$$

**Case II.**  $v = 2y^2 \quad \therefore \frac{\partial v}{\partial y} = 4y$

$$w = 2xyz \quad \therefore \frac{\partial w}{\partial z} = 2xy$$

Substituting the values of  $\frac{\partial v}{\partial y}$  and  $\frac{\partial w}{\partial z}$  in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or  $\frac{\partial u}{\partial x} = -4y - 2xy \text{ or } du = (-4y - 2xy) dx$

Integrating, we get  $u = -4xy - 2y \frac{x^2}{2} + f(y, z) = -4xy - x^2y + f(y, z). \text{ Ans.}$

**Problem 5.8** A fluid flow field is given by

$$V = x^2yi + y^2zj - (2xyz + yz^2)k$$

Prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point  $(2, 1, 3)$ .

**Solution.** For the given fluid flow field  $u = x^2y \quad \therefore \frac{\partial u}{\partial x} = 2xy$

$$v = y^2z \quad \therefore \frac{\partial v}{\partial y} = 2yz$$

$$w = -2xyz - yz^2 \quad \therefore \frac{\partial w}{\partial z} = -2xy - 2yz.$$

For a case of possible steady incompressible fluid flow, the continuity equation (5.4) should be satisfied.

i.e.,  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$

Substituting the values of  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial v}{\partial y}$  and  $\frac{\partial w}{\partial z}$ , we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

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Hence the velocity field  $V = x^2yi + y^2zj - (2xyz + yz^2)k$  is a possible case of fluid flow. **Ans.**

### Velocity at (2, 1, 3)

Substituting the values

$x = 2, y = 1$  and  $z = 3$  in velocity field, we get

$$\begin{aligned} V &= x^2yi + y^2zj - (2xyz + yz^2)k \\ &= 2^2 \times 1i + 1^2 \times 3j - (2 \times 2 \times 1 \times 3 + 1 \times 3^2)k \\ &= 4i + 3j - 21k. \text{ Ans.} \end{aligned}$$

and Resultant velocity

$$= \sqrt{4^2 + 3^2 + (-21)^2} = \sqrt{16 + 9 + 441} = \sqrt{466} = 21.587 \text{ units. Ans.}$$

### Acceleration at (2, 1, 3)

The acceleration components  $a_x, a_y$  and  $a_z$  for steady flow are

$$a_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$a_y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$$

$$a_z = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

$$u = x^2y, \frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 \text{ and } \frac{\partial u}{\partial z} = 0$$

$$v = y^2z, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2yz, \frac{\partial v}{\partial z} = y^2$$

$$w = -2xyz - yz^2, \frac{\partial w}{\partial x} = -2yz, \frac{\partial w}{\partial y} = -2xz - z^2, \frac{\partial w}{\partial z} = -2xy - 2yz.$$

Substituting these values in acceleration components, we get acceleration at (2, 1, 3)

$$\begin{aligned} a_x &= x^2y(2xy) + y^2z(x)^2 - (2xyz + yz^2)(0) \\ &= 2x^3y^2 + x^2y^2z \\ &= 2(2)^31^2 + 2^2 \times 1^2 \times 3 = 2 \times 8 + 12 \\ &= 16 + 12 = 28 \text{ units} \end{aligned}$$

$$\begin{aligned} a_y &= x^2y(0) + y^2z(2yz) - (2xyz + yz^2)(y^2) \\ &= 2y^3z^2 - 2xy^3z - y^3z^2 \\ &= 2 \times 1^3 \times 3^2 - 2 \times 2 \times 1^3 \times 3 - 1^3 \times 3^2 = 18 - 12 - 9 = -3 \text{ units} \end{aligned}$$

$$\begin{aligned} a_z &= x^2y(-2yz) + y^2z(-2xz - z^2) - (2xyz + yz^2)(-2xy - 2yz) \\ &= -2x^2y^2z - 2xy^2z^2 - y^2z^3 + [4x^2y^2z + 2xy^2z^2 + 4xy^2z^2 + 2y^2z^3] \\ &= -2 \times 2^2 \times 1^2 \times 3 - 2 \times 2 \times 1^2 \times 3^2 - 1^2 \times 3^3 \\ &\quad + [4 \times 2^2 \times 1^2 \times 3 + 2 \times 2 \times 1^2 \times 3^2 + 4 \times 2 \times 1^2 \times 3^2 + 2 \times 1^2 \times 3^3] \\ &= -24 - 36 - 27 + [48 + 36 + 72 + 54] \\ &= -24 - 36 - 27 + 48 + 36 + 72 + 54 = 123 \end{aligned}$$

$\therefore$  Acceleration

$$= a_x i + a_y j + a_z k = 28i - 3j + 123k. \text{ Ans.}$$

or      Resultant acceleration =  $\sqrt{28^2 + (-3)^2 + 123^2} = \sqrt{784 + 9 + 15129}$   
 $= \sqrt{15922} = 126.18$  units. Ans.

**Problem 5.9** Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m diameter to 0.2 m diameter over 2 m length. The rate of flow is 20 lit/s. If the rate of flow changes uniformly from 20 l/s to 40 l/s in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.

**Solution.** Given :

Diameter at section 1,  $D_1 = 0.4$  m ;  $D_2 = 0.2$  m,  $L = 2$  m,  $Q = 20$  l/s =  $0.02 \text{ m}^3/\text{s}$  as one litre  
 $= 0.001 \text{ m}^3 = 1000 \text{ cm}^3$

Find (i) Convective acceleration at middle i.e., at A when  $Q = 20$  l/s.

(ii) Total acceleration at A when  $Q$  changes from 20 l/s to 40 l/s in 30 seconds.

**Case I.** In this case, the rate of flow is constant and equal to  $0.02 \text{ m}^3/\text{s}$ . The velocity of flow is in  $x$ -direction only. Hence this is one-dimensional flow and velocity components in  $y$  and  $z$  directions are zero or  $v = 0, z = 0$ .

$$\therefore \text{Convective acceleration } = u \frac{\partial u}{\partial y} \text{ only} \quad \dots(i)$$

Let us find the value of  $u$  and  $\frac{\partial u}{\partial x}$  at a distance  $x$  from inlet

The diameter ( $D_x$ ) at a distance  $x$  from inlet or at section X-X is given by,

$$D_x = 0.4 - \frac{0.4 - 0.2}{2} \times x \\ = (0.4 - 0.1x) \text{ m}$$

The area of cross-section ( $A_x$ ) at section X-X is given by,

$$A_x = \frac{\pi}{4} D_x^2 = \frac{\pi}{4} (0.4 - 0.1x)^2$$

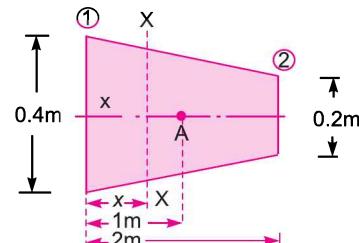


Fig. 5.8

Velocity ( $u$ ) at the section X-X in terms of  $Q$  (i.e., in terms of rate of flow)

$$u = \frac{Q}{\text{Area}} = \frac{Q}{A_x} = \frac{Q}{\frac{\pi}{4} D_x^2} = \frac{4Q}{\pi (0.4 - 0.1x)^2} \\ = \frac{1.273Q}{(0.4 - 0.1x)^2} = 1.273 Q (0.4 - 0.1x)^{-2} \text{ m/s} \quad \dots(ii)$$

To find  $\frac{\partial u}{\partial x}$ , we must differentiate equation (ii) with respect to  $x$ .

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} [1.273 Q (0.4 - 0.1x)^{-2}] \\ &= 1.273 Q (-2)(0.4 - 0.1x)^{-3} \times (-0.1) \quad [\text{Here } Q \text{ is constant}] \\ &= 0.2546 Q (0.4 - 0.1x)^{-3} \end{aligned} \quad \dots(iii)$$

Substituting the value of  $u$  and  $\frac{\partial u}{\partial x}$  in equation (i), we get

$$\begin{aligned} \text{Convective acceleration} &= [1.273 Q (0.4 - 0.1x)^{-2}] \times [0.2546 Q (0.4 - 0.1x)^{-3}] \\ &= 1.273 \times 0.2546 \times Q^2 \times (0.4 - 0.1x)^{-5} \end{aligned}$$

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$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1x)^{-3} \quad [\because Q = 0.02 \text{ m}^3/\text{s}]$$

$\therefore$  Convective acceleration at the middle (where  $x = 1 \text{ m}$ )

$$\begin{aligned} &= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1 \times 1)^{-3} \text{ m/s}^2 \\ &= 1.273 \times 0.2546 \times (0.02)^2 \times (0.3)^{-3} \text{ m/s}^2 \\ &= \mathbf{0.0048 \text{ m/s}^2. \text{ Ans.}} \end{aligned}$$

**Case II.** When  $Q$  changes from  $0.02 \text{ m}^3/\text{s}$  to  $0.04 \text{ m}^3/\text{s}$  in 30 seconds, find the total acceleration at  $x = 1 \text{ m}$  and  $t = 15 \text{ seconds}$ .

Total acceleration = Convective acceleration + Local acceleration at  $t = 15 \text{ seconds}$ .

The rate of flow at  $t = 15 \text{ seconds}$  is given by

$$\begin{aligned} Q &= Q_1 + \frac{Q_2 - Q_1}{30} \times 15 \text{ where } Q_2 = 0.04 \text{ m}^3/\text{s} \text{ and } Q_1 = 0.02 \text{ m}^3/\text{s} \\ &= 0.02 + \frac{(0.04 - 0.02)}{30} \times 15 = 0.03 \text{ m}^3/\text{s} \end{aligned}$$

The velocity ( $u$ ) and gradient  $\left(\frac{\partial u}{\partial x}\right)$  in terms of  $Q$  are given by equations (ii) and (iii) respectively

$$\begin{aligned} \therefore \text{Convective acceleration} &= u \cdot \frac{\partial u}{\partial x} \\ &= [1.273 Q (0.4 - 0.1x)^{-2}] \times [0.2546 Q (0.4 - 0.1x)^{-1}] \\ &= 1.273 \times 0.2546 Q^2 \times (0.4 - 0.1 \times 1)^{-3} \end{aligned}$$

$$\begin{aligned} \therefore \text{Convective acceleration (when } Q = 0.03 \text{ m}^3/\text{s} \text{ and } x = 1 \text{ m}) \\ &= 1.273 \times 0.2546 \times (0.03)^2 \times (0.4 - 0.1 \times 1)^{-3} \\ &= 1.273 \times 0.2546 \times (0.03)^2 \times (0.3)^{-3} \text{ m/s}^2 \\ &= 0.0108 \text{ m/s}^2 \quad \dots(iv) \end{aligned}$$

$$\begin{aligned} \text{Local acceleration} &= \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} [1.273 Q (0.4 - 0.1x)^{-2}] \\ &= 1.273 \times (0.4 - 0.1x)^{-2} \times \frac{\partial Q}{\partial t} \\ &= 1.273 \times (0.4 - 0.1x)^{-2} \times \frac{0.02}{30} \end{aligned}$$

[ $\because$  Local acceleration is at a point where  $x$  is constant but  $Q$  is changing]

Local acceleration (at  $x = 1 \text{ m}$ )

$$\begin{aligned} &= 1.273 \times (0.4 - 0.1 \times 1)^{-2} \times \frac{\partial Q}{\partial t} \\ &= 1.273 \times (0.3)^{-2} \times \frac{0.02}{30} \quad \left[ \because \frac{\partial Q}{\partial t} = \frac{Q_2 - Q_1}{t} = \frac{0.04 - 0.02}{30} = \frac{0.02}{30} \right] \\ &= 0.00943 \text{ m/s}^2 \quad \dots(v) \end{aligned}$$

Hence adding equations (iv) and (v), we get total acceleration.

$\therefore$  Total acceleration = Convective acceleration + Local acceleration

$$= 0.0108 + 0.00943 = \mathbf{0.02023 \text{ m/s}^2. \text{ Ans.}}$$

## ► 5.8 VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

**5.8.1 Velocity Potential Function.** It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by  $\phi$  (Phi). Mathematically, the velocity potential is defined as  $\phi = f(x, y, z)$  for steady flow such that

$$\left. \begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned} \right\} \quad \dots(5.9)$$

where  $u$ ,  $v$  and  $w$  are the components of velocity in  $x$ ,  $y$  and  $z$  directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by

$$\left. \begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{1}{r} \frac{\partial \phi}{\partial \theta} \end{aligned} \right\} \quad \dots(5.9A)$$

where  $u_r$  = velocity component in radial direction (*i.e.*, in  $r$  direction)

and  $u_\theta$  = velocity component in tangential direction (*i.e.*, in  $\theta$  direction)

The continuity equation for an incompressible steady flow is  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ .

Substituting the values of  $u$ ,  $v$  and  $w$  from equation (5.9), we get

$$\frac{\partial}{\partial x} \left( -\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( -\frac{\partial \phi}{\partial z} \right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad \dots(5.10)$$

Equation (5.10) is a Laplace equation.

For two-dimension case, equation (5.10) reduces to  $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ . ...(5.11)

If any value of  $\phi$  that satisfies the Laplace equation, will correspond to some case of fluid flow.

**Properties of the Potential Function.** The rotational components\* are given by

$$\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

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\* Please, refer to equation (5.17) on page 192.

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$$\omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$

$$\omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)$$

Substituting the values, of  $u$ ,  $v$  and  $w$  from equation (5.9) in the above rotational components, we get

$$\omega_z = \frac{1}{2} \left[ \frac{\partial}{\partial x} \left( -\frac{\partial \phi}{\partial y} \right) - \frac{\partial}{\partial y} \left( -\frac{\partial \phi}{\partial x} \right) \right] = \frac{1}{2} \left[ -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} \right]$$

$$\omega_y = \frac{1}{2} \left[ \frac{\partial}{\partial z} \left( -\frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial x} \left( -\frac{\partial \phi}{\partial z} \right) \right] = \frac{1}{2} \left[ -\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z} \right]$$

and

$$\omega_x = \frac{1}{2} \left[ \frac{\partial}{\partial y} \left( -\frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} \left( -\frac{\partial \phi}{\partial y} \right) \right] = \frac{1}{2} \left[ -\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y} \right]$$

If  $\phi$  is a continuous function, then  $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ ;  $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$ ; etc.

$$\therefore \omega_z = \omega_y = \omega_x = 0.$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential ( $\phi$ ) exists, the flow should be irrotational.
2. If velocity potential ( $\phi$ ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

**5.8.2 Stream Function.** It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by  $\psi$  (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as  $\psi = f(x, y)$  such that

$$\left. \begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \frac{\partial \psi}{\partial y} &= -u \end{aligned} \right\} \quad \dots(5.12)$$

and

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r} \quad \dots(5.12A)$$

where  $u_r$  = radial velocity and  $u_\theta$  = tangential velocity

The continuity equation for two-dimensional flow is  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ .

Substituting the values of  $u$  and  $v$  from equation (5.12), we get

$$\frac{\partial}{\partial x} \left( -\frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) = 0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

Hence existence of  $\psi$  means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component  $\omega_z$  is given by  $\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$ .

Substituting the values of  $u$  and  $v$  from equation (5.12) in the above rotational component, we get

$$\omega_z = \frac{1}{2} \left[ \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left( -\frac{\partial \psi}{\partial y} \right) \right] = \frac{1}{2} \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right]$$

For irrotational flow,  $\omega_z = 0$ . Hence above equation becomes as  $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

which is Laplace equation for  $\psi$ .

The properties of stream function ( $\psi$ ) are :

1. If stream function ( $\psi$ ) exists, it is a possible case of fluid flow which may be rotational or irrotational.
2. If stream function ( $\psi$ ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

**5.8.3 Equipotential Line.** A line along which the velocity potential  $\phi$  is constant, is called equipotential line.

For equipotential line  $\phi = \text{Constant}$

$$\therefore d\psi = 0$$

But  $\phi = f(x, y)$  for steady flow

$$\therefore d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= -udx - vdy \quad \left\{ \because \frac{\partial \phi}{\partial x} = -u, \frac{\partial \phi}{\partial y} = -v \right\}$$

$$= -(udx + vdy).$$

For equipotential line,  $d\phi = 0$

$$\text{or } -(udx + vdy) = 0 \text{ or } udx + vdy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{u}{v} \quad \dots(5.13)$$

But  $\frac{dy}{dx}$  = Slope of equipotential line.

**5.8.4 Line of Constant Stream Function**

$\psi = \text{Constant}$

$$\therefore d\psi = 0$$

$$\text{But } d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = +vdx - udy \quad \left\{ \because \frac{\partial \psi}{\partial x} = v; \frac{\partial \psi}{\partial y} = -u \right\}$$

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For a line of constant stream function

$$= d\psi = 0 \text{ or } vdx - udy = 0$$

or

$$\frac{dy}{dx} = \frac{v}{u} \quad \dots(5.14)$$

But  $\frac{dy}{dx}$  is slope of stream line.

From equations (5.13) and (5.14) it is clear that the product of the slope of the equipotential line and the slope of the stream line at the point of intersection is equal to  $-1$ . Thus the equipotential lines are orthogonal to the stream lines at all points of intersection.

**5.8.5 Flow Net.** A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analysing two-dimensional irrotational flow problems.

### 5.8.6 Relation between Stream Function and Velocity Potential Function

From equation (5.9),

we have

$$u = -\frac{\partial \phi}{\partial x} \text{ and } v = -\frac{\partial \phi}{\partial y}$$

From equation (5.12), we have  $u = -\frac{\partial \psi}{\partial y}$  and  $v = \frac{\partial \psi}{\partial x}$

Thus, we have

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Hence

$$\left. \begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned} \right\} \quad \dots(5.15)$$

**Problem 5.10** The velocity potential function ( $\phi$ ) is given by an expression

$$\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$$

(i) Find the velocity components in  $x$  and  $y$  direction.

(ii) Show that  $\phi$  represents a possible case of flow.

**Solution.** Given :  $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$

The partial derivatives of  $\phi$  w.r.t.  $x$  and  $y$  are

$$\frac{\partial \phi}{\partial x} = -\frac{y^3}{3} - 2x + \frac{3x^2y}{3} \quad \dots(1)$$

and

$$\frac{\partial \phi}{\partial y} = -\frac{3xy^2}{3} + \frac{x^3}{3} + 2y \quad \dots(2)$$

(i) The velocity components  $u$  and  $v$  are given by equation (5.9)

$$u = -\frac{\partial \phi}{\partial x} = -\left[-\frac{y^3}{3} - 2x + \frac{3x^2y}{3}\right] = \frac{y^3}{3} + 2x - x^2y$$

$$\therefore u = \frac{y^2}{3} + 2x - x^2y. \text{ Ans.}$$

$$\therefore v = -\frac{\partial \phi}{\partial y} = -\left[-\frac{3xy^2}{3} + \frac{x^3}{3} + 2y\right] = \frac{3xy^2}{3} - \frac{x^3}{3} - 2y = xy^2 - \frac{x^3}{3} - 2y.$$

Ans.

(ii) The given value of  $\phi$ , will represent a possible case of flow if it satisfies the Laplace equation, i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

From equations (1) and (2), we have

Now  $\frac{\partial \phi}{\partial x} = -y^3/3 - 2x + x^2y$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = -2 + 2xy$$

and  $\frac{\partial \phi}{\partial y} = -xy^2 + \frac{x^3}{3} + 2y$

$$\therefore \frac{\partial^2 \phi}{\partial y^2} = -2xy + 2$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (-2 + 2xy) + (-2xy + 2) = 0$$

$\therefore$  Laplace equation is satisfied and hence  $\phi$  represent a possible case of flow. Ans.

**Problem 5.11** The velocity potential function is given by  $\phi = 5(x^2 - y^2)$ .

Calculate the velocity components at the point (4, 5).

**Solution.**

$$\phi = 5(x^2 - y^2)$$

$$\therefore \frac{\partial \phi}{\partial x} = 10x$$

$$\frac{\partial \phi}{\partial y} = -10y.$$

But velocity components  $u$  and  $v$  are given by equation (5.9) as

$$u = -\frac{\partial \phi}{\partial x} = -10x$$

$$v = -\frac{\partial \phi}{\partial y} = -(-10y) = 10y$$

The velocity components at the point (4, 5), i.e., at  $x = 4, y = 5$

$$u = -10 \times 4 = -40 \text{ units. Ans.}$$

$$v = 10 \times 5 = 50 \text{ units. Ans.}$$

**Problem 5.12** A stream function is given by  $\psi = 5x - 6y$ .

Calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

**Solution.**

$$\psi = 5x - 6y$$

$$\therefore \frac{\partial \psi}{\partial x} = 5 \text{ and } \frac{\partial \psi}{\partial y} = -6.$$

But the velocity components  $u$  and  $v$  in terms of stream function are given by equation (5.12) as

$$u = -\frac{\partial \psi}{\partial y} = -(-6) = 6 \text{ units/sec. Ans.}$$

$$v = \frac{\partial \psi}{\partial x} = 5 \text{ units/sec. Ans.}$$

$$\text{Resultant velocity} = \sqrt{u^2 + v^2} = \sqrt{6^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61} = 7.81 \text{ unit/sec}$$

$$\text{Direction is given by, } \tan \theta = \frac{v}{u} = \frac{5}{6} = 0.833$$

$$\therefore \theta = \tan^{-1} 0.833 = 39^\circ 48'. \text{ Ans.}$$

**Problem 5.13** If for a two-dimensional potential flow, the velocity potential is given by

$$\phi = x(2y - 1)$$

determine the velocity at the point  $P (4, 5)$ . Determine also the value of stream function  $\psi$  at the point  $P$ .

**Solution.** Given :  $\phi = x(2y - 1)$

(i) The velocity components in the direction of  $x$  and  $y$  are

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x} [x(2y - 1)] = -[2y - 1] = 1 - 2y$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y} [x(2y - 1)] = -[2x] = -2x$$

At the point  $P (4, 5)$ , i.e., at  $x = 4, y = 5$

$$u = 1 - 2 \times 5 = -9 \text{ units/sec}$$

$$v = -2 \times 4 = -8 \text{ units/sec}$$

$$\therefore \text{Velocity at } P = -9i - 8j$$

$$\text{or Resultant velocity at } P = \sqrt{9^2 + 8^2} = \sqrt{81 + 64} = 12.04 \text{ units/sec} = 12.04 \text{ units/sec. Ans.}$$

(ii) Value of Stream Function at  $P$

$$\text{We know that } \frac{\partial \psi}{\partial y} = -u = -(1 - 2y) = 2y - 1 \quad \dots(i)$$

$$\text{and } \frac{\partial \psi}{\partial x} = v = -2x \quad \dots(ii)$$

Integrating equation (i) w.r.t. 'y', we get

$$\int d\psi = \int (2y - 1) dy \text{ or } \psi = \frac{2y^2}{2} - y + \text{Constant of integration.}$$

The constant of integration is not a function of  $y$  but it can be a function of  $x$ . Let the value of constant of integration is  $k$ . Then

$$\psi = y^2 - y + k. \quad \dots(iii)$$

Differentiating the above equation w.r.t. ' $x$ ', we get

$$\frac{\partial \psi}{\partial x} = \frac{\partial k}{\partial x}.$$

But from equation (ii),  $\frac{\partial \psi}{\partial x} = -2x$

Equating the value of  $\frac{\partial \psi}{\partial x}$ , we get  $\frac{\partial k}{\partial x} = -2x$ .

Integrating this equation, we get  $k = \int -2x dx = -\frac{2x^2}{2} = -x^2$ .

Substituting this value of  $k$  in equation (iii), we get  $\psi = y^2 - y - x^2$ . Ans.

$\therefore$  Stream function  $\psi$  at  $P(4, 5) = 5^2 - 5 - 4^2 = 25 - 5 - 16 = 4$  units. Ans.

**Problem 5.14** The stream function for a two-dimensional flow is given by  $\psi = 2xy$ , calculate the velocity at the point  $P(2, 3)$ . Find the velocity potential function  $\phi$ .

**Solution.** Given :  $\psi = 2xy$

The velocity components  $u$  and  $v$  in terms of  $\psi$  are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(2xy) = -2x$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y.$$

At the point  $P(2, 3)$ , we get  $u = -2 \times 2 = -4$  units/sec

$$v = 2 \times 3 = 6 \text{ units/sec}$$

$\therefore$  Resultant velocity at  $P = \sqrt{u^2 + v^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} = 7.21$  units/sec.

#### Velocity Potential Function $\phi$

We know

$$\frac{\partial \phi}{\partial x} = -u = -(-2x) = 2x \quad \dots(i)$$

$$\frac{\partial \phi}{\partial y} = -v = -2y \quad \dots(ii)$$

Integrating equation (i), we get

$$\int d\phi = \int 2x dx$$

$$\text{or} \quad \phi = \frac{2x^2}{2} + C = x^2 + C \quad \dots(iii)$$

where  $C$  is a constant which is independent of  $x$  but can be a function of  $y$ .

Differentiating equation (iii) w.r.t. ' $y$ ', we get  $\frac{\partial \phi}{\partial y} = \frac{\partial C}{\partial y}$

But from (ii),

$$\frac{\partial \phi}{\partial y} = -2y$$

∴

$$\frac{\partial C}{\partial y} = -2y$$

Integrating this equation, we get  $C = \int -2y dy = -\frac{2y^2}{2} = -y^2$

Substituting this value of  $C$  in equation (iii), we get  $\phi = x^2 - y^2$ . Ans.

**Problem 5.15** Sketch the stream lines represented by  $\psi = x^2 + y^2$ .

Also find out the velocity and its direction at point (1, 2).

**Solution.** Given :  $\psi = x^2 + y^2$

The velocity components  $u$  and  $v$  are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y} (x^2 + y^2) = -2y$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2) = 2x$$

At the point (1, 2), the velocity components are

$$u = -2 \times 2 = -4 \text{ units/sec}$$

$$v = 2 \times 1 = 2 \text{ units/sec}$$

Resultant velocity

$$= \sqrt{u^2 + v^2} = \sqrt{(-4)^2 + 2^2}$$

$$= \sqrt{20} = 4.47 \text{ units/sec}$$

and

$$\tan \theta = \frac{v}{u} = \frac{2}{-4} = \frac{1}{-2}$$

$$\therefore \theta = \tan^{-1} .5 = 26^\circ 34'$$

∴ Resultant velocity makes an angle of  $26^\circ 34'$  with  $x$ -axis.

### Sketch of Stream Lines

$$\psi = x^2 + y^2$$

Let

$$\psi = 1, 2, 3 \text{ and so on.}$$

Then we have

$$1 = x^2 + y^2$$

$$2 = x^2 + y^2$$

$$3 = x^2 + y^2$$

and so on.

Each equation is a equation of a circle. Thus we shall get concentric circles of different diameters as shown in Fig. 5.10.

**Problem 5.16** The velocity components in a two-dimensional flow field for an incompressible fluid are as follows :

$$u = \frac{y^3}{3} + 2x - x^2y \text{ and } v = xy^2 - 2y - x^3/3$$

obtain an expression for the stream function  $\psi$ .

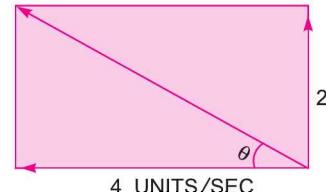


Fig. 5.9

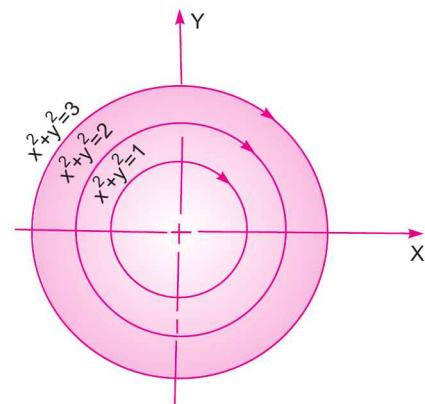


Fig. 5.10

**Solution.** Given :  $u = y^3/3 + 2x - x^2y$   
 $v = xy^2 - 2y - x^3/3$ .

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = xy^2 - 2y - x^3/3 \quad \dots(i)$$

$$\frac{\partial \psi}{\partial y} = -u = -y^3/3 - 2x + x^2y \quad \dots(ii)$$

Integrating (i) w.r.t.  $x$ , we get  $\psi = \int (xy^2 - 2y - x^3/3) dx$

$$\text{or } \psi = \frac{x^2y^2}{2} - 2xy - \frac{x^4}{4 \times 3} + k, \quad \dots(iii)$$

where  $k$  is a constant of integration which is independent of  $x$  but can be a function of  $y$ .

Differentiating equation (iii) w.r.t.  $y$ , we get

$$\frac{\partial \psi}{\partial y} = \frac{2x^2y}{2} - 2x + \frac{\partial k}{\partial y} = x^2y - 2x + \frac{\partial k}{\partial y}$$

But from (ii),  $\frac{\partial \psi}{\partial y} = -y^3/3 - 2x + x^2y$

Comparing the value of  $\frac{\partial \psi}{\partial y}$ , we get  $x^2y - 2x + \frac{\partial k}{\partial y} = -y^3/3 - 2x + x^2y$

$$\therefore \frac{\partial k}{\partial y} = -y^3/3$$

Integrating, we get  $k = \int (-y^3/3) dy = \frac{-y^4}{4 \times 3} = \frac{-y^4}{12}$

Substituting this value in (iii), we get

$$\psi = \frac{x^2y^2}{2} - 2xy - \frac{x^4}{12} - \frac{y^4}{12}. \text{ Ans.}$$

**Problem 5.17** In a two-dimensional incompressible flow, the fluid velocity components are given by  $u = x - 4y$  and  $v = -y - 4x$ .

Show that velocity potential exists and determine its form. Find also the stream function.

**Solution.** Given :  $u = x - 4y$  and  $v = -y - 4x$

$$\therefore \frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

Hence flow is continuous and velocity potential exists.

Let  $\phi = \text{Velocity potential.}$

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Let velocity components in terms of velocity potential is given by

$$\frac{\partial \phi}{\partial x} = -u = -(x - 4y) = -x + 4y \quad \dots(i)$$

and  $\frac{\partial \phi}{\partial y} = -v = -(-y - 4x) = y + 4x \quad \dots(ii)$

Integrating equation (i), we get  $\phi = -\frac{x^2}{2} + 4xy + C \quad \dots(iii)$

where  $C$  is a constant of integration, which is independent of  $x$ .

This constant can be a function of  $y$ .

Differentiating the above equation, i.e., equation (iii) with respect to 'y', we get

$$\frac{\partial \phi}{\partial y} = 0 + 4x + \frac{\partial C}{\partial y}$$

But from equation (iii), we have  $\frac{\partial \phi}{\partial y} = y + 4x$

Equating the two values of  $\frac{\partial \phi}{\partial y}$ , we get

$$4x + \frac{\partial C}{\partial y} = y + 4x \quad \text{or} \quad \frac{\partial C}{\partial y} = y$$

Integrating the above equation, we get

$$C = \frac{y^2}{2} + C_1$$

where  $C_1$  is a constant of integration, which is independent of  $x$  and  $y$ .

Taking it equal to zero, we get  $C = \frac{y^2}{2}$ .

Substituting the value of  $C$  in equation (iii), we get

$$\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2}. \text{ Ans.}$$

### Value of Stream functions

Let  $\psi$  = Stream function

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = -y - 4x \quad \dots(iv)$$

and  $\frac{\partial \psi}{\partial y} = -u = -(x - 4y) = -x + 4y \quad \dots(v)$

Integrating equation (iv) w.r.t.  $x$ , we get

$$\psi = -yx - \frac{4x^2}{2} + k \quad \dots(vi)$$

where  $k$  is a constant of integration which is independent of  $x$  but can be a function of  $y$ .

Differentiating equation (vi) w.r.t.  $y$ , we get  $\frac{\partial \psi}{\partial y} = -x - 0 + \frac{\partial k}{\partial y}$

But from equation (v), we have  $\frac{\partial \psi}{\partial y} = -x + 4y$

Equating the two values of  $\frac{\partial \psi}{\partial y}$ , we get  $-x + \frac{\partial k}{\partial y} = -x + 4y$  or  $\frac{\partial k}{\partial y} = 4y$

Integrating the above equation, we get  $k = \frac{4y^2}{2} = 2y^2$

Substituting the value of  $k$  in equation (vi), we get

$$\psi = -yx - 2x^2 + 2y^2. \text{ Ans.}$$

## ► 5.9 TYPES OF MOTION

A fluid particle while moving may undergo anyone or combination of following four types of displacements :

- (i) Linear Translation or Pure Translation,
- (ii) Linear Deformation,
- (iii) Angular Deformation, and
- (iv) Rotation.

**5.9.1 Linear Translation.** It is defined as the movement of a fluid element in such a way that it moves bodily from one position to another position and the two axes  $ab$  and  $cd$  represented in new positions by  $a'b'$  and  $c'd'$  are parallel as shown in Fig. 5.11 (a).

**5.9.2 Linear Deformation.** It is defined as the deformation of a fluid element in linear direction when the element moves. The axes of the element in the deformed position and un-deformed position are parallel, but their lengths change as shown in Fig. 5.11 (b).

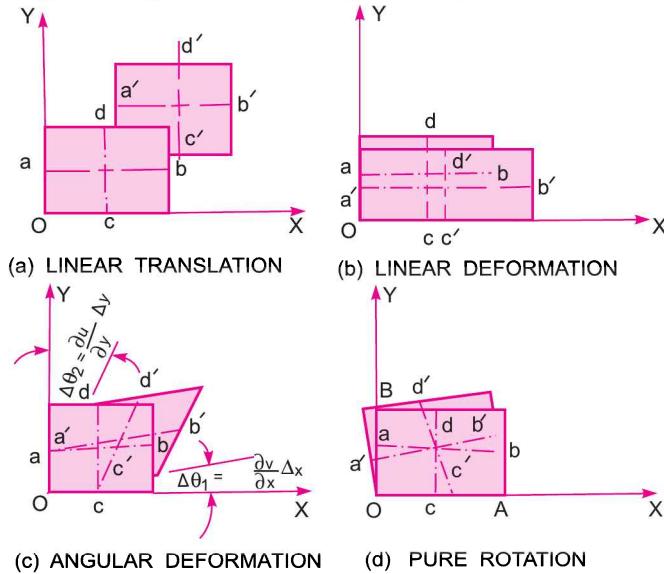


Fig. 5.11. Displacement of a fluid element.

**5.9.3 Angular Deformation or Shear Deformation.** It is defined as the average change in the angle contained by two adjacent sides. Let  $\Delta\theta_1$  and  $\Delta\theta_2$  be the change in angle between two adjacent sides of a fluid element as shown in Fig. 5.11 (c), then angular deformation or shear strain rate

$$= \frac{1}{2} [\Delta\theta_1 + \Delta\theta_2]$$

Now  $\Delta\theta_1 = \frac{\partial v}{\partial x} \times \frac{\Delta x}{\Delta x} = \frac{\partial v}{\partial x}$  and  $\Delta\theta_2 = \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta y} = \frac{\partial u}{\partial y}$ .

∴ **Angular deformation**  $= \frac{1}{2} [\Delta\theta_1 + \Delta\theta_2]$

or **Shear strain rate**  $= \frac{1}{2} \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] \quad \dots(5.16)$

**5.9.4 Rotation.** It is defined as the movement of a fluid element in such a way that both of its axes (horizontal as well as vertical) rotate in the same direction as shown in Fig. 5.11 (d). It is equal

to  $\frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$  for a two-dimensional element in  $x$ - $y$  plane. The rotational components are

$$\left. \begin{aligned} \omega_z &= \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ \omega_x &= \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\ \omega_y &= \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \end{aligned} \right\} \quad \dots(5.17)$$

**5.9.5 Vorticity.** It is defined as the value twice of the rotation and hence it is given as  $2\omega$ .

**Problem 5.18** A fluid flow is given by  $V = 8x^3i - 10x^2yj$ .

Find the shear strain rate and state whether the flow is rotational or irrotational.

**Solution.** Given :  $V = 8x^3i - 10x^2yj$

$$\therefore u = 8x^3, \frac{\partial u}{\partial x} = 24x^2, \frac{\partial u}{\partial y} = 0$$

and  $v = -10x^2y, \frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2$

(i) Shear strain rate is given by equation (5.16) as

$$= \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \frac{1}{2} (-20xy + 0) = -10xy. \text{ Ans.}$$

(ii) Rotation in  $x - y$  plane is given by equation (5.17) or

$$\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} (-20xy - 0) = -10xy$$

As rotation  $\omega_z \neq 0$ . Hence flow is rotational. **Ans.**

**Problem 5.19** The velocity components in a two-dimensional flow are

$$u = y^3/3 + 2x - x^2y \text{ and } v = xy^2 - 2y - x^3/3.$$

Show that these components represent a possible case of an irrotational flow.

**Solution.** Given :  $u = y^3/3 + 2x - x^2y$

$$\therefore \frac{\partial u}{\partial x} = 2 - 2xy$$

$$\frac{\partial u}{\partial y} = \frac{3y^2}{3} - x^2 = y^2 - x^2$$

Also

$$v = xy^2 - 2y - x^3/3$$

$$\therefore \frac{\partial v}{\partial y} = 2xy - 2$$

$$\frac{\partial v}{\partial x} = y^2 - \frac{3x^2}{3} = y^2 - x^2.$$

(i) For a two-dimensional flow, continuity equation is  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Substituting the value of  $\frac{\partial u}{\partial x}$  and  $\frac{\partial v}{\partial y}$ , we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

$\therefore$  It is a possible case of fluid flow.

(ii) Rotation,  $\omega_z$  is given by  $\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} [(y^2 - x^2) - (y^2 - x^2)] = 0$

$\therefore$  Rotation is zero, which means it is case of irrotational flow. **Ans.**

## ► 5.10 VORTEX FLOW

Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known a 'Vortex Flow'. The vortex flow is of two types namely :

1. Forced vortex flow, and
2. Free vortex flow.

**5.10.1 Forced Vortex Flow.** Forced vortex flow is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow, rotates at constant angular velocity,  $\omega$ . The tangential velocity of any fluid particle is given by

$$v = \omega \times r \quad \dots(5.18)$$

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where  $r$  = Radius of fluid particle from the axis of rotation.

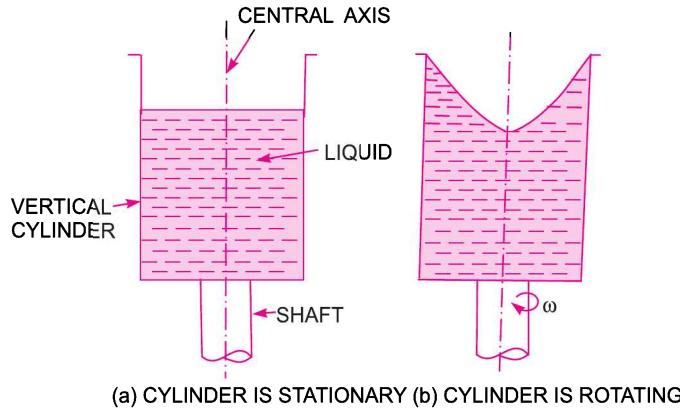


Fig. 5.12 *Forced vortex flow.*

Hence angular velocity  $\omega$  is given by

$$\omega = \frac{v}{r} = \text{Constant.} \quad \dots(5.19)$$

Examples of forced vortex are :

1. A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity  $\omega$ , as shown in Fig. 5.12.
2. Flow of liquid inside the impeller of a centrifugal pump.
3. Flow of water through the runner of a turbine.

**5.10.2 Free Vortex Flow.** When no external torque is required to rotate the fluid mass, that type of flow is called free vortex flow. Thus the liquid in case of free vortex is rotating due to the rotation which is imparted to the fluid previously.

Examples of the free vortex flow are :

1. Flow of liquid through a hole provided at the bottom of a container.
2. Flow of liquid around a circular bend in a pipe.
3. A whirlpool in a river.
4. Flow of fluid in a centrifugal pump casing.

The relation between velocity and radius, in free vortex is obtained by putting the value of external torque equal to zero, or, the time rate of change of angular momentum, i.e., moment of momentum must be zero. Consider a fluid particle of mass 'm' at a radial distance  $r$  from the axis of rotation, having a tangential velocity  $v$ . Then

$$\begin{aligned} \text{Angular momentum} &= \text{Mass} \times \text{Velocity} = m \times v \\ \text{Moment of momentum} &= \text{Momentum} \times r = m \times v \times r \end{aligned}$$

$$\therefore \text{Time rate of change of angular momentum} = \frac{\partial}{\partial t} (mvr)$$

$$\therefore \text{For free vortex } \frac{\partial}{\partial t} (mvr) = 0$$

$$\text{Integrating, we get } mvr = \text{Constant or } vr = \frac{\text{Constant}}{m} = \text{Constant} \quad \dots(5.20)$$

**5.10.3 Equation of Motion for Vortex Flow.** Consider a fluid element  $ABCD$  (shown shaded) in Fig. 5.13 rotating at a uniform velocity in a horizontal plane about an axis perpendicular to the plane of paper and passing through  $O$ .

Let  $r$  = Radius of the element from  $O$ .

$\Delta\theta$  = Angle subtended by the element at  $O$ .

$\Delta r$  = Radial thickness of the element.

$\Delta A$  = Area of cross-section of element.

The forces acting on the element are :

(i) Pressure force,  $p\Delta A$ , on the face  $AB$ .

(ii) Pressure force,  $\left(p + \frac{\partial p}{\partial r} \Delta r\right) \Delta A$  on the face  $CD$ .

(iii) Centrifugal force,  $\frac{mv^2}{r}$  acting in the direction away

from the centre,  $O$ .

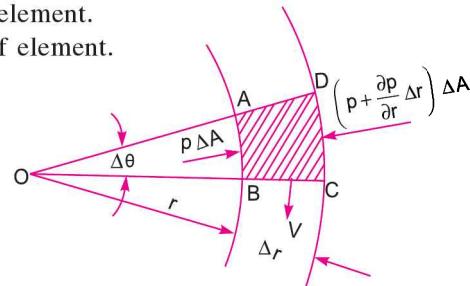


Fig. 5.13

$$\text{Now, the mass of the element} = \text{Mass density} \times \text{Volume} \\ = \rho \times \Delta A \times \Delta r$$

$$\therefore \text{Centrifugal force} = \rho \Delta A \Delta r \frac{v^2}{r}.$$

Equating the forces in the radial direction, we get

$$\left(p + \frac{\partial p}{\partial r} \Delta r\right) \Delta A - p \Delta A = \rho \Delta A \Delta r \frac{v^2}{r}$$

$$\text{or } \frac{\partial p}{\partial r} \Delta r \Delta A = \rho \Delta A \Delta r \frac{v^2}{r}.$$

$$\text{Cancelling } \Delta r \times \Delta A \text{ from both sides, we get } \frac{\partial p}{\partial r} = \rho \frac{v^2}{r} \quad \dots(5.21)$$

Equation (5.21) gives the pressure variation along the radial direction for a forced or free vortex flow in a horizontal plane. The expression  $\frac{\partial p}{\partial r}$  is called pressure gradient in the radial direction. As  $\frac{\partial p}{\partial r}$  is positive, hence pressure increases with the increase of radius ' $r$ '.

The pressure variation in the vertical plane is given by the hydrostatic law, i.e.,

$$\frac{\partial p}{\partial z} = -\rho g \quad \dots(5.22)$$

In equation (5.22),  $z$  is measured vertically in the upward direction.

The pressure,  $p$  varies with respect to  $r$  and  $z$  or  $p$  is a function of  $r$  and  $z$  and hence total derivative of  $p$  is

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz.$$

Substituting the values of  $\frac{\partial p}{\partial r}$  from equation (5.21) and  $\frac{\partial p}{\partial z}$  from equation (5.22), we get

$$dp = \rho \frac{v^2}{r} dr - \rho g dz \quad \dots(5.23)$$

Equation (5.23) gives the variation of pressure of a rotating fluid in any plane.

**5.10.4 Equation of Forced Vortex Flow.** For the forced vortex flow, from equation (5.18), we have

$$v = \omega \times r$$

where  $\omega$  = Angular velocity = Constant.

Substituting the value of  $v$  in equation (5.23), we get

$$dp = \rho \times \frac{\omega^2 r^2}{r} dr - \rho g dz.$$

Consider two points 1 and 2 in the fluid having forced vortex flow as shown in Fig. 5.14. Integrating the above equation for points 1 and 2, we get

$$\int_1^2 dp = \int_1^2 \rho \omega^2 r dr - \int_1^2 \rho g dz$$

or

$$(p_2 - p_1) = \left[ \rho \omega^2 \frac{r^2}{2} \right]_1^2 - \rho g [z]_1^2$$

or

$$\begin{aligned} (p_2 - p_1) &= \frac{\rho \omega^2}{2} [r_2^2 - r_1^2] - \rho g [z_2 - z_1] \\ &= \frac{\rho}{2} [\omega^2 r_2^2 - \omega^2 r_1^2] - \rho g [z_2 - z_1] \\ &= \frac{\rho}{2} [v_2^2 - v_1^2] - \rho g [z_2 - z_1] \quad \left\{ \begin{array}{l} v_2 = \omega r_2 \\ v_1 = \omega r_1 \end{array} \right. \end{aligned}$$

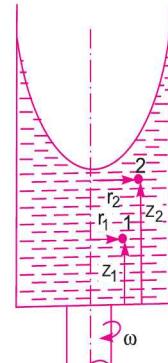


Fig. 5.14

If the points 1 and 2 lie on the free surface of the liquid, then  $p_1 = p_2$  and hence above equation becomes

$$0 = \frac{\rho}{2} [v_2^2 - v_1^2] - \rho g [z_2 - z_1]$$

or

$$\rho g [z_2 - z_1] = \frac{\rho}{2} [v_2^2 - v_1^2]$$

or

$$[z_2 - z_1] = \frac{1}{2g} [v_2^2 - v_1^2].$$

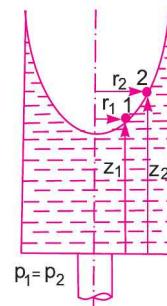


Fig. 5.15

If the point 1 lies on the axis of rotation, then  $v_1 = \omega \times r_1 = \omega \times 0 = 0$ . The above equation becomes as

$$z_2 - z_1 = \frac{1}{2g} v_2^2 = \frac{v_2^2}{2g}$$

Let

$$z_2 - z_1 = Z, \text{ then we have } Z = \frac{v_2^2}{2g} = \frac{\omega^2 \times r_2^2}{2g} \quad \dots(5.24)$$

Thus  $Z$  varies with the square of  $r$ . Hence equation (5.24) is an equation of parabola. This means the free surface of the liquid is a paraboloid.

**Problem 5.20** Prove that in case of forced vortex, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.

**Solution.** Let  $R$  = radius of the cylinder.

$O-O$  = Initial level of liquid in cylinder when the cylinder is not rotating.

$$\therefore \text{Initial height of liquid} = (h + x)$$

$$\therefore \text{Volume of liquid in cylinder} = \pi R^2 \times \text{Height of liquid}$$

$$= \pi R^2 \times (h + x) \quad \dots(i)$$

Let the cylinder is rotated at constant angular velocity  $\omega$ . The liquid will rise at the ends and will fall at the centre.

Let  $y$  = Rise of liquid at the ends from  $O-O$

$x$  = Fall of liquid at the centre from  $O-O$ .

Then volume of liquid

$$\begin{aligned} &= [\text{Volume of cylinder upto level } B-B] \\ &\quad - [\text{Volume of paraboloid}] \\ &= [\pi R^2 \times \text{Height of liquid upto level } B-B] \\ &\quad - \left[ \frac{\pi R^2}{2} \times \text{Height of paraboloid} \right] \\ &= \pi R^2 \times (h + x + y) - \frac{\pi R^2}{2} \times (x + y) \\ &= \pi R^2 \times h + \pi R^2 (x + y) - \frac{\pi R^2}{2} \times (x + y) \\ &= \pi R^2 \times h + \frac{\pi R^2}{2} (x + y) \quad \dots(ii) \end{aligned}$$

Equating (i) and (ii), we get

$$\pi R^2 (h + x) = \pi R^2 \times h + \frac{\pi R^2}{2} (x + y)$$

$$\text{or} \quad \pi R^2 h + \pi R^2 x = \pi R^2 \times h + \frac{\pi R^2}{2} x + \frac{\pi R^2}{2} y$$

$$\text{or} \quad \pi R^2 x - \frac{\pi R^2}{2} x = \frac{\pi R^2}{2} y \quad \text{or} \quad \frac{\pi R^2}{2} x = \frac{\pi R^2}{2} y \quad \text{or} \quad x = y$$

or Fall of liquid at centre = Rise of liquid at the ends.

**Problem 5.21** An open circular tank of 20 cm diameter and 100 cm long contains water upto a height of 60 cm. The tank is rotated about its vertical axis at 300 r.p.m., find the depth of parabola formed at the free surface of water.

**Solution.** Given :

Diameter of cylinder = 20 cm

$$\therefore \text{Radius, } R = \frac{20}{2} = 10 \text{ cm}$$

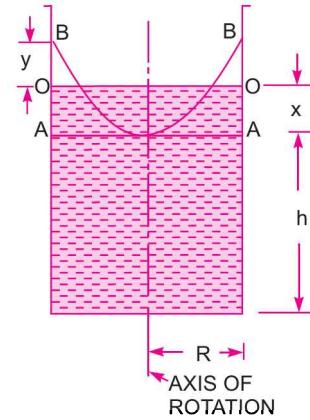


Fig. 5.16

... (ii)

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Height of liquid,  $H = 60 \text{ cm}$   
 Speed,  $N = 300 \text{ r.p.m.}$

Angular velocity,  $\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 300}{60} = 31.41 \text{ rad/sec.}$

Let the depth of parabola  $= Z$

Using equation (5.24),  $Z = \frac{\omega^2 r_2^2}{2g}$ , where  $r_2 = R$   
 $= \frac{\omega^2 R^2}{2g} = \frac{(31.41)^2 \times (10)^2}{2 \times 981} = 50.28 \text{ cm. Ans.}$

**Problem 5.22** An open circular cylinder of 15 cm diameter and 100 cm long contains water upto a height of 80 cm. Find the maximum speed at which the cylinder is to be rotated about its vertical axis so that no water spills.

**Solution.** Given :

Diameter of cylinder  $= 15 \text{ cm}$

$\therefore$  Radius,  $R = \frac{15}{2} = 7.5 \text{ cm}$

Length of cylinder,  $L = 100 \text{ cm}$

Initial height of water  $= 80 \text{ cm.}$

Let the cylinder is rotated at an angular speed of  $\omega$  rad/sec, when the water is about to spill. Then using,

Rise of liquid at ends  $=$  Fall of liquid at centre

But rise of liquid at ends  $=$  Length – Initial height  
 $= 100 - 80 = 20 \text{ cm}$

$\therefore$  Fall of liquid at centre  $= 20 \text{ cm}$

$\therefore$  Height of parabola  $= 20 + 20 = 40 \text{ cm}$

$\therefore Z = 40 \text{ cm}$

Using the relation,  $Z = \frac{\omega^2 R^2}{2g}$ , we get  $40 = \frac{\omega^2 (7.5)^2}{2 \times 981}$

$\therefore \omega^2 = \frac{40 \times 2 \times 981}{7.5 \times 7.5} = 1395.2$

$\therefore \omega = \sqrt{1395.2} = 37.35 \text{ rad/s}$

$\therefore$  Speed,  $N$  is given by  $\omega = \frac{2\pi N}{60}$

or  $N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 37.35}{2 \times \pi} = 356.66 \text{ r.p.m. Ans.}$

**Problem 5.23** A cylindrical vessel 12 cm in diameter and 30 cm deep is filled with water upto the top. The vessel is open at the top. Find the quantity of liquid left in the vessel, when it is rotated about its vertical axis with a speed of (a) 3000 r.p.m., and (b) 600 r.p.m.

**Solution.** Given :

Diameter of cylinder  $= 12 \text{ cm}$

$\therefore$  Radius,  $R = 6 \text{ cm}$

Initial height of water  $= 30 \text{ cm}$

$$\begin{aligned}\text{Initial volume of water} &= \text{Area} \times \text{Initial height of water} \\ &= \frac{\pi}{4} \times 12^2 \times 30 \text{ cm}^3 = 3392.9 \text{ cm}^3\end{aligned}$$

(a) Speed,

$$\therefore \omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.41 \text{ rad/s}$$

$$\text{Height of parabola is given by } Z = \frac{\omega^2 R^2}{2g} = \frac{(31.41)^2 \times 6^2}{2 \times 981} = 18.10 \text{ cm.}$$

As vessel is initially full of water, water will be spilled if it is rotated. Volume of water spilled is equal to the volume of paraboloid.

$$\begin{aligned}\text{But volume of paraboloid} &= [\text{Area of cross-section} \times \text{Height of parabola}] \div 2 \\ &= \frac{\pi}{4} D^2 \times \frac{Z}{2} = \frac{\pi}{4} \times 12^2 \times \frac{18.10}{2} = 1023.53 \text{ cm}^3\end{aligned}$$

$$\begin{aligned}\text{Volume of water left} &= \text{Initial volume} - \text{Volume of water spilled} \\ &= 3392.9 - 1023.53 = 2369.37 \text{ cm}^3. \text{ Ans.}\end{aligned}$$

(b) Speed,

$$\therefore \omega = \frac{2\pi N}{60} = \frac{2\pi \times 600}{60} = 62.82 \text{ rad/s}$$

$$\text{Height of parabola, } Z = \frac{\omega^2 R^2}{2g} = \frac{(62.82)^2 \times 6^2}{2 \times 981} = 72.40 \text{ cm.}$$

As the height of parabola is more than the height of cylinder the shape of imaginary parabola will be as shown in Fig. 5.17.

Let  $r$  = Radius of the parabola at the bottom of the vessel.

$$\begin{aligned}\text{Height of imaginary parabola} &= 72.40 - 30 = 42.40 \text{ cm.}\end{aligned}$$

Volume of water left in the vessel

$$\begin{aligned}&= \text{Volume of water in portions } ABC \text{ and } DEF \\ &= \text{Initial volume of water} \\ &\quad - \text{Volume of paraboloid } AOF \\ &\quad + \text{Volume of paraboloid } COD.\end{aligned}$$

Now volume of paraboloid

$$\begin{aligned}AOF &= \frac{\pi}{4} \times D^2 \times \text{Height of parabola} \\ &= \frac{\pi}{4} \times 12^2 \times \frac{72.4^2}{2} = 4094.12 \text{ cm}^3\end{aligned}$$

For the imaginary parabola ( $COD$ ),  $\omega = 62.82 \text{ rad/sec}$

$$Z = 42.4 \text{ cm}$$

$$r = \text{Radius at the bottom of vessel}$$

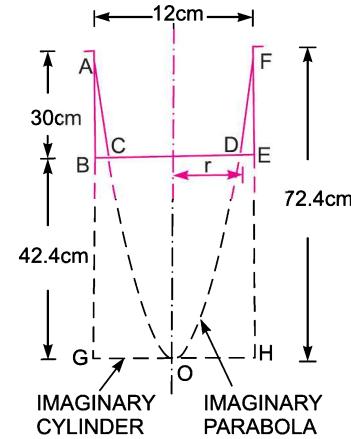


Fig. 5.17

Using the relation  $Z = \frac{\omega^2 r^2}{2g}$ , we get  $42.4 = \frac{62.82^2 \times r^2}{2 \times 981}$

$$\therefore r^2 = \frac{2 \times 981 \times 42.4}{62.82 \times 62.82} = 21.079$$

$$\therefore r = \sqrt{21.079} = 4.59 \text{ cm}$$

$\therefore$  Volume of paraboloid COD

$$\begin{aligned} &= \frac{1}{2} \times \text{Area at the top of the imaginary parabola} \times \text{Height of parabola} \\ &= \frac{1}{2} \times \pi r^2 \times 42.4 = \frac{1}{2} \times \pi \times 4.59^2 \times 42.4 = 1403.89 \text{ cm}^3 \end{aligned}$$

$$\therefore \text{Volume of water left} = 3392.9 - 4094.12 + 1403.89 = 702.67 \text{ cm}^3. \text{ Ans.}$$

**Problem 5.24** An open circular cylinder of 15 cm diameter and 100 cm long contains water upto a height of 70 cm. Find the speed at which the cylinder is to be rotated about its vertical axis, so that the axial depth becomes zero.

**Solution.** Given :

$$\text{Diameter of cylinder} = 15 \text{ cm}$$

$$\therefore \text{Radius, } R = \frac{15}{2} = 7.5 \text{ cm}$$

$$\text{Length of cylinder} = 100 \text{ cm}$$

$$\text{Initial height of water} = 70 \text{ cm.}$$

When axial depth is zero, the depth of paraboloid = 100 cm.

Using the relation,  $Z = \frac{\omega^2 R^2}{2g}$ , we get

$$100 = \frac{\omega^2 \times 7.5^2}{2 \times 9.81}$$

$$\therefore \omega^2 = \frac{100 \times 2 \times 9.81}{7.5 \times 7.5}$$

$$\therefore \omega = \sqrt{\frac{100 \times 2 \times 9.81}{7.5 \times 7.5}} = \frac{442.92}{7.5} = 59.05 \text{ rad/s}$$

$$\therefore \text{Speed, } N \text{ is given by } \omega = \frac{2\pi H}{60}$$

$$\text{or } N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 59.05}{2\pi} = 563.88 \text{ r.p.m. Ans.}$$

**Problem 5.25** For the problem (5.24), find the difference in total pressure force (i) at the bottom of cylinder, and (ii) at the sides of the cylinder due to rotation.

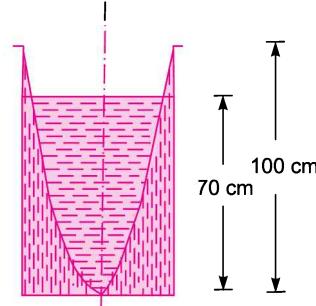


Fig. 5.18

**Solution.** (i) The data is given in Problem 5.24. The difference in total pressure force at the bottom of cylinder is obtained by finding total hydrostatic force at the bottom before rotation and after rotation.

$$\text{Before rotation, } \text{force} = \rho g A \bar{h}$$

$$\text{where } \rho = 1000 \text{ kg/m}^3, A = \text{Area of bottom} = \frac{\pi}{3} D^2 = \frac{\pi}{4} \times (0.15)^2 \text{ m}^2, \bar{h} = 70 \text{ cm} = 0.70 \text{ m}$$

$$\therefore \text{Force} = 1000 \times 9.81 \times \frac{\pi}{4} \times (0.15)^2 \times 0.7 \text{ N} = 121.35 \text{ N}$$

After rotation, the depth of water at the bottom is not constant and hence pressure force due to the height of water, will not be constant. Consider a circular ring of radius  $r$  and width  $dr$  as shown in Fig. 5.19. Let the height of water from the bottom of the tank upto free surface of water at a radius

$$r = Z = \frac{\omega^2 r^2}{2g}.$$

Hydrostatic force on ring at the bottom,

$$\begin{aligned} dF &= \rho g \times \text{Area of ring} \times Z \\ &= 1000 \times 9.81 \times 2\pi r dr \times \frac{\omega^2 r^2}{2g} \\ &= 9810 \times 2 \times \pi r \times \frac{\omega^2 r^2}{2g} \times dr \end{aligned}$$

$\therefore$  Total pressure force at the bottom

$$\begin{aligned} &= \int dF = \int_0^R 9810 \times 2 \times \pi r \times \frac{\omega^2 r^2}{2g} dr \\ &= \int_0^{0.075} 19620 \times \pi \times \frac{\omega^2}{2g} r^3 dr \end{aligned}$$

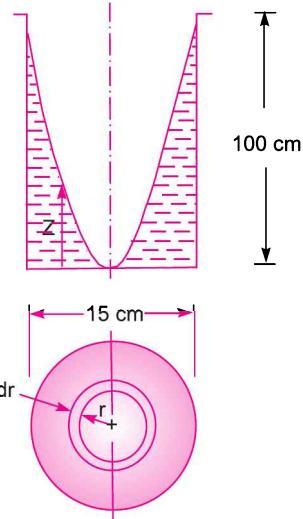


Fig. 5.19

From Problem 5.24,

$$\omega = 59.05 \text{ rad/s}$$

$$R = 7.5 \text{ cm} = .075 \text{ m.}$$

Substituting these values, we get total pressure force

$$\begin{aligned} &= \frac{19620 \times \pi \times (59.05)^2}{2 \times 9.81} \left[ \frac{r^4}{4} \right]_0^{0.075} \\ &= \frac{19620 \times \pi \times (59.05)^2}{2 \times 9.81} \times \frac{(0.075)^4}{4} = 86.62 \text{ N} \end{aligned}$$

$\therefore$  Difference in pressure forces at the bottom

$$121.35 - 86.62 = 34.73 \text{ N. Ans.}$$

(ii) Forces on the sides of the cylinder

$$\text{Before rotation} \quad = \rho g A \bar{h}$$

$$\begin{aligned} \text{where } A &= \text{Surface area of the sides of the cylinder upto height of water} \\ &= \pi D \times \text{Height of water} = \pi \times 0.15 \times 0.70 \text{ m}^2 = 0.33 \text{ m}^2 \end{aligned}$$

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$\bar{h}$  = C.G. of the wetted area of the sides

$$= \frac{1}{2} \times \text{height of water} = \frac{0.70}{2} = 0.35 \text{ m}$$

$\therefore$  Force on the sides before rotation =  $1000 \times 9.81 \times 0.33 \times 0.35 = 1133 \text{ N}$

After rotation, the water is upto the top of the cylinder and hence force on the sides

$$\begin{aligned} &= 1000 \times 9.81 \times \text{Wetted area of the sides} \times \frac{1}{2} \times \text{Height of water} \\ &= 9810 \times \pi D \times 1.0 \times \frac{1}{2} \times 1.0 = 9810 \times \pi \times .15 \times \frac{1}{2} = 2311.43 \text{ N} \end{aligned}$$

$\therefore$  Difference in pressure on the sides

$$2311.43 - 1133 = 1178.43 \text{ N. Ans.}$$

**5.10.5 Closed Cylindrical Vessels.** If a cylindrical vessel is closed at the top, which contains some liquid, the shape of paraboloid formed due to rotation of the vessel will be as shown in Fig. 5.20 for different speed of rotations.

Fig. 5.20 (a) shows the initial stage of the cylinder, when it is not rotated. Fig. 5.20 (b) shows the shape of the paraboloid formed when the speed of rotation is  $\omega_1$ . If the speed is increased further say  $\omega_2$ , the shape of paraboloid formed will be as shown in Fig. 5.20 (c). In this case the radius of the parabola at the top of the vessel is unknown. Also the height of the paraboloid formed corresponding to angular speed  $\omega_2$  is unknown. Thus to solve the two unknowns, we should have two equations. One equation is

$$Z = \frac{\omega^2 r^2}{2g}$$

The second equation is obtained from the fact that for closed vessel, volume of air before rotation is equal to the volume of air after rotation.

Volume of air before rotation = Volume of closed vessel – Volume of liquid in vessel

$$\text{Volume of air after rotation} = \text{Volume of paraboloid formed} = \frac{\pi r^2 \times Z}{2}.$$

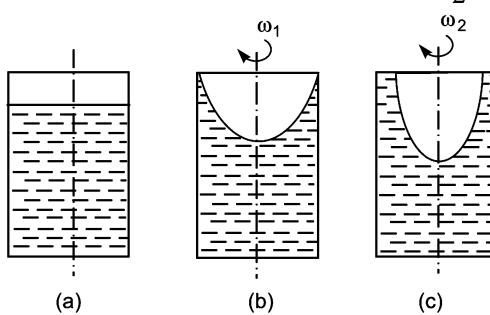


Fig. 5.20

**Problem 5.26** A vessel, cylindrical in shape and closed at the top and bottom, contains water upto a height of 80 cm. The diameter of the vessel is 20 cm and length of vessel is 120 cm. The vessel is rotated at a speed of 400 r.p.m. about its vertical axis. Find the height of paraboloid formed.

**Solution.** Given :

$$\text{Initial height of water} = 80 \text{ cm}$$

$$\text{Diameter of vessel} = 20 \text{ cm}$$

$$\therefore \text{Radius, } R = 10 \text{ cm}$$

$$\text{Length of vessel} = 120 \text{ cm}$$

$$\text{Speed, } N = 400 \text{ r.p.m.}$$

$$\therefore \omega = \frac{2\pi N}{60} = \frac{2\pi \times 400}{60} = 41.88 \text{ rad/s}$$

When the vessel is rotated, let  $Z$

$$= \text{Height of paraboloid formed}$$

$$r = \text{Radius of paraboloid at the top of the vessel}$$

This is the case of closed vessel.

$\therefore$  Volume of air before rotation = Volume of air after rotation

$$\text{or } \frac{\pi}{4} D^2 \times L - \frac{\pi}{4} D^2 \times 80 = \pi r^2 \times \frac{Z}{2}$$

where  $Z$  = Height of paraboloid,  $r$  = Radius of parabola.

$$\text{or } \frac{\pi}{4} D^2 \times 120 - \frac{\pi}{4} D^2 \times 80 = \pi r^2 \times \frac{Z}{2}$$

$$\text{or } \frac{\pi}{4} \times D^2 \times (120 - 80) = \frac{\pi}{4} D^2 \times 40 = \pi r^2 \times \frac{Z}{2}$$

$$\text{or } \frac{\pi}{4} \times 20^2 \times 40 = 4000 \times \pi = \pi r^2 \times \frac{Z}{2}$$

$$\therefore r^2 \times Z = \frac{4000 \times \pi \times 2}{\pi} = 8000 \quad \dots(i)$$

$$\text{Using relation } Z = \frac{\omega^2 r^2}{2g}, \text{ we get } Z = \frac{41.88^2 \times r^2}{2g} = \frac{41.88^2 \times r^2}{2 \times 981} = 0.894 r^2$$

$$\therefore r^2 = \frac{z}{0.894}$$

Substituting this value of  $r^2$  in (i), we get

$$\frac{Z}{0.894} \times Z = 8000$$

$$\therefore Z^2 = 8000 \times 0.894 = 7152$$

$$\therefore Z = \sqrt{7152} = 84.56 \text{ cm. Ans.}$$

### IIInd Method

Let  $Z_1$  = Height of paraboloid, if the vessel would not have been closed at the top, corresponding to speed,

$$N = 400 \text{ r.p.m.}$$

$$\text{or } \omega = 41.88 \text{ rad/s}$$

$$\text{Then } Z_1 = \frac{\omega^2 R^2}{2g} = \frac{41.88^2 \times 10^2}{2 \times 981} = 89.34 \text{ cm.}$$

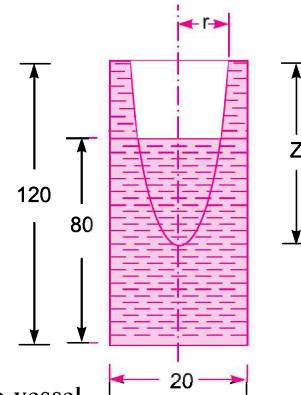


Fig. 5.21

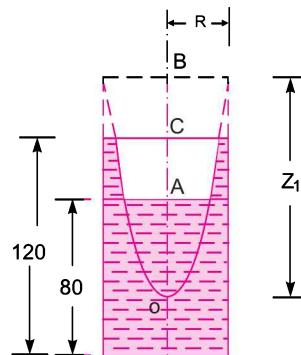


Fig. 5.22

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Half of  $Z_1$  will be below the initial height of water in the vessel

$$\text{i.e., } AO = \frac{Z_1}{2} = \frac{89.34}{2} = 44.67 \text{ cm}$$

But height of paraboloid for closed vessel

$$\begin{aligned} &= CO = CA + AO = (120 - 80) + 44.67 \text{ cm} \\ &= 40 + 44.67 = 84.67 \text{ cm. Ans.} \end{aligned}$$

**Problem 5.27** For the data given in Problem 5.26, find the speed of rotation of the vessel, when axial depth of water is zero.

**Solution.** Given :

$$\begin{array}{ll} \text{Diameter of vessel} & = 20 \text{ cm} \\ \therefore \text{Radius,} & R = 10 \text{ cm} \\ \text{Initial height of water} & = 80 \text{ cm} \\ \text{Length of vessel} & = 120 \text{ cm} \end{array}$$

Let  $\omega$  is the angular speed, when axial depth is zero.

When axial depth is zero, the height of paraboloid is 120 cm and radius of the parabola at the top of the vessel is  $r$ .

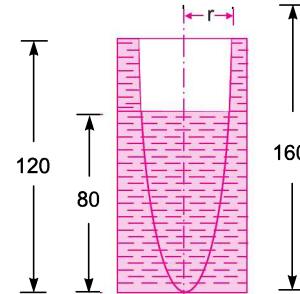


Fig. 5.23

$$\therefore \text{Using the relation, } Z = \frac{\omega^2 r^2}{2g} \text{ or } 120 = \frac{\omega^2 \times r^2}{2 \times 980}$$

$$\therefore \omega^2 r^2 = 2 \times 980 \times 120 = 235200 \quad \dots(i)$$

Volume of air before rotation = Volume of air after paraboloid

$$\therefore \pi R^2 \times (120 - 80) = \text{Volume of paraboloid}$$

$$= \pi r^2 \times \frac{Z}{2}$$

$$\text{or } \pi \times 10^2 \times 40 = \frac{\pi r^2 \times Z}{2} = \frac{\pi r^2}{2} \times 120$$

$$\text{or } r^2 = \frac{\pi \times 10^2 \times 40 \times 2}{\pi \times 120} = \frac{8000}{120} = 66.67$$

Substituting the value of  $r^2$  in equation (i), we get

$$\omega^2 \times 66.67 = 235200$$

$$\omega = \sqrt{\frac{235200}{66.67}} = 59.4 \text{ rad/s}$$

$$\therefore \text{Speed } N \text{ is given by } \omega = \frac{2\pi N}{60}$$

$$\text{or } N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 59.4}{2\pi} = 567.22 \text{ r.p.m. Ans.}$$

**Problem 5.28** The cylindrical vessel of the problem 5.26 is rotated at 700 r.p.m. about its vertical axis. Find the area uncovered at the bottom of the tank.

**Solution.** Given :

$$\begin{array}{ll} \text{Initial height of water} & = 80 \text{ cm} \\ \therefore \text{Diameter of vessel} & = 20 \text{ cm} \\ \therefore \text{Radius,} & R = 10 \text{ cm} \\ \text{Length of vessel} & = 120 \text{ cm} \end{array}$$

Speed,

$$N = 700 \text{ r.p.m.}$$

$$\therefore \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 700}{60} = 73.30 \text{ rad/s.}$$

If the tank is not closed at the top and also is very long, then the height of parabola corresponding to  $\omega = 73.3$  will be

$$= \frac{\omega^2 \times R^2}{2 \times g} = \frac{73.3^2 \times 10^2}{2 \times 980} = 274.12 \text{ cm}$$

From Fig. 5.24,

$$x_1 + 120 + x_2 = 274.12$$

or

$$x_1 + x_2 = 274.12 - 120 = 154.12 \text{ cm} \quad \dots(i)$$

From the parabola,  $KOM$ , we have

$$(120 + x_1) = \frac{\omega^2 r_1^2}{2g} = \frac{73.3^2 \times r_1^2}{2 \times 980} \quad \dots(ii)$$

For the parabola,  $LON$ , we have

$$x_1 = \frac{\omega^2 r_2^2}{2g} = \frac{73.3^2 \times r_2^2}{2 \times 980} \quad \dots(iii)$$

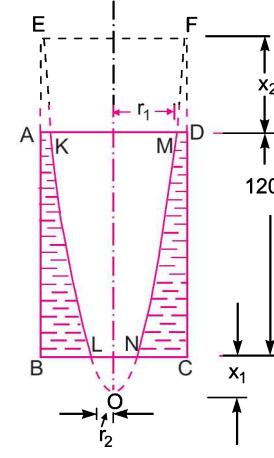


Fig. 5.24

Now, volume of air before rotation = Volume of air after rotation

$$\text{Volume of air before rotation} = \pi R^2 \times (120 - 80) = \pi \times 10^2 \times 40 = 12566.3 \text{ cm}^3 \quad \dots(iv)$$

$$\text{Volume of air after rotation} = \text{Volume of paraboloid } KOM - \text{volume of paraboloid } LON$$

$$= \pi r_1^2 \times \frac{(120 + x_1)}{2} - \pi r_2^2 \times \frac{x_1}{2} \quad \dots(v)$$

Equating (iv) and (v), we get

$$12566.3 = \frac{\pi r_1^2 (120 + x_1)}{2} - \frac{\pi r_2^2 \times x_1}{2} \quad \dots(vi)$$

Substituting the value of  $r_1^2$  from (ii) in (vi), we get

$$12566.3 = \pi \times \frac{(120 + x_1) \times 2 \times 980}{73.3^2} \times \frac{(120 + x_1)}{2} - \frac{\pi r_2^2 \times x_1}{2}$$

$$\left\{ \because \text{From (ii), } r_1^2 = \frac{2 \times 980 \times (120 + x_1)}{(73.3)^2} \right\}$$

$$\text{or} \quad 12566.3 = 0.573 (120 + x_1)^2 - \frac{\pi r_2^2 \times x_1}{2}$$

Substituting the value of  $x_1$  from (iii) in the above equation

$$12566.3 = 0.573 \left( 120 + \frac{73.3^2 \times r_2^2}{2 \times 980} \right)^2 - \frac{\pi r_2^2}{2} \times \frac{73.3^2 \times r_2^2}{2 \times 980}$$

$$= 0.573 (120 + 2.74 r_2^2)^2 - 4.3 \times r_2^2 \times r_2^2$$

$$= 0.573 [120^2 + 2.74^2 r_2^4 + 2 \times 120 \times 2.74 r_2^2] - 4.3 r_2^4$$

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$$= 0.573 [14400 + 7.506 r_2^4 + 657.6 r_2^2] - 4.3 r_2^4$$

$$\frac{12566.3}{0.573} = 21930 = 14400 + 7.506 r_2^4 + 657.6 r_2^2 - 4.3 r_2^4$$

or  $r_2^4 (7.506 - 4.3) + 657.6 r_2^2 + 14400 - 21930 = 0$

or  $3.206 r_2^4 + 657.6 r_2^2 - 7530 = 0$

$$\therefore r_2^2 = \frac{-657.6 \pm \sqrt{657.6^2 - 4 \times (-7530) \times (3.206)}}{2 \times 3.206}$$

$$= \frac{-657.6 \pm \sqrt{432437.76 + 96564.72}}{6.412}$$

$$= \frac{-657.6 \pm 727.32}{6.412} = -215.98 \text{ or } 10.87$$

Negative value is not possible

$$\therefore r_2^2 = 10.87 \text{ cm}^2$$

$$\therefore \text{Area uncovered at the base} = \pi r_2^2 = \pi \times 10.87 = 34.149 \text{ cm}^2. \text{ Ans.}$$

**Problem 5.29** A closed cylindrical vessel of diameter 30 cm and height 100 cm contains water upto a depth of 80 cm. The air above the water surface is at a pressure of 5.886 N/cm<sup>2</sup>. The vessel is rotated at a speed of 250 r.p.m. about its vertical axis. Find the pressure head at the bottom of the vessel : (a) at the centre, and (b) at the edge.

**Solution.** Given :

$$\text{Diameter of vessel} = 30 \text{ cm}$$

$$\therefore \text{Radius, } R = 15 \text{ cm}$$

$$\text{Initial height of water, } H = 80 \text{ cm}$$

$$\text{Length of cylinder, } L = 100 \text{ cm}$$

$$\text{Pressure of air above water} = 5.886 \text{ N/cm}^2$$

$$\text{or } p = 5.886 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

$$\text{Head due to pressure, } h = p/\rho g$$

$$= \frac{5.886 \times 10^4}{1000 \times 9.81} = 6 \text{ m of water}$$

$$\text{Speed, } N = 250 \text{ r.p.m.}$$

$$\therefore \omega = \frac{2\pi N}{60} = \frac{2\pi \times 250}{60} = 26.18 \text{ rad/s}$$

Let  $x_1$  = Height of paraboloid formed, if the vessel is assumed open at the top and it is very long.

$$\text{Then we have } x_1 = \frac{\omega^2 R^2}{2g} = \frac{26.18^2 \times 15^2}{2 \times 981} = 78.60 \text{ cm} \quad \dots(i)$$

Let  $r_1$  is the radius of the actual parabola of height  $x_2$

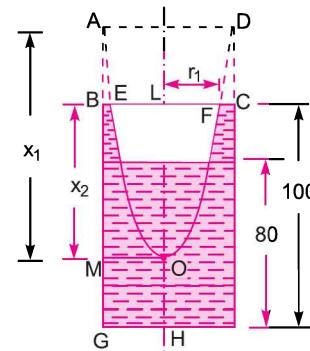


Fig. 5.25

Then  $x_2 = \frac{\omega^2 r_1^2}{2g} = \frac{26.18^2 \times r_1^2}{2 \times 981} = 0.35 r_1^2$  ... (ii)

The volume of air before rotation

$$= \pi R^2 (100 - 80) = \pi \times 15^2 \times 20 = 14137 \text{ cm}^3$$

Volume of air after rotation = Volume of paraboloid EOF

$$= \frac{1}{2} \times \pi r_1^2 \times x_2$$

But volume of air before and after rotation is same.

$$\therefore 14137 = \frac{1}{2} \times \pi r_1^2 \times x_2$$

But from (ii),  $x_2 = 0.35 r_1^2$

$$\therefore 14137 = \frac{1}{2} \times \pi r_1^2 \times 0.35 r_1^2$$

$$\therefore r_1^4 = \frac{2 \times 14137}{\pi \times 0.35}$$

$$r_1 = (25714)^{1/4} = 12.66 \text{ cm}$$

Substituting the value of  $r_1$  in (ii), we get

$$x_2 = 0.35 \times 12.66^2 = 56.1 \text{ cm}$$

### Pressure head at the bottom of the vessel

(a) At the centre. The pressure head at the centre, i.e., at  $H$  = Pressure head due to air + OH

$$= 6.0 + (HL - LO) \quad \left\{ \because OH = LH - LO \right\}$$

$$= 6.0 + (1.0 - 0.561) \quad \left\{ \begin{array}{l} \because HL = 100 \text{ cm} = 1 \text{ m} \\ LO = x_2 = 56.1 \text{ cm} = .561 \text{ m} \end{array} \right\}$$

$$= 6.439 \text{ m of water. Ans.}$$

(b) At the edge, i.e., at

$G$  = Pressure head due to air + height of water above  $G$

$$= 6.0 + AG = 6.0 + (GM + MA) = 6.0 + (HO + x_1)$$

$$= 6.0 + HO + 0.786 \quad \left\{ \because x_1 = 78.6 \text{ cm} = 0.786 \text{ m} \right\}$$

$$= 6.0 + 0.439 + 0.786 \quad \left\{ \begin{array}{l} \because HO = LH - LO = 100 - 56.1 \\ \qquad \qquad \qquad = 43.9 \text{ cm} = 0.439 \text{ m} \end{array} \right\}$$

$$= 7.225 \text{ m of water. Ans.}$$

**Problem 5.30** A closed cylinder of radius  $R$  and height  $H$  is completely filled with water. It is rotated about its vertical axis with a speed of  $\omega$  radians/s. Determine the total pressure exerted by water on the top and bottom of the cylinder.

**Solution.** Given :

$$\text{Radius of cylinder} = R$$

$$\text{Height of cylinder} = H$$

$$\text{Angular speed} = \omega$$

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As the cylinder is closed and completely filled with water, the rise of water level at the ends and depression of water at the centre due to rotation of the vessel, will be prevented. Thus the water will exert force on the complete top of the vessel. Also the pressure will be exerted at the bottom of the cylinder.

**Total Pressure exerted on the top of cylinder.** The top of cylinder is in contact with water and is in horizontal plane. The pressure variation at any radius in horizontal plane is given by equation (5.21)

or

$$\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} = \frac{\rho \omega^2 r^2}{r} = \rho \omega^2 r \quad \{ \because v = \omega \times r \}$$

Integrating, we get

$$\int dp = \int \rho \omega^2 r dr \quad \text{or} \quad p = \frac{\rho \omega^2 r^2}{2} = \frac{\rho}{2} \omega^2 r^2$$

Consider an elementary circular ring of radius  $r$  and width  $dr$  on the top of the cylinder as shown in Fig. 5.26.

$$\text{Area of circular ring} = 2\pi r dr$$

$$\begin{aligned} \therefore \text{Force on the elementary ring} &= \text{Intensity of pressure} \times \text{Area of ring} \\ &= p \times 2\pi r dr \\ &= \frac{\rho}{2} \omega^2 r^2 \times 2\pi r dr. \end{aligned}$$

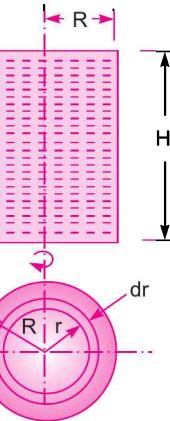


Fig. 5.26

$\therefore$  Total force on the top of the cylinder is obtained by integrating the above equation between the limits 0 and  $R$ .

$$\begin{aligned} \therefore \text{Total force or } F_T &= \int_0^R \frac{\rho}{2} \omega^2 r^2 \times 2\pi r dr = \frac{\rho}{2} \omega^2 \times 2\pi \int_0^R r^3 dr \\ &= \frac{\rho}{2} \omega^2 \times 2\pi \left[ \frac{r^4}{4} \right]_0^R = \frac{\rho}{2} \omega^2 \times 2\pi \times \frac{R^4}{4} \\ &= \frac{\rho \omega^2}{4} \times \pi R^4 \end{aligned} \quad \dots(5.25)$$

Total pressure force on the bottom of cylinder,  $F_B$

$$\begin{aligned} &= \text{Weight of water in cylinder} + \text{total force on the top of cylinder} \\ &= \rho g \times \pi R^2 \times H + \frac{\rho}{4} \omega^2 \times \pi R^4 = \rho g \times \pi R^2 \times H + F_T \end{aligned} \quad \dots(5.26)$$

$\rho$  = Density of water.

**Problem 5.31** A closed cylinder of diameter 200 mm and height 150 mm is completely filled with water. Calculate the total pressure force exerted by water on the top and bottom of the cylinder, if it is rotated about its vertical axis at 200 r.p.m.

**Solution.** Given :

$$\text{Dia. of cylinder} = 200 \text{ mm} = 0.20 \text{ m}$$

$$\text{Radius, } R = 0.1 \text{ m}$$

$$\text{Height of cylinder, } H = 150 \text{ mm} = 0.15 \text{ m}$$

$$\text{Speed, } N = 200 \text{ r.p.m.}$$

$$\therefore \text{Angular speed, } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.94 \text{ rad/s}$$

Total pressure force on the top of the cylinder is given by equation (5.25)

$$F_T = \frac{\rho}{4} \times \omega^2 \times \pi \times R^4 = \frac{1000}{4} \times 20.94^2 \times \pi \times (0.1)^4 = 34.44 \text{ N. Ans.}$$

Now total pressure force on the bottom of the cylinder is given by equation (5.26) as

$$\begin{aligned} F_B &= \rho g \times \pi R^2 \times H + F_T \\ &= 1000 \times 9.81 \times \pi \times (0.1)^2 \times 0.15 + 34.44 \\ &= 46.22 + 34.44 = 80.66 \text{ N. Ans.} \end{aligned}$$

**5.10.6 Equation of Free Vortex Flow.** For the free vortex, from equation (5.20), we have

$$v \times r = \text{Constant} = \text{say } c$$

or

$$v = \frac{c}{r}$$

Substituting the value of  $v$  in equation (5.23), we get

$$dp = \rho \frac{v^2}{r} dr - \rho g dz = \rho \times \frac{c^2}{r^2 \times r} dr - \rho g dz = \rho \times \frac{c^2}{r^3} dr - \rho g dz$$

Consider two points 1 and 2 in the fluid having radius  $r_1$  and  $r_2$  from the central axis respectively as shown in Fig. 5.27. The heights of the points from bottom of the vessel is  $z_1$  and  $z_2$ .

Integrating the above equation for the points 1 and 2, we get

$$\begin{aligned} \int_1^2 dp &= \int_1^2 \frac{\rho c^2}{r^3} dr - \int_1^2 \rho g dz \\ \text{or } p_2 - p_1 &= \rho c^2 \int_1^2 r^{-3} dr - \rho g \int_1^2 dz \\ &= \rho c^2 \left[ \frac{r^{-3+1}}{-2} \right]_1^2 - \rho g [z_2 - z_1] = \frac{\rho c^2}{-2} [r_2^{-2} - r_1^{-2}] - \rho g [z_2 - z_1] \\ &= -\frac{\rho c^2}{2} \left[ \frac{1}{r_2^2} - \frac{1}{r_1^2} \right] - \rho g [z_2 - z_1] = -\frac{\rho}{2} \left[ \frac{c^2}{r_2^2} - \frac{c^2}{r_1^2} \right] - \rho g [z_2 - z_1] \\ &= -\frac{\rho}{2} [v_2^2 - v_1^2] - \rho g [z_2 - z_1] \quad \left\{ \because v_2 = \frac{c}{r_2}, v_1 = \frac{c}{r_1} \right\} \\ &= \frac{\rho}{2} [v_1^2 - v_2^2] - \rho g [z_2 - z_1] \end{aligned}$$

Dividing by  $\rho g$ , we get

$$\begin{aligned} \frac{p_2 - p_1}{\rho g} &= \frac{v_1^2 - v_2^2}{2g} - [z_2 - z_1] \\ \text{or } \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 &= \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 \quad \dots(5.27) \end{aligned}$$

Equation (5.27) is Bernoulli's equation. Hence in case of free vortex flow, Bernoulli's equation is applicable.

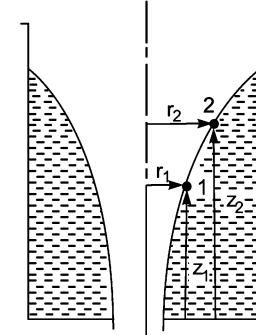


Fig. 5.27

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**Problem 5.32** In a free cylindrical vortex flow, at a point in the fluid at a radius of 200 mm and at a height of 100 mm, the velocity and pressures are 10 m/s and 117.72 kN/m<sup>2</sup> absolute. Find the pressure at a radius of 400 mm and at a height of 200 mm. The fluid is air having density equal to 1.24 kg/m<sup>3</sup>.

**Solution.** At Point 1 : Given :

$$\text{Radius, } r_1 = 200 \text{ mm} = 0.20 \text{ m}$$

$$\text{Height, } z_1 = 100 \text{ mm} = 0.10 \text{ m}$$

$$\text{Velocity, } v_1 = 10 \text{ m/s}$$

$$\text{Pressure, } p_1 = 117.72 \text{ kN/m}^2 = 117.72 \times 10^3 \text{ N/m}^2$$

$$\text{At Point 2 : } r_2 = 400 \text{ mm} = 0.4 \text{ m}$$

$$z_2 = 200 \text{ mm} = 0.2 \text{ m}$$

$$p_2 = \text{pressure at point 2}$$

$$\rho = 1.24 \text{ kg/m}^3$$

For the free vortex from equation (5.20), we have

$$v \times r = \text{constant or } v_1 r_1 = v_2 r_2$$

$$v_2 = \frac{v_1 \times r_1}{r_2} = \frac{10 \times 0.2}{0.4} = 5 \text{ m/s}$$

Now using equation (5.27), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

$$\text{But } \rho = 1.24 \text{ kg/m}^3$$

$$\therefore \frac{117.72 \times 10^3}{1.24 \times 9.81} + \frac{10^2}{2 \times 9.81} + 0.1 = \frac{p_2}{\rho g} + \frac{5^2}{2 \times 9.81} + 0.2$$

$$\text{or } \frac{p_2}{\rho g} = \frac{117.72 \times 10^3}{1.24 \times 9.81} + \frac{10^2}{2 \times 9.81} + 0.1 - \frac{5^2}{2 \times 9.81} - 0.2$$

$$= 9677.4 + 5.096 + 0.1 - 1.274 - 0.2 = 9676.22$$

$$p_2 = 9676.22 \times \rho g = 9676.22 \times 1.24 \times 9.81$$

$$= 117705 \text{ N/m}^2 = 117.705 \times 10^3 \text{ N/m}^2$$

$$= 117.705 \text{ kN/m}^2 (\text{abs.}) = \mathbf{117.705 \text{ kN/m}^2}. \text{ Ans.}$$

## (B) IDEAL FLOW (POTENTIAL FLOW)

### ► 5.11 INTRODUCTION

Ideal fluid is a fluid which is incompressible and inviscid. Incompressible fluid is a fluid for which density ( $\rho$ ) remains constant. Inviscid fluid is a fluid for which viscosity ( $\mu$ ) is zero. Hence a fluid for which density is constant and viscosity is zero, is known as an ideal fluid.

The shear stress is given by,  $\tau = \mu \frac{du}{dy}$ . Hence for ideal fluid the shear stress will be zero as  $\mu = 0$  for ideal fluid. Also the shear force (which is equal to shear stress multiplied by area) will be zero in

case of ideal or potential flow. The ideal fluids will be moving with uniform velocity. All the fluid particles will be moving with the same velocity.

The concept of ideal fluid simplifies the typical mathematical analysis. Fluids such as water and air have low viscosity. Also when the speed of air is appreciably lower than that of sound in it, the compressibility is so low that air is assumed to be incompressible. Hence under certain conditions, certain real fluids such as water and air may be treated like ideal fluids.

### ► 5.12 IMPORTANT CASES OF POTENTIAL FLOW

The following are the important cases of potential flow :

- |                        |                        |
|------------------------|------------------------|
| (i) Uniform flow,      | (ii) Source flow,      |
| (iii) Sink flow,       | (iv) Free-vortex flow, |
| (v) Superimposed flow. |                        |

### ► 5.13 UNIFORM FLOW

In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity. The uniform flow may be :

- |                           |                             |
|---------------------------|-----------------------------|
| (i) Parallel to $x$ -axis | (ii) Parallel to $y$ -axis. |
|---------------------------|-----------------------------|

**5.13.1 Uniform Flow Parallel to  $x$ -Axis.** Fig. 5.27 (a) shows the uniform flow parallel to  $x$ -axis. In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity.

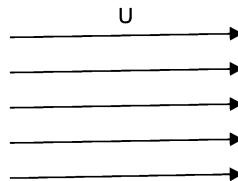


Fig. 5.27 (a)

Let

$U$  = Velocity which is uniform or constant along  $x$ -axis

$u$  and  $v$  = Components of uniform velocity  $U$  along  $x$  and  $y$ -axis.

For the uniform flow, parallel to  $x$ -axis, the velocity components  $u$  and  $v$  are given as

$$u = U \text{ and } v = 0 \quad \dots(5.28)$$

But the velocity  $u$  in terms of stream function is given by,

$$u = \frac{\partial \psi}{\partial y}$$

and in terms of velocity potential the velocity  $u$  is given by,

$$u = \frac{\partial \phi}{\partial x}$$

$$\therefore u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \dots(5.29)$$

$$\text{Similarly, it can be shown that } v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \quad \dots(5.29A)$$

But  $u = U$  from equation (5.28). Substituting  $u = U$  in equation (5.29), we have

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$$\therefore U = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \dots(5.30)$$

or  $U = \frac{\partial \psi}{\partial y}$  and also  $U = \frac{\partial \phi}{\partial x}$

First part gives  $d\psi = U dy$  whereas second part gives  $d\phi = U dx$ .

Integration of these parts gives as

$$\psi = Uy + C_1 \text{ and } \phi = Ux + C_2$$

where  $C_1$  and  $C_2$  are constant of integration.

Now let us plot the stream lines and potential lines for uniform flow parallel to  $x$ -axis.

**Plotting of Stream lines.** For stream lines, the equation is

$$\psi = U \times y + C_1$$

Let  $\psi = 0$ , where  $y = 0$ . Substituting these values in the above equation, we get

$$0 = U \times 0 + C_1 \text{ or } C_1 = 0$$

Hence the equation of stream lines becomes as

$$\psi = U \cdot y \quad \dots(5.31)$$

The stream lines are straight lines parallel to  $x$ -axis and at a distance  $y$  from the  $x$ -axis as shown in Fig. 5.28. In equation (5.31),  $U \cdot y$  represents the volume flow rate (*i.e.*,  $m^3/s$ ) between  $x$ -axis and that stream line at a distance  $y$ .

**Note.** The thickness of the fluid stream perpendicular to the plane is assumed to be unity. Then  $y \times 1$  or  $y$  represents the area of flow. And  $U \cdot y$  represents the product of velocity and area. Hence  $U \cdot y$  represents the volume flow rate.

**Plotting of potential lines.** For potential lines, the equation is

$$\phi = U \cdot x + C_2$$

Let  $\phi = 0$ , where  $x = 0$ . Substituting these values in the above equation, we get  $C_2 = 0$ .

Hence equation of potential lines becomes as

$$\phi = U \cdot x$$

The above equation shows that potential lines are straight lines parallel to  $y$ -axis and at a distance of  $x$  from  $y$ -axis as shown in Fig. 5.29.

Fig. 5.30 shows the plot of stream lines and potential lines for uniform flow parallel to  $x$ -axis. The stream lines and potential lines intersect each other at right angles.

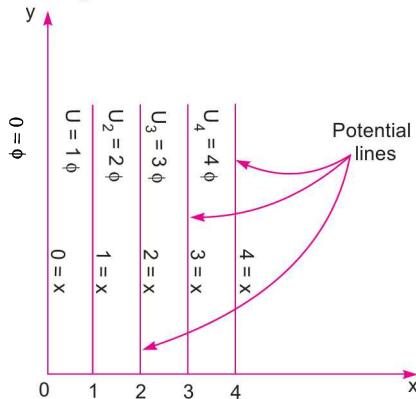


Fig. 5.29

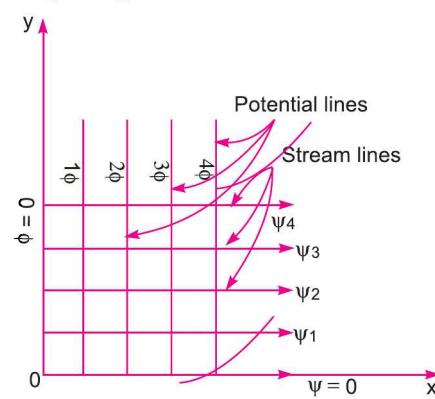


Fig. 5.30

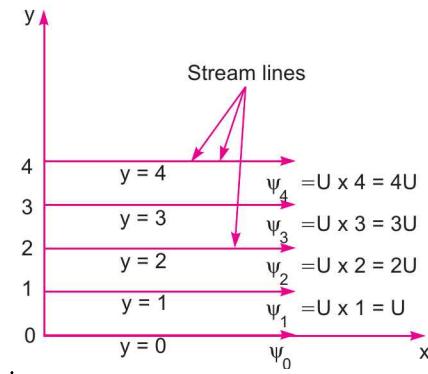


Fig. 5.28  
...(5.32)

**5.13.2 Uniform Potential Flow Parallel to y-Axis.** Fig. 5.31 shows the uniform potential flow parallel to y-axis in which  $U$  is the uniform velocity along y-axis.

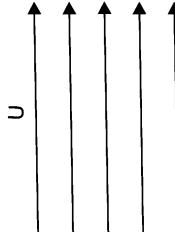


Fig. 5.31

The velocity components  $u, v$  along  $x$ -axis and  $y$ -axis are given by

$$u = 0 \text{ and } v = U \quad \dots(5.33)$$

These velocity components in terms of stream function ( $\psi$ ) and velocity potential function ( $\phi$ ) are given as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \dots(5.34)$$

$$\text{and} \quad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \quad \dots(5.35)$$

But from equation (5.33),  $v = U$ . Substituting  $v = U$  in equation (5.35), we get

$$U = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \quad \text{or} \quad U = -\frac{\partial \psi}{\partial x} \quad \text{and also} \quad U = \frac{\partial \phi}{\partial y}$$

First part gives  $d\psi = -U dx$  whereas second part gives  $d\phi = U dy$ .

Integration of these parts gives as

$$\psi = -U \cdot x + C_1 \text{ and } \phi = U \cdot y + C_2 \quad \dots(5.36)$$

where  $C_1$  and  $C_2$  are constant of integration. Let us now plot the stream lines and potential lines.

**Plotting of Stream lines.** For stream lines, the equation is  $\psi = U \cdot x + C_1$

Let  $\psi = 0$ , where  $x = 0$ . Then  $C_1 = 0$ .

Hence the equation of stream lines becomes as  $\psi = -U \cdot x$   $\dots(5.37)$

The above equation shows that stream lines are straight lines parallel to  $y$ -axis and at a distance of  $x$  from the  $y$ -axis as shown in Fig. 5.32. The  $-ve$  sign shows that the stream lines are in the downward direction.

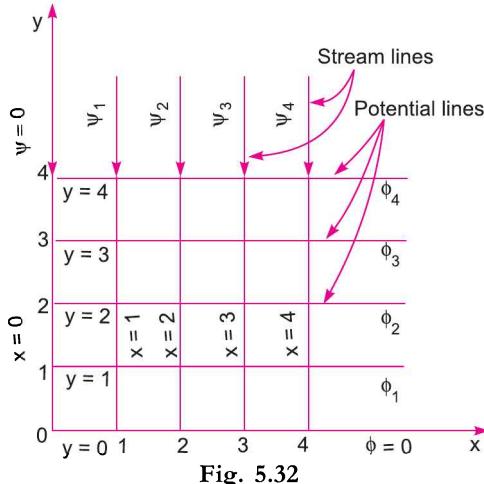


Fig. 5.32

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**Plotting of Potential lines.** For potential lines, the equation is  $\phi = U.y + C_2$

Let  $\phi = 0$ , where  $y = 0$ . Then  $C_2 = 0$ .

Hence equation of potential lines becomes as  $\phi = U.y$  ... (5.38)

The above equation shows that potential lines are straight lines parallel to  $x$ -axis and at a distance of  $y$  from the  $x$ -axis as shown in Fig. 5.32.

### ► 5.14 SOURCE FLOW

The source flow is the flow coming from a point (source) and moving out radially in all directions of a plane at uniform rate. Fig. 5.33 shows a source flow in which the point  $O$  is the source from which the fluid moves radially outward. The strength of a source is defined as the volume flow rate per unit depth. The unit of strength of source is  $\text{m}^2/\text{s}$ . It is represented by  $q$ .

Let  $u_r$  = radial velocity of flow at a radius  $r$  from the source  $O$

$q$  = volume flow rate per unit depth

$r$  = radius

The radial velocity  $u_r$  at any radius  $r$  is given by,

$$u_r = \frac{q}{2\pi r} \quad \dots (5.39)$$

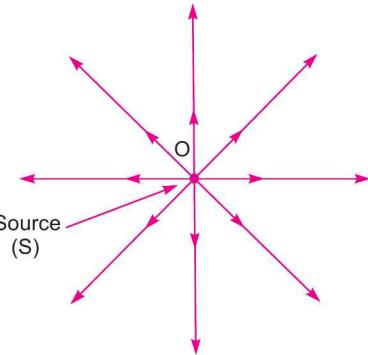


Fig. 5.33 Source flow (Flow away from source)

The above equation shows that with the increase of  $r$ , the radial velocity decreases. And at a large distance away from the source, the velocity will be approximately equal to zero. The flow is in radial direction, hence the tangential velocity  $u_\theta = 0$ .

Let us now find the equation of stream function and velocity potential function for the source flow. As in this case,  $u_\theta = 0$ , the equation of stream function and velocity potential function will be obtained from  $u_r$ .

#### Equation of Stream Function

By definition, the radial velocity and tangential velocity components in terms of stream function are given by

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = - \frac{\partial \psi}{\partial r} \quad [\text{See equation (5.12A)}]$$

But

$$u_r = \frac{q}{2\pi r} \quad [\text{See equation (5.39)}]$$

∴

$$\frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{q}{2\pi r}$$

or

$$d\psi = r \cdot \frac{q}{2\pi r} d\theta = \frac{q}{2\pi} d\theta$$

Integrating the above equation w.r.t.  $\theta$ , we get

$$\psi = \frac{q}{2\pi} \times \theta + C_1, \text{ where } C_1 \text{ is constant of integration.}$$

Let  $\psi = 0$ , when  $\theta = 0$ , then  $C_1 = 0$ .

Hence the equation of stream function becomes as

$$\psi = \frac{q}{2\pi} \cdot \theta \quad \dots(5.40)$$

In the above equation,  $q$  is constant.

The above equation shows that stream function is a function of  $\theta$ . For a given value of  $\theta$ , the stream function  $\psi$  will be constant. And this will be a radial line. The stream lines can be plotted by having different values of  $\theta$ . Here  $\theta$  is taken in radians.

### Plotting of stream lines

When  $\theta = 0$ ,  $\psi = 0$

$$\theta = 45^\circ = \frac{\pi}{4} \text{ radians}, \psi = \frac{q}{2\pi} \cdot \frac{\pi}{4} = \frac{q}{8} \text{ units}$$

$$\theta = 90^\circ = \frac{\pi}{2} \text{ radians}, \psi = \frac{q}{2\pi} \cdot \frac{\pi}{2} = \frac{q}{4} \text{ units}$$

$$\theta = 135^\circ = \frac{3\pi}{4} \text{ radians}, \psi = \frac{q}{2\pi} \cdot \frac{3\pi}{4} = \frac{3q}{8} \text{ units}$$

The stream lines will be radial lines as shown in Fig. 5.34.

### Equation of Potential Function

By definition, the radial and tangential components in terms of velocity function are given by

$$u_r = \frac{\partial \phi}{\partial r} \text{ and } u_\theta = \frac{1}{r} \cdot \frac{\partial \phi}{\partial \theta}$$

[See equation (5.9A)]

$$\text{But from equation (5.39), } u_r = \frac{q}{2\pi r}$$

Equating the two values of  $u_r$ , we get

$$\frac{\partial \phi}{\partial r} = \frac{q}{2\pi r} \text{ or } d\phi = \frac{q}{2\pi r} dr$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{q}{2\pi r} dr$$

or

$$\phi = \frac{q}{2\pi} \int \frac{1}{r} dr \quad \left[ \because \frac{q}{2\pi} \text{ is a constant term} \right]$$

$$= \frac{q}{2\pi} \log_e r \quad \dots(5.41)$$

In the above equation,  $q$  is constant.

The above equation shows, that the velocity potential function is a function of  $r$ . For a given value of  $r$ , the velocity function  $\phi$  will be constant. Hence it will be a circle with origin at the source. The velocity potential lines will be circles with origin at the source as shown in Fig. 5.35.

Let us now find an expression for the pressure in terms of radius.

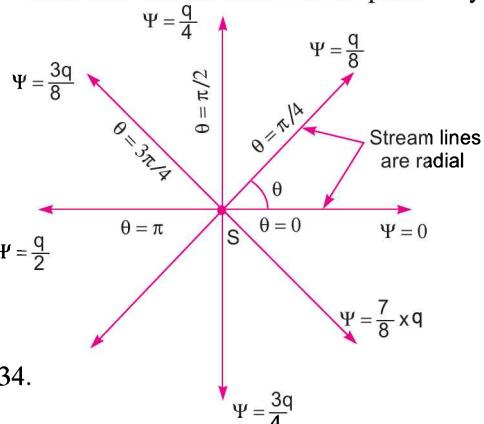


Fig. 5.34 Stream line for source flow.

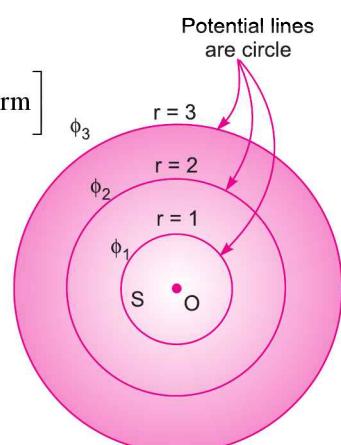


Fig. 5.35 Potential lines for source.

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### Pressure distribution in a plane source flow

The pressure distribution in a plane source flow can be obtained with the help of Bernoulli's equation. Let us assume that the plane of the flow is horizontal. In that case the datum head will be same for two points of flow.

Let  $p$  = pressure at a point 1 which is at a radius  $r$  from the source at point 1

$u_r$  = velocity at point 1

$p_0$  = pressure at point 2, which is at a large distance away from the source. The velocity will be zero at point 2. [Refer to equation (5.39)]

Applying Bernoulli's equation, we get

$$\frac{p}{\rho g} + \frac{u_r^2}{2g} = \frac{p_0}{\rho g} + 0 \quad \text{or} \quad \frac{(p - p_0)}{\rho g} = -\frac{u_r^2}{2g}$$

or

$$(p - p_0) = -\frac{\rho \cdot u_r^2}{2}$$

$$\text{But from equation(5.39), } u_r = \frac{q}{2\pi r}$$

Substituting the value of  $u_r$  in the above equation, we get

$$\begin{aligned} (p - p_0) &= -\left(\frac{\rho}{2}\right) \cdot \left(\frac{q}{2\pi r}\right)^2 \\ &= -\frac{\rho q^2}{8\pi^2 r^2} \end{aligned} \quad \dots(5.42)$$

In the above equation,  $\rho$  and  $q$  are constants.

The above equation shows that the pressure is inversely proportional to the square of the radius from the source.

### ► 5.15 SINK FLOW

The sink flow is the flow in which fluid moves radially inwards towards a point where it disappears at a constant rate. This flow is just opposite to the source flow. Fig. 5.36 shows a sink flow in which the fluid moves radially inwards towards point  $O$ , where it disappears at a constant rate. The pattern of stream lines and equipotential lines of a sink flow is the same as that of a source flow. All the equations derived for a source flow shall hold to good for sink flow also except that in sink flow equations,  $q$  is to be replaced by  $(-q)$ .

**Problem 5.33** Plot the stream lines for a uniform flow of :

- (i) 5 m/s parallel to the positive direction of the  $x$ -axis and
- (ii) 10 m/s parallel to the positive direction of the  $y$ -axis.

**Solution.** (i) The stream function for a uniform flow parallel to the positive direction of the  $x$ -axis is given by equation (5.31) as

$$\psi = U \times y$$

The above equation shows that stream lines are straight lines parallel to the  $x$ -axis at a distance  $y$  from the  $x$ -axis. Here  $U = 5$  m/s and hence above equation becomes as

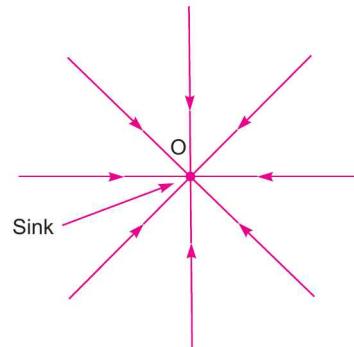


Fig. 5.36 Sink flow  
(Flow toward centre)

$$\psi = 5y$$

For  $y = 0$ , stream function  $\psi = 0$

For  $y = 0.2$ , stream function  $\psi = 5 \times 0.2 = 1$  unit

For  $y = 0.4$ , stream function  $\psi = 5 \times 0.4 = 2$  unit

The other values of stream function can be obtained by substituting the different values of  $y$ . The stream lines are horizontal as shown in Fig. 5.36 (a).

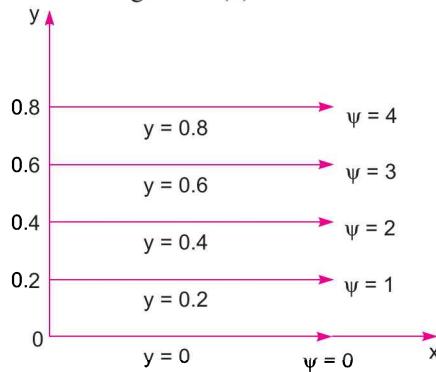


Fig. 5.36 (a)

(ii) The stream function for a uniform flow parallel to the positive direction of the  $y$ -axis is given by equation (5.37) as

$$\psi = -U \times x$$

The above equation shows that stream lines are straight lines parallel to the  $y$ -axis at a distance  $x$  from the  $y$ -axis. Here  $U = 10$  m/s and hence the above equation becomes as

$$\psi = -10 \times x$$

The negative sign shows that the stream lines are in the downward direction.

For  $x = 0$ , the stream function  $\psi = 0$

For  $x = 0.1$ , the stream function  $\psi = -10 \times 0.1 = -1.0$  unit

For  $x = 0.2$ , the stream function  $\psi = -10 \times 0.2 = -2.0$  unit

For  $x = 0.3$ , the stream function  $\psi = -10 \times 0.3 = -3.0$  unit

The other values of stream function can be obtained by substituting the different values of  $x$ . The stream lines are vertical as shown in Fig. 5.36 (b).

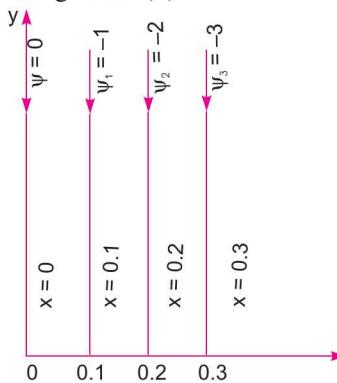


Fig. 5.36 (b)

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**Problem 5.34** Determine the velocity of flow at radii of 0.2 m, 0.4 m and 0.8 m, when the water is flowing radially outward in a horizontal plane from a source at a strength of  $12 \text{ m}^2/\text{s}$ .

**Solution.** Given :

Strength of source,  $q = 12 \text{ m}^2/\text{s}$

The radial velocity  $u_r$  at any radius  $r$  is given by equation (5.39) as

$$u_r = \frac{q}{2\pi r}$$

When  $r = 0.2 \text{ m}$ ,  $u_r = \frac{12}{2\pi \times 0.2} = 9.55 \text{ m/s. Ans.}$

When  $r = 0.4 \text{ m}$ ,  $u_r = \frac{12}{2\pi \times 0.4} = 4.77 \text{ m/s. Ans.}$

When  $r = 0.8 \text{ m}$ ,  $u_r = \frac{12}{2\pi \times 0.8} = 2.38 \text{ m/s. Ans.}$

**Problem 5.35** Two discs are placed in a horizontal plane, one over the other. The water enters at the centre of the lower disc and flows radially outward from a source of strength  $0.628 \text{ m}^2/\text{s}$ . The pressure, at a radius 50 mm, is  $200 \text{ kN/m}^2$ . Find :

(i) pressure in  $\text{kN/m}^2$  at a radius of 500 mm and

(ii) stream function at angles of  $30^\circ$  and  $60^\circ$  if  $\psi = 0$  at  $\theta = 0^\circ$ .

**Solution.** Given :

Source strength,  $q = 0.628 \text{ m}^2/\text{s}$

Pressure at radius 50 mm,  $p_1 = 200 \text{ kN/m}^2 = 200 \times 10^3 \text{ N/m}^2$

(i) Pressure at a radius 500 mm

Let  $p_2$  = pressure at radius 500 mm

$(u_r)_1$  = velocity at radius 50 mm

$(u_r)_2$  = velocity at radius 500 mm

The radial velocity at any radius  $r$  is given by equation (5.39) as

$$u_r = \frac{q}{2\pi r}$$

When  $r = 50 \text{ mm} = 0.05 \text{ m}$ ,  $(u_r)_1 = \frac{0.628}{2\pi \times 0.05} = 1.998 \text{ m/s} \approx 2 \text{ m/s}$

When  $r = 500 \text{ mm} = 0.5 \text{ m}$ ,  $(u_r)_2 = \frac{0.628}{2\pi \times 0.5} = 0.2 \text{ m/s}$

Applying Bernoulli's equation at radius 0.05 m and at radius 0.5 m,

$$\frac{p_1}{\rho g} + \frac{(u_r)_1^2}{2g} = \frac{p_2}{\rho g} + \frac{(u_r)_2^2}{2g}$$

or 
$$\frac{p_1}{\rho} + \frac{(u_r)_1^2}{2} = \frac{p_2}{\rho} + \frac{(u_r)_2^2}{2}$$

$$\text{or } \frac{200 \times 10^3}{1000} + \frac{2^2}{2} = \frac{p_2}{1000} + \frac{0.2^2}{2}$$

$$\text{or } 200 + 2 = \frac{p_2}{1000} + 0.02$$

$$\text{or } \frac{p_2}{1000} = 202 - 0.02 = 201.98$$

$$\therefore p_2 = 201.98 \times 1000 \text{ N/m}^2 = 201.98 \text{ kN/m}^2. \text{ Ans.}$$

(ii) Stream functions at  $\theta = 30^\circ$  and  $\theta = 60^\circ$

For the source flow, the equation of stream function is given by equation (5.40) as

$$\psi = \frac{q}{2\pi} \cdot \theta, \text{ where } \theta \text{ is in radians}$$

$$\text{When } \theta = 30^\circ, \quad \psi = \frac{0.628}{2\pi} \times \frac{30 \times \pi}{180} \quad \left( \because \theta = 30^\circ = \frac{30 \times \pi}{180} \text{ radians} \right)$$

$$= 0.0523 \text{ m}^2/\text{s. Ans.}$$

$$\text{When } \theta = 60^\circ, \quad \psi = \frac{0.628}{2\pi} \times \frac{60\pi}{180} = 0.1046 \text{ m}^2/\text{s. Ans.}$$

## ► 5.16 FREE-VORTEX FLOW

Free-vortex flow is a circulatory flow of a fluid such that its stream lines are concentric circles. For a free-vortex flow,  $u_\theta \propto r = \text{constant}$  (say  $C$ )

Also, circulation around a stream line of an irrotation vortex is

$$\Gamma = 2\pi r \times u_\theta = 2\pi \times C \quad (\because r \times u_\theta = C)$$

where  $u_\theta$  = tangential velocity at any radius  $r$  from the centre.

$$\therefore u_\theta = \frac{\Gamma}{2\pi r}$$

The circulation  $\Gamma$  is taken positive if the free vortex is anticlockwise.

For a free-vortex flow, the velocity components are

$$u_\theta = \frac{\Gamma}{2\pi r} \quad \text{and} \quad u_r = 0$$

### Equation of Stream Function

By definition, the stream function is given by

$$u_\theta = \frac{-\partial \psi}{\partial r} \quad \text{and} \quad u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad [\text{See equation (5.12A)}]$$

In case of free-vortex flow, the radial velocity ( $u_r$ ) is zero. Hence equation of stream function will be obtained from tangential velocity,  $u_\theta$ . The value of  $u_\theta$  is given by

$$u_\theta = \frac{\Gamma}{2\pi r}$$

Equating the two values of  $u_\theta$ , we get

$$-\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r} \quad \text{or} \quad d\psi = -\frac{\Gamma}{2\pi r} dr$$

Integrating the above equation, we get

$$\int d\psi = \int -\frac{\Gamma}{2\pi r} dr = \left(-\frac{\Gamma}{2\pi}\right) \int \frac{1}{r} dr$$

or

$$\psi = \left(-\frac{\Gamma}{2\pi}\right) \log_e r \quad \left(\because \frac{\Gamma}{2\pi} \text{ is a constant term}\right) \dots(5.43)$$

The above equation shows that stream function is a function of radius. For a given value of  $r$ , the stream function is constant. Hence the stream lines are concentric circles as shown in Fig. 5.37.

**Equation of potential function.** By definition, the potential function is given by,

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad \text{and} \quad u_r = \frac{\partial \phi}{\partial r} \quad [\text{See equation (5.9A)}]$$

Here  $u_r = 0$  and  $u_\theta = \frac{\Gamma}{2\pi r}$ . Hence, the equation of potential

function will be obtained from  $u_\theta$ .

Equating the two values of  $u_\theta$ , we get

$$\frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \quad \text{or} \quad d\phi = r \cdot \frac{\Gamma}{2\pi r} \cdot d\theta = \frac{\Gamma}{2\pi} d\theta$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{\Gamma}{2\pi} d\theta \quad \text{or} \quad \phi = \frac{\Gamma}{2\pi} \int d\theta = \frac{\Gamma}{2\pi} \cdot \theta \quad \dots(5.44)$$

The above equation shows that velocity potential function is a function of  $\theta$ . For a given value of  $\theta$ , potential function is a constant. Hence equipotential lines are radial as shown in Fig. 5.38.

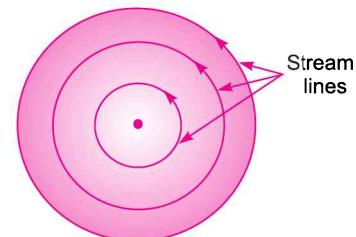


Fig. 5.37

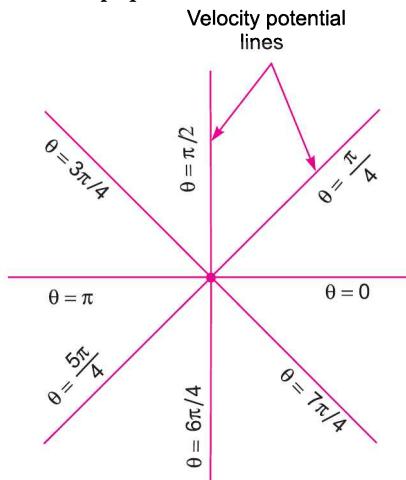


Fig. 5.38 Potential lines are radial.

## ► 5.17 SUPER-IMPOSED FLOW

The flow patterns due to uniform flow, a source flow, a sink flow and a free vortex flow can be super-imposed in any linear combination to get a resultant flow which closely resembles the flow around bodies. The resultant flow will still be potential and ideal. The following are the important super-imposed flow :

- (i) Source and sink pair
- (ii) Doublet (special case of source and sink combination)
- (iii) A plane source in a uniform flow (flow past a half body)
- (iv) A source and sink pair in a uniform flow
- (v) A doublet in a uniform flow.

**5.17.1 Source and Sink Pair.** Fig. 5.39 shows a source and a sink of strength  $q$  and  $(-q)$  placed at  $A$  and  $B$  respectively at equal distance from the point  $O$  on the  $x$ -axis. Thus the source and sink are placed symmetrically on the  $x$ -axis. The source of strength  $q$  is placed at  $A$  and sink of strength  $(-q)$  is placed at  $B$ . The combination of the source and the sink would result in a flownet where stream lines will be circular arcs starting from point  $A$  and ending at point  $B$  as shown in Fig. 5.40.

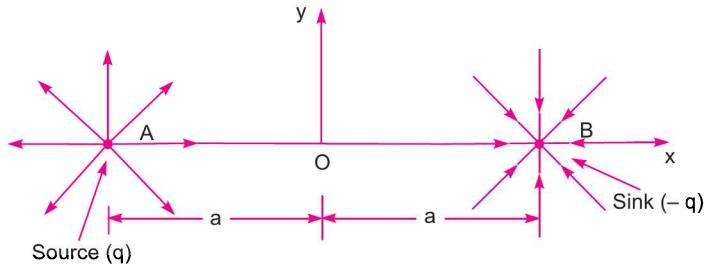


Fig. 5.39 Source and sink pair.

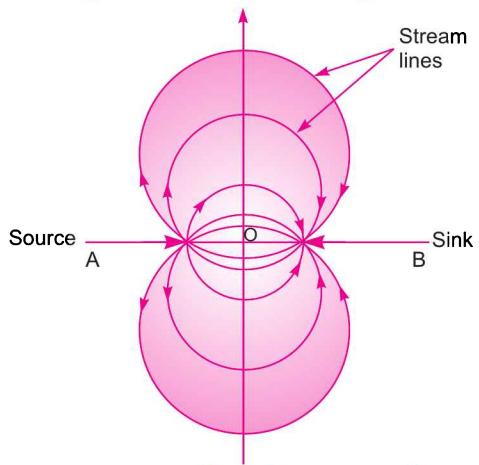


Fig. 5.40 Stream lines for source-sink pair.

### Equation of stream function and potential function

Let  $P$  be any point in the resultant flownet of source and sink as shown in Fig. 5.41.

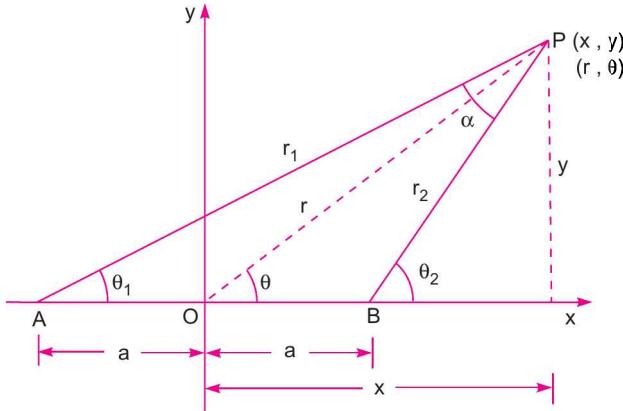


Fig. 5.41

Let  $r, \theta$  = Cylindrical co-ordinates of point  $P$  with respect to origin  $O$

$x, y$  = Corresponding co-ordinates of point  $P$

$r_1, \theta_1$  = Position of point  $P$  with respect to source placed at  $A$

$r_2, \theta_2$  = Position of point  $P$  with respect to sink placed at  $B$

$\alpha$  = Angle subtended at  $P$  by the join of source and sink i.e., angle  $APB$ .

Let us find the equation for the resultant stream function and velocity potential function. The

equation for stream function due to source is given by equation (5.40) as  $\psi_1 = \frac{q \cdot \theta_1}{2\pi}$  whereas due to

sink it is given by  $\psi_2 = \frac{(-q \theta_2)}{2\pi}$ . The equation for resultant stream function ( $\psi$ ) will be the sum of these two stream function.

$\therefore$

$$\begin{aligned}\psi &= \psi_1 + \psi_2 \\ &= \frac{q \theta_1}{2\pi} + \left( \frac{-q \theta_2}{2\pi} \right) = \frac{-q}{2\pi} (\theta_2 - \theta_1) \\ &= \frac{-q}{2\pi} \cdot \alpha \quad [\because \alpha = \theta_2 - \theta_1. \text{ In triangle } ABP, \theta_1 + \alpha + (180^\circ - \theta_2) \\ &\quad = 180^\circ \quad \therefore \alpha = \theta_2 - \theta_1] \\ &= \frac{-q \cdot \alpha}{2\pi} \quad \dots(5.45)\end{aligned}$$

The equation for potential function due to source is given by equation (5.41) as  $\phi_1 = \frac{q}{2\pi} \log_e r_1$  and

due to sink it is given as  $\phi_2 = \frac{-q}{2\pi} \log_e r_2$ . The equation for resultant potential function ( $\phi$ ) will be the sum of these two potential function.

$\therefore$

$$\begin{aligned}\phi &= \phi_1 + \phi_2 \\ &= \frac{q}{2\pi} \log_e r_1 + \left( \frac{-q}{2\pi} \right) \log_e r_2\end{aligned}$$

$$= \frac{q}{2\pi} [\log_e r_1 - \log_e r_2] = \frac{q}{2\pi} \log_e \left( \frac{r_1}{r_2} \right) \quad \dots(5.46)$$

**To prove that resultant stream lines will be circular arc passing through source and sink**  
The resultant stream function is given by equation (5.45) as

$$\psi = \frac{-q \cdot \alpha}{2\pi} \quad \text{or} \quad \frac{-q}{2\pi} (\theta_2 - \theta_1) \quad (\because \alpha = \theta_2 - \theta_1)$$

For a given stream line  $\psi = \text{constant}$ . In the above equation the term  $\frac{q}{2\pi}$  is also constant. This means that  $(\theta_2 - \theta_1)$  or angle  $\alpha$  will also be constant for various positions of  $P$  in the plane.

To satisfy this, the locus of  $P$  must be a circle with  $AB$  as chord, having its centre on  $y$ -axis, as shown in Fig. 5.40.

Consider the equation (5.45) again as

$$\begin{aligned} \psi &= \frac{-q}{2\pi} \alpha = \frac{-q}{2\pi} (\theta_2 - \theta_1) \quad (\because \alpha = \theta_2 - \theta_1) \\ &= \frac{q}{2\pi} (\theta_1 - \theta_2) \\ \text{or} \quad (\theta_1 - \theta_2) &= \frac{2\pi\psi}{q} \end{aligned}$$

Taking tangent to both sides, we get

$$\tan(\theta_1 - \theta_2) = \tan\left(\frac{2\pi\psi}{q}\right) \quad \text{or} \quad \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \cdot \tan\theta_2} = \tan\left(\frac{2\pi\psi}{q}\right) \quad \dots(i)$$

$$\text{But} \quad \tan\theta_1 = \frac{y}{x+a} \quad \text{and} \quad \tan\theta_2 = \frac{y}{x-a} \quad \dots(5.46A)$$

Substituting the values of  $\tan\theta_1$  and  $\tan\theta_2$  in equation (i),

$$\begin{aligned} \frac{\frac{y}{(x+a)} - \frac{y}{(x-a)}}{1 + \frac{y}{(x+a)} \cdot \frac{y}{(x-a)}} &= \tan\left(\frac{2\pi\psi}{q}\right) \\ \text{or} \quad \frac{y(x-a) - y(x+a)}{x^2 - a^2 + y^2} &= \tan\left(\frac{2\pi\psi}{q}\right) \\ \text{or} \quad \frac{-2ay}{x^2 - a^2 + y^2} &= \tan\left(\frac{2\pi\psi}{q}\right) \\ \text{or} \quad \frac{-2ay}{x^2 - a^2 + y^2} &= \frac{1}{\cot\left(\frac{2\pi\psi}{q}\right)} \end{aligned}$$

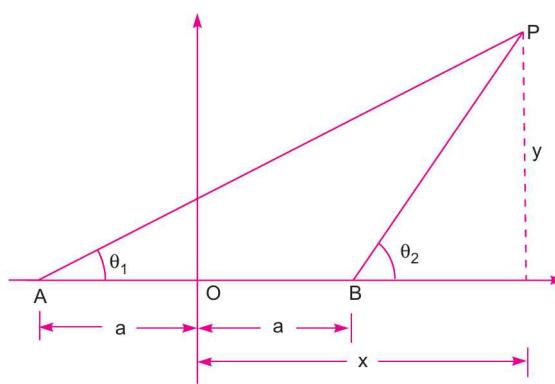


Fig. 5.41 (a)

$$\text{or} \quad x^2 - a^2 + y^2 = -2ay \cot\left(\frac{2\pi\psi}{q}\right)$$

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$$\text{or } x^2 - a^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) = 0$$

$$\text{or } x^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) - a^2 = 0$$

$$\text{or } x^2 + y^2 + 2ay \cot\left(\frac{2\pi\psi}{q}\right) + a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) - a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) - a^2 = 0$$

Adding and subtracting  $a^2 \cot^2\left(\frac{2\pi\psi}{q}\right)$

$$\begin{aligned} \text{or } x^2 + \left[y + a \cot\left(\frac{2\pi\psi}{q}\right)\right]^2 &= a^2 + a^2 \cot^2\left(\frac{2\pi\psi}{q}\right) \\ &= a^2 \left[1 + \cot^2\left(\frac{2\pi\psi}{q}\right)\right] \\ &= a^2 \operatorname{cosec}^2\left(\frac{2\pi\psi}{q}\right) \quad \left[\because 1 + \cot^2\left(\frac{2\pi\psi}{q}\right) = \operatorname{cosec}^2\left(\frac{2\pi\psi}{q}\right)\right] \end{aligned}$$

$$\text{or } x^2 + \left[y + a \cot\left(\frac{2\pi\psi}{q}\right)\right]^2 = \left[a \operatorname{cosec}\left(\frac{2\pi\psi}{q}\right)\right]^2 \quad \dots(5.47)$$

The above is the equation of a circle\* with centre on  $y$ -axis at a distance of  $\pm a \cot\left(\frac{2\pi\psi}{q}\right)$  from the origin. The radius of the circle will be  $a \operatorname{cosec}\left(\frac{2\pi\psi}{q}\right)$ .

Similarly, it can be shown that the potential lines for the source-sink pair will be eccentric non-intersecting circles with their centres on the  $x$ -axis as shown in Fig. 5.41 (b).

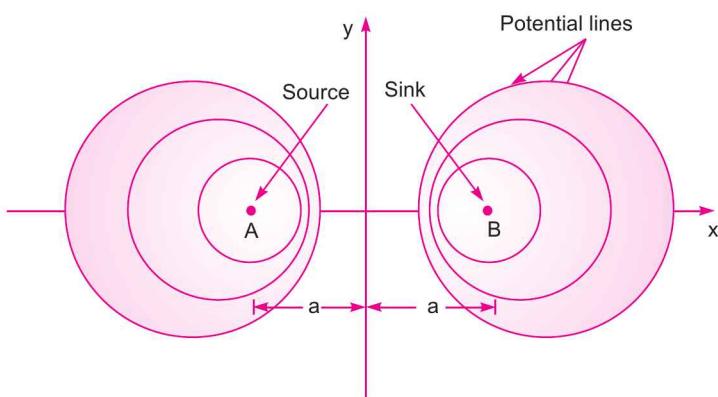


Fig. 5.41 (b) Potential lines for source sink pair (Potential lines are eccentric non-intersecting circles with their centres on x-axis).

\*The equation  $x^2 + y^2 = a^2$  is the equation of a circle with centre at origin and of radius 'a'.

**Problem 5.36** A source and a sink of strength  $4 \text{ m}^2/\text{s}$  and  $8 \text{ m}^2/\text{s}$  are located at  $(-1, 0)$  and  $(1, 0)$  respectively. Determine the velocity and stream function at a point  $P (1, 1)$  which is lying on the flownet of the resultant stream line.

**Solution.** Given :

$$\text{Source strength, } q_1 = 4 \text{ m}^2/\text{s}$$

$$\text{Sink strength, } q_2 = 8 \text{ m}^2/\text{s}$$

Distance of the source and sink from origin,  $a = 1$  unit.

The position of the source, sink and point  $P$  in the flow field is shown in Fig. 5.42.

From Fig. 5.42, it is clear that angle  $\theta_2$  will be  $90^\circ$  and angle  $\theta_1$  can be calculated from right angled triangle  $ABP$ .

The equation for stream function due to source is given by equation (5.40) as  $\psi_1 = \frac{q_1 \times \theta_1}{2\pi}$ ,

whereas due to sink it is given by  $\psi_2 = \frac{-q_2 \times \theta_2}{2\pi}$ . The resultant stream function  $\psi$  is given as

$$\psi = \psi_1 + \psi_2$$

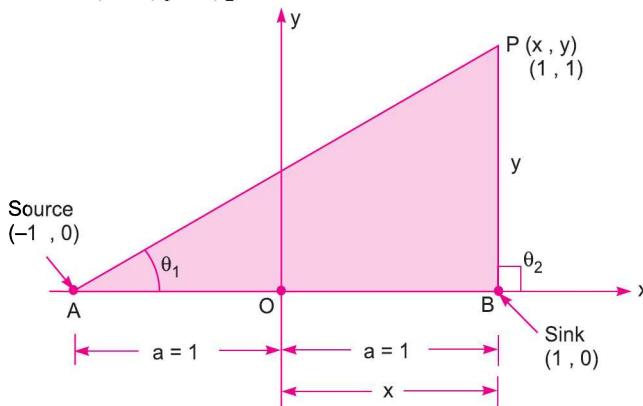


Fig. 5.42

$$= \frac{q_1 \times \theta_1}{2\pi} + \left( \frac{-q_2 \times \theta_2}{2\pi} \right) = \frac{q_1 \times \theta_1}{2\pi} - \frac{q_2 \times \theta_2}{2\pi} \quad \dots(i)$$

Let us find the values of  $\theta_1$  and  $\theta_2$  in radians. From the geometry, it is clear that the triangle  $ABP$  is a right angled triangle with angle  $\theta_2 = 90^\circ = \frac{90}{180} \times \pi = \frac{\pi}{2}$  radians.

$$\text{Also } \tan \theta_1 = \frac{BP}{AB} = \frac{1}{2} = 0.5$$

$$\text{or } \theta_1 = \tan^{-1} 0.5 = 26.56^\circ = 26.56 \times \frac{\pi}{180} \text{ radians} = 0.463$$

Substituting these values in equation (i),

$$\psi = \frac{q_1}{2\pi} \times 0.463 - \frac{q_2}{2\pi} \times \frac{\pi}{2}$$

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$$\begin{aligned}
 &= \frac{\pi}{2\pi} \times 0.463 - \frac{8}{2\pi} \times \frac{\pi}{2} \\
 &= 0.294 - 2.0 = -1.706 \text{ m}^2/\text{s. Ans.}
 \end{aligned}
 \quad (\because q_1 = 4 \text{ m}^2/\text{s}, q_2 = 8 \text{ m}^2/\text{s})$$

To find the velocity at the point  $P$ , let us first find the stream function in terms of  $x$  and  $y$  coordinates. The stream function in terms of  $\theta_1$  and  $\theta_2$  is given by equation (i) above as

$$\psi = \frac{q_1 \times \theta_1}{2\pi} - \frac{q_2 \times \theta_2}{2\pi}$$

The values of  $\theta_1$  and  $\theta_2$  in terms of  $x$ ,  $y$  and  $a$  are given by equation (5.46A) as

$$\tan \theta_1 = \frac{y}{x+a} \quad \text{and} \quad \tan \theta_2 = \frac{y}{(x-a)}$$

or

$$\theta_1 = \tan^{-1} \frac{y}{x+a} \quad \text{and} \quad \theta_2 = \tan^{-1} \frac{y}{(x-a)}$$

Substituting these values of  $\theta_1$  and  $\theta_2$  in equation (i), we get

$$\psi = \frac{q_1}{2\pi} \tan^{-1} \frac{y}{x+a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x-a}$$

The velocity component  $u = \frac{\partial \psi}{\partial y}$  and  $v = -\frac{\partial \psi}{\partial x}$ .

$$\therefore u = \frac{\partial \psi}{\partial y}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial y} \left[ \frac{q_1}{2\pi} \tan^{-1} \frac{y}{x+a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x-a} \right] \\
 &= \frac{q_1}{2\pi} \times \frac{1}{1 + \left( \frac{y}{x+a} \right)^2} \times \frac{1}{(x+a)} - \frac{q_2}{2\pi} \times \frac{1}{1 + \left( \frac{y}{x-a} \right)^2} \times \frac{1}{(x-a)} \\
 &= \frac{q_1}{2\pi} \frac{(x+a)^2}{(x+a)^2 + y^2} \times \frac{1}{(x+a)} - \frac{q_2}{2\pi} \frac{(x-a)^2}{(x-a)^2 + y^2} \times \frac{1}{(x-a)} \\
 &= \frac{q_1}{2\pi} \frac{(x+a)}{(x+a)^2 + y^2} - \frac{q_2}{2\pi} \frac{(x-a)}{(x-a)^2 + y^2}
 \end{aligned}$$

At the point  $P(1, 1)$ , the component  $u$  is obtained by substituting  $x = 1$  and  $y = 1$  in the above equation. The value of  $a$  is also equal to one.

$$\begin{aligned}
 \therefore u &= \frac{q_1}{2\pi} \frac{1+1}{(1+1)^2 + 1^2} - \frac{q_2}{2\pi} \frac{(1-1)}{(1-1)^2 + 1^2} \\
 &= \frac{q_1}{2\pi} \frac{2}{5} - \frac{q_2}{2\pi} \times 0 = \frac{q_1}{2\pi} \times \frac{2}{5} = \frac{4}{2\pi} \times \frac{2}{5} = 0.2544 \text{ m/s}
 \end{aligned}$$

Now

$$\begin{aligned}
 v &= -\frac{\partial \psi}{\partial x} \\
 &= -\frac{\partial}{\partial x} \left[ \frac{q_1}{2\pi} \tan^{-1} \frac{y}{x+a} - \frac{q_2}{2\pi} \tan^{-1} \frac{y}{x-a} \right] \\
 &= -\left[ \frac{q_1}{2\pi} \frac{1}{1+\left(\frac{y}{x+a}\right)^2} \times \frac{y(-1)}{(x+a)^2} \times 1 - \frac{q_2}{2\pi} \times \frac{1}{1+\left(\frac{y}{x-a}\right)^2} \times \frac{y(-1)}{(x-a)^2} \times 1 \right] \\
 &= -\left[ \frac{q_1}{2\pi} \frac{(x+a)^2}{(x+a)^2 + y^2} \times \frac{(-y)}{(x+a)^2} - \frac{q_2}{2\pi} \frac{(x-a)^2}{(x-a)^2 + y^2} \times \frac{(-y)}{(x-a)^2} \right] \\
 &= \frac{q_1}{2\pi} \frac{y}{(x+a)^2 + y^2} - \frac{q_2}{2\pi} \frac{y}{(x-a)^2 + y^2}
 \end{aligned}$$

At the point  $P(1, 1)$ ,

$$\begin{aligned}
 v &= \frac{q_1}{2\pi} \times \frac{1}{(1+1)^2 + 1^2} - \frac{q_2}{2\pi} \times \frac{1}{(1-1)^2 + 1^2} \quad (\because a = 1) \\
 &= \frac{q_1}{2\pi} \times \frac{1}{5} - \frac{q_2}{2\pi} \times \frac{1}{1} \\
 &= \frac{q_1}{2\pi} \times \frac{1}{5} - \frac{q_2}{2\pi} = \frac{4}{2\pi} \times \frac{1}{5} - \frac{8}{2\pi} = 0.1272 - 1.272 = -1.145 \text{ m/s}^2
 \end{aligned}$$

$\therefore$  The resultant velocity,  $V = \sqrt{u^2 + v^2} = \sqrt{0.2544^2 + (-1.145)^2} = 1.174 \text{ m/s. Ans.}$

**Problem 5.37** For the above problem, determine the pressure at  $P(1, 1)$  if the pressure at infinity is zero and density of fluid is  $1000 \text{ kg/m}^3$ .

**Solution.** Given :

$$\begin{aligned}
 \text{Pressure at infinity, } p_0 &= 0 \\
 \text{Density of fluid, } \rho &= 1000 \text{ kg/m}^3
 \end{aligned}$$

The velocity\* of fluid at infinity will be zero. If  $V_0$  = velocity at infinity, then  $V_0 = 0$ .

The resultant velocity of fluid at  $P(1, 1) = 1.174 \text{ m/s}$  (calculated above)

or  $V = 1.174 \text{ m/s.}$

Let  $p$  = pressure at  $P(1, 1)$

Applying Bernoulli's theorem at point at infinity and at point  $P$ , we get

$$\frac{p_0}{\rho g} + \frac{V_0^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g}$$

$$\text{or } 0 + 0 = \frac{p}{\rho g} + \frac{V^2}{2g} \quad \text{or} \quad 0 = \frac{p}{\rho g} + \frac{V^2}{2g} \quad \text{or} \quad 0 = \frac{p}{\rho} + \frac{V^2}{2}$$

$$\text{or } \frac{p}{\rho} = -\frac{V^2}{2} = -\frac{1.174^2}{2} \quad (\because V = 1.174 \text{ m/s})$$

---

\* From equation (5.39), the velocity at a distance ' $r$ ' from source or sink is given by  $u_r = \frac{q}{2\pi r}$ . At infinity,  $r$  is very very large hence velocity is zero.

or

$$p = -\frac{1174^2}{2} \times \rho = -\frac{1174^2 \times 1000}{2} = -689.14 \text{ N/m}^2. \text{ Ans.}$$

**5.17.2 Doublet.** It is a special case of a source and sink pair (both of them are of equal strength) when the two approach each other in such a way that the distance  $2a$  between them approaches zero and the product  $2a \cdot q$  remains constant. This product  $2a \cdot q$  is known as doublet strength and is denoted by  $\mu$ .

$$\therefore \text{Doublet strength, } \mu = 2a \cdot q \quad \dots(5.48)$$

Let  $q$  and  $(-q)$  may be the strength of the source and the sink respectively as shown in Fig. 5.43. Let  $2a$  be the distance between them and  $P$  be any point in the combined field of source and sink.

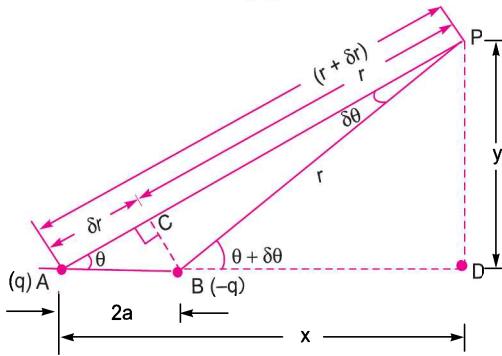


Fig. 5.43

Let  $\theta$  is the angle made by  $P$  at  $A$  whereas  $(\theta + \delta\theta)$  is the angle at  $B$ .

Now the stream function at  $P$ ,

$$\psi = \frac{q\theta}{2\pi} - \frac{q}{2\pi}(\theta + \delta\theta) = -\frac{q}{2\pi}\delta\theta \quad \dots(5.49)$$

From  $B$ , draw  $BC \perp$  on  $AP$ . Let  $AC = \delta r$ ,  $CP = r$  and  $AP = r + \delta r$ . Also angle  $BPC = \delta\theta$ . The angle  $\delta\theta$  is very small. The distance  $BC$  can be taken equal to  $r \times \delta\theta$ . In triangle  $ABC$ , angle  $BCA = 90^\circ$  and hence distance  $BC$  is also equal to  $2a \cdot \sin \theta$ . Equating the two values of  $BC$ , we get

$$r \times \delta\theta = 2a \cdot \sin \theta$$

$$\therefore \delta\theta = \frac{2a \cdot \sin \theta}{r}$$

Substituting the value of  $\delta\theta$  in equation (5.49), we get

$$\begin{aligned} \psi &= -\frac{q}{2\pi} \times \frac{2a \sin \theta}{r} \\ &= -\frac{\mu}{2\pi} \times \frac{\sin \theta}{r} \quad [\because 2a \cdot q = \mu \text{ from equation (5.48)}] \dots(5.50) \end{aligned}$$

In Fig. 5.43, when  $2a \rightarrow 0$ , the angle  $\delta\theta$  subtended by point  $P$  with  $A$  and  $B$  becomes very small. Also  $\delta r \rightarrow 0$  and  $AP$  becomes equal to  $r$ . Then

$$\sin \theta = \frac{PD}{AP} = \frac{y}{r}$$

$$\text{Also } AP^2 = AD^2 + PD^2 \quad \text{or} \quad r^2 = x^2 + y^2$$

Substituting the value of  $\sin \theta$  in equation (5.50), we get

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{r} \times \frac{1}{r} = -\frac{\mu y}{2\pi r^2} = -\frac{\mu y}{2\pi(x^2 + y^2)} \quad (\because r^2 = x^2 + y^2) \quad \dots(5.50A)$$

or  $x^2 + y^2 = -\frac{\mu y}{2\pi\psi}$  or  $x^2 + y^2 + \frac{\mu y}{2\pi\psi} = 0$

The above equation can be written as

$$x^2 + y^2 + 2 \times y \times \frac{\mu}{4\pi\psi} + \left(\frac{\mu}{4\pi\psi}\right)^2 - \left(\frac{\mu}{4\pi\psi}\right)^2 = 0 \quad \left[ \text{Adding and subtracting } \left(\frac{\mu}{4\pi\psi}\right)^2 \right]$$

or  $x^2 + \left(y + \frac{\mu}{4\pi\psi}\right)^2 = \left(\frac{\mu}{4\pi\psi}\right)^2 \quad \dots(5.51)$

The above is the equation of a circle with centre  $\left(0, \frac{\mu}{4\pi\psi}\right)$  and radius  $\frac{\mu}{4\pi\psi}$ . The centre of the circle lies on y-axis at a distance of  $\frac{\mu}{4\pi\psi}$  from x-axis. As the radius of the circle is also equal to  $\frac{\mu}{4\pi\psi}$ , hence the circle will be tangent to the x-axis. Hence stream lines of the doublet will be the family of circles tangent to the x-axis as shown in Fig. 5.44.

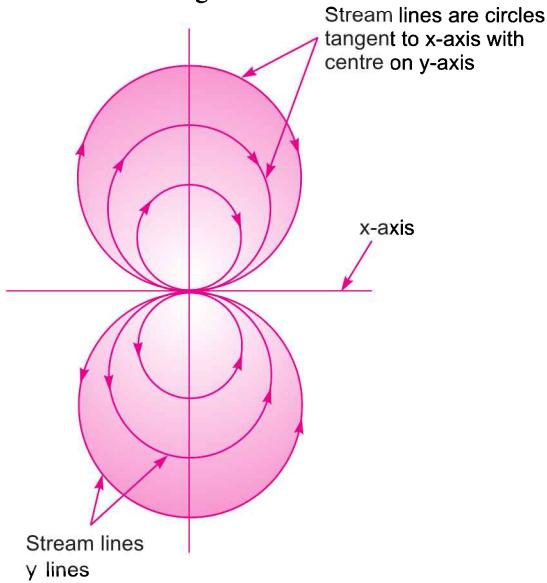


Fig. 5.44 Stream lines for a doublet.

### Potential function at P

Refer to Fig. 5.43. The potential function at  $P$  is given by

$$\phi = \frac{q}{2\pi} \log_e(r + \delta r) + \left(-\frac{q}{2\pi}\right) \log_e r \quad [\text{Refer to equation (5.41)}]$$

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$$\begin{aligned}
 &= \frac{q}{2\pi} \log_e(r + \delta r) - \frac{q}{2\pi} \log_e r = \frac{q}{2\pi} \log_e \left( \frac{r + \delta r}{r} \right) = \frac{q}{2\pi} \log_e \left( 1 + \frac{\delta r}{r} \right)^* \\
 &= \frac{q}{2\pi} \left[ \frac{\delta r}{r} + \left( \frac{\delta r}{r} \right)^2 \times \frac{1}{2} + \dots \right] \\
 &= \frac{q}{2\pi} \cdot \frac{\delta r}{r} \quad \left[ \text{As } \frac{\delta r}{r} \text{ is a small quantity. Hence } \left( \frac{\delta r}{r} \right)^2 \text{ becomes negligible} \right]
 \end{aligned}$$

But in Fig. 5.43, from triangle ABC, we get  $\frac{\delta r}{2a} = \cos \theta$

$$\therefore \delta r = 2a \cos \theta$$

Substituting the value of  $\delta r$ , we get

$$\begin{aligned}
 \phi &= \frac{q}{2\pi} \times \frac{2a \cos \theta}{r} \\
 &= \frac{\mu}{2\pi} \times \frac{\cos \theta}{r} \quad [\because 2a \times q = \mu \text{ from equation (i)}] \dots(5.52)
 \end{aligned}$$

In Fig. 5.43, when  $2a \rightarrow 0$ , the angle  $\delta\theta$  becomes very small.

Also  $\delta r \rightarrow 0$  and AP becomes equal to r. Then

$$\cos \theta = \frac{AD}{AP} = \frac{x}{r}$$

Also  $AP^2 = AD^2 + PD^2$  or  $r^2 = x^2 + y^2$

Substituting the value of  $\cos \theta$  in equation (5.52), we get

$$\begin{aligned}
 \phi &= \frac{\mu}{2\pi} \times \left( \frac{x}{r} \right) \times \frac{1}{r} = \frac{\mu}{2\pi} \times \frac{x}{r^2} \\
 &= \frac{\mu}{2\pi} \times \frac{x}{(x^2 + y^2)} \quad [\because r^2 = x^2 + y^2]
 \end{aligned}$$

$$\text{or } x^2 + y^2 = \frac{\mu}{2\pi} \times \frac{x}{\phi} \quad \text{or } x^2 + y^2 - \frac{\mu}{2\pi} \times \frac{x}{\phi} = 0$$

The above equation can be written as

$$\begin{aligned}
 x^2 - \frac{\mu}{2\pi} \frac{x}{\phi} + \left( \frac{\mu}{4\pi\phi} \right)^2 - \left( \frac{\mu}{4\pi\phi} \right)^2 + y^2 &= 0 \quad \left[ \text{Adding and subtracting } \left( \frac{\mu}{4\pi\phi} \right)^2 \right] \\
 \text{or } \left( x - \frac{\mu}{4\pi\phi} \right)^2 + y^2 &= \left( \frac{\mu}{4\pi\phi} \right)^2 \quad \dots(5.53)
 \end{aligned}$$

The above is the equation of a circle with centre  $\left( \frac{\mu}{4\pi\phi}, 0 \right)$  and radius  $\left( \frac{\mu}{4\pi\phi} \right)$ . The centre of the circle lies on x-axis at a distance of  $\frac{\mu}{4\pi\phi}$  from y-axis. As the radius of the circle is equal to the distance of the centre of the circle from the y-axis, hence the circle will be tangent to the y-axis.

---

\* Expansion of  $\log_e(1 + x) = x + \frac{x^2}{2} + \dots$

Hence the potential lines of a doublet will be a family of circles tangent to the  $y$ -axis with their centres on the  $x$ -axis as shown in Fig. 5.45.

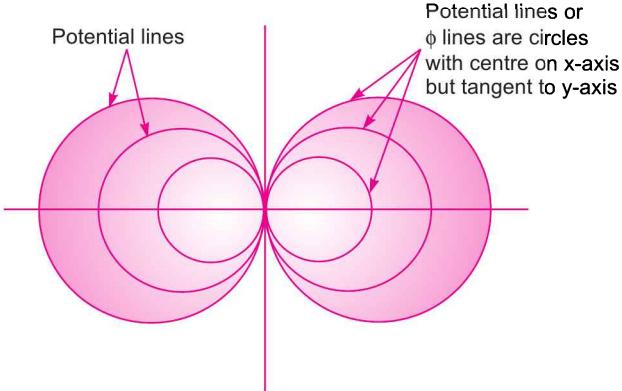


Fig. 5.45 Potential lines for a doublet.

**Problem 5.38** A point  $P(0.5, 1)$  is situated in the flow field of a doublet of strength  $5 \text{ m}^2/\text{s}$ . Calculate the velocity at this point and also the value of the stream function.

**Solution.** Given : Point  $P(0.5, 1)$ . This means  $x = 0.5$  and  $y = 1.0$

Strength of doublet,  $\mu = 5 \text{ m}^2/\text{s}$

(i) Velocity at point  $P$

The velocity at the given point can be obtained if we know the stream function ( $\psi$ ). But stream function is given by equation (5.50A) as

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)}$$

The velocity components  $u$  and  $v$  are obtained from the stream function as

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} \left[ -\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)} \right] \\ &= -\frac{\mu}{2\pi} \frac{\partial}{\partial y} \left[ \frac{y}{(x^2 + y^2)} \right] \quad \left( \because \frac{\mu}{2\pi} \text{ is a constant term} \right) \\ &= -\frac{\mu}{2\pi} \left[ \frac{x^2 - y^2}{(x^2 + y^2)^2} \right] \\ &\quad \left[ \because \frac{\partial}{\partial y} \left[ y(x^2 + y^2)^{-1} \right] = y[-1](x^2 + y^2)^{-2}[2y] + (x^2 + y^2)^{-1}.1 \right] \\ &= \frac{-2y^2}{(x^2 + y^2)^2} + \frac{1}{(x^2 + y^2)} = \frac{-2y^2 + x^2 + y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \end{aligned}$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x} \left[ -\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)} \right]$$

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$$= \frac{\mu}{2\pi} \frac{\partial}{\partial x} \left[ \frac{y}{(x^2 + y^2)} \right] = \frac{\mu}{2\pi} \left[ \frac{-2xy}{(x^2 + y^2)^2} \right]$$

Substituting the values of  $\mu = 5 \text{ m}^2/\text{s}$ ,  $x = 0.5$  and  $y = 1.0$ , we get the velocity components as

$$u = - \frac{\mu}{2\pi} \left[ \frac{x^2 - y^2}{(x^2 + y^2)^2} \right] = - \frac{5}{2\pi} \left[ \frac{0.5^2 - 1^2}{(0.5^2 + 1^2)^2} \right] = - \frac{5}{2\pi} \frac{0.75}{1.25^2} = - 0.382$$

and  $v = \frac{\mu}{2\pi} \left[ \frac{-2xy}{(x^2 + y^2)^2} \right] = \frac{5}{2\pi} \left[ \frac{-2 \times 0.5 \times 1}{(0.5^2 + 1^2)^2} \right] = \frac{5}{2\pi} \left[ \frac{-1}{1.25^2} \right] = - 0.509$

$\therefore$  Resultant velocity,  $V = \sqrt{u^2 + v^2} = \sqrt{(-0.382)^2 + (-0.509)^2} = 0.636 \text{ m/s. Ans.}$

(ii) Value of stream function at point P

$$\begin{aligned} \psi &= - \frac{\mu}{2\pi} \frac{y}{(x^2 + y^2)} = - \frac{5}{2\pi} \times \frac{1.0}{(0.5^2 + 1^2)} = - \frac{5}{2\pi} \times \frac{1}{1.25} \\ &= - 0.636 \text{ m}^2/\text{s. Ans.} \end{aligned}$$

### Solution in polar co-ordinates

The above question can also be done in  $r, \theta$  (i.e., polar) co-ordinates. The stream function in  $r, \theta$  co-ordinates is given by equation (5.50) as

$$\psi = - \frac{\mu}{2\pi} \times \frac{\sin \theta}{r} \quad \dots(i)$$

and velocity components in radial and tangential directions are given as

$$\begin{aligned} u_r &= \frac{1}{r} \times \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left[ - \frac{\mu}{2\pi} \frac{\sin \theta}{r} \right] \\ &= \frac{1}{r} \times \left( - \frac{\mu}{2\pi} \right) \times \frac{1}{r} \frac{\partial}{\partial \theta} (\sin \theta) \\ &\quad \left[ \because \frac{\mu}{2\pi} \text{ is a constant term and also } r \text{ is constant w.r.t. } \theta \right] \\ &= - \frac{\mu}{2\pi} \times \frac{1}{r^2} \cos \theta \quad \dots(ii) \end{aligned}$$

and

$$\begin{aligned} u_\theta &= - \frac{\partial \psi}{\partial r} = - \frac{\partial}{\partial r} \left[ - \frac{\mu}{2\pi} \frac{\sin \theta}{r} \right] \\ &= - \left( - \frac{\mu}{2\pi} \sin \theta \right) \frac{\partial}{\partial r} \left[ \frac{1}{r} \right] = \frac{\mu}{2\pi} \sin \theta (-1) \cdot \frac{1}{r^2} \\ &\quad \left[ \because \frac{\mu \sin \theta}{2\pi} \text{ is a constant w.r.t. } r \right] \\ &= - \frac{\mu}{2\pi} \times \frac{\sin \theta}{r^2} \quad \dots(iii) \end{aligned}$$

Now

$$r = \sqrt{x^2 + y^2} = \sqrt{0.5^2 + 1^2} = \sqrt{1.25}$$

$$\therefore \sin \theta = \frac{y}{r} = \frac{1}{\sqrt{1.25}} = 0.894 \text{ and } \cos \theta = \frac{x}{r} = \frac{0.5}{\sqrt{1.25}} = 0.447$$

Substituting the values of  $r$ ,  $\sin \theta$  and  $\cos \theta$  in above equations (i), (ii) and (iii), we get

$$\psi = -\frac{\mu}{2\pi} \frac{\sin \theta}{r} = -\frac{5}{2\pi} \times \frac{0.894}{\sqrt{1.25}} = -0.636 \text{ m}^2/\text{s. Ans.}$$

$$u_r = -\frac{\mu}{2\pi} \times \frac{1}{r^2} \times \cos \theta = -\frac{5}{2\pi} \times \frac{1}{(1.25)^2} \times 0.447 = -0.2845 \text{ m/s}$$

and

$$u_\theta = -\frac{\mu}{2\pi} \times \frac{\sin \theta}{r^2} = -\frac{5}{2\pi} \times \frac{0.894}{1.25} = -0.569 \text{ m/s}$$

$\therefore$  Resultant velocity,  $V = \sqrt{u_r^2 + u_\theta^2}$

$$= \sqrt{(-0.2845)^2 + (-0.569)^2} = 0.636 \text{ m/s. Ans.}$$

**5.17.3 A Plane Source in a Uniform Flow (Flow Past a Half-Body).** Fig. 5.46 (a) shows a uniform flow of velocity  $U$  and Fig. 5.46 (b) shows a source flow of strength  $q$ . When this uniform flow is flowing over the source flow, a resultant flow will be obtained as shown in Fig. 5.46. This resultant flow is also known as the flow past a half-body. Let the source is placed on the origin  $O$ . Consider a point  $P(x, y)$  lying in the resultant flow field with polar co-ordinates  $r$  and  $\theta$  as shown in Fig. 5.46.

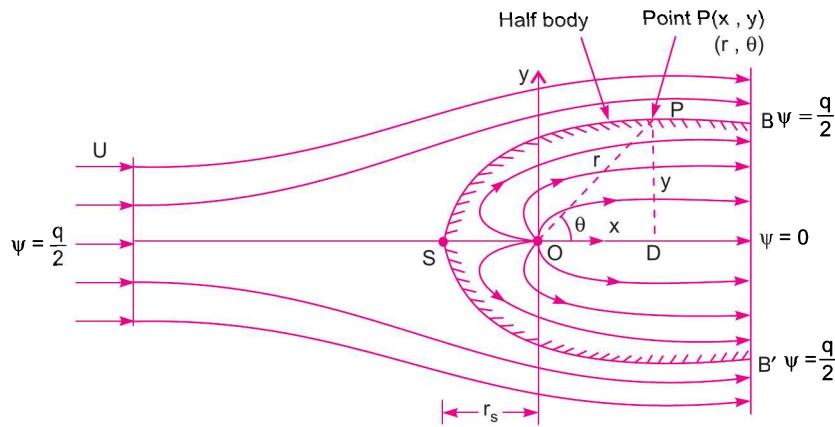
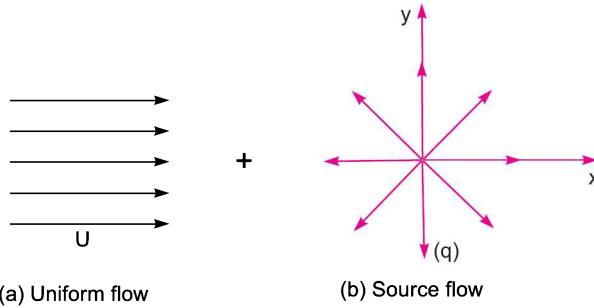


Fig. 5.46 Flow pattern resulting from the combination of a uniform flow and a source.

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The stream function ( $\psi$ ) and potential function ( $\phi$ ) for the resultant flow are obtained as given below :

$\psi$  = Stream function due to uniform flow + stream function due to source

$$= U \cdot y + \frac{q}{2\pi} \theta \quad \dots(5.54)$$

$$= U \cdot r \sin \theta + \frac{q}{2\pi} \theta \quad (\because y = r \sin \theta) \dots(5.54A)$$

and  $\phi$  = Velocity potential function due to uniform flow + Velocity potential function due to source

$$= U \cdot x + \frac{q}{2\pi} \log_e r = U \cdot r \cos \theta + \frac{q}{2\pi} \log_e r \quad \dots(5.54B)$$

The following are the important points for the resultant flow pattern :

(i) *Stagnation point*. On the left side of the source, at the point  $S$  lying on the  $x$ -axis, the velocity of uniform flow and that due to source are equal and opposite to each other. Hence the net velocity of the combined flow field is zero. This point is known as stagnation point and is denoted by  $S$ . The polar co-ordinates of the stagnation point  $S$  are  $r_s$  and  $\pi$ , where  $r_s$  is radial distance of point  $S$  from  $O$ .

The net velocity (or resultant velocity) is zero at the stagnation point  $S$ .

$$\begin{aligned} u_r &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left( U \cdot r \sin \theta + \frac{q}{2\pi} \theta \right) \quad \left[ \because \psi = U \cdot r \sin \theta + \frac{q}{2\pi} \theta \right] \\ &= \frac{1}{r} \left[ U \cdot r \cos \theta + \frac{q}{2\pi} \right] = U \cdot \cos \theta + \frac{q}{2\pi r} \end{aligned}$$

At the stagnation point,  $\theta = \pi$  radians ( $180^\circ$ ) and  $r = r_s$  and net velocity is zero. This means  $u_r = 0$  and  $v_\theta = 0$ . Substituting these values in the above equation, we get

$$0 = U \cdot \cos 180^\circ + \frac{q}{2\pi r_s} \quad [\because u_r = 0, \theta = 180^\circ \text{ and } r = r_s]$$

$$= -U + \frac{q}{2\pi r_s} \quad \text{or} \quad U = \frac{q}{2\pi r_s}$$

$$\text{or} \quad r_s = \frac{q}{2\pi U} \quad \dots(5.55)$$

From the above equation it is clear that position of stagnation point depends upon the free stream velocity  $U$  and source strength  $q$ . At the stagnation point, the value of stream function is obtained from equation (5.54A) as

$$\psi = U \cdot r \sin \theta + \frac{q}{2\pi} \cdot \theta$$

For the stagnation point, the above equation becomes as

$$\begin{aligned} \therefore \psi_s &= U \cdot r_s \sin 180^\circ + \frac{q}{2\pi} \times \theta \\ &= 0 + \frac{q}{2} = \frac{q}{2} \quad [\because \text{At stagnation point, } \theta = \pi \text{ radians} = 180^\circ \text{ and } r = r_s] \end{aligned} \quad \dots(5.56)$$

The above relation gives the equation of stream line passing through stagnation point. We know that no fluid mass crosses a stream line. Hence a stream line is a *virtual solid surface*.

(ii) *Shape of resultant flow.* At the stagnation point  $S$ , the net velocity is zero. The fluid particles that issue from the source cannot proceed further to the left of stagnation point. They are carried along the contour  $BSB'$  that separates the source flow from uniform flow. The curve  $BSB'$  can be regarded as the **solid boundary** of a round nosed body such as a bridge pier around which the uniform flow is forced to pass. The contour  $BSB'$  is called the half body, because it has only the leading point, it trails to infinity at down stream end.

The value of stream function of the stream line passing through stagnation point  $S$  and passing over the solid boundary (*i.e.*, curve  $BSB'$ ) is  $\psi_S = \frac{q}{2}$ .

Thus the composite flow consists of :

(1) flow over a plane half-body (*i.e.*, flow over curve  $BSB'$ ) outside  $\psi = \frac{q}{2}$  and

(2) source flow within the plane half-body.

The plane half-body is described by the dividing stream line,  $\psi = \frac{q}{2}$ .

But the stream function at any point in the combined flow field is given by equation (5.54) as

$$\psi = U \cdot y + \frac{q}{2\pi} \theta$$

If we take  $\psi = \frac{q}{2}$  in the above equation, we will get the equation of the dividing stream line.

$\therefore$  Equation of the dividing stream line (*i.e.*, equation of curve  $BSB'$ ) will be

$$\frac{q}{2} = U \cdot y + \frac{q}{2\pi} \cdot \theta \text{ or } U \cdot y = \frac{q}{2} - \frac{q}{2\pi} \theta = \frac{q}{2} \left(1 - \frac{\theta}{\pi}\right)$$

or

$$y = \frac{q}{2U} \left(1 - \frac{\theta}{\pi}\right) \quad \dots(5.57)$$

From the above equation, the main dimensions of the plane half-body may be obtained. From this equation, it is clear that  $y$  is maximum, when  $\theta = 0$ .

Hence At  $\theta = 0$ ,  $y$  is maximum and  $y_{\max} = \frac{q}{2U}$  ... the maximum ordinate

At  $\theta = \frac{\pi}{2}$ ,  $y = \frac{q}{2U} \left(1 - \frac{\pi}{2} \cdot \frac{1}{\pi}\right) = \frac{q}{4U}$  ... the ordinate above the origin

At  $\theta = \pi$ ,  $y = \frac{q}{2U} \left(1 - \frac{\pi}{\pi}\right) = 0$  ... the leading point of the half-body

At  $\theta = \frac{3\pi}{2}$ ,  $y = \frac{q}{2U} \left(1 - \frac{3\pi}{2\pi}\right) = -\frac{q}{4U}$  ... the ordinate below the origin.

The main dimensions are shown in Fig. 5.47.

(iii) *Resultant velocity at any point*

The velocity components at any point in the flow field are given by

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{d}{d\theta} \left[ U \cdot r \sin \theta + \frac{q}{2\pi} \theta \right]$$

$$= \frac{1}{r} \left[ U \cdot r \cos \theta + \frac{q}{2\pi} \right] = U \cdot \cos \theta + \frac{q}{2\pi r}$$

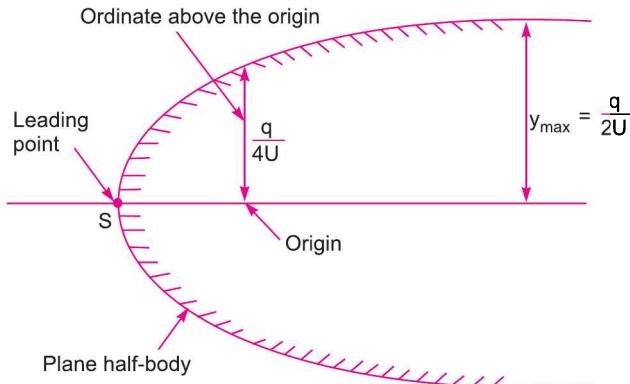


Fig. 5.47

The above equation gives the radial velocity at any point in the flow field. This radial velocity is due to uniform flow and due to source. Due to source the radial velocity is  $\frac{q}{2\pi r}$ . Hence the velocity due to source diminishes with increase in radial distance from the source. At large distance from the source the contribution of source is negligible and hence free stream uniform flow is not influenced by the presence of source.

$$\begin{aligned} u_\theta &= -\frac{\partial \Psi}{\partial r} = -\frac{\partial}{\partial r} \left[ U \cdot r \sin \theta + \frac{q}{2\pi} \theta \right] \\ &= -[U \cdot \sin \theta + 0] = -U \sin \theta \quad \left[ \because \frac{q}{2\pi} \theta \text{ is constant w.r.t. } r \right] \end{aligned}$$

$$\therefore \text{Resultant velocity, } V = \sqrt{u_r^2 + u_\theta^2}$$

#### (iv) Location of stagnation point

At the stagnation point, the velocity components are zero. Hence equating the radial and tangential velocity components to zero, we get

$$u_r = 0 \quad \text{or} \quad U \cos \theta + \frac{q}{2\pi r} = 0 \quad \text{or} \quad U \cos \theta = -\frac{q}{2\pi r}$$

$$\text{or} \quad r \cos \theta = -\frac{q}{2\pi U} \quad \text{But} \quad r \cos \theta = x$$

$$\therefore x = -\frac{q}{2\pi U}$$

$$\begin{aligned} \text{When } u_\theta &= 0 \quad \text{or} \quad -U \sin \theta = 0 \quad \text{or} \quad \sin \theta = 0 \quad \text{as } U \text{ cannot be zero} \\ \text{or } \theta &= 0 \quad \text{or} \quad \pi \quad \text{But } y = r \sin \theta \quad \therefore y = 0 \end{aligned}$$

Hence stagnation point is at  $\left( -\frac{q}{2\pi U}, 0 \right)$ , the leading point of the half-body.

(v) Pressure at any point in flow field

Let  $p_0$  = pressure at infinity where velocity is  $U$

$p$  = pressure at any point  $P$  in the flow field, where velocity is  $V$

Now applying the Bernoulli's equation at a point at infinity and at a point  $P$  in the flow field, we get

$$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g} \quad \text{or} \quad \frac{U^2}{2g} - \frac{V^2}{2g} = \frac{p}{\rho g} - \frac{p_0}{\rho g} = \frac{p - p_0}{\rho g}$$

The pressure co-efficient is defined as

$$\begin{aligned} C_p &= \frac{p - p_0}{\frac{1}{2}\rho U^2} \\ &= \frac{\rho g \left[ \frac{U^2}{2g} - \frac{V^2}{2g} \right]}{\frac{1}{2}\rho U^2} \quad \left[ \because p - p_0 = \rho g \left( \frac{U^2}{2g} - \frac{V^2}{2g} \right) \right] \\ &= \frac{U^2 - V^2}{U^2} = 1 - \left( \frac{V}{U} \right)^2 \end{aligned} \quad \dots(5.58)$$

**Problem 5.39** A uniform flow with a velocity of 3 m/s is flowing over a plane source of strength 30 m<sup>2</sup>/s. The uniform flow and source flow are in the same plane. A point  $P$  is situated in the flow field. The distance of the point  $P$  from the source is 0.5 m and it is at an angle of 30° to the uniform flow. Determine : (i) stream function at point  $P$ , (ii) resultant velocity of flow at  $P$  and (iii) location of stagnation point from the source.

**Solution.** Given : Uniform velocity,  $U = 3$  m/s ; source strength,  $q = 30$  m<sup>2</sup>/s ; co-ordinates of point  $P$  are  $r = 0.5$  m and  $\theta = 30^\circ$ .

(i) Stream function at point  $P$

The stream function at any point in the combined flow field is given by equation (5.54A)

$$\psi = U \cdot r \sin \theta + \frac{q}{2\pi} \theta$$

at point  $P$ ,  $r = 0.5$  m and  $\theta = 30^\circ$  or  $\frac{30}{180} \times \pi$  radians.

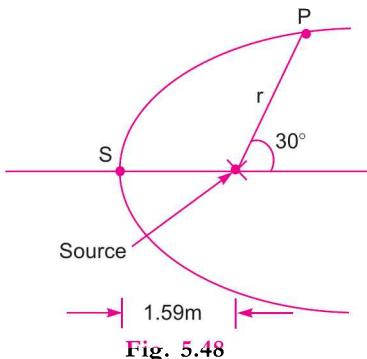


Fig. 5.48

$\therefore$  Stream function at point  $P$ ,

$$\psi = 3 \times 0.5 \times \sin 30^\circ + \frac{30}{2\pi} \times \left( \frac{30}{180} \times \pi \right)$$

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$$= 0.75 + 2.5 = 3.25 \text{ m}^2/\text{s. Ans.}$$

(ii) Resultant velocity at P

The velocity components anywhere in the flow are given by

$$\begin{aligned} u_r &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left[ U \cdot r \sin \theta + \frac{q}{2\pi} \theta \right] \\ &= \frac{1}{r} \left[ U \cdot r \cos \theta + \frac{q}{2\pi} \right] = U \cdot \cos \theta + \frac{q}{2\pi r} \\ &= 3 \times \cos 30^\circ + \frac{30}{2\pi \times 0.5} \quad (\because \text{At } P, r = 0.5, \theta = 30^\circ, q = 30) \\ &= 2.598 + 9.55 = 12.14 \end{aligned}$$

and

$$\begin{aligned} u_\theta &= -\frac{\partial \psi}{\partial r} = -\frac{\partial}{\partial r} \left[ U \cdot r \sin \theta + \frac{q}{2\pi} \cdot \theta \right] \\ &= -U \sin \theta + 0 = -U \sin \theta \\ &= -3 \times \sin 30^\circ = -1.5 \end{aligned}$$

$$\begin{aligned} \therefore \text{ Resultant velocity, } V &= \sqrt{u_r^2 + u_\theta^2} \\ &= \sqrt{12.14^2 + (-1.5)^2} = 12.24 \text{ m/s. Ans.} \end{aligned}$$

(iii) Location of stagnation point

The horizontal distance of the stagnation point S from the source is given by equation (5.55) as

$$r_s = \frac{q}{2\pi U} = \frac{30}{2\pi \times 3} = 1.59 \text{ m. Ans.}$$

The stagnation point will be at a distance of 1.59 m to the left side of the source on the x-axis.

**Problem 5.40** A uniform flow with a velocity of 20 m/s is flowing over a source of strength 10 m<sup>2</sup>/s. The uniform flow and source flow are in the same plane. Obtain the equation of the dividing stream line and sketch the flow pattern.

**Solution.** Given : Uniform velocity,  $U = 20 \text{ m/s}$  ; Source strength,  $q = 10 \text{ m}^2/\text{s}$

(i) Equation of the dividing stream line

The stream function at any point in the combined flow field is given by equation (5.54A)

$$\begin{aligned} \psi &= U \cdot r \sin \theta + \frac{q}{2\pi} \theta \\ &= 20 \times r \sin \theta + \frac{10}{2\pi} \theta \quad (\because U = 20 \text{ m/s and } q = 10 \text{ m}^2/\text{s}) \end{aligned}$$

The value of the stream function for the dividing stream line is  $\psi = \frac{q}{2}$ . Hence substituting  $\psi = \frac{q}{2}$  in the above equation, we get the equation of the dividing stream line.

$$\therefore \frac{q}{2} = 20r \sin \theta + \frac{10}{2\pi} \theta$$

$$\text{or } \frac{10}{2} = 20r \sin \theta + \frac{10}{2\pi} \theta \quad (\because q = 10)$$

or  $5 = 20r \sin \theta + \frac{10}{2\pi} \theta = 20y + \frac{10}{2\pi} \theta \quad (\because r \sin \theta = y)$

$$\therefore 20y = 5 - \frac{10}{2\pi} \theta$$

or  $y = \frac{5}{20} - \frac{10}{2\pi} \times \frac{\theta}{20} = 0.25 - \frac{\theta}{4\pi} \quad \dots(i)$

The above relation gives the equation of the dividing stream line.

From the above equation, for different values of  $\theta$  the value of  $y$  is obtained as :

Value of $\theta$	Value of $y$ from (i)	Remarks
0	0.25 m	Max. half width of body
$\frac{\pi}{2}$	0.125 m	The +ve ordinate above the origin
$\pi$	0	The leading point
$\frac{3\pi}{2}$	- 0.125 m	The -ve ordinate below the origin
$2\pi$	- 0.25 m	The max. -ve ordinate

#### (ii) Sketch of flow pattern

For sketching the flow pattern, let us first find the location of the stagnation point. The horizontal distance of the stagnation point  $S$  from the source is given by the equation,

$$r_s = \frac{q}{2\pi U} = \frac{10}{2\pi \times 20} = 0.0795 \text{ m}$$

Hence the stagnation point lies on the  $x$ -axis at a distance of 0.0795 m or 79.5 mm from the source towards left of the source. The flow pattern is shown in Fig. 5.49.

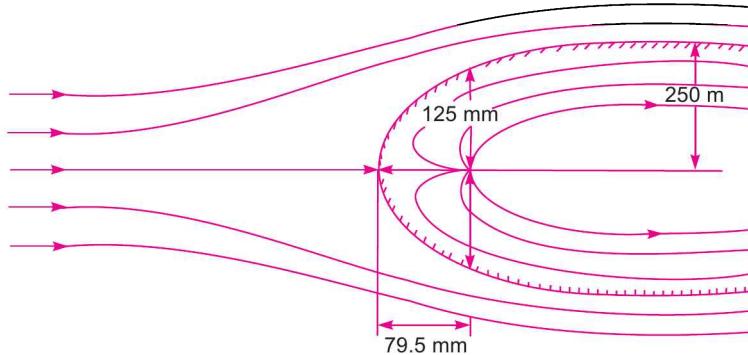


Fig. 5.49

**Problem 5.41** A uniform flow with a velocity of 2 m/s is flowing over a source placed at the origin. The stagnation point occurs at  $(-0.398, 0)$ . Determine :

- (i) Strength of the source,                   (ii) Maximum width of Rankine half-body and
- (iii) Other principal dimensions of the Rankine half-body.

**Solution.** Given :

Uniform velocity,                    $U = 2 \text{ m/s}$

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Co-ordinates of stagnation point = (- 0.398, 0)

This means  $r_s = 0.398$  and stagnation point lies on  $x$ -axis at a distance of 0.398 m towards left of origin. The source is placed at origin.

(i) *Strength of the source*

Let  $q$  = strength of the source

We know that

$$r_s = \frac{q}{2\pi U}$$

or

$$q = 2\pi U \times r_s = 2\pi \times 2 \times 0.398 = 5.0014 \text{ m}^2/\text{s} \approx 5 \text{ m}^2/\text{s. Ans.}$$

(ii) *Maximum width of Rankine half-body*

The main dimensions of the Rankine half-body are obtained from equation (5.57) as

$$y = \frac{q}{2U} \left( 1 - \frac{\theta}{\pi} \right) \quad \dots(i)$$

The value of  $y$  is maximum, when  $\theta = 0$ .

$$\therefore y_{\max} = \frac{q}{2U} \left( 1 - \frac{0}{\pi} \right) = \frac{q}{2U} = \frac{5}{2 \times 2} = 1.25 \text{ m}$$

$\therefore$  Maximum width of Rankine body =  $2 \times y_{\max} = 2 \times 1.25 = 2.5 \text{ m. Ans.}$

(iii) *Other Principal dimensions of Rankine half-body*

Using equation (5.57), we get

$$y = \frac{q}{2U} \left( 1 - \frac{\theta}{\pi} \right)$$

$$\text{At } \theta = \frac{\pi}{2}, \quad y = \frac{q}{2U} \left[ 1 - \frac{\left(\frac{\pi}{2}\right)}{\pi} \right] = \frac{q}{2U} \left[ 1 - \frac{1}{2} \right] = \frac{q}{4U} = \frac{5}{4 \times 2} = 0.625 \text{ m}$$

The above value gives the upper ordinate at the origin, where source is placed.

$\therefore$  Width of body at origin =  $2 \times 0.625 = 1.25 \text{ m}$

At the stagnation point, the width of the body is zero.

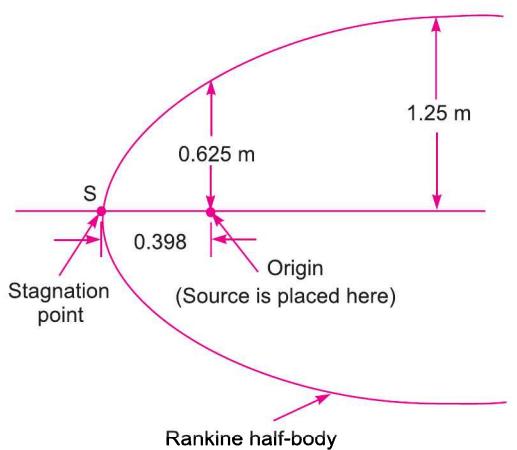


Fig. 5.50

**5.17.4 A Source and Sink Pair in a Uniform Flow (Flow Past a Rankine Oval Body).**

Fig. 5.51 (a) shows a uniform flow of velocity  $U$  and Fig. 5.51 (b) shows a source sink pair of equal strength. When this uniform flow is flowing over the source sink pair, a resultant flow will be obtained as shown in Fig. 5.51 (c). This resultant flow is also known as the flow past a Rankine oval body.

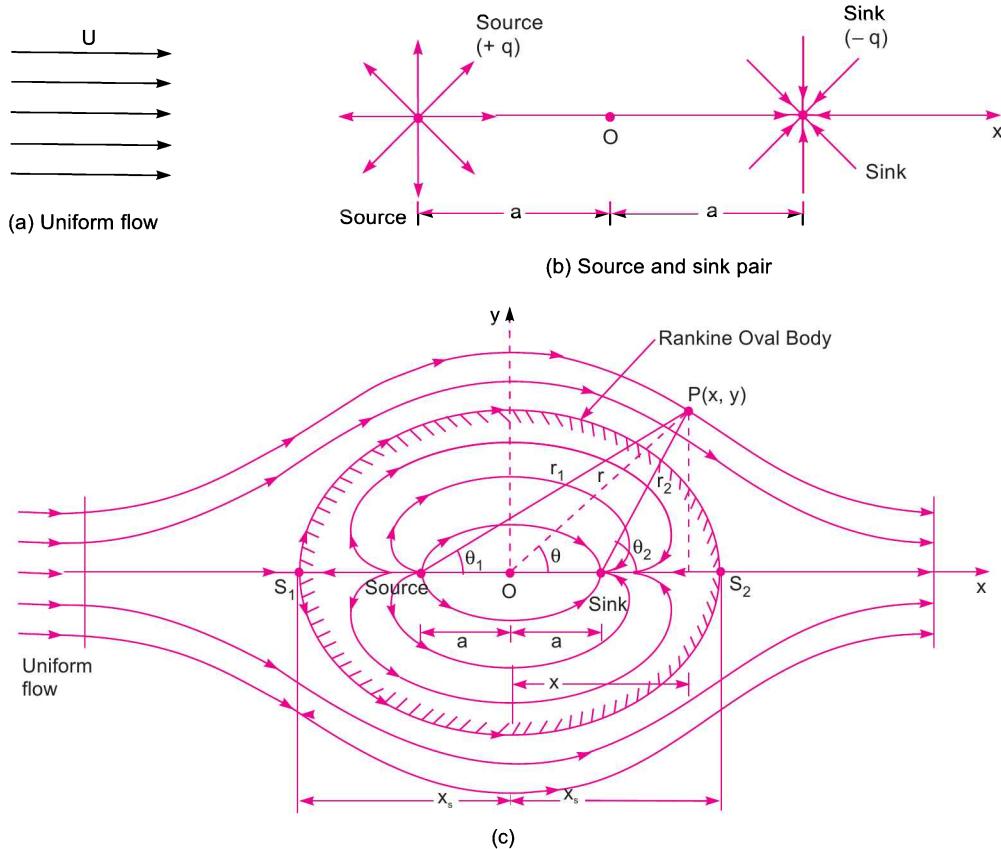


Fig. 5.51

Let  $U$  = Velocity of uniform flow along  $x$ -axis

$q$  = Strength of source

$(-q)$  = Strength of sink

$2a$  = Distance between source and sink which is along  $x$ -axis.

The origin  $O$  of the  $x$ - $y$  co-ordinates is mid-way between source and sink. Consider a point  $P(x, y)$  lying in the resultant flow field. The stream function ( $\psi$ ) and velocity potential function ( $\phi$ ) for the resultant flow field are obtained as given below :

$$\begin{aligned}
 \psi &= \text{Stream function due to uniform flow} + \text{stream function due to source} \\
 &\quad + \text{stream function due to sink} \\
 &= \Psi_{\text{uniform flow}} + \Psi_{\text{source}} + \Psi_{\text{sink}} \\
 &= U \times y + \frac{q}{2\pi} \theta_1 + \frac{(-q)}{2\pi} \times \theta_2 \\
 &\quad (\text{where } \theta_1 \text{ is the angle made by } P \text{ with source along } x\text{-axis and } \theta_2 \text{ with sink})
 \end{aligned}$$

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$$\begin{aligned}
 &= U \times y + \frac{q\theta_1}{2\pi} - \frac{q\theta_2}{2\pi} = U \times y + \frac{q}{2\pi} (\theta_1 - \theta_2) \\
 &= U \times r \sin \theta + \frac{q}{2\pi} (\theta_1 - \theta_2) \quad (\because y = r \sin \theta) \dots(5.59)
 \end{aligned}$$

and

$$\begin{aligned}
 \phi &= \text{potential function due to uniform flow} + \text{potential function due to source} + \text{potential function due to sink} \\
 &= \phi_{\text{uniform flow}} + \phi_{\text{source}} + \phi_{\text{sink}} \\
 &= U \times x + \frac{q}{2\pi} \log_e r_1 + \frac{(-q)}{2\pi} \log_e r_2 \\
 &= U \times r \cos \theta + \frac{q}{2\pi} [\log_e r_1 - \log_e r_2] \quad (\because x = r \cos \theta) \\
 &= U \times r \cos \theta + \frac{q}{2\pi} \left[ \log_e \frac{r_1}{r_2} \right] \quad \dots(5.60)
 \end{aligned}$$

The following are the important points for the resultant flow pattern :

(a) There will be two stagnation points  $S_1$  and  $S_2$ , one to the left of the source and other to the right of the sink. At the stagnation points, the resultant velocity (*i.e.*, velocity due to uniform flow, velocity due to source and velocity due to sink) will be zero. The stagnation point  $S_1$  is to the left of the source and stagnation point  $S_2$  will be to the right of the sink on the  $x$ -axis.

Let  $x_S$  = Distance of the stagnation points from origin  $O$  along  $x$ -axis.

Let us calculate this distance  $x_S$ .

For the stagnation point  $S_1$ ,

(i) Velocity due to uniform flow =  $U$

$$\text{(ii) Velocity due to source} = \frac{q}{2\pi(x_S - a)} \quad \left[ \because \text{The velocity at any radius due to source} = \frac{q}{2\pi r} \right]$$

For  $S_1$ , the radius from source =  $(x_S - a)$

$$\text{(iii) Velocity due to sink} = \frac{-q}{2\pi(x_S + a)} \quad \left[ \because \text{At } S_1, \text{ the radius from sink} = (x_S + a) \right]$$

At point  $S_1$ , the velocity due to uniform flow is in the positive  $x$ -direction whereas due to source and sink are in the  $-ve$   $x$ -direction.

$$\therefore \text{The resultant velocity at } S_1 = U - \frac{q}{2\pi(x_S - a)} - \frac{(-q)}{2\pi(x_S + a)}$$

But the resultant velocity at stagnation point  $S_1$  should be zero.

$$\therefore U - \frac{q}{2\pi(x_S - a)} + \frac{q}{2\pi(x_S + a)} = 0$$

or

$$U = \frac{q}{2\pi(x_S - a)} - \frac{q}{2\pi(x_S + a)}$$

$$= \frac{q}{2\pi} \left[ \frac{1}{(x_s - a)} - \frac{1}{(x_s + a)} \right] = \frac{q}{2\pi} \left[ \frac{(x_s + a) - (x_s - a)}{(x_s - a)(x_s + a)} \right] = \frac{q}{2\pi} \frac{2a}{(x_s^2 - a^2)}$$

or  $x_s^2 - a^2 = \frac{q \cdot a}{\pi U}$

or  $x_s^2 = a^2 + \frac{qa}{\pi U} = a^2 \left[ 1 + \frac{q}{\pi a U} \right]$

$$\therefore x_s = a \sqrt{\left( 1 + \frac{q}{\pi a U} \right)} \quad \dots(5.61)$$

The above equation gives the location of the stagnation point on the  $x$ -axis.

(b) The stream line passing through the stagnation points is having zero velocity and hence can be replaced by a solid body. This solid body is having a shape of oval as shown in Fig. 5.51. There will be two flow fields, one within the oval contour and the other outside the solid body. The flow field within the oval contour will be due to source and sink whereas the flow field outside the body will be due to uniform flow only.

The shape of solid body is obtained from the stream line having stream function equal to zero. But the stream function is given by equation as

$$\psi = U \times r \sin \theta + \frac{q}{2\pi} (\theta_1 - \theta_2)$$

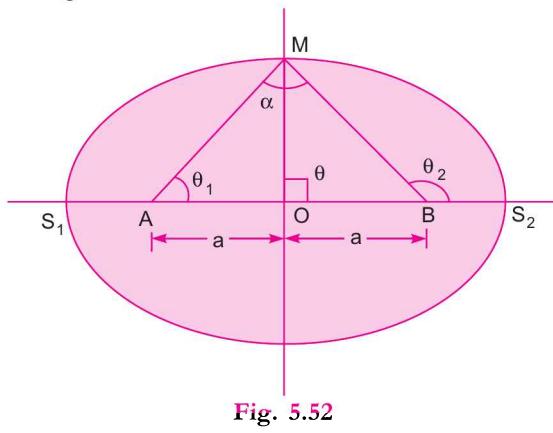
For the shape of solid body,  $\psi = 0$

$$\therefore 0 = U \times r \sin \theta + \frac{q}{2\pi} (\theta_1 - \theta_2)$$

or  $U \times r \sin \theta = - \frac{q}{2\pi} (\theta_1 - \theta_2) = \frac{q}{2\pi} (\theta_2 - \theta_1)$

$$\therefore r = \frac{q}{2\pi} \frac{(\theta_2 - \theta_1)}{U \sin \theta} \quad \dots(5.62)$$

From the above equation, the distances of the surface of the solid body from the origin can be obtained or the shape of the solid body can be obtained. The maximum width of the body ( $y_{\max}$ ) will be equal to  $OM$  as shown in Fig. 5.52.



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From triangle  $AOM$ , we have

$$\tan \theta_1 = \frac{OM}{AO}$$

or

$$OM = AO \tan \theta_1 = a \tan \theta_1$$

or

$$y_{\max} = a \tan \theta_1$$

( $\because OM = y_{\max}$ ) ... (5.63)

Let us find the value of  $\theta_1$ .

When the point  $P$  lies on  $M$ , then  $r = OM$ ,  $\theta = 90^\circ = \frac{\pi}{2}$

and

$$\theta_2 = 180^\circ - \theta_1 = \pi - \theta_1$$

[Refer to Fig. 5.52]

$$[\because AM = BM \therefore \text{Angle } ABM = \text{Angle } BAM = \theta_1]$$

Substituting these values in equation (5.62), we get

$$OM = \frac{q}{2\pi} \frac{((\pi - \theta_1) - \theta_1)}{U \sin \frac{\pi}{2}} = \frac{q}{2\pi} \frac{(\pi - 2\theta_1)}{U}$$

$$\text{or } y_{\max} = \frac{q(\pi - 2\theta_1)}{2\pi U} \quad [\text{where } OM = y_{\max}]$$

$$\text{or } 2\pi U y_{\max} = q(\pi - 2\theta_1) \quad \text{or} \quad \frac{2\pi U y_{\max}}{q} = \pi - 2\theta_1$$

$$\text{or } 2\theta_1 = \pi - \frac{2\pi U y_{\max}}{q} \quad \text{or} \quad \theta_1 = \frac{\pi}{2} - \frac{\pi U y_{\max}}{q}$$

Substituting this value of  $\theta_1$  in equation (5.63), we get

$$y_{\max} = a \tan \left[ \frac{\pi}{2} - \frac{\pi U y_{\max}}{q} \right] = a \cot \left[ \frac{\pi U y_{\max}}{q} \right] \quad \dots(5.64)$$

From the above equation, the value of  $y_{\max}$  is obtained by hit and trial method till L.H.S. = R.H.S. In this equation  $\left( \frac{\pi U y_{\max}}{q} \right)$  is in radians.

The length and width of the Rankine oval is obtained as :

Length,

$$L = 2 \times x_S$$

$$= 2 \times a \sqrt{\left(1 + \frac{q}{\pi a U}\right)} \quad \left[\because x_S = a \sqrt{\left(1 + \frac{q}{\pi a U}\right)}\right] \quad \dots(5.65)$$

and Width,

$$B = 2 \times y_{\max}$$

$$= 2a \cot \left( \frac{\pi U y_{\max}}{q} \right). \quad \dots(5.66)$$

**Problem 5.42** A uniform flow of velocity 6 m/s is flowing along  $x$ -axis over a source and a sink which are situated along  $x$ -axis. The strength of source and sink is  $15 \text{ m}^2/\text{s}$  and they are at a distance of 1.5 m apart. Determine :

- (i) Location of stagnation points,                   (ii) Length and width of the Rankine oval  
(iii) Equation of profile of the Rankine body.

**Solution.** Given : Uniform flow velocity,  $U = 6 \text{ m/s}$   
Strength of source and sink,  $q = 15 \text{ m}^2/\text{s}$   
Distance between source and sink,  $2a = 1.5 \text{ m}$

$$\therefore a = \frac{1.5}{2} = 0.75 \text{ m}$$

(i) Location of stagnation points (Refer to Fig. 5.51)

For finding the location of the stagnation points, the equation (5.61) is used.

$$\therefore x_s = a \sqrt{\left(1 + \frac{q}{\pi a U}\right)} = 0.75 \sqrt{\left(1 + \frac{15}{\pi \times 0.75 \times 6}\right)} = 1.076 \text{ m}$$

The above equation gives the distance of the stagnation points from the origin. There will be two stagnation points.

The distance of stagnation points from the source and sink  $= x_s - a = 1.076 - 0.75 = 0.326 \text{ m. Ans.}$

(ii) Length and width of the Rankine oval

Length,  $L = 2 \times x_s = 2 \times 1.076 = 2.152 \text{ m.}$

Width,  $B = 2 \times y_{\max}$  ... (i)

Let us now find the value of  $y_{\max}$

Using equation (5.64), we get

$$\begin{aligned} y_{\max} &= a \cot \left( \frac{\pi U y_{\max}}{q} \right) = 0.75 \cot \left( \frac{\pi \times 6 \times y_{\max}}{15} \right) = 0.75 \cot (0.4\pi y_{\max}) \\ &= 0.75 \cot \left( 0.4\pi y_{\max} \times \frac{180}{\pi} \right)^{\circ} \end{aligned}$$

$$\left[ \because (0.4\pi y_{\max}) \text{ is in radians and hence } (0.4\pi y_{\max}) \times \frac{180}{\pi} \text{ will be in degrees} \right]$$

$$= 0.75 \cot (72 \times y_{\max})^{\circ}$$

The above equation will be solved by hit and trial method. The value of  $x_s = 1.076$ . But  $x_s$  is equal to length of major axis of Rankine body and  $y_{\max}$  is the length of minor axis of the Rankine body. The length of minor axis will be less than length of major axis. Let us first assume  $y_{\max} = 0.8 \text{ m}$ . Then

$y_{\max}$	L.H.S.	R.H.S.
0.8	0.8	$0.75 \cot (72 \times 0.8)^{\circ} = 0.75 \cot 51.6^{\circ} = 0.475$
0.7	0.7	$0.75 \cot (72 \times 0.7)^{\circ} = 0.75 \cot 50.4^{\circ} = 0.577$
0.6	0.6	$0.75 \cot (72 \times 0.6)^{\circ} = 0.75 \cot 43.2^{\circ} = 0.798$
0.65	0.65	$0.75 \cot (72 \times 0.65)^{\circ} = 0.75 \cot 46.8^{\circ} = 0.704$
0.67	0.67	$0.75 \cot (72 \times 0.67)^{\circ} = 0.75 \cot 48.24^{\circ} = 0.669 \approx 0.67$

From above it is clear that, when  $y_{\max} = 0.67$ , then L.H.S. = R.H.S.

$$\therefore y_{\max} = 0.67 \text{ m}$$

Substituting this value in equation (i), we get

$$\text{Width, } B = 2 \times y_{\max} = 2 \times 0.67 = 1.34 \text{ m. Ans.}$$

(iii) Equation of profile of the Rankine body

The equation of profile of the Rankine body is given by equation (5.62) as

$$r = \frac{q}{2\pi} \frac{(\theta_2 - \theta_1)}{U \sin \theta} = \frac{15}{2\pi} \frac{(\theta_2 - \theta_1)}{6 \times \sin \theta} = \frac{0.398 (\theta_2 - \theta_1)}{\sin \theta}. \text{ Ans.}$$

**5.17.5 A Doublet in a Uniform Flow (Flow Past a Circular Cylinder).** Fig. 5.53 (a) shows a uniform flow of velocity  $U$  in the positive  $x$ -direction and Fig. 5.53 (b) shows a doublet at the origin. Doublet is a special case of a source and a sink combination in which both of equal strength approach each other such that distance between them tends to be zero. When the uniform flow is flowing over the doublet, a resultant flow will be obtained as shown in Fig. 5.53 (c). This resultant flow is known as the flow past a Rankine oval of equal axes or flow past a circular cylinder.

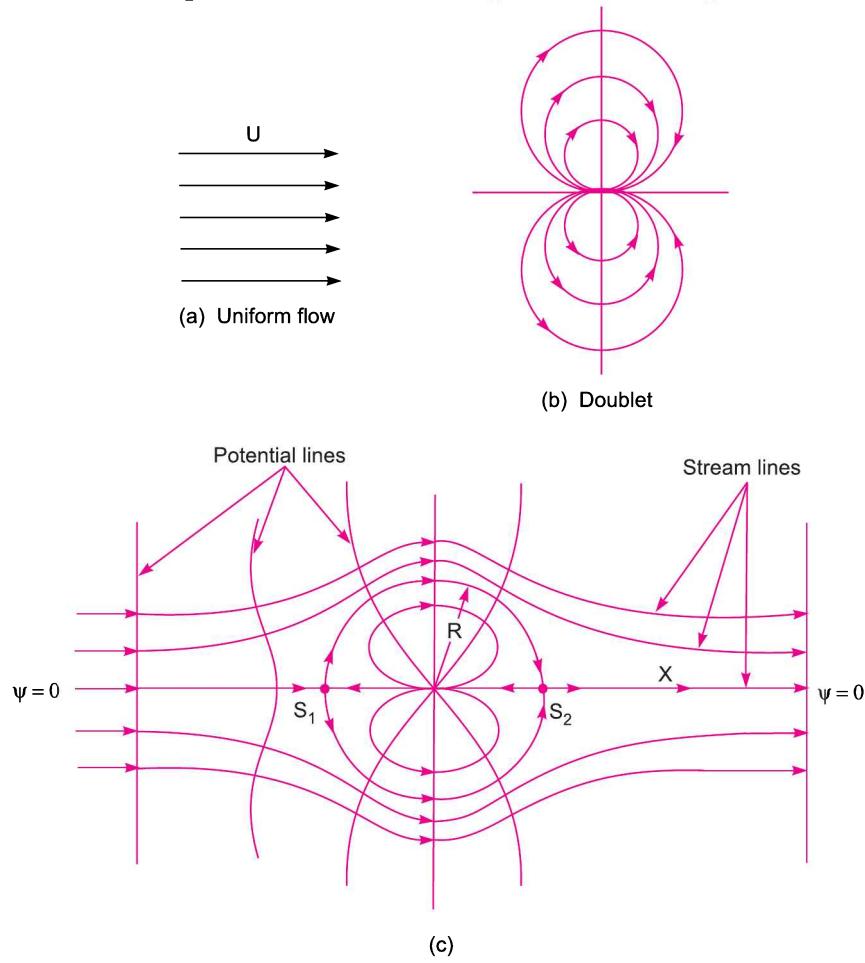


Fig. 5.53

The stream function ( $\psi$ ) and velocity potential function ( $\phi$ ) for the resultant flow is obtained as given below :

$$\psi = \text{stream function due to uniform flow} + \text{stream function due to doublet}$$

$$= U \times y + \left( \frac{-\mu}{2\pi r} \sin \theta \right)$$

$$[\text{Stream function due to doublet is given by equation (5.50) as } = -\frac{\mu}{2\pi r} \sin \theta]$$

$$= U \times r \times \sin \theta - \frac{\mu}{2\pi r} \sin \theta \quad (\because y = r \sin \theta)$$

$$= \left( U \times r - \frac{\mu}{2\pi r} \right) \sin \theta \quad \dots(5.67)$$

and  $\phi = \text{Potential function due to uniform flow} + \text{potential function due to doublet}$

$$= U \times x + \frac{\mu}{2\pi} \times \frac{\cos \theta}{r}$$

$$\left[ \text{From equation (5.52), potential function due to doublet} = \frac{\mu}{2\pi} \times \frac{\cos \theta}{r} \right]$$

$$= U \times r \cos \theta + \frac{\mu}{2\pi} \times \frac{\cos \theta}{r} \quad (\because x = r \cos \theta)$$

$$= \left( U \times r + \frac{\mu}{2\pi r} \right) \cos \theta \quad \dots(5.68)$$

### Shape of Rankine oval of equal axes

To get the profile of the Rankine oval of equal axes, the stream line  $\psi$  is taken as zero. Hence substituting  $\psi = 0$  in equation (5.67), we get

$$0 = \left( U \times r - \frac{\mu}{2\pi r} \right) \sin \theta$$

This means either  $\sin \theta = 0$  or  $U \times r - \frac{\mu}{2\pi r} = 0$

(i) If  $\sin \theta = 0$ , then  $\theta = 0$  and  $\pm \pi$  i.e., a horizontal line through the origin of the doublet. This horizontal line is the  $x$ -axis.

(ii) If  $U \times r - \frac{\mu}{2\pi r} = 0$ , then  $U \times r = \frac{\mu}{2\pi r}$  or  $r^2 = \frac{\mu}{2\pi U}$

or  $r = \sqrt{\frac{\mu}{2\pi U}} = \text{a constant as } \mu \text{ and } U \text{ are constant.}$

Let this constant is equal to  $R$ .

$$\therefore r = \sqrt{\frac{\mu}{2\pi U}} = R$$

This gives that the closed body profile is a circular cylinder of radius  $R$  with centre on doublet. The dividing stream line corresponds to  $\psi = 0$ . This stream line is a circle of radius  $R$ . The stream line  $\psi = 0$  has two stagnation points  $S_1$  and  $S_2$ . At  $S_1$ , the uniform flow splits into two streams that flow along the

circle with radius  $R = \sqrt{\frac{\mu}{2\pi U}}$ , the two branches meet again at the stagnation point  $S_2$  and the flow continues in the downward direction. The uniform flow occurs outside the circle whereas the flow field due to doublet lies entirely within the circle. The stream function for the composite flow is given by equation (5.67) as

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$$\begin{aligned}\psi &= \left( U \times r - \frac{\mu}{2\pi r} \right) \sin \theta = U \left( r - \frac{\mu}{2\pi Ur} \right) \sin \theta \\ &= U \left( r - \frac{R^2}{r} \right) \sin \theta \quad \left( \because \frac{\mu}{2\pi U} = R^2 \right) \dots(5.69)\end{aligned}$$

### Velocity Components ( $u_r$ and $u_\theta$ )

The velocity components at any point in the flow field are given by,

$$\begin{aligned}u_r &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \left[ U \left( r - \frac{R^2}{r} \right) \sin \theta \right] = \frac{1}{r} U \left( r - \frac{R^2}{r} \right) \cos \theta \\ &= U \left( 1 - \frac{R^2}{r^2} \right) \cos \theta \quad \dots(5.70)\end{aligned}$$

and

$$\begin{aligned}u_\theta &= - \frac{\partial \psi}{\partial r} = - \frac{\partial}{\partial r} \left[ U \left( r - \frac{R^2}{r} \right) \sin \theta \right] = - U \left( 1 + \frac{R^2}{r^2} \right) \sin \theta \\ &= - U \left( 1 + \frac{R^2}{r^2} \right) \sin \theta \quad \dots(5.71)\end{aligned}$$

$$\therefore \text{Resultant velocity, } V = \sqrt{u_r^2 + u_\theta^2} \quad \dots(5.72)$$

On the surface of the cylinder,  $r = R$

$$\begin{aligned}u_r &= U \left[ 1 - \frac{R^2}{R^2} \right] \cos \theta \quad [\because \text{In equation (5.70), } r = R] \\ &= 0 \\ \text{and} \quad u_\theta &= - U \left[ 1 + \frac{R^2}{R^2} \right] \sin \theta = - 2U \sin \theta \quad \dots(5.73)\end{aligned}$$

-ve sign shows the clockwise direction of tangential velocity at that point. The value of  $u_\theta$  is maximum, when  $\theta = 90^\circ$  and  $270^\circ$ .

At  $\theta = 0^\circ$  or  $180^\circ$ , the value of  $u_\theta = 0$ . Hence on the surface of the cylinder, the resultant velocity is zero, when  $\theta = 0^\circ$  or  $180^\circ$ . These two points on the surface of cylinder [i.e., at  $\theta = 0^\circ$  and  $180^\circ$ ] where resultant velocity is zero, are known as stagnation points. They are denoted by  $S_1$  and  $S_2$ . Stagnation point  $S_1$  corresponds to  $\theta = 180^\circ$  and  $S_2$  corresponds to  $\theta = 0^\circ$ .

### Pressure distribution on the surface of the cylinder

Let  $p_0$  = pressure at a point in the uniform flow far away from the cylinder and towards the left of the cylinder [i.e., approaching uniform flow]

$U$  = velocity of uniform flow at that point

$p$  = pressure at a point on the surface of the cylinder

$V$  = resultant velocity at that point on the surface of the cylinder. This velocity is equal to  $u_\theta$  as  $u_r$  is zero on the surface of the cylinder.

$$\therefore V = u_\theta = -2U \sin \theta$$

Applying Bernoulli's equation at the above two points,

$$\frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{V^2}{2g}$$

$$\text{or } \frac{p_0}{\rho g} + \frac{U^2}{2g} = \frac{p}{\rho g} + \frac{[-2U \sin \theta]^2}{2g} \quad [\because V = u_\theta = -2U \sin \theta]$$

$$\text{or } \frac{p_0}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{4U^2 \sin^2 \theta}{2}$$

$$\text{or } \frac{p - p_0}{\frac{1}{2} \rho U^2} = \frac{U^2}{2} - \frac{4U^2 \sin^2 \theta}{2} = \frac{1}{2} U^2 (1 - 4 \sin^2 \theta)$$

$$\text{or } \frac{p - p_0}{\frac{1}{2} \rho U^2} = 1 - 4 \sin^2 \theta$$

But  $\frac{p - p_0}{\frac{1}{2} \rho U^2}$  is a dimensionless term and is known as dimensionless pressure co-efficient and is

denoted by  $C_p$ .

$$\therefore C_p = \frac{p - p_0}{\frac{1}{2} \rho U^2} = 1 - 4 \sin^2 \theta$$

### Value of pressure co-efficient for different values of $\theta$

*Value of  $\theta$*                            *Value of  $C_p$*

$$0 \quad 1 - 4 \sin^2 \theta = 1 - 0 = 1$$

$$30^\circ \quad 1 - 4 \sin^2 30^\circ = 1 - 4 \times \left(\frac{1}{2}\right)^2 = 1 - \frac{4}{4} = 1 - 1 = 0$$

$$90^\circ \quad 1 - 4 \sin^2 90^\circ = 1 - 4 \times 1 = 1 - 4 = -3$$

$$150^\circ \quad 1 - 4 \sin^2 150^\circ = 1 - 4 \times \frac{1}{4} = 1 - 1 = 0$$

$$180^\circ \quad 1 - 4 \sin^2 180^\circ = 1 - 0 = 1$$

At  $\theta = 0$  and  $180^\circ$ , there are stagnation points  $S_2$  and  $S_1$  respectively.

At  $\theta = 30^\circ$  and  $150^\circ$ , the pressure co-efficient is zero.

At  $\theta = 90^\circ$ , the pressure co-efficient is  $-3$  (*i.e.*, least pressure)

The variation of pressure co-efficient along the surface of the cylinder for different values of  $\theta$  are shown in Fig. 5.54.

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The positive pressure is acting normal to the surface and towards the surface of the cylinder whereas the negative pressure is acting normal to the surface and away from the surface of the cylinder as shown in Fig. 5.55.

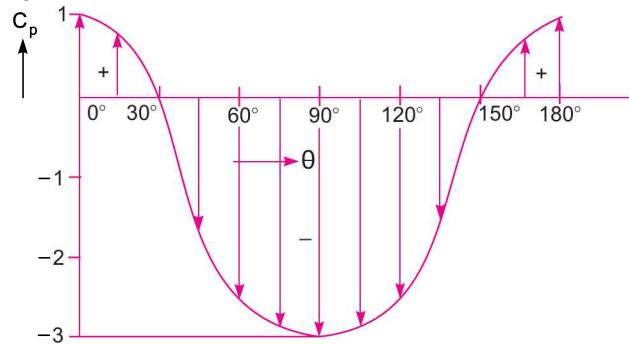


Fig. 5.54

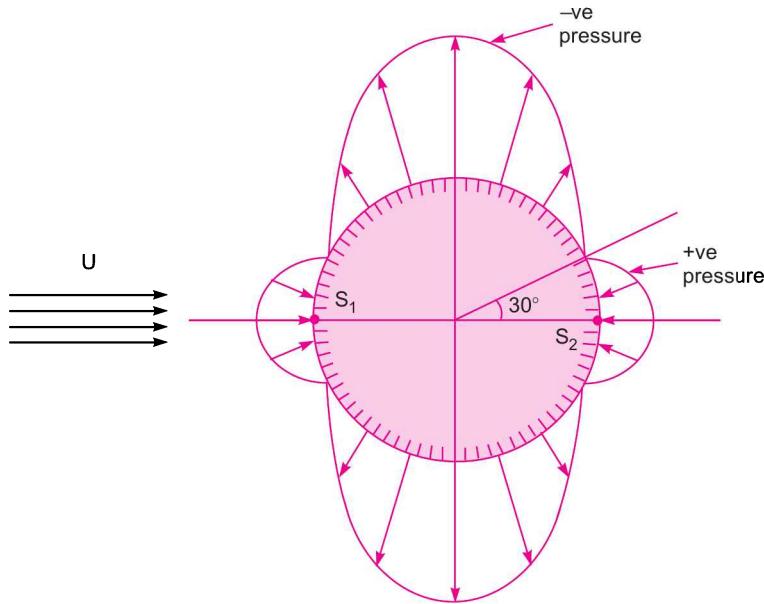


Fig. 5.55

**Problem 5.43** A uniform flow of 12 m/s is flowing over a doublet of strength 18 m<sup>2</sup>/s. The doublet is in the line of the uniform flow. Determine :

- (i) shape of the Rankine oval      (ii) radius of the Rankine circle
- (iii) value of stream line function at Rankine circle
- (iv) resultant velocity at a point on the Rankine circle at an angle of 30° from x-axis
- (v) value of maximum velocity on the Rankine circle and location of the point where velocity is max.

**Solution.** Given :       $U = 12 \text{ m/s}$  ;  $\mu = 18 \text{ m}^2/\text{s}$

(i) *Shape of the Rankine oval*

When a uniform flow is flowing over a doublet and doublet and uniform flow are in line, then the

shape of the Rankine oval will be a circle of radius =  $\sqrt{\frac{\mu}{2\pi U}}$ . Ans.

(ii) Radius of the Rankine circle

$$R = r = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{18}{2\pi \times 12}} = 0.488 \text{ m. Ans.}$$

(iii) Value of stream line function at the Rankine circle

The value of stream line function ( $\psi$ ) at the Rankine circle is zero i.e.,  $\psi = 0$ .

(iv) Resultant velocity on the surface of the circle, when  $\theta = 30^\circ$

On the surface of the cylinder, the radial velocity ( $u_r$ ) is zero. The tangential velocity ( $u_\theta$ ) is given by equation (5.73) as

$$u_\theta = -2U \sin \theta = -2 \times 12 \times \sin 30^\circ = -12 \text{ m/s. Ans.}$$

-ve sign shows the clockwise direction of tangential velocity at that point.

$$\therefore \text{Resultant velocity, } V = \sqrt{u_r^2 + u_\theta^2} = \sqrt{0^2 + (-12)^2} = 12 \text{ m/s. Ans.}$$

(v) Maximum velocity and its location

The resultant velocity at any point on the surface of the cylinder is equal to  $u_\theta$ . But  $u_\theta$  is given by,

$$u_\theta = -2U \sin \theta$$

This velocity will be maximum, when  $\theta = 90^\circ$ .

$$\therefore \text{Max. velocity} = -2U = -2 \times 12 = -24 \text{ m/s. Ans.}$$

**Problem 5.44** A uniform flow of 10 m/s is flowing over a doublet of strength 15 m<sup>2</sup>/s. The doublet is in the line of the uniform flow. The polar co-ordinates of a point P in the flow field are 0.9 m and 30°. Find : (i) stream line function and (ii) the resultant velocity at the point.

**Solution.** Given :  $U = 10 \text{ m/s}$ ;  $\mu = 15 \text{ m}^2/\text{s}$ ;  $r = 0.9 \text{ m}$  and  $\theta = 30^\circ$ .

Let us first find the radius ( $R$ ) of the Rankine circle. This is given by

$$R = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{15}{2\pi \times 10}} = 0.488 \text{ m}$$

The polar co-ordinates of the point P are 0.9 m and 30°.

Hence  $r = 0.9 \text{ m}$  and  $\theta = 30^\circ$ .

As the value of  $r$  is more than the radius of the Rankine circle, hence point P lies outside the cylinder.

(i) Value of stream line function at the point P

The stream line function for the composite flow at any point is given by equation (5.69) as

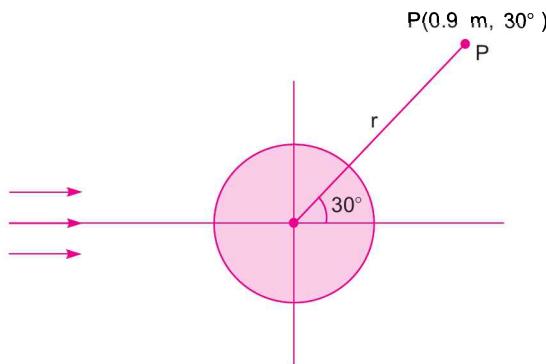


Fig. 5.56

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$$\begin{aligned}
 \psi &= U \left( r - \frac{R^2}{r} \right) \sin \theta \\
 &= 10 \left( 0.9 - \frac{0.488^2}{0.9} \right) \sin 30^\circ (\because r = 0.9 \text{ m}, R = 0.488 \text{ and } \theta = 30^\circ) \\
 &= 10(0.9 - 0.2646) \times \frac{1}{2} = 3.177 \text{ m}^2/\text{s. Ans.}
 \end{aligned}$$

(ii) Resultant velocity at the point P

The radial velocity and tangential velocity at any point in the flow field are given by equations (5.70) and (5.71) respectively.

$$\therefore u_r = U \left( 1 - \frac{R^2}{r^2} \right) \cos \theta = 10 \left( 1 - \frac{0.488^2}{0.9^2} \right) \cos 30^\circ = 611 \text{ m/s}$$

+ve sign shows the radial velocity is outward.

$$\text{and } u_\theta = -U \left( 1 + \frac{R^2}{r^2} \right) \sin \theta = -10 \left( 1 + \frac{0.488^2}{0.9^2} \right) \sin 30^\circ = -6.47 \text{ m/s}$$

-ve sign shows the clockwise direction of tangential velocity.

$\therefore$  Resultant velocity,

$$\begin{aligned}
 V &= \sqrt{u_r^2 + u_\theta^2} \\
 &= \sqrt{6.11^2 + (-6.47)^2} = \sqrt{37.33 + 44.86} \\
 &= 8.89 \text{ m/s. Ans.}
 \end{aligned}$$

### HIGHLIGHTS

- If the fluid characteristics like velocity, pressure, density etc. do not change at a point with respect to time, the fluid flow is called steady flow. If they change w.r.t. time, the fluid flow is called unsteady flow.

$$Or \quad \left( \frac{\partial v}{\partial t} \right) = 0 \text{ for steady flow and } \left( \frac{\partial v}{\partial t} \right) \neq 0 \text{ for unsteady flow.}$$

- If the velocity in a fluid flow does not change with respect to space (length of direction of flow), the flow is said uniform otherwise non-uniform. Thus,

$$\left( \frac{\partial v}{\partial s} \right) = 0 \text{ for uniform flow and } \left( \frac{\partial v}{\partial s} \right) \neq 0 \text{ for non-uniform flow.}$$

- If the Reynolds number in a pipe is less than 2000, the flow is said to be laminar and if Reynold number is more than 4000, the flow is said to be turbulent.
- For compressible flow,  $\rho \neq \text{constant}$   
For incompressible flow,  $\rho = \text{constant}$ .
- Rate of discharge for incompressible fluid (liquid),  $Q = A \times v$ .
- Continuity equation is written as  $A_1 v_1 = A_2 v_2 = A_3 v_3$ .

7. Continuity equation in differential form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ for three-dimensional flow}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ for two-dimensional flow.}$$

8. The components of acceleration in  $x$ ,  $y$  and  $z$  direction are

$$a_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}.$$

9. The components of velocity in  $x$ ,  $y$  and  $z$  direction in terms of velocity potential ( $\phi$ ) are

$$u = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y} \text{ and } w = -\frac{\partial \phi}{\partial z}.$$

10. The stream function ( $\psi$ ) is defined only for two-dimensional flow. The velocity components in  $x$  and  $y$  directions in terms of stream function are  $u = -\frac{\partial \psi}{\partial y}$  and  $v = \frac{\partial \psi}{\partial x}$ .

11. Angular deformation or shear strain rate is given as

$$\text{Shear strain rate} = \frac{1}{2} \left[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right]$$

12. Rotational components of a fluid particle are

$$\omega_z = \frac{1}{2} \left[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right]; \omega_x = \frac{1}{2} \left[ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right]; \omega_y = \frac{1}{2} \left[ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right]$$

13. Vorticity is two times the value of rotation.

14. Flow of a fluid along a curved path is known as vortex flow. If the particles are moving round in curved path with the help of some external torque the flow is called forced vortex flow. And if no external torque is required to rotate the fluid particles, the flow is called free-vortex flow.

15. The relation between tangential velocity and radius :

for forced vortex,  $v = \omega r$ ,

for free vortex,  $v \propto r = \text{constant}$ .

16. The pressure variation along the radial direction for vortex flow along a horizontal plane,  $\frac{\partial p}{\partial r} = \rho \frac{v^2}{r}$

and pressure variation in the vertical plane  $\frac{\partial p}{\partial z} = -\rho g$ .

17. For the forced vortex flow,  $Z = \frac{v^2}{2g} = \frac{\omega^2 r^2}{2g} = \frac{\omega^2 R^2}{2g}$

where  $Z$  = height of paraboloid formed

$\omega$  = angular velocity.

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18. For a forced vortex flow in a open tank.  
Fall of liquid level at centre = Rise of liquid level at the ends.
19. In case of closed cylinder, the volume of air before rotation is equal to the volume of air after rotation.
20. If a close cylindrical vessel completely filled with water is rotated about its vertical axis, the total pressure forces acting on the top and bottom are

$$F_T = \frac{\rho}{4} \omega^2 \pi R^4$$

and  $F_B = F_T + \text{weight of water in cylinder}$

where  $F_T$  = Pressure force on top of cylinder

$F_B$  = Pressure force on the bottom of cylinder

$\omega$  = Angular velocity

$R$  = Radius of the vessel

$$\rho = \text{Density of fluid} = \frac{w}{g}.$$

21. For a free vortex flow the equation is  $\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$ .

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. What are the methods of describing fluid flow ?
2. Explain the terms :
  - (i) Path line,
  - (ii) Streak line,
  - (iii) Stream line, and
  - (iv) Stream tube.
3. Distinguish between :
  - (i) Steady flow and un-steady flow, (ii) Uniform and non-uniform flow,
  - (iii) Compressible and incompressible flow,
  - (iv) Rotational and irrotational flow, (v) Laminar and turbulent flow.
4. Define the following and give one practical example for each :
  - (i) Laminar flow,
  - (ii) Turbulent flow,
  - (iii) Steady flow, and
  - (iv) Uniform flow.
5. Define the equation of continuity. Obtain an expression for continuity equation for a three-dimensional flow. *(R.G.P.V, S 2002)*
6. What do you understand by the terms : (i) Total acceleration, (ii) Convective acceleration, and (iii) Local acceleration ? *(Delhi University, Dec. 2002)*
7. (a) Define the terms :
  - (i) Velocity potential function, and (ii) Stream function.
  - (b) What are the conditions for flow to be irrotational ?
8. What do you mean by equipotential line and a line of constant stream function ?
9. (a) Describe the use and limitations of the flow nets.  
(b) Under what conditions can one draw flow net ?
10. Define the terms :
  - (i) Vortex flow,
  - (ii) Forced vortex flow, and
  - (iii) Free vortex flow.
11. Differentiate between forced vortex and free vortex flow.

12. Derive an expression for the depth of paraboloid formed by the surface of a liquid contained in a cylindrical tank which is rotated at a constant angular velocity  $\omega$  about its vertical axis.
13. Derive an expression for the difference of pressure between two points in a free vortex flow. Does the difference of pressure satisfy Bernoulli's equation ? Can Bernoulli's equation be applied to a forced vortex flow ?
14. Derive, from first principles, the condition for irrotational flow. Prove that, for potential flow, both the stream function and velocity potential function satisfy the Laplace equation.
15. Define velocity potential function and stream function.
16. Under what conditions can one treat real fluid flow as irrotational (as an approximation).
17. Define the following :
  - (i) Steady flow,
  - (ii) Non-uniform flow,
  - (iii) Laminar flow, and
  - (iv) Two-dimensional flow.
18. (a) Distinguish between rotational flow and irrotational flow. Give one example of each  
 (b) Cite two examples of unsteady, non-uniform flow. How can the unsteady flow be transformed to steady flow ?  
(J.N.T. University, S 2002)
19. Explain uniform flow with source and sink. Obtain expressions for stream and velocity potential functions.
20. A point source is a point where an incompressible fluid is imagined to be created and sent out evenly in all directions. Determine its velocity potential and stream function.
21. (i) Explain doublet and define the strength of the doublet  
 (ii) Distinguish between a source and a sink.
22. Sketch the flow pattern of an ideal fluid flow past a cylinder with circulation.
23. Show that in case of forced vortex flow, the rise of liquid level at the ends is equal to the fall of liquid level at the axis of rotation.
24. Differentiate between :
  - (i) Stream function and velocity potential function
  - (ii) Stream line and streak line and
  - (iii) Rotational and irrotational flows.

### (B) NUMERICAL PROBLEMS

1. The diameters of a pipe at the sections 1 and 2 are 15 cm and 20 cm respectively. Find the discharge through the pipe if velocity of water at section 1 is 4 m/s. Determine also the velocity at section 2.  
[Ans. 0.07068 m<sup>3</sup>/s, 2.25 m/s]
2. A 40 cm diameter pipe, conveying water, branches into two pipes of diameters 30 cm and 20 cm respectively. If the average velocity in the 40 cm diameter pipe is 3 m/s. Find the discharge in this pipe. Also determine the velocity in 20 cm pipe if the average velocity in 30 cm diameter pipe is 2 m/s.  
[Ans. 0.3769 m<sup>3</sup>/s, 7.5 m/s]
3. A 30 cm diameter pipe carries oil of sp. gr. 0.8 at a velocity of 2 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.  
[Ans. 4.5 m/s, 113 kg/s]
4. The velocity vector in a fluid flow is given by  $V = 2x^3\mathbf{i} - 5x^2y\mathbf{j} + 4t\mathbf{k}$ .  
 Find the velocity and acceleration of a fluid particle at (1, 2, 3) at time,  $t = 1$ .  
[Ans. 10.95 units, 16.12 units]
5. The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :

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(i)  $u = 4x^2, v = 4xyz$

(ii)  $u = 4x^2 + 3xy, w = z^3 - 4xy - 2yz.$

$$\left[ \text{Ans. (i)} w = -8xz - 2xz^2 + f(x, y) \quad \text{(ii)} v = -8xy - \frac{y^2}{2} + 3yz^2 + f(x, z) \right]$$

Calculate the unknown velocity components so that they satisfy the following equations :

(i)  $u = 2x^2, v = 2xyz, w = ?$       (ii)  $u = 2x^2 + 2xy, w = z^3 - 4xz + 2yz, v = ?$       [Ans. (i)  $w = -1xz - x^2z$ ]

6. A fluid flow is given by :  $V = xy^2i - 2yz^2j - \left( zy^2 - \frac{2z^3}{3} \right) k.$

Prove that it is a case of possible steady incompressible fluid flow.

Calculate the velocity and acceleration at the point [1, 2, 3].      [Ans. 36.7 units, 874.50 units]

7. Find the convective acceleration at the middle of a pipe which converges uniformly from 0.6 m diameter to 0.3 m diameter over 3 m length. The rate of flow is 40 lit/s. If the rate of flow changes uniformly from 40 lit/s to 80 lit/s in 40 seconds, find the total acceleration at the middle of the pipe at 20th second.

$$[\text{Ans. } .0499 \text{ m/s}^2; .11874 \text{ m/s}^2]$$

8. The velocity potential function,  $\phi$ , is given by  $\phi = x^2 - y^2$ . Find the velocity components in  $x$  and  $y$  direction. Also show that  $\phi$  represents a possible case of fluid flow.      [Ans.  $u = 2x$  and  $v = -2y$ ]

9. For the velocity potential function,  $\phi = x^2 - y^2$ , find the velocity components at the point (4, 5).      [Ans.  $u = 8, v = -10$  units]

10. A stream function is given by :  $\psi = 2x - 5y$ . Calculate the velocity components and also magnitude and direction of the resultant velocity at any point.      [Ans.  $u = 5, v = 2$ , Resultant = 5.384 and  $\theta = 21^\circ 48'$ ]

11. If for a two-dimensional potential flow, the velocity potential is given by :  $\phi = 4x(3y - 4)$ , determine the velocity at the point (2, 3). Determine also the value of stream function  $\psi$  at the point (2, 3).

$$\left[ \text{Ans. } 40 \text{ units, } \psi = 6x^2 - 4 \left( \frac{3}{2} y^2 - 4y \right), -18 \right]$$

12. The stream function for a two-dimensional flow is given by  $\psi = 8xy$ , calculate the velocity at the point  $p(4, 5)$ . Find the velocity potential function  $\phi$ .      [Ans.  $u = -32$  units,  $v = 40$  units,  $\phi = 4y^2 - 4x^2$ ]

13. Sketch the stream lines represented by  $\psi = xy$ . Also find out the velocity and its direction at point (2, 3).      [Ans. 3.60 units and  $\theta = 56^\circ 18.6'$  or  $123^\circ 42'$ ]

14. For the velocity components given as :  $u = ay \sin xy, v = ax \sin xy$ .

Obtain an expression for the velocity potential function.      [Ans.  $\phi = a \cos xy$ ]

15. A fluid flow is given by :  $V = 10x^3i - 8x^3yj$ .

Find the shear strain rate and state whether the flow is rotational or irrotational. [Ans.  $-8xy$ , rotational]

16. The velocity components in a two-dimensional flow are :

$$u = 8x^2y - \frac{8}{3}y^3 \text{ and } v = -8xy^3 + \frac{8}{3}x^3.$$

Show that these velocity components represent a possible case of an irrotational flow.

$$\left[ \text{Ans. } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \omega_z = 0 \right]$$

17. An open circular cylinder of 20 cm diameter and 100 cm long contains water upto a height of 80 cm. It is rotated about its vertical axis. Find the speed of rotation when :

(i) no water spills,      (ii) axial depth is zero.      [Ans. (i) 267.51 r.p.m., (ii) 422.98 r.p.m.]

18. A cylindrical vessel 15 cm in diameter and 40 cm long is completely filled with water. The vessel is open at the top. Find the quantity of water left in the vessel, when it is rotated about its vertical axis with a speed of 300 r.p.m.      [Ans.  $4566.3 \text{ cm}^2$ ]

19. An open circular cylinder of 20 cm diameter and 120 cm long contains water upto a height of 80 cm. It is rotated about its vertical axis at 400 r.p.m. Find the difference in total pressure force (i) at the bottom of the cylinder, and (ii) at the sides of the cylinder due to rotation. [Ans. (i) 14.52 N, (ii) 2465.45 N]
20. A closed cylindrical vessel of diameter 15 cm and length 100 cm contains water upto a height of 80 cm. The vessel is rotated at a speed of 500 r.p.m. about its vertical axis. Find the height of paraboloid formed. [Ans. 56.06 cm]
21. For the data given in question 20, find the speed of rotation of the vessel, when axial depth is zero. [Ans. 891.7 r.p.m.]
22. If the cylindrical vessel of question 20, is rotated at 950 r.p.m. about its vertical axis, find the area uncovered at the base of the tank. [Ans. 20.4 cm<sup>2</sup>]
23. A closed cylindrical vessel of diameter 20 cm and height 100 cm contains water upto a height of 70 cm. The air above the water surface is at a pressure of 78.48 kN/m<sup>2</sup>. The vessel is rotated at a speed of 300 r.p.m. about its vertical axis. Find the pressure head at the bottom of the vessel ; (a) at the centre, and (b) at the edge. [Ans. (a) 8.4485 m (b) 8.9515 m]
24. A closed cylinder of diameter 30 cm and height 20 cm is completely filled with water. Calculate the total pressure force exerted by water on the top and bottom of the cylinder, if it is rotated about its vertical axis at 300 r.p.m. [Ans.  $F_T = 392.4$  N,  $F_B = 531$  N]
25. In a free cylindrical vortex flow of water, at a point at a radius of 150 mm the velocity and pressure are 5 m/s and 14.715 N/cm<sup>2</sup>. Find the pressure at a radius of 300 mm. [Ans. 15.65 N/cm<sup>2</sup>]
26. Do the following velocity components represent physically possible flows ?  

$$u = x^2 + z^2 + 5, v = y^2 + z^2, w = 4xyz.$$
 [Ans. No.]
27. State if the flow represented by  $u = 3x + 4y$  and  $v = 2x - 3y$  is rotational or irrotational. [Ans. Rotational]
28. A vessel, cylindrical in shape and closed at the top and bottom, contains water upto a height of 700 mm. The diameter of the vessel is 200 mm and length of vessel is 1.1 m. Find the speed of rotation of the vessel if the axial depth of water is zero.
29. Define rotational and irrotational flow. The stream function and velocity potential for a flow are given by :  

$$\psi = 2xy, \phi = x^2 - y^2.$$
  
 Show that the conditions of continuity and irrotational flow are satisfied.
30. For the steady incompressible flow, are the following values of  $u$  and  $v$  possible ?  
 (i)  $u = 4xy + y^2, v = 6xy + 3x$  and (ii)  $u = 2x^2 + y^2, v = -4xy.$  [Ans. (i) No, (ii) Yes]
31. Define two-dimensional stream function and velocity potential. Show that following stream function :  

$$\psi = 6x - 4y + 7xy + 9$$
  
 represents an irrotational flow. Find its velocity potential. [Ans.  $\phi = 4x + 6y - 3.5x^2 + 3.5y^2 + C$ ]
32. Check if  $\phi = x^2 - y^2 + y$  represents the velocity potential for 2-dimensional irrotational flow. If it does, then determine the stream function  $\psi.$  [Ans. Yes,  $\psi = -2xy + x$ ]
33. If stream function for steady flow is given by  $\psi = (y^2 - x^2),$  determine whether the flow is rotational or irrotational. Then determine the velocity potential  $\phi.$  [Ans. Irrotational,  $\phi = -2xy + C$ ]
34. A pipe (1) 450 mm in diameter branches into two pipes (2) and (3) of diameters 300 mm and 200 mm respectively as shown in Fig. 5.57. If the average velocity in 450 mm diameter pipe is 3 m/s, find : (i) discharge through 450 mm dia. pipe and (ii) velocity in 200 mm diameter pipe if the average velocity in 300 mm pipe is 2.5 m/s. (J.N.T.U., Hyderabad, S 2002)
- [Hint. Given :  $d_1 = 450 \text{ mm} = 0.45 \text{ m}, d_2 = 300 \text{ mm} = 0.3 \text{ m}$   
 $d_3 = 200 \text{ mm} = 0.2 \text{ m}, V_1 = 3 \text{ m/s}, V_2 = 2.5 \text{ m/s}$

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$$(i) \quad Q_1 = A_1 V_1 = \frac{\pi}{4} (0.45^2) \times 3 = 0.477 \text{ m}^3/\text{s.}$$

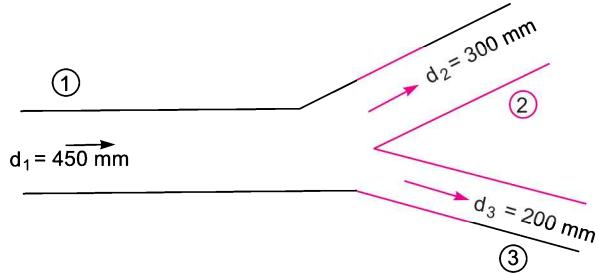


Fig. 5.57

$$(ii) \quad Q_2 = A_2 V_2 = \frac{\pi}{4} (.3^2) \times 2.5 = 0.176 \text{ m}^3/\text{s}$$

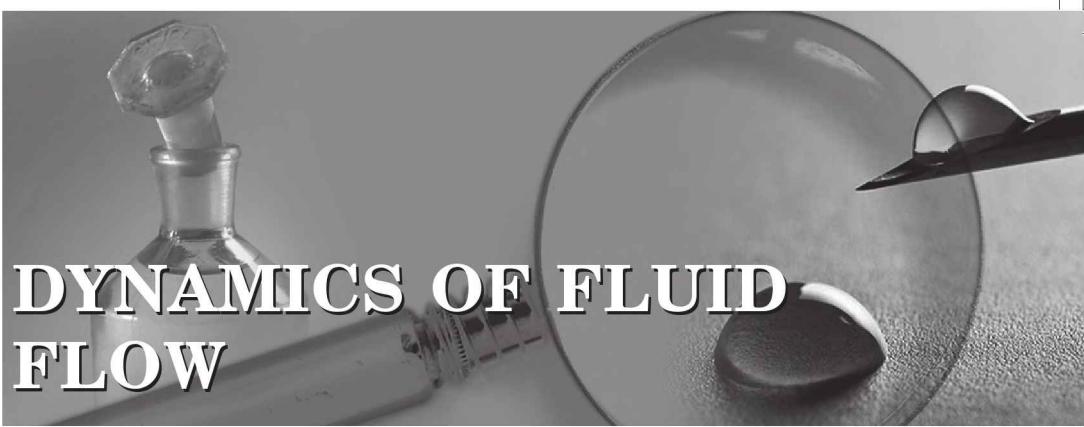
$$\text{But} \quad Q_1 = Q_2 + Q_3 \quad \therefore \quad Q_3 = Q_1 - Q_2 = 0.477 - 0.176 = 0.301$$

$$\text{Also} \quad Q_3 = A_3 \times V_3 = \frac{\pi}{4} (0.2^2) \times V_3$$

$$\therefore \quad V_3 = \frac{Q_3}{\frac{\pi}{4} (0.2^2)} = \frac{0.301}{0.0314} = 9.6 \text{ m/s.}]$$

# 6

## CHAPTER



# DYNAMICS OF FLUID FLOW

## ► 6.1 INTRODUCTION

In the previous chapter, we studied the velocity and acceleration at a point in a fluid flow, without taking into consideration the forces causing the flow. This chapter includes the study of forces causing fluid flow. Thus dynamics of fluid flow is the study of fluid motion with the forces causing flow. The dynamic behaviour of the fluid flow is analysed by the Newton's second law of motion, which relates the acceleration with the forces. The fluid is assumed to be incompressible and non-viscous.

## ► 6.2 EQUATIONS OF MOTION

According to Newton's second law of motion, the net force  $F_x$  acting on a fluid element in the direction of  $x$  is equal to mass  $m$  of the fluid element multiplied by the acceleration  $a_x$  in the  $x$ -direction. Thus mathematically,

$$F_x = m \cdot a_x \quad \dots(6.1)$$

In the fluid flow, the following forces are present :

- (i)  $F_g$ , gravity force.
- (ii)  $F_p$ , the pressure force.
- (iii)  $F_v$ , force due to viscosity.
- (iv)  $F_t$ , force due to turbulence.
- (v)  $F_c$ , force due to compressibility.

Thus in equation (6.1), the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x$$

- (i) If the force due to compressibility,  $F_c$  is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion**.

(ii) For flow, where  $(F_t)$  is negligible, the resulting equations of motion are known as **Navier-Stokes Equation**.

(iii) If the flow is assumed to be ideal, viscous force  $(F_v)$  is zero and equation of motions are known as **Euler's equation of motion**.

### ► 6.3 EULER'S EQUATION OF MOTION

This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as :

Consider a stream-line in which flow is taking place in  $s$ -direction as shown in Fig. 6.1. Consider a cylindrical element of cross-section  $dA$  and length  $ds$ . The forces acting on the cylindrical element are:

1. Pressure force  $pdA$  in the direction of flow.
2. Pressure force  $\left(p + \frac{\partial p}{\partial s} ds\right) dA$  opposite to the direction of flow.
3. Weight of element  $\rho g dA ds$ .

Let  $\theta$  is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of  $s$  must be equal to the mass of fluid element  $\times$  acceleration in the direction  $s$ .

$$\begin{aligned}\therefore & pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos \theta \\ & = \rho dA ds \times a_s \quad \dots(6.2)\end{aligned}$$

where  $a_s$  is the acceleration in the direction of  $s$ .

Now  $a_s = \frac{dv}{dt}$ , where  $v$  is a function of  $s$  and  $t$ .

$$= \frac{\partial v}{\partial s} \frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v \partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{ \because \frac{ds}{dt} = v \right\}$$

If the flow is steady,  $\frac{\partial v}{\partial t} = 0$

$$\therefore a_s = \frac{v \partial v}{\partial s}$$

Substituting the value of  $a_s$  in equation (6.2) and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} ds dA - \rho g dA ds \cos \theta = \rho dA ds \times \frac{\partial v}{\partial s}$$

Dividing by  $\rho dA ds$ ,  $-\frac{\partial p}{\rho \partial s} - g \cos \theta = \frac{v \partial v}{ds}$

or  $\frac{\partial p}{\rho \partial s} + g \cos \theta + v \frac{\partial v}{ds} = 0$

But from Fig. 6.1 (b), we have  $\cos \theta = \frac{dz}{ds}$

$$\therefore \frac{1}{\rho} \frac{dp}{ds} + g \frac{dz}{ds} + \frac{v dv}{ds} = 0 \quad \text{or} \quad \frac{dp}{\rho} + gdz + v dv = 0$$

or  $\frac{dp}{\rho} + gdz + v dv = 0 \quad \dots(6.3)$

Equation (6.3) is known as Euler's equation of motion.

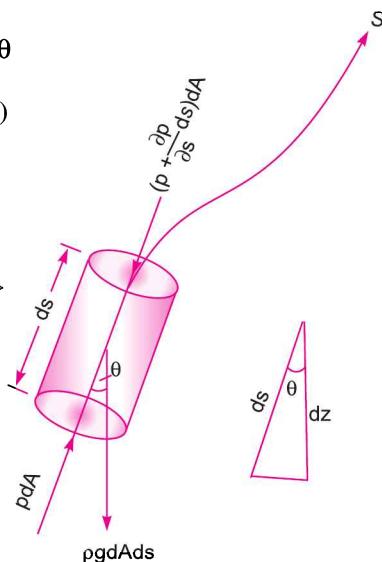


Fig. 6.1 Forces on a fluid element.

## ► 6.4 BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion (6.3) as

$$\int \frac{dp}{\rho} + \int gdz + \int vdv = \text{constant}$$

If flow is incompressible,  $\rho$  is constant and

$$\therefore \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

$$\text{or } \frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

$$\text{or } \frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant} \quad \dots(6.4)$$

Equation (6.4) is a Bernoulli's equation in which

$\frac{p}{\rho g}$  = pressure energy per unit weight of fluid or pressure head.

$v^2/2g$  = kinetic energy per unit weight or kinetic head.

$z$  = potential energy per unit weight or potential head.

## ► 6.5 ASSUMPTIONS

The following are the assumptions made in the derivation of Bernoulli's equation :

- (i) The fluid is ideal, i.e., viscosity is zero (ii) The flow is steady
- (iii) The flow is incompressible (iv) The flow is irrotational.

**Problem 6.1** Water is flowing through a pipe of 5 cm diameter under a pressure of 29.43 N/cm<sup>2</sup> (gauge) and with mean velocity of 2.0 m/s. Find the total head or total energy per unit weight of the water at a cross-section, which is 5 m above the datum line.

**Solution.** Given :

Diameter of pipe	= 5 cm = 0.5 m	
Pressure,	$p = 29.43 \text{ N/cm}^2 = 29.43 \times 10^4 \text{ N/m}^2$	
Velocity,	$v = 2.0 \text{ m/s}$	
Datum head,	$z = 5 \text{ m}$	
Total head	= pressure head + kinetic head + datum head	
Pressure head	$= \frac{p}{\rho g} = \frac{29.43 \times 10^4}{1000 \times 9.81} = 30 \text{ m}$	$\left\{ \rho \text{ for water} = 1000 \frac{\text{kg}}{\text{m}^3} \right\}$
Kinetic head	$= \frac{v^2}{2g} = \frac{2 \times 2}{2 \times 9.81} = 0.204 \text{ m}$	
. . . Total head	$= \frac{p}{\rho g} + \frac{v^2}{2g} + z = 30 + 0.204 + 5 = 35.204 \text{ m. Ans.}$	

**Problem 6.2** A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.

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**Solution.** Given :

$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

∴ Area,

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$$V_1 = 4.0 \text{ m/s}$$

$$D_2 = 0.1 \text{ m}$$

∴

$$A_2 = \frac{\pi}{4} (0.1)^2 = 0.00785 \text{ m}^2$$

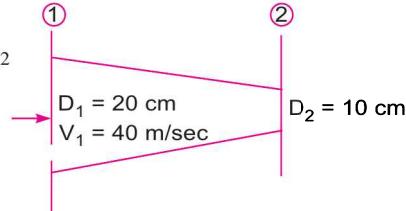


Fig. 6.2

(i) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = 0.815 \text{ m. Ans.}$$

(ii) Velocity head at section 2 = \$V\_2^2/2g\$

To find \$V\_2\$, apply continuity equation at 1 and 2

$$\therefore A_1 V_1 = A_2 V_2 \quad \text{or} \quad V_2 = \frac{A_1 V_1}{A_2} = \frac{0.0314}{0.00785} \times 4.0 = 16.0 \text{ m/s}$$

$$\therefore \text{Velocity head at section 2} = \frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81} = 83.047 \text{ m. Ans.}$$

$$\begin{aligned} \text{(iii) Rate of discharge} &= A_1 V_1 \quad \text{or} \quad A_2 V_2 \\ &= 0.0314 \times 4.0 = 0.1256 \text{ m}^3/\text{s} \\ &= 125.6 \text{ litres/s. Ans.} \end{aligned} \quad \{ \because 1 \text{ m}^3 = 1000 \text{ litres} \}$$

**Problem 6.3** State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's equation from first principle and state the assumptions made for such a derivation.

**Solution. Statement of Bernoulli's Theorem.** It states that in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant. The total energy consists of pressure energy, kinetic energy and potential energy or datum energy. These energies per unit weight of the fluid are :

$$\text{Pressure energy} = \frac{p}{\rho g}$$

$$\text{Kinetic energy} = \frac{v^2}{2g}$$

$$\text{Datum energy} = z$$

Thus mathematically, Bernoulli's theorem is written as

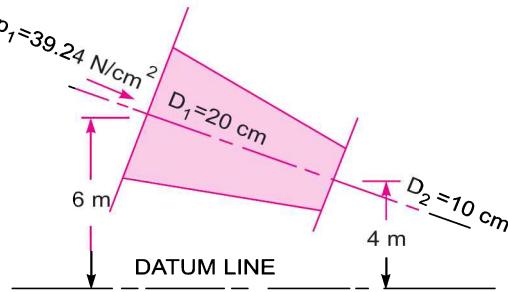
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{Constant.}$$

**Derivation of Bernoulli's theorem.** For derivation of Bernoulli's theorem, Articles 6.3 and 6.4 should be written.

**Assumptions** are given in Article 6.5.

**Problem 6.4** The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 litres/s. The section 1 is 6 m above datum and section 2 is 4 m above datum. If the pressure at section 1 is  $39.24 \text{ N/cm}^2$ , find the intensity of pressure at section 2.

**Solution.** Given :



At section 1,

$$D_1 = 20 \text{ cm} = 0.2 \text{ m}$$

$$A_1 = \frac{\pi}{4} (0.2)^2 = .0314 \text{ m}^2$$

$$p_1 = 39.24 \text{ N/cm}^2 \\ = 39.24 \times 10^4 \text{ N/m}^2$$

$$z_1 = 6.0 \text{ m}$$

At section 2,

$$D_2 = 0.10 \text{ m}$$

$$A_2 = \frac{\pi}{4} (0.1)^2 = .00785 \text{ m}^2$$

$$z_2 = 4 \text{ m}$$

$$p_2 = ?$$

Rate of flow,

$$Q = 35 \text{ lit/s} = \frac{35}{1000} = .035 \text{ m}^3/\text{s}$$

Now

$$Q = A_1 V_1 = A_2 V_2$$

∴

$$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114 \text{ m/s}$$

and

$$V_2 = \frac{Q}{A_2} = \frac{.035}{.00785} = 4.456 \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\text{or } \frac{39.24 \times 10^4}{1000 \times 9.81} + \frac{(1.114)^2}{2 \times 9.81} + 6.0 = \frac{p_2}{1000 \times 9.81} + \frac{(4.456)^2}{2 \times 9.81} + 4.0$$

$$\text{or } 40 + 0.063 + 6.0 = \frac{p_2}{9810} + 1.012 + 4.0$$

$$\text{or } 46.063 = \frac{p_2}{9810} + 5.012$$

$$\therefore \frac{p_2}{9810} = 46.063 - 5.012 = 41.051$$

$$\therefore p_2 = 41.051 \times 9810 \text{ N/m}^2$$

$$= \frac{41.051 \times 9810}{10^4} \text{ N/cm}^2 = 40.27 \text{ N/cm}^2. \text{ Ans.}$$

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**Problem 6.5** Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm<sup>2</sup> and the pressure at the upper end is 9.81 N/cm<sup>2</sup>. Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.

**Solution.** Given :

Section 1,       $D_1 = 300 \text{ mm} = 0.3 \text{ m}$   
 $p_1 = 24.525 \text{ N/cm}^2 = 24.525 \times 10^4 \text{ N/m}^2$

Section 2,       $D_2 = 200 \text{ mm} = 0.2 \text{ m}$   
 $p_2 = 9.81 \text{ N/cm}^2 = 9.81 \times 10^4 \text{ N/m}^2$

Rate of flow       $= 40 \text{ lit/s}$

or       $Q = \frac{40}{1000} = 0.04 \text{ m}^3/\text{s}$

Now

$$A_1 V_1 = A_2 V_2 = \text{rate of flow} = 0.04$$

$$\therefore V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4} D_1^2} = \frac{0.04}{\frac{\pi}{4} (0.3)^2} = 0.5658 \text{ m/s}$$

$$\approx 0.566 \text{ m/s}$$

$$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4} (D_2)^2} = \frac{0.04}{\frac{\pi}{4} (0.2)^2} = 1.274 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\text{or } \frac{24.525 \times 10^4}{1000 \times 9.81} + \frac{.566 \times .566}{2 \times 9.81} + z_1 = \frac{9.81 \times 10^4}{1000 \times 9.81} + \frac{(1.274)^2}{2 \times 9.81} + z_2$$

$$\text{or } 25 + .32 + z_1 = 10 + 1.623 + z_2$$

$$\text{or } 25.32 + z_1 = 11.623 + z_2$$

$$\therefore z_2 - z_1 = 25.32 - 11.623 = 13.697 = 13.70 \text{ m}$$

$\therefore$  Difference in datum head  $= z_2 - z_1 = 13.70 \text{ m. Ans.}$

**Problem 6.6** The water is flowing through a taper pipe of length 100 m having diameters 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 litres/s. The pipe has a slope of 1 in 30. Find the pressure at the lower end if the pressure at the higher level is 19.62 N/cm<sup>2</sup>.

**Solution.** Given :

Length of pipe,       $L = 100 \text{ m}$

Dia. at the upper end,       $D_1 = 600 \text{ mm} = 0.6 \text{ m}$

$$\therefore \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times (.6)^2 = 0.2827 \text{ m}^2$$

$$p_1 = \text{pressure at upper end} = 19.62 \text{ N/cm}^2$$

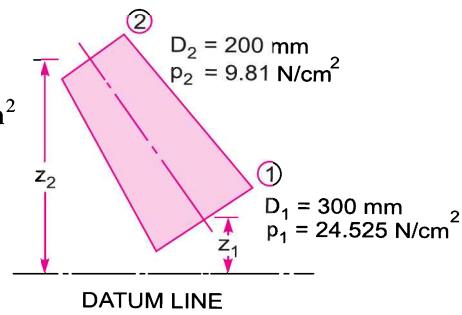
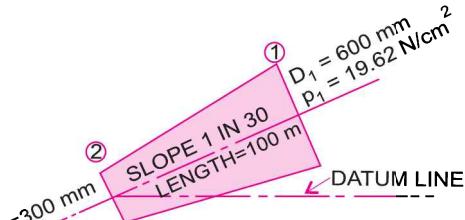


Fig. 6.4

Fig. 6.5



$$= 19.62 \times 10^4 \text{ N/m}^2$$

Dia. at lower end,  $D_2 = 300 \text{ mm} = 0.3 \text{ m}$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$$

$$Q = \text{rate of flow} = 50 \text{ litres/s} = \frac{50}{1000} = 0.05 \text{ m}^3/\text{s}$$

Let the datum line passes through the centre of the lower end.

Then  $z_2 = 0$

$$\text{As slope is 1 in 30 means } z_1 = \frac{1}{30} \times 100 = \frac{10}{3} \text{ m}$$

$$\text{Also we know } Q = A_1 V_1 = A_2 V_2$$

$$\therefore V_1 = \frac{Q}{A_1} = \frac{0.05}{0.07068} = 0.1768 \text{ m/sec} = 0.177 \text{ m/s}$$

$$\text{and } V_2 = \frac{Q}{A_2} = \frac{0.05}{0.07068} = 0.7074 \text{ m/sec} = 0.707 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\text{or } \frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{.177^2}{2 \times 9.81} + \frac{10}{3} = \frac{p_2}{\rho g} + \frac{.707^2}{2 \times 9.81} + 0$$

$$\text{or } 20 + 0.001596 + 3.334 = \frac{p_2}{\rho g} + 0.0254$$

$$\text{or } 23.335 - 0.0254 = \frac{p_2}{1000 \times 9.81}$$

$$\text{or } p_2 = 23.3 \times 9810 \text{ N/m}^2 = 228573 \text{ N/m}^2 = 22.857 \text{ N/cm}^2. \text{ Ans.}$$

## ► 6.6 BERNOULLI'S EQUATION FOR REAL FLUID

The Bernoulli's equation was derived on the assumption that fluid is inviscid (non-viscous) and therefore frictionless. But all the real fluids are viscous and hence offer resistance to flow. Thus there are always some losses in fluid flows and hence in the application of Bernoulli's equation, these losses have to be taken into consideration. Thus the Bernoulli's equation for real fluids between points 1 and 2 is given as

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L \quad \dots(6.5)$$

where  $h_L$  is loss of energy between points 1 and 2.

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**Problem 6.7** A pipe of diameter 400 mm carries water at a velocity of 25 m/s. The pressures at the points A and B are given as 29.43 N/cm<sup>2</sup> and 22.563 N/cm<sup>2</sup> respectively while the datum head at A and B are 28 m and 30 m. Find the loss of head between A and B.

**Solution.** Given :

Dia. of pipe,

$$D = 400 \text{ mm} = 0.4 \text{ m}$$

Velocity,

$$V = 25 \text{ m/s}$$

At point A,

$$p_A = 29.43 \text{ N/cm}^2 = 29.43 \times 10^4 \text{ N/m}^2$$

$$z_A = 28 \text{ m}$$

$$v_A = v = 25 \text{ m/s}$$

∴ Total energy at A,

$$E_A = \frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A$$

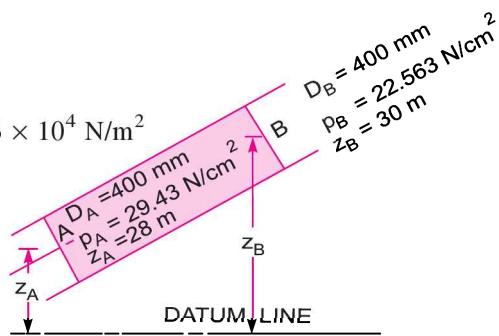


Fig. 6.6

$$= \frac{29.43 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 28$$

$$= 30 + 31.85 + 28 = 89.85 \text{ m}$$

At point B,

$$p_B = 22.563 \text{ N/cm}^2 = 22.563 \times 10^4 \text{ N/m}^2$$

$$z_B = 30 \text{ m}$$

$$v_B = v = v_A = 25 \text{ m/s}$$

∴ Total energy at B,

$$E_B = \frac{p_B}{\rho g} + \frac{v_B^2}{2g} + z_B$$

$$= \frac{22.563 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 30 = 23 + 31.85 + 30 = 84.85 \text{ m}$$

∴ Loss of energy

$$= E_A - E_B = 89.85 - 84.85 = 5.0 \text{ m. Ans.}$$

**Problem 6.8** A conical tube of length 2.0 m is fixed vertically with its smaller end upwards. The velocity of flow at the smaller end is 5 m/s while at the lower end it is 2 m/s. The pressure head at the

smaller end is 2.5 m of liquid. The loss of head in the tube is  $\frac{0.35(v_1 - v_2)^2}{2g}$ , where  $v_1$  is the velocity at the smaller end and  $v_2$  at the lower end respectively. Determine the pressure head at the lower end. Flow takes place in the downward direction.

**Solution.** Let the smaller end is represented by (1) and lower end by (2)

Given :

Length of tube,

$$L = 2.0 \text{ m}$$

$$v_1 = 5 \text{ m/s}$$

$$p_1/\rho g = 2.5 \text{ m of liquid}$$

$$v_2 = 2 \text{ m/s}$$

Loss of head

$$= h_L = \frac{0.35(v_1 - v_2)^2}{2g}$$

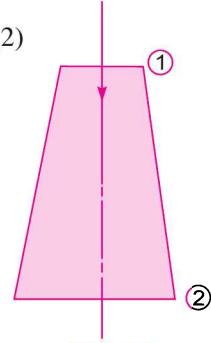


Fig. 6.7

$$= \frac{0.35 [5 - 2]^2}{2g} = \frac{0.35 \times 9}{2 \times 9.81} = 0.16 \text{ m}$$

Pressure head,  $\frac{p_2}{\rho g} = ?$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

Let the datum line passes through section (2). Then  $z_2 = 0$ ,  $z_1 = 2.0$

$$\therefore 2.5 + \frac{5^2}{2 \times 9.81} + 2.0 = \frac{p_2}{\rho g} + \frac{2^2}{2 \times 9.81} + 0 + 0.16$$

$$2.5 + 1.27 + 2.0 = \frac{p_2}{\rho g} + 0.203 + .16$$

or  $\frac{p_2}{\rho g} = (2.5 + 1.27 + 2.0) - (.203 + .16)$

$$= 5.77 - .363 = 5.407 \text{ m of fluid. Ans.}$$

**Problem 6.9** A pipeline carrying oil of specific gravity 0.87, changes in diameter from 200 mm diameter at a position A to 500 mm diameter at a position B which is 4 metres at a higher level. If the pressures at A and B are 9.81 N/cm<sup>2</sup> and 5.886 N/cm<sup>2</sup> respectively and the discharge is 200 litres/s determine the loss of head and direction of flow.

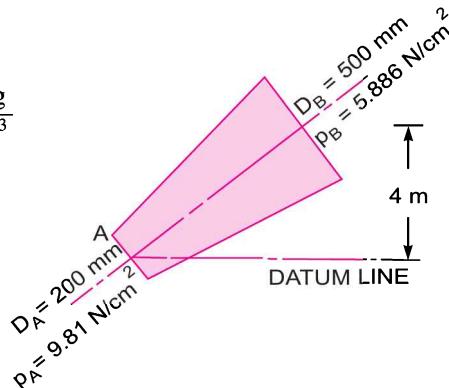
**Solution.** Discharge,  $Q = 200 \text{ lit/s} = 0.2 \text{ m}^3/\text{s}$

Sp. gr. of oil  $= 0.87$

$$\therefore \rho \text{ for oil} = .87 \times 1000 = 870 \frac{\text{kg}}{\text{m}^3}$$

Given : At section A,  $D_A = 200 \text{ mm} = 0.2 \text{ m}$

$$\begin{aligned} \text{Area, } A_A &= \frac{\pi}{4} (D_A)^2 = \frac{\pi}{4} (.2)^2 \\ &= 0.0314 \text{ m}^2 \\ p_A &= 9.81 \text{ N/cm}^2 \\ &= 9.81 \times 10^4 \text{ N/m}^2 \end{aligned}$$



If datum line is passing through A, then

$$Z_A = 0$$

$$V_A = \frac{Q}{A_A} = \frac{0.2}{0.0314} = 6.369 \text{ m/s}$$

At section B,  $D_B = 500 \text{ mm} = 0.50 \text{ m}$

$$\text{Area, } A_B = \frac{\pi}{4} D_B^2 = \frac{\pi}{4} (.5)^2 = 0.1963 \text{ m}^2$$

$$p_B = 5.886 \text{ N/cm}^2 = 5.886 \times 10^4 \text{ N/m}^2$$

Fig. 6.8

$$Z_B = 4.0 \text{ m}$$

$$V_B = \frac{Q}{\text{Area}} = \frac{0.2}{.1963} = 1.018 \text{ m/s}$$

Total energy at A

$$= E_A = \frac{p_A}{\rho g} + \frac{V_A^2}{2g} + Z_A$$

$$= \frac{9.81 \times 10^4}{870 \times 9.81} + \frac{(6.369)^2}{2 \times 9.81} + 0 = 11.49 + 2.067 = 13.557 \text{ m}$$

Total energy at B

$$= E_B = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + Z_B$$

$$= \frac{5.886 \times 10^4}{870 \times 9.81} + \frac{(1.018)^2}{2 \times 9.81} + 4.0 = 6.896 + 0.052 + 4.0 = 10.948 \text{ m}$$

- (i) **Direction of flow.** As  $E_A$  is more than  $E_B$  and hence flow is taking place from A to B. Ans.  
(ii) Loss of head =  $h_L = E_A - E_B = 13.557 - 10.948 = 2.609 \text{ m}$ . Ans.

## ► 6.7 PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is applied in all problems of incompressible fluid flow where energy considerations are involved. But we shall consider its application to the following measuring devices :

1. Venturimeter.
2. Orifice meter.
3. Pitot-tube.

**6.7.1 Venturimeter.** A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It consists of three parts :

(i) A short converging part, (ii) Throat, and (iii) Diverging part. It is based on the Principle of Bernoulli's equation.

### Expression for rate of flow through venturimeter

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing (say water), as shown in Fig. 6.9.

Let  $d_1$  = diameter at inlet or at section (1),

$p_1$  = pressure at section (1)

$v_1$  = velocity of fluid at section (1),

$$a = \text{area at section (1)} = \frac{\pi}{4} d_1^2$$

and  $d_2, p_2, v_2, a_2$  are corresponding values at section (2).

Applying Bernoulli's equation at sections (1) and (2), we get

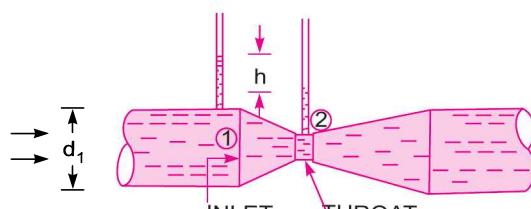


Fig. 6.9 Venturimeter.

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence  $z_1 = z_2$

$$\therefore \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad \text{or} \quad \frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But  $\frac{p_1 - p_2}{\rho g}$  is the difference of pressure heads at sections 1 and 2 and it is equal to  $h$  or  $\frac{p_1 - p_2}{\rho g} = h$

Substituting this value of  $\frac{p_1 - p_2}{\rho g}$  in the above equation, we get

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \quad \dots(6.6)$$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2 v_2}{a_1}$$

Substituting this value of  $v_1$  in equation (6.6)

$$h = \frac{v_2^2}{2g} - \frac{\left(\frac{a_2 v_2}{a_1}\right)^2}{2g} = \frac{v_2^2}{2g} \left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{v_2^2}{2g} \left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

or

$$v_2^2 = 2gh \frac{a_1^2}{a_1^2 - a_2^2}$$

∴

$$v_2 = \sqrt{2gh \frac{a_1^2}{a_1^2 - a_2^2}} = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

∴ Discharge,

$$Q = a_2 v_2 = a_2 \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \quad \dots(6.7)$$

Equation (6.7) gives the discharge under ideal conditions and is called, theoretical discharge. Actual discharge will be less than theoretical discharge.

$$\therefore Q_{\text{act}} = C_d \times \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \quad \dots(6.8)$$

where  $C_d$  = Co-efficient of venturimeter and its value is less than 1.

### Value of 'h' given by differential U-tube manometer

**Case I.** Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

$S_h$  = Sp. gravity of the heavier liquid

$S_o$  = Sp. gravity of the liquid flowing through pipe

$x$  = Difference of the heavier liquid column in U-tube

$$\text{Then } h = x \left[ \frac{S_h}{S_o} - 1 \right] \quad \dots(6.9)$$

**Case II.** If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of  $h$  is given by

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$$h = x \left[ 1 - \frac{S_l}{S_o} \right] \quad \dots(6.10)$$

where  $S_l$  = Sp. gr. of lighter liquid in U-tube

$S_o$  = Sp. gr. of fluid flowing through pipe

$x$  = Difference of the lighter liquid columns in U-tube.

**Case III. Inclined Venturimeter with Differential U-tube manometer.** The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then  $h$  is given as

$$h = \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = x \left[ \frac{S_h}{S_o} - 1 \right] \quad \dots(6.11)$$

**Case IV.** Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of  $h$  is given as

$$h = \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = x \left[ 1 - \frac{S_l}{S_o} \right] \quad \dots(6.12)$$

**Problem 6.10** A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take  $C_d = 0.98$ .

**Solution.** Given :

Dia. at inlet,  $d_1 = 30 \text{ cm}$

$\therefore$  Area at inlet,  $a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$

Dia. at throat,  $d_2 = 15 \text{ cm}$

$\therefore a_2 = \frac{\pi}{4} \times 15^2 = 176.7 \text{ cm}^2$

$C_d = 0.98$

Reading of differential manometer =  $x = 20 \text{ cm}$  of mercury.

$\therefore$  Difference of pressure head is given by (6.9)

$$\text{or } h = x \left[ \frac{S_h}{S_o} - 1 \right]$$

where  $S_h$  = Sp. gravity of mercury = 13.6,  $S_o$  = Sp. gravity of water = 1

$$= 20 \left[ \frac{13.6}{1} - 1 \right] = 20 \times 12.6 \text{ cm} = 252.0 \text{ cm of water.}$$

The discharge through venturimeter is given by eqn. (6.8)

$$\begin{aligned} Q &= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\ &= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 9.81 \times 252} \end{aligned}$$

$$\begin{aligned}
 &= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4} \\
 &= 125756 \text{ cm}^3/\text{s} = \frac{125756}{1000} \text{ lit/s} = \mathbf{125.756 \text{ lit/s. Ans.}}
 \end{aligned}$$

**Problem 6.11** An oil of sp. gr. 0.8 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 25 cm. Calculate the discharge of oil through the horizontal venturimeter. Take  $C_d = 0.98$ .

**Solution.** Given :

$$\text{Sp. gr. of oil, } S_o = 0.8$$

$$\text{Sp. gr. of mercury, } S_h = 13.6$$

Reading of differential manometer,  $x = 25 \text{ cm}$

$$\begin{aligned}
 \therefore \text{Difference of pressure head, } h &= x \left[ \frac{S_h}{S_o} - 1 \right] \\
 &= 25 \left[ \frac{13.6}{0.8} - 1 \right] \text{ cm of oil} = 25 [17 - 1] = 400 \text{ cm of oil.}
 \end{aligned}$$

Dia. at inlet,

$$d_1 = 20 \text{ cm}$$

$$\therefore a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 20^2 = 314.16 \text{ cm}^2$$

$$d_2 = 10 \text{ cm}$$

$$\therefore a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

$$C_d = 0.98$$

$\therefore$  The discharge  $Q$  is given by equation (6.8)

or

$$\begin{aligned}
 Q &= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\
 &= 0.98 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 400} \\
 &= \frac{21421375.68}{\sqrt{98696 - 6168}} = \frac{21421375.68}{304} \text{ cm}^3/\text{s} \\
 &= 70465 \text{ cm}^3/\text{s} = \mathbf{70.465 \text{ litres/s. Ans.}}
 \end{aligned}$$

**Problem 6.12** A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 60 litres/s. Find the reading of the oil-mercury differential manometer. Take  $C_d = 0.98$ .

**Solution.** Given :  $d_1 = 20 \text{ cm}$

$$\therefore a_1 = \frac{\pi}{4} 20^2 = 314.16 \text{ cm}^2$$

$$d_2 = 10 \text{ cm}$$

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$$\therefore a_2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

$$C_d = 0.98$$

$$Q = 60 \text{ litres/s} = 60 \times 1000 \text{ cm}^3/\text{s}$$

$$\text{Using the equation (6.8), } Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$\text{or } 60 \times 1000 = 9.81 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times h} = \frac{1071068.78\sqrt{h}}{304}$$

$$\text{or } \sqrt{h} = \frac{304 \times 60000}{1071068.78} = 17.029$$

$$\therefore h = (17.029)^2 = 289.98 \text{ cm of oil}$$

$$\text{But } h = x \left[ \frac{S_h}{S_o} - 1 \right]$$

where  $S_h$  = Sp. gr. of mercury = 13.6

$S_o$  = Sp. gr. of oil = 0.8

$x$  = Reading of manometer

$$\therefore 289.98 = x \left[ \frac{13.6}{0.8} - 1 \right] = 16x$$

$$\therefore x = \frac{289.98}{16} = 18.12 \text{ cm.}$$

∴ Reading of oil-mercury differential manometer = **18.12 cm. Ans.**

**Problem 6.13** A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of water. The pressure at inlet is  $17.658 \text{ N/cm}^2$  and the vacuum pressure at the throat is 30 cm of mercury. Find the discharge of water through venturimeter. Take  $C_d = 0.98$ .

**Solution.** Given :

$$\text{Dia. at inlet, } d_1 = 20 \text{ cm}$$

$$\therefore a_1 = \frac{\pi}{4} \times (20)^2 = 314.16 \text{ cm}^2$$

$$\text{Dia. at throat, } d_2 = 10 \text{ cm}$$

$$\therefore a_2 = \frac{\pi}{4} \times 10^2 = 78.74 \text{ cm}^2$$

$$p_1 = 17.658 \text{ N/cm}^2 = 17.658 \times 10^4 \text{ N/m}^2$$

$$\rho \text{ for water} \quad = 1000 \frac{\text{kg}}{\text{m}^3} \text{ and } \therefore \frac{p_1}{\rho g} = \frac{17.658 \times 10^4}{9.81 \times 1000} = 18 \text{ m of water}$$

$$\frac{p_2}{\rho g} = -30 \text{ cm of mercury}$$

$$= -0.30 \text{ m of mercury} = -0.30 \times 13.6 = -4.08 \text{ m of water}$$

$$\therefore \text{Differential head} = h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 18 - (-4.08) \\ = 18 + 4.08 = 22.08 \text{ m of water} = 2208 \text{ cm of water}$$

The discharge  $Q$  is given by equation (6.8)

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\ = 0.98 \times \frac{314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 2208} \\ = \frac{50328837.21}{304} \times 165555 \text{ cm}^3/\text{s} = 165.555 \text{ lit/s. Ans.}$$

**Problem 6.14** The inlet and throat diameters of a horizontal venturimeter are 30 cm and 10 cm respectively. The liquid flowing through the meter is water. The pressure intensity at inlet is  $13.734 \text{ N/cm}^2$  while the vacuum pressure head at the throat is 37 cm of mercury. Find the rate of flow. Assume that 4% of the differential head is lost between the inlet and throat. Find also the value of  $C_d$  for the venturimeter.

**Solution.** Given :

$$\text{Dia. at inlet, } d_1 = 30 \text{ cm}$$

$$\therefore a_1 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$$

$$\text{Dia. at throat, } d_2 = 10 \text{ cm}$$

$$\therefore a_2 = \frac{\pi}{4} (10)^2 = 78.54 \text{ cm}^2$$

$$\text{Pressure, } p_1 = 13.734 \text{ N/cm}^2 = 13.734 \times 10^4 \text{ N/m}^2$$

$$\therefore \text{Pressure head, } \frac{p_1}{\rho g} = \frac{13.734 \times 10^4}{1000 \times 9.81} = 14 \text{ m of water}$$

$$\frac{p_2}{\rho g} = -37 \text{ cm of mercury} \\ = \frac{-37 \times 13.6}{100} \text{ m of water} = -5.032 \text{ m of water}$$

$$\text{Differential head, } h = p_1/\rho g - p_2/\rho g \\ = 14.0 - (-5.032) = 14.0 + 5.032 \\ = 19.032 \text{ m of water} = 1903.2 \text{ cm}$$

$$\text{Head lost, } h_f = 4\% \text{ of } h = \frac{4}{100} \times 19.032 = 0.7613 \text{ m}$$

$$\therefore C_d = \sqrt{\frac{h - h_f}{h}} = \sqrt{\frac{19.032 - 0.7613}{19.032}} = 0.98$$

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$$\begin{aligned}
 \therefore \text{Discharge} &= C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}} \\
 &= \frac{0.98 \times 706.85 \times 78.54 \times \sqrt{2 \times 981 \times 1903.2}}{\sqrt{(706.85)^2 - (78.54)^2}} \\
 &= \frac{105132247.8}{\sqrt{499636.9 - 6168}} = 149692.8 \text{ cm}^3/\text{s} = 0.14969 \text{ m}^3/\text{s. Ans.}
 \end{aligned}$$

### PROBLEMS ON INCLINED VENTURI METER

**Problem 6.15** A  $30 \text{ cm} \times 15 \text{ cm}$  venturimeter is inserted in a vertical pipe carrying water, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of  $20 \text{ cm}$ . Find the discharge. Take  $C_d = 0.98$ .

**Solution.** Given :

$$\text{Dia. at inlet, } d_1 = 30 \text{ cm}$$

$$\therefore a_1 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$$

$$\text{Dia. at throat, } d_2 = 15 \text{ cm}$$

$$\therefore a_2 = \frac{\pi}{4} (15)^2 = 176.7 \text{ cm}^2$$

$$h = x \left[ \frac{S_h}{S_o} - 1 \right] = 20 \left[ \frac{13.6}{1.0} - 1.0 \right] = 20 \times 12.6 = 252.0 \text{ cm of water}$$

$$C_d = 0.98$$

$$\text{Discharge,}$$

$$\begin{aligned}
 Q &= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\
 &= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 252} \\
 &= \frac{86067593.36}{\sqrt{499636.3 - 31222.9}} = \frac{86067593.36}{684.4} \\
 &= 125756 \text{ cm}^3/\text{s} = 125.756 \text{ lit/s. Ans.}
 \end{aligned}$$

**Problem 6.16** A  $20 \text{ cm} \times 10 \text{ cm}$  venturimeter is inserted in a vertical pipe carrying oil of sp. gr. 0.8, the flow of oil is in upward direction. The difference of levels between the throat and inlet section is  $50 \text{ cm}$ . The oil mercury differential manometer gives a reading of  $30 \text{ cm}$  of mercury. Find the discharge of oil. Neglect losses.

**Solution.** Dia. at inlet,  $d_1 = 20 \text{ cm}$

$$\therefore a_1 = \frac{\pi}{4} (20)^2 = 314.16 \text{ cm}^2$$

$$\text{Dia. at throat, } d_2 = 10 \text{ cm}$$

$$\therefore a_2 = \frac{\pi}{4} (10)^2 = 78.54 \text{ cm}^2$$

Sp. gr. of oil,  $S_o = 0.8$

Sp. gr. of mercury,  $S_g = 13.6$

Differential manometer reading,  $x = 30 \text{ cm}$

$$\begin{aligned}\therefore h &= \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = x \left[ \frac{S_g}{S_o} - 1 \right] \\ &= 30 \left[ \frac{13.6}{0.8} - 1 \right] = 30 [17 - 1] = 30 \times 16 = 480 \text{ cm of oil}\end{aligned}$$

$$C_d = 1.0$$

The discharge,

$$\begin{aligned}Q &= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\ &= \frac{1.0 \times 314.16 \times 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 480} \text{ cm}^3/\text{s} \\ &= \frac{23932630.7}{304} = 78725.75 \text{ cm}^3/\text{s} = \mathbf{78.725 \text{ litres/s. Ans.}}$$

**Problem 6.17** In a vertical pipe conveying oil of specific gravity 0.8, two pressure gauges have been installed at A and B where the diameters are 16 cm and 8 cm respectively. A is 2 metres above B. The pressure gauge readings have shown that the pressure at B is greater than at A by 0.981 N/cm<sup>2</sup>. Neglecting all losses, calculate the flow rate. If the gauges at A and B are replaced by tubes filled with the same liquid and connected to a U-tube containing mercury, calculate the difference of level of mercury in the two limbs of the U-tube.

**Solution.** Given :

Sp. gr. of oil,

$$S_o = 0.8$$

∴ Density,

$$\rho = 0.8 \times 1000 = 800 \frac{\text{kg}}{\text{m}^3}$$

Dia. at A,

$$D_A = 16 \text{ cm} = 0.16 \text{ m}$$

∴ Area at A,

$$A_1 = \frac{\pi}{4} (0.16)^2 = 0.0201 \text{ m}^2$$

Dia. at B,

$$D_B = 8 \text{ cm} = 0.08 \text{ m}$$

∴ Area at B,

$$A_2 = \frac{\pi}{4} (0.08)^2 = 0.005026 \text{ m}^2$$

(i) Difference of pressures,  $p_B - p_A = 0.981 \text{ N/cm}^2$

$$= 0.981 \times 10^4 \text{ N/m}^2 = \frac{9810 \text{ N}}{\text{m}^2}$$

Difference of pressure head

$$\therefore \frac{p_B - p_A}{\rho g} = \frac{9810}{800 \times 9.81} = 1.25$$

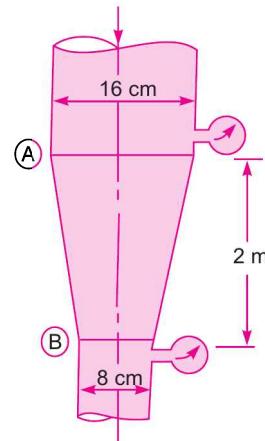


Fig. 6.9 (a)

(∴  $\rho = 800 \text{ kg/m}^3$ )

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Applying Bernoulli's theorem at A and B and taking the reference line passing through section B, we get

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + Z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + Z_B$$

or  $\frac{p_A}{\rho g} - \frac{p_B}{\rho g} - Z_A - Z_B = \frac{V_B^2}{2g} - \frac{V_A^2}{2g}$

or  $\left( \frac{p_A - p_B}{\rho g} \right) + 2.0 - 0.0 = \frac{V_B^2}{2g} - \frac{V_A^2}{2g}$

or  $-1.25 + 2.0 = \frac{V_B^2}{2g} - \frac{V_A^2}{2g} \quad \left( \because \frac{p_B - p_A}{\rho g} = 1.25 \right)$

$$0.75 = \frac{V_B^2}{2g} - \frac{V_A^2}{2g} \quad \dots(i)$$

Now applying continuity equation at A and B, we get

$$V_A \times A_1 = V_B \times A_2$$

or  $V_B = \frac{V_A \times A_1}{A_2} = \frac{V_A \times \frac{\pi}{4}(16)^2}{\frac{\pi}{4}(0.08)^2} = 4V_A$

Substituting the value of  $V_B$  in equation (i), we get

$$0.75 = \frac{16V_A^2}{2g} - \frac{V_A^2}{2g} = \frac{15V_A^2}{2g}$$

$$\therefore V_A = \sqrt{\frac{0.75 \times 2 \times 9.81}{15}} = 0.99 \text{ m/s}$$

∴ Rate of flow,  $Q = V_A \times A_1$   
 $= 0.99 \times 0.0201 = 0.01989 \text{ m}^3/\text{s. Ans.}$

(ii) Difference of level of mercury in the U-tube.

Let  $h$  = Difference of mercury level.

Then  $h = x \left( \frac{S_g}{S_o} - 1 \right)$

where  $h = \left( \frac{p_A}{\rho g} + Z_A \right) - \left( \frac{p_B}{\rho g} + Z_B \right) = \frac{p_A - p_B}{\rho g} + Z_A - Z_B$   
 $= -1.25 + 2.0 - 0$

$$= 0.75 \quad \left( \because \frac{p_B - p_A}{\rho g} = 1.25 \right)$$

∴  $0.75 = x \left[ \frac{13.6}{0.8} - 1 \right] = x \times 16$

∴  $x = \frac{0.75}{16} = 0.04687 \text{ m} = 4.687 \text{ cm. Ans.}$

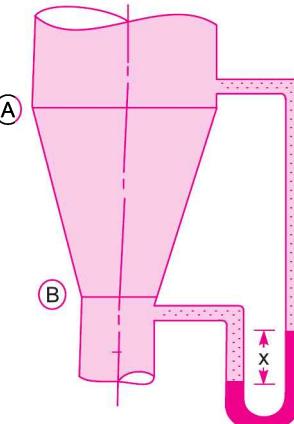


Fig. 6.9 (b)

**Problem 6.18** Find the discharge of water flowing through a pipe 30 cm diameter placed in an inclined position where a venturimeter is inserted, having a throat diameter of 15 cm. The difference of pressure between the main and throat is measured by a liquid of sp. gr. 0.6 in an inverted U-tube which gives a reading of 30 cm. The loss of head between the main and throat is 0.2 times the kinetic head of the pipe.

**Solution.** Dia. at inlet,  $d_1 = 30 \text{ cm}$

$$\therefore a_1 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$$

Dia. at throat,  $d_2 = 15 \text{ cm}$

$$\therefore a_2 = \frac{\pi}{4} (15)^2 = 176.7 \text{ cm}^2$$

Reading of differential manometer,  $x = 30 \text{ cm}$

Difference of pressure head,  $h$  is given by

$$\left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = h$$

Also

$$h = x \left[ 1 - \frac{S_l}{S_o} \right]$$

where  $S_l = 0.6$  and  $S_o = 1.0$

$$= 30 \left[ 1 - \frac{0.6}{1.0} \right] = 30 \times 0.4 = 12.0 \text{ cm of water}$$

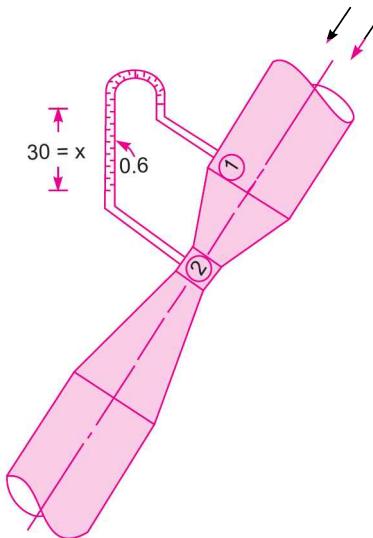


Fig. 6.10

$$\text{Loss of head, } h_L = 0.2 \times \text{kinetic head of pipe} = 0.2 \times \frac{v_1^2}{2g}$$

Now applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_L$$

$$\text{or } \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = h_L$$

$$\text{But } \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = h = 12.0 \text{ cm of water}$$

and

$$h_L = 0.2 \times v_1^2 / 2g$$

$$\therefore 12.0 + \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0.2 \times \frac{v_1^2}{2g}$$

$$\therefore 12.0 + 0.8 \frac{v_1^2}{2g} - \frac{v_2^2}{2g} = 0 \quad \dots(1)$$

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Applying continuity equation at sections (1) and (2), we get

$$a_1 v_1 = a_2 v_2$$

$$\therefore v_1 = \frac{a_2}{a_1} v_2 = \frac{\frac{\pi}{4}(15)^2 v_2}{\frac{\pi}{4}(30)^2} = \frac{v_2}{4}$$

Substituting this value of  $v_1$  in equation (1), we get

$$12.0 + \frac{0.8}{2g} \left( \frac{v_2}{4} \right)^2 - \frac{v_2^2}{2g} = 0 \quad \text{or} \quad 12.0 + \frac{v_2^2}{2g} \left[ \frac{0.8}{16} - 1 \right] = 0$$

$$\text{or} \quad \frac{v_2^2}{2g} [0.05 - 1] = -12.0 \quad \text{or} \quad \frac{0.95 v_2^2}{2g} = 12.0$$

$$\therefore v_2 = \sqrt{\frac{2 \times 981 \times 12.0}{0.95}} = 157.4 \text{ cm/s}$$

$$\therefore \text{Discharge} = a_2 v_2 \\ = 176.7 \times 157.4 \text{ cm}^3/\text{s} = 27800 \text{ cm}^3/\text{s} = 27.8 \text{ litres/s. Ans.}$$

**Problem 6.19** A  $30 \text{ cm} \times 15 \text{ cm}$  venturimeter is provided in a vertical pipe line carrying oil of specific gravity 0.9, the flow being upwards. The difference in elevation of the throat section and entrance section of the venturimeter is 30 cm. The differential U-tube mercury manometer shows a gauge deflection of 25 cm. Calculate :

- (i) the discharge of oil, and
- (ii) the pressure difference between the entrance section and the throat section. Take the co-efficient of discharge as 0.98 and specific gravity of mercury as 13.6.

**Solution.** Given :

$$\text{Dia. at inlet, } d_1 = 30 \text{ cm}$$

$$\therefore \text{Area, } a_1 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$$

$$\text{Dia. at throat, } d_2 = 15 \text{ cm}$$

$$\therefore \text{Area, } a_2 = \frac{\pi}{4} (15)^2 = 176.7 \text{ cm}^2$$

Let section (1) represents inlet and section (2) represents throat. Then  $z_2 - z_1 = 30 \text{ cm}$

$$\text{Sp. gr. of oil, } S_o = 0.9$$

$$\text{Sp. gr. of mercury, } S_g = 13.6$$

$$\text{Reading of diff. manometer, } x = 25 \text{ cm}$$

The differential head,  $h$  is given by

$$\begin{aligned} h &= \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) \\ &= x \left[ \frac{S_g}{S_o} - 1 \right] = 25 \left[ \frac{13.6}{0.9} - 1 \right] = 352.77 \text{ cm of oil} \end{aligned}$$

$$\begin{aligned}
 (i) \text{ The discharge, } Q \text{ of oil} &= C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\
 &= \frac{0.98 \times 706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} = \sqrt{2 \times 981 \times 352.77} \\
 &= \frac{101832219.9}{684.4} = 148790.5 \text{ cm}^3/\text{s} \\
 &= \mathbf{148.79 \text{ litres/s. Ans.}}
 \end{aligned}$$

(ii) Pressure difference between entrance and throat section

$$\begin{aligned}
 h &= \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = 352.77 \\
 \text{or } &\left( \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) + z_1 - z_2 = 352.77
 \end{aligned}$$

$$\text{But } z_2 - z_1 = 30 \text{ cm}$$

$$\therefore \left( \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) - 30 = 352.77$$

$$\therefore \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 352.77 + 30 = 382.77 \text{ cm of oil} = \mathbf{3.8277 \text{ m of oil. Ans.}}$$

$$\text{or } (p_1 - p_2) = 3.8277 \times \rho g$$

$$\begin{aligned}
 \text{But density of oil} &= \text{Sp. gr. of oil} \times 1000 \text{ kg/m}^3 \\
 &= 0.9 \times 1000 = 900 \text{ kg/cm}^3
 \end{aligned}$$

$$\begin{aligned}
 \therefore (p_1 - p_2) &= 3.8277 \times 900 \times 9.81 \frac{\text{N}}{\text{m}^2} \\
 &= \frac{33795}{10^4} \text{ N/cm}^2 = \mathbf{3.3795 \text{ N/cm}^2. \text{ Ans.}}
 \end{aligned}$$

**Problem 6.20** Crude oil of specific gravity 0.85 flows upwards at a volume rate of flow of 60 litre per second through a vertical venturimeter with an inlet diameter of 200 mm and a throat diameter of 100 mm. The co-efficient of discharge of the venturimeter is 0.98. The vertical distance between the pressure tappings is 300 mm.

(i) If two pressure gauges are connected at the tappings such that they are positioned at the levels of their corresponding tapping points, determine the difference of readings in  $\text{N/cm}^2$  of the two pressure gauges.

(ii) If a mercury differential manometer is connected, in place of pressure gauges, to the tappings such that the connecting tube upto mercury are filled with oil, determine the difference in the level of the mercury column.

**Solution.** Given :

$$\text{Specific gravity of oil, } S_o = 0.85$$

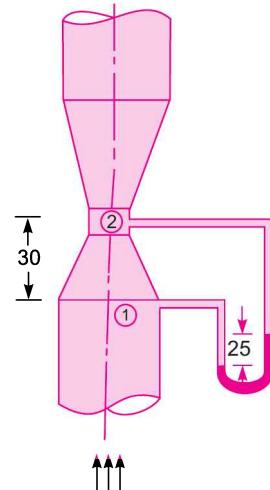


Fig. 6.11

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$$\therefore \text{Density, } \rho = 0.85 \times 1000 = 850 \text{ kg/m}^3$$

Discharge,  $Q = 60 \text{ litre/s}$

$$= \frac{60}{1000} = 0.06 \text{ m}^3/\text{s}$$

Inlet dia,  $d_1 = 200 \text{ mm} = 0.2 \text{ m}$

∴ Area,  $a_1 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$

Throat dia.,  $d_2 = 100 \text{ mm} = 0.1 \text{ m}$

∴ Area,  $a_2 = \frac{\pi}{4} (0.1)^2 = 0.00785 \text{ m}^2$

Value of  $C_d = 0.98$

Let section (1) represents inlet and section (2) represents throat. Then

$$z_2 - z_1 = 300 \text{ mm} = 0.3 \text{ m}$$

(i) Difference of readings in  $\text{N/cm}^2$  of the two pressure gauges  
The discharge  $Q$  is given by,

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

or

$$0.06 = \frac{0.98 \times 0.0314 \times 0.00785}{\sqrt{0.0314^2 - 0.00785^2}} \times \sqrt{2 \times 9.81 \times h}$$

$$= \frac{0.98 \times 0.00024649}{0.0304} \times 4.429 \sqrt{h}$$

$$\therefore \sqrt{h} = \frac{0.06 \times 0.0304}{0.98 \times 0.00024649 \times 4.429} = 1.705$$

$$\therefore h = 1.705^2 = 2.908 \text{ m}$$

But for a vertical venturimeter,  $h = \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right)$

$$\therefore 2.908 = \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = \left( \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) + z_1 - z_2$$

$$\frac{p_1 - p_2}{\rho g} = 2.908 + z_2 - z_1 = 2.908 + 0.3 \quad (\because z_2 - z_1 = 0.3 \text{ m})$$

$$= 3.208 \text{ m of oil}$$

$$\therefore p_1 - p_2 = \rho g \times 3.208$$

$$= 850 \times 9.81 \times 3.208 \text{ N/m}^2 = \frac{850 \times 9.81 \times 3.208}{10^4} \text{ N/cm}^2$$

$$= 2.675 \text{ N/cm}^2. \text{ Ans.}$$

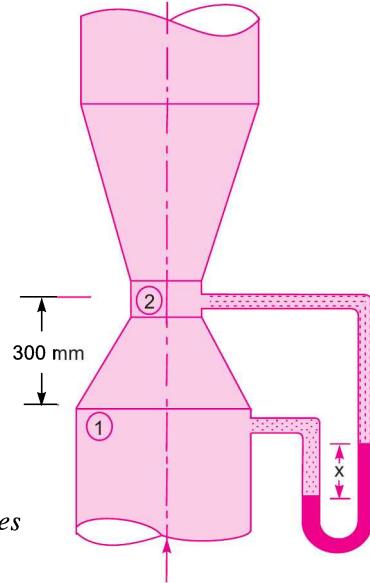


Fig. 6.11 (a)

(ii) Difference in the levels of mercury columns (i.e.,  $x$ )

$$\text{The value of } h \text{ is given by, } h = x \left[ \frac{S_g}{S_o} - 1 \right]$$

$$\therefore 2.908 = x \left[ \frac{13.6}{0.85} - 1 \right] = x [16 - 1] = 15x$$

$$\therefore x = \frac{2.908}{15} = 0.1938 \text{ m} = 19.38 \text{ cm of oil. Ans.}$$

**Problem 6.21** In a 100 mm diameter horizontal pipe a venturimeter of 0.5 contraction ratio has been fixed. The head of water on the metre when there is no flow is 3 m (gauge). Find the rate of flow for which the throat pressure will be 2 metres of water absolute. The co-efficient of discharge is 0.97. Take atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

$$\text{Dia. of pipe, } d_1 = 100 \text{ mm} = 10 \text{ cm}$$

$$\therefore \text{Area, } a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} (10)^2 = 78.54 \text{ cm}^2$$

$$\text{Dia. at throat, } d_2 = 0.5 \times d_1 = 0.5 \times 10 = 5 \text{ cm}$$

$$\therefore \text{Area, } a_2 = \frac{\pi}{4} (5)^2 = 19.635 \text{ cm}^2$$

$$\text{Head of water for no flow} = \frac{p_1}{\rho g} = 3 \text{ m (gauge)} = 3 + 10.3 = 13.3 \text{ m (abs.)}$$

$$\text{Throat pressure head} = \frac{p_2}{\rho g} = 2 \text{ m of water absolute.}$$

$$\therefore \text{Difference of pressure head, } h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 13.3 - 2.0 = 11.3 \text{ m} = 1130 \text{ cm}$$

$$\begin{aligned} \therefore \text{Rate of flow, } Q & \text{ is given by } Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} \\ & = 0.97 \times \frac{78.54 \times 19.635}{\sqrt{(78.54)^2 - (19.635)^2}} \times \sqrt{2 \times 981 \times 1130} \\ & = \frac{2227318.17}{76} = 29306.8 \text{ cm}^3/\text{s} = 29.306 \text{ litres/s. Ans.} \end{aligned}$$

**6.7.2 Orifice Meter or Orifice Plate.** It is a device used for measuring the rate of flow of a fluid through a pipe. It is a cheaper device as compared to venturimeter. It also works on the same principle as that of venturimeter. It consists of a flat circular plate which has a circular sharp edged hole called orifice, which is concentric with the pipe. The orifice diameter is kept generally 0.5 times the diameter of the pipe, though it may vary from 0.4 to 0.8 times the pipe diameter.

A differential manometer is connected at section (1), which is at a distance of about 1.5 to 2.0 times the pipe diameter upstream from the orifice plate, and at section (2), which is at a distance of about half the diameter of the orifice on the downstream side from the orifice plate.

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Let  $p_1$  = pressure at section (1),  
 $v_1$  = velocity at section (1),  
 $a_1$  = area of pipe at section (1), and

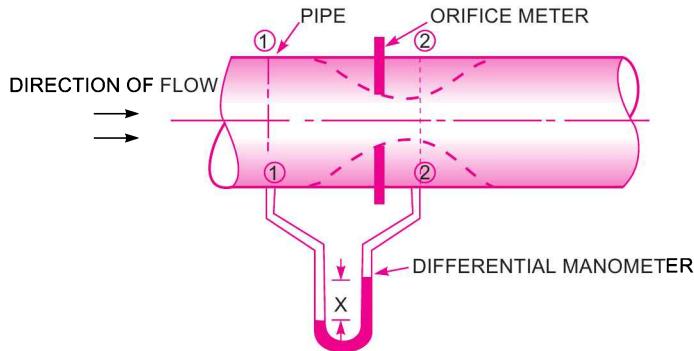


Fig. 6.12. Orifice meter.

$p_2, v_2, a_2$  are corresponding values at section (2). Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

or  $\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$

But  $\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = h = \text{Differential head}$

$$\therefore h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \quad \text{or} \quad 2gh = v_2^2 - v_1^2$$

or  $v_2 = \sqrt{2gh + v_1^2} \quad \dots(i)$

Now section (2) is at the vena-contracta and  $a_2$  represents the area at the vena-contracta. If  $a_0$  is the area of orifice then, we have

$$C_c = \frac{a_2}{a_0}$$

where  $C_c$  = Co-efficient of contraction

$$\therefore a_2 = a_0 \times C_c \quad \dots(ii)$$

By continuity equation, we have

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2}{a_1} v_2 = \frac{a_0 C_c}{a_1} v_2 \quad \dots(iii)$$

Substituting the value of  $v_1$  in equation (i), we get

$$v_2 = \sqrt{2gh + \frac{a_0^2 C_c^2 v_2^2}{a_1^2}}$$

$$\text{or } v_2^2 = 2gh + \left(\frac{a_0}{a_1}\right)^2 C_c^2 v_2^2 \text{ or } v_2^2 \left[1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2\right] = 2gh$$

$$\therefore v_2 = \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$\therefore$  The discharge  $Q = v_2 \times a_2 = v_2 \times a_0 C_c$   $\{ \because a_2 = a_0 C_c \text{ from (ii)} \}$

$$= \frac{a_0 C_c \sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}} \quad \dots(iv)$$

The above expression is simplified by using

$$C_d = C_c \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$$\therefore C_c = C_d \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}}$$

Substituting this value of  $C_c$  in equation (iv), we get

$$\begin{aligned} Q &= a_0 \times C_d \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}} \times \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}} \\ &= \frac{C_d a_0 \sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}} = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}. \end{aligned} \quad \dots(6.13)$$

where  $C_d$  = Co-efficient of discharge for orifice meter.

The co-efficient of discharge for orifice meter is much smaller than that for a venturimeter.

**Problem 6.22** An orifice meter with orifice diameter 10 cm is inserted in a pipe of 20 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter gives readings of 19.62 N/cm<sup>2</sup> and 9.81 N/cm<sup>2</sup> respectively. Co-efficient of discharge for the orifice meter is given as 0.6. Find the discharge of water through pipe.

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**Solution.** Given :

$$\text{Dia. of orifice, } d_0 = 10 \text{ cm}$$

$$\therefore \text{Area, } a_0 = \frac{\pi}{4} (10)^2 = 78.54 \text{ cm}^2$$

$$\text{Dia. of pipe, } d_1 = 20 \text{ cm}$$

$$\therefore \text{Area, } a_1 = \frac{\pi}{4} (20)^2 = 314.16 \text{ cm}^2$$

$$p_1 = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$$

$$\therefore \frac{p_1}{\rho g} = \frac{19.62 \times 10^4}{1000 \times 9.81} = 20 \text{ m of water}$$

$$\text{Similarly } \frac{p_2}{\rho g} = \frac{9.81 \times 10^4}{1000 \times 9.81} = 10 \text{ m of water}$$

$$\therefore h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 20.0 - 10.0 = 10 \text{ m of water} = 1000 \text{ cm of water}$$

$$C_d = 0.6$$

The discharge,  $Q$  is given by equation (6.13)

$$\begin{aligned} Q &= C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh} \\ &= 0.6 \times \frac{78.54 \times 314.16}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2 \times 981 \times 1000} \\ &= \frac{20736838.09}{304} = 68213.28 \text{ cm}^3/\text{s} = \mathbf{68.21 \text{ litres/s. Ans.}} \end{aligned}$$

**Problem 6.23** An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure difference measured by a mercury oil differential manometer on the two sides of the orifice meter gives a reading of 50 cm of mercury. Find the rate of flow of oil of sp. gr. 0.9 when the coefficient of discharge of the orifice meter = 0.64.

**Solution.** Given :

$$\text{Dia. of orifice, } d_0 = 15 \text{ cm}$$

$$\therefore \text{Area, } a_0 = \frac{\pi}{4} (15)^2 = 176.7 \text{ cm}^2$$

$$\text{Dia. of pipe, } d_1 = 30 \text{ cm}$$

$$\therefore \text{Area, } a_1 = \frac{\pi}{4} (30)^2 = 706.85 \text{ cm}^2$$

$$\text{Sp. gr. of oil, } S_o = 0.9$$

$$\text{Reading of diff. manometer, } x = 50 \text{ cm of mercury}$$

$$\therefore \text{Differential head, } h = x \left[ \frac{S_g}{S_o} - 1 \right] = 50 \left[ \frac{13.6}{0.9} - 1 \right] \text{ cm of oil}$$

$$= 50 \times 14.11 = 705.5 \text{ cm of oil}$$

$$C_d = 0.64$$

$\therefore$  The rate of the flow,  $Q$  is given by equation (6.13)

$$\begin{aligned} Q &= C_d \cdot \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh} \\ &= 0.64 \times \frac{176.7 \times 706.85}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 705.5} \\ &= \frac{94046317.78}{684.4} = 137414.25 \text{ cm}^3/\text{s} = \mathbf{137.414 \text{ litres/s. Ans.}} \end{aligned}$$

**6.7.3 Pitot-tube.** It is a device used for measuring the velocity of flow at any point in a pipe or a channel. It is based on the principle that if the velocity of flow at a point becomes zero, the pressure there is increased due to the conversion of the kinetic energy into pressure energy. In its simplest form, the pitot-tube consists of a glass tube, bent at right angles as shown in Fig. 6.13.

The lower end, which is bent through  $90^\circ$  is directed in the upstream direction as shown in Fig. 6.13. The liquid rises up in the tube due to the conversion of kinetic energy into pressure energy.

The velocity is determined by measuring the rise of liquid in the tube.

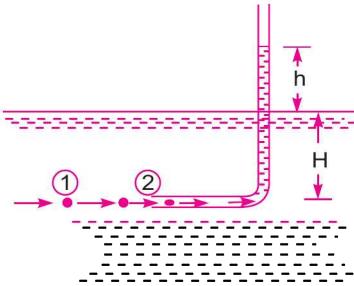


Fig. 6.13 Pitot-tube.

Consider two points (1) and (2) at the same level in such a way that point (2) is just as the inlet of the pitot-tube and point (1) is far away from the tube.

Let

$$p_1 = \text{intensity of pressure at point (1)}$$

$$v_1 = \text{velocity of flow at (1)}$$

$$p_2 = \text{pressure at point (2)}$$

$$v_2 = \text{velocity at point (2), which is zero}$$

$$H = \text{depth of tube in the liquid}$$

$$h = \text{rise of liquid in the tube above the free surface.}$$

Applying Bernoulli's equation at points (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But  $z_1 = z_2$  as points (1) and (2) are on the same line and  $v_2 = 0$ .

$$\frac{p_1}{\rho g} = \text{pressure head at (1)} = H$$

$$\frac{p_2}{\rho g} = \text{pressure head at (2)} = (h + H)$$

Substituting these values, we get

$$\therefore H + \frac{v_1^2}{2g} = (h + H) \quad \therefore h = \frac{v_1^2}{2g} \quad \text{or} \quad v_1 = \sqrt{2gh}$$

This is theoretical velocity. Actual velocity is given by

$$(v_1)_{act} = C_v \sqrt{2gh}$$

where  $C_v$  = Co-efficient of pitot-tube

$$\therefore \text{Velocity at any point } v = C_v \sqrt{2gh} \quad \dots(6.14)$$

**Velocity of flow in a pipe by pitot-tube.** For finding the velocity at any point in a pipe by pitot-tube, the following arrangements are adopted :

1. Pitot-tube along with a vertical piezometer tube as shown in Fig. 6.14.
2. Pitot-tube connected with piezometer tube as shown in Fig. 6.15.
3. Pitot-tube and vertical piezometer tube connected with a differential U-tube manometer as shown in Fig. 6.16.

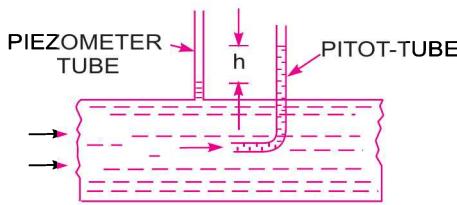


Fig. 6.14

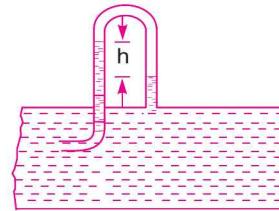


Fig. 6.15

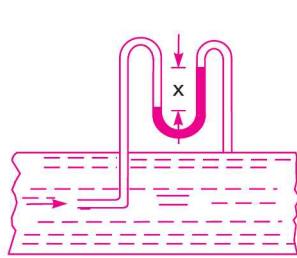


Fig. 6.16

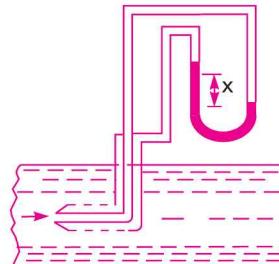


Fig. 6.17

4. Pitot-static tube, which consists of two circular concentric tubes one inside the other with some annular space in between as shown in Fig. 6.17. The outlet of these two tubes are connected to the differential manometer where the difference of pressure head 'h' is measured by knowing the

difference of the levels of the manometer liquid say  $x$ . Then  $h = x \left[ \frac{S_g}{S_o} - 1 \right]$ .

**Problem 6.24** A pitot-static tube placed in the centre of a 300 mm pipe line has one orifice pointing upstream and other perpendicular to it. The mean velocity in the pipe is 0.80 of the central velocity. Find the discharge through the pipe if the pressure difference between the two orifices is 60 mm of water. Take the co-efficient of pitot tube as  $C_v = 0.98$ .

**Solution.** Given :

Dia. of pipe,  $d = 300 \text{ mm} = 0.30 \text{ m}$

Diff. of pressure head,  $h = 60 \text{ mm of water} = .06 \text{ m of water}$

$$C_v = 0.98$$

Mean velocity,  $\bar{V} = 0.80 \times \text{Central velocity}$

Central velocity is given by equation (6.14)

$$= C_v \sqrt{2gh} = 0.98 \times \sqrt{2 \times 9.81 \times .06} = 1.063 \text{ m/s}$$

$$\therefore \bar{V} = 0.80 \times 1.063 = 0.8504 \text{ m/s}$$

Discharge,  $Q = \text{Area of pipe} \times \bar{V}$

$$= \frac{\pi}{4} d^2 \times \bar{V} = \frac{\pi}{4} (0.30)^2 \times 0.8504 = 0.06 \text{ m}^3/\text{s. Ans.}$$

**Problem 6.25** Find the velocity of the flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 100 mm. Take co-efficient of pitot-tube 0.98 and sp. gr. of oil = 0.8.

**Solution.** Given :

$$\text{Diff. of mercury level, } x = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Sp. gr. of oil, } S_o = 0.8$$

$$\text{Sp. gr. of mercury, } S_g = 13.6$$

$$C_v = 0.98$$

$$\text{Diff. of pressure head, } h = x \left[ \frac{S_g}{S_o} - 1 \right] = 0.1 \left[ \frac{13.6}{0.8} - 1 \right] = 1.6 \text{ m of oil}$$

$$\therefore \text{Velocity of flow} = C_v \sqrt{2gh} = 0.98 \sqrt{2 \times 9.81 \times 1.6} = 5.49 \text{ m/s. Ans.}$$

**Problem 6.26** A pitot-static tube is used to measure the velocity of water in a pipe. The stagnation pressure head is 6 m and static pressure head is 5 m. Calculate the velocity of flow assuming the coefficient of tube equal to 0.98.

**Solution.** Given :

$$\text{Stagnation pressure head, } h_s = 6 \text{ m}$$

$$\text{Static pressure head, } h_t = 5 \text{ m}$$

$$\therefore h = 6 - 5 = 1 \text{ m}$$

$$\text{Velocity of flow, } V = C_v \sqrt{2gh} = 0.98 \sqrt{2 \times 9.81 \times 1} = 4.34 \text{ m/s. Ans.}$$

**Problem 6.27** A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.

**Solution.** Given :

$$\text{Diff. of mercury level, } x = 170 \text{ mm} = 0.17 \text{ m}$$

$$\text{Sp. gr. of mercury, } S_g = 13.6$$

$$\text{Sp. gr. of sea-water, } S_o = 1.026$$

$$\therefore h = x \left[ \frac{S_g}{S_o} - 1 \right] = 0.17 \left[ \frac{13.6}{1.026} - 1 \right] = 2.0834 \text{ m}$$

$$\therefore V = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2.0834} = 6.393 \text{ m/s}$$

$$= \frac{6.393 \times 60 \times 60}{1000} \text{ km/hr} = 23.01 \text{ km/hr. Ans.}$$

**Problem 6.28** A pitot-tube is inserted in a pipe of 300 mm diameter. The static pressure in pipe is 100 mm of mercury (vacuum). The stagnation pressure at the centre of the pipe, recorded by the

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pitot-tube is  $0.981 \text{ N/cm}^2$ . Calculate the rate of flow of water through pipe, if the mean velocity of flow is 0.85 times the central velocity. Take  $C_v = 0.98$ .

**Solution.** Given :

$$\text{Dia. of pipe, } d = 300 \text{ mm} = 0.30 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.3)^2 = 0.07068 \text{ m}^2$$

$$\text{Static pressure head} = 100 \text{ mm of mercury (vacuum)}$$

$$= -\frac{100}{1000} \times 13.6 = -1.36 \text{ m of water}$$

$$\text{Stagnation pressure} = 0.981 \text{ N/cm}^2 = 0.981 \times 10^4 \text{ N/m}^2$$

$$\therefore \text{Stagnation pressure head} = \frac{0.981 \times 10^4}{\rho g} = \frac{0.981 \times 10^4}{1000 \times 9.81} = 1 \text{ m}$$

$$\therefore h = \text{Stagnation pressure head} - \text{Static pressure head} \\ = 1.0 - (-1.36) = 1.0 + 1.36 = 2.36 \text{ m of water}$$

$$\therefore \text{Velocity at centre} = C_v \sqrt{2gh} \\ = 0.98 \times \sqrt{2 \times 9.81 \times 2.36} = 6.668 \text{ m/s}$$

$$\text{Mean velocity, } \bar{V} = 0.85 \times 6.668 = 5.6678 \text{ m/s}$$

$$\therefore \text{Rate of flow of water} = \bar{V} \times \text{area of pipe} \\ = 5.6678 \times 0.07068 \text{ m}^3/\text{s} = 0.4006 \text{ m}^3/\text{s. Ans.}$$

### ► 6.8 THE MOMENTUM EQUATION

It is based on the law of conservation of momentum or on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in that direction. The force acting on a fluid mass ' $m$ ' is given by the Newton's second law of motion,

$$F = m \times a$$

where  $a$  is the acceleration acting in the same direction as force  $F$ .

$$\text{But } a = \frac{dv}{dt}$$

$$\therefore F = m \frac{dv}{dt} \\ = \frac{d(mv)}{dt} \quad \{m \text{ is constant and can be taken inside the differential}\}$$

$$\therefore F = \frac{d(mv)}{dt} \quad \dots(6.15)$$

Equation (6.15) is known as the momentum principle.

$$\text{Equation (6.15) can be written as } F \cdot dt = d(mv) \quad \dots(6.16)$$

which is known as the *impulse-momentum equation* and states that the impulse of a force  $F$  acting on a fluid of mass  $m$  in a short interval of time  $dt$  is equal to the change of momentum  $d(mv)$  in the direction of force.

### Force exerted by a flowing fluid on a pipe bend

The impulse-momentum equation (6.16) is used to determine the resultant force exerted by a flowing fluid on a pipe bend.

Consider two sections (1) and (2), as shown in Fig. 6.18.

Let

$v_1$  = velocity of flow at section (1),

$p_1$  = pressure intensity at section (1),

$A_1$  = area of cross-section of pipe at section (1) and

$v_2, p_2, A_2$  = corresponding values of velocity, pressure and area at section (2).

Let  $F_x$  and  $F_y$  be the components of the forces exerted by the flowing fluid on the bend in  $x$ -and  $y$ -directions respectively. Then the force exerted by the bend on the fluid in the directions of  $x$  and  $y$  will be equal to  $-F_x$  and  $-F_y$  but in the opposite directions. Hence component of the force exerted by bend on the fluid in the  $x$ -direction =  $-F_x$  and in the direction of  $y$  =  $-F_y$ . The other external forces acting on the fluid are  $p_1A_1$  and  $p_2A_2$  on the sections (1) and (2) respectively. Then momentum equation in  $x$ -direction is given by

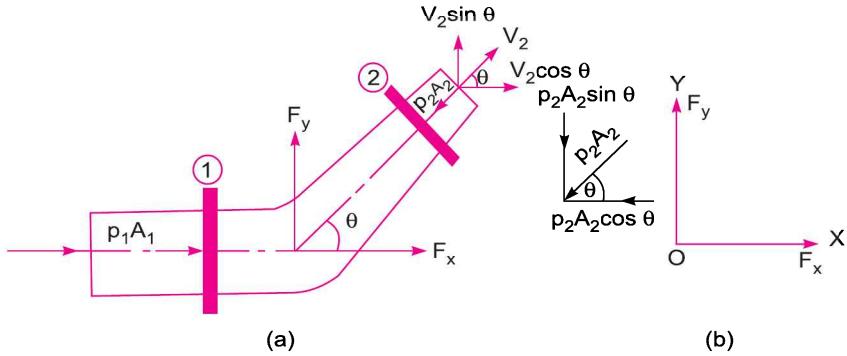


Fig. 6.18 Forces on bend.

Net force acting on fluid in the direction of  $x$  = Rate of change of momentum in  $x$ -direction

$$\begin{aligned} \therefore p_1A_1 - p_2A_2 \cos \theta - F_x &= (\text{Mass per sec}) (\text{change of velocity}) \\ &= \rho Q (\text{Final velocity in the direction of } x) \\ &\quad - \text{Initial velocity in the direction of } x \\ &= \rho Q (V_2 \cos \theta - V_1) \end{aligned} \quad \dots(6.17)$$

$$\therefore F_x = \rho Q (V_1 - V_2 \cos \theta) + p_1A_1 - p_2A_2 \cos \theta \quad \dots(6.18)$$

Similarly the momentum equation in  $y$ -direction gives

$$0 - p_2A_2 \sin \theta - F_y = \rho Q (V_2 \sin \theta - 0) \quad \dots(6.19)$$

$$\therefore F_y = \rho Q (-V_2 \sin \theta) - p_2A_2 \sin \theta \quad \dots(6.20)$$

Now the resultant force ( $F_R$ ) acting on the bend

$$= \sqrt{F_x^2 + F_y^2} \quad \dots(6.21)$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan \theta = \frac{F_y}{F_x} \quad \dots(6.22)$$

**Problem 6.29** A  $45^\circ$  reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 600 mm and 300 mm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet to bend is  $8.829 \text{ N/cm}^2$  and rate of flow of water is 600 litres/s.

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**Solution.** Given :

Angle of bend,

$$\theta = 45^\circ$$

Dia. at inlet,

$$D_1 = 600 \text{ mm} = 0.6 \text{ m}$$

∴ Area,

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.6)^2 = 0.2827 \text{ m}^2$$

Dia. at outlet,

$$D_2 = 300 \text{ mm} = 0.30 \text{ m}$$

∴ Area,

$$A_2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$$

Pressure at inlet,

$$p_1 = 8.829 \text{ N/cm}^2 = 8.829 \times 10^4 \text{ N/m}^2$$

$$Q = 600 \text{ lit/s} = 0.6 \text{ m}^3/\text{s}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.6}{0.2827} = 2.122 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.6}{0.07068} = 8.488 \text{ m/s.}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

But

$$z_1 = z_2$$

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad \text{or} \quad \frac{8.829 \times 10^4}{1000 \times 9.81} + \frac{2.122^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{8.488^2}{2 \times 9.81}$$

$$9 + .2295 = p_2/\rho g + 3.672$$

$$\therefore \frac{p_2}{\rho g} = 9.2295 - 3.672 = 5.5575 \text{ m of water}$$

$$\therefore p_2 = 5.5575 \times 1000 \times 9.81 \text{ N/m}^2 = 5.45 \times 10^4 \text{ N/m}^2$$

Forces on the bend in  $x$ - and  $y$ -directions are given by equations (6.18) and (6.20) as

$$\begin{aligned} F_x &= \rho Q [V_1 - V_2 \cos \theta] + p_1 A_1 - p_2 A_2 \cos \theta \\ &= 1000 \times 0.6 [2.122 - 8.488 \cos 45^\circ] \\ &\quad + 8.829 \times 10^4 \times 0.2827 - 5.45 \times 10^4 \times 0.07068 \times \cos 45^\circ \\ &= -2327.9 + 24959.6 - 2720.3 = 24959.6 - 5048.2 \\ &= 19911.4 \text{ N} \end{aligned}$$

and

$$\begin{aligned} F_y &= \rho Q [-V_2 \sin \theta] - p_2 A_2 \sin \theta \\ &= 1000 \times 0.6 [-8.488 \sin 45^\circ] - 5.45 \times 10^4 \times 0.07068 \times \sin 45^\circ \\ &= -3601.1 - 2721.1 = -6322.2 \text{ N} \end{aligned}$$

-ve sign means  $F_y$  is acting in the downward direction

$$\therefore \text{Resultant force, } F_R = \sqrt{F_x^2 + F_y^2}$$

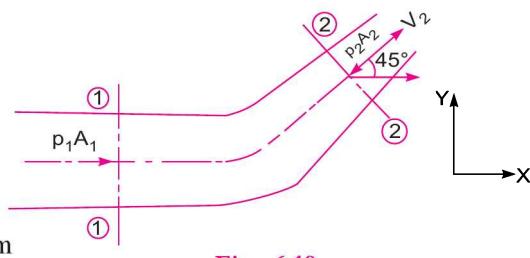


Fig. 6.19

$$= \sqrt{(19911.4)^2 + (-6322.2)^2}$$

$$= 20890.9 \text{ N. Ans.}$$

The angle made by resultant force with  $x$ -axis is given by equation (6.22) or

$$\tan \theta = \frac{F_y}{F_x} = \frac{6322.2}{19911.4} = 0.3175$$

$$\therefore \theta = \tan^{-1} 0.3175 = 17^\circ 36'. \text{ Ans.}$$

**Problem 6.30** 250 litres/s of water is flowing in a pipe having a diameter of 300 mm. If the pipe is bent by  $135^\circ$  (that is change from initial to final direction is  $135^\circ$ ), find the magnitude and direction of the resultant force on the bend. The pressure of water flowing is  $39.24 \text{ N/cm}^2$ .

**Solution.** Given :

$$\text{Pressure, } p_1 = p_2 = 39.24 \text{ N/cm}^2 = 39.24 \times 10^4 \text{ N/m}^2$$

$$\text{Discharge, } Q = 250 \text{ litres/s} = 0.25 \text{ m}^3/\text{s}$$

$$\text{Dia. of bend at inlet and outlet, } D_1 = D_2 = 300 \text{ mm} = 0.3 \text{ m}$$

$$\therefore \text{Area, } A_1 = A_2 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

$$\text{Velocity of water at sections (1) and (2), } V = V_1 = V_2 = \frac{Q}{\text{Area}} = \frac{0.25}{0.07068} = 3.537 \text{ m/s.}$$

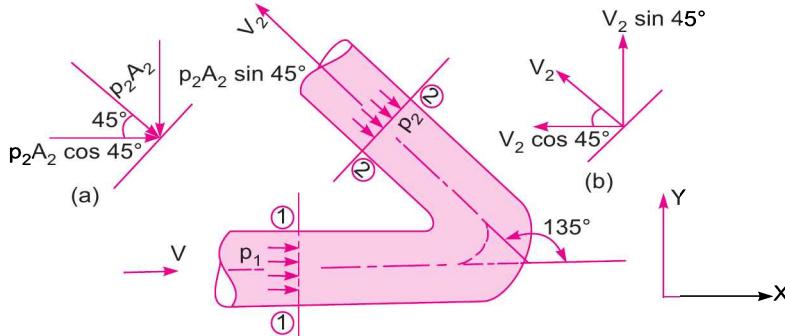


Fig. 6.21

Force along  $x$ -axis

$$= F_x = \rho Q [V_{1x} - V_{2x}] + p_{1x} A_1 + p_{2x} A_2$$

where,  $V_{1x}$  = initial velocity in the direction of  $x = 3.537 \text{ m/s}$

$$V_{2x} = \text{final velocity in the direction of } x = -V_2 \cos 45^\circ = -3.537 \times .7071$$

$$p_{1x} = \text{pressure at section (1) in } x\text{-direction}$$

$$= 39.24 \text{ N/cm}^2 = 39.24 \times 10^4 \text{ N/m}^2$$

$$p_{2x} = \text{pressure at section (2) in } x\text{-direction}$$

$$= p_2 \cos 45^\circ = 39.24 \times 10^4 \times .7071$$

$$\therefore F_x = 1000 \times .25 [3.537 - (-3.537 \times .7071)] + 39.24 \times 10^4 \times 0.07068 + 39.24 \times 10^4 \times 0.07068 \times .7071$$

$$= 1000 \times .25 [3.537 + 3.537 \times .7071] + 39.24 \times 10^4 \times 0.07068 [1 + .7071]$$

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$$= 1509.4 + 47346 = 48855.4 \text{ N}$$

Force along y-axis

$$= F_y = \rho Q [V_{1y} - V_{2y}] + (p_1 A_1)_y + (p_2 A_2)_y$$

where  $V_{1y}$  = initial velocity in y-direction = 0

$V_{2y}$  = final velocity in y-direction =  $-V_2 \sin 45^\circ = 3.537 \times .7071$

$(p_1 A_1)_y$  = pressure force in y-direction = 0

$(p_2 A_2)_y$  = pressure force at (2) in y-direction

$$= -p_2 A_2 \sin 45^\circ = -39.24 \times 10^4 \times .07068 \times .7071$$

$$\therefore F_y = 1000 \times .25[0 - 3.537 \times .7071] + 0 + (-39.24 \times 10^4 \times .07068 \times .7071)$$

$$= -625.2 - 19611.1 = -20236.3 \text{ N}$$

-ve sign means  $F_y$  is acting in the downward direction

$$\therefore \text{Resultant force, } F_R = \sqrt{F_x^2 + F_y^2}$$

$$= \sqrt{48855.4^2 + 20236.3^2}$$

$$= 52880.6 \text{ N. Ans.}$$

The direction of the resultant force  $F_R$ , with the x-axis is given as

$$\tan \theta = \frac{F_y}{F_x} = \frac{20236.3}{48855.4} = 0.4142$$

$$\therefore \theta = 22^\circ 30'. \text{ Ans.}$$

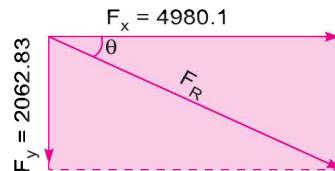


Fig. 6.22

**Problem 6.31** A 300 mm diameter pipe carries water under a head of 20 metres with a velocity of 3.5 m/s. If the axis of the pipe turns through  $45^\circ$ , find the magnitude and direction of the resultant force at the bend.

**Solution.** Given :

Dia. of bend,

$$D = D_1 = D_2 = 300 \text{ mm} = 0.30 \text{ m}$$

∴ Area,

$$A = A_1 = A_2 = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

Velocity,

$$V = V_1 = V_2 = 3.5 \text{ m/s}$$

$$\theta = 45^\circ$$

Discharge,

$$Q = A \times V = 0.07068 \times 3.5 = 0.2475 \text{ m}^3/\text{s}$$

Pressure head

$$= 20 \text{ m of water} \quad \text{or} \quad \frac{p}{\rho g} = 20 \text{ m of water}$$

∴

$$p = 20 \times \rho g = 20 \times 1000 \times 9.81 \text{ N/m}^2 = 196200 \text{ N/m}^2$$

∴ Pressure intensity,

$$p = p_1 = p_2 = 196200 \text{ N/m}^2$$

Now

$$V_{1x} = 3.5 \text{ m/s}, V_{2x} = V_2 \cos 45^\circ = 3.5 \times .7071$$

$$V_{1y} = 0, V_{2y} = V_2 \sin 45^\circ = 3.5 \times .7071$$

$$(p_1 A_1)_x = p_1 A_1 = 196200 \times .07068, (p_1 A_1)_y = 0$$

$$(p_2 A_2)_x = -p_2 A_2 \cos 45^\circ, (p_2 A_2)_y = -p_2 A_2 \sin 45^\circ$$

Force along x-axis,

$$F_x = \rho Q [V_{1x} - V_{2x}] + (p_1 A_1)_x + (p_2 A_2)_x$$

$$= 1000 \times .2475[3.5 - 3.5 \times .7071] + 196200 \times .07068 - p_2 A_2 \times \cos 45^\circ$$

$$\begin{aligned}
 &= 253.68 + 196200 \times .07068 - 196200 \times .07068 \times 0.7071 \\
 &= 253.68 + 13871.34 - 9808.04 = 4316.98 \text{ N}
 \end{aligned}$$

Force along y-axis,

$$\begin{aligned}
 F_y &= \rho Q [V_1 y - V_2 y] + (p_1 A_1)_y + (p_2 A_2)_y \\
 &= 1000 \times .2475[0 - 3.5 \times .7071] + 0 + [- p_2 A_2 \sin 45^\circ] \\
 &= - 612.44 - 196200 \times .07068 \times .7071 \\
 &= - 612.44 - 9808 = - 10420.44 \text{ N}
 \end{aligned}$$

∴ Resultant force

$$F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(4316.98)^2 + (10420.44)^2} = 11279 \text{ N. Ans.}$$

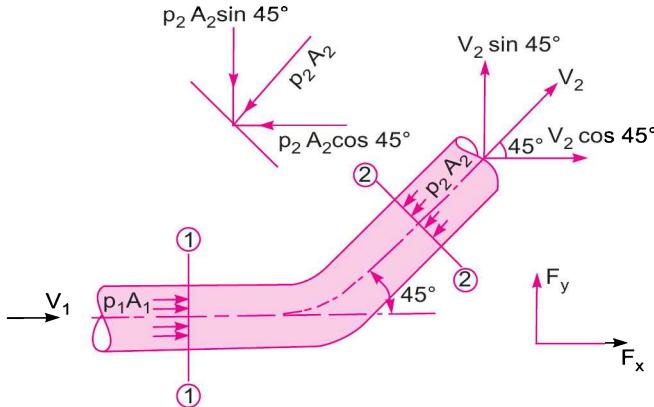


Fig. 6.23

The angle made by  $F_R$  with  $x$ -axis

$$\tan \theta = \frac{F_y}{F_x} = \frac{10420.44}{4316.98} = 2.411$$

$$\therefore \theta = \tan^{-1} 2.411 = 67^\circ 28'. \text{ Ans.}$$

**Problem 6.32** In a  $45^\circ$  bend a rectangular air duct of  $1 \text{ m}^2$  cross-sectional area is gradually reduced to  $0.5 \text{ m}^2$  area. Find the magnitude and direction of the force required to hold the duct in position if the velocity of flow at the  $1 \text{ m}^2$  section is  $10 \text{ m/s}$ , and pressure is  $2.943 \text{ N/cm}^2$ . Take density of air as  $1.16 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Area at section (1), } A_1 = 1 \text{ m}^2$$

$$\text{Area at section (2), } A_2 = 0.5 \text{ m}^2$$

$$\text{Velocity at section (1), } V_1 = 10 \text{ m/s}$$

$$\text{Pressure at section (1), } p_1 = 2.943 \text{ N/cm}^2 = 2.943 \times 10^4 \text{ N/m}^2 = 29430 \text{ N/m}^2$$

$$\text{Density of air, } \rho = 1.16 \text{ kg/m}^3$$

Applying continuity equation at sections (1) and (2)

$$A_1 V_1 = A_2 V_2$$

$$\therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{1}{0.5} \times 10 = 20 \text{ m/s}$$

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Discharge

$$Q = A_1 V_1 = 1 \times 10 = 10 \text{ m}^3/\text{s}$$

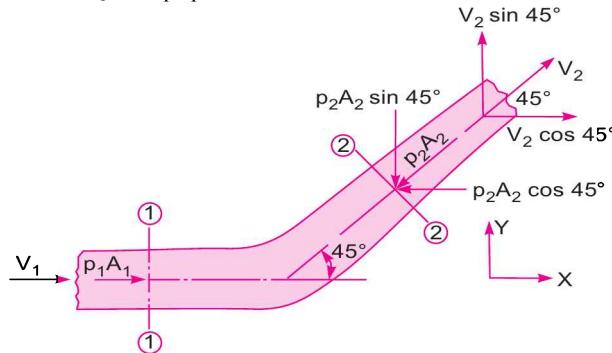


Fig. 6.24

Applying Bernoulli's equation at sections (1) and (2)

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad \{\because Z_1 = Z_2\}$$

$$\text{or } \frac{2.943 \times 10^4}{1.16 \times 9.81} + \frac{10^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{20^2}{2 \times 9.81}$$

$$\therefore \frac{p_2}{\rho g} = \frac{2.943 \times 10^4}{1.16 \times 9.81} + \frac{10^2}{2 \times 9.81} - \frac{20^2}{2 \times 9.81} \\ = 2586.2 + 5.0968 - 20.387 = 2570.90 \text{ m}$$

$$\therefore p_2 = 2570.90 \times 1.16 \times 9.81 = 29255.8 \text{ N}$$

$$\text{Force along } x\text{-axis, } F_x = \rho Q [V_{1x} - V_{2x}] + (p_1 A_1)_x + (p_2 A_2)_x$$

$$\text{where } A_{1x} = 10 \text{ m/s, } V_{2x} = V_2 \cos 45^\circ = 20 \times .7071,$$

$$(p_1 A_1)_x = p_1 A_1 = 29430 \times 1 = 29430 \text{ N}$$

$$\text{and } (p_2 A_2)_x = -p_2 A_2 \cos 45^\circ = -29255.8 \times 0.5 \times .7071$$

$$\therefore F_x = 1.16 \times 10[10 - 20 \times .7071] + 29430 \times 1 - 29255.8 \times .5 \times .7071 \\ = -48.04 + 29430 - 10343.37 = 0 - 19038.59 \text{ N}$$

$$\text{Similarly force along } y\text{-axis, } F_y = \rho Q [V_{1y} - V_{2y}] + (p_1 A_1)_y + (p_2 A_2)_y$$

$$\text{where } V_{1y} = 0, V_{2y} = V_2 \sin 45^\circ = 20 \times .7071 = 14.142$$

$$(p_1 A_1)_y = 0 \text{ and } (p_2 A_2)_y = -p_2 A_2 \sin 45^\circ = -29255.8 \times .5 \times .7071 = -10343.37$$

$$F_y = 1.16 \times 10 [0 - 14.142] + 0 - 10343.37 \\ = -164.05 - 10343.37 = -10507.42 \text{ N}$$

$$\therefore \text{Resultant force, } F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(19038.6)^2 + (10507.42)^2} = 21746.6 \text{ N. Ans.}$$

The direction of  $F_R$  with  $x$ -axis is given as

$$\tan \theta = \frac{F_y}{F_x} = \frac{10507.42}{19038.6} = 0.5519$$

$$\therefore \theta = \tan^{-1} 0.5519 = 28^\circ 53'. \text{ Ans.}$$

$F_R$  is the force exerted on bend. Hence the force required to hold the duct in position is equal to 21746.6 N but it is acting in the opposite direction of  $F_R$ . Ans.

**Problem 6.33** A pipe of 300 mm diameter conveying  $0.30 \text{ m}^3/\text{s}$  of water has a right angled bend in a horizontal plane. Find the resultant force exerted on the bend if the pressure at inlet and outlet of the bend are  $24.525 \text{ N/cm}^2$  and  $23.544 \text{ N/cm}^2$ .

**Solution.** Given :

Dia. of bend,

$$D = 300 \text{ mm} = 0.3 \text{ m}$$

∴ Area,

$$A = A_1 = A_2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$$

∴ Discharge,

$$Q = 0.30 \text{ m}^3/\text{s}$$

∴ Velocity,

$$V = V_1 = V_2 = \frac{Q}{A} = \frac{0.30}{0.07068} = 4.244 \text{ m/s}$$

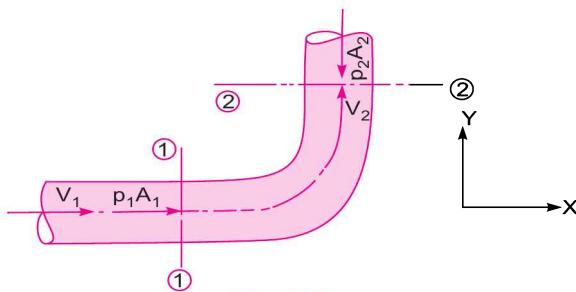


Fig. 6.25

Angle of bend,

$$\theta = 90^\circ$$

$$p_1 = 24.525 \text{ N/cm}^2 = 24.525 \times 10^4 \text{ N/m}^2 = 245250 \text{ N/m}^2$$

$$p_2 = 23.544 \text{ N/cm}^2 = 23.544 \times 10^4 \text{ N/m}^2 = 235440 \text{ N/m}^2$$

Force on bend along  $x$ -axis  $F_x = \rho Q [V_{1x} - V_{2x}] + (p_1 A_1)_x + (p_2 A_2)_x$

where  $\rho = 1000$ ,  $V_{1x} = V_1 = 4.244 \text{ m/s}$ ,  $V_{2x} = 0$

$$(p_1 A_1)_x = p_1 A_1 = 245250 \times 0.07068$$

$$(p_2 A_2)_x = 0$$

∴

$$F_x = 1000 \times 0.30 [4.244 - 0] + 245250 \times 0.07068 + 0$$

$$= 1273.2 + 17334.3 = 18607.5 \text{ N}$$

Force on bend along  $y$ -axis,  $F_y = \rho Q [V_{1y} - V_{2y}] + (p_1 A_1)_y + (p_2 A_2)_y$

where  $V_{1y} = 0$ ,  $V_{2y} = V_2 = 4.244 \text{ m/s}$

$$(p_1 A_1)_y = 0$$
,  $(p_2 A_2)_y = -p_2 A_2 = -235440 \times 0.07068 = -16640.9$

∴

$$F_y = 1000 \times 0.30[0 - 4.244] + 0 - 16640.9$$

$$= -1273.2 - 16640.9 = -17914.1 \text{ N}$$

∴ Resultant force,  $F_R = \sqrt{F_x^2 + F_y^2} = \sqrt{(18607.5)^2 + (17914.1)^2} = 25829.3 \text{ N}$

and

$$\tan \theta = \frac{F_y}{F_x} = \frac{17914.1}{18607.5} = 0.9627$$

∴

$$\theta = 43^\circ 54'. \text{ Ans.}$$

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**Problem 6.34** A nozzle of diameter 20 mm is fitted to a pipe of diameter 40 mm. Find the force exerted by the nozzle on the water which is flowing through the pipe at the rate of 1.2 m<sup>3</sup>/minute.

**Solution.** Given :

$$\text{Dia. of pipe, } D_1 = 40 \text{ mm} = 40 \times 10^{-3} \text{ m} = .04 \text{ m}$$

$$\therefore \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.04)^2 = 0.001256 \text{ m}^2$$

$$\text{Dia. of nozzle, } D_2 = 20 \text{ mm} = 0.02 \text{ m}$$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (.02)^2 = .000314 \text{ m}^2$$

$$\text{Discharge, } Q = 1.2 \text{ m}^3/\text{minute} = \frac{1.2}{60} \text{ m}^3/\text{s} = 0.02 \text{ m}^3/\text{s}$$

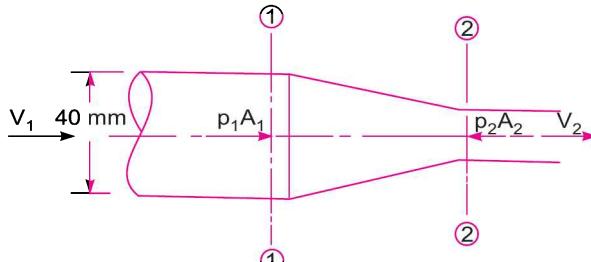


Fig. 6.26

Applying continuity equation at sections (1) and (2),

$$A_1 V_1 = A_2 V_2 = Q$$

$$\therefore V_1 = \frac{Q}{A_1} = \frac{0.2}{.001256} = 15.92 \text{ m/s}$$

$$\text{and } V_2 = \frac{Q}{A_2} = \frac{0.2}{.000314} = 63.69 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\text{Now } z_1 = z_2, \frac{p_2}{\rho g} = \text{atmospheric pressure} = 0$$

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{V_2^2}{2g}$$

$$\therefore \frac{p_1}{\rho g} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} = \frac{(63.69^2)}{2 \times 9.81} - \frac{(15.92^2)}{2 \times 9.81} = 206.749 - 12.917 \\ = 193.83 \text{ m of water}$$

$$\therefore p_1 = 193.83 \times 1000 \times 9.81 \frac{\text{N}}{\text{m}^2} = 1901472 \frac{\text{N}}{\text{m}^2}$$

Let the force exerted by the nozzle on water =  $F_x$

Net force in the direction of  $x$  = rate of change of momentum in the direction of  $x$

$$\therefore p_1 A_1 - p_2 A_2 + F_x = \rho Q (V_2 - V_1)$$

where  $p_2$  = atmospheric pressure = 0 and  $\rho = 1000$

$$\therefore 1901472 \times .001256 - 0 + F_x = 1000 \times 0.02(63.69 - 15.92) \text{ or } 2388.24 + F_x = 916.15$$

$$\therefore F_x = -2388.24 + 916.15 = -1472.09. \text{ Ans.}$$

-ve sign indicates that the force exerted by the nozzle on water is acting from right to left.

**Problem 6.35** The diameter of a pipe gradually reduces from 1 m to 0.7 m as shown in Fig. 6.27. The pressure intensity at the centre-line of 1 m section  $7.848 \text{ kN/m}^2$  and rate of flow of water through the pipe is 600 litres/s. Find the intensity of pressure at the centre-line of 0.7 m section. Also determine the force exerted by flowing water on transition of the pipe.

**Solution.** Given :

Dia. of pipe at section 1,  $D_1 = 1 \text{ m}$

$$\therefore \text{Area, } A_1 = \frac{\pi}{4} (1)^2 = 0.7854 \text{ m}^2$$

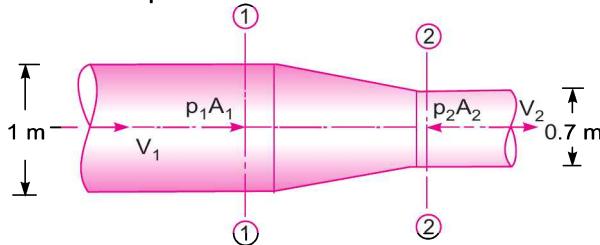


Fig. 6.27

Dia. of pipe at section 2,  $D_2 = 0.7 \text{ m}$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (0.7)^2 = 0.3848 \text{ m}^2$$

Pressure at section 1,  $p_1 = 7.848 \text{ kN/m}^2 = 7848 \text{ N/m}^2$

$$\text{Discharge, } Q = 600 \text{ litres/s} = \frac{600}{1000} = 0.6 \text{ m}^3/\text{s}$$

Applying continuity equation,

$$A_1 V_1 = A_2 V_2 = Q$$

$$\therefore V_1 = \frac{Q}{A_2} = \frac{0.6}{0.7854} = 0.764 \text{ m/s}$$

$$V_2 = \frac{Q}{A_1} = \frac{0.6}{0.3848} = 1.55 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2),

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad \{ \because \text{pipe is horizontal, } \therefore z_1 = z_2 \}$$

$$\text{or } \frac{7848}{1000 \times 9.81} + \frac{(0.764)^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{(1.55)^2}{2 \times 9.81}$$

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$$\therefore \frac{p_2}{\rho g} = 0.8 + \frac{(0.764)^2}{2 \times 9.81} - \frac{(1.55)^2}{2 \times 9.81}$$

$$= 0.8 + 0.0297 - 0.122 = 0.7077 \text{ m of water}$$

$$\therefore p_2 = 0.7077 \times 9.81 \times 1000$$

$$= 6942.54 \text{ N/m}^2 \text{ or } 6.942 \text{ kN/m}^2. \text{ Ans.}$$

Let  $F_x$  = the force exerted by pipe transition on the flowing water in the direction of flow

Then net force in the direction of flow = rate of change of momentum in the direction of flow

$$\text{or } p_1 A_1 - p_2 A_2 + F_x = \rho(V_2 - V_1)$$

$$\therefore 7848 \times 0.7854 - 6942.54 \times 0.3848 + F_x = 1000 \times 0.6[1.55 - 0.764]$$

$$\text{or } 6163.8 - 2671.5 + F_x = 471.56$$

$$\therefore F_x = 471.56 - 6163.8 + 2671.5 = -3020.74 \text{ N}$$

$\therefore$  The force exerted by water on pipe transition

$$= -F_x = -(-3020.74) = 3020.74 \text{ N. Ans.}$$

### ► 6.9 MOMENT OF MOMENTUM EQUATION

Moment of momentum equation is derived from moment of momentum principle which states that the resulting torque acting on a rotating fluid is equal to the rate of change of moment of momentum.

Let

$V_1$  = velocity of fluid at section 1,

$r_1$  = radius of curvature at section 1,

$Q$  = rate of flow of fluid,

$\rho$  = density of fluid,

and

$V_2$  and  $r_2$  = velocity and radius of curvature at section 2

Momentum of fluid at section 1 = mass  $\times$  velocity =  $\rho Q \times V_1/s$

$\therefore$  Moment of momentum per second at section 1,

$$= \rho Q \times V_1 \times r_1$$

Similarly moment of momentum per second of fluid at section 2

$$= \rho Q \times V_2 \times r_2$$

$\therefore$  Rate of change of moment of momentum

$$= \rho Q V_2 r_2 - \rho Q V_1 r_1 = \rho Q [V_2 r_2 - V_1 r_1]$$

According to moment of momentum principle

Resultant torque = rate of change of moment of momentum

or

$$T = \rho Q [V_2 r_2 - V_1 r_1] \quad \dots(6.23)$$

Equation (6.23) is known as moment of momentum equation. This equation is applied :

1. For analysis of flow problems in turbines and centrifugal pumps.

2. For finding torque exerted by water on sprinkler.

**Problem 6.36** A lawn sprinkler with two nozzles of diameter 4 mm each is connected across a tap of water as shown in Fig. 6.28. The nozzles are at a distance of 30 cm and 20 cm from the centre of the tap. The rate of flow of water through tap is  $120 \text{ cm}^3/\text{s}$ . The nozzles discharge water in the downward direction. Determine the angular speed at which the sprinkler will rotate free.

**Solution.** Given :

Dia. of nozzles A and B,

$$D = D_A = D_B = 4 \text{ mm} = .004 \text{ m}$$

∴ Area,

$$A = \frac{\pi}{4} (.004)^2 = .00001256 \text{ m}^2$$

Discharge

$$Q = 120 \text{ cm}^3/\text{s}$$

Assuming the discharge to be equally divided between the two nozzles, we have

$$Q_A = Q_B = \frac{Q}{2} = \frac{120}{2} = 60 \text{ cm}^3/\text{s} = 60 \times 10^{-6} \text{ m}^3/\text{s}$$

∴ Velocity of water at the outlet of each nozzle,

$$V_A = V_B = \frac{Q_A}{A} = \frac{60 \times 10^{-6}}{.00001256} = 4.777 \text{ m/s.}$$

The jet of water coming out from nozzles A and B is having velocity 4.777 m/s. These jets of water will exert force in the opposite direction, i.e., force exerted by the jets will be in the upward direction. The torque exerted will also be in the opposite direction. Hence torque at B will be in the anti-clockwise direction and at A in the clockwise direction. But torque at B is more than the torque at A and hence sprinkle, if free, will rotate in the anti-clockwise direction as shown in Fig. 6.28.

Let  $\omega$  = angular velocity of the sprinkler.

Then absolute velocity of water at A,

$$V_1 = V_A + \omega \times r_A$$

where  $r_A$  = distance of nozzle A from the centre of tap

$$= 20 \text{ cm} = 0.2 \text{ m} \quad \{ \omega \times r_A = \text{tangential velocity due to rotation} \}$$

$$V_1 = (4.777 + \omega \times 0.2) \text{ m/s}$$

Here  $\omega \times r_A$  is added to  $V_A$  as  $V_A$  and tangential velocity due to rotation ( $\omega \times r_A$ ) are in the same direction as shown in Fig. 6.28.

Similarly, absolute velocity of water at B,

$$V_2 = V_B - \text{tangential velocity due to rotation}$$

$$= 4.777 - \omega \times r_B \quad \{ \text{where } r_B = 30 \text{ cm} = 0.3 \text{ m} \}$$

$$= (4.777 - \omega \times 0.3)$$

Now applying equation (6.23), we get

$$\begin{aligned} T &= \rho Q [V_2 r_2 - V_1 r_1] \\ &= \rho Q_A [V_2 r_B - V_1 r_A] \\ &= 1000 \times 60 \times 10^{-6} [(4.777 \times 0.3 \omega) \times 0.3 - (4.777 + 0.2 \omega) \times 0.2] \end{aligned} \quad \begin{array}{l} \text{Here } r_2 = r_B, r_1 = r_A \\ Q = Q_A = Q_B \end{array}$$

The moment of momentum of the fluid entering sprinkler is given zero and also there is no external torque applied on the sprinkler. Hence resultant external torque is zero, i.e.,  $T = 0$

$$\therefore 1000 \times 60 \times 10^{-6} [(4.777 - 0.3 \omega) \times 0.3 - (4.777 + 0.2 \omega) \times 0.2] = 0$$

$$\text{or} \quad (4.777 - 0.3 \omega) \times 0.3 - (4.777 + 0.2 \omega) \times 0.2 = 0$$

$$\text{or} \quad 4.777 \times 0.3 - 0.09 \omega - 4.777 \times 0.2 - 0.04 \omega = 0$$

$$\text{or} \quad 0.1 \times 4.777 = (0.09 + 0.04)\omega = .13 \omega$$

$$\therefore \omega = \frac{4.777}{0.13} = 3.6746 \text{ rad/s. Ans.}$$

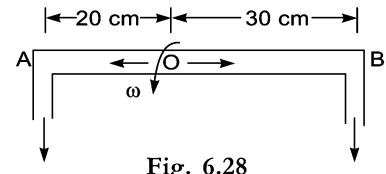


Fig. 6.28

**Problem 6.37** A lawn sprinkler shown in Fig. 6.29 has 0.8 cm diameter nozzle at the end of a rotating arm and discharges water at the rate of 10 m/s velocity. Determine the torque required to hold the rotating arm stationary. Also determine the constant speed of rotation of the arm, if free to rotate.

**Solution.** Dia. of each nozzle = 0.8 cm = .008 m

$$\therefore \text{Area of each nozzle} = \frac{\pi}{4} (.008)^2 = .00005026 \text{ m}^2$$

Velocity of flow at each nozzle = 10 m/s.

$\therefore$  Discharge through each nozzle,

$$\begin{aligned} Q &= \text{Area} \times \text{Velocity} \\ &= .00005026 \times 10 = .0005026 \text{ m}^3/\text{s} \end{aligned}$$

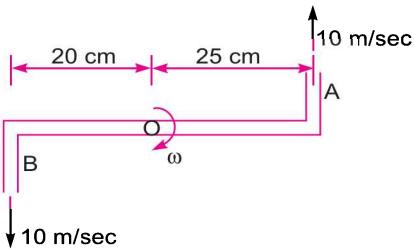


Fig. 6.29

Torque exerted by water coming through nozzle A on the sprinkler = moment of momentum of water through A

$$= r_A \times \rho \times Q \times V_A = 0.25 \times 1000 \times .0005026 \times 10 \text{ clockwise}$$

Torque exerted by water coming through nozzle B on the sprinkler

$$= r_B \times \rho \times Q \times V_B = 0.20 \times 1000 \times .0005026 \times 10 \text{ clockwise}$$

$\therefore$  Total torque exerted by water on sprinkler

$$\begin{aligned} &= .25 \times 1000 \times .0005026 \times 10 + .20 \times 1000 \times .0005026 \times 10 \\ &= 1.2565 + 1.0052 = 2.26 \text{ Nm} \end{aligned}$$

$\therefore$  Torque required to hold the rotating arm stationary = Torque exerted by water on sprinkler  
= 2.26 Nm. Ans.

#### Speed of rotation of arm, if free to rotate

Let  $\omega$  = speed of rotation of the sprinkler

The absolute velocity of flow of water at the nozzles A and B are

$$V_1 = 10.0 - 0.25 \times \omega \text{ and } V_2 = 10.0 - 0.20 \times \omega$$

Torque exerted by water coming out at A, on sprinkler

$$\begin{aligned} &= r_A \times \rho \times Q \times V_1 = 0.25 \times 1000 \times .0005026 \times (10 - 0.25 \omega) \\ &= 0.12565 (10 - 0.25 \omega) \end{aligned}$$

Torque exerted by water coming out at B, on sprinkler

$$\begin{aligned} &= r_B \times \rho \times Q \times V_2 = 0.20 \times 1000 \times .0005026 \times (10.0 - 0.2 \omega) \\ &= 0.10052 (10.0 - 0.2 \omega) \end{aligned}$$

$\therefore$  Total torque exerted by water = 0.12565 (10.0 - 0.25  $\omega$ ) + 0.10052 (10.0 - 0.2  $\omega$ )

Since moment of momentum of the flow entering is zero and no external torque is applied on sprinkler, so the resultant torque on the sprinkler must be zero.

$$\therefore 0.12565 (10.0 - 0.25 \omega) + 0.10052 (10.0 - 0.2 \omega) = 0$$

$$1.2565 - 0.0314 \omega + 1.0052 - 0.0201 \omega = 0$$

$$1.2565 + 1.0052 = \omega (0.0314 + 0.0201)$$

$$2.2617 = 0.0515 \omega$$

$$\therefore \omega = \frac{2.2617}{0.0515} = 43.9 \text{ rad/s. Ans.}$$

and

$$N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 43.9}{2\pi} = 419.2 \text{ r.p.m. Ans.}$$

### ► 6.10 FREE LIQUID JETS

Free liquid jet is defined as the jet of water coming out from the nozzle in atmosphere. The path travelled by the free jet is parabolic.

Consider a jet coming from the nozzle as shown in Fig. 6.30. Let the jet at A, makes an angle  $\theta$  with the horizontal direction. If  $U$  is the velocity of jet of water, then the horizontal component and vertical component of this velocity at A are  $U \cos \theta$  and  $U \sin \theta$ .

Consider another point  $P(x, y)$  on the centre line of the jet. The co-ordinates of  $P$  from A are  $x$  and  $y$ . Let the velocity of jet at  $P$  in the  $x$ - and  $y$ -directions are  $u$  and  $v$ . Let a liquid particle takes time ' $t$ ' to reach from A to P. Then the horizontal and vertical distances travelled by the liquid particle in time ' $t$ ' are :

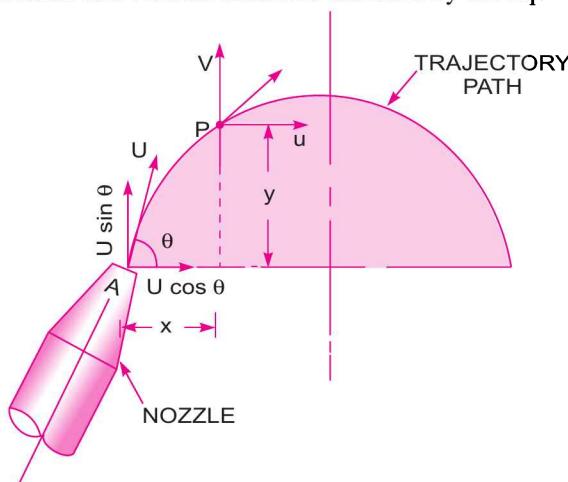


Fig. 6.30 Free liquid jet.

$$\begin{aligned} x &= \text{velocity component in } x\text{-direction} \times t \\ &= U \cos \theta \times t \end{aligned} \quad \dots(i)$$

and

$$\begin{aligned} y &= (\text{vertical component in } y\text{-direction} \times \text{time} - \frac{1}{2} gt^2) \\ &= U \sin \theta \times t - \frac{1}{2} gt^2 \end{aligned} \quad \dots(ii)$$

$\{\because$  Horizontal component of velocity is constant while the vertical distance is affected by gravity}

$$\text{From equation (i), the value of } t \text{ is given as } t = \frac{x}{U \cos \theta}$$

Substituting this value in equation (ii)

$$\begin{aligned} y &= U \sin \theta \times \frac{x}{U \cos \theta} - \frac{1}{2} \times g \times \left( \frac{x}{U \cos \theta} \right)^2 = x \frac{\sin \theta}{\cos \theta} - \frac{gx^2}{2U^2 \cos^2 \theta} \\ &= x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta \quad \left\{ \because \frac{1}{\cos^2 \theta} = \sec^2 \theta \right\} \dots(6.24) \end{aligned}$$

Equation (6.24) gives the variation of  $y$  with the square of  $x$ . Hence this is the equation of a parabola. Thus the path travelled by the free jet in atmosphere is parabolic.

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(i) **Maximum height attained by the jet.** Using the relation  $V_2^2 - V_1^2 = -2gS$ , we get in this case  $V_1 = 0$  at the highest point

$$\begin{aligned}V_1 &= \text{Initial vertical component} \\&= U \sin \theta\end{aligned}$$

-ve sign on right hand side is taken as  $g$  is acting in the downward direction but particles is moving up.

$$\therefore 0 - (U \sin \theta)^2 = -2g \times S$$

where  $S$  is the maximum vertical height attained by the particle.

or

$$-U^2 \sin^2 \theta = -2gS$$

$$\therefore S = \frac{U^2 \sin^2 \theta}{2g} \quad \dots(6.25)$$

(ii) **Time of flight.** It is the time taken by the fluid particle in reaching from  $A$  to  $B$  as shown in Fig. 6.30. Let  $T$  is the time of flight.

Using equation (ii), we have  $y = U \sin \theta \times t - \frac{1}{2} gt^2$

when the particle reaches at  $B$ ,  $y = 0$  and  $t = T$

$$\therefore \text{Above equation becomes as } 0 = U \sin \theta \times T - \frac{1}{2} g \times T^2$$

or

$$0 = U \sin \theta - \frac{1}{2} gT \quad \{\text{Cancelling } T\}$$

or

$$T = \frac{2U \sin \theta}{g} \quad \dots(6.26)$$

(iii) **Time to reach highest point.** The time to reach highest point is half the time of flight. Let  $T^*$  is the time to reach highest point, then

$$T^* = \frac{T}{2} = \frac{2U \sin \theta}{g \times 2} = \frac{U \sin \theta}{g} \quad \dots(6.27)$$

(iv) **Horizontal range of the jet.** The total horizontal distance travelled by the fluid particle is called horizontal range of the jet, i.e., the horizontal distance  $AB$  in Fig. 6.30 is called horizontal range of the jet. Let this range is denoted by  $x^*$ .

Then

$x^*$  = velocity component in  $x$ -direction

$$\begin{aligned}&\times \text{time taken by the particle to reach from } A \text{ to } B \\&= U \cos \theta \times \text{Time of flight}\end{aligned}$$

$$= U \cos \theta \times \frac{2U \sin \theta}{g} \quad \left\{ \because T = \frac{2U \sin \theta}{g} \right\}$$

$$= \frac{U^2}{g} 2 \cos \theta \sin \theta = \frac{U^2}{g} \sin 2\theta \quad \dots(6.28)$$

(v) **Value of  $\theta$  for maximum range.** The range  $x^*$  will be maximum for a given velocity of projection ( $U$ ), when  $\sin 2\theta$  is maximum

or when

$$\sin 2\theta = 1 \text{ or } \sin 2\theta = \sin 90^\circ = 1$$

$$\therefore 2\theta = 90^\circ \text{ or } \theta = 45^\circ$$

$$\text{Then maximum range, } x_{\max}^* = \frac{U^2}{g} \sin^2 \theta = \frac{U^2}{g} \quad \{ \because \sin 90^\circ = 1 \} \dots(6.29)$$

**Problem 6.38** A vertical wall is of 8 m in height. A jet of water is coming out from a nozzle with a velocity of 20 m/s. The nozzle is situated at a distance of 20 m from the vertical wall. Find the angle of projection of the nozzle to the horizontal so that the jet of water just clears the top of the wall.

**Solution.** Given :

$$\text{Height of wall} = 8 \text{ m}$$

$$\text{Velocity of jet, } U = 20 \text{ m/s}$$

$$\text{Distance of jet from wall, } x = 20 \text{ m}$$

$$\text{Let the required angle} = \theta$$

Using equation (6.24), we have

$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

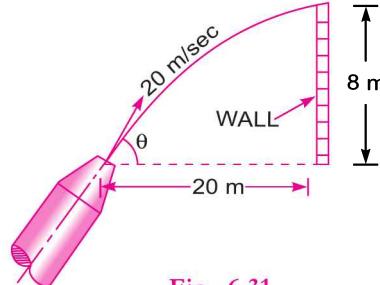


Fig. 6.31

where  $y = 8 \text{ m}$ ,  $x = 20 \text{ m}$ ,  $U = 20 \text{ m/s}$

$$\begin{aligned} 8 &= 20 \tan \theta - \frac{9.81 \times 20^2}{2 \times 20^2} \sec^2 \theta \\ &= 20 \tan \theta - 4.905 \sec^2 \theta \\ &= 20 \tan \theta - 4.905 [1 + \tan^2 \theta] \quad \{\because \sec^2 \theta = 1 + \tan^2 \theta\} \\ &= 20 \tan \theta - 4.905 - 4.905 \tan^2 \theta \end{aligned}$$

$$\text{or } 4.905 \tan^2 \theta - 20 \tan \theta + 8 + 4.905 = 0$$

$$\text{or } 4.905 \tan^2 \theta - 20 \tan \theta + 12.905 = 0$$

$$\therefore \tan \theta = \frac{20 \pm \sqrt{20^2 - 4 \times 12.905 \times 4.905}}{2 \times 4.905} = \frac{20 \pm \sqrt{400 - 253.19}}{9.81}$$

$$= \frac{20 \pm \sqrt{146.81}}{9.81} = \frac{20 \pm 12.116}{9.81} = \frac{32.116}{9.81} \text{ or } \frac{7.889}{9.81}$$

$$\therefore = 3.273 \text{ or } 0.8036$$

$$\therefore \theta = 73^\circ 0.8' \text{ or } 38^\circ 37'. \text{ Ans.}$$

**Problem 6.39** A fire-brigade man is holding a fire stream nozzle of 50 mm diameter as shown in Fig. 6.32. The jet issues out with a velocity of 13 m/s and strikes the window. Find the angle or angles of inclination with which the jet issues from the nozzle. What will be the amount of water falling on the window?

**Solution.** Given :

$$\text{Dia. of nozzle, } d = 50 \text{ mm} = .05 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} (.05)^2 = 0.001963 \text{ m}^2$$

$$\text{Velocity of jet, } U = 13 \text{ m/s.}$$

The jet is coming out from nozzle at A. It strikes the window and let the angle made by the jet at A with horizontal is equal to  $\theta$ .

The co-ordinates of window, with respect to origin at A.

$$x = 5 \text{ m}, y = 7.5 - 1.5 = 6.0 \text{ m}$$

The equation of the jet is given by (6.24) as

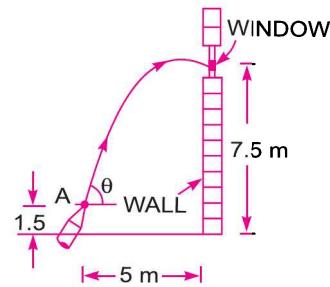


Fig. 6.32

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$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

or  $6.0 = 5 \times \tan \theta - \frac{9.81 \times 5}{2 \times 13^2} [1 + \tan^2 \theta] \quad \{\because \sec^2 \theta = 1 + \tan^2 \theta\}$

or  $6.0 = 5 \tan \theta - .7256 (1 + \tan^2 \theta)$   
 $= 5 \tan \theta - .7256 - .7256 \tan^2 \theta$

or  $0.7256 \tan^2 \theta - 5 \tan \theta + 6 + .7256 = 0$

or  $0.7256 \tan^2 \theta - 5 \tan \theta + 6.7256 = 0$

This is a quadratic equation in  $\tan \theta$ . Hence solution is

$$\begin{aligned} \tan \theta &= \frac{5 \pm \sqrt{5^2 - 4 \times .7256 \times 6.7256}}{2 \times .7256} \\ &= \frac{5 \pm \sqrt{25 - 19.52}}{1.4512} = \frac{5 \pm 2.341}{1.4512} = 5.058 \text{ or } 1.8322 \end{aligned}$$

∴  $\theta = \tan^{-1} 5.058 \text{ or } \tan^{-1} 1.8322 = 78.8^\circ \text{ or } 61.37^\circ. \text{ Ans.}$

Amount of water falling on window = Discharge from nozzle

$$\begin{aligned} &= \text{Area of nozzle} \times \text{Velocity of jet at nozzle} \\ &= 0.001963 \times U = 0.001963 \times 13.0 = 0.0255 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

**Problem 6.40** A nozzle is situated at a distance of 1 m above the ground level and is inclined at an angle of  $45^\circ$  to the horizontal. The diameter of the nozzle is 50 mm and the jet of water from the nozzle strikes the ground at a horizontal distance of 4 m. Find the rate of flow of water.

**Solution.** Given :

Distance of nozzle above ground = 1 m

Angle of inclination,  $\theta = 45^\circ$

Dia. of nozzle,  $d = 50 \text{ mm} = .05 \text{ m}$

$$\therefore \text{Area, } A = \frac{\pi}{4} (.05)^2 = .001963 \text{ m}^2$$

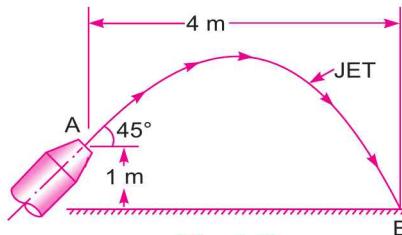


Fig. 6.33

The horizontal distance  $x = 4 \text{ m}$

The co-ordinates of the point B, which is on the centre-line of the jet of water and is situated on the ground, with respect to A (origin) are

$$x = 4 \text{ m and } y = -1.0 \text{ m } \{ \text{From A, point B is vertically down by 1 m} \}$$

The equation of the jet is given by (6.24) as  $y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$

Substituting the known values as

$$-1.0 = 4 \tan 45^\circ - \frac{9.81 \times 4^2}{2U^2} \times \sec^2 45^\circ$$

$$= 4 - \frac{78.48}{U^2} \times (\sqrt{2})^2 \quad \left\{ \sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2} \right\}$$

$$-1.0 = 4 - \frac{78.48 \times 2}{U^2} \quad \text{or} \quad \frac{78.48 \times 2}{U^2} = +4.0 + 1.0 = 5.0$$

$$\therefore U^2 = \frac{78.48 \times 2.0}{5.0} = 31.39$$

$$\therefore U = \sqrt{31.39} = 5.60 \text{ m/s}$$

$$\begin{aligned}\text{Now the rate of flow of fluid} &= \text{Area} \times \text{Velocity of jet} \\ &= A \times U = .001963 \times 5.6 \text{ m}^3/\text{sec} \\ &= 0.01099 \approx .011 \text{ m}^3/\text{s. Ans.}\end{aligned}$$

**Problem 6.41** A window, in a vertical wall, is at a distance of 30 m above the ground level. A jet of water, issuing from a nozzle of diameter 50 mm is to strike the window. The rate of flow of water through the nozzle is 3.5 m<sup>3</sup>/minute and nozzle is situated at a distance of 1 m above ground level. Find the greatest horizontal distance from the wall of the nozzle so that jet of water strikes the window.

**Solution.** Given :

Distance of window from ground level = 30 m

Dia. of nozzle,  $d = 50 \text{ mm} = 0.05 \text{ m}$

$$\therefore \text{Area} \quad A = \frac{\pi}{4}(0.05)^2 = 0.001963 \text{ m}^2$$

The discharge,  $Q = 3.5 \text{ m}^3/\text{minute}$

$$= \frac{3.5}{60} = 0.0583 \text{ m}^3/\text{s}$$

Distance of nozzle from ground = 1 m.

Let the greatest horizontal distance of the nozzle from the wall =  $x$  and let angle of inclination =  $\theta$ . If the jet reaches the window, then the point  $B$  on the window is on the centre-line of the jet. The co-ordinates of  $B$  with respect to  $A$  are

$$x = x, y = 30 - 1.0 = 29 \text{ m}$$

$$\text{The velocity of jet, } U = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{A} = \frac{0.0583}{0.001963} = 29.69 \text{ m/sec}$$

Using the equation (6.34), which is the equation of jet,

$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

or

$$\begin{aligned}29.0 &= x \tan \theta - \frac{9.81x^2}{2 \times (29.69)^2} \sec^2 \theta \\ &= x \tan \theta - 0.0055 \sec^2 \theta \times x^2 \\ &= x \tan \theta - \frac{0.0055 x^2}{\cos^2 \theta} \\ x \tan \theta - .0055 x^2 / \cos^2 \theta - 29 &= 0 \quad \dots(i)\end{aligned}$$

The maximum value of  $x$  with respect to  $\theta$  is obtained, by differentiating the above equation w.r.t.  $\theta$  and substituting the value of  $\frac{dx}{d\theta} = 0$ . Hence differentiating the equation (i) w.r.t.  $\theta$ , we have

$$\left[ x \sec^2 \theta + \tan \theta \times \frac{dx}{d\theta} \right] - 0.0055 \left[ x^2 \times \left( \frac{(-2)}{\cos^3 \theta} \right) (-\sin \theta) + \frac{1}{\cos^2 \theta} \times 2x \frac{dx}{d\theta} \right]$$

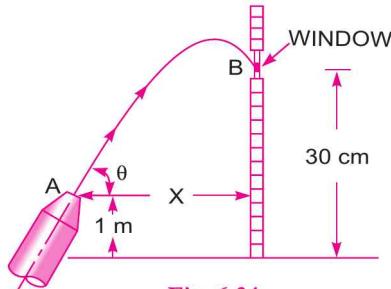


Fig. 6.34

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$$\left\{ \because \frac{d}{d\theta}(x \tan \theta) = x \sec^2 \theta + \tan \theta \frac{dx}{d\theta} \text{ and } \frac{d}{d\theta}\left(\frac{x^2}{\cos^2 \theta}\right) = x^2 \frac{d}{d\theta}\left(\frac{1}{\cos^2 \theta}\right) + \frac{1}{\cos^2 \theta} \frac{d}{d\theta}(x^2) \right\}$$

$$\therefore x \sec^2 \theta + \tan \theta \frac{dx}{d\theta} - .0055 \left[ \frac{2x^2 \sin \theta}{\cos^3 \theta} + \frac{2x}{\cos^2 \theta} \frac{dx}{d\theta} \right] = 0$$

For maximum value of  $x$ , w.r.t.  $\theta$ , we have  $\frac{dx}{d\theta} = 0$

Substituting this value in the above equation, we have

$$x \sec^2 \theta - .0055 \left[ \frac{2x^2 \sin \theta}{\cos^3 \theta} \right] = 0$$

$$\text{or } \frac{x}{\cos^2 \theta} - \frac{.0055 \times 2x^2 \sin \theta}{\cos^3 \theta} = 0 \quad \text{or } x - .011 \times x^2 \frac{\sin \theta}{\cos \theta} = 0$$

$$\text{or } x - .011 x^2 \tan \theta = 0 \quad \text{or } 1 - .011 x \tan \theta = 0$$

$$\text{or } x \tan \theta = \frac{1}{.011} = 90.9 \quad \dots(ii)$$

$$\text{or } x = \frac{90.9}{\tan \theta} \quad \dots(iii)$$

Substituting this value of  $x$  in equation (i), we get

$$\frac{90.9}{\tan \theta} \times \tan \theta - .0055 \times \frac{(90.9)^2}{\tan^2 \theta} \times \frac{1}{\cos^2 \theta} - 29 = 0$$

$$90.9 - \frac{45.445}{\sin^2 \theta} - 29 = 0 \quad \text{or } 61.9 - \frac{45.445}{\sin^2 \theta} = 0$$

$$\text{or } 61.9 = \frac{45.445}{\sin^2 \theta} \quad \text{or } \sin^2 \theta = \frac{45.445}{61.90} = 0.7341$$

$$\therefore \sin \theta = \sqrt{0.7341} = 0.8568$$

$$\therefore \theta = \tan^{-1} 0.8568 = 58^\circ 57.8'$$

Substituting this value of  $\theta$  in equation (iii), we get

$$\begin{aligned} x &= \frac{90.9}{\tan \theta} = \frac{90.9}{\tan 58^\circ 57.8'} = \frac{90.9}{\tan 58.95} = \frac{90.9}{1.66} = 54.759 \text{ m} \\ &= \mathbf{54.76 \text{ m. Ans.}} \end{aligned}$$

## HIGHLIGHTS

1. The study of fluid motion with the forces causing flow is called dynamics of fluid flow, which is analysed by the Newton's second law of motion.
2. Bernoulli's equation is obtained by integrating the Euler's equation of motion. Bernoulli's equation states "For a steady, ideal flow of an incompressible fluid, the total energy which consists of pressure energy, kinetic energy and datum energy, at any point of the fluid is constant". Mathematically,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

where  $\frac{p_1}{\rho g}$  = pressure energy per unit weight = pressure head

$\frac{v_1^2}{2g}$  = kinetic energy per unit weight = kinetic head

$z_1$  = datum energy per unit weight = datum head.

3. Bernoulli's equation for real fluids

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

where  $h_L$  = loss of energy between sections 1 and 2.

4. The discharge,  $Q$ , through a venturimeter or an orifice meter is given by

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

where  $a_1$  = area at the inlet of venturimeter,

$a_2$  = area at the throat of venturimeter,

$C_d$  = co-efficient of venturimeter,

$h$  = difference of pressure head in terms of fluid head flowing through venturimeter.

5. The value of  $h$  is given by differential U-tube manometer

$$h = x \left[ \frac{S_h}{S_o} - 1 \right] \quad \dots \text{(when differential manometer contains heavier liquid)}$$

$$h = x \left[ 1 - \frac{S_l}{S_o} \right] \quad \dots \text{(when differential manometer contains lighter liquid)}$$

$$h = \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = x \left[ \frac{S_h}{S_o} - 1 \right] \quad \dots \text{(for inclined venturimeter in which differential manometer contains heavier liquid)}$$

$$h = \left( \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{p_2}{\rho g} + z_2 \right) = x \left[ 1 - \frac{S_l}{S_o} \right] \quad \dots \text{(for inclined venturimeter in which differential manometer contains lighter liquid)}$$

where  $x$  = difference in the readings of differential manometer,

$S_h$  = sp. gr. of heavier liquid

$S_o$  = sp. gr. of fluid flowing through venturimeter

$S_l$  = sp. gr. of lighter liquid.

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6. Pitot-tube is used to find the velocity of a flowing fluid at any point in a pipe or a channel. The velocity is given by the relation

$$V = C_v \sqrt{2gh}$$

where  $C_v$  = co-efficient of Pitot-tube

$h$  = rise of liquid in the tube above free surface of liquid

$$= x \left[ \frac{S_g}{S_o} - 1 \right] \text{ (for pipes or channels).}$$

7. The momentum equation states that the net force acting on a fluid mass is equal to the change in momentum per second in that direction. This is given as  $F = \frac{d}{dt}(mv)$

The impulse-momentum equation is given by  $F \cdot dt = d(mv)$ .

8. The force exerted by a fluid on a pipe bend in the directions of  $x$  and  $y$  are given by

$$F_x = \frac{\text{mass}}{\text{sec}} (\text{Initial velocity in the direction of } x - \text{Final velocity in } x\text{-direction}) \\ + \text{Initial pressure force in } x\text{-direction} + \text{Final pressure force in } x\text{-direction}$$

and

$$F_y = \rho Q [V_{1y} - V_{2y}] + (p_1 A_1)_y + (p_2 A_2)_y$$

Resultant force,

$$F_R = \sqrt{F_x^2 + F_y^2}$$

and the direction of the resultant with horizontal is  $\tan \theta = \frac{F_y}{F_x}$ .

9. The force exerted by the nozzle on the water is given by  $F_x = \rho Q [V_{2x} - V_{1x}]$  and force exerted by the water on the nozzle is  $-F_x = \rho Q [V_{1x} - V_{2x}]$ .

10. Moment of momentum equation states that the resultant torque acting on a rotating fluid is equal to the rate of change of moment of momentum. Mathematically, it is given by  $T = \rho Q [V_2 r_2 - V_1 r_1]$ .

11. Free liquid jet is the jet of water issuing from a nozzle in atmosphere. The path travelled by the free jet is parabolic. The equation of the jet is given by

$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

where  $x, y$  = co-ordinates of any point on jet w.r.t. to the nozzle

$U$  = velocity of jet of water issuing from nozzle

$\theta$  = inclination of jet issuing from nozzle with horizontal.

12. (i) Maximum height attained by jet =  $\frac{U^2 \sin^2 \theta}{2g}$

$$(ii) \text{ Time of flight, } T = \frac{2U \sin \theta}{g}$$

$$(iii) \text{ Time to reach highest point, } T^* = \frac{T}{2} = \frac{U \sin \theta}{2g}$$

$$(iv) \text{ Horizontal range of the jet, } x^* = \frac{U^2}{g} \sin 2\theta$$

$$(v) \text{ Value of } \theta \text{ for maximum range, } \theta = 45^\circ$$

$$(vi) \text{ Maximum range, } x_{\max}^* = U^2/g.$$

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Name the different forces present in a fluid flow. For the Euler's equation of motion, which forces are taken into consideration.
2. What is Euler's equation of motion ? How will you obtain Bernoulli's equation from it ?
3. Derive Bernoulli's equation for the flow of an incompressible frictionless fluid from consideration of momentum.
4. State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's theorem from first principle and state the assumptions made for such a derivation.
5. What is a venturimeter ? Derive an expression for the discharge through a venturimeter.
6. Explain the principle of venturimeter with a neat sketch. Derive the expression for the rate of flow of fluid through it.
7. Discuss the relative merits and demerits of venturimeter with respect to orifice-meter.

*(Delhi University, Dec. 2002)*

8. Define an orifice-meter. Prove that the discharge through an orifice-meter is given by the relation

$$Q = C_d \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

where  $a_1$  = area of pipe in which orifice-meter is fitted

$a_0$  = area of orifice

*(Technical University of M.P., S 2002)*

9. What is a pitot-tube ? How will you determine the velocity at any point with the help of pitot-tube ?
10. What is the difference between pitot-tube and pitot-static tube ?
11. State the momentum equation. How will you apply momentum equation for determining the force exerted by a flowing liquid on a pipe bend ?
12. What is the difference between momentum equation and impulse momentum equation.
13. Define moment of momentum equation. Where this equation is used.
14. What is a free jet of liquid ? Derive an expression for the path travelled by free jet issuing from a nozzle.
15. Prove that the equation of the free jet of liquid is given by the expression,

$$y = x \tan \theta - \frac{gx^2}{2U^2} \sec^2 \theta$$

where  $x, y$  = co-ordinates of a point on the jet

$U$  = velocity of issuing jet

$\theta$  = inclination of the jet with horizontal.

16. Which of the following statement is correct in case of pipe flow :
  - (a) flow takes place from higher pressure to lower pressure ;
  - (b) flow takes place from higher velocity to lower velocity ;
  - (c) flow takes place from higher elevation to lower elevation ;
  - (d) flow takes place from higher energy to lower energy.
17. Derive Euler's equation of motion along a stream line for an ideal fluid stating clearly the assumptions. Explain how this is integrated to get Bernoulli's equation along a stream-line.
18. State Bernoulli's theorem. Mention the assumptions made. How is it modified while applying in practice? List out its engineering applications.
19. Define continuity equation and Bernoulli's equation.

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20. What are the different forms of energy in a flowing fluid ? Represent schematically the Bernoulli's equation for flow through a tapering pipe and show the position of total energy line and the datum line.
21. Write Euler's equation of motion along a stream line and integrate it to obtain Bernoulli's equation. State all assumptions made.
22. Describe with the help of sketch the construction, operation and use of Pitot-static tube.
23. Starting with Euler's equation of motion along a stream line, obtain Bernoulli's equation by its integration. List all the assumptions made.
24. State the different devices that one can use to measure the discharge through a pipe and also through an open channel. Describe one of such devices with a neat sketch and explain how one can obtain the actual discharge with its help?
25. Derive Bernoulli's equation from fundamentals.

### (B) NUMERICAL PROBLEMS

1. Water is flowing through a pipe of 100 mm diameter under a pressure of  $19.62 \text{ N/cm}^2$  (gauge) and with mean velocity of 3.0 m/s. Find the total head of the water at a cross-section, which is 8 m above the datum line. [Ans. 28.458 m]
2. A pipe, through which water is flowing is having diameters 40 cm and 20 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 5.0 m/s. Find the velocity head at the sections 1 and 2 and also rate of discharge. [Ans. 1.274 m ; 20.387 m ;  $0.628 \text{ m}^3/\text{s}$ ]
3. The water is flowing through a pipe having diameters 20 cm and 15 cm at sections 1 and 2 respectively. The rate of flow through pipe is 40 litres/s. The section 1 is 6 m above datum line and section 2 is 3 m above the datum. If the pressure at section 1 is  $29.43 \text{ N/cm}^2$ , find the intensity of pressure at section 2. [Ans.  $32.19 \text{ N/cm}^2$ ]
4. Water is flowing through a pipe having diameters 30 cm and 15 cm at the bottom and upper end respectively. The intensity of pressure at the bottom end is  $29.43 \text{ N/cm}^2$  and the pressure at the upper end is  $14.715 \text{ N/cm}^2$ . Determine the difference in datum head if the rate of flow through pipe is 50 lit/s. [Ans. 14.618 m]
5. The water is flowing through a taper pipe of length 50 m having diameters 40 cm at the upper end and 20 cm at the lower end, at the rate of 60 litres/s. The pipe has a slope of 1 in 40. Find the pressure at the lower end if the pressure at the higher level is  $24.525 \text{ N/cm}^2$ . [Ans.  $25.58 \text{ N/cm}^2$ ]
6. A pipe of diameter 30 cm carries water at a velocity of 20 m/sec. The pressures at the points A and B are given as  $34.335 \text{ N/cm}^2$  and  $29.43 \text{ N/cm}^2$  respectively, while the datum head at A and B are 25 m and 28 m. Find the loss of head between A and B. [Ans. 2 m]
7. A conical tube of length 3.0 m is fixed vertically with its smaller end upwards. The velocity of flow at the smaller end is 4 m/s while at the lower end it is 2 m/s. The pressure head at the smaller end is 2.0 m of liquid. The loss of head in the tube is  $0.95(v_1 - v_2)^2/2g$ , where  $v_1$  is the velocity at the smaller end and  $v_2$  at the lower end respectively. Determine the pressure head at the lower end. Flow takes place in downward direction. [Ans. 5.56 m of fluid]
8. A pipe line carrying oil of specific gravity 0.8, changes in diameter from 300 mm at a position A to 500 mm diameter to a position B which is 5 m at a higher level. If the pressures at A and B are  $19.62 \text{ N/cm}^2$  and  $14.91 \text{ N/cm}^2$  respectively, and the discharge is 150 litres/s, determine the loss of head and direction of flow. [Ans. 1.45 m, Flow takes place from A to B]
9. A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to inlet and throat is 10 cm of mercury. Determine the rate of flow. Take  $C_d = 0.98$ . [Ans. 88.92 litres/s]

10. An oil of sp. gr. 0.9 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 20 cm. Calculate the discharge of oil through the horizontal venturimeter. Take  $C_d = 0.98$ . [Ans. 59.15 litres/s]
11. A horizontal venturimeter with inlet diameter 30 cm and throat diameter 15 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 50 litres/s, find the reading of the oil-mercury differential manometer. Take  $C_d = 0.98$ . [Ans. 2.489 cm]
12. A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of water. The pressure at inlet is  $14.715 \text{ N/cm}^2$  and vacuum pressure at the throat is 40 cm of mercury. Find the discharge of water through venturimeter. [Ans. 162.539 lit/s]
13. A  $30 \text{ cm} \times 15 \text{ cm}$  venturimeter is inserted in a vertical pipe carrying water, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 30 cm. Find the discharge. Take  $C_d = 0.98$ . [Ans. 154.02 lit/s]
14. If in the problem 13, instead of water, oil of sp. gr. 0.8 is flowing through the venturimeter, determine the rate of flow of oil in litres/s. [Ans. 173.56 lit/s]
15. The water is flowing through a pipe of diameter 30 cm. The pipe is inclined and a venturimeter is inserted in the pipe. The diameter of venturimeter at throat is 15 cm. The difference of pressure between the inlet and throat of the venturimeter is measured by a liquid of sp. gr. 0.8 in an inverted U-tube which gives a reading of 40 cm. The loss of head between the inlet and throat is 0.3 times the kinetic head of the pipe. Find the discharge. [Ans. 22.64 lit/s]
16. A  $20 \times 10 \text{ cm}$  venturimeter is provided in a vertical pipe line carrying oil of sp. gr. 0.8, the flow being upwards. The difference in elevation of the throat section and entrance section of the venturimeter is 50 cm. The differential U-tube mercury manometer shows a gauge deflection of 40 cm. Calculate : (i) the discharge of oil, and (ii) the pressure difference between the entrance section and the throat section. Take  $C_d = 0.98$  and sp. gr. of mercury as 13.6. [Ans. (i) 89.132 lit/s, (ii)  $5.415 \text{ N/cm}^2$ ]
17. In a 200 mm diameter horizontal pipe a venturimeter of 0.5 contraction ratio has been fixed. The head of water on the venturimeter when there is no flow is 4 m (gauge). Find the rate of flow for which the throat pressure will be 4 metres of water absolute. Take  $C_d = 0.97$  and atmospheric pressure head = 10.3 m of water. [Ans. 111.92 lit/s]
18. An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter give readings of  $14.715 \text{ N/cm}^2$  and  $9.81 \text{ N/cm}^2$  respectively. Find the rate of flow of water through the pipe in litres/s. Take  $C_d = 0.6$ . [Ans. 108.434 lit/s]
19. If in problem 18, instead of water, oil of sp. gr. 0.8 is flowing through the orifice meter in which the pressure difference is measured by a mercury oil differential manometer on the two sides of the orifice meter, find the rate of flow of oil when the reading of manometer is 40 cm. [Ans. 122.68 lit/s]
20. The pressure difference measured by the two tappings of a pitot-static tube, one tapping pointing upstream and other perpendicular to the flow, placed in the centre of a pipe line of diameter 40 cm is 10 cm of water. The mean velocity in the pipe is 0.75 times the central velocity. Find the discharge through the pipe. Take co-efficient of pitot-tube as 0.98. [Ans.  $0.1293 \text{ m}^3/\text{s}$ ]
21. Find the velocity of flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 15 cm. Take sp. gr. of oil = 0.8 and co-efficient of pitot-tube as 0.98. [Ans. 6.72 m/s]
22. A sub-marine moves horizontally in sea and has its axis 20 m below the surface of water. A pitot-static tube placed in front of sub-marine and along its axis, is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 20 cm. Find the speed of sub-marine. Take sp. gr. of mercury 13.6 and of sea-water 1.026. [Ans. 24.958 km/hr.]
23. A  $45^\circ$  reducing bend is connected in a pipe line, the diameters at the inlet and outlet of the bend being 40 cm and 20 cm respectively. Find the force exerted by water on the bend if the intensity of pressure at inlet of bend is  $21.58 \text{ N/cm}^2$ . The rate of flow of water is 500 litres/s. [Ans.  $22696.5 \text{ N}$ ;  $20^\circ 3.5'$ ]

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24. The discharge of water through a pipe of diameter 40 cm is 400 litres/s. If the pipe is bend by  $135^\circ$ , find the magnitude and direction of the resultant force on the bend. The pressure of flowing water is  $29.43 \text{ N/cm}^2$ .  
[Ans.  $7063.2 \text{ N}$ ,  $\theta = 22^\circ 29.9'$  with  $x$ -axis clockwise]
25. A 30 cm diameter pipe carries water under a head of 15 metres with a velocity of 4 m/s. If the axis of the pipe turns through  $45^\circ$ , find the magnitude and direction of the resultant force at the bend.  
[Ans.  $8717.5 \text{ N}$ ,  $\theta = 67^\circ 30'$ ]
26. A pipe of 20 cm diameter conveying  $0.20 \text{ m}^3/\text{sec}$  of water has a right angled bend in a horizontal plane. Find the resultant force exerted on the bend if the pressure at inlet and outlet of the bend are  $22.563 \text{ N/cm}^2$  and  $21.582 \text{ N/cm}^2$  respectively.  
[Ans.  $11604.7 \text{ N}$ ,  $\theta = 43^\circ 54.2'$ ]
27. A nozzle of diameter 30 mm is fitted to a pipe of 60 mm diameter. Find the force exerted by the nozzle on the water which is flowing through the pipe at the rate of  $4.0 \text{ m}^3/\text{minute}$ .  
[Ans.  $7057.7 \text{ N}$ ]
28. A lawn sprinkler with two nozzles of diameters 3 mm each is connected across a tap of water. The nozzles are at a distance of 40 cm and 30 cm from the centre of the tap. The rate of water through tap is  $100 \text{ cm}^3/\text{s}$ . The nozzle discharges water in the downward directions. Determine the angular speed at which the sprinkler will rotate free.  
[Ans.  $2.83 \text{ rad/s}$ ]
29. A lawn sprinkler has two nozzles of diameters 8 mm each at the end of a rotating arm and the velocity of flow of water from each nozzle is 12 m/s. One nozzle discharges water in the downward direction, while the other nozzle discharges water vertically up. The nozzles are at a distance of 40 cm from the centre of the rotating arm. Determine the torque required to hold the rotating arm stationary. Also determine the constant speed of rotation of arm, if it is free to rotate.  
[Ans.  $5.78 \text{ Nm}$ ,  $30 \text{ rad/s}$ ]
30. A vertical wall is of 10 m in height. A jet of water is issuing from a nozzle with a velocity of 25 m/s. The nozzle is situated at a horizontal distance of 20 m from the vertical wall. Find the angle of projection of the nozzle to the horizontal so that the jet of water just clears the top of the wall.  
[Ans.  $79^\circ 55'$  or  $36^\circ 41'$ ]
31. A fire-brigade man is holding a fire stream nozzle of 50 mm diameter at a distance of 1 m above the ground and 6 m from a vertical wall. The jet is coming out with a velocity of 15 m/s. This jet is to strike a window, situated at a distance of 10 m above ground in the vertical wall. Find the angle or angles of inclination with the horizontal made by the jet, coming out from the nozzle. What will be the amount of water falling on the window?  
[Ans.  $79^\circ 16.7'$  or  $67^\circ 3.7'$ ;  $0.0294 \text{ m}^3/\text{s}$ ]
32. A window, in a vertical wall, is at a distance of 12 m above the ground level. A jet of water, issuing from a nozzle of diameter 50 mm, is to strike the window. The rate of flow of water through the nozzle is 40 litres/sec. The nozzle is situated at a distance of 1 m above ground level. Find the greatest horizontal distance from the wall of the nozzle so that jet of water strikes the window.  
[Ans. 29.38 m]
33. Explain in brief the working of a pitot-tube. Calculate the velocity of flow of water in a pipe of diameter 300 mm at a point, where the stagnation pressure head is 5 m and static pressure head is 4 m. Given the co-efficient of pitot-tube = 0.97.  
[Ans. 4.3 m/sec]
34. Find the rate of flow of water through a venturimeter fitted in a pipeline of diameter 30 cm. The ratio of diameter of throat and inlet of the venturimeter is \*. The pressure at the inlet of the venturimeter is  $13.734 \text{ N/cm}^2$  (gauge) and vacuum in the throat is 37.5 cm of mercury. The co-efficient of venturimeter is given as 0.98.  
[Ans.  $0.15 \text{ m}^3/\text{s}$ ]
35. A  $30 \text{ cm} \times 15 \text{ cm}$  venturimeter is inserted in a vertical pipe carrying an oil of sp. gr. 0.8, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 30 cm. The difference in the elevation of the throat section and inlet section is 50 cm. Find the rate of flow of oil.
36. A venturimeter is used for measurement of discharge of water in horizontal pipe line. If the ratio of upstream pipe diameter to that of throat is  $2 : 1$ , upstream diameter is 300 mm, the difference in pressure between the throat and upstream is equal to 3 m head of water and loss of head through meter is one-eighth of the throat velocity head, calculate the discharge in the pipe.  
[Ans.  $0.107 \text{ m}^3/\text{s}$ ]
37. A liquid of specific gravity 0.8 is flowing upwards at the rate of  $0.08 \text{ m}^3/\text{s}$ , through a vertical venturimeter with an inlet diameter of 200 mm and throat diameter of 100 mm. The  $C_d = 0.98$  and the vertical distance between pressure tappings is 300 mm. Find :

- (i) the difference in readings of the two pressure gauges, which are connected to the two pressure tappings, and  
 (ii) the difference in the level of the mercury columns of the differential manometer which is connected to the tappings, in place of pressure gauges. [Ans. (i) 42.928 kN/m<sup>2</sup>, (ii) 32.3 cm]  
 [Hint.  $Q = 0.08 \text{ m}^3/\text{s}$ ,  $d_1 = 200 \text{ mm} = 0.2 \text{ m}$ ,  $d_2 = 100 \text{ mm} = 0.1 \text{ m}$ ,

$$C_d = 0.98, z_2 - z_1 = 300 \text{ mm} = 0.3 \text{ m}, a_1 = \frac{\pi}{4}(0.2^2) = 0.0314 \text{ m}^2$$

$$a_2 = \frac{\pi}{4}(0.1^2) = 0.007854 \text{ m}^2. \text{ Using } Q = C_d \frac{a_1 \times a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

Find 'h'. This value of  $h = 5.17 \text{ m}$ .

Now use  $h = \left( \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) + (z_1 - z_2)$ , where  $\rho = 800 \text{ kg/m}^3$ . Find  $(p_1 - p_2)$ .

Now use the formula  $h = x \left[ \frac{S_g}{S_f} - 1 \right]$ ,

where  $h = 5.17 \text{ m}$ ,  $S_g = 13.6$  and  $S_f = 0.8$ . Find the value of  $x$  which will be 32.3 cm.]

38. A venturimeter is installed in a 300 mm diameter horizontal pipe line. The throat pipe rates is 1/3. Water flows through the installation. The pressure in the pipe line is 13.783 N/cm<sup>2</sup> (gauge) and vacuum in the throat is 37.5 cm of mercury. Neglecting head loss in the venturimeter, determine the rate of flow in the pipe line. [Ans. 0.153 m<sup>3</sup>/sec]

$$[\text{Hint. } d_1 = 300 \text{ mm} = 0.3 \text{ m}, d_2 = \frac{1}{3} \times 300 = 100 \text{ mm} = 0.1 \text{ m}, p_1 = 13.783 \text{ N/cm}^2 = 13.783 \times 10^4 \text{ N/m}^2]$$

Hence  $p_1/\rho \times g = 13.783 \times 10^4 / 1000 \times 9.81$   
 $= 14.05 \text{ m}$ ,  $p_2/\rho g = -37.5 \text{ cm of Hg} = -0.375 \times 13.6 \text{ m of water}$   
 $= -5.1 \text{ m of water}$ . Hence  $h = 14.05 - (-5.1) = 19.15 \text{ m of water}$ .

Value of  $C_d = 1.0$ . Now use the formula  $Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$

39. The maximum flow through a 300 mm diameter horizontal main pipe line is 18200 litre/minute. A venturimeter is introduced at a point of the pipe line where the pressure head is 4.6 m of water. Find the smallest dia. of throat so that the pressure at the throat is never negative. Assume co-efficient of meter as unity. [Ans.  $d_2 = 192.4 \text{ mm}$ ]

[Hint.  $d_1 = 300 \text{ mm} = 0.3 \text{ m}$ ,  $Q = 18200 \text{ litres/minute} = 18200/60 = 303.33 \text{ litres/s} = 0.3033 \text{ m}^3/\text{s}$ ,  $p_1/\rho g = 4.6 \text{ m}$ ,  $p_2/\rho g = 0$ . Hence  $h = 4.6 \text{ m}$ ,  $C_d = 1$ .  $d_2$  = dia. at throat. Use formula  $Q = C_d \frac{a_1 \times a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$  and

find the value of  $a_2$ . Then  $a_2 = \frac{\pi}{4} d_2^2$  and find  $d_2$ .]

40. The following are the data given of a change in diameter effected in laying a water supply pipe. The change in diameter is gradual from 20 cm at A to 50 cm at B. Pressures at A and B are 7.848 N/cm<sup>2</sup> and 5.886 N/cm<sup>2</sup> respectively with the end B being 3 m higher than A. If the flow in the pipe line is 200 litre/s, find : (i) direction of flow, (ii) the head lost in friction between A and B.

[Ans. (i) From A to B, (ii) 1.015 m ]

[Hint.  $D_A = 20 \text{ cm} = 0.2 \text{ m}$ ,  $D_B = 50 \text{ cm} = 0.5 \text{ m}$ ,  $p_A = 7.848 \text{ N/cm}^2 = 7.848 \times 10^4 \text{ N/m}^2$   
 $p_B = 5.886 \text{ N/cm}^2 = 5.886 \times 10^4 \text{ N/m}^2$ ,  $Z_A = 0$ ,  $Z_B = 3 \text{ m}$ ,  $Q = 0.2 \text{ m}^3/\text{s}$

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$$V_A = 0.2 \frac{\pi}{4} (2^2) = 6.369 \text{ m/s}, V_B = 0.2 \frac{\pi}{4} (.5^2) = 1.018 \text{ m/s}$$

$$E_A = (p_A/\rho \times g) + \frac{V_A^2}{2g} + Z_A = (7.848 \times 10^4 / 1000 \times 9.81) + (6.369^2 / 2 \times 9.81) + 0 = 10.067 \text{ m}$$

$$E_B = (p_B/\rho \times g) + \frac{V_B^2}{2g} + Z_B = (5.886 \times 10^4 / 1000 \times 9.81) + (1.018^2 / 2 \times 9.81) + 3 = 9.052 \text{ m}$$

41. A venturimeter of inlet diameter 300 mm and throat diameter 150 mm is fixed in a vertical pipe line. A liquid of sp. gr. 0.8 is flowing upward through the pipe line. A differential manometer containing mercury gives a reading of 100 mm when connected at inlet and throat. The vertical difference between inlet and throat is 500 mm. If  $C_d = 0.98$ , then find : (i) rate of flow of liquid in litre per second and (ii) difference of pressure between inlet and throat in N/m<sup>2</sup>. [Ans. (i) 100 litre/s, (ii) 15980 N/m<sup>2</sup>]

42. A venturimeter with a throat diameter of 7.5 cm is installed in a 15 cm diameter pipe. The pressure at the entrance to the meter is 70 kPa (gauge) and it is desired that the pressure at any point should not fall below 2.5 m of absolute water. Determine the maximum flow rate of water through the meter. Take  $C_d = 0.97$  and atmospheric pressure as 100 kPa. (J.N.T.U., Hyderabad S 2002)

[Hint. The pressure at the throat will be minimum. Hence  $\frac{p_2}{\rho g} = 2.5 \text{ m (abs.)}$

$$\text{Given : } d_1 = 15 \text{ cm} \therefore A_1 = \frac{\pi}{4} (15^2) = 176.7 \text{ cm}^2$$

$$d_2 = 7.5 \text{ cm} \therefore A_2 = \frac{\pi}{4} (7.5^2) = 44.175 \text{ cm}^2$$

$$\therefore p_1 = 70 \text{ kPa} = 70 \times 10^3 \text{ N/m}^2 \text{ (gauge)}, p_{atm} = 100 \text{ kPa} = 100 \times 10^3 \text{ N/m}^2$$

$$\therefore p_1 \text{ (abs.)} = 70 \times 10^3 + 100 \times 10^3 = 170 \times 10^3 \text{ N/m}^2 \text{ (abs.)}$$

$$\therefore \frac{p_1}{\rho g} = \frac{170 \times 10^3}{1000 \times 9.81} = 17.33 \text{ m of water (abs.)}$$

$$\therefore h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 17.33 - 2.5 = 14.83 \text{ m of water} = 1483 \text{ cm of water}$$

$$\text{Now } Q = \frac{C_d A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh} = \frac{0.97 \times 176.7 \times 44.175 \times \sqrt{2 \times 981 \times 1483}}{\sqrt{176.7^2 - 44.175^2}} = 75488 \text{ cm}^3/\text{s}$$

$$= 75.488 \text{ litre/s.}]$$

43. Find the discharge of water flowing through a pipe 20 cm diameter placed in an inclined position, where a venturimeter is inserted, having a throat diameter of 10 cm. The difference of pressure between the main and throat is measured by a liquid of specific gravity 0.4 in an inverted U-tube, which gives a reading of 30 cm. The loss of head between the main and throat is 0.2 times the kinetic head of pipe.

(Delhi University, Dec. 2002)

[Hint. Given :  $d_1 = 20 \text{ cm} \therefore a_1 = \frac{\pi}{4} (20^2) = 100 \pi \text{ cm}^2$ ;  $d_2 = 10 \text{ cm} \therefore a_2 = \frac{\pi}{4} (10^2) = 25 \pi \text{ cm}^2$ .

$$x = 30 \text{ cm}, h = x \left(1 - \frac{S_l}{S_o}\right) = 30 \left(1 - \frac{0.4}{1.0}\right) = 18 \text{ cm} = 0.18 \text{ m}$$

$$\text{But } h \text{ is also } = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) \therefore \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 18 \text{ cm} = 0.18 \text{ m}$$

$$h_L = 0.2 \times \frac{V_1^2}{2g}$$

From Bernoulli's equation,  $\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L$

or  $\left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = h_L$

or  $0.18 + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = \frac{0.2 V_1^2}{2g}$   $\left( \because \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = 0.18 \text{ m and } h_L = \frac{0.2 V_1^2}{2g} \right)$

or  $0.18 + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{0.2 V_1^2}{2g} = 0$  or  $0.18 + \frac{0.8 V_1^2}{2g} - \frac{V_2^2}{2g} = 0$

From continuity equation,  $a_1 V_1 = a_2 V_2$  or  $V_2 = \frac{a_1 V_1}{a_2} = \frac{\frac{\pi}{4}(20^2) V_1}{\frac{\pi}{4}(10^2)} = 4V_1$

Now  $0.18 + \frac{0.8 V_1^2}{2g} - \frac{V_2^2}{2g} = 0$  or  $0.18 + \frac{0.8 V_1^2}{2g} - \frac{(4V_1)^2}{2g} = 0$

or  $0.18 + \frac{0.8 V_1^2}{2g} - \frac{16V_1^2}{2g} = 0$  or  $0.18 = \frac{16V_1^2}{2g} - \frac{0.8 V_1^2}{2g} = \frac{15.2V_1^2}{2g}$

$\therefore V_1 = \sqrt{\frac{0.18 \times 2 \times 9.81}{15.2}} = 0.48 \text{ m/s} = 48 \text{ cm/s}$

$\therefore Q = A_1 V_1 = \frac{\pi}{4} (20^2) \times 48 = 15140 \text{ cm}^3/\text{s} = \mathbf{15.14 \text{ litre/s.}}$



# 7

## CHAPTER



# ORIFICES AND MOUTHPIECES

### ► 7.1 INTRODUCTION

Orifice is a small opening of any cross-section (such as circular, triangular, rectangular etc.) on the side or at the bottom of a tank, through which a fluid is flowing. A mouthpiece is a short length of a pipe which is two to three times its diameter in length, fitted in a tank or vessel containing the fluid. Orifices as well as mouthpieces are used for measuring the rate of flow of fluid.

### ► 7.2 CLASSIFICATIONS OF ORIFICES

The orifices are classified on the basis of their size, shape, nature of discharge and shape of the upstream edge. The following are the important classifications :

1. The orifices are classified as **small orifice** or **large orifice** depending upon the size of orifice and head of liquid from the centre of the orifice. If the head of liquid from the centre of orifice is more than five times the depth of orifice, the orifice is called small orifice. And if the head of liquids is less than five times the depth of orifice, it is known as large orifice.
2. The orifices are classified as (i) Circular orifice, (ii) Triangular orifice, (iii) Rectangular orifice and (iv) Square orifice depending upon their cross-sectional areas.
3. The orifices are classified as (i) Sharp-edged orifice and (ii) Bell-mouthing orifice depending upon the shape of upstream edge of the orifices.
4. The orifices are classified as (i) Free discharging orifices and (ii) Drowned or sub-merged orifices depending upon the nature of discharge.

The sub-merged orifices are further classified as (a) Fully sub-merged orifices and (b) Partially sub-merged orifices.

### ► 7.3 FLOW THROUGH AN ORIFICE

Consider a tank fitted with a circular orifice in one of its sides as shown in Fig. 7.1. Let  $H$  be the head of the liquid above the centre of the orifice. The liquid flowing through the orifice forms a jet of liquid whose area of cross-section is less than that of orifice. The area of jet of fluid goes on decreasing and at a section  $C-C$ , the area is minimum. This section is approximately at a distance of half of diameter of the orifice. At this section, the streamlines are straight and parallel to each other and perpendicular to the

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plane of the orifice. This section is called **Vena-contracta**. Beyond this section, the jet diverges and is attracted in the downward direction by the gravity.

Consider two points 1 and 2 as shown in Fig. 7.1. Point 1 is inside the tank and point 2 at the vena-contracta. Let the flow is steady and at a constant head  $H$ . Applying Bernoulli's equation at points 1 and 2.

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But

$$z_1 = z_2$$

∴

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

Now

$$\frac{p_1}{\rho g} = H$$

$$\frac{p_2}{\rho g} = 0 \text{ (atmospheric pressure)}$$

$v_1$  is very small in comparison to  $v_2$  as area of tank is very large as compared to the area of the jet of liquid.

∴

$$H + 0 = 0 + \frac{v_2^2}{2g}$$

∴

$$v_2 = \sqrt{2gH}$$

...(7.1)

This is theoretical velocity. Actual velocity will be less than this value.

### ► 7.4 HYDRAULIC CO-EFFICIENTS

The hydraulic co-efficients are

1. Co-efficient of velocity,  $C_v$
2. Co-efficient of contraction,  $C_c$
3. Co-efficient of discharge,  $C_d$

**7.4.1 Co-efficient of Velocity ( $C_v$ ).** It is defined as the ratio between the actual velocity of a jet of liquid at vena-contracta and the theoretical velocity of jet. It is denoted by  $C_v$  and mathematically,  $C_v$  is given as

$$C_v = \frac{\text{Actual velocity of jet at vena-contracta}}{\text{Theoretical velocity}}$$

$$= \frac{V}{\sqrt{2gH}}, \text{ where } V = \text{actual velocity}, \sqrt{2gH} = \text{Theoretical velocity} \quad \dots(7.2)$$

The value of  $C_v$  varies from 0.95 to 0.99 for different orifices, depending on the shape, size of the orifice and on the head under which flow takes place. Generally the value of  $C_v = 0.98$  is taken for sharp-edged orifices.

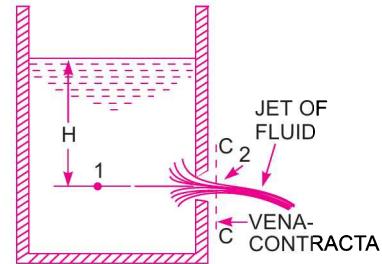


Fig. 7.1 Tank with an orifice.

**7.4.2 Co-efficient of Contraction ( $C_c$ ).** It is defined as the ratio of the area of the jet at vena-contracta to the area of the orifice. It is denoted by  $C_c$ .

Let

$a$  = area of orifice and

$a_c$  = area of jet at vena-contracta.

Then

$$C_c = \frac{\text{area of jet at vena-contracta}}{\text{area of orifice}}$$

$$= \frac{a_c}{a} \quad \dots(7.3)$$

The value of  $C_c$  varies from 0.61 to 0.69 depending on shape and size of the orifice and head of liquid under which flow takes place. In general, the value of  $C_c$  may be taken as 0.64.

**7.4.3 Co-efficient of Discharge ( $C_d$ ).** It is defined as the ratio of the actual discharge from an orifice to the theoretical discharge from the orifice. It is denoted by  $C_d$ . If  $Q$  is actual discharge and  $Q_{th}$  is the theoretical discharge then mathematically,  $C_d$  is given as

$$\begin{aligned} C_d &= \frac{Q}{Q_{th}} = \frac{\text{Actual velocity} \times \text{Actual area}}{\text{Theoretical velocity} \times \text{Theoretical area}} \\ &= \frac{\text{Actual velocity}}{\text{Theoretical velocity}} \times \frac{\text{Actual area}}{\text{Theoretical area}} \\ \therefore C_d &= C_v \times C_c \end{aligned} \quad \dots(7.4)$$

The value of  $C_d$  varies from 0.61 to 0.65. For general purpose the value of  $C_d$  is taken as 0.62.

**Problem 7.1** The head of water over an orifice of diameter 40 mm is 10 m. Find the actual discharge and actual velocity of the jet at vena-contracta. Take  $C_d = 0.6$  and  $C_v = 0.98$ .

**Solution.** Given :

Head,  $H = 10 \text{ cm}$

Dia. of orifice,  $d = 40 \text{ mm} = 0.04 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4}(0.04)^2 = .001256 \text{ m}^2$$

$$C_d = 0.6$$

$$C_v = 0.98$$

$$(i) \quad \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = 0.6$$

But Theoretical discharge =  $V_{th} \times \text{Area of orifice}$

$$V_{th} = \text{Theoretical velocity, where } V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 10} = 14 \text{ m/s}$$

$$\therefore \text{Theoretical discharge} = 14 \times .001256 = 0.01758 \frac{\text{m}^2}{\text{s}}$$

$$\begin{aligned} \therefore \text{Actual discharge} &= 0.6 \times \text{Theoretical discharge} \\ &= 0.6 \times .01758 = \mathbf{0.01054 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

$$(ii) \frac{\text{Actual velocity}}{\text{Theoretical velocity}} = C_v = 0.98$$

$$\therefore \text{Actual velocity} = 0.98 \times \text{Theoretical velocity}$$

$$= 0.98 \times 14 = 13.72 \text{ m/s. Ans.}$$

**Problem 7.2** The head of water over the centre of an orifice of diameter 20 mm is 1 m. The actual discharge through the orifice is 0.85 litre/s. Find the co-efficient of discharge.

**Solution.** Given :

$$\text{Dia. of orifice, } d = 20 \text{ mm} = 0.02 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4}(0.02)^2 = 0.000314 \text{ m}^2$$

$$\text{Head, } H = 1 \text{ m}$$

$$\text{Actual discharge, } Q = 0.85 \text{ litre/s} = 0.00085 \text{ m}^3/\text{s}$$

$$\text{Theoretical velocity, } V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 1} = 4.429 \text{ m/s}$$

$$\therefore \text{Theoretical discharge, } Q_{th} = V_{th} \times \text{Area of orifice}$$

$$= 4.429 \times 0.000314 = 0.00139 \text{ m}^3/\text{s}$$

$$\therefore \text{Co-efficient of discharge} = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{0.00085}{0.00139} = 0.61. \text{ Ans.}$$

## ► 7.5 EXPERIMENTAL DETERMINATION OF HYDRAULIC CO-EFFICIENTS

**7.5.1 Determination of Co-efficient of Discharge ( $C_d$ ).** The water is allowed to flow through an orifice fitted to a tank under a constant head,  $H$  as shown in Fig. 7.2. The water is collected in a measuring tank for a known time,  $t$ . The height of water in the measuring tank is noted down. Then actual discharge through orifice,

$$Q = \frac{\text{Area of measuring tank} \times \text{Height of water in measuring tank}}{\text{Time } (t)}$$

and theoretical discharge = area of orifice  $\times \sqrt{2gH}$

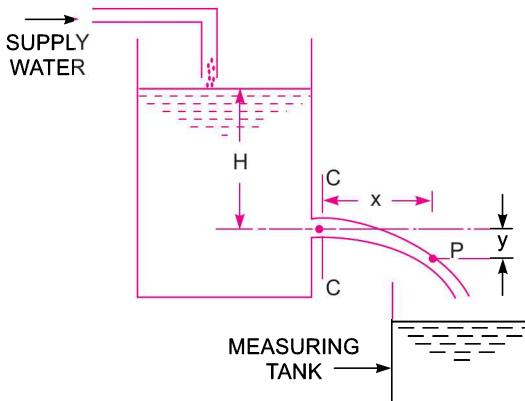


Fig. 7.2 Value of  $C_d$ .

$$\therefore C_d = \frac{Q}{a \times \sqrt{2gH}} \quad \dots(7.5)$$

**7.5.2 Determination of Co-efficient of Velocity ( $C_v$ ).** Let  $C-C$  represents the vena-contracta of a jet of water coming out from an orifice under constant head  $H$  as shown in Fig. 7.2. Consider a liquid particle which is at vena-contracta at any time and takes the position at  $P$  along the jet in time ' $t$ '.

Let  $x$  = horizontal distance travelled by the particle in time ' $t$ '

$y$  = vertical distance between  $P$  and  $C-C$

$V$  = actual velocity of jet at vena-contracta.

Then horizontal distance,  $x = V \times t$  ... (i)

and vertical distance,  $y = \frac{1}{2} g t^2$  ... (ii)

From equation (i),  $t = \frac{x}{V}$

Substituting this value of ' $t$ ' in (ii), we get

$$y = \frac{1}{2} g \times \frac{x^2}{V^2}$$

$$V^2 = \frac{gx^2}{2y}$$

$$\therefore V = \sqrt{\frac{gx^2}{2y}}$$

But theoretical velocity,

$$V_{th} = \sqrt{2gH}$$

$$\begin{aligned} \therefore \text{Co-efficient of velocity, } C_v &= \frac{V}{V_{th}} = \sqrt{\frac{gx^2}{2y}} \times \frac{1}{\sqrt{2gH}} = \sqrt{\frac{x^2}{4yH}} \\ &= \frac{x}{\sqrt{4yH}}. \end{aligned} \quad \dots(7.6)$$

**7.5.3 Determination of Co-efficient of Contraction ( $C_c$ ).** The co-efficient of contraction is determined from the equation (7.4) as

$$C_d = C_v \times C_c$$

$$\therefore C_c = \frac{C_d}{C_v} \quad \dots(7.7)$$

**Problem 7.3** A jet of water, issuing from a sharp-edged vertical orifice under a constant head of 10.0 cm, at a certain point, has the horizontal and vertical co-ordinates measured from the vena-contracta as 20.0 cm and 10.5 cm respectively. Find the value of  $C_v$ . Also find the value of  $C_c$  if  $C_d = 0.60$ .

**Solution.** Given :

Head,  $H = 10.0$  cm

Horizontal distance,  $x = 20.0$  cm

Vertical distance,  $y = 10.5$  cm

$$C_d = 0.6$$

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The value of  $C_v$  is given by equation (7.6) as

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{20.0}{\sqrt{4 \times 10.5 \times 10.0}} = \frac{20}{20.493} = 0.9759 = \mathbf{0.976. Ans.}$$

The value of  $C_c$  is given by equation (7.7) as

$$C_c = \frac{C_d}{C_v} = \frac{0.6}{0.976} = 0.6147 = \mathbf{0.615. Ans.}$$

**Problem 7.4** The head of water over an orifice of diameter 100 mm is 10 m. The water coming out from orifice is collected in a circular tank of diameter 1.5 m. The rise of water level in this tank is 1.0 m in 25 seconds. Also the co-ordinates of a point on the jet, measured from vena-contracta are 4.3 m horizontal and 0.5 m vertical. Find the co-efficients,  $C_d$ ,  $C_v$  and  $C_c$ .

**Solution.** Given :

Head,  $H = 10 \text{ m}$

Dia. of orifice,  $d = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore$  Area of orifice,  $a = \frac{\pi}{4}(0.1)^2 = 0.007853 \text{ m}^2$

Dia. of measuring tank,  $D = 1.5 \text{ m}$

$\therefore$  Area,  $A = \frac{\pi}{4}(1.5)^2 = 1.767 \text{ m}^2$

Rise of water,  $h = 1 \text{ m}$

Time,  $t = 25 \text{ seconds}$

Horizontal distance,  $x = 4.3 \text{ m}$

Vertical distance,  $y = 0.5 \text{ m}$

Now theoretical velocity,  $V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 10} = 14.0 \text{ m/s}$

$\therefore$  Theoretical discharge,  $Q_{th} = V_{th} \times \text{Area of orifice} = 14.0 \times 0.007854 = 0.1099 \text{ m}^3/\text{s}$

Actual discharge,  $Q = \frac{A \times h}{t} = \frac{1.767 \times 1.0}{25} = 0.07068$

$\therefore C_d = \frac{Q}{Q_{th}} = \frac{0.07068}{0.1099} = \mathbf{0.643. Ans.}$

The value of  $C_v$  is given by equation (7.6) as

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{4.3}{\sqrt{4 \times 0.5 \times 10}} = \frac{4.3}{4.472} = \mathbf{0.96. Ans.}$$

$C_c$  is given by equation (7.7) as  $C_c = \frac{C_d}{C_v} = \frac{0.643}{0.96} = \mathbf{0.669. Ans.}$

**Problem 7.5** Water discharge at the rate of 98.2 litres/s through a 120 mm diameter vertical sharp-edged orifice placed under a constant head of 10 metres. A point, on the jet, measured from the vena-contracta of the jet has co-ordinates 4.5 metres horizontal and 0.54 metres vertical. Find the co-efficient  $C_v$ ,  $C_c$  and  $C_d$  of the orifice.

**Solution.** Given :

$$\text{Discharge, } Q = 98.2 \text{ lit/s} = 0.0982 \text{ m}^3/\text{s}$$

$$\text{Dia. of orifice, } d = 120 \text{ mm} = 0.12 \text{ m}$$

$$\therefore \text{Area of orifice, } a = \frac{\pi}{4}(0.12)^2 = 0.01131 \text{ m}^2$$

$$\text{Head, } H = 10 \text{ m}$$

Horizontal distance of a point on the jet from vena-contracta,  $x = 4.5 \text{ m}$   
and vertical distance,  $y = 0.54 \text{ m}$

$$\text{Now theoretical velocity, } V_{th} = \sqrt{2g \times H} = \sqrt{2 \times 9.81 \times 10} = 14.0 \text{ m/s}$$

$$\begin{aligned} \text{Theoretical discharge, } Q_{th} &= V_{th} \times \text{Area of orifice} \\ &= 14.0 \times 0.01131 = 0.1583 \text{ m}^3/\text{s} \end{aligned}$$

$$\text{The value of } C_d \text{ is given by, } C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{Q}{Q_{th}} = \frac{0.0982}{0.1583} = 0.62. \text{ Ans.}$$

The value of  $C_v$  is given by equation (7.6),

$$C_v = \frac{x}{\sqrt{4yH}} = \frac{4.5}{\sqrt{4 \times 0.54 \times 10}} = 0.968. \text{ Ans.}$$

The value of  $C_c$  is given by equation (7.7) as

$$C_c = \frac{C_d}{C_v} = \frac{0.62}{0.968} = 0.64. \text{ Ans.}$$

**Problem 7.6** A 25 mm diameter nozzle discharges  $0.76 \text{ m}^3$  of water per minute when the head is 60 m. The diameter of the jet is 22.5 mm. Determine : (i) the values of co-efficients  $C_c$ ,  $C_v$  and  $C_d$  and (ii) the loss of head due to fluid resistance.

**Solution.** Given :

$$\text{Dia. of nozzle, } D = 25 \text{ mm} = 0.025 \text{ m}$$

$$\text{Actual discharge, } Q_{act} = 0.76 \text{ m}^3/\text{minute} = \frac{0.76}{60} = 0.01267 \text{ m}^3/\text{s}$$

$$\text{Head, } H = 60 \text{ m}$$

$$\text{Dia. of jet, } d = 22.5 \text{ mm} = 0.0225 \text{ m.}$$

(i) Values of co-efficients :

Co-efficient of contraction ( $C_c$ ) is given by,

$$C_c = \frac{\text{Area of jet}}{\text{Area of nozzle}}$$

$$= \frac{\frac{\pi}{4} d^2}{\frac{\pi}{4} D^2} = \frac{d^2}{D^2} = \frac{0.0225^2}{0.025^2} = 0.81. \text{ Ans.}$$

Co-efficient of discharge ( $C_d$ ) is given by,

$$C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}}$$

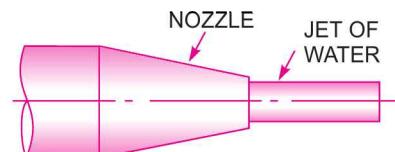


Fig. 7.3

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$$\begin{aligned}
 &= \frac{0.01267}{\text{Theoretical velocity} \times \text{Area of nozzle}} \\
 &= \frac{0.01267}{\sqrt{2gH} \times \frac{\pi}{4} D^2} = \frac{0.01267}{\sqrt{2 \times 9.81 \times 60} \times \frac{\pi}{4} (0.025)^2} \\
 &= \mathbf{0.752. \text{ Ans.}}
 \end{aligned}$$

Co-efficient of velocity ( $C_v$ ) is given by,

$$C_v = \frac{C_d}{C_c} = \frac{0.752}{0.81} = \mathbf{0.928. \text{ Ans.}}$$

(ii) *Loss of head due to fluid resistance :*

Applying Bernoulli's equation at the outlet of nozzle and to the jet of water, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{Loss of head}$$

But  $\frac{p_1}{\rho g} = \frac{p_2}{\rho g} = \text{Atmospheric pressure head}$

$$z_1 = z_2, V_1 = \sqrt{2gH}, V_2 = \text{Actual velocity of jet} = C_v \sqrt{2gH}$$

$$\therefore \frac{(\sqrt{2gH})^2}{2g} = \frac{(C_v \sqrt{2gH})^2}{2g} + \text{Loss of head}$$

or  $H = C_v^2 \times H + \text{Loss of head}$

$$\begin{aligned}
 \therefore \text{Loss of head} &= H - C_v^2 \times H = H(1 - C_v^2) \\
 &= 60(1 - 0.928^2) = 60 \times 0.1388 = \mathbf{8.328 \text{ m. Ans.}}
 \end{aligned}$$

**Problem 7.7** A pipe, 100 mm in diameter, has a nozzle attached to it at the discharge end, the diameter of the nozzle is 50 mm. The rate of discharge of water through the nozzle is 20 litres/s and the pressure at the base of the nozzle is 5.886 N/cm<sup>2</sup>. Calculate the co-efficient of discharge. Assume that the base of the nozzle and outlet of the nozzle are at the same elevation.

**Solution.** Given :

Dia. of pipe,  $D = 100 \text{ mm} = 0.1 \text{ m}$

$$\therefore A_1 = \frac{\pi}{4} (0.1)^2 = .007854 \text{ m}^2$$

Dia. of nozzle,  $d = 50 \text{ mm} = 0.05 \text{ m}$

$$\therefore A_2 = \frac{\pi}{4} (0.05)^2 = .001963 \text{ m}^2$$

Actual discharge,  $Q = 20 \text{ lit/s} = 0.02 \text{ m}^3/\text{s}$

Pressure at the base,  $p_1 = 5.886 \text{ N/cm}^2 = 5.886 \times 10^4 \frac{\text{N}}{\text{m}^2}$

From continuity equation,  $A_1 V_1 = A_2 V_2$

or  $.007854 V_1 = .001963 V_2$

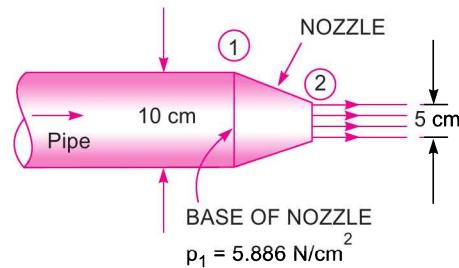


Fig. 7.4

$$\therefore V_1 = \frac{.001963V_2}{.007854} = \frac{V_2}{4}$$

where  $V_1$  and  $V_2$  are theoretical velocity at sections (1) and (2).

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

But

$$z_1 = z_2$$

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g}$$

$$\text{or } \frac{5.886 \times 10^4}{1000 \times 9.81} + \frac{\left(\frac{V_2}{4}\right)^2}{2g} = 0 + \frac{V_2^2}{2g} \quad \left\{ \because \frac{p_2}{\rho g} = \text{Atmospheric pressure} = 0 \right\}$$

$$6.0 + \frac{V_2^2}{2g \times 16} = \frac{V_2^2}{2g}$$

$$\text{or } \frac{V_2^2}{2g} \left[ 1 - \frac{1}{16} \right] = 6.0 \quad \text{or} \quad \frac{V_2^2}{2g} \left[ \frac{15}{16} \right] = 6.0$$

$$\therefore V_2 = \sqrt{6.0 \times 2 \times 9.81 \times \frac{16}{15}} = 11.205 \text{ m/sec}$$

$$\therefore \text{Theoretical discharge} = V_2 \times A_2 = 11.205 \times .001963 = 0.022 \text{ m}^3/\text{s}$$

$$\therefore C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = \frac{0.02}{0.022} = 0.909. \text{ Ans.}$$

**Problem 7.8** A tank has two identical orifices on one of its vertical sides. The upper orifice is 3 m below the water surface and lower one is 5 m below the water surface. If the value of  $C_v$  for each orifice is 0.96, find the point of intersection of the two jets.

**Solution.** Given :

Height of water from orifice (1),  $H_1 = 3 \text{ m}$

From orifice (2),  $H_2 = 5 \text{ m}$

$C_v$  for both = 0.96

Let  $P$  is the point of intersection of the two jets coming from orifices (1) and (2), such that

$x$  = horizontal distance of  $P$

$y_1$  = vertical distance of  $P$  from orifice (1)

$y_2$  = vertical distance of  $P$  from orifice (2)

Then  $y_1 = y_2 + (5 - 3) = y_2 + 2 \text{ m}$

The value of  $C_v$  is given by equation (7.6) as

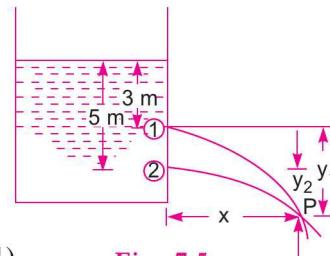


Fig. 7.5

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For orifice (1),  $C_{v_1} = \frac{x}{\sqrt{4y_1 H_1}} = \frac{x}{\sqrt{4y_1 \times 3.0}}$  ... (i)

For orifice (2),  $C_{v_2} = \frac{x}{\sqrt{4y_2 H_2}} = \frac{x}{\sqrt{4 \times y_2 \times 5.0}}$  ... (ii)

As both the orifices are identical

$$\therefore C_{v_1} = C_{v_2}$$

or  $\frac{x}{\sqrt{4y_1 \times 3.0}} = \frac{x}{\sqrt{4y_2 \times 5.0}}$  or  $3y_1 = 5y_2$

But  $y_1 = y_2 + 2.0$

$\therefore 3(y_2 + 2.0) = 5y_2$

$\therefore 2y_2 = 6.0$

$\therefore y_2 = 3.0$

From (ii),  $C_{v_2} = \frac{x}{\sqrt{4y_2 \times H_2}}$

or  $0.96 = \frac{x}{\sqrt{4 \times 3.0 \times 5.0}}$

$\therefore x = 0.96 \times \sqrt{4 \times 3.0 \times 5.0} = 7.436 \text{ m. Ans.}$

**Problem 7.9** A closed vessel contains water upto a height of 1.5 m and over the water surface there is air having pressure  $7.848 \text{ N/cm}^2$  ( $0.8 \text{ kgf/cm}^2$ ) above atmospheric pressure. At the bottom of the vessel there is an orifice of diameter 100 mm. Find the rate of flow of water from orifice. Take  $C_d = 0.6$ .

**Solution.** Given :

Dia. of orifice,  $d = 100 \text{ mm} = 0.1 \text{ m}$

$C_d = 0.6$

Height of water,  $H = 1.5 \text{ m}$

Air pressure,  $p = 7.848 \text{ N/cm}^2 = 7.848 \times 10^4 \text{ N/m}^2$

Applying Bernoulli's equation at sections (1) (water surface) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

Taking datum line passing through (2) which is very close to the bottom surface of the tank. Then  $z_2 = 0$ ,  $z_1 = 1.5 \text{ m}$

Also  $\frac{p_2}{\rho g} = 0$  (atmospheric pressure)

and  $\frac{p_1}{\rho g} = \frac{7.848 \times 10^4}{1000 \times 9.81} = 8 \text{ m of water}$

$\therefore 8 + 0 + 1.5 = 0 + \frac{V_2^2}{2g} + 0$  { $V_1$  is negligible}

$\therefore 9.5 = \frac{V_2^2}{2g}$

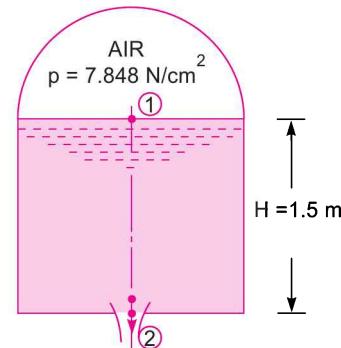


Fig. 7.6

$$\therefore V_2 = \sqrt{2 \times 9.81 \times 9.5} = 13.652 \text{ m/s}$$

$$\therefore \text{Rate of flow of water} = C_d \times a_2 \times V_2$$

$$= 0.6 \times \frac{\pi}{4} (0.1)^2 \times 13.652 \text{ m}^3/\text{s} = 0.0643 \text{ m}^3/\text{s. Ans.}$$

**Problem 7.10** A closed tank partially filled with water upto a height of 0.9 m having an orifice of diameter 15 mm at the bottom of the tank. The air is pumped into the upper part of the tank. Determine the pressure required for a discharge of 1.5 litres/s through the orifice. Take  $C_d = 0.62$ .

**Solution.** Given :

Height of water above orifice,  $H = 0.9 \text{ m}$

Dia. of orifice,  $d = 15 \text{ mm} = 0.015 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} [d^2] = \frac{\pi}{4} (0.015)^2 = 0.0001767 \text{ m}^2$$

Discharge,  $Q = 1.5 \text{ litres/s} = .0015 \text{ m}^3/\text{s}$

$$C_d = 0.62$$

Let  $p$  is intensity of pressure required above water surface in  $\text{N/cm}^2$ .

$$\text{Then pressure head of air} = \frac{p}{\rho g} = \frac{p \times 10^4}{1000 \times 9.81} = \frac{10p}{9.81} \text{ m of water.}$$

If  $V_2$  is the velocity at outlet of orifice, then

$$V_2 = \sqrt{2g \left( H + \frac{p}{\rho g} \right)} = \sqrt{2 \times 9.81 \left( 0.9 + \frac{10p}{9.81} \right)}$$

$$\therefore \text{Discharge } Q = C_d \times a \times \sqrt{2g (H + p/\rho g)}$$

$$.0015 = 0.6 \times 0.0001767 \times \sqrt{2 \times 9.81 (0.9 + p / \rho g)}$$

$$\therefore \sqrt{2 \times 9.81 \left( 0.9 + \frac{10p}{9.81} \right)} = \frac{.0015}{0.6 \times 0.0001767} = 14.148$$

$$\text{or } 2 \times 9.81 \left( 0.9 + \frac{10p}{9.81} \right) = 14.148 \times 14.148$$

$$\therefore \frac{10p}{9.81} = \frac{14.148 \times 14.148}{2 \times 9.81} - 0.9 = 10.202 - 0.9 = 9.302$$

$$\therefore p = \frac{9.302 \times 9.81}{10} = 9.125 \text{ N/cm}^2. \text{ Ans.}$$

## ► 7.6 FLOW THROUGH LARGE ORIFICES

If the head of liquid is less than 5 times the depth of the orifice, the orifice is called large orifice. In case of small orifice, the velocity in the entire cross-section of the jet is considered to be constant and discharge can be calculated by  $Q = C_d \times a \times \sqrt{2gh}$ . But in case of a large orifice, the velocity is not constant over the entire cross-section of the jet and hence  $Q$  cannot be calculated by  $Q = C_d \times a \times \sqrt{2gh}$ .

**7.6.1 Discharge Through Large Rectangular Orifice.** Consider a large rectangular orifice in one side of the tank discharging freely into atmosphere under a constant head,  $H$  as shown in Fig. 7.7.

- Let
- $H_1$  = height of liquid above top edge of orifice
  - $H_2$  = height of liquid above bottom edge of orifice
  - $b$  = breadth of orifice
  - $d$  = depth of orifice =  $H_2 - H_1$
  - $C_d$  = co-efficient of discharge.

Consider an elementary horizontal strip of depth ' $dh$ ' at a depth of ' $h$ ' below the free surface of the liquid in the tank as shown in Fig. 7.7 (b).

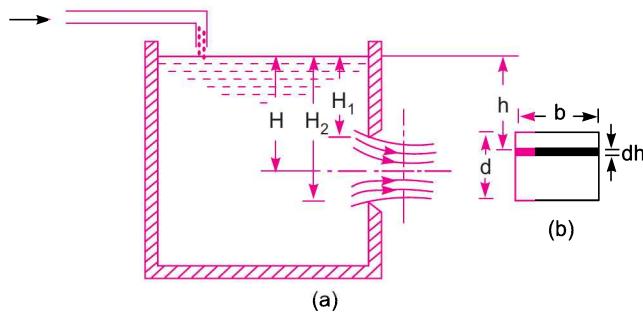


Fig. 7.7 Large rectangular orifice.

$$\therefore \text{Area of strip} = b \times dh$$

and theoretical velocity of water through strip =  $\sqrt{2gh}$ .

$\therefore$  Discharge through elementary strip is given

$$\begin{aligned} dQ &= C_d \times \text{Area of strip} \times \text{Velocity} \\ &= C_d \times b \times dh \times \sqrt{2gh} = C_d b \times \sqrt{2gh} dh \end{aligned}$$

By integrating the above equation between the limits  $H_1$  and  $H_2$ , the total discharge through the whole orifice is obtained

$$\begin{aligned} \therefore Q &= \int_{H_1}^{H_2} C_d \times b \times \sqrt{2gh} dh \\ &= C_d \times b \times \sqrt{2g} \int_{H_1}^{H_2} \sqrt{h} dh = C_d \times b \times \sqrt{2g} \left[ \frac{h^{3/2}}{3/2} \right]_{H_1}^{H_2} \\ &= \frac{2}{3} C_d \times b \sqrt{2g} [H_2^{3/2} - H_1^{3/2}] \quad \dots(7.8) \end{aligned}$$

**Problem 7.11** Find the discharge through a rectangular orifice 2.0 m wide and 1.5 m deep fitted to a water tank. The water level in the tank is 3.0 m above the top edge of the orifice. Take  $C_d = 0.62$ .

**Solution.** Given :

$$\text{Width of orifice, } b = 2.0 \text{ m}$$

$$\text{Depth of orifice, } d = 1.5 \text{ m}$$

$$\text{Height of water above top edge of the orifice, } H_1 = 3 \text{ m}$$

Height of water above bottom edge of the orifice,

$$H_2 = H_1 + d = 3 + 1.5 = 4.5 \text{ m}$$

$$C_d = 0.62$$

Discharge  $Q$  is given by equation (7.8) as

$$\begin{aligned} Q &= \frac{2}{3} C_d \times b \times \sqrt{2g} [H_2^{3/2} - H_1^{3/2}] \\ &= \frac{2}{3} \times 0.62 \times 2.0 \times \sqrt{2+9.81} [4.5^{1.5} - 3^{1.5}] \text{ m}^3/\text{s} \\ &= 3.66[9.545 - 5.196] \text{ m}^3/\text{s} = \mathbf{15.917 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 7.12** A rectangular orifice, 1.5 m wide and 1.0 m deep is discharging water from a tank. If the water level in the tank is 3.0 m above the top edge of the orifice, find the discharge through the orifice. Take the co-efficient of discharging for the orifice = 0.6.

**Solution.** Given :

$$\text{Width of orifice, } b = 1.5 \text{ m}$$

$$\text{Depth of orifice, } d = 1.0 \text{ m}$$

$$H_1 = 3.0 \text{ m}$$

$$H_2 = H_1 + d = 3.0 + 1.0 = 4.0 \text{ m}$$

$$C_d = 0.6$$

Discharge,  $Q$  is given by the equation (7.8) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times b \times \sqrt{2g} [H_2^{3/2} - H_1^{3/2}] \\ &= \frac{2}{3} \times 0.6 \times 1.5 \times \sqrt{2+9.81} [4.0^{1.5} - 3.0^{1.5}] \text{ m}^3/\text{s} \\ &= 2.657 [8.0 - 5.196] \text{ m}^3/\text{s} = \mathbf{7.45 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 7.13** A rectangular orifice 0.9 m wide and 1.2 m deep is discharging water from a vessel. The top edge of the orifice is 0.6 m below the water surface in the vessel. Calculate the discharge through the orifice if  $C_d = 0.6$  and percentage error if the orifice is treated as a small orifice.

**Solution.** Given :

$$\text{Width of orifice, } b = 0.9 \text{ m}$$

$$\text{Depth of orifice, } d = 1.2 \text{ m}$$

$$H_2 = 0.6 \text{ m}$$

$$H_2 = H_1 + d = 0.6 + 1.2 = 1.8 \text{ m}$$

$$C_d = 0.6$$

Discharge  $Q$  is given as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times b \times \sqrt{2g} \times [H_2^{3/2} - H_1^{3/2}] \\ &= \frac{2}{3} \times 0.6 \times 2.9 \times \sqrt{2 \times 9.81} [1.8^{3/2} - 0.6^{3/2}] \text{ m}^3/\text{s} \\ &= 1.5946 [2.4149 - .4647] = \mathbf{3.1097 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

Discharging for a small orifice

$$Q_1 = C_d \times a \times \sqrt{2gh}$$

$$\text{where } h = H_1 + \frac{d}{2} = 0.6 + \frac{1.2}{2} = 1.2 \text{ m and } a = b \times d = 0.9 \times 1.2$$

$$Q_1 = 0.6 \times .9 \times 1.2 \times \sqrt{2 \times 9.81 \times 1.2} = 3.1442 \text{ m}^3/\text{s}$$

$$\% \text{ error} = \frac{Q_1 - Q}{Q} = \frac{3.1442 - 3.1097}{3.1097} = 0.01109 \text{ or } 1.109\%. \text{ Ans.}$$

### ► 7.7 DISCHARGE THROUGH FULLY SUB-MERGED ORIFICE

Fully sub-merged orifice is one which has its whole of the outlet side sub-merged under liquid so that it discharges a jet of liquid into the liquid of the same kind. It is also called totally drowned orifice. Fig. 7.8 shows the fully sub-merged orifice. Consider two points (1) and (2), point 1 being in the reservoir on the upstream side of the orifice and point 2 being at the vena-contracta as shown in Fig. 7.8.

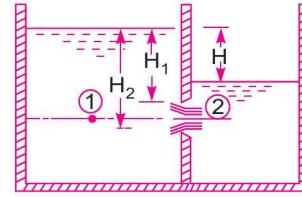


Fig. 7.8 Fully sub-merged orifice.

Let  $H_1$  = Height of water above the top of the orifice on the upstream side,

$H_2$  = Height of water above the bottom of the orifice,

$H$  = Difference in water level,

$b$  = Width of orifice,

$C_d$  = Co-efficient of discharge.

Height of water above the centre of orifice on upstream side

$$= H_1 + \frac{H_2 - H_1}{2} = \frac{H_1 + H_2}{2} \quad \dots(1)$$

Height of water above the centre of orifice on downstream side

$$= \frac{H_1 + H_2}{2} - H \quad \dots(2)$$

Applying Bernoulli's equation at (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} \quad [\because z_1 = z_2]$$

Now  $\frac{p_1}{\rho g} = \frac{H_1 + H_2}{2}$ ,  $\frac{p_2}{\rho g} = \frac{H_1 + H_2}{2} - H$  and  $V_1$  is negligible

$$\therefore \frac{H_1 + H_2}{2} + 0 = \frac{H_1 + H_2}{2} - H + \frac{V_2^2}{2g}$$

$$\therefore \frac{V_2^2}{2g} = H$$

$$\therefore V_2 = \sqrt{2gH}$$

$$\text{Area of orifice} = b \times (H_2 - H_1)$$

$$\therefore \text{Discharge through orifice} = C_d \times \text{Area} \times \text{Velocity}$$

$$= C_d \times b (H_2 - H_1) \times \sqrt{2gH}$$

$$\therefore Q = C_d \times b (H_2 - H_1) \times \sqrt{2gH}. \quad \dots(7.9)$$

**Problem 7.14** Find the discharge through a fully sub-merged orifice of width 2 m if the difference of water levels on both sides of the orifice be 50 cm. The height of water from top and bottom of the orifice are 2.5 m and 2.75 m respectively. Take  $C_d = 0.6$ .

**Solution.** Given :

$$\text{Width of orifice, } b = 2 \text{ m}$$

$$\text{Difference of water level, } H = 50 \text{ cm} = 0.5 \text{ m}$$

$$\text{Height of water from top of orifice, } H_1 = 2.5 \text{ m}$$

$$\text{Height of water from bottom of orifice, } H_2 = 2.75 \text{ m}$$

$$C_d = 0.6$$

Discharge through fully sub-merged orifice is given by equation (7.9)

or

$$Q = C_d \times b \times (H_2 - H_1) \times \sqrt{2gH}$$

$$= 0.6 \times 2.0 \times (2.75 - 2.5) \times \sqrt{2 \times 9.81 \times 0.5} \text{ m}^3/\text{s}$$

$$= 0.9396 \text{ m}^3/\text{s. Ans.}$$

**Problem 7.15** Find the discharge through a totally drowned orifice 2.0 m wide and 1 m deep, if the difference of water levels on both the sides of the orifice be 3 m. Take  $C_d = 0.62$ .

**Solution.** Given :

$$\text{Width of orifice, } b = 2.0 \text{ m}$$

$$\text{Depth of orifice, } d = 1 \text{ m.}$$

Difference of water level on both the sides

$$H = 3 \text{ m}$$

$$C_d = 0.62$$

Discharge through orifice is  $Q = C_d \times \text{Area} \times \sqrt{2gH}$

$$= 0.62 \times b \times d \times \sqrt{2gH}$$

$$= 0.62 \times 2.0 \times 1.0 \times \sqrt{2 \times 9.81 \times 3} \text{ m}^3/\text{s} = 9.513 \text{ m}^3/\text{s. Ans.}$$

## ► 7.8 DISCHARGE THROUGH PARTIALLY SUB-MERGED ORIFICE

Partially sub-merged orifice is one which has its outlet side partially sub-merged under liquid as shown in Fig. 7.9. It is also known as partially drowned orifice. Thus the partially sub-merged orifice has two portions. The upper portion behaves as an orifice discharging free while the lower portion behaves as a sub-merged orifice. Only a large orifice can behave as a partially sub-merged orifice. The total discharge  $Q$  through partially sub-merged orifice is equal to the discharges through free and the sub-merged portions.

Discharge through the sub-merged portion is given by equation (7.9)

$$Q_1 = C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

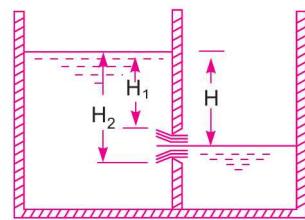


Fig. 7.9 Partially sub-merged orifice.

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Discharge through the free portion is given by equation (7.8) as

$$Q_2 = \frac{2}{3} C_d \times b \times \sqrt{2g} [H_2^{3/2} - H_1^{3/2}]$$

$$\therefore \text{Total discharge } Q = Q_1 + Q_2$$

$$= C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

$$+ \frac{2}{3} C_d \times b \times \sqrt{2g} [H_2^{3/2} - H_1^{3/2}] \dots (7.10)$$

**Problem 7.16** A rectangular orifice of 2 m width and 1.2 m deep is fitted in one side of a large tank. The water level on one side of the orifice is 3 m above the top edge of the orifice, while on the other side of the orifice, the water level is 0.5 m below its top edge. Calculate the discharge through the orifice if  $C_d = 0.64$ .

**Solution.** Given : Width of orifice,  $b = 2 \text{ m}$

Depth of orifice,  $d = 1.2 \text{ m}$

Height of water from top edge of orifice,  $H_1 = 3 \text{ m}$

Difference of water level on both sides,  $H = 3 + 0.5 = 3.5 \text{ m}$

Height of water from the bottom edge of orifice,  $H_2 = H_1 + d = 3 + 1.2 = 4.2 \text{ m}$

The orifice is partially sub-merged. The discharge through sub-merged portion,

$$Q_1 = C_d \times b \times (H_2 - H) \times \sqrt{2gH}$$

$$= 0.64 \times 2.0 \times (4.2 - 3.5) \times \sqrt{2 \times 9.81 \times 3.5} = 7.4249 \text{ m}^3/\text{s}$$

The discharge through free portion is

$$Q_2 = \frac{2}{3} C_d \times b \times \sqrt{2g} [H_2^{3/2} - H_1^{3/2}]$$

$$= \frac{2}{3} \times 0.64 \times 2.0 \times \sqrt{2 \times 9.81} [3.5^{3/2} - 3.0^{3/2}]$$

$$= 3.779 [6.5479 - 5.1961] = 5.108 \text{ m}^3/\text{s}$$

$\therefore$  Total discharge through the orifice is

$$Q = Q_1 + Q_2 = 7.4249 + 5.108 = \mathbf{12.5329 \text{ m}^3/\text{s. Ans.}}$$

### ► 7.9 TIME OF EMPTYING A TANK THROUGH AN ORIFICE AT ITS BOTTOM

Consider a tank containing some liquid upto a height of  $H_1$ . Let an orifice is fitted at the bottom of the tank. It is required to find the time for the liquid surface to fall from the height  $H_1$  to a height  $H_2$ .

Let  $A$  = Area of the tank

$a$  = Area of the orifice

$H_1$  = Initial height of the liquid

$H_2$  = Final height of the liquid

$T$  = Time in seconds for the liquid to fall from  $H_1$  to  $H_2$ .

Let at any time, the height of liquid from orifice is  $h$  and let the liquid surface fall by a small height  $dh$  in time  $dT$ . Then

Volume of liquid leaving the tank in time,  $dT = A \times dh$

Also the theoretical velocity through orifice,  $V = \sqrt{2gh}$

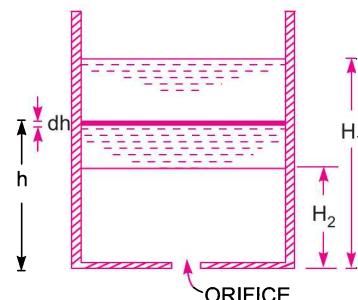


Fig. 7.9. (a)

∴ Discharge through orifice/sec,

$$dQ = C_d \times \text{Area of orifice} \times \text{Theoretical velocity} = C_d \cdot a \cdot \sqrt{2gh}$$

∴ Discharge through orifice in time interval

$$dT = C_d \cdot a \cdot \sqrt{2gh} \cdot dT$$

As the volume of liquid leaving the tank is equal to the volume of liquid flowing through orifice in time  $dT$ , we have

$$A(-dh) = C_d \cdot a \cdot \sqrt{2gh} \cdot dT$$

- ve sign is inserted because with the increase of time, head on orifice decreases.

$$\therefore -Adh = C_d \cdot a \cdot \sqrt{2gh} \cdot dT \text{ or } dT = \frac{-A dh}{C_d \cdot a \cdot \sqrt{2gh}} = \frac{-A(h)^{-1/2}}{C_d \cdot a \cdot \sqrt{2g}} dh$$

By integrating the above equation between the limits  $H_1$  and  $H_2$ , the total time,  $T$  is obtained as

$$\int_0^T dT = \int_{H_1}^{H_2} \frac{-Ah^{-1/2} dh}{C_d \cdot a \cdot \sqrt{2g}} = \frac{-A}{C_d \cdot a \cdot \sqrt{2g}} \int_{H_1}^{H_2} h^{-1/2} dh$$

or

$$\begin{aligned} T &= \frac{-A}{C_d \cdot a \cdot \sqrt{2g}} \left[ \frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_{H_1}^{H_2} = \frac{-A}{C_d \cdot a \cdot \sqrt{2g}} \left[ \frac{\sqrt{h}}{\frac{1}{2}} \right]_{H_1}^{H_2} \\ &= \frac{-2A}{C_d \cdot a \cdot \sqrt{2g}} [\sqrt{H_2} - \sqrt{H_1}] = \frac{2A [\sqrt{H_1} - \sqrt{H_2}]}{C_d \cdot a \cdot \sqrt{2g}} \end{aligned} \quad \dots(7.11)$$

For emptying the tank completely,  $H_2 = 0$  and hence

$$T = \frac{2A \sqrt{H_1}}{C_d \cdot a \cdot \sqrt{2g}}. \quad \dots(7.12)$$

**Problem 7.17** A circular tank of diameter 4 m contains water upto a height of 5 m. The tank is provided with an orifice of diameter 0.5 m at the bottom. Find the time taken by water (i) to fall from 5 m to 2 m (ii) for completely emptying the tank. Take  $C_d = 0.6$ .

**Solution.** Given :

Dia. of tank,

$$D = 4 \text{ m}$$

∴ Area,

$$A = \frac{\pi}{4} (4)^2 = 12.566 \text{ m}^2$$

Dia. of orifice,

$$d = 0.5 \text{ m}$$

∴ Area,

$$a = \frac{\pi}{4} (.5)^2 = 0.1963 \text{ m}^2$$

Initial height of water,

$$H_1 = 5 \text{ m}$$

Final height of water, (i)

$$H_2 = 2 \text{ m} \quad (ii) \quad H_2 = 0$$

**First Case.** When

$$H_2 = 2 \text{ m}$$

Using equation (7.11), we have  $T = \frac{2A}{C_d \cdot a \cdot \sqrt{2g}} [\sqrt{H_1} - \sqrt{H_2}]$

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$$\begin{aligned}
 &= \frac{2 \times 12.566}{0.6 \times .1963 \times \sqrt{2 \times 9.81}} [\sqrt{5} - \sqrt{2.0}] \text{ seconds} \\
 &= \frac{20.653}{0.5217} = 39.58 \text{ seconds. Ans.}
 \end{aligned}$$

**Second Case.** When  $H_2 = 0$

$$\begin{aligned}
 T &= \frac{2A}{C_d \cdot a \cdot \sqrt{2g}} \sqrt{H_1} = \frac{2 \times 12.566 \times \sqrt{5}}{0.6 \times .1963 \times \sqrt{2 \times 9.81}} \\
 &= 107.7 \text{ seconds. Ans.}
 \end{aligned}$$

**Problem 7.18** A circular tank of diameter 1.25 m contains water upto a height of 5 m. An orifice of 50 mm diameter is provided at its bottom. If  $C_d = 0.62$ , find the height of water above the orifice after 1.5 minutes.

**Solution.** Given :

$$\text{Dia. of tank, } D = 1.25 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4}(1.25)^2 = 1.227 \text{ m}^2$$

$$\text{Dia. of orifice, } d = 50 \text{ mm} = .05 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (.05)^2 = .001963 \text{ m}^2$$

$$C_d = 0.62$$

$$\text{Initial height of water, } H_1 = 5 \text{ m}$$

$$\text{Time in seconds, } T = 1.5 \times 60 = 90 \text{ seconds}$$

Let the height of water after 90 seconds =  $H_2$

$$\text{Using equation (7.11), we have } T = \frac{2A[\sqrt{H_1} - \sqrt{H_2}]}{C_d \cdot a \cdot \sqrt{2g}}$$

$$\text{or } 90 = \frac{2 \times 1.227 [\sqrt{5} - \sqrt{H_2}]}{0.62 \times 0.001963 \times \sqrt{2 \times 9.81}} = 455.215 [2.236 - \sqrt{H_2}]$$

$$\therefore \sqrt{H_2} = 2.236 - \frac{90}{455.215} = 2.236 - 0.1977 = 2.0383$$

$$\therefore H_2 = 2.0383 \times 2.0383 = 4.154 \text{ m. Ans.}$$

## ► 7.10 TIME OF EMPTYING A HEMISpherical TANK

Consider a hemispherical tank of radius  $R$  fitted with an orifice of area ' $a$ ' at its bottom as shown in Fig. 7.10. The tank contains some liquid whose initial height is  $H_1$  and in time  $T$ , the height of liquid falls to  $H_2$ . It is required to find the time  $T$ .

Let at any instant of time, the head of liquid over the orifice is  $h$  and at this instant let  $x$  be the radius of the liquid surface. Then

Area of liquid surface,  $A = \pi x^2$

and theoretical velocity of liquid =  $\sqrt{2gh}$ .

Let the liquid level falls down by an amount of  $dh$  in time  $dT$ .

$$\text{tank in time } dT = A \times dh - \pi r^2 \times dh \quad (i)$$

Also volume of liquid flowing through orifice

$$\equiv C_d \times \text{area of orifice} \times \text{velocity} \equiv C_d q_r \sqrt{2gh} \text{ second}$$

$\therefore$  Volume of liquid flowing through orifice in time  $dT$

$$= C_d g \sqrt{2gh} \times dT \quad \dots(ii)$$

From equations (i) and (ii), we get

$$\pi x^2 (-dh) = C_d \cdot a_{\sqrt{2gh}} \cdot dT$$

-ve sign is introduced, because with the increase of  $T$ ,  $h$  will decrease

$$-\pi x^2 dh = C \cdot a \cdot \sqrt{2gh} \cdot dT \quad \dots(iii)$$

But from Fig. 7.10, for  $\Delta OCD$ , we have  $OC = R$

$$DO = R - h$$

$$\therefore CD = x = \sqrt{OC^2 - OD^2} = \sqrt{R^2 - (R-h)^2}$$

$$\therefore x^2 \equiv R^2 - (R - h)^2 \equiv R^2 - (R^2 + h^2 - 2Rh) \equiv 2Rh - h^2$$

Substituting  $x^2$  in equation (iii), we get

$$-\pi(2Rh - h^2)dh = C_d \cdot a \cdot \sqrt{2gh} \cdot dT$$

$$dT = \frac{-\pi (2Rh - h^2) dh}{C_d \cdot a \cdot \sqrt{2gh}} = \frac{-\pi}{C_d \cdot a \cdot \sqrt{2g}} (2Rh - h^2) h^{-1/2} dh$$

$$= \frac{-\pi}{C_d \cdot a \cdot \sqrt{2g}} (2Rh^{1/2} - h^{3/2}) dh$$

The total time  $T$  required to bring the liquid level from  $H_1$  to  $H_2$  is obtained by integrating the above equation between the limits  $H_1$  and  $H_2$ .

$$\therefore T = \int_{H_1}^{H_2} \frac{-\pi}{C_d \cdot a \cdot \sqrt{2g}} (2Rh^{1/2} - h^{3/2}) dh$$

$$= \frac{-\pi}{C_d \cdot a \cdot \sqrt{2g}} \int_{H_1}^{H_2} (2Rh^{1/2} - h^{3/2}) dh$$

$$\begin{aligned}
 &= \frac{-\pi}{C_d \times a \times \sqrt{2g}} \left[ 2R \frac{\frac{h^{1/2+1}}{1/2+1} - \frac{h^{3/2}+1}{3/2+1}}{H_2} \right]_{H_1}^{H_2} \\
 &= \frac{-\pi}{C_d \times a \times \sqrt{2g}} \left[ 2 \times \frac{2}{3} Rh^{3/2} - \frac{2}{5} h^{5/2} \right]_{H_1}^{H_2} \\
 &= \frac{-\pi}{C_d \times a \times \sqrt{2g}} \left[ \frac{4}{3} R(H_2^{3/2} - H_1^{3/2}) - \frac{2}{5}(H_2^{5/2} - H_1^{5/2}) \right] \\
 &= \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[ \frac{4}{3} R(H_1^{3/2} - H_2^{3/2}) - \frac{2}{5}(H_1^{5/2} - H_2^{5/2}) \right] \quad \dots(7.13)
 \end{aligned}$$

For completely emptying the tank,  $H_2 = 0$  and hence

$$T = \frac{\pi}{C_d \cdot a \cdot \sqrt{2g}} \left[ \frac{4}{3} RH_1^{3/2} - \frac{2}{5} H_1^{5/2} \right]. \quad \dots(7.14)$$

**Problem 7.19** A hemispherical tank of diameter 4 m contains water upto a height of 1.5 m. An orifice of diameter 50 mm is provided at the bottom. Find the time required by water (i) to fall from 1.5 m to 1.0 m (ii) for completely emptying the tank. Tank  $C_d = 0.6$ .

**Solution.** Given :

Dia. of hemispherical tank,  $D = 4$  m

∴ Radius,  $R = 2.0$  m

Dia. of orifice,  $d = 50$  mm = 0.05 m

∴ Area,  $a = \frac{\pi}{4}(0.05)^2 = 0.001963 \text{ m}^2$

Initial height of water,  $H_1 = 1.5$  m

$C_d = 0.6$

**First Case.**  $H_2 = 1.0$

Time  $T$  is given by equation (7.13)

$$\begin{aligned}
 \therefore T &= \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[ \frac{4}{3} R(H_1^{3/2} - H_2^{3/2}) - \frac{2}{5}(H_1^{5/2} - H_2^{5/2}) \right] \\
 &= \frac{\pi}{0.6 \times 0.001963 \times \sqrt{2 \times 9.81}} \times \left[ \frac{4}{3} \times 2.0 (1.5^{3/2} - 1.0^{3/2}) - \frac{2}{5} (1.5^{5/2} - 1.0^{5/2}) \right] \\
 &= 602.189 [2.2323 - 0.7022] = 921.4 \text{ second} \\
 &= \mathbf{15 \text{ min } 21.4 \text{ sec. Ans.}}
 \end{aligned}$$

**Second Case.**  $H_2 = 0$  and hence time  $T$  is given by equation (7.14)

$$\begin{aligned}
 \therefore T &= \frac{\pi}{C_d \cdot a \cdot \sqrt{2g}} \left[ \frac{4}{3} RH_1^{3/2} - \frac{2}{5} H_1^{5/2} \right] \\
 &= \frac{\pi}{0.6 \times 0.001963 \sqrt{2 \times 9.81}} \left[ \frac{4}{3} \times 2.0 \times 1.5^{3/2} - \frac{2}{5} \times 1.5^{5/2} \right]
 \end{aligned}$$

$$= 602.189 [4.8989 - 1.1022] \text{ sec} = 2286.33 \text{ sec}$$

**= 38 min 6.33 sec. Ans.**

**Problem 7.20** A hemispherical cistern of 6 m radius is full of water. It is fitted with a 75 mm diameter sharp edged orifice at the bottom. Calculate the time required to lower the level in the cistern by 2 metres. Assume co-efficient of discharge for the orifice is 0.6.

**Solution.** Given :

Radius of hemispherical cistern,  $R = 6 \text{ m}$

Initial height of water,  $H_1 = 6 \text{ m}$

Dia. of orifice,  $d = 75 \text{ mm} = 0.075 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.075)^2 = .004418 \text{ m}^2$$

Fall of height of water = 2 m

$\therefore$  Final height of water,  $H_2 = 6 - 2 = 4 \text{ m}$

$$C_d = 0.6$$

The time  $T$  is given by equation (7.31)

$$\begin{aligned} T &= \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[ \frac{4}{3} R (H_1^{3/2} - H_2^{3/2}) - \frac{2}{5} (H_1^{5/2} - H_2^{5/2}) \right] \\ &= \frac{\pi}{0.6 \times 0.004418 \times \sqrt{2 \times 9.81}} \\ &\quad \times \left[ \frac{4}{3} \times 6 (6.0^{3/2} - 4.0^{3/2}) - \frac{2}{5} (6.0^{5/2} - 4.0^{5/2}) \right] \\ &= 267.56 [8(14.6969 - 8.0) - 0.4 (88.18 - 32.0)] \\ &= 267.56 [53.575 - 22.472] \text{ sec} \\ &= 8321.9 \text{ sec} = \mathbf{2 \text{ hrs } 18 \text{ min } 42 \text{ sec. Ans.}} \end{aligned}$$

**Problem 7.21** A cylindrical tank is having a hemispherical base. The height of cylindrical portion is 5 m and diameter is 4 m. At the bottom of this tank an orifice of diameter 200 mm is fitted. Find the time required to completely emptying the tank. Take  $C_d = 0.6$ .

**Solution.** Given :

Height of cylindrical portion (II) = 5 m

Dia. of tank = 4.0 m

$$\therefore \text{Area, } A = \frac{\pi}{4} (4)^2 = 12.566 \text{ m}^2$$

Dia. of orifice,  $d = 200 \text{ mm} = 0.2 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$$

$$C_d = 0.6$$

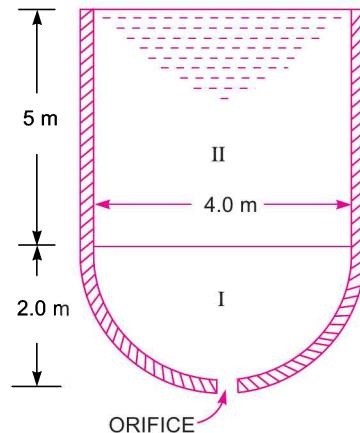


Fig. 7.11

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The tank is splitted in two portions. First portion is a hemispherical tank and second portion is cylindrical tank.

Let  $T_1$  = time for emptying hemispherical portion I.

$T_2$  = time for emptying cylindrical portion II.

Then total time  $T = T_1 + T_2$ .

**For Portion I.**  $H_1 = 2.0 \text{ m}$ ,  $H_2 = 0$ . Then  $T_1$  is given by equation (7.14) as

$$\begin{aligned} T_1 &= \frac{\pi}{C_d \times a \times \sqrt{2g}} \left[ \frac{4}{3} R H_1^{3/2} - \frac{2}{5} H_1^{5/2} \right] \\ &= \frac{\pi}{0.6 \times .0314 \times \sqrt{2 \times 9.81}} \left[ \frac{4}{3} \times 2.0 \times 2.0^{3/2} - \frac{2}{5} \times 2.0^{5/2} \right] \\ &= 37.646 [7.5424 - 2.262] \text{ sec} = 198.78 \text{ sec.} \end{aligned}$$

**For Portion II.**  $H_1 = 2.0 + 5.0 = 7.0 \text{ m}$ ,  $H_2 = 2.0$ . Then  $T_2$  is given by equation (7.11) as

$$T_2 = \frac{2A[\sqrt{H_1} - \sqrt{H_2}]}{C_d \times a \times \sqrt{2g}} = \frac{2 \times 12.566 [\sqrt{7} - \sqrt{2.0}]}{0.6 \times .0314 \times \sqrt{2 \times 9.81}} \text{ sec} = 370.92 \text{ sec}$$

∴ Total time,

$$\begin{aligned} T &= T_1 + T_2 = 198.78 + 370.92 = 569.7 \text{ sec} \\ &= \mathbf{9 \text{ min } 29 \text{ sec. Ans.}} \end{aligned}$$

#### ► 7.11 TIME OF EMPTYING A CIRCULAR HORIZONTAL TANK

Consider a circular horizontal tank of length  $L$  and radius  $R$ , containing liquid upto a height of  $H_1$ . Let an orifice of area ' $a$ ' is fitted at the bottom of the tank. Then the time required to bring the liquid level from  $H_1$  to  $H_2$  is obtained as :

Let at any time, the height of liquid over orifice is ' $h$ ' and in time  $dT$ , let the height falls by an height of ' $dh$ '. Let at this time, the width of liquid surface =  $AC$  as shown in Fig. 7.12.

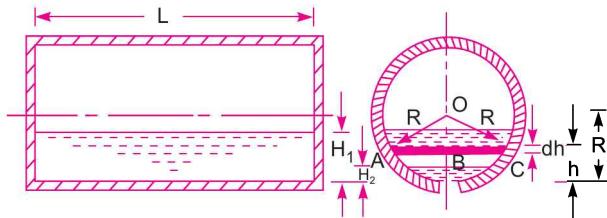


Fig. 7.12

∴ Surface area of liquid =  $L \times AC$

$$\begin{aligned} \text{But } AC &= 2 \times AB = 2 \left[ \sqrt{AO^2 - OB^2} \right] = 2 \left[ \sqrt{R^2 - (R-h)^2} \right] \\ &= 2 \sqrt{R^2 - (R^2 + h^2 - 2Rh)} = 2 \sqrt{2Rh - h^2} \end{aligned}$$

$$\therefore \text{Surface area, } A = L \times 2\sqrt{2Rh - h^2}$$

$\therefore$  Volume of liquid leaving tank in time  $dT$

$$= A \times dh = 2L \sqrt{2Rh - h^2} \times dh \quad \dots(i)$$

Also the volume of liquid flowing through orifice in time  $dT$

$$= C_d \times \text{Area of orifice} \times \text{Velocity} \times dT$$

But the velocity of liquid at the time considered =  $\sqrt{2gh}$

$\therefore$  Volume of liquid flowing through orifice in time  $dT$

$$= C_d \times a \times \sqrt{2gh} \times dT \quad \dots(ii)$$

Equating (i) and (ii), we get

$$2L \sqrt{2Rh - h^2} \times (-dh) = C_d \times a \times \sqrt{2gh} \times dT$$

- ve sign is introduced as with the increase of  $T$ , the height  $h$  decreases,

$$\therefore dT = \frac{-2L \sqrt{2Rh - h^2} dh}{C_d \times a \times \sqrt{2gh}} = \frac{-2L \sqrt{(2R - h)} dh}{C_d \times a \times \sqrt{2g}}$$

[Taking  $\sqrt{h}$  common]

$$\begin{aligned} \therefore \text{Total time, } T &= \int_{H_1}^{H_2} \frac{-2L(2R-h)^{1/2} dh}{C_d \times a \times \sqrt{2g}} \\ &= \frac{-2L}{C_d \times a \times \sqrt{2g}} \int_{H_1}^{H_2} [2R-h]^{1/2} dh \\ &= \frac{-2L}{C_d \times a \times \sqrt{2g}} \left[ \frac{(2R-h)^{1/2+1}}{\frac{1}{2}+1} \times (-1) \right]_{H_1}^{H_2} \\ &= \frac{2L}{C_d \times a \times \sqrt{2g}} \times \frac{2}{3} \times [(2R-h)^{3/2}]_{H_1}^{H_2} \\ &= \frac{4L}{3C_d \times a \times \sqrt{2g}} [(2R-H_2)^{3/2} - (2R-H_1)^{3/2}] \end{aligned} \quad \dots(7.15)$$

For completely emptying the tank,  $H_2 = 0$  and hence

$$T = \frac{4L}{3C_d \times a \times \sqrt{2g}} [(2R)^{3/2} - (2R-H_1)^{3/2}] \quad \dots(7.16)$$

**Problem 7.22** An orifice of diameter 100 mm is fitted at the bottom of a boiler drum of length 5 m and of diameter 2 m. The drum is horizontal and half full of water. Find the time required to empty the boiler, given the value of  $C_d = 0.6$ .

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**Solution.** Given :

$$\text{Dia. of orifice, } d = 100 \text{ mm} = 0.1 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (.1)^2 = .007854 \text{ m}^2$$

$$\text{Length, } L = 5 \text{ m}$$

$$\text{Dia. of drum, } D = 2 \text{ m}$$

$$\therefore \text{Radius, } R = 1 \text{ m}$$

$$\text{Initial height of water, } H_1 = 1 \text{ m}$$

$$\text{Final height of water, } H_2 = 0$$

$$C_d = 0.6$$

For completely emptying the tank,  $T$  is given by equation (7.16)

$$\begin{aligned} \therefore T &= \frac{4L}{3 \times C_d \times a \times \sqrt{2g}} [(2R)^{3/2} - (2R - H_1)^{3/2}] \\ &= \frac{4 \times 5.0}{3 \times .06 \times .007854 \times \sqrt{2 \times 9.81}} [(2 \times 1)^{3/2} - (2 \times 1 - 1)^{3/2}] \\ &= 319.39 [2.8284 - 1.0] = 583.98 \text{ sec} = \mathbf{9 \text{ min } 44 \text{ sec. Ans.}} \end{aligned}$$

**Problem 7.23** An orifice of diameter 150 mm is fitted at the bottom of a boiler drum of length 8 m and of diameter 3 metres. The drum is horizontal and contains water upto a height of 2.4 m. Find the time required to empty the boiler. Take  $C_d = 0.6$ .

**Solution.** Given :

$$\text{Dia. of orifice, } d = 150 \text{ mm} = 0.15 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2$$

$$\text{Length, } L = 8.0 \text{ m}$$

$$\text{Dia. of boiler, } D = 3.0 \text{ m}$$

$$\therefore \text{Radius, } R = 1.5 \text{ m}$$

$$\text{Initial height of water, } H_1 = 2.4 \text{ m}$$

$$\text{Find height of water, } H_2 = 0$$

$$C_d = 0.6.$$

For completely emptying the tank,  $T$  is given by equation (7.16) as

$$\begin{aligned} T &= \frac{4L}{3C_d \times a \times \sqrt{2g}} [(2R)^{3/2} - (2R - H_1)^{3/2}] \\ &= \frac{4 \times 8.0}{3 \times .6 \times .01767 \times \sqrt{2 \times 9.81}} [(2 \times 1.5)^{3/2} - (2 \times 1.5 - 2.4)^{3/2}] \\ &= 227.14 [5.196 - 0.4647] = 1074.66 \text{ sec} \\ &= \mathbf{17 \text{ min } 54.66 \text{ sec. Ans.}} \end{aligned}$$

## ► 7.12 CLASSIFICATION OF MOUTHPIECES

1. The mouthpieces are classified as (i) External mouthpiece or (ii) Internal mouthpiece depending upon their position with respect to the tank or vessel to which they are fitted.
2. The mouthpiece are classified as (i) Cylindrical mouthpiece or (ii) Convergent mouthpiece or (iii) Convergent-divergent mouthpiece depending upon their shapes.
3. The mouthpieces are classified as (i) Mouthpieces running full or (ii) Mouthpieces running free, depending upon the nature of discharge at the outlet of the mouthpiece. This classification is only for internal mouthpieces which are known Borda's or Re-entrant mouthpieces. A mouthpiece is said to be running free if the jet of liquid after contraction does not touch the sides of the mouthpiece. But if the jet after contraction expands and fills the whole mouthpiece it is known as running full.

## ► 7.13 FLOW THROUGH AN EXTERNAL CYLINDRICAL MOUTHPIECE

A mouthpiece is a short length of a pipe which is two or three times its diameter in length. If this pipe is fitted externally to the orifice, the mouthpiece is called external cylindrical mouthpiece and the discharge through orifice increases.

Consider a tank having an external cylindrical mouthpiece of cross-sectional area  $a_1$ , attached to one of its sides as shown in Fig. 7.13. The jet of liquid entering the mouthpiece contracts to form a vena-contracta at a section  $C-C$ . Beyond this section, the jet again expands and fill the mouthpiece completely.

Let  $H$  = Height of liquid above the centre of mouthpiece  
 $v_c$  = Velocity of liquid at  $C-C$  section  
 $a_c$  = Area of flow at vena-contracta  
 $v_1$  = Velocity of liquid at outlet  
 $a_1$  = Area of mouthpiece at outlet  
 $C_c$  = Co-efficient of contraction.

Applying continuity equation at  $C-C$  and (1)-(1), we get

$$a_c \times v_c = a_1 v_1$$

$$\therefore v_c = \frac{a_1 v_1}{a_c} = \frac{v_1}{a_c/a_1}$$

But  $\frac{a_c}{a_1} = C_c$  = Co-efficient of contraction

Taking  $C_c = 0.62$ , we get  $\frac{a_c}{a_1} = 0.62$

$$\therefore v_c = \frac{v_1}{0.62}$$

The jet of liquid from section  $C-C$  suddenly enlarges at section (1)-(1). Due to sudden enlargement,

there will be a loss of head,  $h_L^*$  which is given as  $h_L = \frac{(v_c - v_1)^2}{2g}$

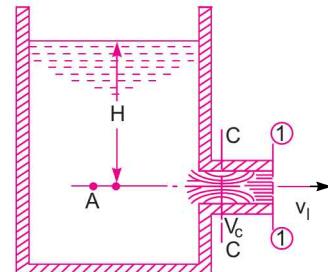


Fig. 7.13 External cylindrical mouthpieces.

\* Please refer Art. 11.4.1 for loss of head due to sudden enlargement.

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$$\text{But } v_c = \frac{v_1}{0.62} \quad \text{hence } h_L = \frac{\left(\frac{v_1}{0.62} - v_1\right)^2}{2g} = \frac{v_1^2}{2g} \left[\frac{1}{0.62} - 1\right]^2 = \frac{0.375 v_1^2}{2g}$$

Applying Bernoulli's equation to point A and (1)-(1)

$$\frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 + h_L$$

where  $z_A = z_1$ ,  $v_A$  is negligible,

$$\frac{p_1}{\rho g} = \text{atmospheric pressure} = 0$$

$$\therefore H + 0 = 0 + \frac{v_1^2}{2g} + .375 \frac{v_1^2}{2g}$$

$$\therefore H = 1.375 \frac{v_1^2}{2g}$$

$$\therefore v_1 = \sqrt{\frac{2gH}{1.375}} = 0.855 \sqrt{2gH}$$

Theoretical velocity of liquid at outlet is  $v_{th} = \sqrt{2gH}$

$\therefore$  Co-efficient of velocity for mouthpiece

$$C_v = \frac{\text{Actual velocity}}{\text{Theoretical velocity}} = \frac{0.855 \sqrt{2gH}}{\sqrt{2gH}} = 0.855.$$

$C_c$  for mouthpiece = 1 as the area of jet of liquid at outlet is equal to the area of mouthpiece at outlet.

$$\text{Thus } C_d = C_c \times C_v = 1.0 \times 0.855 = 0.855$$

Thus the value of  $C_d$  for mouthpiece is more than the value of  $C_d$  for orifice, and so discharge through mouthpiece will be more.

**Problem 7.24** Find the discharge from a 100 mm diameter external mouthpiece, fitted to a side of a large vessel if the head over the mouthpiece is 4 metres.

**Solution.** Given :

$$\text{Dia. of mouthpiece} = 100 \text{ m} = 0.1 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4}(0.1)^2 = 0.007854 \text{ m}^2$$

$$\text{Head, } H = 4.0 \text{ m}$$

$$C_d \text{ for mouthpiece} = 0.855$$

$$\therefore \text{Discharge} = C_d \times \text{Area} \times \text{Velocity} = 0.855 \times a \times \sqrt{2gH}$$

$$= 0.855 \times 0.007854 \times \sqrt{2 \times 9.81 \times 4.0} = 0.05948 \text{ m}^3/\text{s. Ans.}$$

**Problem 7.25** An external cylindrical mouthpiece of diameter 150 mm is discharging water under a constant head of 6 m. Determine the discharge and absolute pressure head of water at vena-contracta. Take  $C_d = 0.855$  and  $C_c$  for vena-contracta = 0.62. Atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

$$\text{Dia. of mouthpiece, } d = 150 \text{ mm} = 0.15 \text{ cm}$$

$$\therefore \text{Area, } a = \frac{\pi}{4}(0.15)^2 = 0.01767 \text{ m}^2$$

$$\text{Head, } H = 6.0 \text{ m}$$

$$C_d = 0.855$$

$$C_c \text{ at vena-contracta} = 0.62$$

$$\text{Atmospheric pressure head, } H_a = 10.3 \text{ m}$$

$$\therefore \text{Discharge} = C_d \times a \times \sqrt{2gH}$$

$$= 0.855 \times 0.01767 \times \sqrt{2 \times 9.81 \times 6.0} = 0.1639 \text{ m}^3/\text{s. Ans.}$$

### Pressure Head at Vena-contracta

Applying Bernoulli's equation at A and C-C, we get

$$\frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c$$

$$\text{But } \frac{p_A}{\rho g} = H_a + H, v_A = 0,$$

$$z_A = z_c$$

$$\therefore H_a + H + 0 = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} = H_c + \frac{v_c^2}{2g}$$

$$\therefore H_c = H_a + H - \frac{v_c^2}{2g}$$

$$\text{But } v_c = \frac{v_1}{0.62}$$

$$\therefore H_c = H_a + H - \left(\frac{v_1}{0.62}\right)^2 \times \frac{1}{2g} = H_a + H - \frac{v_1^2}{2g} \times \frac{1}{(0.62)^2}$$

$$\text{But } H = 1.375 \frac{v_1^2}{2g}$$

$$\therefore \frac{v_1^2}{2g} = \frac{H}{1.375} = 0.7272 H$$

$$\therefore H_c = H_a + H - .7272 H \times \frac{1}{(0.62)^2}$$

$$= H_a + H - 1.89 H = H_a - .89 H$$

$$= 10.3 - .89 \times 6.0 \quad \{ \because H_a = 10.3 \text{ and } H = 6.0 \}$$

$$= 10.3 - 5.34 = 4.96 \text{ m (Absolute). Ans.}$$

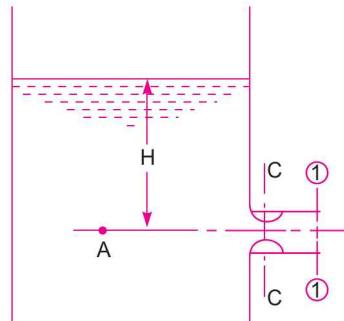


Fig. 7.14

### ► 7.14 FLOW THROUGH A CONVERGENT-DIVERGENT MOUTHPIECE

If a mouthpiece converges upto vena-contracta and then diverges as shown in Fig. 7.15 then that type of mouthpiece is called Convergent-Divergent Mouthpiece. As in this mouthpiece there is no sudden enlargement of the jet, the loss of energy due to sudden enlargement is eliminated. The coefficient of discharge for this mouthpiece is unity. Let  $H$  is the head of liquid over the mouthpiece.

Applying Bernoulli's equation to the free surface of water in tank and section  $C-C$ , we have

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c$$

Taking datum passing through the centre of orifice, we get

$$\frac{p}{\rho g} = H_a, v = 0, z = H, \frac{p_c}{\rho g} = H_c, z_c = 0$$

$$\therefore H_a + 0 + H = H_c + \frac{v_c^2}{2g} + 0 \quad \dots(i)$$

$$\therefore \frac{v_c^2}{2g} = H_a + H - H_c \quad \dots(ii)$$

or

$$v_c = \sqrt{2g(H_a + H - H_c)}$$

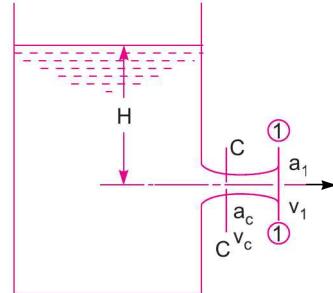


Fig. 7.15 Convergent-divergent mouthpiece.

Now applying Bernoulli's equation at sections  $C-C$  and (1)-(1)

$$\frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1$$

But

$$z_c = z_1 \text{ and } \frac{p_1}{\rho g} = H_a$$

$$\therefore H_c + \frac{v_c^2}{2g} = H_a + \frac{v_1^2}{2g}$$

Also from (i),  $H_c + v_c^2/2g = H + H_a$

$$\therefore H_a + v_1^2/2g = H + H_a$$

$$\therefore v_1 = \sqrt{2gH} \quad \dots(iii)$$

Now by continuity equation,  $a_c v_c = v_1 \times a_1$

$$\begin{aligned} \therefore \frac{a_1}{a_c} &= \frac{v_c}{v_1} = \frac{\sqrt{2g(H_a + H - H_c)}}{\sqrt{2gH}} = \sqrt{\frac{H_a}{H} + 1 - \frac{H_c}{H}} \\ &= \sqrt{1 + \frac{H_a - H_c}{H}} \end{aligned} \quad \dots(7.17)$$

$$\text{The discharge, } Q \text{ is given as } Q = a_c \times \sqrt{2gH} \quad \dots(7.18)$$

where  $a_c$  = area at vena-contracta.

**Problem 7.26** A convergent-divergent mouthpiece having throat diameter of 4.0 cm is discharging water under a constant head of 2.0 m, determine the maximum outer diameter for maximum discharge. Find maximum discharge also. Take  $H_a = 10.3$  m of water and  $H_{sep} = 2.5$  m of water (absolute).

**Solution.** Given :

$$\text{Dia. of throat, } d_c = 4.0 \text{ cm}$$

$$\therefore \text{Area, } a_c = \frac{\pi}{4} (4)^2 = 12.566 \text{ cm}^2$$

$$\text{Constant head, } H = 2.0 \text{ m}$$

$$\text{Find max. dia. at outlet, } d_1 \text{ and } Q_{\max}$$

$$H_a = 10.3 \text{ m}$$

$$H_{sep} = 2.5 \text{ m (absolute)}$$

The discharge,  $Q$  in convergent-divergent mouthpiece depends on the area at throat.

$$\therefore Q_{\max} = a_c \times \sqrt{2gH} = 12.566 \times \sqrt{2 \times 9.81 \times 200} = 7871.5 \text{ cm}^3/\text{s. Ans.}$$

Now ratio of areas at outlet and throat is given by equation (7.17) as

$$\frac{a_1}{a_c} = \sqrt{1 + \frac{H_a - H_c}{H}} = \sqrt{1 + \frac{10.3 - 2.5}{2.0}} \quad \{ \because H_c = H_{sep} = 2.5 \}$$

$$= 2.2135$$

$$\frac{\pi}{4} d_1^2 / \frac{\pi}{4} d_c^2 = 2.2135 \text{ or } \left( \frac{d_1}{d_c} \right)^2 = 2.2135$$

$$\therefore \frac{d_1}{d_c} = \sqrt{2.2135} = 1.4877$$

$$\therefore d_1 = 1.4877 \times d_c = 1.4877 \times 4.0 = 5.95 \text{ cm. Ans.}$$

**Problem 7.27** The throat and exit diameters of convergent-divergent mouthpiece are 5 cm and 10 cm respectively. It is fitted to the vertical side of a tank, containing water. Find the maximum head of a water for steady flow. The maximum vacuum pressure is 8 m of water and take atmospheric pressure = 10.3 m water.

**Solution.** Given :

$$\text{Dia. at throat, } d_c = 5 \text{ cm}$$

$$\text{Dia. at exit, } d_1 = 10 \text{ cm}$$

$$\text{Atmospheric pressure head, } H_a = 10.3 \text{ m}$$

The maximum vacuum pressure will be at a throat only

$$\therefore \text{Pressure head at throat} = 8 \text{ m (vacuum)}$$

$$\text{or } H_c = H_a - 8.0 \text{ (absolute)}$$

$$= 10.3 - 8.0 = 2.3 \text{ m (abs.)}$$

Let maximum head of water over mouthpiece =  $H$  m of water.

The ratio of areas at outlet and throat of a convergent-divergent mouthpiece is given by equation (7.17).

$$\frac{a_1}{a_c} = \sqrt{1 + \frac{H_a - H_c}{H}} \text{ or } \frac{\pi}{4} \left( \frac{d_1}{d_c} \right)^2 = \sqrt{1 + \frac{10.3 - 2.3}{H}}$$

or  $\frac{10^2}{5^2} = 4 = \sqrt{1 + \frac{8}{H}} \text{ or } 16 = 1 + \frac{8}{H} \text{ or } 15 = \frac{8}{H}$

$$\therefore H = \frac{8}{15} = 0.5333 \text{ m of water}$$

$\therefore$  Maximum head of water = **0.533 m. Ans.**

**Problem 7.28** A convergent-divergent mouthpiece is fitted to the side of a tank. The discharge through mouthpiece under a constant head of 1.5 m is 5 litres/s. The head loss in the divergent portion is 0.10 times the kinetic head at outlet. Find the throat and exit diameters, if separation pressure is 2.5 m and atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

$$\text{Constant head, } H = 1.5 \text{ m}$$

$$\text{Discharge, } Q = 5 \text{ litres} = .005 \text{ m}^3/\text{s}$$

$$h_L \text{ or Head loss in divergent} = 0.1 \times \text{kinetic head at outlet}$$

$$H_c \text{ or } H_{sep} = 2.5 \text{ (abs.)}$$

$$H_a = 10.3 \text{ m of water}$$

Find (i) Dia. at throat,  $d_c$

(ii) Dia. at outlet,  $d_1$

(i) **Dia. at throat ( $d_c$ )**. Applying Bernoulli's equation to the free water surface and throat section, we get (See Fig. 7.15).

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_c}{\rho g} + \frac{v_c^2}{2g} + z_c$$

Taking the centre line of mouthpiece as datum, we get

$$H_a + 0 + H = H_c + \frac{v_c^2}{2g}$$

$$\therefore \frac{v_c^2}{2g} = H_a + H - H_c = 10.3 + 1.5 - 2.5 = 9.3 \text{ m of water}$$

$$\therefore v_c = \sqrt{2 \times 9.81 \times 9.3} = 13.508 \text{ m/s}$$

$$\text{Now } Q = a_c \times v_c \text{ or } .005 = \frac{\pi}{4} d_c^2 \times 13.508$$

$$\therefore d_c = \sqrt{\frac{.005 \times 4}{\pi \times 13.508}} = \sqrt{.00047} = .0217 \text{ m} = \mathbf{2.17 \text{ cm. Ans.}}$$

(ii) **Dia. at outlet ( $d_1$ )**. Applying Bernoulli's equation to the free water surface and outlet of mouthpiece (See Fig. 7.15), we get

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 + h_L$$

$$\begin{aligned}
 H_a + 0 + H &= H_a + \frac{v_1^2}{2g} + 0 + 0.1 \times \frac{v_1^2}{2g} & \left\{ \because \frac{P_1}{\rho g} = H_a \right\} \\
 \therefore H &= \frac{v_1^2}{2g} + .1 \times \frac{v_1^2}{2g} = 1.1 \frac{v_1^2}{2g} \\
 \therefore v_1 &= \sqrt{\frac{2gH}{1.1}} = \sqrt{\frac{2 \times 9.81 \times 1.5}{1.1}} = 5.1724 \\
 \text{Now } Q &= A_1 v_1 \text{ or } .005 = \frac{\pi}{4} d_1^2 \times v_1 \\
 \therefore d_1 &= \sqrt{\frac{4 \times .005}{\pi \times v_1}} = \sqrt{\frac{4 \times .005}{\pi \times 5.1724}} = 0.035 \text{ m} = 3.5 \text{ cm. Ans.}
 \end{aligned}$$

### ► 7.15 FLOW THROUGH INTERNAL OR RE-ENTRANT OR BORDA'S MOUTHPIECE

A short cylindrical tube attached to an orifice in such a way that the tube projects inwardly to a tank, is called an internal mouthpiece. It is also called Re-entrant or Borda's mouthpiece. If the length of the tube is equal to its diameter, the jet of liquid comes out from mouthpiece without touching the sides of the tube as shown in Fig. 7.16. The mouthpiece is known as *running free*. But if the length of the tube is about 3 times its diameter, the jet comes out with its diameter equal to the diameter of mouthpiece at outlet as shown in Fig. 7.17. The mouthpiece is said to be *running full*.

(i) **Borda's Mouthpiece Running Free.** Fig. 7.16 shows the Borda's mouthpiece running free.

Let  $H$  = height of liquid above the mouthpiece,

$a$  = area of mouthpiece,

$a_c$  = area of contracted jet in the mouthpiece,

$v_c$  = velocity through mouthpiece.

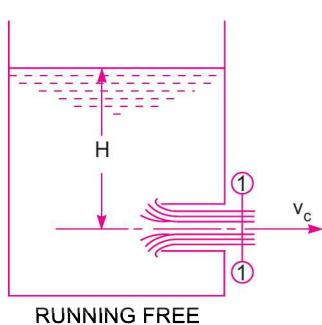


Fig. 7.16

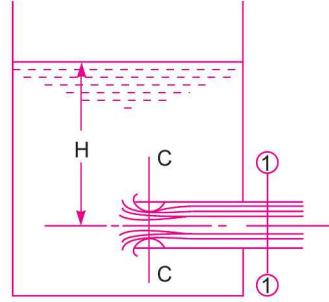


Fig. 7.17

The flow of fluid through mouthpiece is taking place due to the pressure force exerted by the fluid on the entrance section of the mouthpiece. As the area of the mouthpiece is ' $a$ ' hence total pressure force on entrance

$$= \rho g \cdot a \cdot h$$

where  $h$  = distance of C.G. of area ' $a$ ' from free surface =  $H$ .

$$= \rho g \cdot a \cdot H$$

... (i)

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According to Newton's second law of motion, the net force is equal to the rate of change of momentum.

$$\text{Now mass of liquid flowing/sec} = \rho \times a_c \times v_c$$

The liquid is initially at rest and hence initial velocity is zero but final velocity of fluid is  $v_c$ .

$\therefore$  Rate of change of momentum = mass of liquid flowing/sec  $\times$  [final velocity – initial velocity]

$$= \rho a_c \times v_c [v_c - 0] = \rho a_c v_c^2 \quad \dots(iii)$$

Equating (i) and (ii), we get

$$\rho g \cdot a \cdot H = \rho a_c \cdot v_c^2 \quad \dots(iii)$$

Applying Bernoulli's equation to free surface of liquid and section (1)-(1) of Fig. 7.16

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1$$

Taking the centre line of mouthpiece as datum, we have

$$z = H, z_1 = 0, \frac{p}{\rho g} = \frac{p_1}{\rho g} = p_{atmosp.} = 0,$$

$$v_1 = v_c, \quad v = 0$$

$$\therefore 0 + 0 + H = 0 + \frac{v_c^2}{2g} + 0 \quad \text{or} \quad H = \frac{v_c^2}{2g}$$

$$\therefore v_c = \sqrt{2gH}$$

Substituting the value of  $v_c$  in (iii), we get

$$\rho g \cdot a \cdot H = \rho \cdot a_c \cdot 2g \cdot H$$

$$\text{or} \quad a = 2a_c \quad \text{or} \quad \frac{a_c}{a} = \frac{1}{2} = 0.5$$

$$\therefore \text{Co-efficient of contraction, } C_c = \frac{a_c}{a} = 0.5$$

Since there is no loss of head, co-efficient of velocity,  $C_v = 1.0$

$$\therefore \text{Co-efficient of discharge, } C_d = C_c \times C_v = 0.5 \times 1.0 = 0.5$$

$$\therefore \text{Discharge} \quad Q = C_d a \sqrt{2gH} \quad \dots(7.19)$$

$$= 0.5 \times a \sqrt{2gH}$$

(ii) **Borda's Mouthpiece Running Full.** Fig. 7.17 shows Borda's mouthpiece running full.

Let  $H$  = height of liquid above the mouthpiece,

$v_1$  = velocity at outlet or at (1)-(1) of mouthpiece,

$a$  = area of mouthpiece,

$a_c$  = area of the flow at C-C,

$v_c$  = velocity of liquid at vena-contracta or at C-C.

The jet of liquid after passing through C-C, suddenly enlarges at section (1)-(1). Thus there will be a loss of head due to sudden enlargement.

$$\therefore h_L = \frac{(v_c - v_1)^2}{2g} \quad \dots(i)$$

Now from continuity, we have  $a_c \times v_c = a_1 \times v_1$

$$\therefore v_c = \frac{a_1}{a_c} \times v_1 = \frac{v_1}{a_c / a_1} = \frac{v_1}{C_c} = \frac{v_1}{0.5} \quad \{\because C_c = 0.5\}$$

or

$$v_c = 2v_1$$

$$\text{Substituting this value of } v_c \text{ in (i), we get } h_L = \frac{(2v_1 - v_1)^2}{2g} = \frac{v_1^2}{2g}$$

Applying Bernoulli's equation to free surface of water in tank and section (1)-(1), we get

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 + h_L$$

Taking datum line passing through the centre line of mouthpiece

$$0 + 0 + H = 0 + \frac{v_1^2}{2g} + 0 + \frac{v_1^2}{2g}$$

$$\therefore H = \frac{v_1^2}{2g} + \frac{v_1^2}{2g} = \frac{v_1^2}{g}$$

$$\therefore v_1 = \sqrt{gH}$$

Here  $v_1$  is actual velocity as losses have been taken into consideration,

But theoretical velocity,  $v_{th} = \sqrt{2gH}$

$$\therefore \text{Co-efficient of velocity, } C_v = \frac{v_1}{v_{th}} = \frac{\sqrt{gH}}{\sqrt{2gH}} = \frac{1}{\sqrt{2}} = 0.707$$

As the area of the jet at outlet is equal to the area of the mouthpiece, hence co-efficient of contraction = 1

$$\therefore C_d = C_c \times C_v = 1.0 \times 0.707 = 0.707$$

$$\therefore \text{Discharge, } Q = C_d \times a \times \sqrt{2gH} = 0.707 \times a \times \sqrt{2gH} \quad \dots(7.20)$$

**Problem 7.29** An internal mouthpiece of 80 mm diameter is discharging water under a constant head of 8 metres. Find the discharge through mouthpiece, when

(i) The mouthpiece is running free, and (ii) The mouthpiece is running full.

**Solution.** Given :

Dia. of mouthpiece,  $d = 80 \text{ mm} = 0.08 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.08)^2 = 0.005026 \text{ m}^2$$

Constant head,  $H = 4 \text{ m.}$

(i) **Mouthpiece running free.** The discharge,  $Q$  is given by equation (7.19) as

$$\begin{aligned} Q &= 0.5 \times a \times \sqrt{2gH} \\ &= 0.5 \times 0.005026 \times \sqrt{2 \times 9.81 \times 4.0} \\ &= 0.02226 \text{ m}^3/\text{s} = \mathbf{22.26 \text{ litres/s. Ans.}} \end{aligned}$$

(ii) **Mouthpiece running full.** The discharge,  $Q$  is given by equation (7.20) as

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$$\begin{aligned} Q &= 0.707 \times a \times \sqrt{2gH} \\ &= 0.707 \times .005026 \times \sqrt{2 \times 9.81 \times 4.0} \\ &= 0.03147 \text{ m}^3/\text{s} = \mathbf{31.47 \text{ litre/s. Ans.}} \end{aligned}$$

### HIGHLIGHTS

1. Orifice is a small opening on the side or at the bottom of a tank while mouthpiece is a short length of pipe which is two or three times its diameter in length.
2. Orifices as well as mouthpieces are used for measuring the rate of flow of liquid.
3. Theoretical velocity of jet of water from orifice is given by

$$V = \sqrt{2gH}, \text{ where } H = \text{Height of water from the centre of orifice.}$$

4. There are three hydraulic co-efficients namely :

$$(a) \text{ Co-efficient of velocity, } C_v = \frac{\text{Actual velocity at vena - contracta}}{\text{Theoretical velocity}} = \frac{x}{\sqrt{4yH}}$$

$$(b) \text{ Co-efficient of contraction, } C_c = \frac{\text{Area of jet at vena - contracta}}{\text{Area of orifice}}$$

$$(c) \text{ Co-efficient of discharge, } C_d = \frac{\text{Actual discharge}}{\text{Theoretical discharge}} = C_v \times C_c$$

where  $x$  and  $y$  are the co-ordinates of any point of jet of water from vena-contracta.

5. A large orifice is one, where the head of liquid above the centre of orifice is less than 5 times the depth of orifice. The discharge through a large rectangular orifice is

$$Q = \frac{2}{3} C_d \times b \times \sqrt{2g} [H_2^{3/2} - H_1^{3/2}]$$

where  $b$  = Width of orifice,

$C_d$  = Co-efficient of discharge for orifice,

$H_1$  = Height of liquid above top edge of orifice, and

$H_2$  = Height of liquid above bottom edge of orifice.

6. The discharge through fully sub-merged orifice,  $Q = C_d \times b \times (H_2 - H_1) \times \sqrt{2gH}$

where  $b$  = Width of orifice,

$C_d$  = Co-efficient of discharge for orifice,

$H_2$  = Height of liquid above bottom edge of orifice on upstream side,

$H_1$  = Height of liquid above top edge of orifice on upstream side,

$H$  = Difference of liquid levels on both sides of the orifice.

7. Discharge through partially sub-merged orifice,

$$\begin{aligned} Q &= Q_1 + Q_2 \\ &= C_d b (H_2 - H) \times \sqrt{2gH} + 2/3 C_d b \times \sqrt{2g} [H^{3/2} - H_1^{3/2}] \end{aligned}$$

where  $b$  = Width of orifice

$C_d, H_1, H_2$  and  $H$  are having their usual meaning.

8. Time of emptying a tank through an orifice at its bottom is given by,

$$T = \frac{2A [\sqrt{H_1} - \sqrt{H_2}]}{C_d \cdot a \cdot \sqrt{2g}}$$

where  $H_1$  = Initial height of liquid in tank,

$H_2$  = Final height of liquid in tank,

$A$  = Area of tank,

$a$  = Area of orifice,

$C_d$  = Co-efficient of discharge.

If the tank is to be completely emptied, then time  $T$ ,

$$T = \frac{2A\sqrt{H}}{C_d \cdot a \cdot \sqrt{2g}}.$$

9. Time of emptying a hemispherical tank by an orifice fitted at its bottom,

$$T = \frac{\pi}{C_d \cdot a \cdot \sqrt{2g}} \left[ \frac{4}{3} R(H_1^{3/2} - H_2^{3/2}) - \frac{2}{5} (H_1^{5/2} - H_2^{5/2}) \right]$$

and for completely emptying the tank,  $T = \frac{\pi}{C_d \cdot a \cdot \sqrt{2g}} \left[ \frac{4}{3} RH_1^{3/2} - \frac{2}{5} H_1^{5/2} \right]$

where  $R$  = Radius of the hemispherical tank,

$H_1$  = Initial height of liquid,

$H_2$  = Final height of liquid,

$a$  = Area of orifice, and

$C_d$  = Co-efficient of discharge.

10. Time of emptying a circular horizontal tank by an orifice at the bottom of the tank,

$$T = \frac{4L}{3C_d \cdot a \cdot \sqrt{2g}} [(2R - H_2)^{3/2} - (2R - H_1)^{3/2}]$$

and for completely emptying the tank,  $T = \frac{4L}{3C_d \cdot a \cdot \sqrt{2g}} [(2R)^{3/2} - (2R - H_1)^{3/2}]$

where  $L$  = Length of horizontal tank.

11. Co-efficient of discharge for,

(i) External mouthpiece,  $C_d = 0.855$

(ii) Internal mouthpiece, running full,  $C_d = 0.707$

(iii) Internal mouthpiece, running free,  $C_d = 0.50$

(iv) Convergent or convergent-divergent,  $C_d = 1.0$ .

12. For an external mouthpiece, absolute pressure head at vena-contracta

$$H_c = H_a - 0.89 H$$

where  $H_a$  = atmospheric pressure head = 10.3 m of water

$H$  = head of liquid above the mouthpiece.

13. For a convergent-divergent mouthpiece, the ratio of areas at outlet and at vena-contracta is

$$\frac{a_1}{a_c} = \sqrt{1 + \frac{H_a - H_c}{H}}$$

where  $a_1$  = Area of mouthpiece at outlet

$a_c$  = Area of mouthpiece at vena-contracta

$H_a$  = Atmospheric pressure head

$H_c$  = Absolute pressure head at vena-contracta

$H$  = Height of liquid above mouthpiece.

14. In case of internal mouthpieces, if the jet of liquid comes out from mouthpiece without touching its sides it is known as running free. But if the jet touches the sides of the mouthpiece, it is known as running full.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define an orifice and a mouthpiece. What is the difference between the two ?
2. Explain the classification of orifices and mouthpieces based on their shape, size and sharpness ?
3. What are hydraulic co-efficients ? Name them.
4. Define the following co-efficients : (i) Co-efficient of velocity, (ii) Co-efficient of contraction and (iii) Co-efficient of discharge.
5. Derive the expression  $C_d = C_v \times C_c$ .
6. Define vena-contracta.
7. Differentiate between a large and a small orifice. Obtain an expression for discharge through a large rectangular orifice.
8. What do you understand by the terms wholly sub-merged orifice and partially sub-merged orifice ?
9. Prove that the expression for discharge through an external mouthpiece is given by

$$Q = .855 \times a \times v$$

where  $a$  = Area of mouthpiece at outlet and  
 $v$  = Velocity of jet of water at outlet.

10. Distinguish between : (i) External mouthpiece and internal mouthpiece, (ii) Mouthpiece running free and mouthpiece running full.
11. Obtain an expression for absolute pressure head at vena-contracta for an external mouthpiece.
12. What is a convergent-divergent mouthpiece ? Obtain an expression for the ratio of diameters at outlet and at vena-contracta for a convergent-divergent 'mouthpiece' in terms of absolute pressure head at vena-contracta, head of liquid above mouthpiece and atmospheric pressure head.
13. The length of the divergent outlet part in a venturimeter is usually made longer compared with that of the converging inlet part. Why ?
14. Justify the statement, "In a convergent-divergent mouthpiece the loss of head is practically eliminated".

### (B) NUMERICAL PROBLEMS

1. The head of water over an orifice of diameter 50 mm is 12 m. Find the actual discharge and actual velocity of jet at vena-contracta. Take  $C_d = 0.6$  and  $C_v = 0.98$ . [Ans.  $0.018 \text{ m}^3/\text{s}$ ;  $15.04 \text{ m/s}$ ]
2. The head of water over the centre of an orifice of diameter 30 mm is 1.5 m. The actual discharge through the orifice is 2.35 litres/sec. Find the co-efficient of discharge. [Ans. 0.613]
3. A jet of water, issuing from a sharp edged vertical orifice under a constant head of 60 cm, has the horizontal and vertical co-ordinates measured from the vena-contracta at a certain point as 10.0 cm and 0.45 cm respectively. Find the value of  $C_v$ . Also find the value of  $C_v$  if  $C_d = 0.60$ . [Ans. 0.962, 0.623]
4. The head of water over an orifice of diameter 100 mm is 5 m. The water coming out from orifice is collected in a circular tank of diameter 2 m. The rise of water level in circular tank is .45 m in 30 seconds. Also the co-ordinates of a certain point on the jet, measured from vena-contracta are 100 cm horizontal and 5.2 cm vertical. Find the hydraulic co-efficients  $C_d$ ,  $C_v$  and  $C_c$ . [Ans. 0.605, 0.98, 0.617]
5. A tank has two identical orifices in one of its vertical sides. The upper orifice is 4 m below the water surface and lower one 6 m below the water surface. If the value of  $C_v$  for each orifice is 0.98, find the point of intersection of the two jets. [Ans. At a horizontal distance of 9.60 cm]
6. A closed vessel contains water upto a height of 2.0 m and over the water surface there is air having pressure  $8.829 \text{ N/cm}^2$  above atmospheric pressure. At the bottom of the vessel there is an orifice of diameter 15 cm. Find the rate of flow of water from orifice. Take  $C_d = 0.6$ . [Ans.  $0.15575 \text{ m}^3/\text{s}$ ]

7. A closed tank partially filled with water upto a height of 1 m, having an orifice of diameter 20 mm at the bottom of the tank. Determine the pressure required for a discharge of 3.0 litres/s through the orifice. Take  $C_d = 0.62$ . [Ans.  $10.88 \text{ N/cm}^2$ ]
8. Find the discharge through a rectangular orifice 3.0 m wide and 2 m deep fitted to a water tank. The water level in the tank is 4 m above the top edge of the orifice. Take  $C_d = 0.62$  [Ans.  $36.77 \text{ m}^3/\text{s}$ ]
9. A rectangular orifice, 2.0 m wide and 1.5 m deep is discharging water from a tank. If the water level in the tank is 3.0 m above the top edge of the orifice, find the discharge through the orifice. Take  $C_d = 0.6$ . [Ans.  $15.40 \text{ m}^3/\text{s}$ ]
10. A rectangular orifice, 1.0 m wide and 1.5 m deep is discharging water from a vessel. The top edge of the orifice is 0.8 m below the water surface in the vessel. Calculate the discharge through the orifice if  $C_d = 0.6$ . Also calculate the percentage error if the orifice is treated as a small orifice. [Ans. 1.058%]
11. Find the discharge through a fully sub-merged orifice of width 2 m if the difference of water levels on both the sides of the orifice be 800 mm. The height of water from top and bottom of the orifice are 2.5 m and 3 m respectively. Take  $C_d = 0.6$ . [Ans.  $2.377 \text{ m}^3/\text{s}$ ]
12. Find the discharge through a totally drowned orifice 1.5 m wide and 1 m deep, if the difference of water levels on both the sides of the orifice be 2.5 m. Take  $C_d = 0.62$ . [Ans.  $6.513 \text{ m}^3/\text{s}$ ]
13. A rectangular orifice of 1.5 m wide and 1.2 m deep is fitted in one side of a large tank. The water level on one side of the orifice is 2 m above the top edge of the orifice, while on the other side of the orifice, the water level is 0.4 m below its top edge. Calculate the discharge through the orifice if  $C_d = 0.62$ . [Ans.  $7.549 \text{ m}^3/\text{s}$ ]
14. A circular tank of diameter 3 m contains water upto a height of 4 m. The tank is provided with an orifice of diameter 0.4 m at the bottom. Find the time taken by water : (i) to fall from 4 m to 2 m and (ii) for completely emptying the tank. Take  $C_d = 0.6$ . [Ans. (i) 24.8 s, (ii) 84.7 s]
15. A circular tank of diameter 1.5 m contains water upto a height of 4 m. An orifice of 40 mm diameter is provided at its bottom. If  $C_d = 0.62$ , find the height of water above the orifice after 10 minutes. [Ans. 2 m]
16. A hemispherical tank of diameter 4 m contains water upto a height of 2.0 m. An orifice of diameter 50 mm is provided at the bottom. Find the time required by water (i) to fall from 2.0 m to 1.0 m (ii) for completely emptying the tank. Take  $C_d = 0.6$ . [Ans. (i) 30 min 14.34 s, (ii) 52 min 59 s]
17. A hemispherical cistern of 4 m radius is full of water. It is fitted with a 60 mm diameter sharp edged orifice at the bottom. Calculate the time required to lower the level in the cistern by 2 metres. Take  $C_d = 0.6$ . [Ans. 1 hr 58 min 45.9 s]
18. A cylindrical tank is having a hemispherical base. The height of cylindrical portion is 4 m and diameter is 3 m. At the bottom of this tank an orifice of diameter 300 mm is fitted. Find the time required to completely emptying the tank. Take  $C_d = 0.6$ . [Ans. 2 min 7.37 s]
19. An orifice of diameter 200 mm is fitted at the bottom of a boiler drum of length 6 m and of diameter 2 m. The drum is horizontal and half full of water. Find the time required to empty the boiler, given the value of  $C_d = 0.6$ . [Ans. 2 min 55.20 s]
20. An orifice of diameter 150 mm is fitted at the bottom of a boiler drum of length 6 m and of diameter 2 m. The drum is horizontal and contains water upto a height of 1.8 m. Find the time required to empty the boiler. Take  $C_d = 0.6$ . [Ans. 7 min 46.64 s]
21. Find the discharge from a 80 mm diameter external mouthpiece, fitted to a side of a large vessel if the head over the mouthpiece is 6 m. [Ans.  $0.0466 \text{ m}^3/\text{s}$ ]
22. An external cylindrical mouthpiece of diameter 100 mm is discharging water under a constant head of 8 m. Determine the discharge and absolute pressure head of water at vena-contracta. Take  $C_d = 0.855$  and  $C_c$  for vena-contracta = 0.62. Take atmospheric pressure head = 10.3 m of water. [Ans.  $0.084 \text{ m}^3/\text{s}$ ; 3.18 m]
23. A convergent-divergent mouthpiece having throat diameter of 60 mm is discharging water under a constant head of 3.0 m. Determine the maximum outlet diameter for maximum discharge. Find maximum discharge also. Take atmospheric pressure head = 10.3 m of water and separation pressure head = 2.5 m of water absolute. [Ans. 6.88 cm,  $Q_{\max} = 0.01506 \text{ m}^3/\text{s}$ ]

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24. The throat and exit diameter of a convergent-divergent mouthpiece are 40 mm and 80 mm respectively. It is fitted to the vertical side of a tank, containing water. Find the maximum head of water for steady flow. The maximum vacuum pressure is 8 m of water. Take atmospheric pressure head = 10.3 m of water. [Ans. 0.533 m]
25. The discharge through a convergent-divergent mouthpiece fitted to the side of a tank under a constant head of 2 m is 7 litres/s. The head loss in the divergent portion is 0.10 times the kinetic head at outlet. Find the throat and exit diameters, if separation pressure head = 2.5 m and atmospheric pressure head = 10.3 m of water. [Ans. 25.3 mm ; 38.6 mm]
26. An internal mouthpiece of 100 mm diameter is discharging water under a constant head of 5 m. Find the discharge through mouthpiece, when  
(i) the mouthpiece is running free, and (ii) the mouthpiece is running full.  
[Ans. (i) 38.8 litres/s, (ii) 54.86 litres/s]

# 8

## CHAPTER

# NOTCHES AND WEIRS



### ► 8.1 INTRODUCTION

A **notch** is a device used for measuring the rate of flow of a liquid through a small channel or a tank. It may be defined as an opening in the side of a tank or a small channel in such a way that the liquid surface in the tank or channel is below the top edge of the opening.

A **weir** is a concrete or masonry structure, placed in an open channel over which the flow occurs. It is generally in the form of vertical wall, with a sharp edge at the top, running all the way across the open channel. The notch is of small size while the weir is of a bigger size. The notch is generally made of metallic plate while weir is made of concrete or masonry structure.

1. **Nappe or Vein.** The sheet of water flowing through a notch or over a weir is called Nappe or Vein.
2. **Crest or Sill.** The bottom edge of a notch or a top of a weir over which the water flows, is known as the sill or crest.

### ► 8.2 CLASSIFICATION OF NOTCHES AND WEIRS

The notches are classified as :

1. According to the shape of the opening :
  - (a) Rectangular notch,
  - (b) Triangular notch,
  - (c) Trapezoidal notch, and
  - (d) Stepped notch.
2. According to the effect of the sides on the nappe :
  - (a) Notch with end contraction.
  - (b) Notch without end contraction or suppressed notch.

Weirs are classified according to the shape of the opening, the shape of the crest, the effect of the sides on the nappe and nature of discharge. The following are important classifications.

- (a) According to the shape of the opening :
  - (i) Rectangular weir, (ii) Triangular weir, and
  - (iii) Trapezoidal weir (Cipolletti weir)
- (b) According to the shape of the crest :
  - (i) Sharp-crested weir, (ii) Broad-crested weir,
  - (iii) Narrow-crested weir, and (iv) Ogee-shaped weir.

- (c) According to the effect of sides on the emerging nappe :  
 (i) Weir with end contraction, and                   (ii) Weir without end contraction.

### ► 8.3 DISCHARGE OVER A RECTANGULAR NOTCH OR WEIR

The expression for discharge over a rectangular notch or weir is the same.

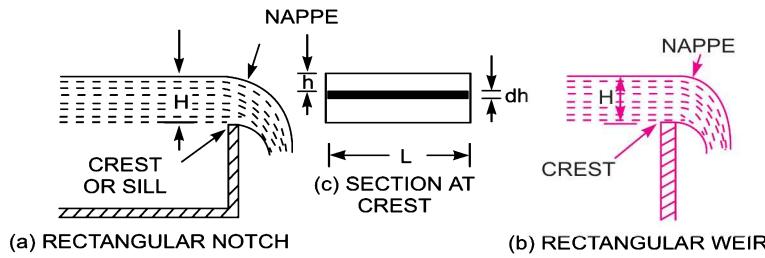


Fig. 8.1 *Rectangular notch and weir.*

Consider a rectangular notch or weir provided in a channel carrying water as shown in Fig. 8.1.

Let

$H$  = Head of water over the crest

$L$  = Length of the notch or weir

For finding the discharge of water flowing over the weir or notch, consider an elementary horizontal strip of water of thickness  $dh$  and length  $L$  at a depth  $h$  from the free surface of water as shown in Fig. 8.1(c).

The area of strip  $= L \times dh$   
 and theoretical velocity of water flowing through strip  $= \sqrt{2gh}$

The discharge  $dQ$ , through strip is

$$\begin{aligned} dQ &= C_d \times \text{Area of strip} \times \text{Theoretical velocity} \\ &= C_d \times L \times dh \times \sqrt{2gh} \end{aligned} \quad \dots(i)$$

where  $C_d$  = Co-efficient of discharge.

The total discharge,  $Q$ , for the whole notch or weir is determined by integrating equation (i) between the limits 0 and  $H$ .

$$\begin{aligned} \therefore Q &= \int_0^H C_d \cdot L \cdot \sqrt{2gh} \cdot dh = C_d \times L \times \sqrt{2g} \int_0^H h^{1/2} dh \\ &= C_d \times L \times \sqrt{2g} \left[ \frac{h^{1/2+1}}{\frac{1}{2}+1} \right]_0^H = C_d \times L \times \sqrt{2g} \left[ \frac{h^{3/2}}{3/2} \right]_0^H \\ &= \frac{2}{3} C_d \times L \times \sqrt{2g} [H]^{3/2}. \end{aligned} \quad \dots(8.1)$$

**Problem 8.1** Find the discharge of water flowing over a rectangular notch of 2 m length when the constant head over the notch is 300 mm. Take  $C_d = 0.60$ .

**Solution.** Given :

Length of the notch,  $L = 2.0$  m

Head over notch,  $H = 300 \text{ m} = 0.30 \text{ m}$

$$C_d = 0.60$$

Discharge,  $Q = \frac{2}{3} C_d \times L \times \sqrt{2g} [H^{3/2}]$

$$= \frac{2}{3} \times 0.6 \times 2.0 \times \sqrt{2 \times 9.81} \times [0.30]^{1.5} \text{ m}^3/\text{s}$$

$$= 3.5435 \times 0.1643 = 0.582 \text{ m}^3/\text{s. Ans.}$$

**Problem 8.2** Determine the height of a rectangular weir of length 6 m to be built across a rectangular channel. The maximum depth of water on the upstream side of the weir is 1.8 m and discharge is 2000 litres/s. Take  $C_d = 0.6$  and neglect end contractions.

**Solution.** Given :

Length of weir,  $L = 6 \text{ m}$

Depth of water,  $H_1 = 1.8 \text{ m}$

Discharge,  $Q = 2000 \text{ lit/s} = 2 \text{ m}^3/\text{s}$

$$C_d = 0.6$$

Let  $H$  is height of water above the crest of weir, and  $H_2$  = height of weir (Fig. 8.2)

The discharge over the weir is given by the equation (8.1) as

$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g} H^{3/2}$$

or

$$2.0 = \frac{2}{3} \times 0.6 \times 6.0 \times \sqrt{2 \times 9.81} \times H^{3/2}$$

$$= 10.623 H^{3/2}$$

$$\therefore H^{3/2} = \frac{2.0}{10.623}$$

$$\therefore H = \left( \frac{2.0}{10.623} \right)^{2/3} = 0.328 \text{ m}$$

$$\begin{aligned} \therefore \text{Height of weir, } H_2 &= H_1 - H \\ &= \text{Depth of water on upstream side} - H \\ &= 1.8 - .328 = 1.472 \text{ m. Ans.} \end{aligned}$$

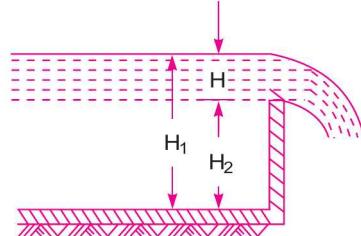


Fig. 8.2

**Problem 8.3** The head of water over a rectangular notch is 900 mm. The discharge is 300 litres/s. Find the length of the notch, when  $C_d = 0.62$ .

**Solution.** Given :

Head over notch,  $H = 90 \text{ cm} = 0.9 \text{ m}$

Discharge,  $Q = 300 \text{ lit/s} = 0.3 \text{ m}^3/\text{s}$

$$C_d = 0.62$$

Let length of notch  $= L$

Using equation (8.1), we have

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

or

$$0.3 = \frac{2}{3} \times 0.62 \times L \times \sqrt{2 \times 9.81} \times (0.9)^{3/2}$$

$$= 1.83 \times L \times 0.8538$$

$$\therefore L = \frac{0.3}{1.83 \times 0.8538} = .192 \text{ m} = 192 \text{ mm. Ans.}$$

#### ► 8.4 DISCHARGE OVER A TRIANGULAR NOTCH OR WEIR

The expression for the discharge over a triangular notch or weir is the same. It is derived as :

Let  $H$  = head of water above the V-notch

$\theta$  = angle of notch

Consider a horizontal strip of water of thickness ' $dh$ ' at a depth of  $h$  from the free surface of water as shown in Fig. 8.3.

From Fig. 8.3 (b), we have

$$\tan \frac{\theta}{2} = \frac{AC}{OC} = \frac{AC}{(H-h)}$$

$$\therefore AC = (H-h) \tan \frac{\theta}{2}$$

$$\text{Width of strip} = AB = 2AC = 2(H-h) \tan \frac{\theta}{2}$$

$$\therefore \text{Area of strip} = 2(H-h) \tan \frac{\theta}{2} \times dh$$

The theoretical velocity of water through strip =  $\sqrt{2gh}$

$\therefore$  Discharge, through the strip,

$$dQ = C_d \times \text{Area of strip} \times \text{Velocity (theoretical)}$$

$$= C_d \times 2(H-h) \tan \frac{\theta}{2} \times dh \times \sqrt{2gh}$$

$$= 2C_d (H-h) \tan \frac{\theta}{2} \times \sqrt{2gh} \times dh$$

$$\therefore \text{Total discharge, } Q = \int_0^H 2C_d (H-h) \tan \frac{\theta}{2} \times \sqrt{2gh} \times dh$$

$$= 2C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \int_0^H (H-h) h^{1/2} dh$$

$$= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \int_0^H (Hh^{1/2} - h^{3/2}) dh$$

$$= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \left[ \frac{Hh^{3/2}}{3/2} - \frac{h^{5/2}}{5/2} \right]_0^H$$

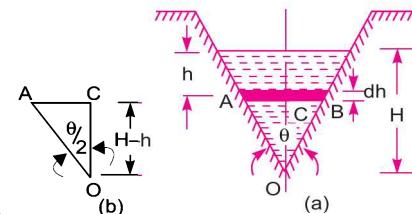


Fig. 8.3 The triangular notch.

$$\begin{aligned}
&= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \left[ \frac{2}{3} H \cdot H^{3/2} - \frac{2}{5} H^{5/2} \right] \\
&= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \left[ \frac{2}{3} H^{5/2} - \frac{2}{5} H^{5/2} \right] \\
&= 2 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \left[ \frac{4}{15} H^{5/2} \right] \\
&= \frac{8}{15} C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2} \quad \dots(8.2)
\end{aligned}$$

For a right-angled V-notch, if  $C_d = 0.6$

$$\theta = 90^\circ, \therefore \tan \frac{\theta}{2} = 1$$

$$\begin{aligned}
\text{Discharge, } Q &= \frac{8}{15} \times 0.6 \times 1 \times \sqrt{2 \times 9.81} \times H^{5/2} \quad \dots(8.3) \\
&= 1.417 H^{5/2}.
\end{aligned}$$

**Problem 8.4** Find the discharge over a triangular notch of angle  $60^\circ$  when the head over the V-notch is 0.3 m. Assume  $C_d = 0.6$ .

**Solution.** Given :

$$\begin{aligned}
\text{Angle of V-notch, } \theta &= 60^\circ \\
\text{Head over notch, } H &= 0.3 \text{ m} \\
C_d &= 0.6
\end{aligned}$$

Discharge,  $Q$  over a V-notch is given by equation (8.2)

$$\begin{aligned}
Q &= \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2} \\
&= \frac{8}{15} \times 0.6 \tan \frac{60^\circ}{2} \times \sqrt{2 \times 9.81} \times (0.3)^{5/2} \\
&= 0.8182 \times 0.0493 = \mathbf{0.040 \text{ m}^3/\text{s. Ans.}}
\end{aligned}$$

**Problem 8.5** Water flows over a rectangular weir 1 m wide at a depth of 150 mm and afterwards passes through a triangular right-angled weir. Taking  $C_d$  for the rectangular and triangular weir as 0.62 and 0.59 respectively, find the depth over the triangular weir.

**Solution.** Given :

$$\begin{aligned}
\text{For rectangular weir, length, } L &= 1 \text{ m} \\
\text{Depth of water, } H &= 150 \text{ mm} = 0.15 \text{ m} \\
C_d &= 0.62 \\
\text{For triangular weir, } \theta &= 90^\circ \\
C_d &= 0.59
\end{aligned}$$

Let depth over triangular weir =  $H_1$

The discharge over the rectangular weir is given by equation (8.1) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \\ &= \frac{2}{3} \times 0.62 \times 1.0 \times \sqrt{2 \times 9.81} \times (0.15)^{3/2} \text{ m}^3/\text{s} = 0.10635 \text{ m}^3/\text{s} \end{aligned}$$

The same discharge passes through the triangular right-angled weir. But discharge,  $Q$ , is given by equation (8.2) for a triangular weir as

$$\begin{aligned} Q &= \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2} \\ \therefore 0.10635 &= \frac{8}{15} \times 0.59 \times \tan \frac{90^\circ}{2} \times \sqrt{2g} \times H_1^{5/2} \quad \{ \because \theta = 90^\circ \text{ and } H = H_1 \} \\ &= \frac{8}{15} \times 0.59 \times 1 \times 4.429 \times H_1^{5/2} = 1.3936 H_1^{5/2} \\ \therefore H_1^{5/2} &= \frac{0.10635}{1.3936} = 0.07631 \\ \therefore H_1 &= (0.07631)^{0.4} = \mathbf{0.3572 \text{ m. Ans.}} \end{aligned}$$

**Problem 8.5A** Water flows through a triangular right-angled weir first and then over a rectangular weir of 1 m width. The discharge co-efficients of the triangular and rectangular weirs are 0.6 and 0.7 respectively. If the depth of water over the triangular weir is 360 mm, find the depth of water over the rectangular weir.

**Solution.** Given :

For triangular weir :  $\theta = 90^\circ, C_d = 0.6, H = 360 \text{ mm} = 0.36 \text{ m}$

For rectangular weir :  $L = 1 \text{ m}, C_d = 0.7, H = ?$

The discharge for a triangular weir is given by equation (8.2) as

$$\begin{aligned} Q &= \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2} \\ &= \frac{8}{15} \times 0.6 \times \tan \left( \frac{90^\circ}{2} \right) \times \sqrt{2 \times 9.81} \times (0.36)^{5/2} = 0.1102 \text{ m}^3/\text{s} \end{aligned}$$

The same discharge is passing through the rectangular weir. But discharge for a rectangular weir is given by equation (8.1) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \\ \text{or } 0.1102 &= \frac{2}{3} \times 0.7 \times 1 \times \sqrt{2 \times 9.81} \times H^{3/2} = 2.067 H^{3/2} \\ \text{or } H^{3/2} &= \frac{0.1102}{2.067} = 0.0533 \\ \therefore H &= (0.0533)^{2/3} = 0.1415 \text{ m} = \mathbf{141.5 \text{ mm. Ans.}} \end{aligned}$$

**Problem 8.6** A rectangular channel 2.0 m wide has a discharge of 250 litres per second, which is measured by a right-angled V-notch weir. Find the position of the apex of the notch from the bed of the channel if maximum depth of water is not to exceed 1.3 m. Take  $C_d = 0.62$ .

**Solution.** Given :

Width of rectangular channel,  $L = 2.0$  m

Discharge,  $Q = 250 \text{ lit/s} = 0.25 \text{ m}^3/\text{s}$

Depth of water in channel  $= 1.3$  m

Let the height of water over V-notch  $= H$

The rate of flow through V-notch is given by equation (8.2) as

$$Q = \frac{8}{15} \times C_d \times \sqrt{2g} \times \tan \frac{\theta}{2} \times H^{5/2}$$

where  $C_d = 0.62$ ,  $\theta = 90^\circ$

$$\therefore Q = \frac{8}{15} \times .62 \times \sqrt{2 \times 9.81} \times \tan \frac{90^\circ}{2} \times H^{5/2}$$

$$\text{or } 0.25 = \frac{8}{15} \times .62 \times 4.429 \times 1 \times H^{5/2}$$

$$\text{or } H^{5/2} = \frac{.25 \times 15}{8 \times .62 \times 4.429} = 0.1707$$

$$\therefore H = (.1707)^{2/5} = (.1707)^{0.4} = 0.493 \text{ m}$$

Position of apex of the notch from the bed of channel

$=$  depth of water in channel–height of water over V-notch

$= 1.3 - .493 = 0.807 \text{ m. Ans.}$

## ► 8.5 ADVANTAGES OF TRIANGULAR NOTCH OR WEIR OVER RECTANGULAR NOTCH OR WEIR

A triangular notch or weir is preferred to a rectangular weir or notch due to following reasons :

1. The expression for discharge for a right-angled V-notch or weir is very simple.
2. For measuring low discharge, a triangular notch gives more accurate results than a rectangular notch.
3. In case of triangular notch, only one reading, i.e.,  $H$  is required for the computation of discharge.
4. Ventilation of a triangular notch is not necessary.

## ► 8.6 DISCHARGE OVER A TRAPEZOIDAL NOTCH OR WEIR

As shown in Fig. 8.4, a trapezoidal notch or weir is a combination of a rectangular and triangular notch or weir. Thus the total discharge will be equal to the sum of discharge through a rectangular weir or notch and discharge through a triangular notch or weir.

Let  $H$  = Height of water over the notch

$L$  = Length of the crest of the notch

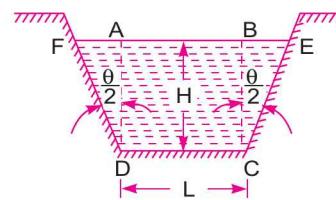


Fig. 8.4 The trapezoidal notch.

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$C_{d_1}$  = Co-efficient of discharge for rectangular portion  $ABCD$  of Fig. 8.4.

$C_{d_2}$  = Co-efficient of discharge for triangular portion [FAD and BCE]

The discharge through rectangular portion  $ABCD$  is given by (8.1)

$$\text{or } Q_1 = \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g} \times H^{3/2}$$

The discharge through two triangular notches  $FDA$  and  $BCE$  is equal to the discharge through a single triangular notch of angle  $\theta$  and it is given by equation (8.2) as

$$Q_2 = \frac{8}{15} \times C_{d_2} \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$\therefore$  Discharge through trapezoidal notch or weir  $FDCEF = Q_1 + Q_2$

$$= \frac{2}{3} C_{d_1} L \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan \theta/2 \times \sqrt{2g} \times H^{5/2}. \quad \dots(8.4)$$

**Problem 8.7** Find the discharge through a trapezoidal notch which is 1 m wide at the top and 0.40 m at the bottom and is 30 cm in height. The head of water on the notch is 20 cm. Assume  $C_d$  for rectangular portion = 0.62 while for triangular portion = 0.60.

**Solution.** Given :

Top width,  $AE = 1 \text{ m}$

Base width,  $CD = L = 0.4 \text{ m}$

Head of water,  $H = 0.20 \text{ m}$

For rectangular portion,  $C_{d_1} = 0.62$

For triangular portion,  $C_{d_2} = 0.60$

From  $\Delta ABC$ , we have

$$AE = 1 \text{ m}$$

$$CD = L = 0.4 \text{ m}$$

$$H = 0.20 \text{ m}$$

$$C_{d_1} = 0.62$$

$$C_{d_2} = 0.60$$

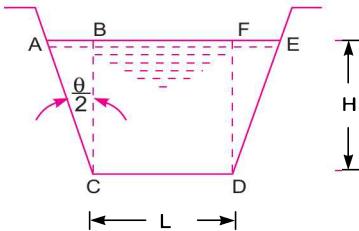


Fig. 8.5

$$\tan \frac{\theta}{2} = \frac{AB}{BC} = \frac{(AE - CD)/2}{H}$$

$$= \frac{(1.0 - 0.4)/2}{0.3} = \frac{0.6/2}{0.3} = \frac{0.3}{0.3} = 1$$

Discharge through trapezoidal notch is given by equation (8.4)

$$\begin{aligned} Q &= \frac{2}{3} C_{d_1} \times L \times \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2} \\ &= \frac{2}{3} \times 0.62 \times 0.4 \times \sqrt{2 \times 9.81} \times (0.2)^{3/2} + \frac{8}{15} \times 0.60 \times 1 \times \sqrt{2 \times 9.81} \times (0.2)^{5/2} \\ &= 0.06549 + 0.02535 = 0.09084 \text{ m}^3/\text{s} = \mathbf{90.84 \text{ litres/s. Ans.}} \end{aligned}$$

### ► 8.7 DISCHARGE OVER A STEPPED NOTCH

A stepped notch is a combination of rectangular notches. The discharge through stepped notch is equal to the sum of the discharges through the different rectangular notches.

Consider a stepped notch as shown in Fig. 8.6.

Let  $H_1$  = Height of water above the crest of notch 1,

$L_1$  = Length of notch 1,

$H_2, L_2$  and  $H_3, L_3$  are corresponding values for notches 2 and 3 respectively.

$C_d$  = Co-efficient of discharge for all notches

$$\therefore \text{Total discharge } Q = Q_1 + Q_2 + Q_3$$

$$\text{or } Q = \frac{2}{3} \times C_d \times L_1 \times \sqrt{2g} [H_1^{3/2} - H_2^{3/2}]$$

$$+ \frac{2}{3} C_d \times L_2 \times \sqrt{2g} [H_2^{3/2} - H_3^{3/2}] + \frac{2}{3} C_d \times L_3 \times \sqrt{2g} \times H_3^{3/2}. \quad \dots(8.5)$$

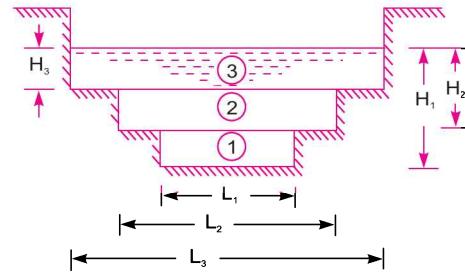


Fig. 8.6 The stepped notch.

**Problem 8.8** Fig. 8.7 shows a stepped notch. Find the discharge through the notch if  $C_d$  for all section = 0.62.

**Solution.** Given :

$$L_1 = 40 \text{ cm}, L_2 = 80 \text{ cm},$$

$$L_3 = 120 \text{ cm}$$

$$H_1 = 50 + 30 + 15 = 95 \text{ cm},$$

$$H_2 = 80 \text{ cm}, H_3 = 50 \text{ cm},$$

$$C_d = 0.62$$

$$\text{Total discharge, } Q = Q_1 + Q_2 + Q_3$$

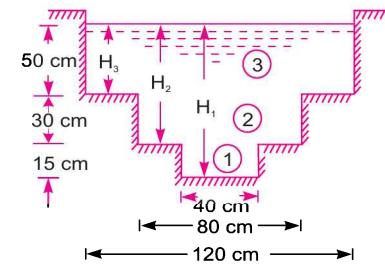


Fig. 8.7

$$\text{where } Q_1 = \frac{2}{3} \times C_d \times L_1 \times \sqrt{2g} [H_1^{3/2} - H_2^{3/2}]$$

$$= \frac{2}{3} \times 0.62 \times 40 \times \sqrt{2 \times 981} \times [95^{3/2} - 80^{3/2}]$$

$$= 732.26[925.94 - 715.54] = 154067 \text{ cm}^3/\text{s} = 154.067 \text{ lit/s}$$

$$Q_2 = \frac{2}{3} \times C_d \times L_2 \times \sqrt{2g} \times [H_2^{3/2} - H_3^{3/2}]$$

$$= \frac{2}{3} \times 0.62 \times 80 \times \sqrt{2 \times 981} \times [80^{3/2} - 50^{3/2}]$$

$$= 1464.52[715.54 - 353.55] \text{ cm}^3/\text{s} = 530141 \text{ cm}^3/\text{s} = 530.144 \text{ lit/s}$$

$$\text{and } Q_3 = \frac{2}{3} \times C_d \times L_3 \times \sqrt{2g} \times H_3^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 120 \times \sqrt{2 \times 981} \times 50^{3/2} = 776771 \text{ cm}^3/\text{s} = 776.771 \text{ lit/s}$$

$$\therefore Q = Q_1 + Q_2 + Q_3 = 154.067 + 530.144 + 776.771 \\ = 1460.98 \text{ lit/s. Ans.}$$

► **8.8 EFFECT ON DISCHARGE OVER A NOTCH OR WEIR DUE TO ERROR IN THE MEASUREMENT OF HEAD**

For an accurate value of the discharge over a weir or notch, an accurate measurement of head over the weir or notch is very essential as the discharge over a triangular notch is proportional to  $H^{5/2}$  and in case of rectangular notch it is proportional to  $H^{3/2}$ . A small error in the measurement of head, will affect the discharge considerably. The following cases of error in the measurement of head will be considered :

- (i) For Rectangular Weir or Notch.
- (ii) For Triangular Weir or Notch.

**8.8.1 For Rectangular Weir or Notch.** The discharge for a rectangular weir or notch is given by equation (8.1) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \\ &= KH^{3/2} \end{aligned} \quad \dots(i)$$

$$\text{where } K = \frac{2}{3} C_d \times L \times \sqrt{2g}$$

Differentiating the above equation, we get

$$dQ = K \times \frac{3}{2} H^{1/2} dH \quad \dots(ii)$$

$$\text{Dividing (ii) by (i), } \frac{dQ}{Q} = \frac{K \times \frac{3}{2} \times H^{1/2} dH}{KH^{3/2}} = \frac{3}{2} \frac{dH}{H} \quad \dots(8.6)$$

Equation (8.6) shows that an error of 1% in measuring  $H$  will produce 1.5% error in discharge over a rectangular weir or notch.

**8.8.2 For Triangular Weir or Notch.** The discharge over a triangular weir or notch is given by equation (8.2) as

$$\begin{aligned} Q &= \frac{8}{15} C_d \cdot \tan \frac{\theta}{2} \sqrt{2g} \times H^{5/2} \\ &= KH^{5/2} \end{aligned} \quad \dots(iii)$$

$$\text{where } K = \frac{8}{15} C_d \cdot \tan \frac{\theta}{2} \sqrt{2g}$$

Differentiating equation (iii), we get

$$dQ = K \frac{5}{2} H^{3/2} \times dH \quad \dots(iv)$$

$$\text{Dividing (iv) by (iii), we get } \frac{dQ}{Q} = \frac{K \frac{5}{2} H^{3/2} dH}{KH^{5/2}} = \frac{5}{2} \frac{dH}{H} \quad \dots(8.7)$$

Equation (8.7) shows that an error of 1% in measuring  $H$  will produce 2.5% error in discharge over a triangular weir or notch.

**Problem 8.9** A rectangular notch 40 cm long is used for measuring a discharge of 30 litres per second. An error of 1.5 mm was made, while measuring the head over the notch. Calculate the percentage error in the discharge. Take  $C_d = 0.60$ .

**Solution.** Given :

$$\begin{aligned} \text{Length of notch, } & L = 40 \text{ cm} \\ \text{Discharge, } & Q = 30 \text{ lit/s} = 30000 \text{ cm}^3/\text{s} \\ \text{Error in head, } & dH = 1.5 \text{ mm} = 0.15 \text{ cm} \\ & C_d = 0.60 \end{aligned}$$

Let the height of water over rectangular notch =  $H$

The discharge through a rectangular notch is given by (8.1)

$$\text{or } Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$\text{or } 30000 = \frac{2}{3} \times 0.60 \times 40 \times \sqrt{2 \times 981} \times H^{3/2}$$

$$\text{or } H^{3/2} = \frac{3 \times 30000}{2 \times 0.60 \times 40 \times \sqrt{2 \times 981}} = 42.33$$

$$\therefore H = (42.33)^{2/3} = 12.16 \text{ cm}$$

Using equation (8.6), we get

$$\frac{dQ}{Q} = \frac{3}{2} \frac{dH}{H} = \frac{3}{2} \times \frac{0.15}{12.16} = 0.0185 = 1.85\%. \text{ Ans.}$$

**Problem 8.10** A right-angled V-notch is used for measuring a discharge of 30 litres/s. An error of 1.5 mm was made while measuring the head over the notch. Calculate the percentage error in the discharge. Take  $C_d = 0.62$ .

**Solution.** Given :

$$\begin{aligned} \text{Angle of V-notch, } & \theta = 90^\circ \\ \text{Discharge, } & Q = 30 \text{ lit/s} = 30000 \text{ cm}^3/\text{s} \\ \text{Error in head, } & dH = 1.5 \text{ mm} = 0.15 \text{ cm} \\ & C_d = 0.62 \end{aligned}$$

Let the head over the V-notch =  $H$

The discharge  $Q$  through a triangular notch is given by equation (8.2)

$$Q = \frac{8}{15} C_d \cdot \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

$$\text{or } 30000 = \frac{8}{15} \times 0.62 \times \tan \left( \frac{90^\circ}{2} \right) \times \sqrt{2 \times 981} \times H^{5/2}$$

$$= \frac{8}{15} \times 0.62 \times 1 \times 44.29 \times H^{5/2}$$

$$\therefore H^{5/2} = \frac{30000 \times 15}{8 \times 0.62 \times 44.29} = 2048.44$$

$$\therefore H = (2048.44)^{2/5} = 21.11 \text{ cm}$$

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Using equation (8.7), we get

$$\frac{dQ}{Q} = \frac{5}{2} \frac{dH}{H} = 2.5 \times \frac{0.15}{21.11} = 0.01776 = 1.77\%. \text{ Ans.}$$

**Problem 8.11** The head of water over a triangular notch of angle  $60^\circ$  is 50 cm and co-efficient of discharge is 0.62. The flow measured by it is to be within an accuracy of 1.5% up or down. Find the limiting values of the head.

**Solution.** Given :

$$\begin{aligned}\text{Angle of } V\text{-notch,} & \quad \theta = 60^\circ \\ \text{Head of water,} & \quad H = 50 \text{ cm} \\ C_d & = 0.62\end{aligned}$$

$$\frac{dQ}{Q} = \pm 1.5\% = \pm 0.015$$

The discharge  $Q$  over a triangular notch is

$$\begin{aligned}Q &= \frac{8}{15} C_d \sqrt{2g} \tan \frac{\theta}{2} H^{5/2} \\ &= \frac{8}{15} \times 0.62 \times \sqrt{2 \times 981} \times \tan \frac{60^\circ}{2} \times (50)^{5/2} \\ &= 14.64 \times 0.5773 \times 17677.67 = 149405.86 \text{ cm}^3/\text{s}\end{aligned}$$

Now applying equation (8.7), we get

$$\begin{aligned}\frac{dQ}{Q} &= \frac{5}{2} \frac{dH}{H} \text{ or } \pm .015 = 2.5 \frac{dH}{H} \text{ or } \frac{dH}{H} = \pm \frac{.015}{2.5} \\ \therefore dH &= \pm \frac{.015}{2.5} \times H = \pm \frac{.015}{2.5} \times 50 = \pm 0.3\end{aligned}$$

$\therefore$  The limiting values of the head

$$\begin{aligned}&= H \pm dH = 50 \pm 0.3 = 50.3 \text{ cm, } 49.7 \text{ cm} \\ &= \mathbf{50.3 \text{ cm and } 49.7 \text{ cm. Ans.}}$$

### ► 8.9. (a) TIME REQUIRED TO EMPTY A RESERVOIR OR A TANK WITH A RECTANGULAR WEIR OR NOTCH

Consider a reservoir or tank of uniform cross-sectional area  $A$ . A rectangular weir or notch is provided in one of its sides.

Let  $L$  = Length of crest of the weir or notch

$C_d$  = Co-efficient of discharge

$H_1$  = Initial height of liquid above the crest of notch

$H_2$  = Final height of liquid above the crest of notch

$T$  = Time required in seconds to lower the height of liquid from  $H_1$  to  $H_2$ .

Let at any instant, the height of liquid surface above the crest of weir or notch be  $h$  and in a small time  $dT$ , let the liquid surface falls by ' $dh$ '. Then,

$$-Adh = Q \times dT$$

-ve sign is taken, as with the increase of  $T$ ,  $h$  decreases.

But  $Q = \frac{2}{3} C_d \times L \times \sqrt{2g} \times h^{3/2}$

$$\therefore -Adh = \frac{2}{3} C_d \times L \times \sqrt{2g} \cdot h^{3/2} \times dT \text{ or } dT = \frac{-Adh}{\frac{2}{3} C_d \times L \times \sqrt{2g} \times h^{3/2}}$$

The total time  $T$  is obtained by integrating the above equation between the limits  $H_1$  and  $H_2$ .

$$\int_0^T dT = \int_{H_1}^{H_2} \frac{-Adh}{\frac{2}{3} C_d \times L \times \sqrt{2g} \times h^{3/2}}$$

or

$$\begin{aligned} T &= \frac{-A}{\frac{2}{3} C_d \times L \times \sqrt{2g}} \int_{H_1}^{H_2} h^{-3/2} dh = \frac{-3A}{2 C_d \times L \times \sqrt{2g}} \left[ \frac{h^{-3/2+1}}{-\frac{3}{2} + 1} \right]_{H_1}^{H_2} \\ &= \frac{-3A}{2 C_d \times L \times \sqrt{2g}} \left[ \frac{h^{-1/2}}{-\frac{1}{2}} \right]_{H_1}^{H_2} = \frac{-3A}{2 C_d \times L \times \sqrt{2g}} \left( -\frac{2}{1} \right) \left[ \frac{1}{\sqrt{h}} \right]_{H_1}^{H_2} \\ &= \frac{3A}{C_d \times L \times \sqrt{2g}} \left[ \frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}} \right]. \end{aligned} \quad \dots(8.8)$$

### (b) TIME REQUIRED TO EMPTY A RESERVOIR OR A TANK WITH A TRIANGULAR WEIR OR NOTCH

Consider a reservoir or tank of uniform cross-sectional area  $A$ , having a triangular weir or notch in one of its sides.

Let  $\theta$  = Angle of the notch

$C_d$  = Co-efficient of discharge

$H_1$  = Initial height of liquid above the apex of notch

$H_2$  = Final height of liquid above the apex of notch

$T$  = Time required in seconds, to lower the height from  $H_1$  to  $H_2$  above the apex of the notch.

Let at any instant, the height of liquid surface above the apex of weir or notch be  $h$  and in a small time  $dT$ , let the liquid surface falls by 'dh'. Then

$$-Adh = Q \times dT$$

-ve sign is taken, as with the increase of  $T$ ,  $h$  decreases.

And  $Q$  for a triangular notch is

$$Q = \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \sqrt{2g} \times h^{5/2}$$

$$\therefore -Adh = \frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times h^{5/2} \times dT$$

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$$\therefore dT = \frac{Adh}{\frac{8}{15} \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times h^{5/2}}$$

The total time  $T$  is obtained by integrating the above equation between the limits  $H_1$  and  $H_2$ .

$$\begin{aligned} \therefore \int_0^T dT &= \int_{H_1}^{H_2} \frac{-Adh}{\frac{8}{15} C_d \tan \frac{\theta}{2} \sqrt{2g} h^{5/2}} \\ \text{or } T &= \frac{-A}{\frac{8}{15} C_d \times \tan \frac{\theta}{2} \times \sqrt{2g}} \int_{H_1}^{H_2} h^{-5/2} dh \\ &= \frac{-15A}{8 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g}} \left[ \frac{h^{-3/2}}{-\frac{3}{2}} \right]_{H_1}^{H_2} \\ &= \frac{-15A}{8 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g}} \times \left( -\frac{2}{3} \right) \left[ \frac{1}{h^{3/2}} \right]_{H_1}^{H_2} \\ &= \frac{5A}{4 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g}} \left[ \frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}} \right]. \end{aligned} \quad \dots(8.9)$$

**Problem 8.12** Find the time required to lower the water level from 3 m to 2 m in a reservoir of dimension  $80 \text{ m} \times 80 \text{ m}$ , by a rectangular notch of length 1.5 m. Take  $C_d = 0.62$ .

**Solution.** Given :

$$\text{Initial height of water, } H_1 = 3 \text{ m}$$

$$\text{Final height of water, } H_2 = 2 \text{ m}$$

$$\text{Dimension of reservoir} = 80 \text{ m} \times 80 \text{ m}$$

$$\text{or Area, } A = 80 \times 80 = 6400 \text{ m}^2$$

$$\text{Length of notch, } L = 1.5 \text{ m, } C_d = 0.62$$

Using the relation given by the equation (8.8)

$$\begin{aligned} T &= \frac{3A}{C_d \times L \times \sqrt{2g}} \left[ \frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}} \right] \\ &= \frac{3 \times 6400}{0.62 \times 1.5 \times \sqrt{2 \times 9.81}} \left[ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right] \\ &= 4661.35 [0.7071 - 0.5773] \text{ seconds} \\ &= 605.04 \text{ seconds} = \mathbf{10 \text{ min } 5 \text{ sec. Ans.}} \end{aligned}$$

**Problem 8.13** If in problem 8.12, instead of a rectangular notch, a right-angled V-notch is used, find the time required. Take all other data same.

**Solution.** Given :

$$\begin{aligned}\text{Angle of notch, } & \theta = 90^\circ \\ \text{Initial height of water, } & H_1 = 3 \text{ m} \\ \text{Final height of water, } & H_2 = 2 \text{ m} \\ \text{Area of reservoir, } & A = 80 \times 80 = 6400 \text{ m}^2 \\ C_d & = 0.62\end{aligned}$$

Using the relation given by equation (8.9)

$$\begin{aligned}T &= \frac{5A}{4 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g}} \left[ \frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}} \right] \\ &= \frac{5 \times 6400}{4 \times .62 \times \tan \frac{90^\circ}{2} \times \sqrt{2 \times 9.81}} \left[ \frac{1}{2^{1.5}} - \frac{1}{3^{1.5}} \right] \quad \left\{ \because \tan \frac{90^\circ}{2} = 1 \right\} \\ &= 2913.34 \times \left[ \frac{1}{2.8284} - \frac{1}{5.1961} \right] \\ &= 2913.34 [0.3535 - 0.1924] \text{ seconds} \\ &= 469.33 \text{ seconds} = \mathbf{7 \text{ min } 49.33 \text{ sec. Ans.}}$$

**Problem 8.14** A right-angled V-notch is inserted in the side of a tank of length 4 m and width 2.5 m. Initial height of water above the apex of the notch is 30 cm. Find the height of water above the apex if the time required to lower the head in tank from 30 cm to final height is 3 minutes. Take  $C_d = 0.60$ .

**Solution.** Given :

$$\begin{aligned}\text{Angle of notch, } & \theta = 90^\circ \\ \text{Area of tank, } & A = \text{Length} \times \text{width} = 4 \times 2.5 = 10.0 \text{ m}^2 \\ \text{Initial height of water, } & H_1 = 30 \text{ cm} = 0.3 \text{ m} \\ \text{Time, } & T = 3 \text{ min} = 3 \times 60 = 180 \text{ seconds} \\ C_d & = 0.60\end{aligned}$$

Let the final height of water above the apex of notch =  $H_2$

Using the relation given by equation (8.9)

$$\begin{aligned}T &= \frac{5A}{4 \times C_d \times \tan \frac{\theta}{2} \times \sqrt{2g}} \left[ \frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}} \right] \\ 180 &= \frac{5 \times 10}{4 \times .60 \times \tan \left( \frac{90^\circ}{2} \right) \times \sqrt{2 \times 9.81}} \left[ \frac{1}{H_2^{3/2}} - \frac{1}{(0.3)^{3/2}} \right] \\ &= \frac{50}{4 \times .60 \times 1 \times 4.429} \left[ \frac{1}{H_2^{3/2}} - \frac{1}{(0.3)^{3/2}} \right]\end{aligned}$$

$$\text{or } \frac{1}{H_2^{1.5}} - \frac{1}{0.3^{1.5}} = \frac{180 \times 4 \times 0.60 \times 4.429}{50} = 38.266.$$

$$\text{or } \frac{1}{H_2^{1.5}} - 6.0858 = 38.266$$

$$\therefore \frac{1}{H_2^{1.5}} = 38.266 + 6.0858 = 44.35 \text{ or } H_2^{1.5} = \frac{1}{44.35} = 0.0225$$

$$\therefore H_2 = (0.0225)^{1/1.5} = (0.0225)^{.6667} = 0.0822 \text{ m} = 8.22 \text{ cm. Ans.}$$

## ► 8.10 VELOCITY OF APPROACH

Velocity of approach is defined as the velocity with which the water approaches or reaches the weir or notch before it flows over it. Thus if  $V_a$  is the velocity of approach, then an additional head  $h_a$

equal to  $\frac{V_a^2}{2g}$  due to velocity of approach, is acting on the water flowing over the notch. Then initial height of water over the notch becomes  $(H + h_a)$  and final height becomes equal to  $h_a$ . Then all the formulae are changed taking into consideration of velocity of approach.

The velocity of approach,  $V_a$  is determined by finding the discharge over the notch or weir neglecting velocity of approach. Then dividing the discharge by the cross-sectional area of the channel on the upstream side of the weir or notch, the velocity of approach is obtained. Mathematically,

$$V_a = \frac{Q}{\text{Area of channel}}$$

This velocity of approach is used to find an additional head  $\left( h_a = \frac{V_a^2}{2g} \right)$ . Again the discharge is

calculated and above process is repeated for more accurate discharge.

Discharge over a rectangular weir, with velocity of approach

$$= \frac{2}{3} \times C_d \times L \times \sqrt{2g} [(H_1 + h_a)^{3/2} - h_a^{3/2}] \quad \dots(8.10)$$

**Problem 8.15** Water is flowing in a rectangular channel of 1 m wide and 0.75 m deep. Find the discharge over a rectangular weir of crest length 60 cm, if the head of water over the crest of weir is 20 cm and water from channel flows over the weir. Take  $C_d = 0.62$ . Neglect end contractions. Take velocity of approach into consideration.

**Solution.** Given :

$$\text{Area of channel, } A = \text{Width} \times \text{depth} = 1.0 \times 0.75 = 0.75 \text{ m}^2$$

$$\text{Length of weir, } L = 60 \text{ cm} = 0.6 \text{ m}$$

$$\text{Head of water, } H_1 = 20 \text{ cm} = 0.2 \text{ m}$$

$$C_d = 0.62$$

Discharge over a rectangular weir without velocity of approach is given by

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H_1^{3/2}$$

$$= \frac{2}{3} \times 0.62 \times 0.6 \times \sqrt{2 \times 9.81} \times (0.2)^{3/2} \text{ m}^3/\text{s}$$

$$= 1.098 \times 0.0894 = 0.0982 \text{ m}^3/\text{s}$$

Velocity of approach,  $V_a = \frac{Q}{A} = \frac{0.0982}{0.75} = 0.1309 \text{ m/s}$

$\therefore$  Additional head,  $h_a = \frac{V_a^2}{2g} = (.1309)^2/2 \times 9.81 = .0008733 \text{ m}$

Then discharge with velocity of approach is given by equation (8.10)

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} [(H_1 + h_a)^{3/2} - h_a^{3/2}] \\ &= \frac{2}{3} \times 0.62 \times 0.6 \times \sqrt{2 \times 9.81} [(0.2 + .00087)^{3/2} - (.00087)^{3/2}] \\ &= 1.098 [0.09002 - .00002566] \\ &= 1.098 \times 0.09017 = \mathbf{.09881 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 8.16** Find the discharge over a rectangular weir of length 100 m. The head of water over the weir is 1.5 m. The velocity of approach is given as 0.5 m/s. Take  $C_d = 0.60$ .

**Solution.** Given :

Length of weir,  $L = 100 \text{ m}$

Head of water,  $H_1 = 1.5 \text{ m}$

Velocity of approach,  $V_a = 0.5 \text{ m/s}$

$C_d = 0.60$

$\therefore$  Additional head,  $h_a = \frac{V_a^2}{2g} = \frac{0.5 \times 0.5}{2 \times 9.81} = 0.0127 \text{ m}$

The discharge,  $Q$  over a rectangular weir due to velocity of approach is given by equation (8.10)

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} [(H_1 + h_a)^{3/2} - h_a^{3/2}] \\ &= \frac{2}{3} \times 0.6 \times 100 \times \sqrt{2 \times 9.81} [(1.5 + .0127)^{3/2} - .0127^{3/2}] \\ &= 177.16 [1.5127^{3/2} - .0127^{3/2}] \\ &= 177.16 [1.8605 - .00143] = \mathbf{329.35 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 8.17** A rectangular weir of crest length 50 cm is used to measure the rate of flow of water in a rectangular channel of 80 cm wide and 70 cm deep. Determine the discharge in the channel if the water level is 80 mm above the crest of weir. Take velocity of approach into consideration and value of  $C_d = 0.62$ .

**Solution.** Given :

Length of weir,  $L = 50 \text{ cm} = 0.5 \text{ m}$

Area of channel,  $A = \text{Width} \times \text{depth} = 80 \text{ cm} \times 70 \text{ cm} = 0.80 \times 0.70 = 0.56 \text{ m}^2$

Head over weir,  $H = 80 \text{ mm} = 0.08 \text{ m}$

$C_d = 0.62$

The discharge over a rectangular weir without velocity of approach is given by equation (8.1)

$$\begin{aligned}
 Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \\
 &= \frac{2}{3} \times 0.62 \times 0.5 \times \sqrt{2 \times 9.81} \times (0.08)^{3/2} \text{ m}^3/\text{s} \\
 &= 0.9153 \times .0226 = .0207 \text{ m}^3/\text{s}
 \end{aligned}$$

Velocity of approach,  $V_a = \frac{Q}{A} = \frac{.0207}{0.56} = .0369 \text{ m/s}$

$\therefore$  Head due to  $V_a$ ,  $h_a = V_a^2/2g = \frac{(.0369)^2}{2 \times 9.81} = .0000697 \text{ m}$

Discharge with velocity of approach is

$$\begin{aligned}
 Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} [(H_1 + h_a)^{3/2} - h_a^{3/2}] \\
 &= \frac{2}{3} \times 0.62 \times 0.5 \times \sqrt{2 \times 9.81} [(0.08 + .0000697)^{3/2} - .0000697^{3/2}] \\
 &= 0.9153 \times [.0800697^{1.5} - .0000697^{1.5}] \\
 &= .9153 [.02265 - .000000582] = \mathbf{0.2073 \text{ m}^3/\text{s. Ans.}}
 \end{aligned}$$

**Problem 8.18** A suppressed rectangular weir is constructed across a channel of 0.77 m width with a head of 0.39 m and the crest 0.6 m above the bed of the channel. Estimate the discharge over it. Consider velocity of approach and assume  $C_d = 0.623$ .

**Solution.** Given :

Width of channel,  $b = 0.77 \text{ m}$

Head over weir,  $H = 0.39 \text{ m}$

Height of crest from bed of channel = 0.6 m

$\therefore$  Depth of channel =  $0.6 + 0.39 = 0.99$

Value of  $C_d = 0.623$

Suppressed weir means that the width of channel is equal to width of weir i.e., there is no end contraction.

$\therefore$  Width of channel = Width of weir = 0.77 m

Now area of channel,  $A = \text{Width of channel} \times \text{Depth of channel}$   
 $= 0.77 \times 0.99$

The discharge over a rectangular weir without velocity of approach is given by equation (8.1).

$$\begin{aligned}
 \therefore Q &= \frac{2}{3} \times C_d \times b \times \sqrt{2g} \times H^{3/2} \quad (\because \text{Here } b = L) \\
 &= \frac{2}{3} \times 0.623 \times 0.77 \times \sqrt{2 \times 9.81} \times 0.39^{3/2} = 0.345 \text{ m}^3/\text{s}
 \end{aligned}$$

Now velocity of approach,  $V_a = \frac{Q}{\text{Area of channel}} = \frac{0.345}{0.77 \times 0.99} = 0.4526 \text{ m/s}$

Head due to velocity of approach,

$$h_a = \frac{V_a^2}{2g} = \frac{0.4526^2}{2 \times 9.81} = 0.0104 \text{ m}$$

Now the discharge with velocity of approach is given by,

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times b \times \sqrt{2g} [(H + h_a)^{3/2} - h_a^{3/2}] \\ &= \frac{2}{3} \times 0.623 \times 0.77 \times \sqrt{2 \times 9.81} [(0.39 + 0.0104)^{3/2} - (0.0104)^{3/2}] \\ &= \frac{2}{3} \times 0.623 \times 0.77 \times 4.43 [0.2533 - 0.00106] \\ &= \mathbf{0.3573 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 8.19** A sharp crested rectangular weir of 1 m height extends across a rectangular channel of 3 m width. If the head of water over the weir is 0.45 m, calculate the discharge. Consider velocity of approach and assume  $C_d = 0.623$ .

**Solution.** Given :

Width of channel,	$b = 3 \text{ m}$
Height of weir	$= 1 \text{ m}$
Head of water over weir,	$H = 0.45 \text{ m}$
$\therefore$ Depth of channel	$= \text{Height of weir} + \text{Head of water over weir}$ $= 1 + 0.45 = 1.45 \text{ m}$
Value of	$C_d = 0.623$

The discharge over a rectangular weir without velocity of approach is given by equation (8.1) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times b \times \sqrt{2g} \times H^{3/2} \\ &= \frac{2}{3} \times 0.623 \times 3 \times \sqrt{2 \times 9.81} \times 0.45^{3/2} = 1.665 \text{ m}^3/\text{s} \end{aligned}$$

Now velocity of approach is given by

$$\begin{aligned} V_a &= \frac{Q}{\text{Area of channel}} \\ &= \frac{1.665}{\text{Width of channel} \times \text{Depth of channel}} = \frac{1.665}{3 \times 1.45} = 0.382 \text{ m/s} \end{aligned}$$

Head due to velocity of approach is given by,

$$h_a = \frac{V_a^2}{2g} = \frac{0.382^2}{2 \times 9.81} = 0.0074 \text{ m}$$

Now the discharge with velocity of approach is given by,

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times b \times \sqrt{2g} [(H + h_a)^{3/2} - (h_a)^{3/2}] \\ &= \frac{2}{3} \times 0.623 \times 3 \times \sqrt{2 \times 9.81} [(0.45 + 0.0074)^{3/2} - (0.0074)^{3/2}] \\ &= \mathbf{1.703 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

### ► 8.11 EMPIRICAL FORMULAE FOR DISCHARGE OVER RECTANGULAR WEIR

The discharge over a rectangular weir is given by

$$Q = \frac{2}{3} C_d \sqrt{2g} \times L \times [H^{3/2}] \text{ without velocity of approach} \quad \dots(i)$$

$$= \frac{2}{3} C_d \sqrt{2g} \times L \times [(H + h_a)^{3/2} - h_a^{3/2}] \text{ with velocity of approach} \quad \dots(ii)$$

Equations (i) and (ii) are applicable to the weir or notch for which the crest length is equal to the width of the channel. This type of weir is called *Suppressed weir*. But if the weir is not suppressed, the effect of end contraction will be taken into account.

(a) **Francis's Formula.** Francis on the basis of his experiments established that end contraction decreases the effective length of the crest of weir and hence decreases the discharge. Each end contraction reduces the crest length by  $0.1 \times H$ , where  $H$  is the head over the weir. For a rectangular weir there are two end contractions only and hence effective length

$$L = (L - 0.2 H)$$

and

$$Q = \frac{2}{3} \times C_d \times [L - 0.2 \times H] \times \sqrt{2g} H^{3/2}$$

If

$$C_d = 0.623, g = 9.81 \text{ m/s}^2, \text{ then}$$

$$\begin{aligned} Q &= \frac{2}{3} \times .623 \times \sqrt{2 \times 9.81} \times [L - 0.2 \times H] \times H^{3/2} \\ &= 1.84 [L - 0.2 \times H] H^{3/2} \end{aligned} \quad \dots(8.11)$$

If end contractions are suppressed, then

$$H = 1.84 L H^{3/2} \quad \dots(8.12)$$

If velocity of approach is considered, then

$$Q = 1.84 L [(H + h_a)^{3/2} - h_a^{3/2}] \quad \dots(8.13)$$

(b) **Bazin's Formula.** On the basis of results of a series of experiments, Bazin's proposed the following formula for the discharge over a rectangular weir as

$$Q = m \times L \times \sqrt{2g} \times H^{3/2} \quad \dots(8.14)$$

$$\text{where } m = \frac{2}{3} \times C_d = 0.405 + \frac{.003}{H}$$

$H$  = height of water over the weir

If velocity of approach is considered, then

$$Q = m_1 \times L \times \sqrt{2g} [(H + h_a)^{3/2}] \quad \dots(8.15)$$

$$\text{where } m_1 = 0.405 + \frac{.003}{(H + h_a)}.$$

**Problem 8.20** The head of water over a rectangular weir is 40 cm. The length of the crest of the weir with end contraction suppressed is 1.5 m. Find the discharge using the following formulae : (i) Francis's Formula and (ii) Bazin's Formula.

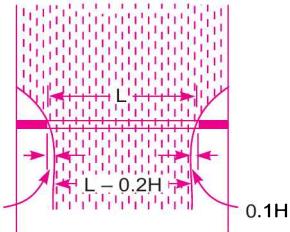


Fig. 8.8

**Solution.** Given :

$$\text{Head of water, } H = 40 \text{ cm} = 0.40 \text{ m}$$

$$\text{Length of weir, } L = 1.5 \text{ m}$$

(i) Francis's Formula for end contraction suppressed is given by equation (8.12).

$$\begin{aligned} Q &= 1.84 L \times H^{3/2} = 1.84 \times 1.5 \times (0.40)^{3/2} \\ &= 0.6982 \text{ m}^3/\text{s} \end{aligned}$$

(ii) Bazin's Formula is given by equation (8.14)

$$Q = m \times L \times \sqrt{2g} \times H^{3/2}$$

$$\text{where } m = 0.405 + \frac{.003}{H} = 0.405 + \frac{.003}{.40} = 0.4125$$

$$\begin{aligned} \therefore Q &= .4125 \times 1.5 \times \sqrt{2 \times 9.81} \times (0.4)^{3/2} \\ &= \mathbf{0.6932 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 8.21** A weir 36 metres long is divided into 12 equal bays by vertical posts, each 60 cm wide. Determine the discharge over the weir if the head over the crest is 1.20 m and velocity of approach is 2 metres per second.

**Solution.** Given :

$$\text{Length of weir, } L_1 = 36 \text{ m}$$

$$\text{Number of bays, } = 12$$

For 12 bays, no. of vertical post = 11

$$\text{Width of each post} = 60 \text{ cm} = 0.6 \text{ m}$$

$$\therefore \text{Effective length, } L = L_1 - 11 \times 0.6 = 36 - 6.6 = 29.4 \text{ m}$$

$$\text{Head on weir, } H = 1.20 \text{ m}$$

$$\text{Velocity of approach, } V_a = 2 \text{ m/s}$$

$$\therefore \text{Head due to } V_a, h_a = \frac{V_a^2}{2g} = \frac{2^2}{2 \times 9.81} = 0.2038 \text{ m}$$

$$\begin{aligned} \text{Number of end contraction, } n &= 2 \times 12 && \{\text{Each bay has two end contractions}\} \\ &= 24 \end{aligned}$$

$\therefore$  Discharge by Francis Formula with end contraction and velocity of approach is

$$\begin{aligned} Q &= 1.84 [L - 0.1 \times n(H + h_a)][(H + h_a)^{3/2} - h_a^{3/2}] \\ &= 1.84[29.4 - 0.1 \times 24(1.20 + 0.2038)] \times [(1.2 + 0.2038)^{1.5} - 0.2038^{1.5}] \\ &= 1.84[29.4 - 3.369][1.663 - 0.092] \\ &= \mathbf{75.246 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 8.22** A discharge of 2000 m<sup>3</sup>/s is to pass over a rectangular weir. The weir is divided into a number of openings each of span 10 m. If the velocity of approach is 4 m/s, find the number of openings needed in order the head of water over the crest is not to exceed 2 m.

**Solution.** Given :

$$\text{Total discharge, } Q = 2000 \text{ m}^3/\text{s}$$

$$\text{Length of each opening, } L = 10$$

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Velocity of approach,  $V_a = 4 \text{ m/s}$

Head over weir,  $H = 2 \text{ m}$

Let number of openings  $= N$

Head due to velocity of approach,

$$h_a = \frac{V_a^2}{2g} = \frac{4 \times 4}{2 \times 9.81} = 0.8155 \text{ m}$$

For each opening, number of end contractions are two. Hence discharge for each opening considering velocity of approach is given by Francis Formula

$$\begin{aligned} i.e., \quad Q &= 1.84[L - 0.1 \times 2 \times (H + h_a)][(H + h_a)^{3/2} - h_a^{3/2}] \\ &= 1.84[10.0 - 0.2 \times (2 + .8155)][2.8155^{1.5} - .8155^{1.5}] \\ &= 17.363[4.7242 - 0.7364] = 69.24 \text{ m}^3/\text{s} \end{aligned}$$

$$\begin{aligned} \therefore \text{Number of opening} &= \frac{\text{Total discharge}}{\text{Discharge for one opening}} = \frac{2000}{69.24} \\ &= 28.88 \text{ (say 29)} = \mathbf{29. Ans.} \end{aligned}$$

#### ► 8.12 CIPOLLETTI WEIR OR NOTCH

Cipolletti weir is a trapezoidal weir, which has side slopes of 1 horizontal to 4 vertical as shown in Fig. 8.9. Thus in  $\Delta ABC$ ,

$$\tan \frac{\theta}{2} = \frac{AB}{BC} = \frac{H/4}{H} = \frac{1}{4}$$

$$\therefore \frac{\theta}{2} = \tan^{-1} \frac{1}{4} = 14^\circ 2'.$$

By giving this slope to the sides, an increase in discharge through the triangular portions  $ABC$  and  $DEF$  of the weir is obtained. If this slope is not provided the weir would be a rectangular one, and due to end contraction, the discharge would decrease. Thus in case of cipolletti weir, the factor of end contraction is not required which is shown below.

The discharge through a rectangular weir with two end contractions is

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times (L - 0.2 H) \sqrt{2g} \times H^{3/2} \\ &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} H^{3/2} - \frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2} \end{aligned}$$

Thus due to end contraction, the discharge decreases by  $\frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$ . This decrease in discharge can be compensated by giving such a slope to the sides that the discharge through two triangular portions is equal to  $\frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$ . Let the slope is given by  $\theta/2$ . The discharge through a V-notch of angle  $\theta$  is given by

$$= \frac{8}{15} \times C_d \times \sqrt{2g} \times \tan \frac{\theta}{2} H^{5/2}$$

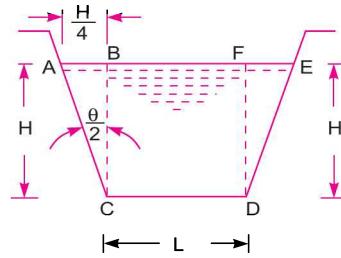


Fig. 8.9 The cipolletti weir.

Thus

$$\frac{8}{15} \times C_d \times \sqrt{2g} \times \tan \frac{\theta}{2} H^{5/2} = \frac{2}{15} \times C_d \times \sqrt{2g} \times H^{5/2}$$

$$\therefore \tan \frac{\theta}{2} = \frac{2}{15} \times \frac{15}{8} = \frac{1}{4} \quad \text{or} \quad \theta/2 = \tan^{-1} \frac{1}{4} = 14^\circ 2'.$$

Thus discharge through cipolletti weir is

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} H^{3/2} \quad \dots(8.16)$$

If velocity of approach,  $V_a$  is to be taken into consideration,

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} [(H + h_a)^{3/2} - h_a^{3/2}] \quad \dots(8.17)$$

**Problem 8.23** Find the discharge over a cipolletti weir of length 2.0 m when the head over the weir is 1 m. Take  $C_d = 0.62$ .

**Solution.** Given :

$$\text{Length of weir, } L = 20 \text{ m}$$

$$\text{Head over weir, } H = 1.0 \text{ m}$$

$$C_d = 0.62$$

Using equation (8.16), the discharge is given as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \\ &= \frac{2}{3} \times 0.62 \times 2.0 \times \sqrt{2 \times 9.81} \times (1)^{3/2} = 3.661 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

**Problem 8.24** A cipolletti weir of crest length 60 cm discharges water. The head of water over the weir is 360 mm. Find the discharge over the weir if the channel is 80 cm wide and 50 cm deep. Take  $C_d = 0.60$ .

**Solution.** Given :

$$C_d = 0.60$$

$$\text{Length of weir, } L = 60 \text{ cm} = 0.60 \text{ m}$$

$$\text{Head of water, } H = 360 \text{ mm} = 0.36 \text{ m}$$

$$\text{Channel width} = 80 \text{ cm} = 0.80 \text{ m}$$

$$\text{Channel depth} = 50 \text{ cm} = 0.50 \text{ m}$$

$$A = \text{cross-sectional area of channel} = 0.8 \times 0.5 = 0.4 \text{ m}^2$$

To find velocity of approach, first determine discharge over the weir as

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2}$$

$$\text{The velocity of approach, } V_a = \frac{Q}{A}$$

$$\therefore Q = \frac{2}{3} \times 0.60 \times 0.60 \times \sqrt{2 \times 9.81} \times (0.36)^{3/2} \text{ m}^3/\text{s} = 0.2296 \text{ m}^3/\text{s}$$

$$\therefore V_a = \frac{0.2296}{0.40} = 0.574 \text{ m/s}$$

Head due to velocity of approach,

$$h_a = V_a^2/2g = \frac{(0.574)^2}{2 \times 9.81} = 0.0168 \text{ m}$$

Thus the discharge is given by equation (8.17) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} [(H + h_a)^{1.5} - h_a^{1.5}] \\ &= \frac{2}{3} \times 0.60 \times .6 \times \sqrt{2 \times 9.81} [(0.36 + 0.0168)^{1.5} - (0.0168)^{1.5}] \\ &= 1.06296 \times [0.2313 - 0.002177] = 0.2435 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

### ► 8.13 DISCHARGE OVER A BROAD-CRESTED WEIR

A weir having a wide crest is known as broad-crested weir.

Let  $H$  = height of water above the crest

$L$  = length of the crest

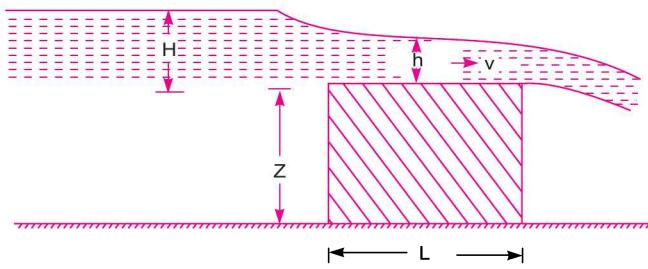


Fig. 8.10 Broad-crested weir.

If  $2L > H$ , the weir is called broad-crested weir

If  $2L < H$ , the weir is called a narrow-crested weir

Fig. 8.10 shows a broad-crested weir.

Let  $h$  = head of water at the middle of weir which is constant

$v$  = velocity of flow over the weir

Applying Bernoulli's equation to the still water surface on the upstream side and running water at the end of weir,

$$0 + 0 + H = 0 + \frac{v^2}{2g} + h$$

$$\therefore \frac{v^2}{2g} = H - h$$

$$\therefore v = \sqrt{2g(H - h)}$$

$\therefore$  The discharge over weir  $Q = C_d \times \text{Area of flow} \times \text{Velocity}$

$$= C_d \times L \times h \times \sqrt{2g(H - h)}$$

$$= C_d \times L \times \sqrt{2g(Hh^2 - h^3)} \quad \dots(8.18)$$

The discharge will be maximum, if  $(Hh^2 - h^3)$  is maximum

$$\text{or } \frac{d}{dh} (Hh^2 - h^3) = 0 \text{ or } 2h \times H - 3h^2 = 0 \text{ or } 2H = 3h$$

$$\therefore h = \frac{2}{3} H$$

$Q_{\max}$  will be obtained by substituting this value of  $h$  in equation (8.18) as

$$\begin{aligned} Q_{\max} &= C_d \times L \times \sqrt{2g} \left[ H \times \left( \frac{2}{3} H \right)^2 - \left( \frac{2}{3} H \right)^3 \right] \\ &= C_d \times L \times \sqrt{2g} \sqrt{H \times \frac{4}{9} \times H^2 - \frac{8}{27} H^3} \\ &= C_d \times L \times \sqrt{2g} \sqrt{\frac{4}{9} H^3 - \frac{8}{27} H^3} = C_d \times L \times \sqrt{2g} \sqrt{\frac{(12-8)H^3}{27}} \\ &= C_d \times L \times \sqrt{2g} \sqrt{\frac{4}{27} H^3} = C_d \times L \times \sqrt{2g} \times 0.3849 \times H^{3/2} \\ &= .3849 \times \sqrt{2 \times 9.81} \times C_d \times L \times H^{3/2} = 1.7047 \times C_d \times L \times H^{3/2} \\ &= 1.705 \times C_d \times L \times H^{3/2}. \end{aligned} \quad \dots(8.19)$$

#### ► 8.14 DISCHARGE OVER A NARROW-CRESTED WEIR

For a narrow-crested weir,  $2L < H$ . It is similar to a rectangular weir or notch hence,  $Q$  is given by

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \quad \dots(8.20)$$

#### ► 8.15 DISCHARGE OVER AN Ogee WEIR

Fig. 8.11 shows an Ogee weir, in which the crest of the weir rises upto maximum height of  $0.115 \times H$  (where  $H$  is the height of water above inlet of the weir) and then falls as shown in Fig. 8.11. The discharge for an Ogee weir is the same as that of a rectangular weir, and it is given by

$$Q = \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \quad \dots(8.21)$$

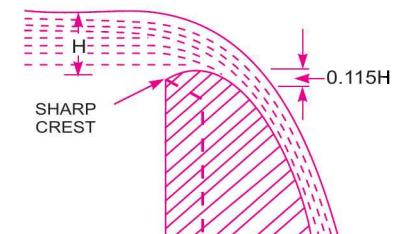


Fig. 8.11 An Ogee weir.

#### ► 8.16 DISCHARGE OVER SUB-MERGED OR DROWNED WEIR

When the water level on the downstream side of a weir is above the crest of the weir, then the weir is called to be a sub-merged or drowned weir. Fig. 8.12 shows a sub-merged weir. The total discharge, over the weir is obtained by dividing the weir into two parts. The portion between upstream and downstream water surface may be treated as free weir and portion between downstream water surface and crest of weir as a drowned weir.

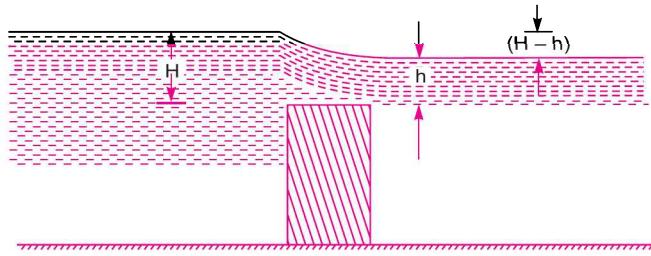


Fig. 8.12 Sub-merged weir.

Let  $H$  = height of water on the upstream side of the weir

$h$  = height of water on the downstream side of the weir

Then

$Q_1$  = discharge over upper portion

$$= \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g} [H - h]^{3/2}$$

$Q_2$  = discharge through drowned portion

$$\begin{aligned} &= C_{d_2} \times \text{Area of flow} \times \text{Velocity of flow} \\ &= C_{d_2} \times L \times h \times \sqrt{2g(H - h)} \end{aligned}$$

∴ Total discharge,

$$Q = Q_1 + Q_2$$

$$= \frac{2}{3} C_{d_1} \times L \times \sqrt{2g} [H - h]^{3/2} + C_{d_2} \times L \times h \times \sqrt{2g(H - h)} \dots(8.22)$$

**Problem 8.25** (a) A broad-crested weir of 50 m length, has 50 cm height of water above its crest. Find the maximum discharge. Take  $C_d = 0.60$ . Neglect velocity of approach. (b) If the velocity of approach is to be taken into consideration, find the maximum discharge when the channel has a cross-sectional area of  $50 \text{ m}^2$  on the upstream side.

**Solution.** Given :

$$\text{Length of weir}, \quad L = 50 \text{ m}$$

$$\text{Head of water}, \quad H = 50 \text{ cm} = 0.5 \text{ m}$$

$$C_d = 0.60$$

(i) **Neglecting velocity of approach.** Maximum discharge is given by equation (8.19) as

$$\begin{aligned} Q_{\max} &= 1.705 \times C_d \times L \times H^{3/2} \\ &= 1.705 \times 0.60 \times 50 \times (.5)^{3/2} = 18.084 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

(ii) **Taking velocity of approach into consideration**

$$\text{Area of channel}, \quad A = 50 \text{ m}^2$$

$$\text{Velocity of approach}, \quad V_a = \frac{Q}{A} = \frac{18.084}{50} = 0.36 \text{ m/s}$$

$$\therefore \text{Head due to } V_a, \quad h_a = \frac{V_a^2}{2g} = \frac{0.36 \times .36}{2 \times 9.81} = .0066 \text{ m}$$

Maximum discharge,  $Q_{\max}$  is given by

$$\begin{aligned} Q_{\max} &= 1.705 \times C_d \times L \times [(H + h_a)^{3/2} - h_a^{3/2}] \\ &= 1.705 \times 0.6 \times 50 \times [(0.50 + 0.0066)^{1.5} - (0.0066)^{1.5}] \\ &= 51.15[0.3605 - 0.000536] = 18.412 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

**Problem 8.26** An Ogee weir 5 metres long has a head of 40 cm of water. If  $C_d = 0.6$ , find the discharge over the weir.

**Solution.** Given :

$$\text{Length of weir, } L = 5 \text{ m}$$

$$\text{Head of water, } H = 40 \text{ cm} = 0.40 \text{ m}$$

$$C_d = 0.6$$

Discharge over Ogee weir is given by equation (8.21) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_d \times L \times \sqrt{2g} \times H^{3/2} \\ &= \frac{2}{3} \times 0.60 \times 5.0 \times \sqrt{2 \times 9.81} \times (0.4)^{3/2} = 2.2409 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

**Problem 8.27** The heights of water on the upstream and downstream side of a sub-merged weir of 3 m length are 20 cm and 10 cm respectively. If  $C_d$  for free and drowned portions are 0.6 and 0.8 respectively, find the discharge over the weir.

**Solution.** Given :

$$\text{Height of water on upstream side, } H = 20 \text{ cm} = 0.20 \text{ m}$$

$$\text{Height of water on downstream side, } h = 10 \text{ cm} = 0.10 \text{ m}$$

$$\text{Length of weir, } L = 3 \text{ m}$$

$$C_{d_1} = 0.6$$

$$C_{d_2} = 0.8$$

Total discharge  $Q$  is the sum of discharge through free portion and discharge through the drowned portion. This is given by equation (8.22) as

$$\begin{aligned} Q &= \frac{2}{3} \times C_{d_1} \times L \times \sqrt{2g} [H - h]^{3/2} + C_{d_2} \times L \times h \times \sqrt{2g(H - h)} \\ &= \frac{2}{3} \times 0.6 \times 3 \times \sqrt{2 \times 9.81} [0.20 - 0.10]^{1.5} + 0.8 \times 3 \times 0.10 \times \sqrt{2 \times 9.81 (0.2 - 0.1)} \\ &= 0.168 + 0.336 = 0.504 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

## HIGHLIGHTS

1. A notch is a device used for measuring the rate of flow of a liquid through a small channel. A weir is a concrete or masonry structure placed in the open channel over which the flow occurs.
2. The discharge through a rectangular notch or weir is given by

$$Q = \frac{2}{3} C_d \times L \times H^{3/2}$$

where  $C_d$  = Co-efficient of discharge,

$L$  = Length of notch or weir,

$H$  = Head of water over the notch or weir.

3. The discharge over a triangular notch or weir is given by

$$Q = \frac{8}{15} C_d \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

where  $\theta$  = total angle of triangular notch.

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4. The discharge through a trapezoidal notch or weir is equal to the sum of discharge through a rectangular notch and the discharge through a triangular notch. It is given as

$$Q = \frac{2}{3} C_{d_1} \times L \times \sqrt{2g} \times H^{3/2} + \frac{8}{15} C_{d_2} \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{5/2}$$

where  $C_{d_1}$  = co-efficient of discharge for rectangular notch,

$C_{d_2}$  = co-efficient of discharge for triangular notch,

$\theta/2$  = slope of the side of trapezoidal notch.

5. The error in discharge due to the error in the measurement of head over a rectangular and triangular notch or weir is given by

$$\frac{dQ}{Q} = \frac{3}{2} \frac{dH}{H} \quad \dots \text{For a rectangular weir or notch}$$

$$= \frac{5}{2} \frac{dH}{H} \quad \dots \text{For a triangular weir or notch}$$

where  $Q$  = discharge through rectangular or triangular notch or weir

$H$  = head over the notch or weir.

6. The time required to empty a reservoir or a tank by a rectangular or a triangular notch is given by

$$H = \frac{3A}{C_d L \sqrt{2g}} \left[ \frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}} \right] \quad \dots \text{By a rectangular notch}$$

$$= \frac{5A}{4C_d \tan \frac{\theta}{2} \times \sqrt{2g}} \left[ \frac{1}{H_2^{3/2}} - \frac{1}{H_1^{3/2}} \right] \quad \dots \text{By a triangular notch}$$

where  $A$  = cross-sectional area of a tank or a reservoir

$H_1$  = initial height of liquid above the crest or apex of notch

$H_2$  = final height of liquid above the crest or apex of notch.

7. The velocity with which the water approaches the weir or notch is called the velocity of approach. It is denoted by  $V_a$  and is given by

$$V_a = \frac{\text{Discharge over the notch or weir}}{\text{Cross-sectional area of channel}}$$

8. The head due to velocity of approach is given by  $h_a = \frac{V_a^2}{2g}$ .

9. Discharge over a rectangular weir, with velocity of approach,

$$Q = \frac{2}{3} C_d L \sqrt{2g} [(H_1 + h_a)^{3/2} - h_a^{3/2}]$$

10. Francis's Formula for a rectangular weir is given by

$$\begin{aligned} Q &= 1.84 [L - 0.2 H] H^{3/2} && \dots \text{For two end contractions} \\ &= 1.84 L H^{3/2} && \dots \text{If end contractions are suppressed} \\ &= 1.84 L [(H + h_a)^{3/2} - h_a^{3/2}] && \dots \text{If velocity of approach is considered} \end{aligned}$$

where  $L$  = length of weir,

$H$  = height of water above the crest of the weir,

$h_a$  = head due to velocity of approach.

11. Bazin's Formula for discharge over a rectangular weir,

$$Q = m L \sqrt{2g} H^{3/2} \quad \dots \text{without velocity of approach}$$

$$= m L \sqrt{2g} [(H + h_a)^{3/2}] \quad \dots \text{with velocity of approach}$$

$$\text{where } m = \frac{2}{3} C_d = 0.405 + \frac{.003}{H}$$

... without velocity of approach

$$= 0.405 + \frac{.003}{(H + h_a)}$$

... with velocity of approach.

12. A trapezoidal weir, with side slope or 1 horizontal to 4 vertical, is called Cipolletti weir. The discharge through Cipolletti weir is given by

$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g} H^{3/2} \quad \text{... without velocity of approach}$$

$$= \frac{2}{3} C_d \times L \times \sqrt{2g} [(H + h_a)^{3/2} - h_a^{3/2}] \quad \text{... with velocity of approach.}$$

13. The discharge over a broad-crested weir is given by,

$$Q = C_d L \sqrt{2g(Hh^2 - h^3)}$$

where  $H$  = height of water above the crest  
 $h$  = head of water at the middle of the weir which is constant  
 $L$  = length of the weir.

- 14.** The condition for maximum discharge over a broad-crested weir is  $h = \frac{2}{3} H$

and maximum discharge is given by  $Q_{max} = 1.705 C_d L H^{3/2}$ .

- 15.** The discharge over an Ogee weir is given by  $Q = \frac{2}{3} C_d L \times \sqrt{2g} \times H^{3/2}$ .

16. The discharge over sub-merged or drowned weir is given by

$Q$  = discharge over upper portion + discharge through downed portion

$$= \frac{2}{3} C_{d_1} L \times \sqrt{2g} (H-h)^{3/2} + C_{d_2} L h \times \sqrt{2g(H-h)}$$

where  $H$  = height of water on the upstream side of the weir,  
 $h$  = height of water on the downstream side of the weir.

## **EXERCISE**

## (A) THEORETICAL PROBLEMS

1. Define the terms : notch, weir, nappe and crest.
  2. How are the weirs and notches classified ?
  3. Find an expression for the discharge over a rectangular weir in terms of head of water over the crest of the weir.
  4. Prove that the discharge through a triangular notch or weir is given by

$$Q = \frac{8}{15} C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} H^{3/2}$$

where  $H$  = head of water over the notch or weir  
 $\theta$  = angle of notch or weir.

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5. What are the advantages of triangular notch or weir over rectangular notch ?
6. Prove that the error in discharge due to the error in the measurement of head over a rectangular notch is given by

$$\frac{dQ}{Q} = \frac{3}{2} \frac{dH}{H}$$

where  $Q$  = discharge through rectangular notch  
and  $H$  = head over the rectangular notch.

7. Find an expression for the time required to empty a tank of area of cross-section  $A$ , with a rectangular notch.
8. What do you understand by 'Velocity of Approach' ? Find an expression for the discharge over a rectangular weir with velocity of approach.
9. Define 'end contraction' of a weir. What is the effect of end contraction on the discharge through a weir ?
10. What is a Cipolletti Weir ? Prove that the discharge through Cipolletti weir is given by

$$Q = \frac{2}{3} C_d L \sqrt{2g} H^{3/2}$$

where  $L$  = length of weir, and  $H$  = head of water over weir.

11. Differentiate between Broad-crested weir and Narrow-crested weir. Find the condition for maximum discharge over a Broad-crested weir and hence derive an expression for maximum discharge over a broad-crested weir.
12. What do you mean by a drowned weir ? How will you determine the discharge for the drowned weir ?
13. Discuss 'end contraction' of a weir.
14. State the different devices that can be used to measure the discharge through a pipe also through an open channel. Describe one of such devices with a neat sketch and explain how one can obtain the actual discharge with its help.
15. What is the difference between a notch and a weir ?
16. Define velocity of approach. How does the velocity of approach affect the discharge over a weir ?

### (B) NUMERICAL PROBLEMS

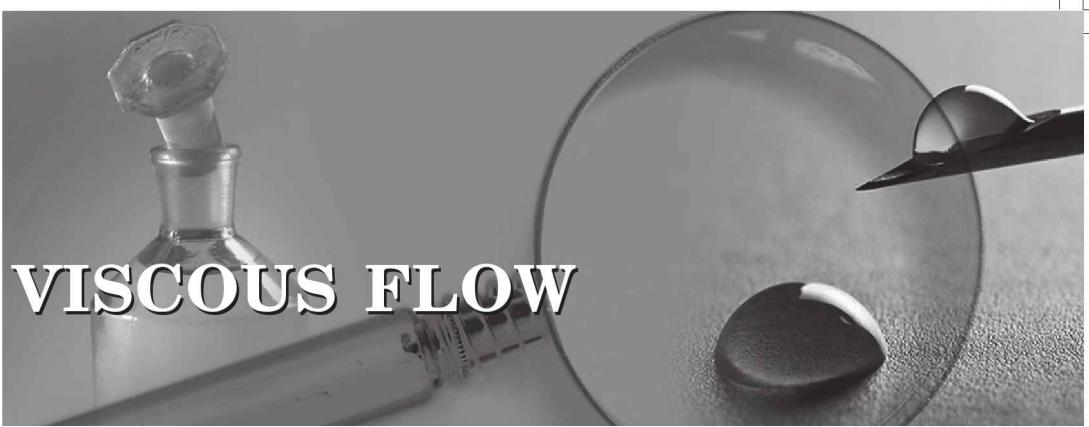
1. Find the discharge of water flowing over rectangular notch of 3 m length when the constant head of water over the notch is 40 cm. Take  $C_d = 0.6$ . [Ans.  $1.344 \text{ m}^3/\text{s}$ ]
2. Determine the height of a rectangular weir of length 5 m to be built across a rectangular channel. The maximum depth of water on the upstream side of the weir is 1.5 m and discharge is  $1.5 \text{ m}^3$  per second. Take  $C_d = 0.6$  and neglect end contractions. [Ans.  $1.194 \text{ m}$ ]
3. Find the discharge over a triangular notch of angle  $60^\circ$  when the head over the triangular notch is 0.20 m. Take  $C_d = 0.6$ . [Ans.  $0.0164 \text{ m}^3/\text{s}$ ]
4. A rectangular channel 1.5 m wide has a discharge of 200 litres per second, which is measured by a right-angled V-notch weir. Find the position of the apex of the notch from the bed of the channel if maximum depth of water is not to exceed 1 m. Take  $C_d = 0.62$ . [Ans.  $.549 \text{ m}$ ]
5. Find the discharge through a trapezoidal notch which is 1.2 m wide at the top and 0.50 m at the bottom and is 40 cm in height. The head of water on the notch is 30 cm. Assume  $C_d$  for rectangular portion as 0.62 while for triangular portion = 0.60. [Ans.  $0.22 \text{ m}^3/\text{s}$ ]
6. A rectangular notch 50 cm long is used for measuring a discharge of 40 litres per second. An error of 2 mm was made in measuring the head over the notch. Calculate the percentage error in the discharge. Take  $C_d = 0.6$ . [Ans. 2.37%]
7. A right-angled V-notch is used for measuring a discharge of 30 litres/s. An error of 2 mm was made in measuring the head over the notch. Calculate the percentage error in the discharge. Take  $C_d = 0.62$ . [Ans. 2.37%]

8. Find the time required to lower the water level from 3 m to 1.5 m in a reservoir of dimension 70 m  $\times$  70 m, by a rectangular notch of length 2.0 m. Take  $C_d = 0.60$ . [Ans. 11 min 1 s]
9. If in the problem 8, instead of a rectangular notch, a right angled V-notch is used, find the time required. Take all other data same. [Ans. 13 min 31 s]
10. Water is flowing in a rectangular channel of 1.2 m wide and 0.8 m deep. Find the discharge over a rectangular weir of crest length 70 cm if the head of water over the crest of weir is 25 cm and water from channel flows over the weir. Take  $C_d = 0.60$ . Neglect end contractions but consider velocity of approach. [Ans. 0.1557 m<sup>3</sup>/s]
11. Find the discharge over a rectangular weir of length 80 m. The head of water over the weir is 1.2 m. The velocity of approach is given as 1.5 m/s. Take  $C_d = 3.6$ . [Ans. 208.11 m<sup>3</sup>/s]
12. The head of water over a rectangular weir is 50 cm. The length of the crest of the weir with end contraction suppressed is 1.4 m. Find the discharge using following formulae : (i) Francis's Formula and (ii) Bazin's Formula. [Ans. (i) 0.91 m<sup>3</sup>/s, (ii) .901 m<sup>3</sup>/s]
13. A discharge of 1500 m<sup>3</sup>/s is to pass over a rectangular weir. The weir is divided into a number of openings each of span 7.5 m. If the velocity of approach is 3 m/s, find the number of openings needed in order the head of water over the crest is not to exceed 1.8. [Ans. 37.5 say 38]
14. Find the discharge over a cipolletti weir of length 1.8 m when the head over the weir is 1.2 m. Take  $C_d = 0.62$  [Ans. 4.331 m<sup>3</sup>/s]
15. (a) A broad-crested weir of length 40 m, has 400 mm height of water above its crest. Find the maximum discharge. Take  $C_d = 0.6$ . Neglect velocity of approach. [Ans. 10.352 m<sup>3</sup>/s]  
(b) If the velocity of approach is to be taken into consideration, find the maximum discharge when the channel has a cross-sectional area of 40 m<sup>2</sup> on the upstream side. [Ans. 10.475 m<sup>3</sup>/s]
16. An Ogee weir 4 m long has a head of 500 mm of water. If  $C_d = 0.6$ , find the discharge over the weir. [Ans. 2.505 m<sup>3</sup>/s]
17. The heights of water on the upstream and downstream side of a sub-merged weir of length 3.5 m are 300 mm and 150 mm respectively. If  $C_d$  for free and drowned portion are 0.6 and 0.8 respectively, find the discharge over the weir. [Ans. 1.0807 m<sup>3</sup>/s]



# 9

## CHAPTER



### ► 9.1 INTRODUCTION

This chapter deals with the flow of fluids which are viscous and flowing at very low velocity. At low velocity the fluid moves in layers. Each layer of fluid slides over the adjacent layer. Due to relative velocity between two layers the velocity gradient  $\frac{du}{dy}$  exists and hence a shear stress  $\tau = \mu \frac{du}{dy}$  acts on the layers.

The following cases will be considered in this chapter :

1. Flow of viscous fluid through circular pipe.
2. Flow of viscous fluid between two parallel plates.
3. Kinetic energy correction and momentum correction factors.
4. Power absorbed in viscous flow through
  - (a) Journal bearings, (b) Foot-step bearings, and (c) Collar bearings.

### ► 9.2 FLOW OF VISCOUS FLUID THROUGH CIRCULAR PIPE

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum velocity to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through the circular pipe will be viscous or laminar, if the Reynolds number ( $R_e^*$ ) is less than 2000. The expression for Reynold number is given by

$$R_e = \frac{\rho V D}{\mu}$$

where  $\rho$  = Density of fluid flowing through pipe

$V$  = Average velocity of fluid

$D$  = Diameter of pipe and

$\mu$  = Viscosity of fluid.

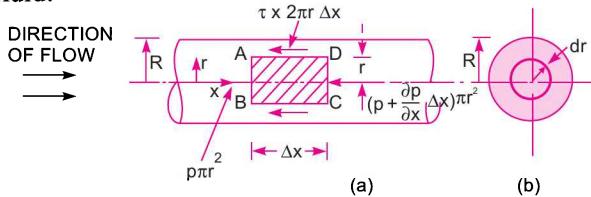


Fig. 9.1 Viscous flow through a pipe.

\* For derivation, please refer to Art. 12.8.1.

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Consider a horizontal pipe of radius  $R$ . The viscous fluid is flowing from left to right in the pipe as shown in Fig. 9.1 (a). Consider a fluid element of radius  $r$ , sliding in a cylindrical fluid element of radius  $(r + dr)$ . Let the length of fluid element be  $\Delta x$ . If ' $p$ ' is the intensity of pressure on the face  $AB$ , then the intensity of pressure on face  $CD$  will be  $\left( p + \frac{\partial p}{\partial x} \Delta x \right)$ . Then the forces acting on the fluid element are :

1. The pressure force,  $p \times \pi r^2$  on face  $AB$ .

2. The pressure force,  $\left( p + \frac{\partial p}{\partial x} \Delta x \right) \pi r^2$  on face  $CD$ .

3. The shear force,  $\tau \times 2\pi r \Delta x$  on the surface of fluid element. As there is no acceleration, hence the summation of all forces in the direction of flow must be zero i.e.,

$$p\pi r^2 - \left( p + \frac{\partial p}{\partial x} \Delta x \right) \pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or 
$$-\frac{\partial p}{\partial x} \Delta x \pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or 
$$-\frac{\partial p}{\partial x} \cdot r - 2\tau = 0$$

$$\therefore \tau = -\frac{\partial p}{\partial x} \frac{r}{2} \quad \dots(9.1)$$

The shear stress  $\tau$  across a section varies with ' $r$ ' as  $\frac{\partial p}{\partial x}$  across a section is constant. Hence shear stress distribution across a section is linear as shown in Fig. 9.2 (a).

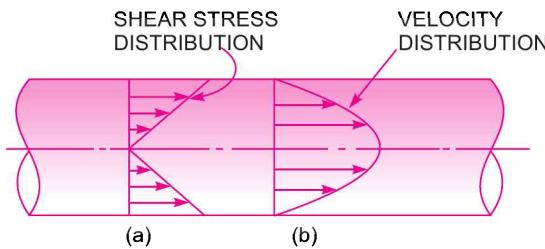


Fig. 9.2 Shear stress and velocity distribution across a section.

(i) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress  $\tau = \mu \frac{du}{dy}$  is substituted in equation (9.1).

But in the relation  $\tau = \mu \frac{du}{dy}$ ,  $y$  is measured from the pipe wall. Hence

$$y = R - r \quad \text{and} \quad dy = -dr$$

$$\therefore \tau = \mu \frac{du}{-dr} = -\mu \frac{du}{dr}$$

Substituting this value in (9.1), we get

$$-\mu \frac{du}{dr} = -\frac{\partial p}{\partial x} \frac{r}{2} \quad \text{or} \quad \frac{du}{dr} = \frac{1}{2\mu} \frac{\partial p}{\partial x} r$$

Integrating this above equation w.r.t. 'r', we get

$$u = \frac{1}{4\mu} \frac{\partial p}{\partial x} r^2 + C \quad \dots(9.2)$$

where  $C$  is the constant of integration and its value is obtained from the boundary condition that at  $r = R$ ,  $u = 0$ .

$$\therefore 0 = \frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 + C$$

$$\therefore C = -\frac{1}{4\mu} \frac{\partial p}{\partial x} R^2$$

Substituting this value of  $C$  in equation (9.2), we get

$$\begin{aligned} u &= \frac{1}{4\mu} \frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 \\ &= -\frac{1}{4\mu} \frac{\partial p}{\partial x} [R^2 - r^2] \end{aligned} \quad \dots(9.3)$$

In equation (9.3), values of  $\mu$ ,  $\frac{\partial p}{\partial x}$  and  $R$  are constant, which means the velocity,  $u$  varies with the square of  $r$ . Thus equation (9.3) is an equation of parabola. This shows that the velocity distribution across the section of a pipe is parabolic. This velocity distribution is shown in Fig. 9.2 (b).

(ii) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when  $r = 0$  in equation (9.3). Thus maximum velocity,  $U_{\max}$  is obtained as

$$U_{\max} = -\frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 \quad \dots(9.4)$$

The average velocity,  $\bar{u}$ , is obtained by dividing the discharge of the fluid across the section by the area of the pipe ( $\pi R^2$ ). The discharge ( $Q$ ) across the section is obtained by considering the flow through a circular ring element of radius  $r$  and thickness  $dr$  as shown in Fig. 9.1 (b). The fluid flowing per second through this elementary ring

$$dQ = \text{velocity at a radius } r \times \text{area of ring element}$$

$$= u \times 2\pi r dr$$

$$= -\frac{1}{4\mu} \frac{\partial p}{\partial x} [R^2 - r^2] \times 2\pi r dr$$

$$\therefore Q = \int_0^R dQ = \int_0^R -\frac{1}{4\mu} \frac{\partial p}{\partial x} (R^2 - r^2) \times 2\pi r dr$$

$$= \frac{1}{4\mu} \left( \frac{-\partial p}{\partial x} \right) \times 2\pi \int_0^R (R^2 - r^2) r dr$$

$$= \frac{1}{4\mu} \left( \frac{-\partial p}{\partial x} \right) \times 2\pi \int_0^R (R^2 r - r^3) dr$$

$$= \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) \times 2\pi \left[ \frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R = \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) \times 2\pi \left[ \frac{R^4}{2} - \frac{R^4}{4} \right]$$

$$= \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) \times 2\pi \times \frac{R^4}{4} = \frac{\pi}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^4$$

$$\therefore \text{Average velocity, } \bar{u} = \frac{Q}{\text{Area}} = \frac{\frac{\pi}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^4}{\pi R^2}$$

$$\text{or } \bar{u} = \frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2 \quad \dots(9.5)$$

Dividing equation (9.4) by equation (9.5),

$$\frac{U_{\max}}{\bar{u}} = \frac{-\frac{1}{4\mu} \frac{\partial p}{\partial x} R^2}{\frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2} = 2.0$$

$\therefore$  Ratio of maximum velocity to average velocity = 2.0.

### (iii) Drop of Pressure for a given Length (L) of a pipe

From equation (9.5), we have

$$\bar{u} = \frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2 \quad \text{or} \quad \left( -\frac{\partial p}{\partial x} \right) = \frac{8\mu \bar{u}}{R^2}$$

Integrating the above equation w.r.t.  $x$ , we get

$$\begin{aligned} - \int_2^1 dp &= \int_2^1 \frac{8\mu \bar{u}}{R^2} dx \\ \therefore - [p_1 - p_2] &= \frac{8\mu \bar{u}}{R^2} [x_1 - x_2] \quad \text{or} \quad (p_1 - p_2) = \frac{8\mu \bar{u}}{R^2} [x_2 - x_1] \\ &= \frac{8\mu \bar{u}}{R^2} L \quad \{ \because x_2 - x_1 = L \text{ from Fig. 9.3} \} \\ &= \frac{8\mu \bar{u} L}{(D/2)^2} \quad \left\{ \because R = \frac{D}{2} \right\} \end{aligned}$$

$$\text{or } (p_1 - p_2) = \frac{32\mu \bar{u} L}{D^2}, \quad \text{where } p_1 - p_2 \text{ is the drop of pressure.}$$

$$\therefore \text{Loss of pressure head} = \frac{p_1 - p_2}{\rho g}$$

$$\therefore \frac{p_1 - p_2}{\rho g} = h_f = \frac{32\mu \bar{u} L}{\rho g D^2} \quad \dots(9.6)$$

Equation (9.6) is called **Hagen Poiseuille Formula**.

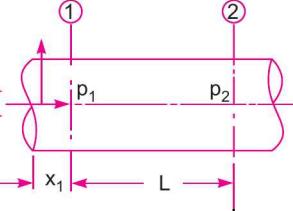


Fig. 9.3

**Problem 9.1** A crude oil of viscosity 0.97 poise and relative density 0.9 is flowing through a horizontal circular pipe of diameter 100 mm and of length 10 m. Calculate the difference of pressure at the two ends of the pipe, if 100 kg of the oil is collected in a tank in 30 seconds.

**Solution.** Given :  $\mu = 0.97 \text{ poise} = \frac{0.97}{10} = 0.097 \text{ Ns/m}^2$

Relative density = 0.9

$\therefore \rho_0$ , or density,  $= 0.9 \times 1000 = 900 \text{ kg/m}^3$

Dia. of pipe,  $D = 100 \text{ mm} = 0.1 \text{ m}$

$L = 10 \text{ m}$

Mass of oil collected,  $M = 100 \text{ kg}$

Time,  $t = 30 \text{ seconds}$

Calculate difference of pressure or  $(p_1 - p_2)$ .

The difference of pressure  $(p_1 - p_2)$  for viscous or laminar flow is given by

$$p_1 - p_2 = \frac{32\mu\bar{u}L}{D^2}, \text{ where } \bar{u} = \text{average velocity} = \frac{Q}{\text{Area}}$$

Now, mass of oil/sec  $= \frac{100}{30} \text{ kg/s}$   
 $= \rho_0 \times Q = 900 \times Q$   $(\because \rho_0 = 900)$

$\therefore \frac{100}{30} = 900 \times Q$

$\therefore Q = \frac{100}{30} \times \frac{1}{900} = 0.0037 \text{ m}^3/\text{s}$

$\therefore \bar{u} = \frac{Q}{\text{Area}} = \frac{0.0037}{\frac{\pi}{4} D^2} = \frac{0.0037}{\frac{\pi}{4} (0.1)^2} = 0.471 \text{ m/s.}$

For laminar or viscous flow, the Reynolds number ( $R_e$ ) is less than 2000. Let us calculate the Reynolds number for this problem.

Reynolds number,  $R_e^* = \frac{\rho V D}{\mu}$

where  $\rho = \rho_0 = 900$ ,  $V = \bar{u} = 0.471$ ,  $D = 0.1 \text{ m}$ ,  $\mu = 0.097$

$\therefore R_e = 900 \times \frac{0.471 \times 0.1}{0.097} = 436.91$

As Reynolds number is less than 2000, the flow is laminar.

$$\therefore p_1 - p_2 = \frac{32\mu\bar{u}L}{D^2} = \frac{32 \times 0.097 \times 0.471 \times 10}{(0.1)^2} \text{ N/m}^2$$

$$= 1462.28 \text{ N/m}^2 = 1462.28 \times 10^{-4} \text{ N/cm}^2 = \mathbf{0.1462 \text{ N/cm}^2. \text{ Ans.}}$$

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\* For derivation, please refer to Art. 12.8.1

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**Problem 9.2** An oil of viscosity  $0.1 \text{ Ns/m}^2$  and relative density 0.9 is flowing through a circular pipe of diameter 50 mm and of length 300 m. The rate of flow of fluid through the pipe is 3.5 litres/s. Find the pressure drop in a length of 300 m and also the shear stress at the pipe wall.

**Solution.** Given : Viscosity,  $\mu = 0.1 \text{ Ns/m}^2$

Relative density  $= 0.9$

$\therefore \rho_0$  or density of oil  $= 0.9 \times 1000 = 900 \text{ kg/m}^3$  ( $\because$  Density of water  $= 1000 \text{ kg/m}^3$ )

$D = 50 \text{ mm} = .05 \text{ m}$

$L = 300 \text{ m}$

$$Q = 3.5 \text{ litres/s} = \frac{3.5}{1000} = .0035 \text{ m}^3/\text{s}$$

Find (i) Pressure drop,  $p_1 - p_2$

(ii) Shear stress at pipe wall,  $\tau_0$

$$(i) \text{ Pressure drop } (p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2}, \text{ where } \bar{u} = \frac{Q}{\text{Area}} = \frac{.0035}{\frac{\pi}{4}D^2} = \frac{.0035}{\frac{\pi}{4}(.05)^2} = 1.782 \text{ m/s}$$

$$\text{The Reynolds number } (R_e) \text{ is given by, } R_e = \frac{\rho V D}{\mu}$$

where  $\rho = 900 \text{ kg/m}^3$ ,  $V = \text{average velocity} = \bar{u} = 1.782 \text{ m/s}$

$$\therefore R_e = 900 \times \frac{1.782 \times .05}{0.1} = 801.9$$

As Reynolds number is less than 2000, the flow is viscous or laminar

$$\therefore p_1 - p_2 = \frac{32 \times 0.1 \times 1.782 \times 3000}{(.05)^2}$$

$$= 684288 \text{ N/m}^2 = 68428 \times 10^{-4} \text{ N/cm}^2 = \mathbf{68.43 \text{ N/cm}^2. Ans.}$$

(ii) Shear Stress at the pipe wall ( $\tau_0$ )

The shear stress at any radius  $r$  is given by the equation (9.1)

$$\text{i.e., } \tau = -\frac{\partial p}{\partial x} \frac{r}{2}$$

$\therefore$  Shear stress at pipe wall, where  $r = R$  is given by

$$\tau_0 = \frac{-\partial p}{\partial x} \frac{R}{2}$$

$$\text{Now } \frac{-\partial p}{\partial x} = \frac{-(p_2 - p_1)}{x_2 - x_1} = \frac{p_1 - p_2}{x_2 - x_1} = \frac{p_1 - p_2}{L}$$

$$= \frac{684288}{300} \frac{\text{N/m}^2}{\text{m}} = 2280.96 \text{ N/m}^3$$

$$\text{and } R = \frac{D}{2} = \frac{.05}{2} = .025 \text{ m}$$

$$\tau_0 = 2280.96 \times \frac{.025}{2} \frac{\text{N}}{\text{m}^2} = \mathbf{28.512 \text{ N/m}^2. Ans.}$$

**Problem 9.3** A laminar flow is taking place in a pipe of diameter 200 mm. The maximum velocity is 1.5 m/s. Find the mean velocity and the radius at which this occurs. Also calculate the velocity at 4 cm from the wall of the pipe.

**Solution.** Given : Dia. of pipe,  $D = 200 \text{ mm} = 0.20 \text{ m}$

$$U_{\max} = 1.5 \text{ m/s}$$

Find (i) Mean velocity,  $\bar{u}$

(ii) Radius at which  $\bar{u}$  occurs

(iii) Velocity at 4 cm from the wall.

(i) **Mean velocity,  $\bar{u}$**

Ratio of  $\frac{U_{\max}}{\bar{u}} = 2.0 \quad \text{or} \quad \frac{1.5}{\bar{u}} = 2.0 \quad \therefore \quad \bar{u} = \frac{1.5}{2.0} = 0.75 \text{ m/s. Ans.}$

(ii) **Radius at which  $\bar{u}$  occurs**

The velocity,  $u$ , at any radius ' $r$ ' is given by (9.3)

$$u = -\frac{1}{4\mu} \frac{\partial p}{\partial x} [R^2 - r^2] = -\frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 \left[ 1 - \frac{r^2}{R^2} \right]$$

But from equation (9.4)  $U_{\max}$  is given by

$$U_{\max} = -\frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 \quad \therefore \quad u = U_{\max} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad \dots(1)$$

Now, the radius  $r$  at which  $u = \bar{u} = 0.75 \text{ m/s}$

$$\therefore 0.75 = 1.5 \left[ 1 - \left( \frac{r}{D/2} \right)^2 \right]$$

$$= 1.5 \left[ 1 - \left( \frac{r}{0.2/2} \right)^2 \right] = 1.5 \left[ 1 - \left( \frac{r}{0.1} \right)^2 \right]$$

$$\therefore \frac{0.75}{1.5} = 1 - \left( \frac{r}{0.1} \right)^2$$

$$\therefore \left( \frac{r}{0.1} \right)^2 = 1 - \frac{0.75}{1.5} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore \frac{r}{0.1} = \sqrt{\frac{1}{2}} = \sqrt{0.5}$$

$$\therefore r = 0.1 \times \sqrt{0.5} = 0.1 \times 0.707 = 0.0707 \text{ m} \\ = 70.7 \text{ mm. Ans.}$$

(iii) **Velocity at 4 cm from the wall**

$$r = R - 4.0 = 10 - 4.0 = 6.0 \text{ cm} = 0.06 \text{ m}$$

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∴ The velocity at a radius = 0.06 m

or 4 cm from pipe wall is given by equation (1)

$$= U_{\max} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] = 1.5 \left[ 1 - \left( \frac{0.06}{0.1} \right)^2 \right]$$

$$= 1.5[1.0 - .36] = 1.5 \times .64 = 0.96 \text{ m/s. Ans.}$$

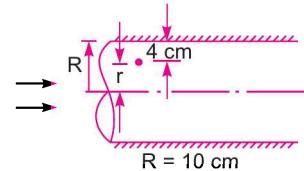


Fig. 9.4

**Problem 9.4** Crude oil of  $\mu = 1.5 \text{ poise}$  and relative density 0.9 flows through a 20 mm diameter vertical pipe. The pressure gauges fixed 20 m apart read  $58.86 \text{ N/cm}^2$  and  $19.62 \text{ N/cm}^2$  as shown in Fig. 9.5. Find the direction and rate of flow through the pipe.

**Solution.** Given :  $\mu = 1.5 \text{ poise} = \frac{1.5}{10} = 0.15 \text{ Ns/m}^2$

Relative density = 0.9

∴ Density of oil =  $0.9 \times 1000 = 900 \text{ kg/m}^3$

Dia. of pipe,  $D = 20 \text{ mm} = 0.02 \text{ m}$

$L = 20 \text{ m}$

$$p_A = 58.86 \text{ N/cm}^2 = 58.86 \times 10^4 \text{ N/m}^2$$

$$p_B = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2.$$

Find (i) Direction of flow

(ii) Rate of flow.

(i) **Direction of flow.** To find the direction of flow, the total energy  $\left( \frac{p}{\rho g} + \frac{v^2}{2g} + Z \right)$  at the lower end

A and at the upper end B is to be calculated. The direction of flow will be given from the higher energy to the lower energy. As here the diameter of the pipe is same and hence kinetic energy at A and B will

be same. Hence to find the direction of flow, calculate  $\left( \frac{p}{\rho g} + Z \right)$  at A and B.

Taking the level at A as datum. The value of  $\left( \frac{p}{\rho g} + Z \right)$  at

$$A = \frac{p_A}{\rho g} + Z_A$$

$$= \frac{6 \times 10^4 \times 9.81}{900 \times 9.81} + 0 \quad \{ \because r = 900 \text{ kg/cm}^2 \}$$

$$= 66.67 \text{ m}$$

$$\text{The value of } \left( \frac{p}{\rho g} + Z \right) \text{ at } B = \frac{p_B}{\rho g} + Z_B$$

$$= \frac{2 \times 10^4 \times 9.81}{900 \times 9.81} + 20 = 22.22 + 20 = 42.22 \text{ m}$$

As the value of  $\left( \frac{p}{\rho g} + Z \right)$  is higher at A and hence flow takes place from A to B. Ans.

(ii) **Rate of flow.** The loss of pressure head for viscous flow through circular pipe is given by

$$h_f = \frac{32\mu u L}{\rho g D^2}$$

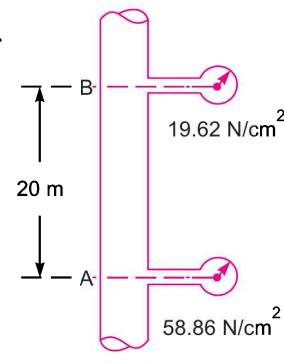


Fig. 9.5

For a vertical pipe  $h_f = \text{Loss of piezometric head}$

$$= \left( \frac{p_A}{\rho g} + Z_A \right) - \left( \frac{p_B}{\rho g} + Z_B \right) = 66.67 - 42.22 = 24.45 \text{ m}$$

$$\therefore 24.45 = \frac{32 \times 0.15 \times \bar{u} \times 20.0}{900 \times 9.81 \times (.02)^2}$$

$$\text{or } \bar{u} = \frac{24.45 \times 900 \times 9.81 \times .0004}{32 \times 0.15 \times 20.0} = 0.889 \approx 0.9 \text{ m/s.}$$

The Reynolds number should be calculated. If Reynolds number is less than 2000, the flow will be laminar and the above expression for loss of pressure head for laminar flow can be used.

$$\text{Now Reynolds number} = \frac{\rho V D}{\mu}$$

where  $\rho = 900 \text{ kg/m}^3$  and  $V = \bar{u}$

$$\therefore \text{Reynolds number} = 900 \times \frac{0.9 \times .02}{0.15} = 108$$

As Reynolds number is less than 2000, the flow is laminar.

$$\begin{aligned} \therefore \text{Rate of flow} &= \text{average velocity} \times \text{area} \\ &= \bar{u} \times \frac{\pi}{4} D^2 = 0.9 \times \frac{\pi}{4} \times (.02)^2 \text{ m}^3/\text{s} = 2.827 \times 10^{-4} \text{ m}^3/\text{s} \\ &= \mathbf{0.2827 \text{ litres/s. Ans.}} \quad (\because 10^{-3} \text{ m}^3 = 1 \text{ litre}) \end{aligned}$$

**Problem 9.5** A fluid of viscosity  $0.7 \text{ Ns/m}^2$  and specific gravity 1.3 is flowing through a circular pipe of diameter 100 mm. The maximum shear stress at the pipe wall is given as  $196.2 \text{ N/m}^2$ , find (i) the pressure gradient, (ii) the average velocity, and (iii) Reynolds number of the flow.

$$\text{Solution. Given : } \mu = 0.7 \frac{\text{Ns}}{\text{m}^2}$$

$$\text{Sp. gr.} = 1.3$$

$$\therefore \text{Density} = 1.3 \times 1000 = 1300 \text{ kg/m}^3$$

$$\text{Dia. of pipe, } D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Shear stress, } \tau_0 = 196.2 \text{ N/m}^2$$

$$\text{Find (i) Pressure gradient, } \frac{dp}{dx}$$

$$(ii) \text{ Average velocity, } \bar{u}$$

$$(iii) \text{ Reynolds number, } R_e$$

$$(i) \text{ Pressure gradient, } \frac{dp}{dx}$$

The maximum shear stress ( $\tau_0$ ) is given by

$$\tau_0 = - \frac{\partial p}{\partial x} \frac{R}{2} \text{ or } 196.2 = - \frac{\partial p}{\partial x} \times \frac{D}{4} = - \frac{\partial p}{\partial x} \times \frac{0.1}{4}$$

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$$\therefore \frac{\partial p}{\partial x} = - \frac{196.2 \times 4}{0.1} = - 7848 \text{ N/m}^2 \text{ per m}$$

$\therefore$  Pressure Gradient = **- 7848 N/m<sup>2</sup> per m.** Ans.

(ii) Average velocity,  $\bar{u}$

$$\begin{aligned} \bar{u} &= \frac{1}{2} U_{\max} = \frac{1}{2} \left[ -\frac{1}{4\mu} \frac{\partial p}{\partial x} R^2 \right] & \left\{ \because U_{\max} = -\frac{1}{8\mu} \frac{\partial p}{\partial x} R^2 \right\} \\ &= \frac{1}{8\mu} \times \left( -\frac{\partial p}{\partial x} \right) R^2 \\ &= \frac{1}{8 \times 0.7} \times (7848) \times (.05)^2 & \left\{ \because R = \frac{D}{2} = \frac{1}{2} = .05 \right\} \\ &= 3.50 \text{ m/s} \end{aligned}$$

(iii) Reynolds number,  $R_e$

$$\begin{aligned} R_e &= \frac{\bar{u} \times D}{v} = \frac{\bar{u} \times D}{\mu / \rho} = \frac{\rho \times \bar{u} \times D}{\mu} \\ &= 1300 \times \frac{3.50 \times 0.1}{0.7} = \mathbf{650.00.} \text{ Ans.} \end{aligned}$$

**Problem 9.6** What power is required per kilometre of a line to overcome the viscous resistance to the flow of glycerine through a horizontal pipe of diameter 100 mm at the rate of 10 litres/s? Take  $\mu = 8$  poise and kinematic viscosity ( $v$ ) = 6.0 stokes.

**Solution.** Given :

Length of pipe,  $L = 1 \text{ km} = 1000 \text{ m}$

Dia. of pipe,  $D = 100 \text{ mm} = 0.1 \text{ m}$

Discharge,  $Q = 10 \text{ litres/s} = \frac{10}{1000} \text{ m}^3/\text{s} = .01 \text{ m}^3/\text{s}$

Viscosity,  $\mu = 8 \text{ poise} = \frac{8}{10} \frac{\text{Ns}}{\text{m}^2} = 0.8 \text{ N s/m}^2$

Kinematic Viscosity,  $v = 6.0 \text{ stokes}$   $\left( \because 1 \text{ poise} = \frac{1}{10} \text{ Ns/m}^2 \right)$   
 $= 6.0 \text{ cm}^2/\text{s} = 6.0 \times 10^{-4} \text{ m}^2/\text{s}$

Loss of pressure head is given by equation (9.6) as  $h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$

Power required =  $W \times h_f$  watts  $\dots(i)$

where  $W = \text{weight of oil flowing per sec} = \rho g \times Q$

Substituting the values of  $W$  and  $h_f$  in equation (i),

Power required  $= (\rho g \times Q) \times \frac{(32 \mu \bar{u} L)}{\rho g D^2} \text{ watts} = \frac{Q \times 32 \mu \bar{u} L}{D^2}$  (cancelling  $\rho g$ )

But  $\bar{u} = \frac{Q}{\text{Area}} = \frac{.01}{\frac{\pi}{4} D^2} = \frac{.01 \times 4}{\pi \times (.1)^2} = 1.273 \text{ m/s}$

$$\therefore \text{Power required} = \frac{.01 \times 32 \times 0.8 \times 1.273 \times 1000}{(.1)^2}$$

$$= 32588.8 \text{ W} = \mathbf{32.588 \text{ kW. Ans.}}$$

### ► 9.3 FLOW OF VISCOUS FLUID BETWEEN TWO PARALLEL PLATES

In this case also, the shear stress distribution, the velocity distribution across a section ; the ratio of maximum velocity to average velocity and difference of pressure head for a given length of parallel plates, are to be calculated.

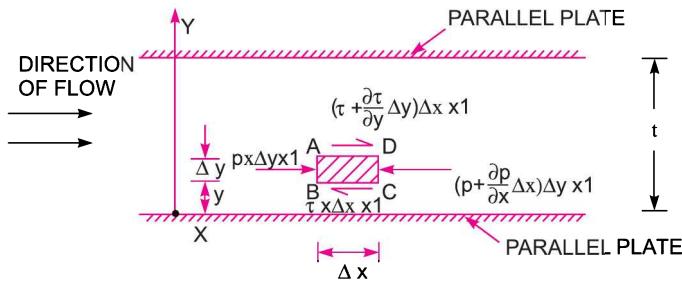


Fig. 9.6 Viscous flow between two parallel plates.

Consider two parallel fixed plates kept at a distance 't' apart as shown in Fig. 9.6. A viscous fluid is flowing between these two plates from left to right. Consider a fluid element of length  $\Delta x$  and thickness  $\Delta y$  at a distance  $y$  from the lower fixed plate. If  $p$  is the intensity of pressure on the face  $AB$  of the

fluid element then intensity of pressure on the face  $CD$  will be  $\left(p + \frac{\partial p}{\partial x} \Delta x\right)$ . Let  $\tau$  is the shear stress

acting on the face  $BC$  then the shear stress on the face  $AD$  will be  $\left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right)$ . If the width of the element in the direction perpendicular to the paper is unity then the forces acting on the fluid element are :

1. The pressure force,  $p \times \Delta y \times 1$  on face  $AB$ .
2. The pressure force,  $\left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y \times 1$  on face  $CD$ .
3. The shear force,  $\tau \times \Delta x \times 1$  on face  $BC$ .
4. The shear force,  $\left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right) \Delta x \times 1$  on face  $AD$ .

For steady and uniform flow, there is no acceleration and hence the resultant force in the direction of flow is zero.

$$\therefore p\Delta y \times 1 - \left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y \times 1 - \tau\Delta x \times 1 + \left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right) \Delta x \times 1 = 0$$

$$\text{or } -\frac{\partial p}{\partial x} \Delta x \Delta y + \frac{\partial \tau}{\partial x} \Delta y \Delta x = 0$$

$$\text{Dividing by } \Delta x \Delta y, \text{ we get } -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \quad \text{or} \quad \frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y} \quad \dots(9.7)$$

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(i) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress  $\tau = \mu \frac{du}{dy}$  from Newton's law of viscosity for laminar flow is substituted in equation (9.7).

$$\therefore \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \mu \frac{du}{dy} \right) = \mu \frac{\partial^2 u}{\partial y^2}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

Integrating the above equation w.r.t.  $y$ , we get

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1 \quad \left\{ \because \frac{\partial p}{\partial x} \text{ is constant} \right\}$$

$$\text{Integrating again} \quad u = \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{y^2}{2} + C_1 y + C_2 \quad \dots(9.8)$$

where  $C_1$  and  $C_2$  are constants of integration. Their values are obtained from the two boundary conditions that is (i) at  $y = 0, u = 0$  (ii) at  $y = t, u = 0$ .

The substitution of  $y = 0, u = 0$  in equation (9.8) gives

$$0 = 0 + C_1 \times 0 + C_2 \text{ or } C_2 = 0$$

The substitution of  $y = t, u = 0$  in equation (9.8) gives

$$0 = \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{t^2}{2} + C_1 \times t + 0$$

$$\therefore C_1 = -\frac{1}{\mu} \frac{\partial p}{\partial x} \frac{t^2}{2 \times t} = -\frac{1}{2\mu} \frac{\partial p}{\partial x} t$$

Substituting the values of  $C_1$  and  $C_2$  in equation (9.8)

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + y \left( -\frac{1}{2\mu} \frac{\partial p}{\partial x} t \right)$$

$$\text{or} \quad u = -\frac{1}{2\mu} \frac{\partial p}{\partial x} [ty - y^2] \quad \dots(9.9)$$

In the above equation,  $\mu, \frac{\partial p}{\partial x}$  and  $t$  are constant. It means  $u$  varies with the square of  $y$ . Hence equation (9.9) is a equation of a parabola. Hence velocity distribution across a section of the parallel plate is parabolic. This velocity distribution is shown in Fig. 9.7 (a). Shear stress distribution is shown in Fig. 9.7 (b).

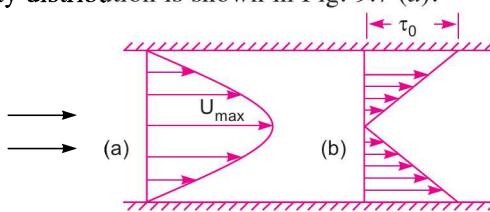


Fig. 9.7 Velocity distribution and shear stress distribution across a section of parallel plates.

(ii) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when  $y = t/2$ . Substituting this value in equation (9.9), we get

$$\begin{aligned} U_{\max} &= -\frac{1}{2\mu} \frac{\partial p}{\partial x} \left[ t \times \frac{t}{2} - \left( \frac{t}{2} \right)^2 \right] \\ &= -\frac{1}{2\mu} \frac{\partial p}{\partial x} \left[ \frac{t^2}{2} - \frac{t^2}{4} \right] = -\frac{1}{2\mu} \frac{\partial p}{\partial x} \frac{t^2}{4} = -\frac{1}{8\mu} \frac{\partial p}{\partial x} t^2 \end{aligned} \quad \dots(9.10)$$

The average velocity,  $\bar{u}$ , is obtained by dividing the discharge ( $Q$ ) across the section by the area of the section ( $t \times 1$ ). And the discharge  $Q$  is obtained by considering the rate of flow of fluid through the strip of thickness  $dy$  and integrating it. The rate of flow through strip is

$$\begin{aligned} dQ &= \text{Velocity at a distance } y \times \text{Area of strip} \\ &= -\frac{1}{2\mu} \frac{\partial p}{\partial x} [ty - y^2] \times dy \times 1 \\ \therefore Q &= \int_0^t dQ = \int_0^t -\frac{1}{2\mu} \frac{\partial p}{\partial x} [ty - y^2] dy \\ &= -\frac{1}{2\mu} \frac{\partial p}{\partial x} \left[ \frac{ty^2}{2} - \frac{y^3}{3} \right]_0^t = \frac{1}{2\mu} \frac{\partial p}{\partial x} \left[ \frac{t^3}{2} - \frac{t^3}{3} \right] \\ &= -\frac{1}{2\mu} \frac{\partial p}{\partial x} \frac{t^3}{6} = -\frac{1}{12\mu} \frac{\partial p}{\partial x} t^3 \\ \therefore \bar{u} &= \frac{Q}{\text{Area}} = -\frac{\frac{1}{12\mu} \frac{\partial p}{\partial x} \cdot t^3}{t \times 1} = -\frac{1}{12\mu} \frac{\partial p}{\partial x} t^2 \end{aligned} \quad \dots(9.11)$$

Dividing equation (9.10) by equation (9.11), we get

$$\frac{U_{\max}}{\bar{u}} = \frac{-\frac{1}{8\mu} \frac{\partial p}{\partial x} t^2}{-\frac{1}{12\mu} \frac{\partial p}{\partial x} t^2} = \frac{12}{8} = \frac{3}{2} \quad \dots(9.12)$$

(iii) **Drop of Pressure head for a given Length.** From equation (9.11), we have

$$\bar{u} = -\frac{1}{12\mu} \frac{\partial p}{\partial x} t^2 \quad \text{or} \quad \frac{\partial p}{\partial x} = -\frac{12\mu \bar{u}}{t^2}$$

Integrating this equation w.r.t.  $x$ , we get

$$\int_2^1 dp = \int_2^1 -\frac{12\mu \bar{u}}{t^2} dx$$

$$\text{or} \quad p_1 - p_2 = -\frac{12\mu \bar{u}}{t^2} [x_1 - x_2] = \frac{12\mu \bar{u}}{t^2} [x_2 - x_1]$$

or

$$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2} \quad [ \because x_1 - x_2 = L ]$$

If  $h_f$  is the drop of pressure head, then

$$h_f = \frac{p_1 - p_2}{\rho g} = \frac{12\mu\bar{u}L}{\rho g t^2} \quad \dots(9.13)$$

(iv) **Shear Stress Distribution.** It is obtained by substituting the value of  $u$  from equation (9.9) into

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$\therefore \tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial}{\partial y} \left[ -\frac{1}{2\mu} \frac{\partial p}{\partial x} (ty - y^2) \right] = \mu \left[ -\frac{1}{2\mu} \frac{\partial p}{\partial x} (t - 2y) \right]$$

$$\tau = -\frac{1}{2} \frac{\partial p}{\partial x} [ t - 2y ] \quad \dots(9.14)$$

In equation (9.14),  $\frac{\partial p}{\partial x}$  and  $t$  are constant. Hence  $\tau$  varies linearly with  $y$ . The shear stress distribution

is shown in Fig. 9.7 (b). Shear stress is maximum, when  $y = 0$  or  $t$  at the walls of the plates. Shear stress is zero, when  $y = t/2$  that is at the centre line between the two plates. Max. shear stress ( $\tau_0$ ) is given by

$$\tau_0 = -\frac{1}{2} \frac{\partial p}{\partial x} t. \quad \dots(9.15)$$

**Problem 9.7** Calculate : (i) the pressure gradient along flow, (ii) the average velocity, and (iii) the discharge for an oil of viscosity  $0.02 \text{ Ns/m}^2$  flowing between two stationary parallel plates  $1 \text{ m}$  wide maintained  $10 \text{ mm}$  apart. The velocity midway between the plates is  $2 \text{ m/s}$ .

**Solution.** Given :

$$\text{Viscosity, } \mu = .02 \text{ N s/m}^2$$

$$\text{Width, } b = 1 \text{ m}$$

$$\text{Distance between plates, } t = 10 \text{ mm} = .01 \text{ m}$$

$$\text{Velocity midway between the plates, } U_{\max} = 2 \text{ m/s.}$$

$$(i) \text{ Pressure gradient } \left( \frac{dp}{dx} \right)$$

$$\text{Using equation (9.10), } U_{\max} = -\frac{1}{8\mu} \frac{dp}{dx} t^2 \quad \text{or} \quad 2.0 = -\frac{1}{8 \times .02} \left( \frac{dp}{dx} \right) (.01)^2$$

$$\therefore \frac{dp}{dx} = -\frac{2.0 \times 8 \times .02}{.01 \times .01} = -3200 \text{ N/m}^2 \text{ per m. Ans.}$$

$$(ii) \text{ Average velocity } (\bar{u})$$

$$\text{Using equation (9.12), } \frac{U_{\max}}{\bar{u}} = \frac{3}{2} \quad \therefore \quad \bar{u} = \frac{2 U_{\max}}{3} = \frac{2 \times 2}{3} = 1.33 \text{ m/s. Ans.}$$

$$(iii) \text{ Discharge } (Q) = \text{Area of flow} \times \bar{u} = b \times t \times \bar{u} = 1 \times .01 \times 1.33 = .0133 \text{ m}^3/\text{s. Ans.}$$

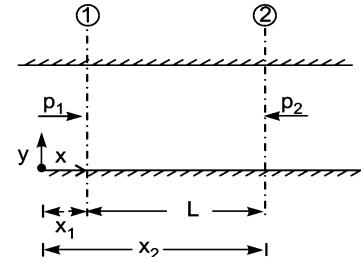


Fig. 9.8

**Problem 9.8** Determine (i) the pressure gradient, (ii) the shear stress at the two horizontal parallel plates and (iii) the discharge per metre width for the laminar flow of oil with a maximum velocity of 2 m/s between two horizontal parallel fixed plates which are 100 mm apart. Given  $\mu = 2.4525 \text{ N s/m}^2$ .

**Solution.** Given :

$$U_{\max} = 2 \text{ m/s}, t = 100 \text{ mm} = 0.1 \text{ m}, \mu = 2.4525 \text{ N s/m}^2$$

Find (i) Pressure gradient,  $\frac{dp}{dx}$

(ii) Shear stress at the wall,  $\tau_0$

(iii) Discharge per metre width,  $Q$ .

(i) **Pressure gradient**,  $\frac{dp}{dx}$

Maximum velocity,  $U_{\max}$ , is given by equation (9.10)

$$U_{\max} = -\frac{1}{8\mu} \frac{dp}{dx} t^2$$

Substituting the values

$$\text{or } 2.0 = -\frac{1}{8 \times 2.4525} \times \frac{dp}{dx} \times (0.1)^2$$

$$\therefore \frac{dp}{dx} = -\frac{2.0 \times 8 \times 2.4525}{0.1 \times 0.1} = -3924 \text{ N/m}^2 \text{ per m. Ans.}$$

(ii) **Shear stress at the wall**,  $\tau_0$

$$\tau_0 \text{ is given by equation (9.15) as } \tau_0 = -\frac{1}{2} \frac{dp}{dx} \times t = -\frac{1}{2} (-3924) \times 0.1 = 196.2 \text{ N/m}^2. \text{ Ans.}$$

(iii) **Discharge per metre width**,  $Q$

$$= \text{Mean velocity} \times \text{Area}$$

$$= \frac{2}{3} U_{\max} \times (t \times 1) = \frac{2}{3} \times 2.0 \times 0.1 \times 1 = 0.133 \text{ m}^3/\text{s. Ans.}$$

**Problem 9.9** An oil of viscosity 10 poise flows between two parallel fixed plates which are kept at a distance of 50 mm apart. Find the rate of flow of oil between the plates if the drop of pressure in a length of 1.2 m be 0.3 N/cm<sup>3</sup>. The width of the plates is 200 mm.

**Solution.** Given :

$$\mu = 10 \text{ poise}$$

$$= \frac{10}{10} \text{ N s/m}^2 = 1 \text{ N s/m}^2 \quad \left( \because 1 \text{ poise} = \frac{1}{10} \frac{\text{Ns}}{\text{m}^2} \right)$$

$$t = 50 \text{ mm} = 0.05 \text{ m}$$

$$p_1 - p_2 = 0.3 \text{ N/m}^2 = 0.3 \times 10^4 \text{ N/m}^2$$

$$L = 1.20 \text{ m}$$

$$\text{Width, } B = 200 \text{ mm} = 0.20 \text{ m.}$$

Find  $Q$ , rate of flow

The difference of pressure is given by equation (9.13)

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$$p_1 - p_2 = \frac{12\bar{\mu}uL}{t^2}$$

Substituting the values, we get

$$0.3 \times 10^4 = 12 \times 1.0 \times \frac{\bar{u} \times 1.20}{.05 \times .05}$$

$$\therefore \bar{u} = \frac{0.3 \times 10^4 \times 1.0 \times .05 \times .05}{12 \times 1.20} = 0.52 \text{ m/s}$$

$$\begin{aligned} \therefore \text{Rate of flow} &= \bar{u} \times \text{Area} = 0.52 \times (B \times t) \\ &= 0.52 \times 0.20 \times .05 \text{ m}^3/\text{s} = .0052 \text{ m}^3/\text{s} \\ &= 0.0052 \times 10^3 \text{ litre/s} = \mathbf{5.2 \text{ litre/s. Ans.}} \end{aligned}$$

**Problem 9.10** Water at  $15^\circ\text{C}$  flows between two large parallel plates at a distance of 1.6 mm apart. Determine (i) the maximum velocity (ii) the pressure drop per unit length and (iii) the shear stress at the walls of the plates if the average velocity is 0.2 m/s. The viscosity of water at  $15^\circ\text{C}$  is given as 0.01 poise.

**Solution.** Given :  $t = 1.6 \text{ mm} = 1.6 \times 10^{-3} \text{ m}$   
 $= 0.0016 \text{ m}$

$$\bar{u} = 0.2 \text{ m/sec, } \mu = .01 \text{ poise} = \frac{.01}{10} = 0.001 \text{ N s/m}^2$$

(i) **Maximum velocity**,  $U_{\max}$  is given by equation (9.12)

$$\text{i.e., } U_{\max} = \frac{3}{2} \bar{u} = 1.5 \times 0.2 = \mathbf{0.3 \text{ m/s. Ans.}}$$

(ii) **The pressure drop**,  $(p_1 - p_2)$  is given by equation (9.13)

$$p_1 - p_2 = \frac{12\bar{\mu}uL}{t^2}$$

$$\text{or pressure drop per unit length} = \frac{12\bar{\mu}u}{t^2}$$

$$\text{or } \frac{\partial p}{\partial x} = 12 \times \frac{.01}{10} \times \frac{0.2}{(.0016)^2} = 937.44 \text{ N/m}^2 \text{ per m.}$$

(iii) **Shear stress at the walls** is given by equation (9.15)

$$\tau_0 = -\frac{1}{2} \frac{\partial p}{\partial x} \times t = \frac{1}{2} \times 937.44 \times .0016 = \mathbf{0.749 \text{ N/m}^2. \text{ Ans.}}$$

**Problem 9.11** There is a horizontal crack 40 mm wide and 2.5 mm deep in a wall of thickness 100 mm. Water leaks through the crack. Find the rate of leakage of water through the crack if the difference of pressure between the two ends of the crack is  $0.02943 \text{ N/cm}^2$ . Take the viscosity of water equal to 0.01 poise.

**Solution.** Given :

$$\text{Width of crack, } b = 40 \text{ mm} = 0.04 \text{ m}$$

$$\text{Depth of crack, } t = 2.5 \text{ mm} = .0025 \text{ m}$$

$$\text{Length of crack, } L = 100 \text{ mm} = 0.1 \text{ m}$$

$$p_1 - p_2 = 0.02943 \text{ N/cm}^2 = 0.02943 \times 10^4 \text{ N/m}^2 = 294.3 \text{ N/m}^2$$

$$\mu = .01 \text{ poise} = \frac{.01}{10} \frac{\text{Ns}}{\text{m}^2}$$

Find rate of leakage ( $Q$ )

$(p_1 - p_2)$  is given by equation (9.13) as

$$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2} \quad \text{or} \quad 294.3 = 12 \times \frac{.01}{10} \times \frac{\bar{u} \times 0.1}{(.0025 \times .0025)}$$

$$\therefore \bar{u} = \frac{294.3 \times 10 \times .0025 \times .0025}{12 \times .01 \times 0.1} = 1.5328 \text{ m/s}$$

$\therefore \text{Rate of leakage} = \bar{u} \times \text{area of cross-section of crack}$

$$\begin{aligned} &= 1.538 \times (b \times t) \\ &= 1.538 \times .04 \times .0025 \text{ m}^3/\text{s} = 1.538 \times 10^{-4} \text{ m}^3/\text{s} \\ &= 1.538 \times 10^{-4} \times 10^3 \text{ litre/s} = \mathbf{0.1538 \text{ litre/s. Ans.}} \end{aligned}$$

**Problem 9.12** The radial clearance between a hydraulic plunger and the cylinder walls is 0.1 mm; the length of the plunger is 300 mm and diameter 100 mm. Find the velocity of leakage and rate of leakage past the plunger at an instant when the difference of the pressure between the two ends of the plunger is 9 m of water. Take  $\mu = 0.0127$  poise.

**Solution.** Given :

The flow through the clearance area will be the same as the flow between two parallel surfaces,

$$t = 0.1 \text{ mm} = 0.0001 \text{ m}$$

$$L = 300 \text{ mm} = 0.3 \text{ m}$$

Diameter,

$$D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Difference of pressure} = \frac{p_1 - p_2}{\rho g} = 9 \text{ m of water}$$

$$\therefore p_1 - p_2 = 9 \times 1000 \times 9.81 \text{ N/m}^2 = 88290 \text{ N/m}^2$$

$$\text{Viscosity, } \mu = .0127 \text{ poise} = \frac{.0127}{10} \frac{\text{Ns}}{\text{m}^2}$$

Find (i) Velocity of leakage, i.e., mean velocity  $\bar{u}$

(ii) Rate of leakage,  $Q$

(i) **Velocity of leakage ( $\bar{u}$ )**. The average velocity ( $\bar{u}$ ) is given by equation (9.11)

$$\begin{aligned} \bar{u} &= - \frac{1}{12\mu} \frac{\partial p}{\partial x} t^2 \\ &= \frac{1}{12 \times \frac{.0127}{10}} \times \frac{p_1 - p_2}{L} \times (.0001) \times (.0001) \\ &= \frac{1}{12 \times .0127} \times \frac{88290}{0.3} \times (.0001) \times (.0001) \\ &= .193 \text{ m/s} = \mathbf{19.3 \text{ cm/s. Ans.}} \end{aligned}$$

(ii) Rate of leakage,  $Q$ 

$$\begin{aligned}
 Q &= \bar{u} \times \text{area of flow} \\
 &= 0.193 \times \pi D \times t \text{ m}^3/\text{s} = 0.193 \times \pi \times .1 \times .0001 \text{ m}^3/\text{s} \\
 &= 6.06 \times 10^{-6} \text{ m}^3/\text{s} = 6.06 \times 10^{-6} \times 10^3 \text{ litre/s} \\
 &= 6.06 \times 10^{-3} \text{ litre/s. Ans.}
 \end{aligned}$$

► **9.4 KINETIC ENERGY CORRECTION AND MOMENTUM CORRECTION FACTORS**

Kinetic energy correction factor is defined as the ratio of the kinetic energy of the flow per second based on actual velocity across a section to the kinetic energy of the flow per second based on average velocity across the same section. It is denoted by  $\alpha$ . Hence mathematically,

$$\alpha = \frac{\text{K.E./sec based on actual velocity}}{\text{K.E./sec based on average velocity}} \quad \dots(9.16)$$

**Momentum Correction Factor.** It is defined as the ratio of momentum of the flow per second based on actual velocity to the momentum of the flow per second based on average velocity across a section. It is denoted by  $\beta$ . Hence mathematically,

$$\beta = \frac{\text{Momentum per second based on actual velocity}}{\text{Momentum per second based on average velocity}}. \quad \dots(9.17)$$

**Problem 9.13** Show that the momentum correction factor and energy correction factor for laminar flow through a circular pipe are  $4/3$  and  $2.0$  respectively.

**Solution.** (i) Momentum Correction Factor or  $\beta$

The velocity distribution through a circular pipe for laminar flow at any radius  $r$  is given by equation (9.3)

$$\text{or } u = \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) (R^2 - r^2) \quad \dots(1)$$

Consider an elementary area  $dA$  in the form of a ring at a radius  $r$  and of width  $dr$ , then

$$dA = 2\pi r dr$$

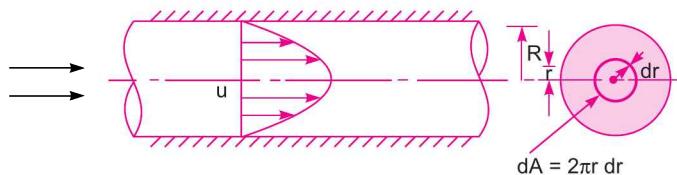


Fig. 9.9

Rate of fluid flowing through the ring

$$\begin{aligned}
 &= dQ = \text{velocity} \times \text{area of ring element} \\
 &= u \times 2\pi r dr
 \end{aligned}$$

Momentum of the fluid through ring per second

$$\begin{aligned}
 &= \text{mass} \times \text{velocity} \\
 &= \rho \times dQ \times u = \rho \times 2\pi r dr \times u \times u = 2\pi\rho u^2 r dr
 \end{aligned}$$

∴ Total actual momentum of the fluid per second across the section

$$= \int_0^R 2\pi\rho u^2 r dr$$

Substituting the value of  $u$  from (1)

$$\begin{aligned}
 &= 2\pi\rho \int_0^R \left[ \frac{1}{4\mu} \left( \frac{-\partial p}{\partial x} \right) (R^2 - r^2) \right]^2 r dr \\
 &= 2\pi\rho \left[ \frac{1}{4\mu} \left( \frac{-\partial p}{\partial x} \right) \right]^2 \int_0^R [R^2 - r^2]^2 r dr \\
 &= 2\pi\rho \frac{1}{(16\mu^2)} \left( \frac{\partial p}{\partial x} \right)^2 \int_0^R (R^4 + r^4 - 2R^2r^2) r dr \\
 &= \frac{\pi\rho}{8\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 \int_0^R (R^4r + r^5 - 2R^2r^3) dr \\
 &= \frac{\pi\rho}{8\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 \left[ \frac{R^4r^2}{2} + \frac{r^6}{6} - \frac{2R^2r^4}{4} \right]_0^R = \frac{\pi\rho}{8\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 \left[ \frac{R^6}{2} + \frac{R^6}{6} - \frac{2R^6}{4} \right] \\
 &= \frac{\pi\rho}{8\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 \frac{6R^6 + 2R^6 - 6R^6}{12} \\
 &= \frac{\pi\rho}{8\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 \times \frac{R^6}{6} = \frac{\pi\rho}{48\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 R^6
 \end{aligned} \tag{2}$$

Momentum of the fluid per second based on average velocity

$$\begin{aligned}
 &= \frac{\text{mass of fluid}}{\text{sec}} \times \text{average velocity} \\
 &= \rho A \bar{u} \times \bar{u} = \rho A \bar{u}^2
 \end{aligned}$$

where  $A$  = Area of cross-section =  $\pi R^2$ ,  $\bar{u}$  = average velocity =  $\frac{U_{\max}}{2}$

$$\begin{aligned}
 &= \frac{1}{2} \times \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) R^2 && \left\{ \because U_{\max} = \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) R^2 \right\} \\
 &= \frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2
 \end{aligned}$$

$\therefore$  Momentum/sec based on average velocity

$$\begin{aligned}
 &= \rho \times \pi R^2 \times \left[ \frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2 \right]^2 = \rho \times \pi R^2 \times \frac{1}{64\mu^2} \left( -\frac{\partial p}{\partial x} \right)^2 R^4 \\
 &= \frac{\rho \pi \left( -\frac{\partial p}{\partial x} \right)^2 R^6}{64\mu^2}
 \end{aligned} \tag{3}$$

$$\therefore \beta = \frac{\text{Momentum / sec based on actual velocity}}{\text{Momentum / sec based on average velocity}}$$

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$$= \frac{\frac{\pi \rho}{48\mu^2} \left( \frac{\partial p}{\partial x} \right)^2 R^6}{\frac{\pi \rho}{64\mu^2} \left( -\frac{\partial p}{\partial x} \right)^2 R^6} = \frac{64}{48} = \frac{4}{3}. \text{ Ans.}$$

(ii) **Energy Correction Factor,  $\alpha$ .** Kinetic energy of the fluid flowing through the elementary ring of radius ' $r$ ' and of width ' $dr$ ' per sec

$$\begin{aligned} &= \frac{1}{2} \times \text{mass} \times u^2 = \frac{1}{2} \times \rho dQ \times u^2 \\ &= \frac{1}{2} \times \rho \times (u \times 2\pi r dr) \times u^2 = \frac{1}{2} \rho \times 2\pi r u^3 dr = \pi \rho r u^3 dr \end{aligned}$$

$\therefore$  Total actual kinetic energy of flow per second

$$\begin{aligned} &= \int_0^R \pi \rho r u^3 dr = \int_0^R \pi \rho r \left[ \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) (R^2 - r^2) \right]^3 dr \\ &= \pi \rho \times \left[ \frac{1}{4\mu} \left( -\frac{\partial p}{\partial x} \right) \right]^3 \int_0^R [R^2 - r^2]^3 r dr \\ &= \pi \rho \times \frac{1}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \int_0^R (R^6 - r^6 - 3R^4r^2 + 3R^2r^4)r dr \\ &= \frac{\pi \rho}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \int_0^R (R^6r - r^7 - 3R^4r^3 + 3R^2r^5) dr \\ &= \frac{\pi \rho}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \left[ \frac{R^6r^2}{2} - \frac{r^8}{8} - \frac{3R^4r^4}{4} + \frac{3R^2r^6}{6} \right]_0^R \\ &= \frac{\pi \rho}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \left[ \frac{R^8}{2} - \frac{R^8}{8} - \frac{3R^8}{4} + \frac{3R^8}{6} \right] \\ &= \frac{\pi \rho}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 R^8 \left[ \frac{12 - 3 - 18 + 12}{24} \right] \\ &= \frac{\pi \rho}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \frac{R^8}{8} \quad \dots(4) \end{aligned}$$

Kinetic energy of the flow based on average velocity

$$= \frac{1}{2} \times \text{mass} \times \bar{u}^2 = \frac{1}{2} \times \rho A \bar{u} \times \bar{u}^2 = \frac{1}{2} \times \rho A \bar{u}^3$$

Substituting the value of  $A = \pi R^2$

and  $\bar{u} = \frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2$

∴ Kinetic energy of the flow/sec

$$\begin{aligned}
 &= \frac{1}{2} \times \rho \times \pi R^2 \times \left[ \frac{1}{8\mu} \left( -\frac{\partial p}{\partial x} \right) R^2 \right]^3 \\
 &= \frac{1}{2} \times \rho \times \pi R^2 \times \frac{1}{64 \times 8\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \times R^6 \\
 &= \frac{\rho \pi}{128 \times 8\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \times R^8
 \end{aligned} \quad \dots(5)$$

$$\therefore \alpha = \frac{\text{K.E./sec based on actual velocity}}{\text{K.E./sec based on average velocity}} = \frac{\text{Equation (4)}}{\text{Equation (5)}}$$

$$\begin{aligned}
 &= \frac{\pi \rho}{64\mu^3} \left( -\frac{\partial p}{\partial x} \right)^3 \times \frac{R^8}{8} = \frac{128 \times 8}{64 \times 8} = \mathbf{2.0. \text{ Ans.}}
 \end{aligned}$$

## ► 9.5 POWER ABSORBED IN VISCOS FLOW

For the lubrication of the machine parts, an oil is used. Flow of oil in bearings is an example of viscous flow. If a highly viscous oil is used for lubrication of bearings, it will offer great resistance and thus a greater power loss will take place. But if a light oil is used, a required film between the rotating part and stationary metal surface will not be possible. Hence, the wear of the two surface will take place. Hence an oil of correct viscosity should be used for lubrication. The power required to overcome the viscous resistance in the following cases will be determined :

1. Viscous resistance of Journal Bearings,
2. Viscous resistance of Foot-step Bearings,
3. Viscous resistance of Collar Bearings.

**9.5.1 Viscous Resistance of Journal Bearings.** Consider a shaft of diameter  $D$  rotating in a journal bearing. The clearance between the shaft and journal bearing is filled with a viscous oil. The oil film in contact with the shaft rotates at the same speed as that of shaft while the oil film in contact with journal bearing is stationary. Thus the viscous resistance will be offered by the oil to the rotating shaft.

Let

$N$  = speed of shaft in r.p.m.

$t$  = thickness of oil film

$L$  = length of oil film

$$\therefore \text{Angular speed of the shaft, } \omega = \frac{2\pi N}{60}$$

$$\therefore \text{Tangential speed of the shaft} = \omega \times R \text{ or } V = \frac{2\pi N}{60} \times \frac{D}{2} = \frac{\pi D N}{60}$$

The shear stress in the oil is given by,  $\tau = \mu \frac{du}{dy}$

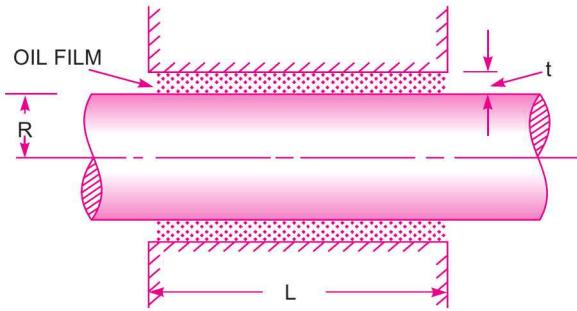


Fig. 9.10 Journal bearing.

As the thickness of oil film is very small, the velocity distribution in the oil film can be assumed as linear.

Hence

$$\frac{du}{dy} = \frac{V - 0}{t} = \frac{V}{t} = \frac{\pi DN}{60 \times t}$$

∴

$$\tau = \mu \frac{\pi DN}{60 \times t}$$

∴ Shear force or viscous resistance =  $\tau \times$  Area of surface of shaft

$$= \frac{\mu \pi DN}{60t} \times \pi DL = \frac{\mu \pi^2 D^2 NL}{60t}$$

∴ Torque required to overcome the viscous resistance,

$$T = \text{Viscous resistance} \times \frac{D}{2}$$

$$= \frac{\mu \pi^2 D^2 NL}{60t} \times \frac{D}{2} = \frac{\mu \pi^2 D^3 NL}{120t}$$

∴ Power absorbed in overcoming the viscous resistance

$$*P = \frac{2\pi NT}{60} = \frac{2\pi N}{60} \times \frac{\mu \pi^2 D^3 NL}{120t}$$

$$= \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t} \text{ watts. Ans.} \quad \dots(9.18)$$

**Problem 9.14** A shaft having a diameter of 50 mm rotates centrally in a journal bearing having a diameter of 50.15 mm and length 100 mm. The angular space between the shaft and the bearing is filled with oil having viscosity of 0.9 poise. Determine the power absorbed in the bearing when the speed of rotation is 60 r.p.m.

**Solution.** Given :

Dia. of shaft,	$D = 50 \text{ mm or } .05 \text{ m}$
Dia. of bearing,	$D_1 = 50.15 \text{ mm or } 0.05015 \text{ m}$
Length,	$L = 100 \text{ mm or } 0.1 \text{ m}$

\*Power,  $P = T \times \omega = T \times \frac{2\pi N}{60} = \frac{2\pi NT}{60}$  watts  $= \frac{2\pi NT}{60,000} \text{ kW.}$

$$\mu \text{ of oil} = 0.9 \text{ poise} = \frac{0.9}{10} \frac{\text{Ns}}{\text{m}^2}$$

$$N = 600 \text{ r.p.m.}$$

Power = ?

$$\therefore \text{Thickness of oil film, } t = \frac{D_1 - D}{2} = \frac{50.15 - 50}{2}$$

$$= \frac{0.15}{2} = 0.075 \text{ mm} = 0.075 \times 10^{-3} \text{ m}$$

$$\text{Tangential speed of shaft, } V = \frac{\pi DN}{60} = \frac{\pi \times 0.05 \times 600}{60} = 0.5 \times \pi \text{ m/s}$$

$$\text{Shear stress } \tau = \mu \frac{du}{dy} = \mu \frac{V}{t} = \frac{0.9}{10} \times \frac{0.5 \times \pi}{0.075 \times 10^{-3}} = 1883.52 \text{ N/m}^2$$

$$\therefore \text{Shear force (F)} = \tau \times \text{Area} = 1883.52 \times \pi D \times L \\ = 1883.52 \times \pi \times .05 \times 0.1 = 29.586 \text{ N}$$

$$\text{Resistance torque } T = F \times \frac{D}{2} = 29.586 \times \frac{.05}{2} = 0.7387 \text{ Nm}$$

$$\text{Power} = \frac{2\pi NT}{60} = \frac{2\pi \times 600 \times 0.7387}{60} = 46.41 \text{ W. Ans.}$$

**Problem 9.15** A shaft of 100 mm, diameter rotates at 60 r.p.m. in a 200 mm long bearing. Taking that the two surfaces are uniformly separated by a distance of 0.5 mm and taking linear velocity distribution in the lubricating oil having dynamic viscosity of 4 centipoises, find the power absorbed in the bearing.

**Solution.** Given :

$$\text{Dia. of shaft, } D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Length of bearing, } L = 200 \text{ mm} = 0.2 \text{ m}$$

$$t = 0.5 \text{ mm} = .5 \times 10^{-3} \text{ m}$$

$$\mu = 4 \text{ centipoise} = .04 \text{ poise} = \frac{0.04}{10} \frac{\text{Ns}}{\text{m}^2}$$

$$N = 60 \text{ r.p.m.}$$

Find power absorbed

Using equation (9.18),

$$P = \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t}$$

$$= \frac{.04}{10} \times \frac{\pi^3 \times (.1)^3 \times (60)^2 \times 0.2}{60 \times 60 \times 0.5 \times 10^{-3}} = 4.961 \times 10^{-2} \text{ W. Ans.}$$

**Problem 9.16** A shaft of diameter 0.35 m rotates at 200 r.p.m. inside a sleeve 100 mm long. The dynamic viscosity of lubricating oil in the 2 mm gap between sleeve and shaft is 8 poises. Calculate the power lost in the bearing.

**Solution.** Given :

$$\text{Dia. of shaft, } D = 0.35 \text{ m}$$

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Speed of shaft,  $N = 200 \text{ r.p.m.}$

Length of sleeve,  $L = 100 \text{ mm} = 0.1 \text{ m}$

Distance between sleeve and shaft,  $t = 2 \text{ mm} = 2 \times 10^{-3} \text{ m}$

Viscosity,  $\mu = 8 \text{ poise} = \frac{8}{10} \frac{\text{Ns}}{\text{m}^2}$

The power lost in the bearing is given by equation (9.18) as

$$P = \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t} \text{ watts}$$

$$= \frac{8}{10} \times \frac{\pi^3 \times (.35)^3 \times (200)^2 \times 0.1}{60 \times 60 \times 2 \times 10^{-3}} = 590.8 \text{ W} = 0.59 \text{ kW. Ans.}$$

**Problem 9.17** A sleeve, in which a shaft of diameter 75 mm, is running at 1200 r.p.m., is having a radial clearance of 0.1 mm. Calculate the torque resistance if the length of sleeve is 100 mm and the space is filled with oil of dynamic viscosity 0.96 poise.

**Solution.** Given :

Dia. of shaft,  $D = 75 \text{ mm} = 0.075 \text{ m}$

$N = 1200 \text{ r.p.m.}$

$t = 0.1 \text{ mm} = 0.1 \times 10^{-3} \text{ m}$

Length of sleeve,  $L = 100 \text{ mm} = 0.1 \text{ m}$

$\mu = 0.96 \text{ poise} = \frac{0.96}{10} \frac{\text{Ns}}{\text{m}^2}$

Tangential velocity of shaft,  $V = \frac{\pi D N}{60} = \frac{\pi \times 0.075 \times 1200}{60} = 4.712 \text{ m/s}$

Shear stress,  $\tau = \mu \frac{V}{t} = \frac{0.96}{10} \times \frac{4.712}{0.1 \times 10^{-3}} = 4523.5 \text{ N/m}^2$

Shear force,  $F = \tau \times \pi D L$   
 $= 4523.5 \times \pi \times 0.075 \times 0.1 = 106.575 \text{ N}$

$\therefore$  Torque resistance  $= F \times \frac{D}{2}$   
 $= 106.575 \times \frac{0.075}{2} = 3.996 \text{ Nm. Ans.}$

**Problem 9.18** A shaft of 100 mm diameter runs in a bearing of length 200 mm with a radial clearance of 0.025 mm at 30 r.p.m. Find the velocity of the oil, if the power required to overcome the viscous resistance is 183.94 watts.

**Solution.** Given :

$D = 100 \text{ mm} = 0.1 \text{ m}$

$L = 200 \text{ mm} = 0.2 \text{ m}$

$t = 0.025 \text{ mm} = 0.025 \times 10^{-3} \text{ m}$

$N = 30 \text{ r.p.m.} ; \text{H.P.} = 0.25$

Find viscosity of oil,  $\mu$ .

The h.p. is given by equation (9.18) as

$$P = \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t} \quad \text{or} \quad 183.94 = \frac{\mu \pi^3 \times (1)^3 \times (30)^2 \times 0.2}{60 \times 60 \times 0.025 \times 10^{-3}}$$

$$\therefore \mu = \frac{183.94 \times 60 \times 60 \times 0.025 \times 10^{-3}}{\pi^3 \times 0.001 \times 900 \times 0.2} \frac{\text{Ns}}{\text{m}^2}$$

$$= 2.96 \frac{\text{Ns}}{\text{m}^2} = 2.96 \times 10 = \mathbf{29.6 \text{ poise. Ans.}}$$

**9.5.2 Viscous Resistance of Foot-Step Bearing.** Fig. 9.11 shows the foot-step bearing, in which a vertical shaft is rotating. An oil film between the bottom surface of the shaft and bearing is provided, to reduce the wear and tear. The viscous resistance is offered by the oil to the shaft. In this case the radius of the surface of the shaft in contact with oil is not constant as in the case of the journal bearing. Hence, viscous resistance in foot-step bearing is calculated by considering an elementary circular ring of radius  $r$  and thickness  $dr$  as shown in Fig. 9.11.

Let

$N$  = speed of the shaft

$t$  = thickness of oil film

$R$  = radius of the shaft

Area of the elementary ring =  $2\pi r dr$

$$\text{Now shear stress is given by } t = \mu \frac{du}{dy} = \mu \frac{V}{t}$$

where  $V$  is the tangential velocity of shaft at radius  $r$  and is equal to

$$\omega \times r = \frac{2\pi N}{60} \times r$$

$\therefore$  Shear force on the ring =  $dF = \tau \times \text{area of elementary ring}$

$$= \mu \times \frac{2\pi N}{60} \times \frac{r}{t} \times 2\pi r dr = \frac{\mu}{15} \frac{\pi^2 N r^2}{t} dr$$

$\therefore$  Torque required to overcome the viscous resistance,

$$dT = dF \times r$$

$$= \frac{\mu}{15t} \pi^2 N r^2 dr \times r = \frac{\mu}{15t} \pi^2 N r^3 dr \quad \dots(9.19)$$

$\therefore$  Total torque required to overcome the viscous resistance,

$$T = \int_0^R dT = \int_0^R \frac{\mu}{15t} \pi^2 N r^3 dr$$

$$= \frac{\mu}{15t} \pi^2 N \int_0^R r^3 dr = \frac{\mu}{15t} \pi^2 N \left[ \frac{r^4}{4} \right]_0^R = \frac{\mu}{15t} \pi^2 N \frac{R^4}{4}$$

$$= \frac{\mu}{60t} \pi^2 N R^4 \quad \dots(9.19A)$$

$\therefore$  Power absorbed,

$$P = \frac{2\pi N T}{60} \text{ watts}$$

$$= \frac{2\pi N}{60} \times \frac{\mu}{60t} \pi^2 N R^4 = \frac{\mu \pi^3 N^2 R^4}{60 \times 30t} \quad \dots(9.20)$$

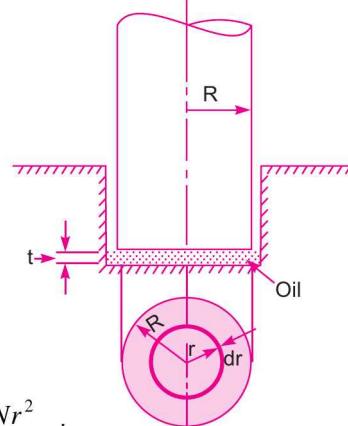


Fig. 9.11 Foot-step bearing.

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**Problem 9.19** Find the torque required to rotate a vertical shaft of diameter 100 mm at 750 r.p.m. The lower end of the shaft rests in a foot-step bearing. The end of the shaft and surface of the bearing are both flat and are separated by an oil film of thickness 0.5 mm. The viscosity of the oil is given 1.5 poise.

**Solution.** Given :

$$\text{Dia. of shaft, } D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\therefore R = \frac{D}{2} = \frac{0.1}{2} = 0.05 \text{ m}$$

$$N = 750 \text{ r.p.m.}$$

$$\text{Thickness of oil film, } t = 0.5 \text{ mm} = 0.0005 \text{ m}$$

$$\mu = 1.5 \text{ poise} = \frac{1.5}{10} \frac{\text{Ns}}{\text{m}^2}$$

The torque required is given by equation (9.19) as

$$\begin{aligned} T &= \frac{\mu}{60t} \pi^2 N R^4 \text{ Nm} \\ &= \frac{1.5}{10} \times \frac{\pi^2 \times 750 \times (0.05)^4}{60 \times 0.0005} = 0.2305 \text{ Nm. Ans.} \end{aligned}$$

**Problem 9.20** Find the power required to rotate a circular disc of diameter 200 mm at 1000 r.p.m. The circular disc has a clearance of 0.4 mm from the bottom flat plate and the clearance contains oil of viscosity 1.05 poise.

**Solution.** Given :

$$\text{Dia. of disc, } D = 200 \text{ mm} = 0.2 \text{ m}$$

$$\therefore R = \frac{D}{2} = \frac{0.2}{2} = 0.1 \text{ m}$$

$$N = 1000 \text{ r.p.m.}$$

$$\text{Thickness of oil film, } t = 0.4 \text{ mm} = 0.0004 \text{ m}$$

$$\mu = 1.05 \text{ poise} = \frac{1.05}{10} \text{ N s/m}^2$$

The power required to rotate the disc is given by equation (9.20) as

$$\begin{aligned} P &= \frac{\mu \pi^3 N^2 R^4}{60 \times 30 \times t} \text{ watts} \\ &= \frac{1.05}{10} \times \frac{\pi^3 \times 1000^2 \times (0.1)^4}{60 \times 30 \times 0.0004} = 452.1 \text{ W. Ans.} \end{aligned}$$

**9.5.3 Viscous Resistance of Collar Bearing.** Fig. 9.12 shows the collar bearing, where the face of the collar is separated from bearing surface by an oil film of uniform thickness.

Let

$N$  = Speed of the shaft in r.p.m.

$R_1$  = Internal radius of the collar

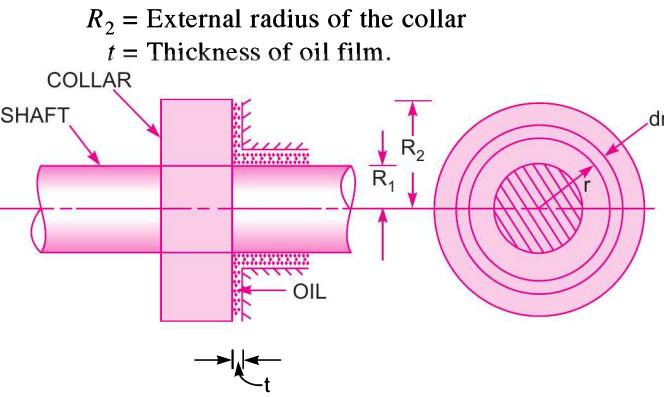


Fig. 9.12 Collar bearing.

Consider an elementary circular ring of radius ' $r$ ' and width  $dr$  of the bearing surface. Then the torque ( $dT$ ) required to overcome the viscous resistance on the elementary circular ring is the same as given by equation (9.19A) or

$$dT = \frac{\mu}{15t} \pi^2 N r^3 dr$$

∴ Total torque, required to overcome the viscous resistance, on the whole collar is

$$\begin{aligned} T &= \int_{R_1}^{R_2} dT = \int_{R_1}^{R_2} \frac{\mu}{15t} \pi^2 N r^3 dr = \frac{\mu}{15t} \pi^2 N \left[ \frac{r^4}{4} \right]_{R_1}^{R_2} \\ &= \frac{\mu}{15t \times 4} \pi^2 N [R_2^4 - R_1^4] = \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4] \end{aligned} \quad \dots(9.21)$$

∴ Power absorbed in overcoming viscous resistance

$$\begin{aligned} P &= \frac{2\pi N T}{60} = \frac{2\pi N}{60} \times \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4] \\ &= \frac{\mu \pi^3 N^2}{60 \times 30t} [R_2^4 - R_1^4] \text{ watts.} \end{aligned} \quad \dots(9.22)$$

**Problem 9.21** A collar bearing having external and internal diameters 150 mm and 100 mm respectively is used to take the thrust of a shaft. An oil film of thickness 0.25 mm is maintained between the collar surface and the bearing. Find the power lost in overcoming the viscous resistance when the shaft rotates at 300 r.p.m. Take  $\mu = 0.91$  poise.

**Solution.** Given :

$$\text{External Dia. of collar, } D_2 = 150 \text{ mm} = 0.15 \text{ m}$$

$$\therefore R_2 = \frac{D_2}{2} = \frac{0.15}{2} = 0.075 \text{ m}$$

$$\text{Internal Dia. of collar, } D_1 = 100 \text{ mm} = 0.1 \text{ m}$$

$$\therefore R_1 = \frac{D_1}{2} = \frac{0.1}{2} = 0.05 \text{ m}$$

$$\begin{aligned} \text{Thickness of oil film, } t &= 0.25 \text{ mm} = 0.00025 \text{ m} \\ N &= 300 \text{ r.p.m.} \end{aligned}$$

$$\mu = 0.91 \text{ poise} = \frac{0.91}{10} \frac{\text{Ns}}{\text{m}^2}$$

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The power required is given by equation (9.22) or

$$\begin{aligned} P &= \frac{\mu\pi^3 N^2}{60 \times 30 \times t} [R_2^4 - R_1^4] \\ &= \frac{0.91}{10} \times \frac{\pi^3 \times 300^2 \times [0.075^4 - .05^4]}{60 \times 30 \times .00025} \\ &= 564314 [.00003164 - .00000625] \\ &= 564314 \times .00002539 = \mathbf{14.327 \text{ W. Ans.}} \end{aligned}$$

**Problem 9.22** The external and internal diameters of a collar bearing are 200 mm and 150 mm respectively. Between the collar surface and the bearing, an oil film of thickness 0.25 mm and of viscosity 0.9 poise, is maintained. Find the torque and the power lost in overcoming the viscous resistance of the oil when the shaft is running at 250 r.p.m.

**Solution.** Given :

$$\begin{aligned} D_2 &= 200 \text{ mm} = 0.2 \text{ m} \\ \therefore R_2 &= \frac{D_2}{2} = \frac{0.2}{2} = 0.1 \text{ m} \\ D_1 &= 150 \text{ mm} = 0.15 \text{ m} \\ \therefore R_1 &= \frac{D_1}{2} = \frac{0.15}{2} = 0.075 \text{ m} \\ t &= 0.25 \text{ mm} = .00025 \text{ m} \\ \mu &= 0.9 \text{ poise} = \frac{0.9}{10} \frac{\text{Ns}}{\text{m}^2} \end{aligned}$$

Torque required is given by equation (9.21)

$$\begin{aligned} \therefore T &= \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4] = \frac{0.9}{10} \times \frac{\pi^2 \times 250 [0.1^4 - 0.075^4]}{60 \times 0.00025} \text{ Nm} \\ &= 14804.4 [.0001 - .00003164] = \mathbf{1.0114 \text{ Nm. Ans.}} \end{aligned}$$

$\therefore$  Power lost in viscous resistance

$$= \frac{2\pi NT}{60} = \frac{2\pi \times 250 \times 1.0114}{60} = \mathbf{26.48 \text{ W. Ans.}}$$

## ► 9.6 LOSS OF HEAD DUE TO FRICTION IN VISCOUS FLOW

The loss of pressure head,  $h_f$  in a pipe of diameter  $D$ , in which a viscous fluid of viscosity  $\mu$  is flowing with a velocity  $\bar{u}$  is given by Hagen Poiseuille formula i.e., by equation (9.6) as

$$h_f = \frac{32\mu\bar{u}L}{\rho g D^2} \quad \dots(i)$$

where  $L$  = length of pipe

The loss of head due to friction\* is given by

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{D \times 2g} = \frac{4 \cdot f \cdot L \cdot \bar{u}^2}{D \times 2g} \quad \dots(ii)$$

{  $\because$  velocity in pipe is always average velocity  $\therefore V = \bar{u}$  }

\*For derivation, please refer to Art. 10.3.1.

where  $f$  = co-efficient of friction between the pipe and fluid.

$$\text{Equating (i) and (ii), we get } \frac{32\mu\bar{u}L}{\rho g D^2} = \frac{4 \cdot f \cdot L \cdot \bar{u}^2}{D \times 2g}$$

$$f = \frac{32\mu\bar{u}L \times D \times 2g}{4 \cdot L \cdot \bar{u}^2 \cdot \rho g \cdot D^2} = \frac{16\mu}{\bar{u} \cdot \rho \cdot D}$$

$$= 16 \times \frac{\mu}{\rho V D} = 16 \times \frac{1}{R_e}$$

where  $\frac{\mu}{\rho V D} = \frac{1}{R_e}$  and  $R_e$  = Reynolds number  $= \frac{\rho V D}{\mu}$

$$\therefore f = \frac{16}{R_e}. \quad \dots(9.23)$$

**Problem 9.23** Water is flowing through a 200 mm diameter pipe with coefficient of friction  $f = 0.04$ . The shear stress at a point 40 mm from the pipe axis is  $0.00981 \text{ N/cm}^2$ . Calculate the shear stress at the pipe wall.

**Solution.** Given :

$$\text{Dia. of pipe, } D = 200 \text{ mm} = 0.20 \text{ m}$$

$$\text{Co-efficient of friction, } f = 0.04$$

$$\text{Shear stress at } r = 40 \text{ mm, } \tau = 0.00981 \text{ N/cm}^2$$

$$\text{Let the shear stress at pipe wall} = \tau_0.$$

First find whether the flow is viscous or not. The flow will be viscous if Reynolds number  $R_e$  is less than 2000.

$$\text{Using equation (9.23), we get } f = \frac{16}{R_e} \quad \text{or} \quad .04 = \frac{16}{R_e}$$

$$\therefore R_e = \frac{16}{.04} = 400$$

This means flow is viscous. The shear stress in case of viscous flow through a pipe is given by the equation (9.1) as

$$\tau = - \frac{\partial p}{\partial x} \frac{r}{2}$$

But  $\frac{\partial p}{\partial x}$  is constant across a section. Across a section, there is no variation of  $x$  and there is no variation of  $p$ .

$$\therefore \tau \propto r$$

At the pipe wall, radius = 100 mm and shear stress is  $\tau_0$

$$\therefore \frac{\tau}{r} = \frac{\tau_0}{100} \quad \text{or} \quad \frac{0.00981}{40} = \frac{\tau_0}{100}$$

$$\therefore \tau_0 = \frac{100 \times 0.00981}{40} = 0.0245 \text{ N/cm}^2. \text{ Ans.}$$

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**Problem 9.24** A pipe of diameter 20 cm and length 10<sup>4</sup> m is laid at a slope of 1 in 200. An oil of sp. gr. 0.9 and viscosity 1.5 poise is pumped up at the rate of 20 litres per second. Find the head lost due to friction. Also calculate the power required to pump the oil.

**Solution.** Given :

$$\text{Dia. of pipe, } D = 20 \text{ cm} = 0.2 \text{ m}$$

$$\text{Length of pipe, } L = 10000 \text{ m}$$

$$\text{Slope of pipe, } i = 1 \text{ in } 200 = \frac{1}{200}$$

$$\text{Sp. gr. of oil, } S = 0.9$$

$$\therefore \text{Density of oil, } \rho = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$\text{Viscosity of oil, } \mu = 1.5 \text{ poise} = \frac{1.5}{10} \frac{\text{Ns}}{\text{m}^2}$$

$$\text{Discharge, } Q = 20 \text{ litre/s} = 0.02 \text{ m}^3/\text{s} \quad \{ \because 1000 \text{ litres} = 1 \text{ m}^3 \}$$

$$\therefore \text{Velocity of flow, } \bar{u} = \frac{Q}{\text{Area}} = \frac{0.020}{\frac{\pi}{4} D^2} = \frac{0.020}{\frac{\pi}{4} (0.2)^2} = 0.6366 \text{ m/s}$$

$$\therefore R_e = \text{Reynolds number}$$

$$= \frac{\rho V D}{\mu} = \frac{900 \times 0.6366 \times 0.2}{1.5} \frac{1}{10}$$

$$= \frac{900 \times 0.6366 \times 0.2 \times 10}{1.5} \quad \{ \because V = \bar{u} = 0.6366 \}$$

$$= 763.89$$

As the Reynolds number is less than 2000, the flow is viscous. The co-efficient of friction for viscous flow is given by equation (9.23) as

$$f = \frac{16}{R_e} = \frac{16}{763.89} = 0.02094$$

$$\therefore \text{Head lost due to friction, } h_f = \frac{4 \cdot f \cdot L \cdot \bar{u}^2}{D \times 2g}$$

$$= \frac{4 \times 0.02094 \times 10000 \times (0.6366)^2}{0.2 \times 2 \times 9.81} \text{ m} = 86.50 \text{ m. Ans.}$$

Due to slope of pipe 1 in 200, the height through which oil is to be raised by pump

$$= \text{Slope} \times \text{Length of pipe}$$

$$= i \times L = \frac{1}{200} \times 10000 = 50 \text{ m}$$

$\therefore$  Total head against which pump is to work,

$$H = h_f + i \times L = 86.50 + 50 = 136.50 \text{ m}$$

$\therefore$  Power required to pump the oil

$$= \frac{\rho g \cdot Q \cdot H}{1000} = \frac{900 \times 9.81 \times 0.20 \times 136.50}{1000} = 24.1 \text{ kW. Ans.}$$

### ► 9.7 MOVEMENT OF PISTON IN DASH-POT

Consider a piston moving in a vertical dash-pot containing oil as shown in Fig. 9.13.

Let  $D$  = Diameter of piston,

$L$  = Length of piston,

$W$  = Weight of piston,

$\mu$  = Viscosity of oil,

$V$  = Velocity of piston,

$\bar{u}$  = Average velocity of oil in the clearance,

$t$  = Clearance between the dash-pot and piston,

$\Delta p$  = Difference of pressure intensities between the two ends of the piston.

The flow of oil through clearance is similar to the viscous flow between two parallel plates. The difference of pressure for parallel plates for length ' $L$ ' is given by

$$\Delta p = \frac{12\mu\bar{u}L}{t^2} \quad \dots(i)$$

Also the difference of pressure at the two ends of piston is given by,

$$\Delta p = \frac{\text{Weight of piston}}{\text{Area of piston}} = \frac{W}{\frac{\pi}{4}D^2} = \frac{4W}{\pi D^2} \quad \dots(ii)$$

$$\text{Equating (i) and (ii), we get } \frac{12\mu\bar{u}L}{t^2} = \frac{4W}{\pi D^2}$$

$$\therefore \bar{u} = \frac{4W}{\pi D^2} \times \frac{t^2}{12\mu L} = \frac{Wt^2}{3\pi\mu LD^2} \quad \dots(iii)$$

$V$  is the velocity of piston or the velocity of oil in dash-pot in contact with piston. The rate of flow of oil in dash-pot

$$= \text{velocity} \times \text{area of dash-pot} = V \times \frac{\pi}{4} D^2$$

Rate of flow through clearance = velocity through clearance  $\times$  area of clearance =  $\bar{u} \times \pi D \times t$

Due to continuity equation, rate of flow through clearance must be equal to rate of flow through dash-pot.

$$\therefore \bar{u} \times \pi D \times t = V \times \frac{\pi}{4} D^2$$

$$\therefore \bar{u} = V \times \frac{\pi}{4} D^2 \times \frac{1}{\pi D \times t} = \frac{VD}{4t} \quad \dots(iv)$$

Equating the value of  $\bar{u}$  from (iii) and (iv), we get

$$\frac{Wt^2}{3\pi\mu LD^2} = \frac{VD}{4t}$$

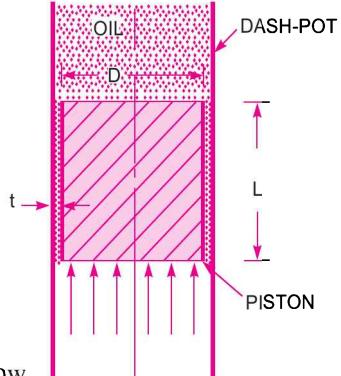


Fig. 9.13

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$$\mu = \frac{4t^3 W}{3\pi LD^3 V} = \frac{4Wt^3}{3\pi LD^3 V}. \quad \dots(9.24)$$

**Problem 9.25** An oil dash-pot consists of a piston moving in a cylinder having oil. This arrangement is used to damp out the vibrations. The piston falls with uniform speed and covers 5 cm in 100 seconds. If an additional weight of 1.36 N is placed on the top of the piston, it falls through 5 cm in 86 seconds with uniform speed. The diameter of the piston is 7.5 cm and its length is 10 cm. The clearance between the piston and the cylinder is 0.12 cm which is uniform throughout. Find the viscosity of oil.

**Solution.** Given :

Distance covered by piston due to self weight, = 5 cm

Time taken, = 100 sec

Additional weight, = 1.36 N

Time taken to cover 5 cm due to additional weight, = 86 sec

Dia. of piston,  $D = 7.5 \text{ cm} = 0.075 \text{ m}$

Length of piston,  $L = 10 \text{ cm} = 0.1 \text{ m}$

Clearance,  $t = 0.12 \text{ cm} = 0.0012 \text{ m}$

Let the viscosity of oil =  $\mu$

$W$  = Weight of piston,

$V$  = Velocity of piston without additional weight,

$V^*$  = Velocity of piston with additional weight.

Using equation (9.24), we have

$$\mu = \frac{4Wt^3}{3\pi D^3 LV} = \frac{4[W + 1.36]t^3}{3\pi D^3 LV^*}$$

or

$$\frac{W}{V} = \frac{W + 1.36}{V^*} \quad \left( \text{Cancelling } \frac{4Wt^3}{3\pi D^3 L} \right)$$

or

$$\frac{V}{V^*} = \frac{W}{W + 1.36} \quad \dots(i)$$

But

$V$  = Velocity of piston due to self weight of piston

$$= \frac{\text{Distance covered}}{\text{Time taken}} = \frac{5}{100} \text{ cm/s}$$

Similarly,

$$V^* = \frac{\text{Distance covered due to self weight + additional weight}}{\text{Time taken}}$$

$$= \frac{5}{86} \text{ cm/s}$$

$$\therefore \frac{V}{V^*} = \frac{5}{100} \times \frac{86}{5} = 0.86 \quad \dots(ii)$$

Equating (i) and (ii), we get  $\frac{W}{W + 1.36} = 0.86$

$$\text{or} \quad W = 0.86 W + .86 \times 1.36$$

$$\text{or} \quad W - 0.86 W = 0.14 W = .86 \times 1.36$$

$$\therefore W = \frac{0.86 \times 1.36}{0.14} = 8.354 \text{ N}$$

Using equation (9.24), we get  $\mu = \frac{4Wt^3}{3\pi D^3 LV}$

$$= \frac{4 \times 8.354 \times (0.0012)^3}{3\pi \times (0.075)^3 \times 10 \times \left(\frac{5}{100} \times \frac{1}{100}\right)} \left\{ \because V = \frac{5}{100} \text{ cm/s} = \frac{5}{100} \times \frac{1}{100} \text{ m/s} \right\}$$

$$= 0.29 \text{ N s/m}^2 = 0.29 \times 10 \text{ poise} = 2.9 \text{ poise. Ans.}$$

## ► 9.8 METHODS OF DETERMINATION OF CO-EFFICIENT OF VISCOSITY

The following are the experimental methods of determining the co-efficient of viscosity of a liquid:

1. Capillary tube method,
2. Falling sphere resistance method,
3. By rotating cylinder method, and
4. Orifice type viscometer.

The apparatus used for determining the viscosity of a liquid is called viscometer.

**9.8.1 Capillary Tube Method.** In capillary tube method, the viscosity of a liquid is calculated by measuring the pressure difference for a given length of the capillary tube. The Hagen Poiseuille law is used for calculating viscosity.

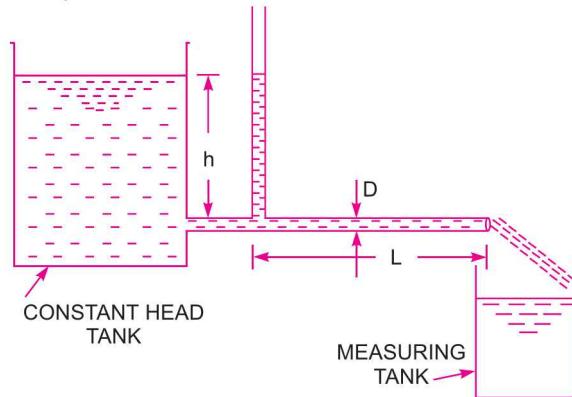


Fig. 9.14 Capillary tube viscometer.

Fig. 9.14 shows the capillary tube viscometer. The liquid whose viscosity is to be determined is filled in a constant head tank. The liquid is maintained at constant temperature and is allowed to pass through the capillary tube from the constant head tank. Then, the liquid is collected in a measuring tank for a given time. Then the rate of liquid collected in the tank per second is determined. The pressure head 'h' is measured at a point far away from the tank as shown in Fig. 9.14.

Then  $h$  = Difference of pressure head for length  $L$ .

The pressure at outlet is atmospheric.

Let

$D$  = Diameter of capillary tube,

$L$  = Length of tube for which difference of pressure head is known,

$\rho$  = Density of fluid,

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and

$\mu$  = Co-efficient of viscosity.

Using Hagen Poiseuille's Formula,  $h = \frac{32\mu\bar{u}L}{\rho g D^2}$

But

$$\bar{u} = \frac{Q}{\text{Area}} = \frac{Q}{\frac{\pi}{4}D^2}$$

where  $Q$  is rate of liquid flowing through tube.

$$h = \frac{32\mu \times \frac{Q}{\frac{\pi}{4}D^2} \times L}{\rho g D^2} = \frac{128\mu Q \cdot L}{\pi \rho g D^4}$$

or

$$\mu = \frac{\pi \rho g h \bar{D}^4}{128 Q \cdot L} \quad \dots(9.25)$$

Measurement of  $D$  should be done very accurately.

### 9.8.2 Falling Sphere Resistance Method.

**Theory.** This method is based on Stoke's law, according to which the drag force,  $F$  on a small sphere moving with a constant velocity,  $U$  through a viscous fluid of viscosity,  $\mu$  for viscous conditions is given by

$$F = 3\pi\mu U d \quad \dots(i)$$

where  $d$  = diameter of sphere

$U$  = velocity of sphere.

When the sphere attains a constant velocity  $U$ , the drag force is the difference between the weight of sphere and buoyant force acting on it.

Let

$L$  = distance travelled by sphere in viscous fluid,

$t$  = time taken by sphere to cover distance  $L$ ,

$\rho_s$  = density of sphere,

$\rho_f$  = density of fluid,

$W$  = weight of sphere,

and  $F_B$  = buoyant force acting on sphere.

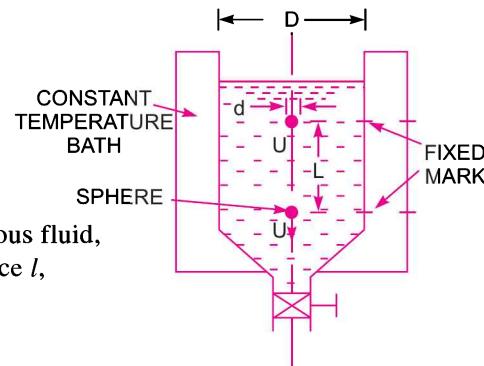


Fig. 9.15 Falling sphere resistance method.

Then constant velocity of sphere,  $U = \frac{L}{t}$

Weight of sphere,  $W = \text{volume} \times \text{density of sphere} \times g$

$$= \frac{\pi}{6} d^3 \times \rho_s \times g \quad \left\{ \because \text{volume of sphere} = \frac{\pi}{6} d^3 \right\}$$

and buoyant force,  $F_B$  = weight of fluid displaced

= volume of liquid displaced  $\times$  density of fluid  $\times g$

$$= \frac{\pi}{6} d^3 \times \rho_f \times g \quad \{ \text{volume of liquid displaced} = \text{volume of sphere} \}$$

For equilibrium,

$$\text{Drag force} = \text{Weight of sphere} - \text{buoyant force}$$

or

$$F = W - F_B$$

Substituting the values of  $F$ ,  $W$  and  $F_B$ , we get

$$3\pi\mu Ud = \frac{\pi}{6} d^3 \times \rho_s \times g - \frac{\pi}{6} d^3 \times \rho_f \times g = \frac{\pi}{6} d^3 \times g [\rho_s - \rho_f]$$

or

$$\mu = \frac{\pi}{6} \frac{d^3 \times g [\rho_s - \rho_f]}{3\pi Ud} = \frac{gd^2}{18U} [\rho_s - \rho_f] \quad \dots(9.26)$$

where  $\rho_f$  = Density of liquid

Hence in equation (9.26), the values of  $d$ ,  $U$ ,  $\rho_s$  and  $\rho_f$  are known and hence the viscosity of liquid can be determined.

**Method.** Thus this method consists of a tall vertical transparent cylindrical tank, which is filled with the liquid whose viscosity is to be determined. This tank is surrounded by another transparent tank to keep the temperature of the liquid in the cylindrical tank to be constant.

A spherical ball of small diameter ' $d$ ' is placed on the surface of liquid. Provision is made to release this ball. After a short distance of travel, the ball attains a constant velocity. The time to travel a known vertical distance between two fixed marks on the cylindrical tank is noted to calculate the constant velocity  $U$  of the ball. Then with the known values of  $d$ ,  $\rho_s$ ,  $\rho_f$  the viscosity  $\mu$  of the fluid is calculated by using equation (9.26).

**9.8.3 Rotating Cylinder Method.** This method consists of two concentric cylinders of radii  $R_1$  and  $R_2$  as shown in Fig. 9.16. The narrow space between the two cylinders is filled with the liquid whose viscosity is to be determined. The inner cylinder is held stationary by means of a torsional spring while outer cylinder is rotated at constant angular speed  $\omega$ . The torque  $T$  acting on the inner cylinder is measured by the torsional spring. The torque on the inner cylinder must be equal and opposite to the torque applied on the outer cylinder.

The torque applied on the outer cylinder is due to viscous resistance provided by liquid in the annular space and at the bottom of the inner cylinder.

Let  $\omega$  = angular speed of outer cylinder.

Tangential (peripheral) speed of outer cylinder

$$= \omega \times R_2$$

Tangential velocity of liquid layer in contact with outer cylinder will be equal to the tangential velocity of outer cylinder.

$\therefore$  Velocity of liquid layer with outer cylinder =  $\omega \times R_2$

Velocity of liquid layer with inner cylinder = 0

$\therefore$  Velocity gradient over the radial distance ( $R_2 - R_1$ )

$$= \frac{du}{dy} = \frac{\omega R_2 - 0}{R_2 - R_1} = \frac{\omega R_2}{R_2 - R_1}$$

$\therefore$  Shear stress ( $\tau$ )

$$= \mu \frac{du}{dy} = \mu \frac{\omega R_2}{(R_2 - R_1)}$$

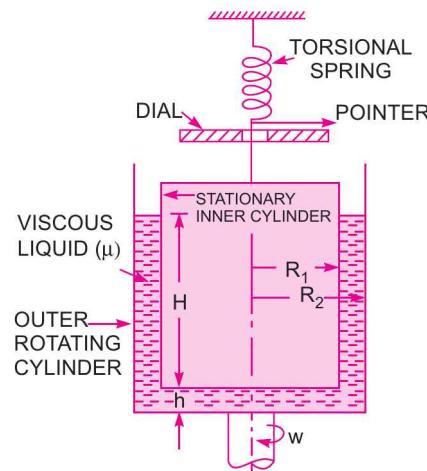


Fig. 9.16 Rotating cylinder viscometer.

$\{\because$  Inner cylinder is stationary }

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$$\begin{aligned}
 \therefore \text{Shear force } (F) &= \text{shear stress} \times \text{area of surface} \\
 &= \tau \times 2\pi R_1 H \\
 &\quad \{ \because \text{shear stress is acting on surface area} = 2\pi R_1 \times H \} \\
 &= \mu \frac{\omega R_2}{(R_2 - R_1)} \times 2\pi R_1 H
 \end{aligned}$$

The torque  $T_1$  on the inner cylinder due to shearing action of the liquid in the annular space is

$$\begin{aligned}
 T_1 &= \text{shear force} \times \text{radius} \\
 &= \mu \frac{\omega R_2}{(R_2 - R_1)} \times 2\pi R_1 H \times R_1 \\
 &= \frac{2\pi\mu\omega H R_1^2 R_2}{(R_2 - R_1)} \quad \dots(i)
 \end{aligned}$$

If the gap between the bottom of the two cylinders is ' $h$ ', then the torque applied on inner cylinder ( $T_2$ ) is given by equation (9.19A) as

$$T_2 = \frac{\mu}{60t} \pi^2 N R^4$$

But here

$$R = R_1, t = h \text{ then } T_2 = \frac{\mu}{60h} \pi^2 N R_1^4$$

$$\omega = \frac{2\pi N}{60} \text{ or } N = \frac{60\omega}{2\pi}$$

$$\therefore T_2 = \frac{\mu}{60h} \times \pi^2 \times \frac{60\omega}{2\pi} \times R_1^4 = \frac{\pi\mu\omega}{2h} R_1^4 \quad \dots(ii)$$

$\therefore$  Total torque  $T$  acting on the inner cylinder is

$$T = T_1 + T_2$$

$$\begin{aligned}
 &= \frac{2\pi\mu\omega H R_1^2 R_2}{(R_2 - R_1)} + \frac{\pi\mu\omega}{2h} R_1^4 = 2\pi\mu R_1^2 \left[ \frac{R_2 H}{R_2 - R_1} + \frac{R_1^2}{4h} \right] \times \omega \\
 \therefore \mu &= \frac{2(R_2 - R_1)hT}{\pi R_1^2 \omega [4HhR_2 + R_1^2 (R_2 - R_1)]} \quad \dots(9.27)
 \end{aligned}$$

where  $T$  = torque measured by the strain of the torsional spring,

$R_1, R_2$  = radii of inner and outer cylinder,

$h$  = clearance at the bottom of cylinders,

$H$  = height of liquid in annular space,

$\mu$  = co-efficient of viscosity to be determined.

Hence, the value of  $\mu$  can be calculated from equation (9.27).

**9.8.4 Orifice Type Viscometer.** In this method, the time taken by a certain quantity of the liquid whose viscosity is to be determined, to flow through a short capillary tube is noted down. The co-efficient of viscosity is then obtained by comparing with the co-efficient of viscosity of a liquid whose viscosity is known or by the use conversion factors.

Viscometers such as Saybolt, Redwood or Engler are usually used. The principle for all the three viscometer is same. In the United Kingdom, Redwood viscometer is used while in U.S.A., Saybolt viscometer is commonly used.

Fig. 9.17 shows that Saybolt viscometer, which consists of a tank at the bottom of which a short capillary tube is fitted. In this tank the liquid whose viscosity is to be determined is filled. This tank is surrounded by another tank, called constant temperature bath. The liquid is allowed to flow through capillary tube at a standard temperature. The time taken by 60 c.c. of the liquid to flow through the capillary tube is noted down. The initial height of liquid in the tank is previously adjusted to a standard height. From the time measurement, the kinematic viscosity of liquid is known from the relation,

$$\nu = At - \frac{B}{t}$$

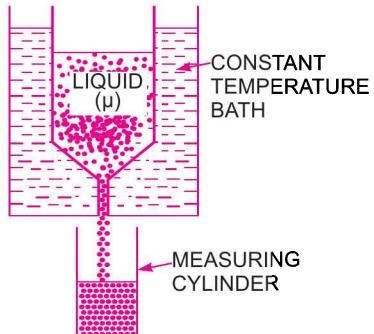


Fig. 9.17 Saybolt viscometer.

where  $A = 0.24$ ,  $B = 190$ ,  $t$  = time noted in seconds,  $\nu$  = kinematic viscosity in stokes.

**Problem 9.26** The viscosity of an oil of sp. gr. 0.9 is measured by a capillary tube of diameter 50 mm. The difference of pressure head between two points 2 m apart is 0.5 m of water. The mass of oil collected in a measuring tank is 60 kg in 100 seconds. Find the viscosity of oil.

**Solution.** Given :

$$\text{Sp. gr. of oil} = 0.9$$

$$\text{Dia. of capillary tube, } D = 50 \text{ mm} = 5 \text{ cm} = 0.05 \text{ m}$$

$$\text{Length of tube, } L = 2 \text{ m}$$

$$\text{Difference of pressure head, } h = 0.5 \text{ m}$$

$$\text{Mass of oil, } M = 60 \text{ kg}$$

$$\text{Time, } t = 100 \text{ s}$$

$$\text{Mass of oil per second} = \frac{60}{100} = 0.6 \text{ kg/s}$$

$$\text{Density of oil, } \rho = \text{sp. gr. of oil} \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$\therefore \text{Discharge, } Q = \frac{\text{Mass of oil / s}}{\text{Density}} = \frac{0.6}{900} \text{ m}^3/\text{s} = 0.000667 \text{ m}^3/\text{s}$$

Using equation (9.25), we get viscosity,

$$\mu = \frac{\pi \rho g h D^4}{128 Q \cdot L} \quad [\text{here } h = h_f = 0.5]$$

$$= \frac{\pi \times 900 \times 9.81 \times 0.5 \times (0.05)^4}{128 \times 0.000667 \times 2.0} = 0.5075 \text{ (SI Units) N s/m}^2$$

$$= 0.5075 \times 10 \text{ poise} = 5.075 \text{ poise. Ans.}$$

**Problem 9.27** A capillary tube of diameter 2 mm and length 100 mm is used for measuring viscosity of a liquid. The difference of pressure between the two ends of the tube is 0.6867 N/cm<sup>2</sup> and the viscosity of liquid is 0.25 poise. Find the rate of flow of liquid through the tube.

**Solution.** Given :

$$\text{Dia. of capillary tube, } D = 2 \text{ m} = 2 \times 10^{-3} \text{ m}$$

$$\text{Length of tube, } L = 100 \text{ mm} = 10 \text{ cm} = 0.1 \text{ m}$$

$$\text{Difference of pressure, } \Delta p = 0.6867 \text{ N/cm}^2 = 0.6867 \times 10^4 \text{ N/m}^2$$

$$\therefore \text{Difference of pressure head, } h = \frac{\Delta p}{\rho g} = \frac{0.6867 \times 10^4}{\rho g}$$

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Viscosity,  $\mu = 0.25 \text{ poise}$

$$= \frac{0.25}{10} \text{ Ns/m}^2$$

Let the rate of flow of liquid =  $Q$

$$\text{Using equation (9.25), we get } \mu = \frac{\pi \rho g h D^4}{128 \cdot Q \cdot L} = \pi \rho g \times \frac{\frac{0.6867 \times 10^4}{\rho g} \times (2 \times 10^{-3})^4}{128 \times Q \times 0.1}$$

$$\text{or } \frac{0.25}{10} = \frac{\pi \times 0.6867 \times 10^4 \times (2 \times 10^{-3})^4}{128 \times Q \times 0.1}$$

$$\begin{aligned} \text{or } Q &= \frac{\pi \times 0.6867 \times 10^4 \times 2^4 \times 10^{-12} \times 10}{128 \times 0.1 \times 0.25} \text{ m}^3/\text{s} \\ &= 107.86 \times 10^{-8} \text{ m}^3/\text{s} = 107.86 \times 10^{-8} \times 10^6 \text{ cm}^3/\text{s} \\ &= 107.86 \times 10^{-2} \text{ cm}^3/\text{s} = \mathbf{1.078 \text{ cm}^3/\text{s. Ans.}} \end{aligned}$$

**Problem 9.28** A sphere of diameter 2 mm falls 150 mm in 20 seconds in a viscous liquid. The density of the sphere is  $7500 \text{ kg/m}^3$  and of liquid is  $900 \text{ kg/m}^3$ . Find the co-efficient of viscosity of the liquid.

**Solution.** Given :

$$\text{Dia. of sphere, } d = 2 \text{ mm} = 2 \times 10^{-3} \text{ m}$$

$$\text{Distance travelled by sphere} = 150 \text{ mm} = 0.15 \text{ m}$$

$$\text{Time taken, } t = 20 \text{ seconds}$$

$$\text{Velocity of sphere, } U = \frac{\text{Distance}}{\text{Time}} = \frac{0.15}{20} = .0075 \text{ m/s}$$

$$\text{Density of sphere, } \rho_s = 7500 \text{ kg/m}^3$$

$$\text{Density of liquid, } \rho_f = 900 \text{ kg/m}^3$$

$$\begin{aligned} \text{Using relation (9.26), we get } \mu &= \frac{gd^2}{18U} [\rho_s - \rho_f] = \frac{9.81 \times [2 \times 10^{-3}]^2}{18 \times 0.0075} [7500 - 900] \\ &= \frac{9.81 \times 4 \times 10^{-6} \times 6600}{18 \times 0.0075} = 1.917 \frac{\text{Ns}}{\text{m}^2} \\ &= 1.917 \times 10 = \mathbf{19.17 \text{ poise. Ans.}} \end{aligned}$$

**Problem 9.29** Find the viscosity of a liquid of sp. gr. 0.8, when a gas bubble of diameter 10 mm rises steadily through the liquid at a velocity of 1.2 cm/s. Neglect the weight of the bubble.

**Solution.** Given :

$$\text{Sp. gr. of liquid} = 0.8$$

$$\therefore \text{Density of liquid, } \rho_f = 0.8 \times 1000 = 800 \text{ kg/m}^3$$

$$\text{Dia. of gas bubble, } D = 10 \text{ mm} = 1 \text{ cm} = 0.01 \text{ m}$$

$$\text{Velocity of bubble, } U = 1.2 \text{ cm/s} = .012 \text{ m/s}$$

As weight of bubble is neglected and density of bubble

$$\rho_s = 0$$

Now using the relation,  $\mu = \frac{gd^2}{18U} [\rho_s - \rho_f]$  which is for a falling sphere.

For a rising bubble, the relation will become as

$$\mu = \frac{gd^2}{18U} [\rho_f - \rho_s]$$

$$\text{Substituting the values, we get } \mu = \frac{9.81 \times .01 \times .01}{18 \times .012} [800 - 0] \frac{\text{Ns}}{\text{m}^2} = 3.63 \frac{\text{Ns}}{\text{m}^2}$$

$$= 3.63 \times 10 = 36.3 \text{ poise. Ans.}$$

**Problem 9.30** The viscosity of a liquid is determined by rotating cylinder method, in which case the inner cylinder of diameter 20 cm is stationary. The outer cylinder of diameter 20.5 cm, contains the liquid upto a height of 30 cm. The clearance at the bottom of the two cylinders is 0.5 cm. The outer cylinder is rotated at 400 r.p.m. The torque registered on the torsion meter attached to the inner cylinder is 5.886 Nm. Find the viscosity of fluid.

**Solution.** Given :

$$\text{Dia. of inner cylinder, } D_1 = 20 \text{ cm}$$

$$\therefore \text{Radius of inner cylinder, } R_1 = 10 \text{ cm} = 0.1 \text{ m}$$

$$\text{Dia. of outer cylinder, } D_2 = 20.5 \text{ cm}$$

$$\therefore \text{Radius of outer cylinder, } R_2 = \frac{20.5}{2} = 10.25 \text{ cm} = .1025 \text{ m}$$

Height of liquid from bottom of outer cylinder = 30 cm

Clearance at the bottom of two cylinders,  $h = 0.5 \text{ cm} = .005 \text{ m}$

$\therefore$  Height of inner cylinder immersed in liquid

$$= 30 - h = 30 - 0.5 = 29.5 \text{ m}$$

or

$$H = 29.5 \text{ cm} = .295 \text{ m}$$

$$\text{Speed of outer cylinder, } N = 400 \text{ r.p.m.}$$

$$\therefore \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 400}{60} = 41.88$$

$$\text{Torque measured, } T = 5.886 \text{ Nm}$$

$$\text{Using equation (9.27), we get } \mu = \frac{2(R_2 - R_1) \times h \times T}{\pi R_1^2 \omega [4HhR_2 + R_1^2(R_2 - R_1)]}$$

$$= \frac{2(.1025 - .1) \times .005 \times 5.886}{\pi \times (.1)^2 \times 41.88 [4 \times .295 \times .005 \times .1025 + .1^2 (.1025 - .1)]}$$

$$= \frac{2 \times .0025 \times .005 \times 5.886}{\pi \times .01 \times 41.88 [.0006047 - .000025]}$$

$$= 0.19286 \text{ Ns/m}^2 = 0.19286 \times 10 = 1.9286 \text{ poise. Ans.}$$

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**Problem 9.31** A sphere of diameter 1 mm falls through 335 m in 100 seconds in a viscous fluid. If the relative densities of the sphere and the liquid are 7.0 and 0.96 respectively, determine the dynamic viscosity of the liquid.

**Solution.** Given :

$$\text{Dia. of sphere, } d = 1 \text{ mm} = 0.001 \text{ m}$$

$$\text{Distance travelled by sphere} = 335 \text{ mm} = 0.335 \text{ m}$$

$$\text{Time taken, } t = 100 \text{ seconds}$$

$$\therefore \text{Velocity of sphere, } U = \frac{\text{Distance}}{\text{Time}} = \frac{0.335}{100} = 0.00335 \text{ m/sec}$$

$$\text{Relative density of sphere} = 7$$

$$\therefore \text{Density of sphere, } \rho_s = 7 \times 1000 = 7000 \text{ kg/m}^3$$

$$\text{Relative density of liquid} = 0.96$$

$$\therefore \text{Density of liquid, } \rho_f = 0.96 \times 1000 = 960 \text{ kg/m}^3$$

$$\begin{aligned} \text{Using the relation (9.26), we get } \mu &= \frac{gd^2}{18U} [\rho_s - \rho_f] = \frac{9.81 \times 0.001^2}{18 \times 0.00335} [7000 - 960] \\ &= \frac{0.00000981 \times 6040}{18 \times 0.00335} = 0.981 \text{ Ns/m}^2 \\ &= 0.981 \times 10 = \mathbf{9.81 \text{ poise. Ans.}} \end{aligned}$$

**Problem 9.32** Determine the fall velocity of 0.06 mm sand particle (specific gravity = 2.65) in water at 20°C, take  $\mu = 10^{-3}$  kg/ms.

**Solution.** Given :

$$\text{Dia. of sand particle, } d = 0.06 \text{ mm} = 0.06 \times 10^{-3} \text{ m}$$

$$\text{Specific gravity of sand} = 2.65$$

$$\therefore \text{Density of sand, } \rho_s = 2.65 \times 1000 \text{ kg/m}^3 \quad (\because \rho \text{ for water in S.I. unit} = 1000 \text{ kg/m}^3) \\ = 2650 \text{ kg/m}^3$$

$$\text{Viscosity of water, } \mu^* = 10^{-3} \text{ kg/ms} = 10^{-3} \text{ Ns/m}^2 \quad \left[ \because \frac{\text{Ns}}{\text{m}^2} = \left( \text{kg} \times \frac{\text{m}}{\text{s}^2} \right) \times \frac{\text{s}}{\text{m}^2} = \frac{\text{kg}}{\text{ms}} \right]$$

$$\text{Density of water, } \rho_f = 1000 \text{ kg/m}^3$$

Sand particle is just like a sphere.

For equilibrium of sand particle,

$$\text{Drag force} = \text{Weight of sand particle} - \text{buoyant force}$$

$$\text{or } F_D = W - F_B \quad \dots(i)$$

$$\text{But } F_D = 3\pi\mu \times U \times d, \text{ where } U = \text{Velocity of particle}$$

$$= 3\pi \times 10^{-3} \times U \times 0.06 \times 10^{-3} \text{ N}$$

$$W = \text{Weight of sand particle}$$

$$= \frac{\pi}{6} \times d^3 \times \rho_s \times g = \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 2650 \times 9.81 \text{ N}$$

$$F_B = \text{Buoyant force} = \text{Weight of water displaced}$$

\*Viscosity in S.I. unit = N s/m<sup>2</sup>. But 1 N = 1 kg × 1 m/s<sup>2</sup>

Hence viscosity =  $\left( \frac{1 \text{ kg} \times 1 \text{ m}}{\text{s}^2} \right) \times \frac{\text{s}}{\text{m}^2} = \text{kg/ms. Hence kg/ms} = \frac{\text{Ns}}{\text{m}^2}$ .

$$= \frac{\pi}{6} \times d^3 \times \rho_f \times g = \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 1000 \times 9.81 \text{ N}$$

Substituting the above values in equation (i), we get

$$3\pi \times 10^{-3} \times U \times 0.06 \times 10^{-3} = \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 2650 \times 9.81 - \frac{\pi}{6} \times (0.06 \times 10^{-3})^3 \times 1000 \times 9.81$$

Cancelling  $(\pi \times 0.06 \times 10^{-3})^2$  throughout, we get

$$\begin{aligned} 3 \times U &= \frac{1}{6} \times 0.06^2 \times 10^{-3} \times 2650 \times 9.81 - \frac{1}{6} \times 0.06^2 \times 10^{-3} \times 1000 \times 9.81 \\ &= \frac{1}{6} \times 0.06^2 \times 10^{-3} \times 9.81 (2650 - 1000) \\ &= \frac{1}{6} \times 0.0036 \times 10^{-3} \times 9.81 \times 1650 = 0.009712 \end{aligned}$$

$$\therefore U = 0.009712/3 = \mathbf{0.00323 \text{ m/sec. Ans.}}$$

## HIGHLIGHTS

1. A flow is said to be viscous if the Reynolds number is less than 2000, or the fluid flows in layers.
2. For the viscous flow through circular pipes,

$$(i) \text{ Shear stress} \dots \tau = - \frac{\partial p}{\partial x} \frac{r}{2}$$

$$(ii) \text{ Velocity} \dots u = - \frac{1}{4\mu} \frac{\partial p}{\partial x} [R^2 - r^2]$$

$$(iii) \text{ Ratio of velocities} \frac{U_{\max}}{\bar{u}} = 2.0$$

$$(iv) \text{ Loss of pressure head, } h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$$

where  $\frac{\partial p}{\partial x}$  = pressure gradient,  $r$  = radius at any point,

$R$  = radius of the pipe,  $U_{\max}$  = maximum velocity or velocity at  $r = 0$ ,

$\bar{u}$  = average velocity =  $\frac{Q}{\pi R^2}$ ,  $\mu$  = co-efficient of viscosity,

$D$  = diameter of the pipe.

3. For the viscous flow between two parallel plates,

$$u = - \frac{1}{2\mu} \frac{\partial p}{\partial x} (ty - y^2) \dots \text{Velocity distribution}$$

$$\frac{U_{\max}}{\bar{u}} = 1.5 \dots \text{Ratio of maximum and average velocity}$$

$$h_f = \frac{12\mu\bar{u}L}{\rho g t^2} \dots \text{Loss of pressure head}$$

$$\tau = - \frac{1}{2} \frac{\partial p}{\partial x} [t - 2y] \dots \text{Shear stress distribution}$$

where  $t$  = thickness or distance between two plates,

$y$  = distance in the vertical direction from the lower plate,

$\tau$  = shear stress at any point in flow.

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4. The kinetic energy correction factor  $\alpha$  is given as

$$\alpha = \frac{\text{K.E. per second based on actual velocity}}{\text{K.E. per second based on average velocity}}$$

$$= 2.0 \dots \text{for a circular pipe.}$$

5. Momentum correction factor,  $\beta$  is given by

$$\beta = \frac{\text{Momentum per second based on actual velocity}}{\text{Momentum per second based on average velocity}}$$

$$= \frac{4}{3} \dots \text{for a circular pipe.}$$

6. For the viscous resistance of Journal Bearing.

$$V = \frac{\pi DN}{60}, \frac{du}{dy} = \frac{V}{t} = \frac{\pi DN}{60t}$$

$$\tau = \frac{\mu \pi dN}{60t}, \text{ Shear force} = \frac{\mu \pi^2 D^2 NL}{60t}$$

$$\text{Torque, } T = \frac{\mu \pi^2 D^3 NL}{120t} \text{ and power} = \frac{\mu \pi^3 D^3 N^2 L}{60 \times 60 \times t}$$

where  $L$  = length of bearing,  $N$  = speed of shaft

$t$  = clearance between the shaft and bearing.

7. For the Foot-Step Bearing, the shear force, torque and h.p. absorbed are given as :

$$\text{Shear force, } F = \frac{\mu}{15} \frac{\pi^2 N}{t} \frac{R^3}{3}$$

$$\text{Torque, } T = \frac{\mu}{60t} \pi^2 N R^4$$

$$\text{Power} = \frac{\mu \pi^3 N^2 R^4}{60 \times 30 \times t}$$

where  $R$  = radius of the shaft,  $N$  = speed of the shaft.

8. For the collar bearing the torque and power absorbed are given as

$$T = \frac{\mu}{60t} \pi^2 N [R_2^4 - R_1^4], \quad P = \frac{\mu \pi^3 N^2}{60 \times 30t} [R_2^4 - R_1^4]$$

where  $R_1$  = internal radius of the collar,  
 $t$  = thickness of oil film,

$R_2$  = external radius of the collar,  
 $P$  = power in watts.

9. For the viscous flow the co-efficient of friction is given by,  $f = \frac{16}{R_e}$

$$\text{where } R_e = \text{the Reynolds number} = \frac{\rho VD}{\mu} = \frac{VD}{\nu}.$$

10. The co-efficient of viscosity is determined by dash-pot arrangement as  $\mu = \frac{4Wt^3}{3\pi LD^3 V}$

where  $W$  = weight of the piston,  
 $L$  = length of the piston,  
 $V$  = velocity of the piston.

$t$  = clearance between dash-pot and piston,  
 $D$  = diameter of the piston,

11. The co-efficient of viscosity of a liquid is also determined experimentally by the following method :

$$(i) \text{ Capillary tube method, } \mu = \frac{\pi \rho g h D^4}{128 Q L}$$

$$(ii) \text{ Falling sphere method, } \mu = \frac{g d^2 [\rho_s - \rho_f]}{18 U}$$

$$(iii) \text{ Rotating cylinder method, } \mu = \frac{2 (R_2 - R_1) h T}{\pi R_1^2 \omega [4 H h R_2 + R_1^2 (R_2 - R_1)]}$$

where  $w$  = specific weight of fluid,

$L$  = length of the tube,

$D$  = diameter of the capillary tube,

$Q$  = rate of flow of fluid through capillary tube,

$d$  = diameter of the sphere,

$\rho_s$  = density of sphere,

$\rho_f$  = density of fluid,

$U$  = velocity of sphere,

$R_2$  = radius of outer rotating cylinder,

$R_1$  = radius of inner stationary cylinder,

$T$  = torque.

## EXERCISE

### (A) THEORETICAL PROBLEMS

- Define the terms : Viscosity, kinematic viscosity, velocity gradient and pressure gradient.
- What do you mean by 'Viscous Flow'?
- Derive an expression for the velocity distribution for viscous flow through a circular pipe. Also sketch the velocity distribution and shear stress distribution across a section of the pipe.
- Prove that the maximum velocity in a circular pipe for viscous flow is equal to two times the average velocity of the flow.  
*(Delhi University, December 2002)*
- Find an expression for the loss of head of a viscous fluid flowing through a circular pipe.
- What is Hagen Poiseuille's Formula ? Derive an expression for Hagen Poiseuille's Formula.
- Prove that the velocity distribution for viscous flow between two parallel plates when both plates are fixed across a section is parabolic in nature. Also prove that maximum velocity is equal to one and a half times the average velocity.
- Show that the difference of pressure head for a given length of the two parallel plates which are fixed and through which viscous fluid is flowing is given by

$$h_f = \frac{12 \mu \bar{u} L}{\rho g t^2}$$

where  $\mu$  = Viscosity of fluid,

$\bar{u}$  = Average velocity,

$t$  = Distance between the two parallel plates,

$L$  = Length of the plates.

- Define the terms : Kinetic energy correction factor and momentum correction factor.
- Prove that for viscous flow through a circular pipe the kinetic energy correction factor is equal to 2 while momentum correction factor =  $\frac{4}{3}$ .
- A shaft is rotating in a journal bearing. The clearance between the shaft and the bearing is filled with a viscous oil. Find an expression for the power absorbed in overcoming viscous resistance.
- Prove that power absorbed in overcoming viscous resistance in foot-step bearing is given by

$$P = \frac{\mu \pi^3 N^2 R^4}{60 \times 30 t}$$

where  $R$  = Radius of the shaft,

$N$  = Speed of the shaft,

$t$  = Clearance between shaft and foot-step bearing,

$\mu$  = Viscosity of fluid.

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13. Show that the value of the co-efficient of friction for viscous flow through a circular pipe is given by,

$$f = \frac{16}{R_e}, \text{ where } R_e = \text{Reynolds number.}$$

14. Prove that the co-efficient of viscosity by the dash-pot arrangement is given by,

$$\mu = \frac{4Wt^3}{3\pi LD^3V}$$

where  $W$  = Weight of the piston,  $t$  = Clearance between dash-pot and piston,

$L$  = Length of piston,  $D$  = Diameter of piston,

$V$  = Velocity of piston.

15. What are the different methods of determining the co-efficient of viscosity of a liquid ? Describe any two method in details.

16. Prove that the loss of pressure head for the viscous flow through a circular pipe is given by

$$h_f = \frac{32\mu \bar{u}L}{\rho g d^2}$$

where  $\bar{u}$  = Average velocity,  $w$  = Specific weight.

17. For a laminar steady flow, prove that the pressure gradient in a direction of motion is equal to the shear gradient normal to the direction of motion.

18. Describe Reynolds experiments to demonstrate the two types of flow.

19. For the laminar flow through a circular pipe, prove that :

- (i) the shear stress variation across the section of the pipe is linear and  
(ii) the velocity variation is parabolic.

## (B) NUMERICAL PROBLEMS

1. A crude oil of viscosity 0.9 poise and sp. gr. 0.8 is flowing through a horizontal circular pipe of diameter 80 mm and of length 15 m. Calculate the difference of pressure at the two ends of the pipe, if 50 kg of the oil is collected in a tank in 15 seconds. [Ans. 0.559 N/cm<sup>2</sup>]

2. A viscous flow is taking place in a pipe of diameter 100 mm. The maximum velocity is 2 m/s. Find the mean velocity and the radius at which this occurs. Also calculate the velocity at 30 mm from the wall of the pipe. [Ans. 1 m/s,  $r = 35.35$  mm,  $u = 1.68$  m/s]

3. A fluid of viscosity 0.5 poise and specific gravity 1.20 is flowing through a circular pipe of diameter 100 mm. The maximum shear stress at the pipe wall is given as 147.15 N/m<sup>2</sup>, find : (a) the pressure gradient, (b) the average velocity, and (c) the Reynolds number of the flow.

[Ans. (a) – 64746 N/m<sup>2</sup> per m, (b) 3.678 m/s, (c) 882.72]

4. Determine (a) the pressure gradient, (b) the shear stress at the two horizontal parallel plates and (c) the discharge per metre width for the laminar flow of oil with a maximum velocity of 1.5 m/s between two horizontal parallel fixed plates which are 80 mm apart. Take viscosity of oil as  $\frac{1.962 \text{ Ns}}{\text{m}^2}$ .

[Ans. (a) – 3678.7 N/m<sup>2</sup> per m, (b) 147.15 N/m<sup>2</sup>, (c) .08 m<sup>3</sup>/s]

5. Water is flowing between two large parallel plates which are 2.0 mm apart. Determine : (a) maximum velocity, (b) the pressure drop per unit length and (c) the shear stress at walls of the plate if the average velocity is 0.4 m/s. Take viscosity of water as 0.01 poise.

[Ans. (a) 0.6 m/s, (b) 1199.7 N/m<sup>2</sup> per m, (c) 1.199 N/m<sup>3</sup>]

6. There is a horizontal crack 50 mm wide and 3 mm deep in a wall of thickness 150 mm. Water leaks through the crack. Find the rate of leakage of water through the crack if the difference of pressure between the two ends of the crack is 245.25 N/m<sup>2</sup>. Take the viscosity of water as 0.01 poise. [Ans. 183.9 cm<sup>3</sup>/s]

7. A shaft having a diameter of 10 cm rotates centrally in a journal bearing having a diameter of 10.02 cm and length 20 cm. The annular space between the shaft and the bearing is filled with oil having viscosity of 0.8 poise. Determine the power absorbed in the bearing when the speed of rotation is 500 r.p.m.  
**[Ans. 343.6 W]**
8. A shaft 150 mm diameter runs in a bearing of length 300 mm, with a radial clearance of 0.04 mm at 40 r.p.m. Find the viscosity of the oil, if the power required to overcome the viscous resistance is 220.725 W.  
**[Ans. 6.32 poise]**
9. Find the torque required to rotate a vertical shaft of diameter 8 cm at 800 r.p.m. The lower end of the shaft rests in a foot-step bearing. The end of the shaft and surface of the bearing are both flat and are separated by an oil film of thickness 0.075 cm. The viscosity of the oil is given as 1.2 poise. **[Ans. 0.0538 Nm]**
10. A collar bearing having external and internal diameters 20 cm and 10 cm respectively is used to take the thrust of a shaft. An oil film of thickness 0.03 cm is maintained between the collar surface and the bearing. Find the power lost in overcoming the viscous resistance when the shaft rotates at 250 r.p.m. Take  $\mu = 0.9$  poise.  
**[Ans. 30.165 W]**
11. Water is flowing through a 150 mm diameter pipe with a co-efficient of friction  $f = .05$ . The shear stress at a point 40 mm from the pipe wall is  $0.01962 \text{ N/cm}^2$ . Calculate the shear stress at the pipe wall.  
**[Ans. 0.04198 N/cm<sup>2</sup>]**
12. An oil dash-pot consists of a piston moving in a cylinder having oil. The piston falls with uniform speed and covers 4.5 cm in 80 seconds. If an additional weight of 1.5 N is placed on the top of the piston, it falls through 4.5 cm in 70 seconds with uniform speed. The diameter of the piston is 10 cm and its length is 15 cm. The clearance between the piston and the cylinder is 0.15 cm, which is uniform throughout. Find the viscosity of oil.  
**[Ans. 0.177 poise]**
13. The viscosity of oil of sp. gr. 0.8 is measured by a capillary tube of diameter 40 mm. The difference of pressure head between two points 1.5 m apart is 0.3 m of water. The mass of oil collected in a measuring tank is 40 kg in 120 seconds. Find the viscosity of the oil.  
**[Ans. 2.36 poise]**
14. A capillary tube of diameter 4 mm and length 150 mm is used for measuring viscosity of a liquid. The difference of pressure between the two ends of the tube is  $0.7848 \text{ N/cm}^2$  and the viscosity of the liquid is 0.2 poise. Find the rate of flow of liquid through the tube.  
**[Ans.  $16.43 \text{ cm}^3/\text{s}$ ]**
15. A sphere of diameter 3 mm falls 100 mm in 1.5 seconds in a viscous liquid. The density of the sphere is  $7000 \text{ kg/m}^3$  and of liquid is  $800 \text{ kg/m}^3$ . Find the co-efficient of viscosity of the liquid. **[Ans. 45.61 poise]**
16. The viscosity of a liquid is determined by rotating cylinder method, in which case the inner cylinder of diameter 25 cm is stationary. The outer cylinder of diameter 25.5 cm contains the liquid upto a height of 40 cm. The clearance at the bottom of the two cylinders is 0.6 cm. The outer cylinder is rotated at 300 r.p.m. The torque registered on the torsion metre attached to the inner cylinder is 4.905 Nm. Find the viscosity of liquid.  
**[Ans. .77 poise]**
17. Calculate : (a) the pressure gradient along the flow, (b) the average velocity, and (c) the discharge for an oil of viscosity  $0.02 \text{ Ns/m}^2$  flowing between two stationary parallel plates 1 m wide maintained 10 mm apart. The velocity midway between the plates is 2.5 m/s.  
**[Ans. (a)  $-4000 \text{ N/m}^2$  per m, (b)  $1.667 \text{ m/s}$ , (c)  $.01667 \text{ m}^3/\text{s}$ ]**
18. Calculate :  
(i) the pressure gradient along the flow,  
(ii) the average velocity, and  
(iii) the discharge for an oil of viscosity  $0.03 \text{ N s/m}^2$  flowing between two stationary plates which are parallel and are at 10 mm apart. Width of plates is 2 m. The velocity midway between the plates is 2.0 m/s.
19. A cylinder of 100 mm diameter, 0.15 m length and weighing 10 N slides axially in a vertical pipe of 104 mm dia. If the space between cylinder surface and pipe wall is filled with liquid of viscosity  $\mu$  and the cylinder slides downwards at a velocity of 0.45 m/s, determine  $\mu$ .  
**[Hint.  $D = 100 \text{ mm} = 0.1 \text{ m}$ ,  $L = 0.15 \text{ m}$ ,  $W = 10 \text{ N}$ ,  $D_p = 1.4 \text{ mm} = 0.104 \text{ m}$ ,  $V = 0.45 \text{ m/s}$ . Hence  $t = (0.104 - 0.1)/2 = 0.002 \text{ m}$ .**

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$$\mu = \frac{4Wt^3}{3\pi D^3 LV} = \frac{4 \times 10 \times 0.002^3}{3\pi \times 0.1^3 \times 0.15 \times .45} = 503 \times 10^{-6} \text{ N s/m}^2]$$

20. A liquid is pumped through a 15 cm diameter and 300 m long pipe at the rate of 20 tonnes per hour. The density of liquid is 910 kg/m<sup>3</sup> and kinematic viscosity = 0.002 m<sup>2</sup>/s. Determine the power required and show that the flow is viscous.

[Hint.  $D = 15 \text{ cm} = 0.15 \text{ m}$ ,  $L = 300 \text{ m}$ ,  $W = 20 \text{ tonnes/hr}$

$$= 20 \times 1000 \text{ kgf}/60 \times 60 \text{ sec} = 5.555 \text{ kgf/sec} = 5.555 \times 9.81 \text{ N/s.}$$

$$Q = \frac{W}{\rho g} = \frac{5.555 \times 9.81}{910 \times 9.81} = 0.0061 \text{ m}^3/\text{s.} V = \frac{Q}{A} = \frac{0.0061}{\frac{\pi}{4}(15^2)} \\ = 0.345 \text{ m/s, } v = 0.002 \text{ m}^2/\text{s.}$$

Now

$$R_e = \frac{\rho V D}{\mu} = \frac{V \times D}{v} = \frac{0.345 \times 0.15}{0.002} = 25.87$$

which is less than 2000. Hence flow is viscous.

$$h_f = 32 \mu LV / \rho g D^2, \text{ where } v = \frac{\mu}{\rho} \therefore \mu = v \times \rho = 0.002 \times 910 = 1.82$$

$$\text{Hence, } h_f = \frac{32 \times 1.82 \times 300 \times 0.345}{(910 \times 9.81 \times 0.15^2)} = 30$$

$$\therefore P = \rho g Q h_f / 1000 = 910 \times 9.81 \times 0.0061 \times 30 / 1000 = 1.633 \text{ kW.}]$$

21. An oil of specific gravity 0.9 and viscosity 10 poise is flowing through a pipe of diameter 110 mm. The velocity at the centre is 2 m/s, find : (i) pressure gradient in the direction of flow, (ii) shear stress at the pipe wall ; (iii) Reynolds number, and (iv) velocity at a distance of 30 mm from the wall.

[Hint.  $\rho = 900 \text{ kg/m}^3$  ;  $\mu = 10 \text{ poise} = 1 \text{ N s/m}^2$  ;  $D = 110 \text{ mm} = 0.11 \text{ m}$ ,

$$U_{\max} = 2 \text{ m/s; } \bar{u} = 1 \text{ m/s; } U_{\max} = \frac{1}{4\mu} \left( \frac{-dp}{dx} \right) R^2$$

$$(i) \left( \frac{-dp}{dx} \right) = \frac{4\mu \times U_{\max}}{R^2} = \frac{4 \times 1 \times 2}{0.055^2} = 2644.6 \text{ N/m}^3;$$

$$(ii) \tau_0 = \left( \frac{-dp}{dx} \right) \times \frac{R}{2} = 2644.6 \times \frac{0.055}{2} = 72.72 \text{ N/m}^2;$$

$$(iii) R_e = \frac{\rho \times \bar{u} \times D}{\mu} = \frac{900 \times 1 \times 0.11}{1} = 99; \text{ and}$$

$$(iv) u = \frac{1}{4\mu} \left( \frac{-dp}{dx} \right) (R^2 - r^2) = \frac{1}{4 \times 1} (2644.6) (0.055^2 - 0.025^2) = 1.586 \text{ m/s.}]$$

22. Determine (i) the pressure gradient, (ii) the shear stress at the two horizontal plates, (iii) the discharge per metre width for laminar flow of oil with a maximum velocity of 2 m/s between two plates which are 150 mm apart. Given :  $\mu = 2.5 \text{ N s/m}^2$ .

(Delhi University, December 2002)

[Hint.  $U_{\max} = 2 \text{ m/s, } t = 150 \text{ mm} = 0.15 \text{ m, } \mu = 2.5 \text{ N s/m}^2$

$$(i) U_{\max} = - \frac{1}{8\mu} \frac{dp}{dx} t^2 \quad \therefore \frac{dp}{dx} = \frac{-8\mu U_{\max}}{t^2} = \frac{-8 \times 2.5 \times 2}{0.15^2} = -1777.77 \text{ N/m}^2.$$

$$(ii) \tau_0 = - \frac{1}{2} \frac{dp}{dx} \times t = - \frac{1}{2} (-1777.77) \times 0.15 = 133.33 \text{ N/m}^2.$$

$$(iii) Q = \text{Mean velocity} \times \text{Area} = \left( \frac{2}{3} U_{\max} \right) \times (t \times 1) = \left( \frac{2}{3} \times 2 \right) \times (0.15 \times 1) = 0.2 \text{ m}^3/\text{s.}]$$

# 10

## CHAPTER



### ► 10.1 INTRODUCTION

The laminar flow has been discussed in chapter 9. In laminar flow the fluid particles move along straight parallel path in layers or laminae, such that the paths of individual fluid particles do not cross those of neighbouring particles. Laminar flow is possible only at low velocities and when the fluid is highly viscous. But when the velocity is increased or fluid is less viscous, the fluid particles do not move in straight paths. The fluid particles move in random manner resulting in general mixing of the particles. This type of flow is called turbulent flow.

A laminar flow changes to turbulent flow when (i) velocity is increased or (ii) diameter of a pipe is increased or (iii) the viscosity of fluid is decreased. O. Reynold was first to demonstrate that the transition from laminar to turbulent depends not only on the mean velocity but on the quantity  $\frac{\rho V D}{\mu}$ . This quantity  $\frac{\rho V D}{\mu}$  is a dimensionless quantity and is called Reynolds number ( $R_e$ ). In case of circular pipe if  $R_e < 2000$  the flow is said to be laminar and if  $R_e > 4000$ , the flow is said to be turbulent. If  $R_e$  lies between 2000 to 4000, the flow changes from laminar to turbulent.

### ► 10.2 REYNOLDS EXPERIMENT

The type of flow is determined from the Reynolds number *i.e.*,  $\frac{\rho V \times d}{\mu}$ . This was demonstrated by O. Reynold in 1883. His apparatus is shown in Fig. 10.1.

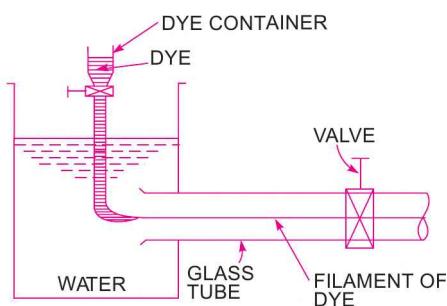


Fig. 10.1 *Reynold apparatus.*

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The apparatus consists of :

- (i) A tank containing water at constant head,
- (ii) A small tank containing some dye,
- (iii) A glass tube having a bell-mouthing entrance at one end and a regulating valve at other ends.

The water from the tank was allowed to flow through the glass tube. The velocity of flow was varied by the regulating valve. A liquid dye having same specific weight as water was introduced into the glass tube as shown in Fig. 10.1.

The following observations were made by Reynold :

(i) When the velocity of flow was low, the dye filament in the glass tube was in the form of a straight line. This straight line of dye filament was parallel to the glass tube, which was the case of laminar flow as shown in Fig. 10.2 (a).

(ii) With the increase of velocity of flow, the dye filament was no longer a straight-line but it became a wavy one as shown in Fig. 10.2 (b). This shows that flow is no longer laminar.

(iii) With further increase of velocity of flow, the wavy dye-filament broke-up and finally diffused in water as shown in Fig. 10.2 (c). This means that the fluid particles of the dye at this higher velocity are moving in random fashion, which shows the case of turbulent flow. Thus in case of turbulent flow the mixing of dye-filament and water is intense and flow is irregular, random and disorderly.

In case of laminar flow, the loss of pressure head was found to be proportional to the velocity but in case of turbulent flow, Reynold observed that loss of head is approximately proportional to the square of velocity. More exactly the loss of head,  $h_f \propto V^n$ , where  $n$  varies from 1.75 to 2.0

### ► 10.3 FRICTIONAL LOSS IN PIPE FLOW

When a liquid is flowing through a pipe, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity. This viscous action causes loss of energy which is usually known as frictional loss.

On the basis of his experiments, William Froude gave the following laws of fluid friction for turbulent flow.

The frictional resistance for turbulent flow is :

- (i) proportional to  $V^n$ , where  $n$  varies from 1.5 to 2.0,
- (ii) proportional to the density of fluid,
- (iii) proportional to the area of surface in contact,
- (iv) independent of pressure,
- (v) dependent on the nature of the surface in contact.

**10.3.1 Expression for Loss of Head Due to Friction in Pipes.** Consider a uniform horizontal pipe, having steady flow as shown in Fig. 10.3. Let 1-1 and 2-2 are two sections of pipe.

Let  $p_1$  = pressure intensity at section 1-1,

$V_1$  = velocity of flow at section 1-1,

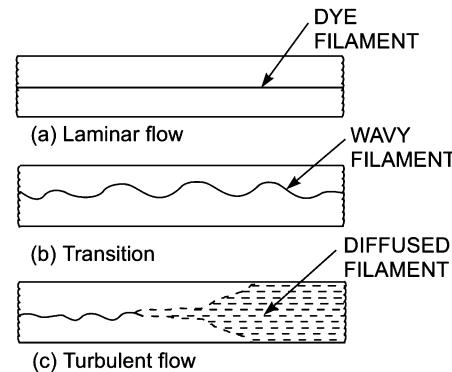


Fig. 10.2 Different stages of filament.

$L$  = length of the pipe between sections 1-1 and 2-2,  
 $d$  = diameter of pipe,  
 $f'$  = frictional resistance per unit wetted area per unit velocity,  
 $h_f$  = loss of head due to friction,  
and  $p_2, V_2$  = are values of pressure intensity and velocity at section 2-2.

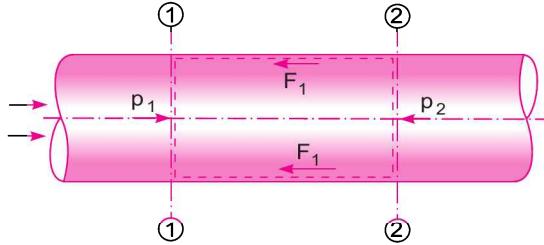


Fig. 10.3 Uniform horizontal pipe.

Applying Bernoulli's equations between sections 1-1 and 2-2,  
Total head at 1-1 = Total head at 2-2 + loss of head due to friction between 1-1 and 2-2

$$\text{or } \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f$$

But  $z_1 = z_2$  as pipe is horizontal

$V_1 = V_2$  as dia. of pipe is same at 1-1 and 2-2

$$\therefore \frac{p_1}{\rho g} = \frac{p_2}{\rho g} + h_f \text{ or } h_f = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \quad \dots(i)$$

But  $h_f$  is the head lost due to friction and hence intensity of pressure will be reduced in the direction of flow by frictional resistance.

Now frictional resistance = frictional resistance per unit wetted area per unit velocity  $\times$  wetted area  $\times$  velocity<sup>2</sup>

$$\text{or } F_1 = f' \times \pi d L \times V^2 \quad [\because \text{wetted area} = \pi d \times L, \text{velocity} = V = V_1 = V_2] \\ = f' \times P \times L \times V^2 \quad [\because \pi d = \text{Perimeter} = P] \dots(ii)$$

The forces acting on the fluid between sections 1-1 and 2-2 are :

1. pressure force at section 1-1 =  $p_1 \times A$

where  $A$  = Area of pipe

2. pressure force at section 2-2 =  $p_2 \times A$

3. frictional force  $F_1$  as shown in Fig. 10.3.

Resolving all forces in the horizontal direction, we have

$$p_1 A - p_2 A - F_1 = 0 \quad \dots(10.1)$$

$$\text{or } (p_1 - p_2)A = F_1 = f' \times P \times L \times V^2 \quad [\because \text{From (ii), } F_1 = f' P L V^2]$$

$$\text{or } p_1 - p_2 = \frac{f' \times P \times L \times V^2}{A}$$

But from equation (i),  $p_1 - p_2 = \rho g h_f$

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Equating the value of  $(p_1 - p_2)$ , we get

$$\rho g h_f = \frac{f' \times P \times L \times V^2}{A}$$

or

$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2 \quad \dots(iii)$$

$$\text{In equation (iii), } \frac{P}{A} = \frac{\text{Wetted perimeter}}{\text{Area}} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$$

$$\therefore h_f = \frac{f'}{\rho g} \times \frac{4}{d} \times L \times V^2 = \frac{f'}{\rho g} \times \frac{4LV^2}{d} \quad \dots(iv)$$

Putting  $\frac{f'}{\rho} = \frac{f}{2}$ , where  $f$  is known as co-efficient of friction.

$$\text{Equation (iv), becomes as } h_f = \frac{4 \cdot f}{2g} \cdot \frac{LV^2}{d} = \frac{4f \cdot L \cdot V^2}{d \times 2g} \quad \dots(10.2)$$

Equation (10.2) is known as Darcy-Weisbach equation. This equation is commonly used for finding loss of head due to friction in pipes.

Sometimes equation (10.2) is written as

$$h_f = \frac{f \cdot L \cdot V^2}{d \times 2g} \quad \dots(10.2A)$$

Then  $f$  is known as friction factor.

**10.3.2 Expression for Co-efficient of Friction in Terms of Shear Stress.** The equation (10.1) gives the forces acting on a fluid between sections 1-1 and 2-2 of Fig. 10.3 in horizontal direction as

$$p_1 A - p_2 A - F_1 = 0$$

or

$$\begin{aligned} (p_1 - p_2) A &= F_1 = \text{force due to shear stress } \tau_0 \\ &= \text{shear stress} \times \text{surface area} \\ &= \tau_0 \times \pi d \times L \end{aligned}$$

or

$$(p_1 - p_2) \frac{\pi}{4} d^2 = \tau_0 \times \pi d \times L \quad \left\{ \because A = \frac{\pi}{4} d^2 \right\}$$

Cancelling  $\pi d$  from both sides, we have

$$(p_1 - p_2) \frac{d}{4} = \tau_0 \times L$$

or

$$(p_1 - p_2) = \frac{4\tau_0 \times L}{d} \quad \dots(10.3)$$

$$\text{Equation (10.2) can be written as } h_f = \frac{p_1 - p_2}{\rho g} = \frac{4f \cdot L \cdot V^2}{d \times 2g}$$

$$\text{or } (p_1 - p_2) = \frac{4f \cdot L \cdot V^2}{d \times 2g} \times \rho g \quad \dots(10.4)$$

Equating the value of  $(p_1 - p_2)$  in equations (10.3) and (10.4),

$$\frac{4\tau_0 \times L}{d} = \frac{4f \cdot L \cdot V^2}{d \times 2g} \times \rho g$$

$$\text{or } \tau_0 = \frac{fV^2 \times \rho g}{2g} = \frac{fV^2}{2g} \times \rho g$$

$$\text{or } \tau_0 = f \frac{\rho V^2}{2} \quad \dots(10.5)$$

$$\therefore f = \frac{2\tau_0}{\rho V^2}. \quad \dots(10.6)$$

#### ► 10.4 SHEAR STRESS IN TURBULENT FLOW

The shear stress in viscous flow is given by Newton's law of viscosity as

$$\tau_v = \mu \frac{du}{dy}, \quad \text{where } \tau_v = \text{shear stress due to viscosity.}$$

Similar to the expression for viscous shear, J. Boussinesq expressed the turbulent shear in mathematical form as

$$\tau_t = \eta \frac{d\bar{u}}{dy} \quad \dots(10.7)$$

where  $\tau_t$  = shear stress due to turbulence

$\eta$  = eddy viscosity

$\bar{u}$  = average velocity at a distance  $y$  from boundary.

The ratio of  $\eta$  (eddy viscosity) and  $\rho$  (mass density) is known as kinematic eddy viscosity and is denoted by  $\epsilon$  (epsilon). Mathematically it is written as

$$\epsilon = \frac{\eta}{\rho} \quad \dots(10.8)$$

If the shear stress due to viscous flow is also considered, then the total shear stress becomes as

$$\tau = \tau_v + \tau_t = \mu \frac{du}{dy} + \eta \frac{d\bar{u}}{dy} \quad \dots(10.9)$$

The value of  $\eta = 0$  for laminar flow. For other cases the value of  $\eta$  may be several thousand times the value of  $\mu$ . To find shear stress in turbulent flow, equation (10.7) given by Boussinesq is used. But as the value of  $\eta$  (eddy viscosity) cannot be predicted, this equation is having limited use.

**10.4.1 Reynolds Expression for Turbulent Shear Stress.** Reynolds in 1886 developed an expression for turbulent shear stress between two layers of a fluid at a small distance apart, which is given as

$$\tau = \rho u' v' \quad \dots(10.10)$$

where  $u'$ ,  $v'$  = fluctuating component of velocity in the direction of  $x$  and  $y$  due to turbulence.

As  $u'$  and  $v'$  are varying and hence  $\tau$  will also vary. Hence to find the shear stress, the time average on both the sides of the equation (10.10) is taken. Then equation (10.10) becomes as

$$\bar{\tau} = \overline{\rho u' v'} \quad \dots(10.11)$$

The turbulent shear stress given by equation (10.11) is known as Reynold stress.

**10.4.2 Prandtl Mixing Length Theory for Turbulent Shear Stress.** In equation (10.11), the turbulent shear stress can only be calculated if the value of  $u' v'$  is known. But it is very difficult to measure  $u' v'$ . To overcome this difficulty, L. Prandtl in 1925, presented a mixing length hypothesis which can be used to express turbulent shear stress in terms of measurable quantities.

According to Prandtl, the mixing length  $l$ , is that distance between two layers in the transverse direction such that the lumps of fluid particles from one layer could reach the other layer and the particles are mixed in the other layer in such a way that the momentum of the particles in the direction of  $x$  is same. He also assumed that the velocity fluctuation in the  $x$ -direction  $u'$  is related to the mixing length  $l$  as

$$u' = l \frac{du}{dy}$$

and  $v'$ , the fluctuation component of velocity in  $y$ -direction is of the same order of magnitude as  $u'$  and hence

$$v' = l \frac{du}{dy}$$

$$\text{Now } \overline{u' \times v'} \text{ becomes as } \overline{u' v'} = \left( l \frac{du}{dy} \right) \times \left( l \frac{du}{dy} \right) = l^2 \left( \frac{du}{dy} \right)^2$$

Substituting the value of  $\overline{u' v'}$  in equation (10.11), we get the expression for shear stress in turbulent flow due to Prandtl as

$$\bar{\tau} = \rho l^2 \left( \frac{du}{dy} \right)^2 \quad \dots(10.12)$$

Thus the total shear stress at any point in turbulent flow is the sum of shear stress due to viscous shear and turbulent shear and can be written as

$$\bar{\tau} = \mu \frac{du}{dy} + \rho l^2 \left( \frac{du}{dy} \right)^2 \quad \dots(10.13)$$

But the viscous shear stress is negligible except near the boundary. Equation (10.13) is used for most of turbulent fluid flow problems for determining shear stress in turbulent flow.

## ► 10.5 VELOCITY DISTRIBUTION IN TURBULENT FLOW IN PIPES

In case of turbulent flow, the total shear stress at any point is the sum of viscous shear stress and turbulent shear stress. Also the viscous shear stress is negligible except near the boundary. Hence it can be assumed that the shear stress in turbulent flow is given by equation (10.12). From this equation, the velocity distribution can be obtained if the relation between  $l$ , the mixing length and  $y$  is known. Prandtl assumed that the mixing length,  $l$  is a linear function of the distance  $y$  from the pipe wall i.e.,  $l = ky$ , where  $k$  is a constant, known as Karman constant and  $= 0.4$ .

Substituting the value of  $l$  in equation (10.12), we get

$$\bar{\tau} \text{ or } \tau = \rho \times (ky)^2 \times \left( \frac{du}{dy} \right)^2$$

or  $\tau = \rho k^2 y^2 \left( \frac{du}{dy} \right)^2 \text{ or } \left( \frac{du}{dy} \right)^2 = \tau / \rho k^2 y^2$

or  $\frac{du}{dy} = \sqrt{\frac{\tau}{\rho k^2 y^2}} = \frac{1}{ky} \sqrt{\frac{\tau}{\rho}}$  ... (10.14)

For small values of  $y$  that is very close to the boundary of the pipe, Prandtl assumed shear stress  $\tau$  to be constant and approximately equal to  $\tau_0$  which presents the turbulent shear stress at the pipe boundary. Substituting  $\tau = \tau_0$  in equation (10.14), we get

$$\frac{du}{dy} = \frac{1}{ky} \sqrt{\frac{\tau_0}{\rho}} \quad \dots (10.15)$$

In equation (10.15),  $\sqrt{\frac{\tau_0}{\rho}}$  has the dimensions  $\sqrt{\frac{ML^{-1}T^{-2}}{ML^{-3}}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$ . But  $\frac{L}{T}$  is velocity and hence  $\sqrt{\frac{\tau_0}{\rho}}$  has the dimension of velocity, which is known as shear velocity and is denoted by  $u_*$ .

Thus  $\sqrt{\frac{\tau_0}{\rho}} = u_*$ , then equation (10.15) becomes  $\frac{du}{dy} = \frac{1}{ky} u_*$ .

For a given case of turbulent flow,  $u_*$  is constant. Hence integrating above equation, we get

$$u = \frac{u_*}{k} \log_e y + C \quad \dots (10.16)$$

where  $C$  = constant of integration.

Equation (10.16) shows that in turbulent flow, the velocity varies directly with the logarithm of the distance from the boundary or in other words the velocity distribution in turbulent flow is logarithmic in nature. To determine the constant of integration,  $C$  the boundary condition that at  $y = R$  (radius of pipe),  $u = u_{\max}$  is substituted in equation (10.16).

Hence  $u_{\max} = \frac{u_*}{k} \log_e R + C \quad \therefore \quad C = u_{\max} - \frac{u_*}{k} \log_e R$

Substituting the value of  $C$  in equation (10.16), we get

$$\begin{aligned} u &= \frac{u_*}{k} \log_e y + u_{\max} - \frac{u_*}{k} \log_e R = u_{\max} + \frac{u_*}{k} (\log_e y - \log_e R) \\ &= u_{\max} + \frac{u_*}{0.4} \log_e (y/R) \quad [\because k = 0.4 = \text{Karman constant}] \\ &= u_{\max} + 2.5 u_* \log_e (y/R) \end{aligned} \quad \dots (10.17)$$

Equation (10.17) is called 'Prandtl's universal velocity distribution equation for turbulent flow in pipes. This equation is applicable to smooth as well as rough pipe boundaries. Equation (10.17) is also written as

$$u_{\max} - u = -2.5 u_* \log_e (y/R) = 2.5 u_* \log_e (R/y)$$

Dividing by  $u_*$ , we get

$$\frac{u_{\max} - u}{u_*} = 2.5 \log_e (R/y) = 2.5 \times 2.3 \log_{10} (R/y) \quad [\because \log_e (R/y) = 2.3 \log_{10} (R/y)]$$

or

$$\frac{u_{\max} - u}{u_*} = 5.75 \log_{10} (R/y) \quad \dots(10.18)$$

In equation (10.18), the difference between the maximum velocity  $u_{\max}$ , and local velocity  $u$  at any point i.e.,  $(u_{\max} - u)$  is known as ‘velocity defect’.

**10.5.1 Hydrodynamically Smooth and Rough Boundaries.** Let  $k$  is the average height of the irregularities projecting from the surface of a boundary as shown in Fig. 10.4. If the value of  $k$  is large for a boundary then the boundary is called rough boundary and if the value of  $k$  is less, then boundary is known as smooth boundary, in general. This is the classification of rough and smooth boundary based on boundary characteristics. But for proper classification, the flow and fluid characteristics are also to be considered.

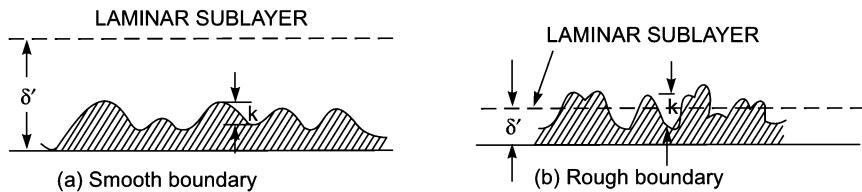


Fig. 10.4 *Smooth and rough boundaries.*

For turbulent flow analysis along a boundary, the flow is divided in two portions. The first portion consists of a thin layer of fluid in the immediate neighbourhood of the boundary, where viscous shear stress predominates while the shear stress due to turbulence is negligible. This portion is known as laminar sub-layer. The height upto which the effect of viscosity predominates in this zone is denoted by  $\delta'$ . The second portion of flow, where shear stress due to turbulence are large as compared to viscous stress is known as turbulent zone.

If the average height  $k$  of the irregularities, projecting from the surface of a boundary is much less than  $\delta'$ , the thickness of laminar sub-layer as shown in Fig. 10.4 (a), the boundary is called smooth boundary. This is because, outside the laminar sub-layer the flow is turbulent and eddies of various size present in turbulent flow try to penetrate the laminar sub-layer and reach the surface of the boundary. But due to great thickness of laminar sub-layer the eddies are unable to reach the surface irregularities and hence the boundary behaves as a smooth boundary. This type of boundary is called hydrodynamically smooth boundary.

Now, if the Reynolds number of the flow is increased then the thickness of laminar sub-layer will decrease. If the thickness of laminar sub-layer becomes much smaller than the average height  $k$  of irregularities of the surface as shown in Fig. 10.4 (b), the boundary will act as rough boundary. This is because the irregularities of the surface are above the laminar sub-layer and the eddies present in turbulent zone will come in contact with the irregularities of the surface and lot of energy will be lost. Such a boundary is called hydrodynamically rough boundary.

From Nikuradse's experiment :

1. If  $\frac{k}{\delta'}$  is less than 0.25 or  $\frac{k}{\delta'} < 0.25$ , the boundary is called smooth boundary.

2. If  $\frac{k}{\delta'} > 6.0$ , the boundary is rough,
3. If  $0.25 < \left(\frac{k}{\delta'}\right) < 6.0$ , the boundary is in transition.

In terms of roughness Reynolds number  $\frac{u_* k}{v}$  :

1. If  $\frac{u_* k}{v} < 4$ , boundary is considered smooth,
2. If  $\frac{u_* k}{v}$  lies between 4 and 100, boundary is in transition stage, and
3. If  $\frac{u_* k}{v} > 100$ , the boundary is rough.

**10.5.2 Velocity Distribution for Turbulent Flow in Smooth Pipes.** The velocity distribution for turbulent flow in smooth or rough pipe is given by equation (10.16) as

$$u = \frac{u_*}{k} \log_e y + C$$

It may be seen that at  $y = 0$ , the velocity  $u$  at wall is  $-\infty$ . This means that velocity  $u$  is positive at some distance far away from the wall and  $-\infty$  (minus infinity) at the wall. Hence at some finite distance from wall, the velocity will be equal to zero. Let this distance from pipe wall is  $y'$ . Now the constant  $C$  is determined from the boundary condition *i.e.*, at  $y = y'$ ,  $u = 0$ . Hence above equation becomes as

$$0 = \frac{u_*}{k} \log_e y' + C \text{ or } C = -\frac{u_*}{k} \log_e y'$$

Substituting the value of  $C$  in the above equation, we get

$$u = \frac{u_*}{k} \log_e y - \frac{u_*}{k} \log_e y' = \frac{u_*}{k} \log_e (y/y')$$

Substituting the value of  $k = 0.4$ , we get

$$u = \frac{u_*}{0.4} \log_e (y/y') = 2.5 u_* \log_e (y/y')$$

$$\frac{u}{u_*} = 2.5 \times 2.3 \log_{10} (y/y') \quad [\because \log_e (y/y') = 2.3 \log_{10} (y/y')]$$

or 
$$\frac{u}{u_*} = 5.75 \log_{10} (y/y') \quad \dots(10.19)$$

For the smooth boundary, there exists a laminar sub-layer as shown in Fig. 10.4 (a). The velocity distribution in the laminar sub-layer is parabolic in nature. Thus in the laminar sub-layer, logarithmic velocity distribution does not hold good. Thus it can be assumed that  $y'$  is proportional to  $\delta'$ , where  $\delta'$  is the thickness of laminar sub-layer. From Nikuradse's experiment the value of  $y'$  is given as

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$$y' = \frac{\delta'}{107}$$

where  $\delta' = \frac{11.6v}{u_*}$ , where  $v$  = kinematic viscosity of fluid.

$$\therefore y' = \frac{11.6v}{u_*} \times \frac{1}{107} = \frac{0.108v}{u_*}$$

Substituting this value of  $y'$  in equation (10.19), we obtain

$$\begin{aligned} \frac{u}{u_*} &= 5.75 \log_{10} \left( \frac{y}{\frac{0.108v}{u_*}} \right) \\ &= 5.75 \log_{10} \left( \frac{yu_*}{0.108v} \right) = 5.75 \log_{10} \left( \frac{u_*y}{v} \times 9.259 \right) \\ &= 5.75 \log_{10} \frac{u_*y}{v} + 5.75 \log_{10} 9.259 \quad \left[ \because \frac{1}{0.108} = 9.259 \right] \\ &= 5.75 \log_{10} \frac{u_*y}{v} + 5.55 \end{aligned} \quad \dots(10.20)$$

**10.5.3 Velocity Distribution for Turbulent Flow in Rough Pipes.** In case of rough boundaries, the thickness of laminar sub-layer is very small as shown in Fig. 10.4 (b). The surface irregularities are above the laminar sub-layer and hence the laminar sub-layer is completely destroyed. Thus  $y'$  can be considered proportional to the height of protrusions  $k$ . Nikuradse's experiment shows

the value of  $y'$  for pipes coated with uniform sand (rough pipes) as  $y' = \frac{k}{30}$ .

Substituting this value of  $y'$  in equation (10.19), we get

$$\begin{aligned} \frac{u}{u_*} &= 5.75 \log_{10} \left( \frac{y}{k/30} \right) = 5.75 [\log_{10}(y/k) \times 30] \\ &= 5.75 \log_{10}(y/k) + 5.75 \log_{10}(30.0) = 5.75 \log_{10}(y/k) + 8.5 \end{aligned} \quad \dots(10.21)$$

**Problem 10.1** A pipe-line carrying water has average height of irregularities projecting from the surface of the boundary of the pipe as 0.15 mm. What type of boundary is it? The shear stress developed is  $4.9 \text{ N/m}^2$ . The kinematic viscosity of water is .01 stokes.

**Solution.** Given :

Average height of irregularities,  $k = 0.15 \text{ mm} = 0.15 \times 10^{-3} \text{ m}$

Shear stress developed,  $\tau_0 = 4.9 \text{ N/m}^2$

Kinematic viscosity,  $v = 0.01 \text{ stokes} = .01 \text{ cm}^2/\text{s} = .01 \times 10^{-4} \text{ m}^2/\text{s}$

Density of water,  $\rho = 1000 \text{ kg/m}^3$

$$\text{Shear velocity, } u_* = \sqrt{\tau_0 / \rho} = \sqrt{\frac{4.9}{1000}} = \sqrt{0.0049} = 0.07 \text{ m/s}$$

$$\text{Roughness Reynold number} = \frac{u_* k}{v} = \frac{0.07 \times 0.15 \times 10^{-3}}{0.01 \times 10^{-4}} = 10.5.$$

Since  $\frac{u_* k}{v}$  lies between 4 and 100 and hence pipe surface behaves as in transition.

**Problem 10.2** A rough pipe is of diameter 8.0 cm. The velocity at a point 3.0 cm from wall is 30% more than the velocity at a point 1 cm from pipe wall. Determine the average height of the roughness.

**Solution.** Given :

$$\text{Dia. of rough pipe, } D = 8 \text{ cm} = .08 \text{ m}$$

$$\text{Let velocity of flow at 1 cm from pipe wall} = u$$

$$\text{Then velocity of flow at 3 cm from pipe wall} = 1.3 u$$

The velocity distribution for rough pipe is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5, \text{ where } k = \text{height of roughness.}$$

For a point, 1 cm from pipe wall, we have

$$\frac{u}{u_*} = 5.75 \log_{10} (1.0/k) + 8.5 \quad \dots(i)$$

For a point, 3 cm from pipe wall, velocity is 1.3  $u$  and hence

$$\frac{1.3u}{u_*} = 5.75 \log_{10} (3.0/k) + 8.5 \quad \dots(ii)$$

$$\text{Dividing (ii) by (i), we get } 1.3 = \frac{5.75 \log_{10} (3.0/k) + 8.5}{5.75 \log_{10} (1.0/k) + 8.5}$$

$$\text{or } 1.3[5.75 \log_{10} (1.0/k) + 8.5] = 5.75 \log_{10} (3.0/k) + 8.5$$

$$\text{or } 7.475 \log_{10} (1.0/k) + 11.05 = 5.75 \log_{10} (3.0/k) + 8.5$$

$$\text{or } 7.475 \log_{10} (1.0/k) - 5.75 \log_{10} (3.0/k) = 8.5 - 11.05 = -2.55$$

$$\text{or } 7.475 [\log_{10} 1.0 - \log_{10} k] - 5.75 [\log_{10} 3.0 - \log_{10} k] = -2.55$$

$$\text{or } 7.475 [0 - \log_{10} k] - 5.75 [0.4771 - \log_{10} k] = -2.55$$

$$\text{or } -7.475 \log_{10} k - 2.7433 + 5.75 \log_{10} k = -2.55$$

$$\text{or } -1.725 \log_{10} k = 2.7433 - 2.55 = 0.1933$$

$$\text{or } \log_{10} k = \frac{0.1933}{-1.725} = -0.1120 = -1.888$$

$$k = .7726 \text{ cm. Ans.}$$

**Problem 10.3** A smooth pipe of diameter 80 mm and 800 m long carries water at the rate of  $0.480 \text{ m}^3/\text{minute}$ . Calculate the loss of head, wall shearing stress, centre line velocity, velocity and shear stress at 30 mm from pipe wall. Also calculate the thickness of laminar sub-layer. Take kinematic viscosity of water as 0.015 stokes. Take the value of co-efficient of friction ' $f$ ' from the relation given as

$$f = \frac{.0791}{(R_e)^{1/4}}, \text{ where } R_e = \text{Reynolds number.}$$

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**Solution.** Given :

$$\text{Dia. of smooth pipe, } d = 80 \text{ mm} = .08 \text{ m}$$

$$\text{Length of pipe, } L = 800 \text{ m}$$

$$\text{Discharge, } Q = 0.048 \text{ m}^3/\text{minute} = \frac{0.48}{60} = .008 \text{ m}^3/\text{s}$$

$$\text{Kinematic viscosity, } v = .015 \text{ stokes} = .015 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Density of water, } \rho = 1000 \text{ kg/m}^3$$

$$\text{Mean velocity, } V = \frac{Q}{\text{Area}} = \frac{0.008}{\frac{\pi}{4}(.08)^2} = 1.591 \text{ m/s}$$

$$\therefore \text{Reynolds number, } R_e = \frac{V \times d}{v} = \frac{1.591 \times 0.08}{.015 \times 10^{-4}} = 8.485 \times 10^4$$

As the Reynolds number is more than 4000, the flow is turbulent.

$$\text{Now the value of 'f' is given by } f = \frac{.0791}{R_e^{1/4}} = \frac{.0791}{(8.485 \times 10^4)^{1/4}} = .004636$$

(i) Head lost is given by equation (10.2) as

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} = \frac{4 \times .004636 \times 800 \times 1.591^2}{.08 \times 2 \times 9.81} = 23.42 \text{ m. Ans.}$$

(ii) Wall shearing stress,  $\tau_0$  is given by equation (10.5) as

$$\tau_0 = \frac{f \rho V^2}{2} = .004636 \times \frac{1000}{2} \times 1.591^2 = 5.866 \text{ N/m}^2. \text{ Ans.}$$

(iii) Centre-line velocity,  $u_{\max}$  for smooth pipe is given by equation (10.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{v} + 5.55 \quad \dots(i)$$

$$\text{where } u_* \text{ is shear velocity and } u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{5.866}{1000}} = 0.0765 \text{ m/s}$$

$$\text{The velocity will be maximum when } y = \frac{d}{2} = \frac{.08}{2} = .04 \text{ m.}$$

Hence at  $y = .04 \text{ m}$ ,  $u = u_{\max}$ . Substituting these values in (i), we get

$$\begin{aligned} \frac{u_{\max}}{0.0765} &= 5.75 \log_{10} \frac{0.0765 \times 0.04}{.015 \times 10^{-4}} + 5.55 \\ &= 5.75 \log_{10} 2040 + 5.55 \\ &= 5.75 \times 3.309 + 5.55 = 19.03 + 5.55 = 24.58 \\ \therefore u_{\max} &= 0.0765 \times 24.58 = 1.88 \text{ m/s. Ans.} \end{aligned}$$

(iv) The shear stress,  $\tau$  at any point is given by

$$\tau = - \frac{\partial p}{\partial x} \frac{r}{2} \quad \dots(A)$$

where  $r$  = distance from centre of pipe  
and hence shear stress at pipe wall where  $r = R$  is

$$\tau_0 = - \frac{\partial p}{\partial x} \frac{R}{2} \quad \dots(B)$$

Dividing equation (A) by equation (B), we get

$$\frac{\tau}{\tau_0} = \frac{r}{R}$$

$$\therefore \text{Shear stress} \quad \tau = \frac{\tau_0 r}{R}$$

A point 30 mm from pipe wall is having  $r = 4 - 3 = 1 \text{ cm} = .01 \text{ m}$

$$\therefore \tau \text{ at } (r = .01 \text{ m}) = \frac{\tau_0 \times .01}{.04} = \frac{5.866}{4} = 1.4665 \text{ N/m}^2. \text{ Ans.}$$

Velocity at a point 3 cm from pipe wall means  $y = 3 \text{ cm} = .03 \text{ m}$

and is given by equation (10.20) as  $\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{v} + 5.55$ , where  $u_* = .0765$ ,  $y = .03$

$$\begin{aligned} \therefore \frac{u}{.0765} &= 5.75 \log_{10} \frac{.0765 \times .03}{.015 \times 10^{-4}} + 5.55 \\ &= 5.75 \log_{10} 1530 + 5.55 = 23.86 \end{aligned}$$

$$\therefore u = 0.0765 \times 23.86 = 1.825 \text{ m/s. Ans.}$$

(v) Thickness of laminar sub-layer is given by

$$\begin{aligned} \delta' &= \frac{11.6 \times v}{u_*} = \frac{11.6 \times .015 \times 10^{-4}}{.0765} = 2.274 \times 10^{-4} \text{ m} \\ &= 2.274 \times 10^{-2} \text{ cm} = .02274 \text{ cm. Ans.} \end{aligned}$$

**Problem 10.4** Determine the wall shearing stress in a pipe of diameter 100 mm which carries water. The velocities at the pipe centre and 30 mm from the pipe centre are 2 m/s and 1.5 m/s respectively. The flow in pipe is given as turbulent.

**Solution.** Given :

$$\text{Dia. of pipe, } D = 100 \text{ mm} = 0.10 \text{ m}$$

$$\therefore \text{Radius, } R = \frac{0.10}{2} = 0.05 \text{ m}$$

Velocity at centre,  $u_{\max} = 2 \text{ m/s}$

Velocity at 30 mm or 0.03 m from centre = 1.5 m/s

$$\therefore \text{Velocity (at } r = 0.03 \text{ m), } u = 1.5 \text{ m/s}$$

Let the wall shearing stress  $= \tau_0$

For turbulent flow, the velocity distribution in terms of centre line velocity ( $u_{\max}$ ) is given by equation (10.18) as

$$\frac{u_{\max} - u}{u_*} = 5.75 \log_{10} \left( \frac{R}{y} \right)$$

where  $u = 1.5 \text{ m/s}$  at  $y = (R - r) = 0.05 - 0.03 = .02 \text{ m}$

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$$\therefore \frac{2.0 - 1.5}{u_*} = 5.75 \log_{10} \frac{.05}{.02} = 2.288 \text{ or } \frac{0.5}{u_*} = 2.288$$

$$\therefore u_* = \frac{0.5}{2.288} = 0.2185 \text{ m/s}$$

Using the relation  $u_* = \sqrt{\tau_0 / \rho}$ , where  $\rho$  for water = 1000 kg/m<sup>3</sup>

$$\therefore 0.2185 = \sqrt{\frac{\tau_0}{1000}} \text{ or } \frac{\tau_0}{1000} = 0.2185^2 = 0.0477$$

$$\text{or } \tau_0 = 0.0477 \times 1000 = 47.676 \text{ N/m}^2. \text{ Ans.}$$

**10.5.4 Velocity Distribution for Turbulent Flow in Terms of Average Velocity.** The average velocity  $\bar{U}$ , through the pipe is obtained by first finding the total discharge  $Q$  and then dividing the total discharge by the area of the pipe.

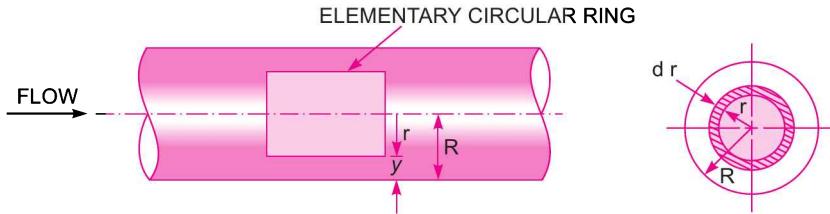


Fig. 10.5 Average velocity for turbulent flow.

Consider an elementary circular ring of radius 'r' and thickness  $dr$  as shown in Fig. 10.5. The distance of the ring from pipe wall is  $y = (R - r)$ , where  $R$  = radius of pipe.

Then the discharge,  $dQ$ , through the ring is given by

$$dQ = \text{area of ring} \times \text{velocity} \\ = 2\pi r dr \times u = u \times 2\pi r dr$$

$$\text{Total discharge, } Q = \int_0^R dQ = \int_0^R u \times 2\pi r dr \quad \dots(10.22)$$

(a) **For smooth pipes.** For smooth pipes, the velocity distribution is given by equation (10.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_* y}{v} + 5.5$$

$$\text{or } u = \left[ 5.75 \log_{10} \frac{u_* y}{v} + 5.5 \right] \times u_*$$

$$\text{But } y = (R - r)$$

$$\therefore u = \left[ 5.75 \log_{10} \frac{u_* (R - r)}{v} + 5.5 \right] \times u_*$$

Substituting the value of  $u$  in equation (10.22), we get

$$Q = \int_0^R \left[ 5.75 \log_{10} \frac{u_* (R - r)}{v} + 5.5 \right] u_* \times 2\pi r dr$$

$$\therefore \text{Average velocity, } \bar{U} = \frac{Q}{\text{Area}} = \frac{Q}{\pi R^2}$$

$$= \frac{1}{\pi R^2} \int_0^R \left[ 5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.5 \right] u_* 2\pi r dr$$

Integration of the above equation and subsequent simplification gives the average velocity for turbulent flow in smooth pipes as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{v} + 1.75 \quad \dots(10.23)$$

(b) **For rough pipes.** For rough pipes, the velocity at any point in turbulent flow is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

But

$$y = (R - r)$$

$$\therefore \frac{u}{u_*} = 5.75 \log_{10} \left( \frac{R-r}{k} \right) + 8.5$$

or

$$u = u_* \left[ 5.75 \log_{10} \left( \frac{R-r}{k} \right) + 8.5 \right]$$

Substituting the value of  $u$  in equation (10.22), we get

$$\begin{aligned} Q &= \int_0^R u_* \left[ 5.75 \log_{10} \left( \frac{R-r}{k} \right) + 8.5 \right] 2\pi r dr \\ \therefore \text{Average velocity, } \bar{U} &= \frac{Q}{\pi R^2} = \frac{\int_0^R u_* \left[ 5.75 \log_{10} \left( \frac{R-r}{k} \right) + 8.5 \right] 2\pi r dr}{\pi R^2} \end{aligned}$$

Integration of the above equation and subsequent simplification will give the following relation for average velocity,  $\bar{U}$  for turbulent flow in rough pipe as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{R}{k} + 4.75 \quad \dots(10.24)$$

(c) **Difference of the velocity at any point and average velocity for smooth and rough pipes.**  
The velocity at any point for turbulent flow for smooth pipes is given by equation (10.20) as

$$\frac{u}{u_*} = 5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.5 \quad [\because y = R - r]$$

and the average velocity is given by equation (10.23) as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{u_* R}{v} + 1.75$$

$\therefore$  Difference of velocity  $u$  and  $\bar{U}$  for smooth pipe is obtained as

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$$\frac{u}{u_*} - \frac{\bar{U}}{u_*} = \left[ 5.75 \log_{10} \frac{u_*(R-r)}{v} + 5.5 \right] - \left[ 5.75 \log_{10} \frac{u_* R}{v} + 1.75 \right]$$

or

$$\begin{aligned} \frac{u - \bar{U}}{u_*} &= 5.75 \left[ \log_{10} \frac{u_*(R-r)}{v} - \log_{10} \frac{u_* R}{v} \right] + 5.5 - 1.75 \\ &= 5.75 \log_{10} \left[ \frac{u_*(R-r)}{v} \div \frac{u_* R}{v} \right] + 3.75 \\ &= 5.75 \log_{10} \left( \frac{R-r}{v} \right) + 3.75 \\ &= 5.75 \log_{10} (y/R) + 3.75 \end{aligned} \quad \dots(10.25) [\because R-r=y]$$

Similarly the velocity,  $u$  at any point for rough pipe is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

and average velocity is given by equation (10.24) as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} (R/k) + 4.75$$

$\therefore$  Difference of velocity  $u$  and  $\bar{U}$  for rough pipe is given by

$$\begin{aligned} \frac{u}{u_*} - \frac{\bar{U}}{u_*} &= [5.75 \log_{10} (y/k) + 8.5] - [5.75 \log_{10} (R/k) + 4.75] \\ &= 5.75 \log_{10} [(y/k) \div (R/k)] + 8.5 - 4.75 \\ \text{or } \frac{u - \bar{U}}{u_*} &= 5.75 \log_{10} (y/R) + 3.75 \end{aligned} \quad \dots(10.26)$$

Equations (10.25) and (10.26) are the same. This shows that the difference of velocity at any point and the average velocity will be the same in case of smooth as well as rough pipes.

**Problem 10.5** Determine the distance from the pipe wall at which the local velocity is equal to the average velocity for turbulent flow in pipes.

**Solution.** Given :

Local velocity at a point = average velocity

or

$$u = \bar{U}$$

For a smooth or rough pipe, the difference of velocity at any point and average velocity is given by equation (10.25) or equation (10.26) as

$$\frac{u - \bar{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75$$

Substituting the given condition i.e.,  $u = \bar{U}$ , we get

$$\frac{\bar{U} - \bar{U}}{u_*} = 0 = 5.75 \log_{10} (y/R) + 3.75 \text{ or } 5.75 \log_{10} (y/R) = 3.75$$

or  $\log_{10}(y/R) = -\frac{3.75}{5.75} = -0.6521 = -1.3479$

$\therefore y/R = 0.22279 \approx 0.2228$  or  $y = .2228$  R. Ans.

**Problem 10.6** For turbulent flow in a pipe of diameter 300 mm, find the discharge when the centre-line velocity is 2.0 m/s and the velocity at a point 100 mm from the centre as measured by pitot-tube is 1.6 m/s.

**Solution.** Given :

Dia. of pipe,  $D = 300 \text{ mm} = 0.3 \text{ m}$

$\therefore$  Radius,  $R = \frac{0.3}{2} = 0.15 \text{ m}$

Velocity at centre,  $u_{\max} = 2.0 \text{ m/s}$

Velocity (at  $r = 100 \text{ mm} = 0.1 \text{ m}$ ),  $u = 1.6 \text{ m/s}$

Now  $y = R - r = 0.15 - 0.10 = 0.05 \text{ m}$

$\therefore$  Velocity (at  $r = 0.1 \text{ m}$  or at  $y = 0.05 \text{ m}$ ),  $u = 1.6 \text{ m/s}$

The velocity in terms of centre-line velocity is given by equation (10.18) as

$$\frac{u_{\max} - u}{u_*} = 5.75 \log_{10}(R/y)$$

Substituting the values, we get  $\frac{2.0 - 1.6}{u_*} = 5.75 \log_{10} \frac{0.15}{0.05}$   
 $= 5.75 \log_{10} 3.0 = 2.7434$

$$\left[ \begin{array}{l} \because y = 0.05 \text{ m} \\ R = 0.15 \text{ m} \end{array} \right]$$

or  $\frac{0.4}{u_*} = 2.7434$

$\therefore u_* = \frac{0.4}{2.7434} = 0.1458 \text{ m/s}$  ... (i)

Using equation (10.26) which gives relation between velocity at any point and average velocity, we have

$$\frac{u - \bar{U}}{u_*} = 5.75 \log_{10}(y/R) + 3.75$$

at  $y = R$ , velocity  $u$  becomes  $= u_{\max}$

$\therefore \frac{u_{\max} - \bar{U}}{u_*} = 5.75 \log_{10}(R/R) + 3.75 = 5.75 \times 0 + 3.75 = 3.75$

But  $u_{\max} = 2.0$  and  $u_*$  from (i) = 0.1458

$\therefore \frac{2.0 - \bar{U}}{0.1458} = 3.75$

or  $\bar{U} = 2.0 - 0.1458 \times 3.75 = 2.0 - 0.5467 = 1.4533 \text{ m/s}$

$\therefore$  Discharge,  $Q = \text{Area} \times \text{average velocity}$

$$= \frac{\pi}{4} D^2 \times \bar{U} = \frac{\pi}{4} (0.3)^2 \times 1.4533 = 0.1027 \text{ m}^3/\text{s. Ans.}$$

**10.5.5 Velocity Distribution for Turbulent Flow in Smooth Pipes by Power Law.** The velocity distribution for turbulent flow as given by equations (10.18), (10.20) and (10.21) are logarithmic in nature. These equations are not convenient to use. Nikuradse carried out experiments for different Reynolds number to determine the velocity distribution law in smooth pipes. He expressed the velocity distribution in exponential form as

$$\frac{u}{u_{\max}} = (y/R)^{1/n} \quad \dots(10.27)$$

where exponent  $\frac{1}{n}$  depends on Reynolds number

The value of  $\left(\frac{1}{n}\right)$  decreases, with increasing Reynolds number.

$$\text{For } R_e = 4 \times 10^3, \quad \frac{1}{n} = \frac{1}{6}$$

$$R_e = 1.1 \times 10^5, \quad \frac{1}{n} = \frac{1}{7}$$

$$R_e \geq 2 \times 10^6, \quad \frac{1}{n} = \frac{1}{10}$$

Thus if  $\frac{1}{n} = \frac{1}{7}$ , the velocity distribution law becomes as

$$\frac{u}{u_{\max}} = \left(\frac{y}{R}\right)^{1/7} \quad \dots(10.28)$$

Equation (10.28) is known as 1/7th power law of velocity distribution for smooth pipes.

## ► 10.6 RESISTANCE OF SMOOTH AND ROUGH PIPES

The loss of head, due to friction in pipes is given by equation (10.2) as

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$$

In this equation, the value of co-efficient of friction,  $f$  should be known accurately for predicting the loss of head due to friction in pipes. On the basis of dimensional analysis, it can be shown that the pressure loss in a straight pipe of diameter  $D$ , length  $L$ , roughness  $k$ , average velocity of flow  $\bar{U}$ , viscosity and density of fluid  $\mu$  and  $\rho$  is

$$\Delta p = \frac{\rho \bar{U}^2}{2} \phi \left[ R_e, \frac{k}{D}, \frac{L}{D} \right] \quad \text{or} \quad \frac{\Delta p}{\rho \bar{U}^2} = \phi \left[ R_e, \frac{k}{D}, \frac{L}{D} \right]$$

Experimentally it was found that pressure drop is a function of  $\frac{L}{D}$  to the first power and hence

$$\frac{\frac{\Delta p}{\rho \bar{U}^2}}{2} = \frac{L}{D} \phi \left[ R_e, \frac{k}{D} \right] \quad \text{or} \quad \frac{\Delta p \times D}{L \rho \bar{U}^2} = \frac{2}{2} \phi \left[ R_e, \frac{k}{D} \right]$$

The term of the right hand side is called co-efficient of friction  $f$ . Thus  $f = \phi \left[ R_e, \frac{k}{D} \right]$

This equation shows that friction co-efficient is a function of Reynolds number and  $k/D$  ratio, where  $k$  is the average height of pipe wall roughness protrusions.

(a) **Variation of ' $f$ ' for Laminar Flow.** In viscous flow chapter, it is shown that co-efficient of friction ' $f$ ' for laminar flow in pipes is given by

$$f = \frac{16}{R_e} \quad \dots(10.29)$$

Thus friction co-efficient is only a function of Reynolds number in case of laminar flow. It is independent of ( $k/D$ ) ratio.

(b) **Variation of ' $f$ ' for Turbulent Flow.** For turbulent flow, the co-efficient of friction is a function of  $R_e$  and  $k/D$  ratio. For relative roughness ( $k/D$ ), in the turbulent flow the boundary may be smooth or rough and hence the value of ' $f$ ' will be different for these boundaries.

(i) **' $f$ ' for smooth pipes.** For turbulent flow in smooth pipes, co-efficient of friction is a function of Reynolds number only. The value of laminar sub-layer in case of smooth pipe is large as compared to the average height of surface roughness  $k$ . The value of ' $f$ ' for smooth pipe for Reynolds number varying from 4000 to 100000 is given by the relation

$$f = \frac{0.791}{(R_e)^{1/4}} \quad \dots(10.30)$$

The equation (10.30) is given by Blasius.

The value of ' $f$ ' for  $R_e > 10^5$  is obtained from equation (10.23) which gives the velocity distribution for smooth pipe in terms of average velocity ( $\bar{U}$ ) as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} \frac{(u_* R)}{v} + 1.75 \quad \dots(10.31)$$

From equation (10.6), we have  $f = \frac{2\tau_0}{\rho V^2}$ , where  $V$  = average velocity

$$\therefore f = \frac{2\tau_0}{\rho \bar{U}^2} = \frac{2}{\bar{U}^2} \left( \sqrt{\frac{\tau_0}{\rho}} \right)^2 = \frac{2}{\bar{U}^2} \times u_*^2 \quad \left[ \because \sqrt{\frac{\tau_0}{\rho}} = u_* \right]$$

$$\therefore u_*^2 = \frac{f \bar{U}^2}{2}$$

$$\text{or} \quad u_* = \bar{U} \sqrt{\frac{f}{2}} \quad \dots(10.31A)$$

Substituting the value of  $u_*$  in equation (10.31), we get

$$\frac{\bar{U}}{\bar{U} \sqrt{f/2}} = 5.75 \log_{10} \left( \frac{\sqrt{f/2}}{v} \right) R + 1.75$$

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or

$$\frac{1}{\sqrt{f/2}} = 5.75 \log_{10} \left( \frac{\bar{U}R}{v} \sqrt{f/2} \right) + 1.75$$

Taking  $R = D/2$  and simplifying, the above equation is written as

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10} \left( \frac{\bar{U}D}{v} \sqrt{4f} \right) - 0.91$$

But  $\frac{\bar{U}D}{v} = R_e$  and hence above equation is written as

$$\frac{1}{\sqrt{4f}} = 2.03 \log_{10} (R_e \sqrt{4f}) - 0.91 \quad \dots(10.32)$$

Equation (10.32) is valid upto  $R_e = 4 \times 10^6$

Nikuradse's experimental result for turbulent flow in smooth pipe for 'f' is

$$\frac{1}{\sqrt{4f}} = 2.0 \log_{10} (R_e \sqrt{4f}) - 0.8 \quad \dots(10.33)$$

This is applicable upto  $R_e = 4 \times 10^7$ . But the equation (10.33) is solved by hit and trial method. The value of 'f' (i.e., co-efficient of friction) can alternately be obtained as

$$f = .0008 + \frac{.05525}{(R_e)^{0.237}} \quad \dots(10.34)$$

The value of 'f' [i.e., friction factor which is used in equation (10.2A)] is given by

$$f = 0.0032 + \frac{0.221}{(R_e)^{0.237}} \quad \dots(10.34A)$$

(ii) **Value of 'f' for rough pipes.** For turbulent flow in rough pipes, the co-efficient of friction is a function of relative roughness ( $k/D$ ) and it is independent of Reynolds number. This is because the value of laminar sub-layer for rough pipes is very small as compared to the height of surface roughness. The average velocity for rough pipes is given by (10.24) as

$$\frac{\bar{U}}{u_*} = 5.75 \log_{10} (R/k) + 4.75$$

But

$$u_* = \bar{U} \sqrt{f/2}$$

Substituting the value of  $u_*$  in the above equation, we get

$$\frac{\bar{U}}{\bar{U} \sqrt{f/2}} = 5.75 \log_{10} (R/k) + 4.75$$

which is simplified to the form as  $\frac{1}{\sqrt{4f}} = 2.03 \log_{10} (R/k) + 1.68 \quad \dots(10.35)$

But Nikuradse's experimental result gave for rough pipe the following relation for 'f' as

$$\frac{1}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74 \quad \dots(10.36)$$

(c) **Value of 'f' for commercial pipes.** The value of 'f' for commercial pipes such as pipes made of metal, concrete and wood is obtained from Nikuradse's experimental data for smooth and rough

pipes. According to Colebrook, by subtracting  $2 \log_{10} (R/k)$  from both sides of equations (10.33) and (10.36), the value of 'f' is obtained for commercial smooth and rough pipes as :

### 1. Smooth pipes

$$\begin{aligned}\frac{1}{\sqrt{4f}} - 2 \log_{10} (R/k) &= 2 \log_{10} (R_e \sqrt{4f}) - 0.8 - 2 \log_{10} (R/k) \\ &= 2 \log_{10} \left( \frac{R_e \sqrt{4f}}{R/k} \right) - 0.8\end{aligned}\quad \dots(10.37)$$

### 2. Rough pipes

$$\begin{aligned}\frac{1}{\sqrt{4f}} - 2 \log_{10} (R/k) &= 2 \log_{10} (R/k) + 1.74 - 2 \log_{10} (R/k) \\ &= 1.74.\end{aligned}\quad \dots(10.38)$$

**Problem 10.7** For the problem 10.6, find the co-efficient of friction and the average height of roughness projections.

**Solution.** From the solution of problem 10.6, we have

$$\begin{aligned}R &= 0.15 \text{ m} \\ u_* &= 0.1458 \text{ m/s} \\ \overline{U} &= 1.4533 \text{ m/s}\end{aligned}$$

For co-efficient of friction, we know that

$$u_* = \overline{U} \sqrt{f/2}$$

or  $0.1458 = 1.4533 \sqrt{f/2}$

or  $\sqrt{f/2} = \frac{0.1458}{1.4533} = 0.1$

$\therefore f = 2.0 \times (.1)^2 = .02. \text{ Ans.}$

Height of roughness projection is obtained from equation (10.36) as

$$\frac{1}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74$$

Substituting the values of  $R$  and  $f$ , we get

$$\frac{1}{\sqrt{4 \times 0.2}} = 2 \log_{10} \left( \frac{0.15}{k} \right) + 1.74 \quad \text{or} \quad 3.5355 = 2 \log_{10} \left( \frac{.15}{k} \right) + 1.74$$

or  $\log_{10} \left( \frac{.15}{k} \right) = \frac{3.5355 - 1.74}{2} = 0.8977 = \log_{10} 7.90$

$\therefore \frac{0.15}{k} = 7.90$

$\therefore k = \frac{0.15}{7.90} = 0.01898 \text{ m} = 18.98 \text{ mm. Ans.}$

**Problem 10.8** Water is flowing through a rough pipe of diameter 500 mm and length 4000 m at the rate of  $0.5 \text{ m}^3/\text{s}$ . Find the power required to maintain this flow. Take the average height of roughness as  $k = 0.40 \text{ mm}$ .

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**Solution.** Given :

$$\text{Dia. of rough pipe, } D = 500 \text{ mm} = 0.50 \text{ m}$$

$$\therefore \text{Radius, } R = \frac{D}{2} = 0.25 \text{ m}$$

$$\text{Length of pipe, } L = 4000 \text{ m}$$

$$\text{Discharge, } Q = 0.5 \text{ m}^3/\text{s}$$

$$\text{Average height of roughness, } k = 0.40 \text{ mm} = 0.4 \times 10^{-3} \text{ m}$$

First find the value of co-efficient of friction. Then calculate the head lost due to friction and then power required.

For a rough pipe, the value of ' $f$ ' is given by the equation (10.36) as

$$\begin{aligned} \frac{1}{\sqrt{4f}} &= 2 \log_{10} (R/k) + 1.74 = 2 \log_{10} \left( \frac{.25}{.4 \times 10^{-3}} \right) + 1.74 \\ &= 2 \log_{10} (625.0) + 1.74 = 5.591 + 1.74 = 7.331 \end{aligned}$$

$$\text{or } \sqrt{4f} = \frac{1}{7.331} = 0.1364 \text{ or } f = (0.1364)^2/4 = .00465$$

$$\text{Also the average velocity, } \bar{U} = \frac{\text{Discharge}}{\text{Area}} = \frac{0.5}{\frac{\pi}{4} D^2} = \frac{0.5}{\frac{\pi}{4} (.5)^2} = 2.546$$

$$\begin{aligned} \therefore \text{Head lost due to friction, } h_f &= \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} = \frac{4 \times .00465 \times 4000 \times 2.546^2}{0.5 \times 2 \times 9.81} \\ &= 49.16 \text{ m} \quad [\because V = \bar{U} = 2.546, d = D = 0.5] \end{aligned}$$

$$\begin{aligned} \therefore \text{Power required, } P &= \frac{W \times h_f}{1000} = \frac{w \cdot Q \cdot h_f}{1000} = \frac{\rho \times g \times Q \times h_f}{1000} \text{ kW} \\ &= \frac{1000 \times 9.81 \times 0.5 \times 49.16}{1000} = \mathbf{241.13 \text{ kW. Ans.}} \end{aligned}$$

**Problem 10.9** A smooth pipe of diameter 400 mm and length 800 m carries water at the rate of  $0.04 \text{ m}^3/\text{s}$ . Determine the head lost due to friction, wall shear stress, centre-line velocity and thickness of laminar sub-layer. Take the kinematic viscosity of water as 0.018 stokes.

**Solution.** Given :

$$\text{Dia. of pipe, } D = 400 \text{ mm} = 0.40 \text{ m}$$

$$\therefore \text{Radius, } R = \frac{D}{2} = 0.20 \text{ m}$$

$$\text{Length of pipe, } L = 800 \text{ m}$$

$$\text{Discharge, } Q = 0.04 \text{ m}^3/\text{s}$$

$$\text{Kinematic viscosity, } v = 0.018 \text{ stokes} = 0.018 \text{ cm}^2/\text{s} = 0.018 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Average velocity, } \bar{U} = \frac{Q}{\text{Area}} = \frac{0.04}{\frac{\pi}{4} (0.4)^2} = 0.3183 \text{ m/s}$$

$$\therefore \text{Reynolds number, } R_e = \frac{V \times D}{v} = \frac{\bar{U} \times D}{v} = \frac{0.3183 \times 0.4}{0.018 \times 10^{-4}} = 7.073 \times 10^4$$

The flow is turbulent.

The co-efficient of friction 'f' is obtained from equation (10.30) as

$$f = \frac{0.0791}{(R_e)^{1/4}} = \frac{0.0791}{(7.073 \times 10^4)^{1/4}} = \frac{0.0791}{16.30} = .00485$$

$$(i) \text{ Head lost due to friction, } h_f = \frac{4 \cdot f \cdot L \cdot V^2}{D \times 2g} = \frac{4 \cdot f \cdot L \cdot \bar{U}^2}{D \times 2g}$$

$$= \frac{4 \times .00485 \times 800 \times (.3183)^2}{0.40 \times 2 \times 9.81} = \mathbf{0.20 \text{ m. Ans.}}$$

(ii) Wall shear stress ( $\tau_0$ ) is given by equation (10.5) as

$$\tau_0 = \frac{f \cdot \rho \cdot V^2}{2} = \frac{f \cdot \rho \cdot \bar{U}^2}{2} \quad [\because V = \bar{U}]$$

$$= 0.00485 \times 1000 \times \frac{(.3184)^2}{2.0} \text{ N/m}^2 = \mathbf{0.245 \text{ N/m}^2. Ans.}$$

(iii) The centre-line velocity ( $u_{\max}$ ) for smooth pipe is given by equation (10.20) as in which  $u = u_{\max}$  at  $y = R$

$$\therefore \frac{u_{\max}}{u_*} = 5.75 \log_{10} \frac{u_* R}{v} + 5.55 \quad [\text{Put in equation (10.20), } u = u_{\max} \text{ at } y = R]$$

where the shear velocity  $u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.245}{1000}} = \sqrt{0.000245} = 0.0156 \text{ m/s}$

Substituting the values of  $u_*$ ,  $R$  and  $v$  in the above equation, we get

$$\frac{u_{\max}}{0.0156} = 5.75 \log_{10} \frac{0.0156 \times 0.20}{0.018 \times 10^{-4}} + 5.55 = 24.173$$

or  $u_{\max} = 24.173 \times 0.0156 = \mathbf{0.377 \text{ m/s. Ans.}}$

(iv) The thickness of laminar sub-layer ( $\delta'$ ) is given by

$$\delta' = \frac{11.6 \times v}{u_*} = \frac{11.6 \times 0.018 \times 10^{-4}}{0.0156} = .001338 \text{ m} = \mathbf{1.338 \text{ mm. Ans.}}$$

**Problem 10.10** A rough pipe of diameter 400 mm and length 1000 m carries water at the rate of  $0.4 \text{ m}^3/\text{s}$ . The wall roughness is 0.012 mm. Determine the co-efficient of friction, wall shear stress, centre-line velocity and velocity at a distance of 150 mm from the pipe wall.

**Solution.** Given :

Dia. of rough pipe,  $D = 400 \text{ mm} = 0.4 \text{ m}$

∴ Radius,  $R = \frac{D}{2} = \frac{0.4}{2} = 0.20 \text{ m}$

Length of pipe,  $L = 1000 \text{ m}$

Discharge,  $Q = 0.4 \text{ m}^3/\text{s}$

Wall roughness,  $k = 0.012 \text{ mm} = 0.012 \times 10^{-3} \text{ m}$

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(i) The value of co-efficient of friction ' $f$ ' for rough pipe is given by the equation (10.36) as

$$\frac{1.0}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74$$

or

$$\begin{aligned}\frac{1.0}{\sqrt{4f}} &= 2 \log_{10} \left( \frac{0.20}{.012 \times 10^{-3}} \right) + 1.74 \\ &= 2 \log_{10} (16666.67) + 1.74 = 10.183 \\ \therefore 4f &= \left( \frac{1}{10.183} \right)^2 = .00964 \\ \therefore f &= \frac{.00964}{4.0} = \mathbf{.00241. Ans.}\end{aligned}$$

(ii) Centre-line velocity ( $u_{\max}$ ) for rough pipe is given by equation (10.21) in which  $u$  is made  $= u_{\max}$  at  $y = R$  and hence

$$\frac{u_{\max}}{u_*} = 5.75 \log_{10} (R/k) + 8.5 \quad \dots(i)$$

where shear velocity,  $u_* = \sqrt{\frac{\tau_0}{\rho}}$

and  $\tau_0$  = wall shear stress  $= \frac{f \cdot \rho \cdot V^2}{2}$

where  $V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{\frac{\pi}{4} D^2} = \frac{Q}{\frac{\pi}{4} (.4)^2} = \mathbf{3.183 \text{ m/s. Ans.}}$

(iii)  $\therefore \tau_0 = \frac{f \cdot \rho \cdot V^2}{2} = .00241 \times 1000 \times \frac{3.183^2}{2.0} = \mathbf{12.2 \text{ N/m}^2. Ans.}$

$\therefore u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{12.2}{1000}} = 0.11 \text{ m/s}$

Substituting the value of  $u_*$ ,  $R$ ,  $k$  in equation (i), we get

$$\frac{u_{\max}}{0.11} = 5.75 \log_{10} \left( \frac{0.2}{.012 \times 10^{-3}} \right) + 8.5 = 32.77$$

$\therefore u_{\max} = 32.77 \times 0.11 = \mathbf{3.60 \text{ m/s. Ans.}}$

(iv) Velocity ( $u$ ) at a distance  $y = 150 \text{ mm} = 0.15 \text{ m}$

The velocity ( $u$ ) at any point for rough pipe is given by equation (10.21) as

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

where  $u_* = 0.11 \text{ m/s}$  and  $y = 0.15 \text{ m}$ ,  $k = 0.012 \times 10^{-3} \text{ m}$

$$\therefore \frac{u}{0.11} = 5.75 \log_{10} \left( \frac{0.15}{0.012 \times 10^{-3}} \right) + 8.5 = 32.05$$

$$\therefore u = 32.05 \times 0.11 = 3.52 \text{ m/s. Ans.}$$

**Problem 10.11** A smooth pipe line of 100 mm diameter carries 2.27 m<sup>3</sup> per minute of water at 20°C with kinematic viscosity of 0.0098 stokes. Calculate the friction factor, maximum velocity as well as shear stress at the boundary.

**Solution.** Given :

$$\text{Dia. of pipe, } D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\therefore \text{Radius of pipe, } R = 0.05 \text{ m}$$

$$\text{Discharge, } Q = 2.27 \text{ m}^3/\text{min} = \frac{2.27}{60} \text{ m}^3/\text{s} = 0.0378 \text{ m}^3/\text{s}$$

$$\text{Kinematic viscosity, } v = 0.0098 \text{ stokes} = 0.0098 \text{ cm}^2/\text{s} = 0.0098 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Now average velocity is given by } \bar{U} = \frac{Q}{\text{Area}} = \frac{0.0378}{\frac{\pi}{4}(0.1)^2} = \frac{0.0378 \times 4}{\pi \times 0.01} = 4.817 \text{ m/s}$$

$$\therefore \text{Reynolds number is given by, } R_e = \frac{\bar{U} \times D}{v} = \frac{4.817 \times 0.1}{0.0098 \times 10^{-4}} = 4.9154 \times 10^5.$$

The flow is turbulent and  $R_e$  is more than  $10^5$ . Hence for smooth pipe, the co-efficient of friction 'f' is obtained from equation (10.33) as

$$\frac{1}{\sqrt{4f}} = 2.0 \log_{10} (R_e \sqrt{4f}) - 0.8$$

or

$$\begin{aligned} \frac{1}{\sqrt{4f}} &= 2.0 \log_{10} (4.9154 \times 10^5 \times \sqrt{4f}) - 0.8 \\ &= 2.0 [\log_{10} 4.9154 \times 10^5 + \log_{10} \sqrt{4f}] - 0.8 \\ &= 2.0 [5.6915 + \log_{10} \sqrt{4f}] - 0.8 = 2 \times 5.6915 + 2 \log_{10} \sqrt{4f} - 0.8 \\ &= 11.3830 + \log_{10} (\sqrt{4f})^2 - 0.8 = 11.383 + \log_{10} (4f) - 0.8 \end{aligned}$$

or

$$\frac{1}{\sqrt{4f}} - \log_{10} (4f) = 11.383 - 0.8 = 10.583 \quad \dots(i)$$

(i) *Friction factor*

Now, friction factor ( $f^*$ ) =  $4 \times$  co-efficient of friction =  $4f$

Substituting the value of '4f' in equation (i), we get

$$\frac{1}{\sqrt{f^*}} - \log_{10} f^* = 10.583 \quad \dots(ii)$$

The above equation is solved by hit and trial method.

Let  $f^* = 0.1$ , then L.H.S. of equation (ii), becomes as

$$\text{L.H.S.} = \frac{1}{\sqrt{0.1}} - \log_{10} 0.1 = 3.16 - (-1.0) = 4.16$$

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Let  $f^* = 0.01$ , then L.H.S. of equation (ii), becomes as

$$\text{L.H.S.} = \frac{1}{\sqrt{0.01}} - \log_{10} 0.01 = 10 - (-2) = 12$$

But for exact solution, L.H.S. should be 10.583. Hence value of  $f^*$  lies between 0.1 and 0.01.

Let  $f^* = 0.013$  then L.H.S. of equation (ii), becomes as

$$\text{L.H.S.} = \frac{1}{\sqrt{0.013}} - \log_{10} 0.013 = 8.77 - (-1.886) = 8.77 + 1.886 = 10.656$$

which is approximately equal to 10.583.

Hence the value of  $f^*$  is equal to 0.013.

$\therefore$  Friction factor,  $f^* = 0.013$ . Ans.

(ii) Maximum velocity ( $u_{max}$ )

Now we know that  $f^* = 4f$

$$\therefore \text{Co-efficient of friction, } f = \frac{f^*}{4} = \frac{0.013}{4} = 0.00325$$

Now the shear velocity ( $u_*$ ) in terms of co-efficient of friction and average velocity is given by equation (10.31A) as

$$u_* = \bar{U} \sqrt{\frac{f}{2}} = 4.817 \times \sqrt{\frac{0.00325}{2}} = 4.817 \times 0.0403 = 0.194$$

For smooth pipe, the velocity at any point is given by equation (10.20)

$$u = u_* \left[ 5.75 \log_{10} \frac{u_* \times y}{v} + 5.55 \right]$$

The velocity will be maximum at the centre of the pipe,

where  $y = R = 0.05$

i.e., radius of pipe. Hence the above equation becomes as

$$\begin{aligned} U_{max} &= u_* \left[ 5.75 \log_{10} \frac{u_* \times R}{v} + 5.55 \right] \\ &= 0.194 \left[ 5.75 \log_{10} \frac{0.194 \times 0.05}{0.0098 \times 10^{-4}} + 5.55 \right] \\ &= 0.194 [22.974 + 5.55] = 5.528 \text{ m/s. Ans.} \end{aligned}$$

(iii) Shear stress at the boundary ( $\tau_0$ )

$$\text{We know that } u_* = \sqrt{\frac{\tau_0}{\rho}} \text{ or } u_*^2 = \frac{\tau_0}{\rho}$$

$$\therefore \tau_0 = \rho u_*^2 = 1000 \times 0.194^2 = 37.63 \text{ N/m}^2. \text{ Ans.}$$

**Problem 10.12** Hydrodynamically smooth pipe carries water at the rate of 300 l/s at 20°C ( $\rho = 1000 \text{ kg/m}^3$ ,  $v = 10^{-6} \text{ m}^2/\text{s}$ ) with a head loss of 3 m in 100 m length of pipe. Determine the pipe diameter. Use  $f = 0.0032 + \frac{0.221}{(R_e)^{0.237}}$  equation for  $f$ , where  $h_f = \frac{f \times L \times V^2}{D \times 2g}$  and  $R_e = \frac{\rho V D}{\mu}$ .

$$f = 0.0032 + \frac{0.221}{(R_e)^{0.237}}$$

**Solution.** Given :

$$\text{Discharge, } Q = 300 \text{ l/s} = 0.3 \text{ m}^3/\text{s}$$

$$\text{Density, } \rho = 1000 \text{ kg/m}^3$$

$$\text{Kinematic viscosity, } v = 10^{-6} \text{ m}^2/\text{s}$$

$$\text{Head loss, } h_f = 3 \text{ m}$$

$$\text{Length of pipe, } L = 100 \text{ m}$$

$$\text{Value of friction factor, } f = 0.0032 + \frac{0.221}{(R_e)^{0.237}}$$

$$\begin{aligned} \text{Reynolds number, } R_e &= \frac{\rho V D}{\mu} = \frac{V \times D}{v} \\ &= \frac{V \times D}{10^{-6}} = V \times D \times 10^6 \end{aligned} \quad \left( \because \frac{\mu}{\rho} = v \right)$$

Find : Diameter of pipe.

Let  $D$  = Diameter of pipe

Head loss in terms of friction factor is given as

$$h_f = \frac{f \times L \times V^2}{D \times 2g}$$

$$\text{or } 3 = \frac{f \times 100 \times V^2}{D \times 2 \times 9.81} \quad (\because h_f = 3, L = 100 \text{ m})$$

$$\text{or } f = \frac{3 \times D \times 2 \times 9.81}{100 V^2} \text{ or } f = \frac{0.5886 D}{V^2} \quad \dots(i)$$

Now

$$Q = A \times V$$

$$\text{or } 0.3 = \frac{\pi}{4} D^2 \times V \text{ or } D^2 \times V = \frac{4 \times 0.3}{\pi} = 0.382$$

$$\therefore V = \frac{0.382}{D^2} \quad \dots(ii)$$

$$\text{Also } f = 0.0032 + \frac{0.221}{(R_e)^{0.237}}$$

$$\begin{aligned} \text{or } \frac{0.5886 D}{V^2} &= 0.0032 + \frac{0.221}{(V \times D \times 10^6)^{0.237}} \\ &\quad \left( \because \text{From equation (i), } f = \frac{0.5886 D}{V^2} \text{ and } R_e = V \times D \times 10^6 \right) \end{aligned}$$

$$\begin{aligned} \text{or } \frac{0.5886 D}{\left(\frac{0.382}{D^2}\right)^2} &= 0.0032 + \frac{0.221}{\left(\frac{0.382}{D^2} \times D \times 10^6\right)^{0.237}} \\ &\quad \left( \because \text{From equation (ii), } V = \frac{0.382}{D^2} \right) \end{aligned}$$

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or

$$\frac{0.5886 \times D^5}{0.382^2} = 0.0032 + \frac{0.221}{\frac{(0.382 \times 10^6)^{0.237}}{D^{0.237}}}$$

or

$$4.033 D^5 = 0.0032 + 0.0105 \times D^{0.237}$$

or

$$4.033 D^5 - 0.0105 D^{0.237} - 0.0032 = 0$$

... (iii)

The above equation (iii) will be solved by hit and trial method.

(i) Assume  $D = 1$  m, then L.H.S. of equation (iii), becomes as

$$\begin{aligned} \text{L.H.S.} &= 4.033 \times 1^5 - 0.0105 \times 1^{0.237} - 0.0032 \\ &= 4.033 - 0.0105 - 0.0032 = 4.0193 \end{aligned}$$

By increasing the value of  $D$  more than 1 m, the L.H.S. will go on increasing. Hence decrease the value of  $D$ .

(ii) Assume  $D = 0.3$  m, then L.H.S. of equation (iii),

$$\begin{aligned} \text{becomes as L.H.S.} &= 4.033 \times 0.3^5 - 0.0105 \times 0.3^{0.237} - 0.0032 \\ &= 0.0098 - 0.00789 - 0.0032 = -0.00129 \end{aligned}$$

As this value is negative, the value of  $D$  will be slightly more than 0.3.

(iii) Assume  $D = 0.306$  m, then L.H.S. of equation (iii), becomes as

$$\begin{aligned} \text{L.H.S.} &= 4.033 \times 0.306^5 - 0.0105 \times 0.306^{0.237} - 0.0032 \\ &= 0.0108 - 0.00793 - 0.0032 = -0.00033 \end{aligned}$$

This value of L.H.S. is approximately equal to zero. Actually the value of  $D$  will be slightly more than 0.306 m say **0.308 m. Ans.**

**Problem 10.13** Water is flowing through a rough pipe of diameter 600 mm at the rate of 600 litres/second. The wall roughness is 3 mm. Find the power lost for 1 km length of pipe.

**Solution.** Given :

$$\text{Dia. of pipe, } D = 600 \text{ mm} = 0.6 \text{ m}$$

$$\therefore \text{Radius of pipe, } R = \frac{0.6}{2} = 0.3 \text{ m}$$

$$\text{Discharge, } Q = 600 \text{ litre/s} = 0.6 \text{ m}^3/\text{s}$$

$$\text{Wall roughness, } k = 3 \text{ mm} = 3 \times 10^{-3} \text{ m} = 0.003 \text{ m}$$

$$\text{Length of pipe, } L = 1 \text{ km} = 1000 \text{ m}$$

For rough pipes, the co-efficient of friction in terms of wall roughness,  $k$  is given by equation (10.36) as

$$\frac{1}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74 = 2 \log_{10} \left( \frac{0.3}{0.003} \right) + 1.74 = 5.74$$

$$\text{or } \sqrt{4f} = \frac{1}{5.74} = 0.1742 \text{ or } 4f = (0.1742)^2 = 0.03035$$

$$\text{The head loss due to friction is given by, } h_f = \frac{4f \times L \times V^2}{D \times 2g}$$

where  $V = \frac{Q}{A} = \frac{0.6}{\frac{\pi}{4}(0.6^2)} = 2.122 \text{ m/s}$

$$h_f = \frac{0.03035 \times 1000 \times 2.122^2}{0.6 \times 2 \times 9.81} = 11.6 \text{ m}$$

The power\* lost is given by,  $P = \frac{\rho g \times Q \times h_f}{1000} = \frac{1000 \times 9.81 \times 0.6 \times 11.6}{1000} \text{ kW} = 68.27 \text{ kW. Ans.}$

## HIGHLIGHTS

- If the Reynold number is less than 2000 in a pipe, the flow is laminar while if the Reynold number is more than 4000, the flow is turbulent in pipes.
- Loss of pressure head in a laminar flow is proportional to the mean velocity of flow, while in case of turbulent flow it is approximately proportional to the square of velocity.
- Expression for head loss due to friction in pipes is given by Darcy-Weisbach equation,

$$\begin{aligned} h_f &= \frac{4 \times f \times L \times V^2}{d \times 2g}, \text{ where } f = \text{co-efficient of friction} \\ &= \frac{f \times L \times V^2}{d \times 2g}, \text{ where } f = \text{friction factor} \end{aligned}$$

- Co-efficient of friction is expressed in terms of shear stress as  $= \frac{2\tau_0}{\rho V^2}$

where  $V$  = mean velocity of flow,  $\rho$  = mass density of fluid.

- Shear stress in turbulent flow is sum of shear stress due to viscosity and shear stress due to turbulence, i.e.,

$$\begin{aligned} \tau &= \tau_v + \tau_t, \quad \text{where} \quad \tau_v = \text{shear stress due to viscosity} \\ &\qquad \qquad \qquad \tau_t = \text{shear stress due to turbulence} \\ &= \mu \frac{d\bar{u}}{dy} + \eta \frac{d\bar{u}}{dy} \end{aligned}$$

- Turbulent shear stress by Reynolds is given as  $\tau = \rho u'v'$   
where  $u'$  and  $v'$  = fluctuating component of velocity.

- The expression for shear stress in turbulent flow due to Prandtl is  $\bar{\tau} = \rho l^2 \left( \frac{du}{dy} \right)^2$ , where  $l$  = mixing length.

- The velocity distribution in the turbulent flow for pipes is given by the expression

$$u = u_{max} + 2.5 u_* \log_e (y/R)$$

where  $u_{max}$  = is the centre-line velocity,  
 $y$  = distance from the pipe wall,  
 $R$  = radius of the pipe,

and  $u_*$  = shear velocity which is equal to  $\sqrt{\frac{\tau_0}{\rho}}$ .

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\* Power =  $\rho g \times Q \times h_f$  watt =  $\frac{\rho g \times Q \times h_f}{1000}$  kW.

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9. Velocity defect is the difference between the maximum velocity ( $u_{\max}$ ) and local velocity ( $u$ ) at any point and is given by  $(u_{\max} - u) = 5.75 \times u_* \log_{10}(R/y)$ .
10. The boundary is known as hydrodynamically smooth if  $k$ , the average height of the irregularities projecting from the surface of the boundary is small compared to the thickness of the laminar sub-layer ( $\delta'$ ) and boundary is rough if  $k$  is large in comparison with the thickness of the sub-layer.

or if  $\frac{k}{\delta'} < 0.25$ , the boundary is smooth ; if  $\frac{k}{\delta'} > 6.0$ , the boundary is rough

and if  $\frac{k}{\delta'}$  lie between 0.25 to 6.0, the boundary is in transition.

11. Velocity distribution for turbulent flow is

$$\begin{aligned}\frac{u}{u_*} &= 5.75 \log_{10} \frac{u_* y}{v} + 5.55 \text{ for smooth pipes} \\ &= 5.75 \log_{10} (y/k) + 8.5 \text{ for rough pipes}\end{aligned}$$

where  $u$  = velocity at any point in the turbulent flow,

$$u_* = \text{shear velocity and } = \sqrt{\frac{\tau_0}{\rho}}, v = \text{kinematic viscosity of fluid},$$

$y$  = distance from pipe wall, and  $k$  = roughness factor.

12. Velocity distribution in terms of average velocity is

$$\begin{aligned}\frac{\bar{U}}{u_*} &= 5.75 \log_{10} \frac{u_* R}{v} + 1.75 \text{ for smooth pipes,} \\ &= 5.75 \log_{10} R/k + 4.75 \text{ for rough pipes.}\end{aligned}$$

13. Difference of local velocity and average velocity for smooth and rough pipes is

$$\frac{u - \bar{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75.$$

14. The co-efficient of friction is given by

$$\begin{aligned}f &= \frac{16}{R_e} \dots \text{for laminar flow,} \\ &= \frac{0.0791}{(R_e)^{1/4}} \text{ for turbulent flow in smooth pipes for } R_e \geq 4000 \text{ by } \leq 10^5 \\ &= .0008 + \frac{.05525}{(R_e)^{257}} \text{ for } R_e \leq 10^5 \text{ but } \geq 4 \times 10^7\end{aligned}$$

$$\frac{1}{\sqrt{4f}} = 2 \log_{10} (R/k) + 1.74 \text{ for rough pipes where } R_e = \text{Reynolds number.}$$

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. What do you understand by turbulent flow ? What factor decides the type of flow in pipes ?
2. (a) Derive an expression for the loss of head due to friction in pipes.  
(b) Derive Darcy-Weisbach equation. (J.N.T.U., Hyderabad, S 2002)
3. Explain the term co-efficient of friction. On what factors does this co-efficient depend ?

4. Obtain an expression for the co-efficient of friction in the terms of shear stress.
5. What do you mean by Prandtl mixing Length Theory ? Find an expression for shear stress due to Prandtl.
6. Derive an expression for Prandtl's universal velocity distribution for turbulent flow in pipes. Why this velocity distribution is called universal ?
7. What is a velocity defect ? Derive an expression for velocity defect in pipes.
8. How would you distinguish between hydrodynamically smooth and rough boundaries ?
9. Obtain an expression for the velocity distribution for turbulent flow in smooth pipes.
10. Show that velocity distribution for turbulent flow through rough pipe is given by

$$\frac{u}{u_*} = 5.75 \log_{10} (y/k) + 8.5$$

where  $u_*$  = shear velocity,  $y$  = distance from pipe wall,  $k$  = roughness factor.

11. Obtain an expression for velocity distribution in terms of average velocity for (a) smooth pipes and (b) rough pipes.
12. Prove that the difference of local velocity and average velocity for turbulent flow through rough or smooth pipes is given by

$$\frac{u - \bar{U}}{u_*} = 5.75 \log_{10} (y/R) + 3.75.$$

13. Obtain an expression for velocity distribution in turbulent flow for (i) smooth pipes and (ii) rough pipes.  
(Delhi University, December, 2002)

### (B) NUMERICAL PROBLEMS

1. A pipe-line carrying water has average height of irregularities projecting from the surface of the boundary of the pipe as 0.20 mm. What type of the boundary is it ? The shear stress development is  $7.848 \text{ N/m}^2$ . Take value of kinematic viscosity for water as 0.01 stokes. [Ans. Boundary is in transition]
2. Determine the average height of the roughness for a rough pipe of diameter 10.0 cm when the velocity at a point 4 cm from wall is 40% more than the velocity at a point 1 cm from pipe wall. [Ans. 0.94 cm]
3. A smooth pipe of diameter 10 cm and 1000 m long carries water at the rate of  $0.70 \text{ m}^2/\text{minute}$ . Calculate the loss of head, wall shearing stress, centre line velocity, velocity and shear stress at 3 cm from pipe wall. Also calculate the thickness of the laminar sub-layer. Take kinematic viscosity of water as 0.015 stokes and value of co-efficient of friction 'f' as

$$f = \frac{0.791}{(R_e)^{1/4}}, \text{ where } R_e = \text{Reynolds number.}$$

[Ans. 20.05 m,  $4.9 \text{ N/m}^2$  ;  $1.774 \text{ m/s}$  ;  $1.65 \text{ m/s}$  ;  $19.62 \text{ N/m}^2$  ; 0.248 mm]

4. The velocities of water through a pipe of diameter 10 cm, are 4 m/s and 3.5 m/s at the centre of the pipe and 2 cm from the pipe centre respectively. Determine the wall shearing stress in the pipe for turbulent flow. [Ans.  $15.66 \text{ kgf/m}^2$ ]
5. For turbulent flow in a pipe of diameter 200 mm, find the discharge when the centre-line velocity is 30 m/s and velocity at a point 80 mm from the centre as measured by pitot-tube is 2.0 m/s. [Ans. 64.9 litres/s]
6. For problem 5, find the co-efficient of friction and the average height of roughness projections. [Ans. 0.029, 25.2 mm]
7. Water is flowing through a rough pipe of diameter 40 cm and length 3000 m at the rate of  $0.4 \text{ m}^3/\text{s}$ . Find the power required to maintain this flow. Take the average height of roughness as  $K = 0.3 \text{ mm}$ . [Ans. 278.5 kN]

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8. A smooth pipe of diameter 300 mm and length 600 m carries water at rate of  $0.04 \text{ m}^2/\text{s}$ . Determine the head lost due to friction, wall shear stress, centre-line velocity and thickness of laminar sub-layer. Take the kinematic viscosity of water as 0.018 stokes. [Ans. 0.588 m,  $0.72 \text{ N/cm}^2$ ,  $0.665 \text{ m/s}$ ,  $0.779 \text{ mm}$ ]
9. A rough pipe of diameter 300 mm and length 800 m carries water at the rate of  $0.4 \text{ m}^2/\text{s}$ . The wall roughness is 0.015 mm. Determine the co-efficient of friction, wall shear stress, centre line velocity and velocity at a distance of 100 mm from the pipe wall.

[Ans.  $f = .00263$ ,  $\tau_0 = 42.08 \text{ N/cm}^2$ ,  $u_{\max} = 6.457 \text{ m/s}$ ,  $u = 6.249 \text{ m/s}$ ]

10. Determine the distance from the centre of the pipe, at which the local velocity is equal to the average velocity for turbulent flow in pipes.

[Ans.  $0.7772 R$ ]

# 11

## CHAPTER

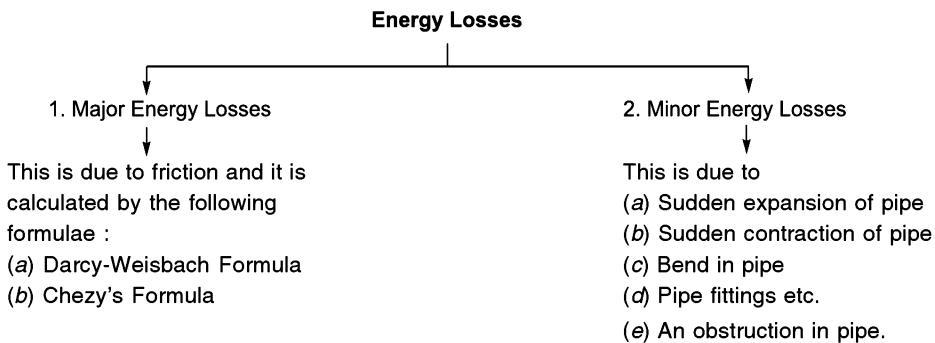


### ► 11.1 INTRODUCTION

In chapters 9 and 10, laminar flow and turbulent flow have been discussed. We have seen that when the Reynolds number is less than 2000 for pipe flow, the flow is known as laminar flow whereas when the Reynolds number is more than 4000, the flow is known as turbulent flow. In this chapter, the turbulent flow of fluids through pipes running full will be considered. If the pipes are partially full as in the case of sewer lines, the pressure inside the pipe is same and equal to atmospheric pressure. Then the flow of fluid in the pipe is not under pressure. This case will be taken in the chapter of flow of water through open channels. Here we will consider flow of fluids through pipes under pressure only.

### ► 11.2 LOSS OF ENERGY IN PIPES

When a fluid is flowing through a pipe, the fluid experiences some resistance due to which some of the energy of fluid is lost. This loss of energy is classified as :



### ► 11.3 LOSS OF ENERGY (OR HEAD) DUE TO FRICTION

(a) **Darcy-Weisbach Formula.** The loss of head (or energy) in pipes due to friction is calculated from Darcy-Weisbach equation which has been derived in chapter 10 and is given by

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} \quad \dots(11.1)$$

where  $h_f$  = loss of head due to friction

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$f$  = co-efficient of friction which is a function of Reynolds number

$$= \frac{16}{R_e} \text{ for } R_e < 2000 \text{ (viscous flow)}$$

$$= \frac{0.079}{R_e^{1/4}} \text{ for } R_e \text{ varying from 4000 to } 10^6$$

$L$  = length of pipe,

$V$  = mean velocity of flow,

$d$  = diameter of pipe.

(b) **Chezy's Formula for loss of head due to friction in pipes.** Refer to chapter 10 article 10.3.1 in which expression for loss of head due to friction in pipes is derived. Equation (iii) of article 10.3.1, is

$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2 \quad \dots(11.2)$$

where  $h_f$  = loss of head due to friction,  $P$  = wetted perimeter of pipe,

$A$  = area of cross-section of pipe,  $L$  = length of pipe,

and  $V$  = mean velocity of flow.

Now the ratio of  $\frac{A}{P} \left( = \frac{\text{Area of flow}}{\text{Perimeter (wetted)}} \right)$  is called hydraulic mean depth or hydraulic radius and

is denoted by  $m$ .

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{\frac{\pi}{4}d^2}{\pi d} = \frac{d}{4}$$

Substituting  $\frac{A}{P} = m$  or  $\frac{P}{A} = \frac{1}{m}$  in equation (11.2), we get

$$h_f = \frac{f'}{\rho g} \times L \times V^2 \times \frac{1}{m} \text{ or } V^2 = h_f \times \frac{\rho g}{f'} \times m \times \frac{1}{L} = \frac{\rho g}{f'} \times m \times \frac{h_f}{L}$$

$$\therefore V = \sqrt{\frac{\rho g}{f'} \times m \times \frac{h_f}{L}} = \sqrt{\frac{\rho g}{f'}} \sqrt{m \frac{h_f}{L}} \quad \dots(11.3)$$

Let  $\sqrt{\frac{\rho g}{f'}} = C$ , where  $C$  is a constant known as Chezy's constant and  $\frac{h_f}{L} = i$ , where  $i$  is loss of head per unit length of pipe.

Substituting the values of  $\sqrt{\frac{\rho g}{f'}}$  and  $\sqrt{\frac{h_f}{L}}$  in equation (11.3), we get

$$V = C \sqrt{mi} \quad \dots(11.4)$$

Equation (11.4) is known as Chezy's formula. Thus the loss of head due to friction in pipe from Chezy's formula can be obtained if the velocity of flow through pipe and also the value of  $C$  is known. The value of  $m$  for pipe is always equal to  $d/4$ .

**Problem 11.1** Find the head lost due to friction in a pipe of diameter 300 mm and length 50 m, through which water is flowing at a velocity of 3 m/s using (i) Darcy formula, (ii) Chezy's formula for which  $C = 60$ .

Take  $\nu$  for water = 0.01 stoke.

**Solution.** Given :

Dia. of pipe,	$d = 300 \text{ mm} = 0.30 \text{ m}$
Length of pipe,	$L = 50 \text{ m}$
Velocity of flow,	$V = 3 \text{ m/s}$
Chezy's constant,	$C = 60$
Kinematic viscosity,	$\nu = 0.01 \text{ stoke} = 0.01 \text{ cm}^2/\text{s}$ $= 0.01 \times 10^{-4} \text{ m}^2/\text{s}$ .

(i) **Darcy Formula** is given by equation (11.1) as

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$$

where 'f' = co-efficient of friction is a function of Reynolds number,  $R_e$

$$\text{But } R_e \text{ is given by} \quad R_e = \frac{V \times d}{\nu} = \frac{3.0 \times 0.30}{0.01 \times 10^{-4}} = 9 \times 10^5$$

$$\therefore \text{Value of } f = \frac{0.079}{R_e^{1/4}} = \frac{0.079}{(9 \times 10^5)^{1/4}} = .00256$$

$$\therefore \text{Head lost, } h_f = \frac{4 \times .00256 \times 50 \times 3^2}{0.3 \times 2.0 \times 9.81} = .7828 \text{ m. Ans.}$$

(ii) **Chezy's Formula.** Using equation (11.4)

$$V = C \sqrt{mi}$$

$$\text{where } C = 60, m = \frac{d}{4} = \frac{0.30}{4} = 0.075 \text{ m}$$

$$\therefore 3 = 60 \sqrt{0.075 \times i} \text{ or } i = \left( \frac{3}{60} \right)^2 \times \frac{1}{0.075} = 0.0333$$

$$\text{But } i = \frac{h_f}{L} = \frac{h_f}{50}$$

$$\text{Equating the two values of } i, \text{ we have } \frac{h_f}{50} = .0333$$

$$\therefore h_f = 50 \times .0333 = 1.665 \text{ m. Ans.}$$

**Problem 11.2** Find the diameter of a pipe of length 2000 m when the rate of flow of water through the pipe is 200 litres/s and the head lost due to friction is 4 m. Take the value of  $C = 50$  in Chezy's formulae.

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**Solution.** Given :

Length of pipe,

$$L = 2000 \text{ m}$$

Discharge,

$$Q = 200 \text{ litre/s} = 0.2 \text{ m}^3/\text{s}$$

Head lost due to friction,  $h_f = 4 \text{ m}$

Value of Chezy's constant,  $C = 50$

Let the diameter of pipe =  $d$

$$\text{Velocity of flow, } V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{\frac{\pi}{4} d^2} = \frac{0.2}{\frac{\pi}{4} d^2} = \frac{0.2 \times 4}{\pi d^2}$$

$$\text{Hydraulic mean depth, } m = \frac{d}{4}$$

$$\text{Loss of head per unit length, } i = \frac{h_f}{L} = \frac{4}{2000} = .002$$

Chezy's formula is given by equation (11.4) as  $V = C \sqrt{mi}$

Substituting the values of  $V$ ,  $m$ ,  $i$  and  $C$ , we get

$$\frac{0.2 \times 4}{\pi d^2} = 50 \sqrt{\frac{d}{4} \times .002} \text{ or } \sqrt{\frac{d}{4} \times .002} = \frac{0.2 \times 4}{\pi d^2 \times 50} = \frac{.00509}{d^2}$$

$$\text{Squaring both sides, } \frac{d}{4} \times .002 = \frac{.00509^2}{d^4} = \frac{.0000259}{d^4} \text{ or } d^5 = \frac{4 \times .0000259}{.002} = 0.0518$$

$$\therefore d = \sqrt[5]{0.0518} = (0.0518)^{1/5} = 0.553 \text{ m} = 553 \text{ mm. Ans.}$$

**Problem 11.3** A crude oil of kinematic viscosity 0.4 stoke is flowing through a pipe of diameter 300 mm at the rate of 300 litres per sec. Find the head lost due to friction for a length of 50 m of the pipe.

**Solution.** Given :

Kinematic viscosity,  $v = 0.4 \text{ stoke} = 0.4 \text{ cm}^2/\text{s} = .4 \times 10^{-4} \text{ m}^2/\text{s}$

Dia. of pipe,  $d = 300 \text{ mm} = 0.30 \text{ m}$

Discharge,  $Q = 300 \text{ litres/s} = 0.3 \text{ m}^3/\text{s}$

Length of pipe,  $L = 50 \text{ m}$

$$\text{Velocity of flow, } V = \frac{Q}{\text{Area}} = \frac{0.3}{\frac{\pi}{4}(0.3)^2} = 4.24 \text{ m/s}$$

$$\therefore \text{Reynolds number, } R_e = \frac{V \times d}{v} = \frac{4.24 \times 0.30}{0.4 \times 10^{-4}} = 3.18 \times 10^4$$

As  $R_e$  lies between 4000 and 100000, the value of  $f$  is given by

$$f = \frac{.079}{(R_e)^{1/4}} = \frac{.079}{(3.18 \times 10^4)^{1/4}} = .00591$$

$$\therefore \text{Head lost due to friction, } h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} = \frac{4 \times 0.00591 \times 50 \times 4.24^2}{0.3 \times 2 \times 9.81} = 3.61 \text{ m. Ans.}$$

**Problem 11.4** An oil of sp. gr. 0.7 is flowing through a pipe of diameter 300 mm at the rate of 500 litres/s. Find the head lost due to friction and power required to maintain the flow for a length of 1000 m. Take  $\nu = .29$  stokes.

**Solution.** Given :

Sp. gr. of oil,

$$S = 0.7$$

Dia. of pipe,

$$d = 300 \text{ mm} = 0.3 \text{ m}$$

Discharge,

$$Q = 500 \text{ litres/s} = 0.5 \text{ m}^3/\text{s}$$

Length of pipe,

$$L = 1000 \text{ m}$$

Velocity,

$$V = \frac{Q}{\text{Area}} = \frac{0.5}{\frac{\pi}{4} d^2} = \frac{0.5 \times 4}{\pi \times 0.3^2} = 7.073 \text{ m/s}$$

$$\therefore \text{Reynolds number, } R_e = \frac{V \times d}{\nu} = \frac{7.073 \times 0.3}{0.29 \times 10^{-4}} = 7.316 \times (10)^4$$

$$\therefore \text{Co-efficient of friction, } f = \frac{0.79}{R_e^{1/4}} = \frac{0.79}{(7.316 \times 10^4)^{1/4}} = .0048$$

$$\therefore \text{Head lost due to friction, } h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .0048 \times 1000 \times 7.073^2}{0.3 \times 2 \times 9.81} = 163.18 \text{ m}$$

$$\text{Power required} = \frac{\rho g \cdot Q \cdot h_f}{1000} \text{ kW}$$

where  $\rho$  = density of oil =  $0.7 \times 1000 = 700 \text{ kg/m}^3$

$$\therefore \text{Power required} = \frac{700 \times 9.81 \times 0.5 \times 163.18}{1000} = 560.28 \text{ kW. Ans.}$$

**Problem 11.5** Calculate the discharge through a pipe of diameter 200 mm when the difference of pressure head between the two ends of a pipe 500 m apart is 4 m of water. Take the value of ' $f$ ' = 0.009 in the formula  $h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$ .

**Solution.** Given :

Dia. of pipe,

$$d = 200 \text{ mm} = 0.20 \text{ m}$$

Length of pipe,

$$L = 500 \text{ m}$$

Difference of pressure head,

$$h_f = 4 \text{ m of water}$$

$$f = .009$$

$$\text{Using equation (11.1), we have } h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$$

$$\text{or } 4.0 = \frac{4 \times .009 \times 500 \times V^2}{0.2 \times 2 \times 9.81} \text{ or } V^2 = \frac{4.0 \times 0.2 \times 2 \times 9.81}{4.0 \times .009 \times 500} = 0.872$$

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$$\therefore V = \sqrt{0.872} = 0.9338 \approx 0.934 \text{ m/s}$$

$\therefore$  Discharge,  $Q = \text{velocity} \times \text{area}$

$$= 0.934 \times \frac{\pi}{4} d^2 = 0.934 \times \frac{\pi}{4} (0.2)^2$$

$$= 0.0293 \text{ m}^3/\text{s} = 29.3 \text{ litres/s. Ans.}$$

**Problem 11.6** Water is flowing through a pipe of diameter 200 mm with a velocity of 3 m/s. Find the head lost due to friction for a length of 5 m if the co-efficient of friction is given by  $f = 0.02 + \frac{0.9}{R_e^{0.3}}$ , where  $R_e$  is Reynolds number. The kinematic viscosity of water = .01 stoke.

**Solution.** Given :

$$\text{Dia. of pipe, } d = 200 \text{ mm} = 0.20 \text{ m}$$

$$\text{Velocity, } V = 3 \text{ m/s}$$

$$\text{Length, } L = 5 \text{ m}$$

$$\text{Kinematic viscosity, } v = 0.01 \text{ stoke} = .01 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\therefore \text{Reynolds number, } R_e = \frac{V \times d}{v} = \frac{3 \times 0.20}{.01 \times 10^{-4}} = 6 \times 10^5$$

$$\begin{aligned} \text{Value of } f &= .02 + \frac{0.9}{R_e^{0.3}} = .02 + \frac{0.9}{(6 \times 10^5)^{0.3}} = .02 + \frac{0.09}{54.13} \\ &= .02 + .00166 = 0.02166 \end{aligned}$$

$$\therefore \text{Head lost due to friction, } h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4.0 \times 0.02166 \times 5.0 \times 3^2}{0.20 \times 2.0 \times 9.81} \\ = 0.993 \text{ m of water. Ans.}$$

**Problem 11.7** An oil of sp. gr. 0.9 and viscosity 0.06 poise is flowing through a pipe of diameter 200 mm at the rate of 60 litres/s. Find the head lost due to friction for a 500 m length of pipe. Find the power required to maintain this flow.

**Solution.** Given :

$$\text{Sp. gr. of oil} = 0.9$$

$$\text{Viscosity, } \mu = 0.06 \text{ poise} = \frac{0.06}{10} \text{ Ns/m}^2$$

$$\text{Dia. of pipe, } d = 200 \text{ mm} = 0.2 \text{ m}$$

$$\text{Discharge, } Q = 60 \text{ litres/s} = 0.06 \text{ m}^3/\text{s}$$

$$\text{Length, } L = 500 \text{ m}$$

$$\text{Density, } \rho = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

$$\therefore \text{Reynolds number, } R_e = \frac{\rho V d}{\mu} = 900 \times \frac{V \times 0.2}{0.06} \\ = 900 \times \frac{60 \times 0.2}{0.06} = 120000$$

$$\text{where } V = \frac{Q}{\text{Area}} = \frac{0.06}{\frac{\pi}{4} d^2} = \frac{0.06}{\frac{\pi}{4} (0.2)^2} = 1.909 \text{ m/s} \approx 1.91 \text{ m/s}$$

$$\therefore R_e = 900 \times \frac{1.91 \times 0.2 \times 10}{0.06} = 57300$$

As  $R_e$  lies between 4000 and  $10^5$ , the value of co-efficient of friction,  $f$  is given by

$$f = \frac{0.079}{R_e^{0.25}} = \frac{0.079}{(57300)^{0.25}} = .0051$$

$$\therefore \text{Head lost due to friction, } h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .0051 \times 500 \times 1.91^2}{0.2 \times 2 \times 9.81}$$

$$= 9.48 \text{ m of water. Ans.}$$

$$\therefore \text{Power required} = \frac{\rho g \cdot Q \cdot h_f}{1000} = \frac{900 \times 9.81 \times 0.06 \times 9.48}{1000} = 5.02 \text{ kW. Ans.}$$

#### ► 11.4 MINOR ENERGY (HEAD) LOSSES

The loss of head or energy due to friction in a pipe is known as major loss while the loss of energy due to change of velocity of the following fluid in magnitude or direction is called minor loss of energy. The minor loss of energy (or head) includes the following cases :

1. Loss of head due to sudden enlargement,
2. Loss of head due to sudden contraction,
3. Loss of head at the entrance of a pipe,
4. Loss of head at the exit of a pipe,
5. Loss of head due to an obstruction in a pipe,
6. Loss of head due to bend in the pipe,
7. Loss of head in various pipe fittings.

In case of long pipe the above losses are small as compared with the loss of head due to friction and hence they are called minor losses and even may be neglected without serious error. But in case of a short pipe, these losses are comparable with the loss of head due to friction.

**11.4.1 Loss of Head Due to Sudden Enlargement.** Consider a liquid flowing through a pipe which has sudden enlargement as shown in Fig. 11.1. Consider two sections (1)-(1) and (2)-(2) before and after the enlargement.

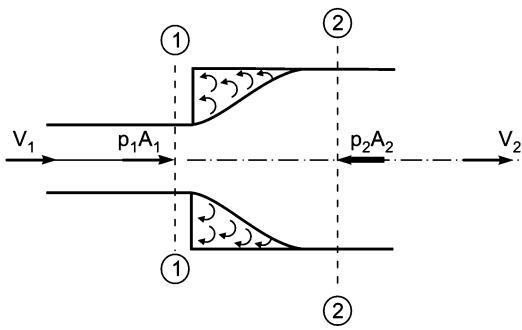


Fig. 11.1 Sudden enlargement.

Let  $p_1$  = pressure intensity at section 1-1,

$V_1$  = velocity of flow at section 1-1,

$A_1$  = area of pipe at section 1-1,

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$p_2$ ,  $V_2$  and  $A_2$  = corresponding values at section 2-2.

Due to sudden change of diameter of the pipe from  $D_1$  to  $D_2$ , the liquid flowing from the smaller pipe is not able to follow the abrupt change of the boundary. Thus the flow separates from the boundary and turbulent eddies are formed as shown in Fig. 11.1. The loss of head (or energy) takes place due to the formation of these eddies.

Let  $p'$  = pressure intensity of the liquid eddies on the area ( $A_2 - A_1$ )

$h_e$  = loss of head due to sudden enlargement

Applying Bernoulli's equation at sections 1-1 and 2-2,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{loss of head due to sudden enlargement}$$

But  $z_1 = z_2$  as pipe is horizontal

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_e$$

$$\text{or } h_e = \left( \frac{p_1}{\rho g} - \frac{p_2}{\rho g} \right) + \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right) \quad \dots(i)$$

Consider the control volume of liquid between sections 1-1 and 2-2. Then the force acting on the liquid in the control volume in the direction of flow is given by

$$F_x = p_1 A_1 + p'(A_2 - A_1) - p_2 A_2$$

But experimentally it is found that  $p' = p_1$

$$\therefore F_x = p_1 A_1 + p_1 (A_2 - A_1) - p_2 A_2 = p_1 A_2 - p_2 A_2 \\ = (p_1 - p_2) A_2 \quad \dots(ii)$$

Momentum of liquid/sec at section 1-1 = mass  $\times$  velocity

$$= \rho A_1 V_1 \times V_1 = \rho A_1 V_1^2$$

Momentum of liquid/sec at section 2-2 =  $\rho A_2 V_2 \times V_2 = \rho A_2 V_2^2$

$$\therefore \text{Change of momentum/sec} = \rho A_2 V_2^2 - \rho A_1 V_1^2$$

But from continuity equation, we have

$$A_1 V_1 = A_2 V_2 \text{ or } A_1 = \frac{A_2 V_2}{V_1}$$

$$\therefore \text{Change of momentum/sec} = \rho A_2 V_2^2 - \rho \times \frac{A_2 V_2}{V_1} \times V_1^2 = \rho A_2 V_2^2 - \rho A_2 V_1 V_2 \\ = \rho A_2 [V_2^2 - V_1 V_2] \quad \dots(iii)$$

Now net force acting on the control volume in the direction of flow must be equal to the rate of change of momentum or change of momentum per second. Hence equating (ii) and (iii)

$$(p_1 - p_2) A_2 = \rho A_2 [V_2^2 - V_1 V_2]$$

$$\text{or } \frac{p_1 - p_2}{\rho} = V_2^2 - V_1 V_2$$

$$\text{Dividing by } g \text{ on both sides, we have } \frac{p_1 - p_2}{\rho g} = \frac{V_2^2 - V_1 V_2}{g} \text{ or } \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = \frac{V_2^2 - V_1 V_2}{g}$$

Substituting the value of  $\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right)$  in equation (i), we get

$$\begin{aligned} h_e &= \frac{V_2^2 - V_1 V_2}{g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = \frac{2V_2^2 - 2V_1 V_2 + V_1^2 - V_2^2}{2g} \\ &= \frac{V_2^2 + V_1^2 - 2V_1 V_2}{2g} = \left(\frac{V_1 - V_2}{2g}\right)^2 \\ \therefore h_e &= \frac{(V_1 - V_2)^2}{2g}. \end{aligned} \quad \dots(11.5)$$

**11.4.2 Loss of Head due to Sudden Contraction.** Consider a liquid flowing in a pipe which has a sudden contraction in area as shown in Fig. 11.2. Consider two sections 1-1 and 2-2 before and after contraction. As the liquid flows from large pipe to smaller pipe, the area of flow goes on decreasing and becomes minimum at a section C-C as shown in Fig. 11.2. This section C-C is called Vena-contracta. After section C-C, a sudden enlargement of the area takes place. The loss of head due to sudden contraction is actually due to sudden enlargement from Vena-contracta to smaller pipe.

Let  $A_c$  = Area of flow at section C-C

$V_c$  = Velocity of flow at section C-C

$A_2$  = Area of flow at section 2-2

$V_2$  = Velocity of flow at section 2-2

$h_c$  = Loss of head due to sudden contraction.

Now  $h_c$  = actual loss of head due to enlargement from section C-C to section 2-2 and is given by equation (11.5) as

$$= \frac{(V_c - V_2)^2}{2g} = \frac{V_2^2}{2g} \left[ \frac{V_c}{V_2} - 1 \right]^2 \quad \dots(i)$$

From continuity equation, we have

$$A_c V_c = A_2 V_2 \quad \text{or} \quad \frac{V_c}{V_2} = \frac{A_2}{A_c} = \frac{1}{(A_c / A_2)} = \frac{1}{C_c} \quad \left[ \because C_c = \frac{A_c}{A_2} \right]$$

Substituting the value of  $\frac{V_c}{V_2}$  in (i), we get

$$h_c = \frac{V_2^2}{2g} \left[ \frac{1}{C_c} - 1 \right]^2 \quad \dots(11.6)$$

$$= \frac{k V_2^2}{2g}, \text{ where } k = \left[ \frac{1}{C_c} - 1 \right]^2$$

If the value of  $C_c$  is assumed to be equal to 0.62, then

$$k = \left[ \frac{1}{0.62} - 1 \right]^2 = 0.375$$

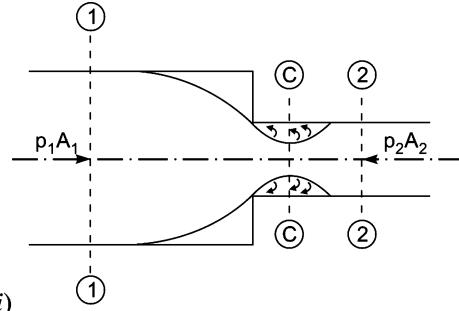


Fig. 11.2 Sudden contraction.

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Then  $h_c$  becomes as  $h_c = \frac{kV_2^2}{2g} = 0.375 \frac{V_2^2}{2g}$

If the value of  $C_c$  is not given then the head loss due to contraction is taken as

$$= 0.5 \frac{V_2^2}{2g} \text{ or } h_c = 0.5 \frac{V_2^2}{2g}. \quad \dots(11.7)$$

**Problem 11.8** Find the loss of head when a pipe of diameter 200 mm is suddenly enlarged to a diameter of 400 mm. The rate of flow of water through the pipe is 250 litres/s.

**Solution.** Given :

Dia. of smaller pipe,  $D_1 = 200 \text{ mm} = 0.20 \text{ m}$

$$\therefore \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (0.2)^2 = 0.03141 \text{ m}^2$$

Dia. of large pipe,  $D_2 = 400 \text{ mm} = 0.4 \text{ m}$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} \times (0.4)^2 = 0.12564 \text{ m}^2$$

Discharge,  $Q = 250 \text{ litres/s} = 0.25 \text{ m}^3/\text{s}$

$$\text{Velocity, } V_1 = \frac{Q}{A_1} = \frac{0.25}{0.03141} = 7.96 \text{ m/s}$$

$$\text{Velocity, } V_2 = \frac{Q}{A_2} = \frac{0.25}{0.12564} = 1.99 \text{ m/s}$$

Loss of head due to enlargement is given by equation (11.5) as

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2g} = 1.816 \text{ m of water. Ans.}$$

**Problem 11.9** At a sudden enlargement of a water main from 240 mm to 480 mm diameter, the hydraulic gradient rises by 10 mm. Estimate the rate of flow. (J.N.T.U., S 2002)

**Solution.** Given :

Dia. of smaller pipe,  $D_1 = 240 \text{ mm} = 0.24 \text{ m}$

$$\therefore \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (0.24)^2$$

Dia. of large pipe,  $D_2 = 480 \text{ mm} = 0.48 \text{ m}$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (0.48)^2$$

$$\text{Rise of hydraulic gradient*, i.e., } \left( z_2 + \frac{p_2}{\rho g} \right) - \left( \frac{p_1}{\rho g} + z_1 \right) = 10 \text{ mm} = \frac{10}{1000} = \frac{1}{100} \text{ m}$$

Let the rate of flow =  $Q$

Applying Bernoulli's equation to both sections, i.e., smaller pipe section, and large pipe section.

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{Head loss due to enlargement} \quad \dots(i)$$

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\* Please refer Art. 11.5.1.

But head loss due to enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g} \quad \dots(ii)$$

From continuity equation, we have  $A_1 V_1 = A_2 V_2$

$$\therefore V_1 = \frac{A_2 V_2}{A_1} = \frac{\frac{\pi}{4} D_2^2 \times V_2}{\frac{\pi}{4} D_1^2} = \left( \frac{D_2}{D_1} \right)^2 \times V_2 = \left( \frac{.48}{.24} \right)^2 \times V_2 = 2^2 \times V_2 = 4V_2$$

Substituting this value in (ii), we get

$$h_e = \frac{(4V_2 - V_2)^2}{2g} = \frac{(3V_2)^2}{2g} = \frac{9V_2^2}{2g}$$

Now substituting the value of  $h_e$  and  $V_1$  in equation (i),

$$\frac{p_1}{\rho g} + \frac{(4V_2)^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \frac{9V_2^2}{2g}$$

$$\text{or } \frac{16V_2^2}{2g} - \frac{V_2^2}{2g} - \frac{9V_2^2}{2g} = \left( \frac{p_2}{\rho g} + z_2 \right) - \left( \frac{p_1}{\rho g} + z_1 \right)$$

$$\text{But hydraulic gradient rise} = \left( \frac{p_2}{\rho g} + z_2 \right) - \left( \frac{p_1}{\rho g} + z_1 \right) = \frac{1}{100}$$

$$\therefore \frac{16V_2^2}{2g} - \frac{V_2^2}{2g} - \frac{9V_2^2}{2g} = \frac{1}{100} \text{ or } \frac{6V_2^2}{2g} = \frac{1}{100}$$

$$\therefore V_2 = \sqrt{\frac{2 \times 9.81}{6 \times 100}} = 0.1808 \approx 0.181 \text{ m/s}$$

$$\therefore \text{Discharge, } Q = A_2 \times V_2$$

$$= \frac{\pi}{4} D_2^2 \times V_2 = \frac{\pi}{4} (.48)^2 \times .181 = 0.03275 \text{ m}^3/\text{s}$$

$$= 32.75 \text{ litres/s. Ans.}$$

**Problem 11.10** The rate of flow of water through a horizontal pipe is  $0.25 \text{ m}^3/\text{s}$ . The diameter of the pipe which is  $200 \text{ mm}$  is suddenly enlarged to  $400 \text{ mm}$ . The pressure intensity in the smaller pipe is  $11.772 \text{ N/cm}^2$ . Determine :

- (i) loss of head due to sudden enlargement, (ii) pressure intensity in the large pipe,
- (iii) power lost due to enlargement.

**Solution.** Given :

$$\text{Discharge, } Q = 0.25 \text{ m}^3/\text{s}$$

$$\text{Dia. of smaller pipe, } D_1 = 200 \text{ mm} = 0.20 \text{ m}$$

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$$\therefore \text{Area, } A_1 = \frac{\pi}{4} (0.2)^2 = 0.03141 \text{ m}^2$$

Dia. of large pipe,  $D_2 = 400 \text{ mm} = 0.40 \text{ m}$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (0.4)^2 = 0.12566 \text{ m}^2$$

Pressure in smaller pipe,  $p_1 = 11.772 \text{ N/cm}^2 = 11.772 \times 10^4 \text{ N/m}^2$

$$\text{Now velocity, } V_1 = \frac{Q}{A_1} = \frac{0.25}{0.03141} = 7.96 \text{ m/s}$$

$$\text{Velocity, } V_2 = \frac{Q}{A_2} = \frac{0.25}{0.12566} = 1.99 \text{ m/s}$$

(i) Loss of head due to sudden enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2 \times 9.81} = 1.816 \text{ m. Ans.}$$

(ii) Let the pressure intensity in large pipe =  $p_2$ .

Then applying Bernoulli's equation before and after the sudden enlargement,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_e$$

But

$$z_1 = z_2 \quad (\text{Given horizontal pipe})$$

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_e \text{ or } \frac{p_2}{\rho g} = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - h_e$$

$$= \frac{11.772 \times 10^4}{1000 \times 9.81} + \frac{7.96^2}{2 \times 9.81} - \frac{1.99^2}{2 \times 9.81} - 1.816$$

$$= 12.0 + 3.229 - 0.2018 - 1.8160$$

$$= 15.229 - 2.0178 = 13.21 \text{ m of water}$$

$$\therefore p_2 = 13.21 \times \rho g = 13.21 \times 1000 \times 9.81 \text{ N/m}^2$$

$$= 13.21 \times 1000 \times 9.81 \times 10^{-4} \text{ N/cm}^2 = 12.96 \text{ N/cm}^2. \text{ Ans.}$$

(iii) Power lost due to sudden enlargement,

$$P = \frac{\rho g \cdot Q \cdot h_e}{1000} = \frac{1000 \times 9.81 \times 0.25 \times 1.816}{1000} = 4.453 \text{ kW. Ans.}$$

**Problem 11.11** A horizontal pipe of diameter 500 mm is suddenly contracted to a diameter of 250 mm. The pressure intensities in the large and smaller pipe is given as 13.734 N/cm<sup>2</sup> and 11.772 N/cm<sup>2</sup> respectively. Find the loss of head due to contraction if  $C_c = 0.62$ . Also determine the rate of flow of water.

**Solution.** Given :

Dia. of large pipe,  $D_1 = 500 \text{ mm} = 0.5 \text{ m}$

$$\text{Area, } A_1 = \frac{\pi}{4} (0.5)^2 = 0.1963 \text{ m}^2$$

Dia. of smaller pipe,  $D_2 = 250 \text{ mm} = 0.25 \text{ m}$

∴ Area,  $A_2 = \frac{\pi}{4} (0.25)^2 = 0.04908 \text{ m}^2$

Pressure in large pipe,  $p_1 = 13.734 \text{ N/cm}^2 = 13.734 \times 10^4 \text{ N/m}^2$

Pressure in smaller pipe,  $p_2 = 11.772 \text{ N/cm}^2 = 11.772 \times 10^4 \text{ N/m}^2$

$C_c = 0.62$

$$\text{Head lost due to contraction} = \frac{V_2^2}{2g} \left[ \frac{1}{C_c} - 1.0 \right]^2 = \frac{V_2^2}{2g} \left[ \frac{1}{0.62} - 1.0 \right]^2 = 0.375 \frac{V_2^2}{2g}$$

From continuity equation, we have  $A_1 V_1 = A_2 V_2$

or  $V_1 = \frac{A_2 V_2}{A_1} = \frac{\frac{\pi}{4} D_2^2 \times V_2}{\frac{\pi}{4} D_1^2} = \left( \frac{D_2}{D_1} \right)^2 \times V_2 = \left( \frac{0.25}{0.50} \right)^2 V_2 = \frac{V_2}{4}$

Applying Bernoulli's equation before and after contraction,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_c$$

But  $z_1 = z_2$  (pipe is horizontal)

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c$$

But  $h_c = 0.375 \frac{V_2^2}{2g}$  and  $V_1 = \frac{V_2}{4}$

Substituting these values in the above equation, we get

$$\frac{13.734 \times 10^4}{9.81 \times 1000} + \frac{\left(\frac{V_2}{4}\right)^2}{2g} = \frac{11.772 \times 10^4}{1000 \times 9.81} + \frac{V_2^2}{2g} + 0.375 \frac{V_2^2}{2g}$$

or  $14.0 + \frac{V_2^2}{16 \times 2g} = 12.0 + 1.375 \frac{V_2^2}{2g}$

or  $14 - 12 = 1.375 \frac{V_2^2}{2g} - \frac{1}{16} \frac{V_2^2}{2g} = 1.3125 \frac{V_2^2}{2g}$

or  $2.0 = 1.3125 \times \frac{V_2^2}{2g}$  or  $V_2 = \sqrt{\frac{2.0 \times 2 \times 9.81}{1.3125}} = 5.467 \text{ m/s.}$

(i) Loss of head due to contraction,  $h_c = 0.375 \frac{V_2^2}{2g} = \frac{0.375 \times (5.467)^2}{2 \times 9.81} = 0.571 \text{ m. Ans.}$

(ii) Rate of flow of water,  $Q = A_2 V_2 = 0.04908 \times 5.467 = 0.2683 \text{ m}^3/\text{s} = 268.3 \text{ lit/s. Ans.}$

**Problem 11.12** If in the problem 11.11, the rate of flow of water is 300 litres/s, other data remaining the same, find the value of co-efficient of contraction,  $C_c$ .

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**Solution.** Given :

$$D_1 = 0.5 \text{ m}, D_2 = 0.25 \text{ m}, p_1 = 13.734 \times 10^4 \text{ N/m}^2,$$

$$p_2 = 11.772 \times 10^4 \text{ N/m}^2, Q = 300 \text{ lit/s} = 0.3 \text{ m}^3/\text{s}$$

Also from Problem 11.11,  $V_1 = \frac{V_2}{4}$ , where  $V_1 = \frac{Q}{A_1} = \frac{0.30}{\frac{\pi}{4}(0.5)^2} = 1.528 \text{ m/s}$

$$\therefore V_2 = 4 \times V_1 = 4 \times 1.528 = 6.112 \text{ m/s}$$

From Bernoulli's equation, we have

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c. \quad [\because z_1 = z_2]$$

or  $\frac{13.734 \times 10^4}{9.81 \times 1000} + \frac{(1.528)^2}{2 \times 9.81} = \frac{11.772 \times (10)^4}{9.81 \times 1000} + \frac{(6.112)^2}{2 \times 9.81} + h_c$

or  $14.0 + 0.119 = 12.0 + 1.904 + h_c$   
 $14.119 = 13.904 + h_c$

$$\therefore h_c = 14.119 - 13.904 = 0.215$$

But from equation (11.6),  $h_c = \frac{V_2^2}{2g} \left[ \frac{1}{C_c} - 1 \right]^2$

Hence equating the two values of  $h_c$ , we get

$$\frac{V_2^2}{2g} \left[ \frac{1}{C_c} - 1 \right]^2 = 0.215$$

$$V_2 = 6.112, \therefore \frac{6.112^2}{2 \times 9.81} \left[ \frac{1}{C_c} - 1 \right]^2 = 0.215$$

or  $\left[ \frac{1}{C_c} - 1 \right]^2 = \frac{0.215 \times 2.0 \times 9.81}{6.112 \times 6.112} = 0.1129$

or  $\frac{1.0}{C_c} - 1.0 = \sqrt{0.1129} = 0.336 \text{ or } \frac{1.0}{C_c} = 1.0 + 0.336 = 1.336$

$$\therefore C_c = \frac{1.0}{1.336} = \mathbf{0.748. Ans.}$$

**Problem 11.13** A 150 mm diameter pipe reduces abruptly to 100 mm diameter. If the pipe carries water at 30 litres per second, calculate the pressure loss across the contraction. Take the co-efficient of contraction as 0.6.

**Solution.** Given :

$$\text{Dia. of large pipe, } D_1 = 150 \text{ mm} = 0.15 \text{ m}$$

$$\text{Area of large pipe, } A_1 = \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2$$

$$\text{Dia. of smaller pipe, } D_2 = 100 \text{ mm} = 0.10 \text{ m}$$

$$\text{Area of smaller pipe, } A_2 = \frac{\pi}{4} (0.10)^2 = 0.007854 \text{ m}^2$$

$$\text{Discharge, } Q = 30 \text{ litres/s} = 0.03 \text{ m}^3/\text{s}$$

$$\text{Co-efficient of contraction, } C_c = 0.6$$

$$\text{From continuity equation, we have } A_1 V_1 = A_2 V_2 = Q$$

$$\therefore V_1 = \frac{Q}{A_1} = \frac{0.03}{0.01767} = 1.697 \text{ m/s}$$

$$\text{and } V_2 = \frac{Q}{A_2} = \frac{0.03}{0.007854} = 3.82 \text{ m/s}$$

Applying Bernoulli's equation before and after contraction,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 + h_c \quad \dots(i)$$

$$\text{But } Z_1 = Z_2$$

and  $h_c$ , the head loss due to contraction is given by equation (11.6) as

$$h_c = \frac{V_2^2}{2g} \left[ \frac{1}{C_c} - 1 \right]^2 = \frac{3.82^2}{2 \times 9.81} \left[ \frac{1}{0.6} - 1 \right]^2 = 0.33$$

Substituting these values in equation (i), we get

$$\frac{p_1}{\rho g} + \frac{1.697^2}{2 \times 9.81} = \frac{p_2}{\rho g} + \frac{3.82^2}{2 \times 9.81} + 0.33$$

$$\text{or } \frac{p_1}{\rho g} + 0.1467 = \frac{p_2}{\rho g} + 0.7438 + 0.33$$

$$\therefore \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 0.7438 + 0.33 - 0.1467 = 0.9271 \text{ m of water}$$

$$\therefore (p_1 - p_2) = \rho g \times 0.9271 = 1000 \times 9.81 \times 0.9271 \text{ N/m}^2 \\ = 0.909 \times 10^4 \text{ N/m}^2 = 0.909 \text{ N/cm}^2$$

$$\therefore \text{Pressure loss across contraction} \\ = p_1 - p_2 = 0.909 \text{ N/cm}^2. \text{ Ans.}$$

**Problem 11.14** In Fig. 11.3 below, when a sudden contraction is introduced in a horizontal pipe line from 50 cm to 25 cm, the pressure changes from  $10,500 \text{ kg/m}^2$  ( $103005 \text{ N/m}^2$ ) to  $6900 \text{ kg/m}^2$  ( $67689 \text{ N/m}^2$ ). Calculate the rate of flow. Assume co-efficient of contraction of jet to be 0.65.

Following this if there is a sudden enlargement from 25 cm to 50 cm and if the pressure at the 25 cm section is  $6900 \text{ kg/m}^2$  ( $67689 \text{ N/m}^2$ ) what is the pressure at the 50 cm enlarged section ?

**Solution.** Given :

$$\text{Dia. of large pipe, } D_1 = 50 \text{ cm} = 0.5 \text{ m}$$

$$\text{Area, } A_1 = \frac{\pi}{4} (0.5)^2 = 0.1963 \text{ m}^2$$

$$\text{Dia. of smaller pipe, } D_2 = 25 \text{ cm} = 0.25 \text{ m}$$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (0.25)^2 = 0.04908 \text{ m}^2$$

$$\text{Pressure in large pipe, } p_1 = 10500 \text{ kg/m}^2 \text{ or } 103005 \text{ N/m}^2$$

$$\text{Pressure in smaller pipe, } p_2 = 6900 \text{ kg/m}^2 \text{ or } 67689 \text{ N/m}^2$$

$$\text{Co-efficient of contraction, } C_c = 0.65$$

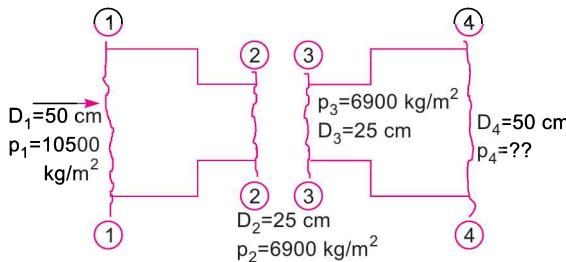


Fig. 11.3

Head lost due to contraction is given by equation (11.6),

$$h_c = \frac{V_2^2}{2g} \left( \frac{1}{C_c} - 1.0 \right)^2 = \frac{V_2^2}{2g} \left( \frac{1}{0.65} - 1 \right)^2 = 0.2899 \frac{V_2^2}{2g} \quad \dots(i)$$

From continuity equation, we have

$$\begin{aligned} A_1 V_1 &= A_2 V_2 \text{ or } V_1 = \frac{A_2 V_2}{A_1} = \frac{\frac{\pi}{4} D_2^2 \times V_2}{\frac{\pi}{4} D_1^2} \\ &= \left( \frac{D_2}{D_1} \right)^2 \times V_2 = \left( \frac{0.25}{0.50} \right)^2 \times V_2 = \frac{V_2}{4} \end{aligned} \quad \dots(ii)$$

Applying Bernoulli's equation at sections 1-1 and 2-2,

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 + h_c$$

But

$$Z_1 = Z_2 \text{ (as pipe is horizontal)}$$

$$\therefore \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_c$$

Substituting the values of  $p_1$ ,  $p_2$ ,  $h_c$  and  $V_1$ , we get

$$\frac{103005}{1000 \times 9.81} + \frac{(V_2/4)^2}{2g} = \frac{67689}{1000 \times 9.81} + \frac{V_2^2}{2g} + .2899 \frac{V_2^2}{2g}$$

$$\text{or } 10.5 + \frac{V_2^2}{16 \times 2g} = 6.9 + 1.2899 \frac{V_2^2}{2g}$$

$$\text{or } 10.5 - 6.9 = 1.2899 \frac{V_2^2}{2g} - \frac{1}{16} \times \frac{V_2^2}{2g} = 1.2274 \frac{V_2^2}{2g}$$

$$\text{or } 3.6 = 1.2274 \times \frac{V_2^2}{2g}$$

$$\therefore V_2 = \sqrt{\frac{3.6 \times 2 \times 9.81}{1.2274}} = 7.586 \text{ m/s}$$

$$(i) \text{ Rate of flow of water, } Q = A_2 V_2 = 0.04908 \times 7.586 \\ = 0.3723 \text{ m}^3/\text{s or } 372.3 \text{ lit/s. Ans.}$$

(ii) Applying Bernoulli's equation to sections 3-3 and 4-4,

$$\frac{p_3}{\rho g} + \frac{V_3^2}{2g} + Z_3 = \frac{p_4}{\rho g} + \frac{V_4^2}{2g} + Z_4 + \text{head loss due to sudden enlargement (} h_e \text{)}$$

$$\text{But } p_3 = 6900 \text{ kg/m}^2, \text{ or } 67689 \text{ N/m}^2$$

$$V_3 = V_2 = 7.586 \text{ m/s}$$

$$V_4 = V_1 = \frac{V_2}{4} = \frac{7.586}{4} = 1.8965$$

$$Z_3 = Z_4$$

And head loss due to sudden enlargement is given by equation (11.5) as

$$h_e = \frac{(V_3 - V_4)^2}{2g} = \frac{(7.586 - 1.8965)^2}{2 \times 9.81} = 1.65 \text{ m}$$

Substituting these values in Bernoulli's equation, we get

$$\frac{67689}{1000 \times 9.81} + \frac{7.586^2}{2 \times 9.81} = \frac{p_4}{1000 \times 9.81} + \frac{1.8965^2}{2 \times 9.81} + 1.65$$

$$\text{or } 6.9 + 2.933 = \frac{p_4}{1000 \times 9.81} + 0.183 + 1.65$$

$$\therefore \frac{p_4}{1000 \times 9.81} = 6.9 + 2.933 - 0.183 - 1.65 = 9.833 - 1.833 = 8.00$$

$$\therefore p_4 = 8 \times 1000 \times 9.81 = 78480 \text{ N/m}^2. \text{ Ans.}$$

**11.4.3 Loss of Head at the Entrance of a Pipe.** This is the loss of energy which occurs when a liquid enters a pipe which is connected to a large tank or reservoir. This loss is similar to the loss of head due to sudden contraction. This loss depends on the form of entrance. For a sharp edge entrance, this loss is slightly more than a rounded or bell mouthed entrance. In practice the value of loss of head at the entrance (or inlet) of a pipe with sharp cornered entrance is taken  $= 0.5 \frac{V^2}{2g}$ , where  $V$  = velocity of liquid in pipe.

This loss is denoted by  $h_i$

$$\therefore h_i = 0.5 \frac{V^2}{2g} \quad \dots(11.8)$$

**11.4.4 Loss of Head at the Exit of Pipe.** This is the loss of head (or energy) due to the velocity of liquid at outlet of the pipe which is dissipated either in the form of a free jet (if outlet of the pipe is free) or it is lost in the tank or reservoir (if the outlet of the pipe is connected to the tank or reservoir). This loss is equal to  $\frac{V^2}{2g}$ , where  $V$  is the velocity of liquid at the outlet of pipe. This loss is denoted  $h_o$ .

$$\therefore h_o = \frac{V^2}{2g} \quad \dots(11.9)$$

where  $V$  = velocity at outlet of pipe.

**11.4.5 Loss of Head Due to an Obstruction in a Pipe.** Whenever there is an obstruction in a pipe, the loss of energy takes place due to reduction of the area of the cross-section of the pipe at the place where obstruction is present. There is a sudden enlargement of the area of flow beyond the obstruction due to which loss of head takes place as shown in Fig. 11.3 (a)

Consider a pipe of area of cross-section  $A$  having an obstruction as shown in Fig. 11.3 (a).

Let  $a$  = Maximum area of obstruction

$A$  = Area of pipe

$V$  = Velocity of liquid in pipe

Then  $(A - a)$  = Area of flow of liquid at section 1-1.

As the liquid flows and passes through section 1-1, a vena-contracta is formed beyond section 1-1, after which the stream of liquid widens again and velocity of flow at section 2-2 becomes uniform and equal to velocity,  $V$  in the pipe. This situation is similar to the flow of liquid through sudden enlargement.

Let  $V_c$  = Velocity of liquid at vena-contracta.

Then loss of head due to obstruction = loss of head due to enlargement from vena-contracta to section 2-2.

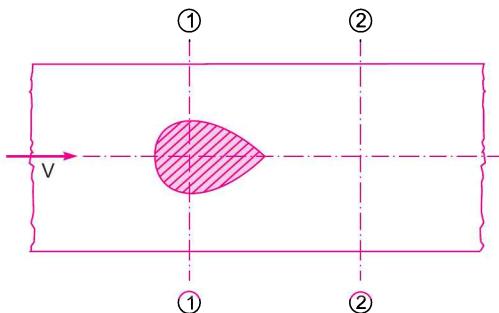


Fig. 11.3 (a) An obstruction in a pipe.

$$= \frac{(V_c - V)^2}{2g} \quad \dots(i)$$

From continuity, we have  $a_c \times V_c = A \times V$  ...(ii)

where  $a_c$  = area of cross-section at vena-contracta

If  $C_c$  = co-efficient of contraction

Then  $C_c = \frac{\text{area at vena - contracta}}{(A - a)} = \frac{a_c}{(A - a)}$

$$\therefore a_c = C_c \times (A - a)$$

Substituting this value in (ii), we get

$$C_c \times (A - a) \times V_c = A \times V \quad \therefore \quad V_c = \frac{A \times V}{C_c (A - a)}$$

Substituting this value of  $V_c$  in equation (i), we get

$$\text{Head loss due to obstruction} = \frac{(V_c - V)^2}{2g} = \frac{\left( \frac{A \times V}{C_c (A - a)} - V \right)^2}{2g} = \frac{V^2}{2g} \left( \frac{A}{C_c (A - a)} - 1 \right)^2 \quad \dots(11.10)$$

**11.4.6 Loss of Head due to Bend in Pipe.** When there is any bend in a pipe, the velocity of flow changes, due to which the separation of the flow from the boundary and also formation of eddies takes place. Thus the energy is lost. Loss of head in pipe due to bend is expressed as

$$h_b = \frac{kV^2}{2g}$$

where  $h_b$  = loss of head due to bend,  $V$  = velocity of flow,  $k$  = co-efficient of bend

The value of  $k$  depends on

- (i) Angle of bend,
- (ii) Radius of curvature of bend,
- (iii) Diameter of pipe.

**11.4.7 Loss of Head in Various Pipe Fittings.** The loss of head in the various pipe fittings such as valves, couplings etc., is expressed as

$$= \frac{kV^2}{2g} \quad \dots(11.11)$$

where  $V$  = velocity of flow,  $k$  = co-efficient of pipe fitting.

**Problem 11.15** Water is flowing through a horizontal pipe of diameter 200 mm at a velocity of 3 m/s. A circular solid plate of diameter 150 mm is placed in the pipe to obstruct the flow. Find the loss of head due to obstruction in the pipe if  $C_c = 0.62$ .

**Solution.** Given :

Dia. of pipe,  $D = 200 \text{ mm} = 0.20 \text{ m}$

Velocity,  $V = 3.0 \text{ m/s}$

Area of pipe,  $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.2)^2 = 0.03141 \text{ m}^2$

Dia. of obstruction,  $d = 150 \text{ mm} = 0.15 \text{ m}$

$\therefore$  Area of obstruction,  $a = \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2$

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$$C_c = 0.62$$

The head lost due to obstruction is given by equation (11.10) as

$$\begin{aligned} &= \frac{V^2}{2g} \left( \frac{A}{C_c(A-a)} - 1.0 \right)^2 \\ &= \frac{3 \times 3}{2 \times 9.81} \left[ \frac{0.03141}{0.62 [0.03141 - 0.01767]} - 1.0 \right]^2 \\ &= \frac{9}{2 \times 9.81} [3.687 - 1.0]^2 = 3.311 \text{ m. Ans.} \end{aligned}$$

**Problem 11.16** Determine the rate of flow of water through a pipe of diameter 20 cm and length 50 m when one end of the pipe is connected to a tank and other end of the pipe is open to the atmosphere. The pipe is horizontal and the height of water in the tank is 4 m above the centre of the pipe. Consider all minor losses and take  $f = .009$  in the formula  $h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$ .

**Solution.** Dia. of pipe,  $d = 20 \text{ cm} = 0.20 \text{ m}$

Length of pipe,  $L = 50 \text{ m}$

Height of water,  $H = 4 \text{ m}$

Co-efficient of friction,  $f = .009$

Let the velocity of water in pipe  $= V \text{ m/s.}$

Applying Bernoulli's equation at the top of the water surface in the tank and at the outlet of pipe, we have  
[Taking point 1 on the top and point 2 at the outlet of pipe].

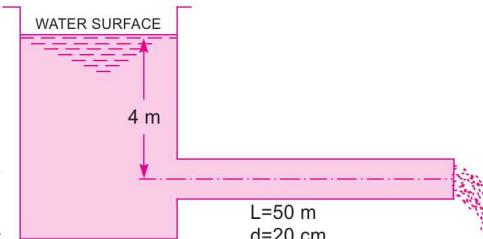


Fig. 11.4

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{all losses}$$

Considering datum line passing through the centre of pipe

$$0 + 0 + 4.0 = 0 + \frac{V_2^2}{2g} + 0 + (h_i + h_f)$$

or

$$4.0 = \frac{V_2^2}{2g} + h_i + h_f$$

But the velocity in pipe =  $V$ ,  $\therefore V = V_2$

$$\therefore 4.0 = \frac{V^2}{2g} + h_i + h_f \quad \dots(i)$$

From equation (11.8),  $h_i = 0.5 \frac{V^2}{2g}$  and  $h_f$  from equation (11.1) is given as

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$$

Substituting these values, we have

$$\begin{aligned} 4.0 &= \frac{V^2}{2g} + \frac{0.5 V^2}{2g} + \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} \\ &= \frac{V^2}{2g} \left[ 1.0 + 0.5 + \frac{4 \times .009 \times 50}{0.2} \right] = \frac{V^2}{2g} [1.0 + 0.5 + 9.0] \\ &= 10.5 \times \frac{V^2}{2g} \\ \therefore V &= \sqrt{\frac{4 \times 2 \times 9.81}{10.5}} = 2.734 \text{ m/sec} \\ \therefore \text{Rate of flow, } Q &= A \times V = \frac{\pi}{4} \times (0.2)^2 \times 2.734 = 0.08589 \text{ m}^3/\text{s} \\ &= 85.89 \text{ litres/s. Ans.} \end{aligned}$$

**Problem 11.17** A horizontal pipe line 40 m long is connected to a water tank at one end and discharges freely into the atmosphere at the other end. For the first 25 m of its length from the tank, the pipe is 150 mm diameter and its diameter is suddenly enlarged to 300 mm. The height of water level in the tank is 8 m above the centre of the pipe. Considering all losses of head which occur, determine the rate of flow. Take  $f = .01$  for both sections of the pipe.

**Solution.** Given :

Total length of pipe,	$L = 40 \text{ m}$
Length of 1st pipe,	$L_1 = 25 \text{ m}$
Dia. of 1st pipe,	$d_1 = 150 \text{ mm} = 0.15 \text{ m}$
Length of 2nd pipe,	$L_2 = 40 - 25 = 15 \text{ m}$
Dia. of 2nd pipe,	$d_2 = 300 \text{ mm} = 0.3 \text{ m}$
Height of water,	$H = 8 \text{ m}$
Co-efficient of friction,	$f = 0.01$

Applying Bernoulli's theorem to the free surface of water in the tank and outlet of pipe as shown in Fig. 11.5 and taking reference line passing through the centre of pipe.

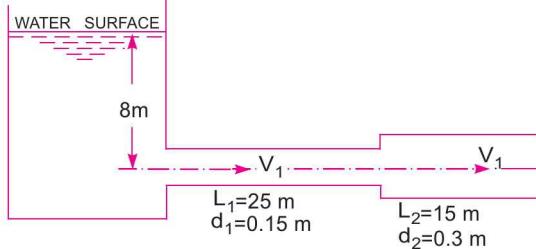


Fig. 11.5

$$0 + 0 + 8 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + 0 + \text{all losses}$$

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or  $8.0 = 0 + \frac{V_2^2}{2g} + h_i + h_{f_1} + h_e + h_{f_2}$  ... (i)

where  $h_i$  = loss of head at entrance =  $0.5 \frac{V_1^2}{2g}$

$$h_{f_1} = \text{head lost due to friction in pipe 1} = \frac{4 \times f \times L_1 \times V_1^2}{d_1 \times 2g}$$

$$h_e = \text{loss head due to sudden enlargement} = \frac{(V_1 - V_2)^2}{2g}$$

$$h_{f_2} = \text{Head lost due to friction in pipe 2} = \frac{4 \times f \times L_2 \times V_2^2}{d_2 \times 2g}$$

But from continuity equation, we have

$$A_1 V_1 = A_2 V_2$$

$$\therefore V_1 = \frac{A_2 V_2}{A_1} = \frac{\frac{\pi}{4} d_2^2 \times V_2}{\frac{\pi}{4} d_1^2} = \left( \frac{d_2}{d_1} \right)^2 \times V_2 = \left( \frac{0.3}{0.15} \right)^2 \times V_2 = 4V_2$$

Substituting the value of  $V_1$  in different head losses, we have

$$h_i = \frac{0.5 V_1^2}{2g} = \frac{0.5 \times (4V_2)^2}{2g} = \frac{8V_2^2}{2g}$$

$$h_{f_1} = \frac{4 \times 0.01 \times 25 \times (4V_2)^2}{0.15 \times 2 \times g} = \frac{4 \times 0.01 \times 25 \times 16}{0.15} \times \frac{V_2^2}{2g} = 106.67 \frac{V_2^2}{2g}$$

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(4V_2 - V_2)^2}{2g} = \frac{9V_2^2}{2g}$$

$$h_{f_2} = \frac{4 \times 0.01 \times 15 \times V_2^2}{0.3 \times 2g} = \frac{4 \times 0.01 \times 15}{0.3} \times \frac{V_2^2}{2g} = 2.0 \times \frac{V_2^2}{2g}$$

Substituting the values of these losses in equation (i), we get

$$8.0 = \frac{V_2^2}{2g} + \frac{8V_2^2}{2g} + 106.67 \frac{V_2^2}{2g} + \frac{9V_2^2}{2g} + 2 \times \frac{V_2^2}{2g}$$

$$= \frac{V_2^2}{2g} [1 + 8 + 106.67 + 9 + 2] = 126.67 \frac{V_2^2}{2g}$$

$$\therefore V_2 = \sqrt{\frac{8.0 \times 2 \times g}{126.67}} = \sqrt{\frac{8.0 \times 2 \times 9.81}{126.67}} = \sqrt{1.2391} = 1.113 \text{ m/s}$$

$$\therefore \text{Rate of flow, } Q = A_2 \times V_2 = \frac{\pi}{4} (0.3)^2 \times 1.113 = 0.07867 \text{ m}^3/\text{s} = \mathbf{78.67 \text{ litres/s. Ans.}}$$

**Problem 11.18** Determine the difference in the elevations between the water surfaces in the two tanks which are connected by a horizontal pipe of diameter 300 mm and length 400 m. The rate of flow of water through the pipe is 300 litres/s. Consider all losses and take the value of  $f = .008$ .

**Solution.** Given :

$$\text{Diameter of pipe, } d = 300 \text{ mm} = 0.30 \text{ m}$$

$$\text{Length, } L = 400 \text{ m}$$

$$\text{Discharge, } Q = 300 \text{ lit/s} = 0.3 \text{ m}^3/\text{s}$$

$$\text{Co-efficient of friction, } f = 0.008$$

$$\text{Velocity, } V = \frac{Q}{\text{Area}} = \frac{0.3}{\frac{\pi}{4} \times (0.3)^2} = 4.244 \text{ m/s}$$

Let the two tanks are connected by a pipe as shown in Fig. 11.6.

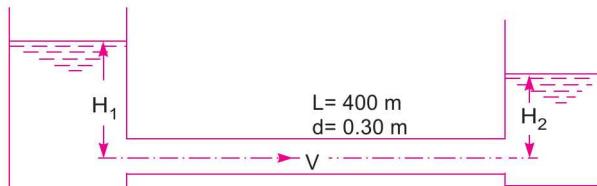


Fig. 11.6

Let  $H_1$  = height of water in 1st tank above the centre of pipe

$H_2$  = height of water in 2nd tank above the centre of pipe

Then difference in elevations between water surfaces =  $H_1 - H_2$

Applying Bernoulli's equation to the free surface of water in the two tanks, we have

$$\begin{aligned} H_1 &= H_2 + \text{losses} \\ &= H_2 + h_i + H_{f_i} + h_o \end{aligned} \quad \dots(i)$$

$$\text{where } h_i = \text{Loss of head at entrance} = 0.5 \frac{V^2}{2g} = \frac{0.5 \times 4.244^2}{2 \times 9.81} = 0.459 \text{ m}$$

$$h_{f_i} = \text{Loss of head due to friction} = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times 0.008 \times 400 \times 4.244^2}{0.3 \times 2 \times 9.81} = 39.16 \text{ m}$$

$$h_o = \text{Loss of head at outlet} = \frac{V^2}{2g} = \frac{4.244^2}{2 \times 9.81} = 0.918 \text{ m}$$

Substituting these values in (i), we get

$$H_1 = H_2 + 0.459 + 39.16 + 0.918 = H_2 + 40.537$$

$$\therefore H_1 - H_2 = \text{Difference in elevations}$$

$$= 40.537 \text{ m. Ans.}$$

**Problem 11.19** The friction factor for turbulent flow through rough pipes can be determined by

$$\text{Karman-Prandtl equation, } \frac{1}{\sqrt{f}} = 2 \log_{10} (R_0/k) + 1.74$$

where  $f$  = friction factor,  $R_0$  = pipe radius,  $k$  = average roughness.

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Two reservoirs with a surface level difference of 20 metres are to be connected by 1 metre diameter pipe 6 km long. What will be the discharge when a cast iron pipe of roughness  $k = 0.3 \text{ mm}$  is used? What will be the percentage increase in the discharge if the cast iron pipe is replaced by a steel pipe of roughness  $k = 0.1 \text{ mm}$ ? Neglect all local losses.

**Solution.** Given :

Difference in levels,

$$h = 20 \text{ m}$$

Dia. of pipe,

$$d = 1.0 \text{ m}$$

$$\therefore \text{Radius, } R_0 = 0.5 \text{ m} = 500 \text{ mm}$$

Length of pipe,

$$L = 6 \text{ km} = 6 \times 1000 = 6000 \text{ m}$$

Roughness of cast iron pipe,

$$k = 0.3 \text{ mm}$$

Roughness of steel pipe,

$$k = 0.1 \text{ mm}$$

**1st Case. Cast Iron Pipe.** First find the value of friction factor using

$$\frac{1}{\sqrt{f}} = 2 \log_{10} (R_0/k) + 1.74 \quad \dots(i)$$

$$= 2 \log_{10} (500/0.3) + 1.74 = 8.1837$$

$$\therefore f = \left( \frac{1}{8.1837} \right)^2 = 0.0149$$

Local losses are to be neglected. This means only head loss due to friction is to be considered. Head loss due to friction is

$$20 = \frac{f \times L \times V^2}{d \times 2g}$$

[Here  $f$  is the friction factor and not co-efficient of friction

$\therefore$  Friction factor =  $4 \times$  co-efficient of friction]

$$20 = \frac{0.0149 \times 6000 \times V^2}{1.0 \times 2 \times 9.81} = 4.556 V^2$$

$$\therefore V = \sqrt{\frac{20}{4.556}} = 2.095 \text{ m/s}$$

$$\therefore \text{Discharge, } Q_1 = V \times \text{Area} = 2.095 \times \frac{\pi}{4} \times d^2 = 2.095 \times \frac{\pi}{4} \times 1^2 = 1.645 \text{ m}^3/\text{s}$$

**2nd Case. Steel Pipe.**  $k = 0.1 \text{ mm}, R_0 = 500 \text{ mm}$

Substituting these values in equation (i), we get

$$\frac{1}{\sqrt{f}} = 2 \log_{10} (500/0.1) + 1.74 = 9.1379$$

$$\therefore f = \left( \frac{1}{9.1379} \right)^2 = 0.0119$$

$$\text{Head loss due to friction, } 20 = \frac{f \times L \times V^2}{d \times 2g} \text{ or } 20 = \frac{0.0119 \times 6000 \times V^2}{1.0 \times 2 \times 9.81} = 3.639 V^2$$

$$\therefore V = \sqrt{\frac{20}{3.639}} = 2.344 \text{ m/s}$$

$$\therefore \text{Discharge, } Q_2 = V \times \text{Area} = 2.344 \times \frac{\pi}{4} \times 1^2 = 1.841 \text{ m}^3/\text{s}$$

$$\text{percentage increase in the discharge} = \frac{Q_2 - Q_1}{Q_1} \times 100 = \frac{(1.841 - 1.645)}{1.645} \times 100 = 11.91\%. \text{ Ans.}$$

**Problem 11.20** Design the diameter of a steel pipe to carry water having kinematic viscosity  $\nu = 10^{-6} \text{ m}^2/\text{s}$  with a mean velocity of 1 m/s. The head loss is to be limited to 5 m per 100 m length of pipe. Consider the equivalent sand roughness height of pipe  $k_s = 45 \times 10^{-4} \text{ cm}$ . Assume that the Darcy Weisbach friction factor over the whole range of turbulent flow can be expressed as

$$f = 0.0055 \left[ 1 + \left( 20 \times 10^3 \frac{k_s}{D} + \frac{10^6}{R_e} \right)^{1/3} \right]$$

where  $D$  = Diameter of pipe and  $R_e$  = Reynolds number.

**Solution.** Given :

$$\text{Kinematic viscosity, } \nu = 10^{-6} \text{ m}^2/\text{s}$$

$$\text{Mean velocity, } V = 1 \text{ m/s}$$

$$\text{Head loss, } h_f = 5 \text{ m in a length } L = 100 \text{ m}$$

$$\text{Value of } k_s = 45 \times 10^{-4} \text{ cm} = 45 \times 10^{-6} \text{ m}$$

$$\text{Value of } f = 0.0055 \left[ 1 + \left( 20 \times 10^3 \frac{k_s}{D} + \frac{10^6}{R_e} \right)^{1/3} \right] \quad \dots(i)$$

$$\text{Using Darcy Weisbach equation, } h_f = \frac{4 \times f \times L \times V^2}{D \times 2g}$$

$$\text{or } f = \frac{h_f \times D \times 2g}{4 \times L \times V^2} = \frac{5 \times D \times 2 \times 9.81}{4 \times 100 \times 1^2} = 0.2452 D$$

Now the Reynolds number is given by,

$$R_e = \frac{\rho V D}{\mu} = \frac{V \times D}{\nu} \quad \left( \because \nu = \frac{\mu}{\rho} \right)$$

$$= \frac{1 \times D}{10^{-6}} = 10^6 D$$

Substituting the values of  $f$ ,  $R_e$  and  $k_s$  in equation (i), we get

$$0.2452 D = 0.0055 \left[ 1 + \left( 20 \times 10^3 \times \frac{45 \times 10^{-6}}{D} + \frac{10^6}{10^6 D} \right)^{1/3} \right]$$

$$\text{or } \frac{0.2452}{0.0055} D = \left[ 1 + \left( \frac{0.9}{D} + \frac{1}{D} \right)^{1/3} \right]$$

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or  $44.58 D = \left[ 1 + \left( \frac{1.9}{D} \right)^{1/3} \right]$  or  $44.58 D - 1 = \left( \frac{1.9}{D} \right)^{1/3}$

or  $(44.58 D - 1)^3 = \frac{1.9}{D}$  or  $D (44.58 D - 1)^3 = 1.9$  ... (ii)

Equation (ii) is solved by hit and trial method.

(i) Let  $D = 0.1$  m, then L.H.S. of equation (ii) becomes as

$$\text{L.H.S.} = 0.1 (44.58 \times 0.1 - 1)^3 = 0.1 \times 3.458^3 = 4.135$$

This is more than the R.H.S.

(ii) Let  $D = 0.08$  m, then L.H.S. of equation (ii) becomes as

$$\text{L.H.S.} = 0.08 (44.58 \times 0.08 - 1)^3 = 0.08 (2.5664)^3 = 1.352$$

This is less than the R.H.S.

Hence value of  $D$  lies between 0.1 and 0.08

(iii) Let  $D = 0.085$ , then L.H.S. of equation (ii) becomes as

$$\text{L.H.S.} = 0.085 (44.58 \times 0.085 - 1)^3 = 1.844$$

This value is slightly less than R.H.S. Hence increase the value of  $D$  slightly.

(iv) Let  $D = 0.0854$  m, then L.H.S. of equation (ii) becomes as

$$\text{L.H.S.} = 0.0854 (44.58 \times 0.0854 - 1)^3 = 1.889$$

This value is nearly equal to R.H.S.

$\therefore$  Correct value of  $D = 0.0854$  m. Ans.

**Problem 11.21** A pipe line AB of diameter 300 mm and of length 400 m carries water at the rate of 50 litres/s. The flow takes place from A to B where point B is 30 metres above A. Find the pressure at A if the pressure at B is 19.62 N/cm<sup>2</sup>. Take  $f = .008$ .

**Solution.** Given :

$$\text{Dia. of pipe, } d = 300 \text{ mm} = 0.30 \text{ m}$$

$$\text{Length of pipe, } L = 400 \text{ m}$$

$$\text{Discharge, } Q = 50 \text{ litres/s} = 0.05 \text{ m}^3/\text{s}$$

$$\therefore \text{Velocity, } V = \frac{Q}{\text{Area}} = \frac{0.05}{\frac{\pi}{4} d^2} = \frac{0.05}{\frac{\pi}{4} \times (0.3)^2} = 0.7074 \text{ m/s}$$

$$\text{Pressure at B, } p_B = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$$

$$f = .008$$

Applying Bernoulli's equations at points A and B and taking datum line passing through A, we have

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + h_f$$

But

$$V_A = V_B$$

[ $\because$  Dia. is same]

$$z_A = 0, z_B = 30$$

and

$$h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$$

$$\begin{aligned} \therefore \frac{p_A}{\rho g} + 0 &= \frac{19.62 \times 10^4}{1000 \times 9.81} + 30 + \frac{4 \times .008 \times 400 \times .7074^2}{0.3 \times 2 \times 9.81} \\ &= 20 + 30 + 1.088 = 51.088 \\ \therefore p_A &= 51.088 \times 1000 \times 9.81 \text{ N/m}^2 \\ &= \frac{51.088 \times 1000 \times 9.81}{10^4} \\ &= 50.12 \text{ N/cm}^2. \text{ Ans.} \end{aligned}$$

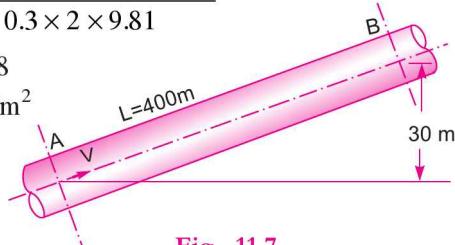


Fig. 11.7

## ► 11.5 HYDRAULIC GRADIENT AND TOTAL ENERGY LINE

The concept of hydraulic gradient line and total energy line is very useful in the study of flow of fluids through pipes. They are defined as :

**11.5.1 Hydraulic Gradient Line.** It is defined as the line which gives the sum of pressure head ( $\frac{p}{w}$ ) and datum head ( $z$ ) of a flowing fluid in a pipe with respect to some reference line or it is the line which is obtained by joining the top of all vertical ordinates, showing the pressure head ( $p/w$ ) of a flowing fluid in a pipe from the centre of the pipe. It is briefly written as H.G.L. (Hydraulic Gradient Line).

**11.5.2 Total Energy Line.** It is defined as the line which gives the sum of pressure head, datum head and kinetic head of a flowing fluid in a pipe with respect to some reference line. It is also defined as the line which is obtained by joining the tops of all vertical ordinates showing the sum of pressure head and kinetic head from the centre of the pipe. It is briefly written as T.E.L. (Total Energy Line).

**Problem 11.22** For the problem 11.16, draw the Hydraulic Gradient Line (H.G.L.) and Total Energy Line (T.E.L.).

**Solution.** Given :

$$L = 50 \text{ m}, d = 200 \text{ mm} = 0.2 \text{ m}$$

$$H = 4 \text{ m}, f = .009$$

Velocity,  $V$  through pipe is calculated in problem 11.16 and its value is  $V = 2.734 \text{ m/s}$

Now,  $h_i = \text{Head lost at entrance of pipe}$

$$= 0.5 \frac{V^2}{2g} + \frac{0.5 \times 2.734^2}{2 \times 9.81} = 0.19 \text{ m}$$

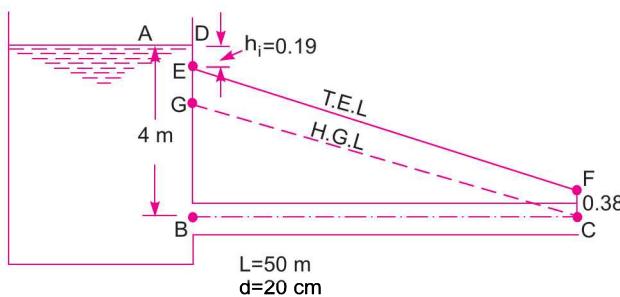


Fig. 11.8

and  $h_f = \text{Head loss due to friction}$

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$$= \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times 0.009 \times 50 \times (2.734)^2}{0.2 \times 2 \times 9.81} = 3.428 \text{ m.}$$

(a) **Total Energy Line (T.E.L.).** Consider three points, *A*, *B* and *C* on the free surface of water in the tank, at the inlet of the pipe and at the outlet of the pipe respectively as shown in Fig. 11.8. Let us find total energy at these points, taking the centre of pipe as reference line.

1. Total energy at *A* =  $\frac{p}{\rho g} + \frac{V^2}{2g} + z = 0 + 0 + 4.0 = 4 \text{ m}$
2. Total energy at *B* = Total energy at *A* -  $h_i = 4.0 - 0.19 = 3.81 \text{ m}$
3. Total energy at *C* =  $\frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c = 0 + \frac{V^2}{2g} + 0 = \frac{2.734^2}{2 \times 9.81} = 0.38 \text{ m.}$

Hence total energy line will coincide with free surface of water in the tank. At the inlet of the pipe, it will decrease by  $h_i$  ( $= 0.19 \text{ m}$ ) from free surface and at outlet of pipe total energy is  $0.38 \text{ m}$ . Hence in Fig. 11.8,

- (i) Point *D* represents total energy at *A*
- (ii) Point *E*, where  $DE = h_i$ , represents total energy at inlet of the pipe
- (iii) Point *F*, where  $CF = 0.38$  represents total energy at outlet of pipe.

Join *D* to *E* and *E* to *F*. Then *DEF* represents the total energy line.

(b) **Hydraulic Gradient Line (H.G.L.).** H.G.L. gives the sum of  $(p/w + z)$  with reference to the datum-line. Hence hydraulic gradient line is obtained by subtracting  $\frac{V^2}{2g}$  from total energy line. At outlet of the pipe, total energy =  $\frac{V^2}{2g}$ . By subtracting  $\frac{V^2}{2g}$  from total energy at this point, we shall get point *C*, which lies on the centre line of pipe. From *C*, draw a line *CG* parallel to *EF*. Then *CG* represents the hydraulic gradient line.

**Problem 11.23** For the problem 11.17, draw the hydraulic gradient and total energy line.

**Solution.** Refer to problem 11.17.

Given :  $L_1 = 25 \text{ m}$ ,  $d_1 = 0.15 \text{ m}$

$L_2 = 15 \text{ m}$ ,  $d_2 = 0.3 \text{ m}$ ,  $f = .01$ ,  $H = 8 \text{ m}$

The velocity  $V_2$  as calculated in problem 11.17 is

$$V_2 = 1.113 \text{ m/s}$$

$$V_1 = 4V_2 = 4 \times 1.113 = 4.452 \text{ m/s}$$

The various head losses are  $h_i = 0.5 \times \frac{V_1^2}{2g} = \frac{0.5 \times 4.452^2}{2 \times 9.81} = 0.50 \text{ m}$

$$h_{f_1} = \frac{4f \times L_1 \times V_1^2}{d_1 \times 2g} = \frac{4 \times .01 \times 25 \times (4.452)^2}{0.15 \times 2 \times 9.81} = 6.73 \text{ m}$$

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(4.452 - 1.11)^2}{2 \times 9.81} = 0.568 \text{ m}$$

$$h_{f_2} = \frac{4 \times f \times L_2 \times V_2^2}{d_2 \times 2g} = \frac{4 \times 0.01 \times 15 \times (1.113)^2}{0.3 \times 2 \times 9.81} = 0.126 \text{ m}$$

$$h_o = \frac{V_2^2}{2g} = \frac{1.113^2}{2 \times 9.81} = 0.063 \text{ m}$$

Also  $V_1^2/2g = \frac{4.452^2}{2 \times 9.81} = 1.0 \text{ m.}$

### Total Energy Line

- (i) Point A lies on free surface of water.
- (ii) Take  $AB = h_i = 0.5 \text{ m.}$
- (iii) From B, draw a horizontal line. Take BL equal to the length of pipe, i.e.,  $L_1$ . From L draw a vertical line downward.
- (iv) Cut the line LC =  $h_{f_1} = 6.73 \text{ m.}$
- (v) Join the point B to C. From C, take a line CD vertically downward equal to  $h_e = 0.568 \text{ m.}$
- (vi) From D, draw DM horizontal and from point F which is lying on the centre of the pipe, draw a vertical line in the upward direction, meeting at M. From M, take a distance ME =  $h_{f_2} = 0.126 \text{ m.}$

Join DE.

Then line ABCDE represents the total energy line.

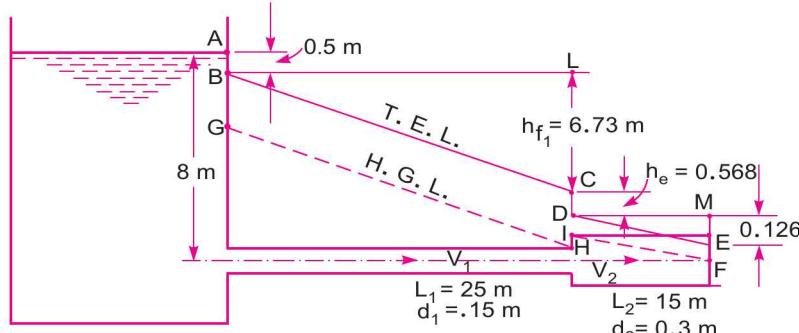


Fig. 11.9

### Hydraulic Gradient Line (H.G.L.)

- (i) From B, take  $BG = \frac{V_1^2}{2g} = 1.0 \text{ m.}$
- (ii) Draw the line GH parallel to the line BC.
- (iii) From F, draw a line FI parallel to the line ED.
- (iv) Join the point H and I.

Then the line GHIF represents the hydraulic gradient line (H.G.L.).

**Problem 11.24** For Problem 11.18, draw the hydraulic gradient and total energy line.

**Solution.** Refer to Problem 11.18,

Given :  $d = 300 \text{ mm} = 0.3 \text{ m}$

$L = 400 \text{ m}, Q = 300 \text{ litres/s} = 0.3 \text{ m}^3/\text{s}$

$f = .008$

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Let  $H_1 = 50 \text{ m}$ . But  $H_1 - H_2 = 40.537 \text{ m}$  (Calculated in Problem 11.18)  
 $\therefore H_2 = 50 - 40.537 = 9.463 \text{ m}$ .

The calculated losses are :

- (i)  $h_i = 0.459 \text{ m}$
- (ii)  $h_{f_1} = 39.16 \text{ m}$
- (iii)  $h_o = 0.918 \text{ m}$
- (a) T.E.L.
- (i) Point A is on the free surface of water in 1st tank. From A, take  $AB = h_i = 0.459 \text{ m}$ .
- (ii) Draw a horizontal line BF. Take BF equal to the length of pipe. From F, draw a vertical line in the downward direction. Cut FC =  $h_{f_1} = 39.16 \text{ m}$ .
- (iii) Join BC. From C take CD =  $h_o = 0.918 \text{ m}$ . The point D should coincide with free surface of water in 2nd tank. Then line ABCD is the total energy line.

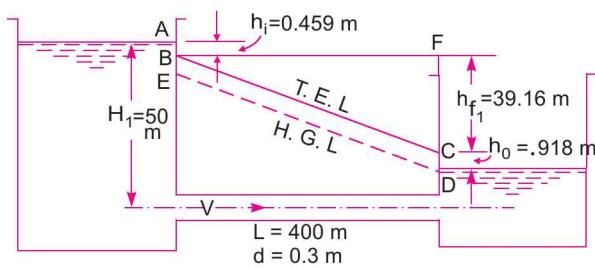


Fig. 11.10

- (b) H.G.L. From D, draw a line DE parallel to line BC. Then DE is the H.G.L.  
 Or

From B, take  $BE = \frac{V^2}{2g} = 0.918 \text{ m}$  and from E draw a line ED parallel to BC. The point D should

coincide with free surface of water in the 2nd tank. Then line ED represents the H.G.L.

**Problem 11.25** The rate of flow of water pumped into a pipe ABC, which is 200 m long, is 20 litres/s. The pipe is laid on an upward slope of 1 in 40. The length of the portion AB is 100 m and its diameter is 100 mm, while the length of the portion BC is also 100 m but its diameter is 200 mm. The change of diameter at B is sudden. The flow is taking place from A to C, where the pressure at A is  $19.62 \text{ N/cm}^2$  and end C is connected to a tank. Find the pressure at C and draw the hydraulic gradient and total energy line. Take  $f = .008$ .

**Solution.** Given :

Length of pipe,  $ABC = 200 \text{ m}$

Discharge,  $Q = 20 \text{ litres/s} = 0.02 \text{ m}^3/\text{s}$

Slope of pipe,  $i = 1 \text{ in } 40 = \frac{1}{40}$

Length of pipe,  $AB = 100 \text{ m}$ , Dia. of pipe AB = 100 mm = 0.1 m

Length of pipe,  $BC = 100 \text{ m}$ , Dia. of pipe BC = 200 mm = 0.2 m

Pressure at A,  $p_A = 19.62 \text{ N/cm}^2 = 19.62 \times 10^4 \text{ N/m}^2$

Co-efficient of friction,  $f = .008$

$$\text{Velocity of water in pipe AB, } V_1 = \frac{\text{Discharge}}{\text{Area of AB}} = \frac{0.02}{\frac{\pi}{4}(0.1)^2} = 2.54 \text{ m/s}$$

$$\text{Velocity of water in pipe } BC, V_2 = \frac{Q}{\text{Area of } BC} = \frac{0.02}{\frac{\pi}{4}(0.2)^2} = 0.63 \text{ m/s}$$

Applying Bernoulli's equation to points A and C,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{total loss from } A \text{ to } C \quad \dots(i)$$

Total loss from A to C = Loss due to friction in pipe AB + loss of head due to enlargement at B + loss of head due to friction in pipe BC.  $\dots(ii)$

Now loss of head due to friction in pipe AB,

$$h_{f_1} = \frac{4fLV^2}{d \times 2g} = \frac{4 \times 0.008 \times 100 \times (2.54)^2}{0.1 \times 2 \times 9.81} = 10.52 \text{ m}$$

Loss of head due to friction in pipe BC,

$$h_{f_2} = \frac{4 \times 0.008 \times 100 \times (0.63)^2}{0.2 \times 2 \times 9.81} = 0.323 \text{ m}$$

Loss of head due to enlargement at B,

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(2.54 - 0.63)^2}{2 \times 9.81} = 0.186 \text{ m}$$

$$\therefore \text{Total loss from } A \text{ to } C = h_{f_1} + h_e + h_{f_2} = 10.52 + 0.186 + 0.323 = 11.029 \approx 11.03 \text{ m}$$

Substituting this value in (i), we get

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + 11.03 \quad \dots(iii)$$

Taking datum line passing through A, we have

$$z_A = 0$$

$$z_c = \frac{1}{40} \times \text{total length of pipe} = \frac{1}{40} \times 200 = 5 \text{ m}$$

$$\text{Also } p_A = 19.62 \times 10^4 \text{ N/m}^2$$

$$V_A = V_1 = 2.54 \text{ m/s}, V_c = V_2 = 0.63 \text{ m/s}$$

Substituting these values in (iii), we get

$$\frac{19.62 \times (10)^4}{1000 \times 9.81} + \frac{(2.54)^2}{2 \times 9.81} + 0 = \frac{p_c}{\rho g} + \frac{(0.63)^2}{2 \times 9.81} + 5.0 + 11.03$$

$$\text{or } 20 + 0.328 = \frac{p_c}{\rho g} + 0.02 + 5.0 + 11.03$$

$$\therefore 20.328 = \frac{p_c}{\rho g} + 16.05$$

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$$\therefore \frac{p_c}{\rho g} = 20.328 - 16.05 = 4.278 \text{ m}$$

or

$$p_c = 4.278 \times 1000 \times 9.81 \text{ N/m}^2 \\ = \frac{4.278 \times 1000 \times 9.81}{10^4} \text{ N/cm}^2 = 4.196 \text{ N/cm}^2. \text{ Ans.}$$

### Hydraulic Gradient and Total Energy Line

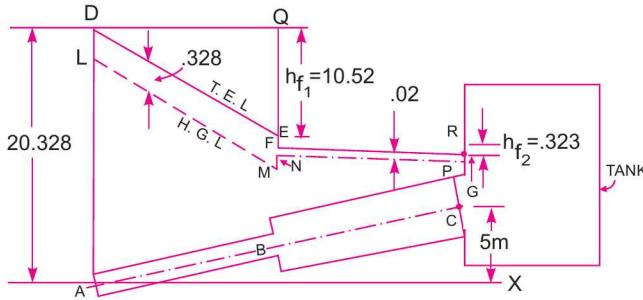


Fig. 11.11

**Pipe AB.** Assuming the datum line passing through A, then total energy at A

$$= \frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{(2.54)^2}{2 \times 9.81} + 0 = 20 + 0.328 \\ = 20.328 \text{ m}$$

Total energy at B

$$= \text{Total energy at } A - h_{f_1} = 20.328 - 10.52 = 9.808 \text{ m}$$

Also

$$V_c^2/2g = \frac{(0.63)^2}{2 \times 9.81} = 0.02.$$

**Total Energy Line.** Draw a horizontal line AX as shown in Fig. 11.11. The centre-line of the pipe is drawn in such a way that slope of pipe is 1 in 40. Thus the point C will be at a height of  $\frac{1}{40} \times 200 = 5 \text{ m}$  from the line AX. Now draw a vertical line AD equal to total energy at A, i.e.,  $AD = 20.328 \text{ m}$ . From point D, draw a horizontal line and from point B, a vertical line, meeting at Q. From Q, take vertical distance  $QE = h_{f_1} = 10.52 \text{ m}$ . Join DE. From E, take  $EF = h_e = 0.186 \text{ m}$ . From F, draw a horizontal line and from C, a vertical line meeting at R. From R take  $RG = h_{f_2} = 0.323 \text{ m}$ . Join F to G. Then DEFG represents the total energy line.

**Hydraulic Gradient Line.** Draw the line  $LM$  parallel to the line  $DE$  at a distance in the downward direction equal to 0.328 m. Also draw the line  $PN$  parallel to the line  $GF$  at a distance of  $\frac{V_c^2}{2g} = 0.02$ .

Join point M to N. Then line  $LMNP$  represents the hydraulic gradient line.

**Problem 11.26** A pipe line, 300 mm in diameter and 3200 m long is used to pump up 50 kg per second of an oil whose density is  $950 \text{ kg/m}^3$  and whose kinematic viscosity is 2.1 stokes. The centre of the pipe line at the upper end is 40 m above than that at the lower end. The discharge at the upper end is atmospheric. Find the pressure at the lower end and draw the hydraulic gradient and the total energy line.

**Solution.** Given :

$$\text{Dia. of pipe, } d = 300 \text{ mm} = 0.3 \text{ m}$$

$$\text{Length of pipe, } L = 3200 \text{ m}$$

$$\text{Mass, } M = 50 \text{ kg/s} = \rho \cdot Q$$

$$\therefore \text{Discharge, } Q = \frac{50}{\rho} = \frac{50}{950} = 0.0526 \text{ m}^3/\text{s}$$

$$\therefore \text{Density, } \rho = 950 \text{ kg/m}^3$$

$$\text{Kinematic viscosity, } v = 2.1 \text{ stokes} = 2.1 \text{ cm}^2/\text{s}$$

$$= 2.1 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Height of upper end} = 40 \text{ m}$$

$$\text{Pressure at upper end} = \text{atmospheric} = 0$$

$$\text{Reynolds number, } R_e = \frac{V \times d}{v}, \text{ where } V = \frac{\text{Discharge}}{\text{Area}} = \frac{0.0526}{\frac{\pi}{4}(0.3)^2} = 0.744 \text{ m/s}$$

$$\therefore R_e = \frac{0.744 \times 0.30}{2.1 \times 10^{-4}} = 1062.8$$

$$\therefore \text{Co-efficient of friction, } f = \frac{16}{R_e} = \frac{16}{1062.8} = 0.015$$

$$\text{Head lost due to friction, } h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$$

$$= \frac{4 \times 0.015 \times 3200 \times (0.744)^2}{0.3 \times 2 \times 9.81} = 18.05 \text{ m of oil}$$

Applying the Bernoulli's equation at the lower and upper end of the pipe and taking datum line passing through the lower end, we have

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f$$

But  $z_1 = 0$ ,  $z_2 = 40 \text{ m}$ ,  $V_1 = V_2$  as diameter is same

$$p_2 = 0, h_f = 18.05 \text{ m}$$

$\therefore$  Substituting these values, we have

$$\frac{p_1}{\rho g} = 40 + 18.05 = 58.05 \text{ m of oil}$$

$$\therefore p_1 = 58.05 \times \rho g = 58.05 \times 950 \times 9.81 \quad [\because \rho \text{ for oil} = 950]$$

$$= 540997 \text{ N/m}^2 = \frac{540997}{10^{-4}} \text{ N/cm}^2 = 54.099 \text{ N/cm}^2. \text{ Ans.}$$

**H.G.L. and T.E.L.**

$$\frac{V^2}{2g} = \frac{(0.744)^2}{2 \times 9.81} = 0.0282 \text{ m}$$

$$\frac{p_1}{\rho g} = 58.05 \text{ m of oil}$$

$$\frac{p_2}{\rho g} = 0$$

Draw a horizontal line  $AX$  as shown in Fig. 11.12. From  $A$ , draw the centre line of the pipe in such a way that point  $C$  is a distance of 40 m above the horizontal line. Draw a vertical line  $AB$  through  $A$  such that  $AB = 58.05 \text{ m}$ . Join  $B$  with  $C$ . Then  $BC$  is the hydraulic gradient line.

Draw a line  $DE$  parallel to  $BC$  at a height of 0.0282 m above the hydraulic gradient line. Then  $DE$  is the total energy line.

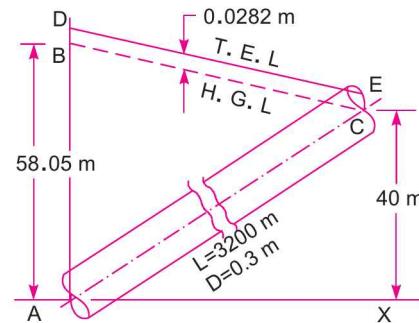


Fig. 11.12

## ► 11.6 FLOW THROUGH SYPHON

Syphon is a long bent pipe which is used to transfer liquid from a reservoir at a higher elevation to another reservoir at a lower level when the two reservoirs are separated by a hill or high level ground as shown in Fig. 11.13.

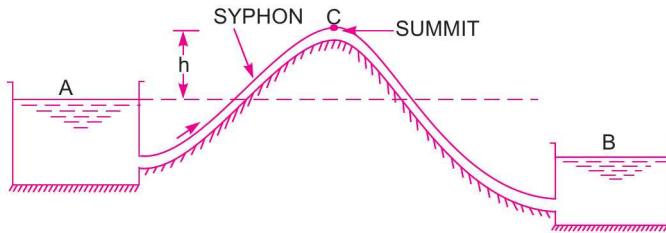


Fig. 11.13

The point  $C$  which is at the highest of the syphon is called the summit. As the point  $C$  is above the free surface of the water in the tank  $A$ , the pressure at  $C$  will be less than atmospheric pressure. Theoretically, the pressure at  $C$  may be reduced to  $-10.3 \text{ m}$  of water but in actual practice this pressure is only  $-7.6 \text{ m}$  of water or  $10.3 - 7.6 = 2.7 \text{ m}$  of water absolute. If the pressure at  $C$  becomes less than 2.7 m of water absolute, the dissolved air and other gases would come out from water and collect at the summit. The flow of water will be obstructed. Syphon is used in the following cases :

1. To carry water from one reservoir to another reservoir separated by a hill or ridge.
2. To take out the liquid from a tank which is not having any outlet.
3. To empty a channel not provided with any outlet sluice.

**Problem 11.27** A syphon of diameter 200 mm connects two reservoirs having a difference in elevation of 20 m. The length of the syphon is 500 m and the summit is 3.0 m above the water level in the upper reservoir. The length of the pipe from upper reservoir to the summit is 100 m. Determine the discharge through the syphon and also pressure at the summit. Neglect minor losses. The co-efficient of friction,  $f = .005$ .

**Solution.** Given :

Dia. of syphon,

$$d = 200 \text{ mm} = 0.20 \text{ m}$$

Difference in level of two reservoirs,

$$H = 20 \text{ m}$$

Length of syphon,

$$L = 500 \text{ m}$$

Height of summit from upper reservoir,  $h = 3.0 \text{ m}$   
 Length of syphon upto summit,  $L_1 = 100 \text{ m}$   
 Co-efficient of friction,  $f = .005$

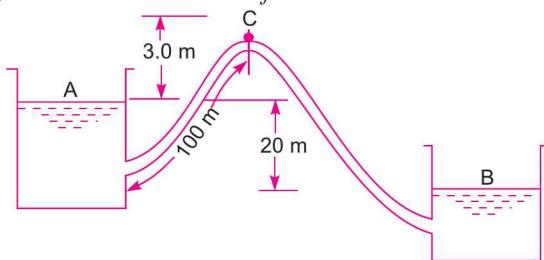


Fig. 11.14

If minor losses are neglected then the loss of head takes place only due to friction.

Applying Bernoulli's equation to points A and B,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + \text{Loss of head due to friction from } A \text{ to } B$$

or

$$0 + 0 + z_A = 0 + 0 + z_B + h_f \quad [\because p_A = p_B = \text{atmospheric pressure}, V_A = V_B = 0]$$

∴

$$z_A - z_B = h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$$

But

$$z_A - z_B = 20 \text{ m}$$

∴

$$20 = \frac{4 \times .005 \times 500 \times V^2}{0.20 \times 2 \times 9.81} = 2.548 V^2$$

∴

$$V = \sqrt{\frac{20}{2.548}} = 2.80 \text{ m/s}$$

∴ Discharge,

$$Q = \text{Velocity} \times \text{Area}$$

$$= 2.80 \times \frac{\pi}{4} (.2)^2 = 0.0879 \text{ m}^3/\text{s} = 87.9 \text{ litres/s. Ans.}$$

**Pressure at Summit.** Applying Bernoulli's equation to points A and C,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{Loss of head due to friction between } A \text{ and } C$$

or

$$0 + 0 + 0 = \frac{p_c}{\rho g} + \frac{V^2}{2g} + 3.0 + h_{f_i} \quad [\text{Taking datum line passing through } A]$$

∴

$$\begin{aligned} 0 &= \frac{p_c}{\rho g} + \frac{2.8^2}{2 \times 9.81} + 3.0 + \frac{4 \times .005 \times 100 \times (2.8)^2}{0.2 \times 2 \times 9.81} \quad [V_c = V = 2.80] \\ &= \frac{p_c}{\rho g} + 0.399 + 3.0 + 4.00 = \frac{p_c}{\rho g} + 7.399 \end{aligned}$$

∴

$$\frac{p_c}{\rho g} = -7.399 \text{ m of water. Ans.}$$

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**Problem 11.28** A siphon of diameter 200 mm connects two reservoirs having a difference in elevation of 15 m. The total length of the siphon is 600 m and the summit is 4 m above the water level in the upper reservoir. If the separation takes place at 2.8 m of water absolute, find the maximum length of siphon from upper reservoir to the summit. Take  $f = .004$  and atmospheric pressure = 10.3 m of water.

**Solution.** Given :

$$\text{Dia. of siphon, } d = 200 \text{ mm} = 0.2 \text{ m}$$

$$\text{Difference of level in two reservoirs} = 15 \text{ m}$$

$$\text{Total length of pipe} = 600 \text{ m}$$

$$\text{Height of summit from upper reservoir} = 4 \text{ m}$$

$$\text{Pressure head at summit, } \frac{p_c}{\rho g} = 2.8 \text{ m of water absolute}$$

$$\text{Atmospheric pressure head, } \frac{p_c}{\rho g} = 10.3 \text{ m of water absolute}$$

$$\text{Co-efficient of friction, } f = .004$$

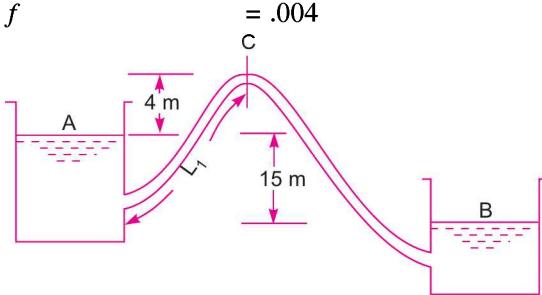


Fig. 11.15 (a)

Applying Bernoulli's equation to points A and C and taking the datum line passing through A,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{Loss of head due to friction between A and C}$$

Substituting the values of pressures in terms of absolute, we have

$$10.3 + 0 + 0 = 2.8 + \frac{V^2}{2g} + 4.0 + h_{f_i} \quad [ \because V_c = \text{velocity in pipe} = V ]$$

$$\therefore h_{f_i} = 10.3 - 2.8 - 4.0 - \frac{V^2}{2g} = 3.5 - \frac{V^2}{2g} \quad \dots(i)$$

Applying Bernoulli's equation to points A and B and taking datum line passing through B,

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + \text{Loss of head due to friction from A to B}$$

$$\text{But } \frac{p_A}{\rho g} = \frac{p_B}{\rho g} = \text{atmospheric pressure}$$

$$V_A = 0, V_B = 0, z_A = 15, z_B = 0$$

$$\therefore 0 + 0 + 15 = 0 + 0 + 0 + h_f$$

$$\therefore h_f = 15 \text{ or } \frac{4 \times f \times L \times V^2}{d \times 2g} = 15$$

$$\text{or } \frac{4 \times .004 \times 600 \times V^2}{0.2 \times 2 \times 9.81} = 15 \text{ or } V = \sqrt{\frac{15 \times 0.2 \times 2 \times 9.81}{4 \times .004 \times 600}} = 2.47 \text{ m/s}$$

Substituting this value of  $V$  in equation (i), we get

$$h_{f_1} = 3.5 - \frac{2.47^2}{2 \times 9.81} = 3.5 - 0.311 = 3.189 \text{ m} \quad \dots(ii)$$

$$\text{But } h_{f_2} = \frac{4 \times f \times L_1 \times V^2}{d \times 2g} \quad \dots(iii)$$

where  $L_1$  = inlet leg of syphon or length of syphon from upper reservoir to the summit.

$$h_{f_2} = \frac{4 \times .004 \times L_1 \times (2.47)^2}{0.2 \times 2 \times 9.81} = 0.0248 \times L_1$$

Substituting this value in equation (ii),

$$0.0248 L_1 = 3.189$$

$$\therefore L_1 = \frac{3.189}{0.0248} = 128.58 \text{ m. Ans.}$$

**Problem 11.29** A syphon of diameter 200 mm connects two reservoirs whose water surface level differ by 40 m. The total length of the pipe is 8000 m. The pipe crosses a ridge. The summit of ridge is 8 m above the level of water in the upper reservoir. Determine the minimum depth of the pipe below the summit of the ridge, if the absolute pressure head at the summit of syphon is not to fall below 3.0 m of water. Take  $f = 0.006$  and atmospheric pressure head = 10.3 m of water. The length of syphon from the upper reservoir to the summit is 500 m. Find the discharge also.

**Solution.** Given :

$$\text{Dia. of syphon, } d = 200 \text{ mm} = 0.20 \text{ m}$$

$$\text{Difference in levels of two reservoirs, } H = 40 \text{ m}$$

$$\text{Total length of pipe, } L = 8000 \text{ m}$$

$$\text{Height of ridge summit from water level in upper reservoir} = 8 \text{ m}$$

$$\text{Let the depth of the pipe below the summit of ridge} = x \text{ m}$$

$$\therefore \text{Height of syphon from water surface in the upper reservoir} = (8 - x) \text{ m}$$

$$\text{Pressure head at } C, \quad \frac{P_c}{\rho g} = 3.0 \text{ m of water absolute}$$

$$\text{Atmospheric pressure head, } \frac{P_a}{\rho g} = 10.3 \text{ m of water}$$

$$\text{Co-efficient of friction } f = .006$$

$$\text{Length of syphon from upper reservoir to the summit, } L_1 = 500 \text{ m}$$

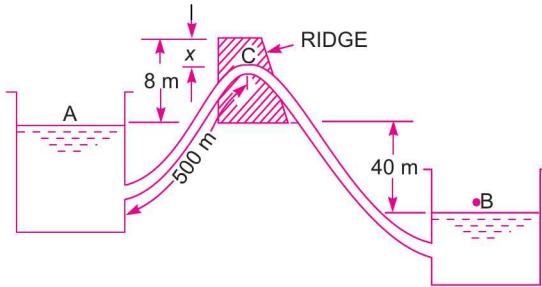


Fig. 11.15 (b)

Applying Bernoulli's equation to points A and B and taking datum line passing through B, we have

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + z_B + \text{head loss due to friction A to } B$$

or  $0 + 0 + 40 = 0 + 0 + 0 + \frac{4 \times f \times L \times V^2}{d \times 2g}$

$$\therefore 40 = \frac{4 \times 0.006 \times 8000 \times V^2}{0.2 \times 2 \times 9.81}$$

$$\therefore V = \sqrt{\frac{40 \times 0.2 \times 2 \times 9.81}{4 \times .006 \times 8000}} = 0.904 \text{ m/s}$$

Now applying Bernoulli's equation to points A and C and assuming datum line passing through A, we have

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z_A = \frac{p_c}{\rho g} + \frac{V_c^2}{2g} + z_c + \text{head loss due to friction from A to } C$$

Substituting  $\frac{p_A}{\rho g}$  and  $\frac{p_c}{\rho g}$  in terms of absolute pressure

$$10.3 + 0 + 0 = 3.0 + \frac{V^2}{2g} + (8 - x) + \frac{4 \times f \times L_1 \times V^2}{d \times 2g}$$

or  $10.3 = 3.0 + \frac{(0.904)^2}{2 \times 9.81} + (8 - x) + \frac{4 \times .006 \times 500 \times (0.904)^2}{0.2 \times 2 \times 9.81}$   
 $= 3.0 + 0.041 + (8 - x) + 2.499 = 13.54 - x$   
 $\therefore x = 13.54 - 10.3 = 3.24 \text{ m. Ans.}$

Discharge,  $Q = \text{Area} \times \text{Velocity} = \frac{\pi}{4} \times (.2)^2 \times 0.904 = 0.0283 \text{ m}^3/\text{s. Ans.}$

### ► 11.7 FLOW THROUGH PIPES IN SERIES OR FLOW THROUGH COMPOUND PIPES

Pipes in series or compound pipes are defined as the pipes of different lengths and different diameters connected end to end (in series) to form a pipe line as shown in Fig. 11.16.

Let,  $L_1, L_2, L_3$  = length of pipes 1, 2 and 3 respectively

$d_1, d_2, d_3$  = diameter of pipes 1, 2, 3 respectively

$V_1, V_2, V_3$  = velocity of flow through pipes 1, 2, 3

$f_1, f_2, f_3$  = co-efficient of friction for pipes 1, 2, 3

$H$  = difference of water level in the two tanks.

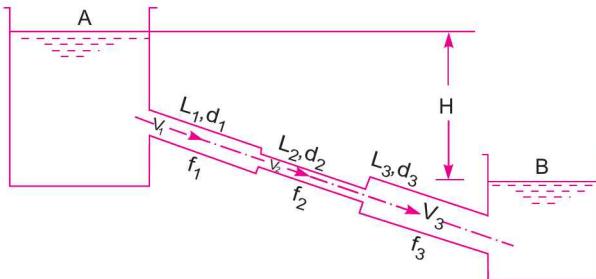


Fig. 11.16

The discharge passing through each pipe is same.

$$\therefore Q = A_1 V_1 = A_2 V_2 = A_3 V_3$$

The difference in liquid surface levels is equal to the sum of the total head loss in the pipes.

$$\begin{aligned} \therefore H &= \frac{0.5 V_1^2}{2g} + \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} + \frac{0.5 V_2^2}{2g} + \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} \\ &\quad + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3 L_3 V_3^2}{d_3 \times 2g} + \frac{V_3^2}{2g} \dots(11.12) \end{aligned}$$

If minor losses are neglected, then above equation becomes as

$$H = \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} + \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} + \frac{4f_3 L_3 V_3^2}{d_3 \times 2g} \dots(11.13)$$

If the co-efficient of friction is same for all pipes

i.e.,  $f_1 = f_2 = f_3 = f$ , then equation (11.13) becomes as

$$\begin{aligned} H &= \frac{4f L_1 V_1^2}{d_1 \times 2g} + \frac{4f L_2 V_2^2}{d_2 \times 2g} + \frac{4f L_3 V_3^2}{d_3 \times 2g} \\ &= \frac{4f}{2g} \left[ \frac{L_1 V_1^2}{d_1} + \frac{L_2 V_2^2}{d_2} + \frac{L_3 V_3^2}{d_3} \right] \dots(11.14) \end{aligned}$$

**Problem 11.30** The difference in water surface levels in two tanks, which are connected by three pipes in series of lengths 300 m, 170 m and 210 m and of diameters 300 mm, 200 mm and 400 mm respectively, is 12 m. Determine the rate of flow of water if co-efficient of friction are .005, .0052 and .0048 respectively, considering : (i) minor losses also (ii) neglecting minor losses.

**Solution.** Given :

Difference of water level,  $H = 12$  m

Length of pipe 1,  $L_1 = 300$  m and dia.,  $d_1 = 300$  mm = 0.3 m

Length of pipe 2,  $L_2 = 170$  m and dia.,  $d_2 = 200$  mm = 0.2 m

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Length of pipe 3,  $L_3 = 210 \text{ m}$  and dia.,  $d_3 = 400 \text{ mm} = 0.4 \text{ m}$

Also,  $f_1 = .005, f_2 = .0052$  and  $f_3 = .0048$

(i) Considering Minor Losses. Let  $V_1, V_2$  and  $V_3$  are the velocities in the 1st, 2nd and 3rd pipe respectively.

From continuity, we have  $A_1 V_1 = A_2 V_2 = A_3 V_3$

$$\therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_2^2} V_1 = \frac{d_1^2}{d_2^2} V_1 = \left(\frac{0.3}{0.2}\right)^2 \times V_1 = 2.25 V_1$$

and

$$V_3 = \frac{A_1 V_1}{A_3} = \frac{d_1^2}{d_3^2} V_1 = \left(\frac{0.3}{0.4}\right)^2 V_1 = 0.5625 V_1$$

Now using equation (11.12), we have

$$H = \frac{0.5 V_1^2}{2g} + \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} + \frac{0.5 V_2^2}{2g} + \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} + \frac{(V_2 - V_3)^2}{2g} + \frac{4f_3 L_3 V_3^2}{d_3 \times 2g} + \frac{V_3^2}{2g}$$

$$\begin{aligned} \text{Substituting } V_2 \text{ and } V_3, \quad 12.0 &= \frac{0.5 V_1^2}{2g} + \frac{4 \times .005 \times 300 \times V_1^2}{0.3 \times 2g} + \frac{0.5 \times (2.25 V_1)^2}{2g} \\ &+ 4 \times 0.0052 \times 170 \times \frac{(2.25 V_1)^2}{0.2 \times 2g} + \frac{(2.25 V_1 - 0.5625 V_1)^2}{2g} + \frac{4 \times .0048 \times 210 \times (0.5625 V_1)^2}{0.4 \times 2g} + \frac{(0.5625 V_1)^2}{2g} \end{aligned}$$

or

$$\begin{aligned} 12.0 &= \frac{V_1^2}{2g} [0.5 + 20.0 + 2.53 + 89.505 + 2.847 + 3.189 + 0.316] \\ &= \frac{V_1^2}{2g} [118.887] \end{aligned}$$

$$\therefore V_1 = \sqrt{\frac{12 \times 2 \times 9.81}{118.887}} = 1.407 \text{ m/s}$$

$\therefore$  Rate of flow,  $Q = \text{Area} \times \text{Velocity} = A_1 \times V_1$

$$\begin{aligned} &= \frac{\pi}{4} (d_1)^2 \times V_1 = \frac{\pi}{4} (.3)^2 \times 1.407 = 0.09945 \text{ m}^3/\text{s} \\ &= \mathbf{99.45 \text{ litres/s. Ans.}} \end{aligned}$$

(ii) Neglecting Minor Losses. Using equation (11.13), we have

$$H = \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} + \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} + \frac{4f_3 L_3 V_3^2}{d_3 \times 2g}$$

$$\text{or } 12.0 = \frac{V_1^2}{2g} \left[ \frac{4 \times .005 \times 300}{0.3} + \frac{4 \times .0052 \times 170 \times (2.25)^2}{0.2} + \frac{4 \times .0048 \times 210 \times (0.5625)^2}{0.4} \right]$$

$$= \frac{V_1^2}{2g} [20.0 + 89.505 + 3.189] = \frac{V_1^2}{2g} \times 112.694$$

$$\therefore V_1 = \sqrt{\frac{2 \times 9.81 \times 12.0}{112.694}} = 1.445 \text{ m/s}$$

$$\therefore \text{Discharge, } Q = V_1 \times A_1 = 1.445 \times \frac{\pi}{4} (.3)^2 = 0.1021 \text{ m}^3/\text{s} = 102.1 \text{ litres/s. Ans.}$$

**Problem 11.30 (A).** Three pipes of 400 mm, 200 mm and 300 mm diameters have lengths of 400 m, 200 m, and 300 m respectively. They are connected in series to make a compound pipe. The ends of this compound pipe are connected with two tanks whose difference of water levels is 16 m. If co-efficient of friction for these pipes is same and equal to 0.005, determine the discharge through the compound pipe neglecting first the minor losses and then including them.

**Solution.** Given :

Difference of water levels,  $H = 16 \text{ m}$

Length and dia. of pipe 1,  $L_1 = 400 \text{ m}$  and  $d_1 = 400 \text{ mm} = 0.4 \text{ m}$

Length and dia. of pipe 2,  $L_2 = 200 \text{ m}$  and  $d_2 = 200 \text{ mm} = 0.2 \text{ m}$

Length and dia. of pipe 3,  $L_3 = 300 \text{ m}$  and  $d_3 = 300 \text{ mm} = 0.3 \text{ m}$

Also  $f_1 = f_2 = f_3 = 0.005$

(i) Discharge through the compound pipe first neglecting minor losses.

Let  $V_1$ ,  $V_2$  and  $V_3$  are the velocities in the 1st, 2nd and 3rd pipe respectively.

From continuity, we have  $A_1 V_1 = A_2 V_2 = A_3 V_3$

$$\therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_2^2} \times V_1 = \frac{d_1^2}{d_2^2} V_1 = \left(\frac{0.4}{0.2}\right)^2 V_1 = 4V_1$$

$$\text{and } V_3 = \frac{A_1 V_1}{A_3} = \frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_3^2} \times V_1 = \frac{d_1^2}{d_3^2} V_1 = \left(\frac{0.4}{0.2}\right)^2 V_1 = 1.77 V_1$$

Now using equation (11.13), we have

$$H = \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} + \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} + \frac{4f_3 L_3 V_3^2}{d_3 \times 2g}$$

$$\begin{aligned} \text{or } 16 &= \frac{4 \times 0.005 \times 400 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 200 \times (4V_1)^2}{0.2 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 300}{0.3 \times 2 \times 9.81} \times (1.77 V_1)^2 \\ &= \frac{V_1^2}{2 \times 9.81} \left( \frac{4 \times 0.005 \times 400}{0.4} + \frac{4 \times 0.005 \times 200 \times 16}{0.2} + \frac{4 \times 0.005 \times 300 \times 3.157}{0.3} \right) \end{aligned}$$

$$16 = \frac{V_1^2}{2 \times 9.81} (20 + 320 + 63.14) = \frac{V_1^2}{2 \times 9.81} \times 403.14$$

$$\therefore V_1 = \sqrt{\frac{16 \times 2 \times 9.81}{403.14}} = 0.882 \text{ m/s}$$

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$$\therefore \text{Discharge, } Q = A_1 \times V_1 = \frac{\pi}{4} (0.4)^2 \times 0.882 = \mathbf{0.1108 \text{ m}^3/\text{s. Ans.}}$$

(ii) Discharge through the compound pipe considering minor losses also.

*Minor losses are :*

$$(a) \text{ At inlet, } h_i = \frac{0.5 V_1^2}{2g}$$

(b) Between 1st pipe and 2nd pipe, due to contraction,

$$h_c = \frac{0.5 V_2^2}{2g} = \frac{0.5(4V_1^2)}{2g} \quad (\because V_2 = 4V_1)$$

$$= \frac{0.5 \times 16 \times V_1^2}{2g} = 8 \times \frac{V_1^2}{2g}$$

(c) Between 2nd pipe and 3rd pipe, due to sudden enlargement,

$$h_e = \frac{(V_2 - V_3)^2}{2g} = \frac{(4V_1 - 1.77V_1)^2}{2g} \quad (\because V_3 = 1.77 V_1)$$

$$= (2.23)^2 \times \frac{V_1^2}{2g} = 4.973 \frac{V_1^2}{2g}$$

$$(d) \text{ At the outlet of 3rd pipe, } h_o = \frac{V_3^2}{2g} = \frac{(1.77V_1)^2}{2g} = 1.77^2 \times \frac{V_1^2}{2g} = 3.1329 \frac{V_1^2}{2g}$$

$$\text{The major losses are } = \frac{4f_1 \times L_1 \times V_1^2}{d_1 \times 2g} + \frac{4f_2 \times L_2 \times V_2^2}{d_2 \times 2g} + \frac{4f_3 \times L_3 \times V_3^2}{d_3 \times 2g}$$

$$= \frac{4 \times 0.005 \times 400 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 200 \times (4V_1)^2}{0.2 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 300 \times (1.77V_1)^2}{0.3 \times 2 \times 9.81}$$

$$= 403.14 \times \frac{V_1^2}{2 \times 9.81}$$

$\therefore$  Sum of minor losses and major losses

$$= \left[ \frac{0.5 V_1^2}{2g} + 8 \times \frac{V_1^2}{2g} + 4.973 \frac{V_1^2}{2g} + 3.1329 \frac{V_1^2}{2g} \right] + 403.14 \frac{V_1^2}{2g}$$

$$= 419.746 \frac{V_1^2}{2g}$$

But total loss must be equal to  $H$  (or 16 m)

$$\therefore 419.746 \times \frac{V_1^2}{2g} = 16 \quad \therefore V_1 = \sqrt{\frac{16 \times 2 \times 9.81}{419.746}} = 0.864 \text{ m/s}$$

$$\therefore \text{Discharge, } Q = A_1 V_1 = \frac{\pi}{4} (0.4)^2 \times 0.864 = \mathbf{0.1085 \text{ m}^3/\text{s. Ans.}}$$

### ► 11.8 EQUIVALENT PIPE

This is defined as the pipe of uniform diameter having loss of head and discharge equal to the loss of head and discharge of a compound pipe consisting of several pipes of different lengths and diameters. The uniform diameter of the equivalent pipe is called equivalent size of the pipe. The length of equivalent pipe is equal to sum of lengths of the compound pipe consisting of different pipes.

Let  $L_1$  = length of pipe 1 and  $d_1$  = diameter of pipe 1

$L_2$  = length of pipe 2 and  $d_2$  = diameter of pipe 2

$L_3$  = length of pipe 3 and  $d_3$  = diameter of pipe 3

$H$  = total head loss

$L$  = length of equivalent pipe

$d$  = diameter of the equivalent pipe

Then  $L = L_1 + L_2 + L_3$

Total head loss in the compound pipe, neglecting minor losses

$$H = \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} + \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} + \frac{4f_3 L_3 V_3^2}{d_3 \times 2g} \quad \dots(11.14A)$$

Assuming  $f_1 = f_2 = f_3 = f$

$$\text{Discharge, } Q = A_1 V_1 = A_2 V_2 = A_3 V_3 = \frac{\pi}{4} d_1^2 V_1 = \frac{\pi}{4} d_2^2 V_2 = \frac{\pi}{4} d_3^2 V_3$$

$$\therefore V_1 = \frac{4Q}{\pi d_1^2}, V_2 = \frac{4Q}{\pi d_2^2} \text{ and } V_3 = \frac{4Q}{\pi d_3^2}$$

Substituting these values in equation (11.14A), we have

$$H = \frac{4f L_1 \left( \frac{4Q}{\pi d_1^2} \right)^2}{d_1 \times 2g} + \frac{4f L_2 \left( \frac{4Q}{\pi d_2^2} \right)^2}{d_2 \times 2g} + \frac{4f L_3 \left( \frac{4Q}{\pi d_3^2} \right)^2}{d_3 \times 2g}$$

$$= \frac{4 \times 16 f Q^2}{\pi^2 \times 2g} \left[ \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} \right] \quad \dots(11.15)$$

$$\text{Head loss in the equivalent pipe, } H = \frac{4f \cdot L \cdot V^2}{d \times 2g} \quad [\text{Taking same value of } f \text{ as in compound pipe}]$$

$$\text{where } V = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4} d^2} = \frac{4Q}{\pi d^2}$$

$$\therefore H = \frac{4 f \cdot L \cdot \left( \frac{4Q}{\pi d^2} \right)^2}{d \times 2g} = \frac{4 \times 16 Q^2 f}{\pi^2 \times 2g} \left[ \frac{L}{d^5} \right] \quad \dots(11.16)$$

Head loss in compound pipe and in equivalent pipe is same hence equating equations (11.15) and (11.16), we have

$$\frac{4 \times 16 f Q^2}{\pi^2 \times 2g} \left[ \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} \right] = \frac{4 \times 16 Q^2 f}{\pi^2 \times 2g} \left[ \frac{L}{d^5} \right]$$

or  $\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} = \frac{L}{d^5}$  or  $\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$  ... (11.17)

Equation (11.17) is known as Dupuit's equation. In this equation  $L = L_1 + L_2 + L_3$  and  $d_1, d_2$  and  $d_3$  are known. Hence the equivalent size of the pipe, i.e., value of  $d$  can be obtained.

**Problem 11.31** Three pipes of lengths 800 m, 500 m and 400 m and of diameters 500 mm, 400 mm and 300 mm respectively are connected in series. These pipes are to be replaced by a single pipe of length 1700 m. Find the diameter of the single pipe.

**Solution.** Given :

Length of pipe 1,  $L_1 = 800$  m and dia.,  $d_1 = 500$  mm = 0.5 m

Length of pipe 2,  $L_2 = 500$  m and dia.,  $d_2 = 400$  mm = 0.4 m

Length of pipe 3,  $L_3 = 400$  m and dia.,  $d_3 = 300$  mm = 0.3 m

Length of single pipe,  $L = 1700$  m

Let the diameter of equivalent single pipe =  $d$

$$\text{Applying equation (11.17), } \frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

or  $\frac{1700}{d^5} = \frac{800}{0.5^5} + \frac{500}{0.4^5} + \frac{400}{0.3^5} = 25600 + 48828.125 + 164609 = 239037$

$\therefore d^5 = \frac{1700}{239037} = .007118$

$\therefore d = (.007118)^{0.2} = 0.3718 = 371.8 \text{ mm. Ans.}$

## ► 11.9 FLOW THROUGH PARALLEL PIPES

Consider a main pipe which divides into two or more branches as shown in Fig. 11.17 and again join together downstream to form a single pipe, then the branch pipes are said to be connected in parallel. The discharge through the main is increased by connecting pipes in parallel.

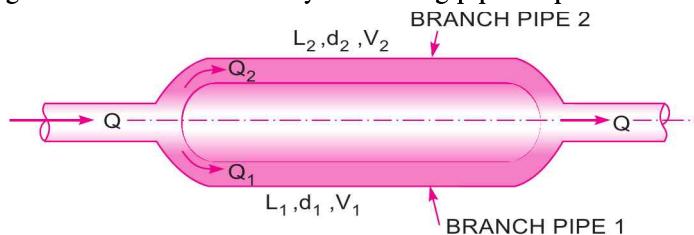


Fig. 11.17

The rate of flow in the main pipe is equal to the sum of rate of flow through branch pipes. Hence from Fig. 11.17, we have

$$Q = Q_1 + Q_2 \quad \dots(11.18)$$

In this, arrangement, the loss of head for each branch pipe is same.

$\therefore$  Loss of head for branch pipe 1 = Loss of head for branch pipe 2

$$\text{or } \frac{4f_1 L_1 V_1^2}{d_1 \times 2g} = \frac{4f_2 L_2 V_2^2}{d_2 \times 2g} \quad \dots(11.19)$$

$$\text{If } f_1 = f_2, \text{ then } \frac{L_1 V_1^2}{d_1 \times 2g} = \frac{L_2 V_2^2}{d_2 \times 2g} \quad \dots(11.20)$$

**Problem 11.32** A main pipe divides into two parallel pipes which again forms one pipe as shown in Fig. 11.17. The length and diameter for the first parallel pipe are 2000 m and 1.0 m respectively, while the length and diameter of 2nd parallel pipe are 2000 m and 0.8 m. Find the rate of flow in each parallel pipe, if total flow in the main is 3.0 m<sup>3</sup>/s. The co-efficient of friction for each parallel pipe is same and equal to .005.

**Solution.** Given :

$$\begin{aligned} \text{Length of pipe 1,} & \quad L_1 = 2000 \text{ m} \\ \text{Dia. of pipe 1,} & \quad d_1 = 1.0 \text{ m} \\ \text{Length of pipe 2,} & \quad L_2 = 2000 \text{ m} \\ \text{Dia. of pipe 2,} & \quad d_2 = 0.8 \text{ m} \\ \text{Total flow,} & \quad Q = 3.0 \text{ m}^3/\text{s} \\ & \quad f_1 = f_2 = f = .005 \end{aligned}$$

$$\text{Let } \begin{aligned} Q_1 &= \text{discharge in pipe 1} \\ Q_2 &= \text{discharge in pipe 2} \end{aligned}$$

$$\text{From equation (11.18), } Q = Q_1 + Q_2 = 3.0 \quad \dots(i)$$

Using equation (11.19), we have

$$\frac{4f_1 L_1 V_1^2}{d_1 \times 2g} = \frac{4f_2 L_2 V_2^2}{d_2 \times 2g}$$

$$\frac{4 \times .005 \times 2000 \times V_1^2}{1.0 \times 2 \times 9.81} = \frac{4 \times .005 \times 2000 \times V_2^2}{0.8 \times 2 \times 9.81}$$

$$\text{or } \frac{V_1^2}{1.0} = \frac{V_2^2}{0.8} \text{ or } V_1^2 = \frac{V_2^2}{0.8}$$

$$\therefore V_1 = \frac{V_2}{\sqrt{0.8}} = \frac{V_2}{.894} \quad \dots(ii)$$

$$\text{Now } Q_1 = \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} (1)^2 \times \frac{V_2}{.894} \quad \left[ \because V_1 = \frac{V_2}{.894} \right]$$

$$\text{and } Q_2 = \frac{\pi}{4} d_2^2 \times V_2 = \frac{\pi}{4} (.8)^2 \times V_2 = \frac{\pi}{4} \times .64 \times V_2$$

Substituting the value of  $Q_1$  and  $Q_2$  in equation (i), we get

$$\frac{\pi}{4} \times \frac{V_2}{.894} + \frac{\pi}{4} \times .64 V_2 = 3.0 \text{ or } 0.8785 V_2 + 0.5026 V_2 = 3.0$$

$$\text{or } V_2 [0.8785 + 0.5026] = 3.0 \text{ or } V = \frac{3.0}{1.3811} = 2.17 \text{ m/s.}$$

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Substituting this value in equation (ii),

$$V_1 = \frac{V_2}{.894} = \frac{2.17}{0.894} = 2.427 \text{ m/s}$$

Hence

$$Q_1 = \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} \times 1^2 \times 2.427 = 1.906 \text{ m}^3/\text{s. Ans.}$$

$$\therefore Q_2 = Q - Q_1 = 3.0 - 1.906 = 1.094 \text{ m}^3/\text{s. Ans.}$$

**Problem 11.33** A pipe line of 0.6 m diameter is 1.5 km long. To increase the discharge, another line of the same diameter is introduced parallel to the first in the second half of the length. Neglecting minor losses, find the increase in discharge if  $4f = 0.04$ . The head at inlet is 300 mm.

**Solution.** Given :

$$\text{Dia. of pipe line, } D = 0.6 \text{ m}$$

$$\text{Length of pipe line, } L = 1.5 \text{ km} = 1.5 \times 1000 = 1500 \text{ m}$$

$$4f = 0.04 \text{ or } f = .01$$

$$\text{Head at inlet, } h = 300 \text{ mm} = 0.3 \text{ m}$$

$$\text{Head at outlet, } = \text{atmospheric head} = 0$$

$$\therefore \text{Head loss, } h_f = 0.3 \text{ m}$$

$$\text{Length of another parallel pipe, } L_1 = \frac{1500}{2} = 750 \text{ m}$$

$$\text{Dia. of another parallel pipe, } d_1 = 0.6 \text{ m}$$

Fig. 11.18 shows the arrangement of pipe system.

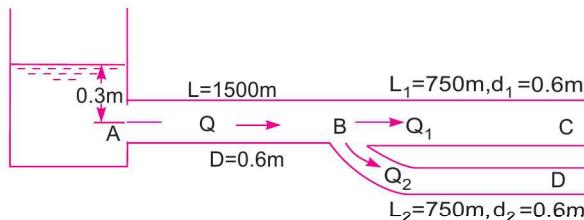


Fig. 11.18

**1st Case.** Discharge for a single pipe of length 1500 m and dia. = 0.6 m.

$$\text{This head lost due to friction in single pipe is } h_f = \frac{4fLV^{*2}}{d \times 2g}$$

where  $V^*$  = velocity of flow for single pipe

$$\text{or } 0.3 = \frac{4 \times .01 \times 1500 \times V^{*2}}{0.6 \times 2g}$$

$$\therefore V^* = \sqrt{\frac{0.3 \times 0.6 \times 2 \times 9.81}{4 \times .01 \times 1500}} = 0.2426 \text{ m/s}$$

$$\therefore \text{Discharge, } Q^* = V^* \times \text{Area} = 0.2426 \times \frac{\pi}{4} (.6)^2 = 0.0685 \text{ m}^3/\text{s} \quad \dots(i)$$

**2nd Case.** When an additional pipe of length 750 m and diameter 0.6 m is connected in parallel with the last half length of the pipe.

Let  $Q_1$  = discharge in 1st parallel pipe

$Q_2$  = discharge in 2nd parallel pipe

$$\therefore Q = Q_1 + Q_2$$

where  $Q$  = discharge in main pipe when pipes are parallel.

But as the length and diameters of each parallel pipe is same

$$\therefore Q_1 = Q_2 = Q/2$$

Consider the flow through pipe ABC or ABD

Head loss through ABC = Head lost through AB + head lost through BC ...(ii)

But head lost due to friction through ABC = 0.3 m given

Head loss due to friction through AB =  $\frac{4 \times f \times 750 \times V^2}{0.6 \times 2 \times 9.81}$ , where  $V$  = velocity of flow through AB

$$= \frac{Q}{\text{Area}} = \frac{Q}{\frac{\pi}{4}(0.6)^2} = \frac{40}{\pi \times .36}$$

$\therefore$  Head loss due to friction through AB

$$= \frac{4 \times .01 \times 750}{0.6 \times 2 \times 9.81} \times \left( \frac{4Q}{\pi \times .36} \right)^2 = 31.87 Q^2$$

Head loss due to friction through BC

$$\begin{aligned} &= \frac{4 \times f \times L_1 \times V_1^2}{d_1 \times 2g} \\ &= \frac{4 \times .01 \times 750}{0.6 \times 2 \times 9.81} \times \left[ \frac{Q}{2 \times \frac{\pi}{4}(0.6)^2} \right] \quad \left[ \because V_1 = \frac{\text{Distance}}{\frac{\pi}{4}(0.6)^2} = \frac{Q}{2 \times \frac{\pi}{4} \times (0.6)^2} \right] \\ &= \frac{4 \times .01 \times 750}{0.6 \times 2 \times 9.81} \times \frac{16}{4 \times \pi^2 \times .36^2} Q^2 = 7.969 Q^2 \end{aligned}$$

Substituting these values in equation (ii), we get

$$0.3 = 31.87 Q^2 + 7.969 Q^2 = 39.839 Q^2$$

$$\therefore Q = \sqrt{\frac{0.3}{39.839}} = 0.0867 \text{ m}^3/\text{s}$$

$\therefore$  Increase in discharge =  $Q - Q^* = 0.0867 - 0.0685 = 0.0182 \text{ m}^3/\text{s}$ . Ans.

**Problem 11.34** A pumping plant forces water through a 600 mm diameter main, the friction head being 27 m. In order to reduce the power consumption, it is proposed to lay another main of appropriate diameter along the side of the existing one, so that two pipes may work in parallel for the entire length and reduce the friction head to 9.6 m only. Find the diameter of the new main if, with the exception of diameter, it is similar to the existing one in every respect.

**Solution.** Given :

Dia. of single main pipe,  $d = 600 \text{ mm} = 0.6 \text{ m}$

Friction head,  $h_f = 27 \text{ m}$

Friction head for two parallel pipes  $= 9.6 \text{ m}$

**1st Case.**

For a single main [Fig. 11.19 (a)]

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$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} \text{ or } 27.0 = \frac{4 \times f \times L \times V^2}{0.6 \times 2 \times 9.81}$$

$$\therefore fLV^2 = \frac{27.0 \times 0.6 \times 2 \times 9.81}{4} = \frac{317.844}{5} = 79.461, \text{ where } V = \frac{Q}{A}$$

$$\therefore f \cdot L \cdot \frac{Q^2}{A^2} = 79.461 \quad \dots(i)$$

**2nd Case.** Two pipes are in parallel [Fig. 11.19 (b)]

Loss of head in any one pipe = 9.6 m

$$\therefore \text{For 1st pipe, } h_{f_1} = \frac{4 \cdot f \cdot L \cdot V_1^2}{d_1 \times 2g} = 9.6$$

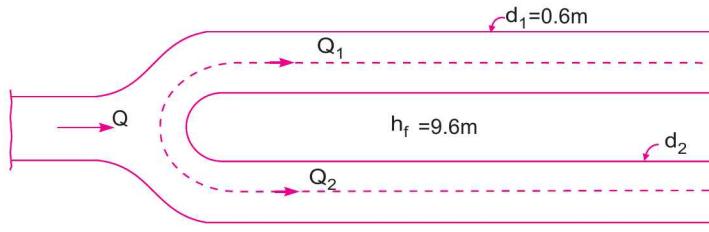
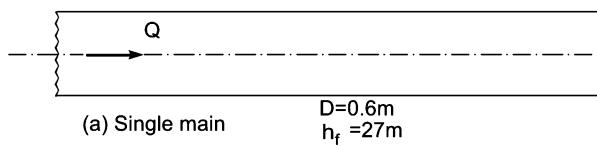


Fig. 11.19

But

$$L_1 = L, V_1 = \frac{Q_1}{A_1} = \frac{Q_1}{A} \quad \left[ \because A_1 = A = \frac{\pi}{4} (0.6)^2 \right]$$

$$d_1 = d = 0.6$$

$$\therefore \frac{4 \cdot f \cdot L}{0.6 \times 2 \times 9.81} \frac{Q_1^2}{A^2} = 0.6$$

or

$$f \times L \times \frac{Q_1^2}{A^2} = \frac{9.6 \times 0.6 \times 2 \times 9.81}{4} = 28.2528 \quad \dots(ii)$$

For the 2nd pipe,

$$h_{f_2} = \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} = 9.6, \quad \text{where } L_2 = L, V_2 = \frac{Q_2}{A_2}$$

$$\therefore \frac{4f \times L \times Q_2^2}{d_2 \times 2g \times A_2^2} = 9.6$$

or

$$\frac{f \times L \times Q_2^2}{d_2 \times A_2^2} = \frac{9.6 \times 2 \times 9.81}{4} = 47.088 \quad \dots(iii)$$

Dividing equation (i) by equation (ii), we get

$$\frac{Q^2}{Q_1^2} = \frac{79.461}{28.2528} = 2.8125$$

$$\therefore \frac{Q}{Q_1} = \sqrt{2.8125} = 1.667$$

$$\therefore Q_1 = \frac{Q}{1.667} = .596 Q$$

But  $Q_1 + Q_2 = Q$

$$\therefore Q_2 = Q - Q_1 = Q - .596 Q = 0.404 Q$$

Dividing equation (ii) by equation (iii),

$$\frac{Q_1^2 \times d_2 \times A_2^2}{A^2 \times Q_2^2} = \frac{28.2528}{47.088} = 0.6$$

But  $A_2 = \frac{\pi}{4} d_2^2$  and  $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.6)^2 = \frac{\pi}{4} \times .36$

$$\therefore \frac{Q_1^2}{Q_2^2} \times \frac{d_2 \times \left(\frac{\pi}{4}\right)^2 \times d_2^4}{\left(\frac{\pi}{4}\right)^2 \times (.36)^2} = 0.6 \quad \text{or} \quad \left(\frac{.596 Q}{.404 Q}\right)^2 \times \frac{d_2^5}{.36^2} = 0.6$$

or  $d_2^5 = 0.6 \times .36^2 \times \left(\frac{.404}{.596}\right)^2 = 0.03537$

$$\therefore d_2 = (0.03537)^{1/5} = 0.5125 \text{ m} = \mathbf{512.5 \text{ mm. Ans.}}$$

**Problem 11.35** A pipe of diameter 20 cm and length 2000 m connects two reservoirs, having difference of water levels as 20 m. Determine the discharge through the pipe.

If an additional pipe of diameter 20 cm and length 1200 m is attached to the last 1200 m length of the existing pipe, find the increase in the discharge. Take  $f = .015$  and neglect minor losses.

**Solution.** Given :

$$\text{Dia. of pipe, } d = 20 \text{ cm} = 0.20 \text{ m}$$

$$\text{Length of pipe, } L = 2000 \text{ m}$$

$$\text{Difference of water levels, } H = 20 \text{ m}$$

$$\text{Co-efficient of friction, } f = 0.015$$

**1st Case.** When a single pipe connects the two reservoirs

$$H = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g} = \frac{4f \cdot L}{d \times 2g} \left( \frac{Q}{\frac{\pi}{4} d^2} \right)^2 \quad \left[ \because V = \frac{Q}{\frac{\pi}{4} d^2} \right]$$

$$= \frac{32f \cdot L \cdot Q^2}{\pi^2 \times g \times d^5}$$

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or

$$20 = \frac{32 \times .015 \times 2000 \times Q^2}{\pi^2 \times 9.81 \times (0.2)^5} = 30985.07 Q^2$$

$$\therefore Q = \sqrt{\frac{20}{30985.07}} = 0.0254 \text{ m}^3/\text{s. Ans.}$$

### 2nd Case.

Let  $Q_1$  = discharge through pipe  $CD$ ,

$Q_2$  = discharge through pipe  $DE$ ,

$Q_3$  = discharge through pipe  $DF$ .

Length of pipe  $CD$ ,  $L_1 = 800 \text{ m}$  and its dia.,  $d_1 = 0.20 \text{ m}$

Length of pipe  $DE$ ,  $L_2 = 1200 \text{ m}$  and its dia.,  $d_2 = 0.20 \text{ m}$

Length of pipe  $DF$ ,  $L_3 = 1200 \text{ m}$  and its dia.,  $d_3 = 0.20 \text{ m}$ .

Since the diameters and lengths of the pipes  $DE$  and  $DF$  are equal. Hence  $Q_2$  will be equal to  $Q_3$ .

Also for parallel pipes, we have

$$Q_1 = Q_2 + Q_3 = Q_2 + Q_2 = 2Q_2 \quad [\because Q_2 = Q_3]$$

$$\therefore Q_2 = \frac{Q_1}{2}$$

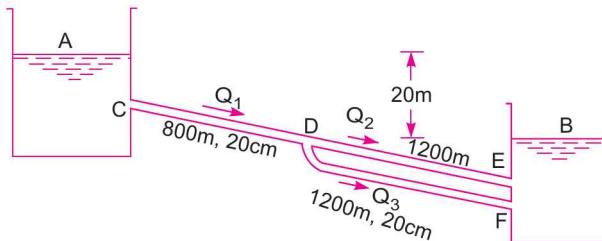


Fig. 11.20

Applying Bernoulli's equation to points  $A$  and  $B$  and taking the flow through  $CDE$ , we have

$$20 = \frac{4f \cdot L_1 \cdot V_1^2}{d_1 \times 2g} + \frac{4f \cdot L_2 \cdot V_2^2}{d_2 \times 2g}$$

$$\text{where } V_1 = \frac{Q_1}{\frac{\pi}{4}(0.2)^2} = \frac{4Q_1}{\pi \times 0.04}, V_2 = \frac{Q_2}{\frac{\pi}{4}(0.2)^2} = \frac{4Q_2}{\pi \times 0.04} = \frac{4 \times \frac{Q_1}{2}}{\pi \times 0.04} = \frac{2Q_1}{\pi \times 0.04}$$

$$\begin{aligned} &= \frac{4 \times 0.015 \times 800}{0.2 \times 2 \times 9.81} \times \left( \frac{4Q_1}{\pi \times 0.04} \right)^2 + \frac{4 \times 0.015 \times 1200}{0.2 \times 2 \times 9.81} \times \left( \frac{2Q_1}{\pi \times 0.04} \right)^2 \\ &= 12394 Q_1^2 + 4647 Q_1^2 = 17041 Q_1^2 \end{aligned}$$

$$\therefore Q_1 = \sqrt{\frac{20}{17041}} = 0.0342 \text{ m}^3/\text{s}$$

Increase in discharge =  $Q_1 - Q = 0.0342 - 0.0254 = 0.0088 \text{ m}^3/\text{s. Ans.}$

**Problem 11.36** Two pipes have a length  $L$  each. One of them has a diameter  $D$ , and the other a diameter  $d$ . If the pipes are arranged in parallel, the loss of head, when a total quantity of water  $Q$  flows through them is  $h$ , but, if the pipes are arranged in series and the same quantity  $Q$  flows through them, the loss of head is  $H$ . If  $d = \frac{D}{2}$ , find the ratio of  $H$  to  $h$ , neglecting secondary losses and assuming the pipe co-efficient  $f$  has a constant value.

**Solution.** Given :

$$\text{Length of pipe 1, } L_1 = L \text{ and its dia. } d_1 = D$$

$$\text{Length of pipe 2, } L_2 = L \text{ and its dia., } d_2 = d$$

$$\text{Total discharge} = Q$$

Head loss when pipes are arranged in parallel =  $h$

Head loss when pipes are arranged in series =  $H$

$$d = \frac{D}{2} \text{ and } f \text{ is constant}$$

**1st Case.** When pipes are connected to parallel

$$Q = Q_1 + Q_2 \quad \dots(i)$$

Loss of head in each pipe =  $h$

$$\text{For pipe } AB, \frac{4fL_1V_1^2}{d_1 \times 2g} = h, \text{ where } V_1 = \frac{Q_1}{A_1} = \frac{Q_1}{\frac{\pi}{4}D^2} = \frac{4Q_1}{\pi D^2}$$

$$d_1 = D$$

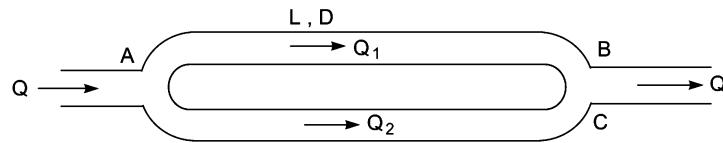


Fig. 11.21

$$\therefore \frac{4fL \times \left(\frac{4Q_1}{\pi D^2}\right)^2}{D \times 2g} = h \text{ or } \frac{32fLQ_1^2}{\pi^2 D^5 \times g} = h \quad \dots(ii)$$

$$\text{For pipe } AC, \frac{32fLQ_2^2}{\pi^2 d^5 \times g} = h \quad \dots(iii)$$

$$\therefore \frac{32fLQ_1^2}{\pi^2 D^5 g} = \frac{32fLQ_2^2}{\pi^2 d^5 g} \text{ or } \frac{Q_1^2}{D^5} = \frac{Q_2^2}{d^5}$$

$$\text{or } \left(\frac{Q_1}{Q_2}\right)^2 = \frac{D^5}{d^5} = \frac{(2d)^5}{d^5} = 2^5 = 32 \quad [\because D = 2d]$$

$$\therefore \frac{Q_1}{Q_2} = \sqrt{32} = 5.657 \text{ or } Q_1 = 5.657 Q_2$$

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Substituting the values of  $Q_1$  in equation (i), we get

$$Q = 5.657 Q_2 \quad Q_2 = 6.657 Q_2$$

$$\therefore Q_2 = \frac{Q}{6.657} = 0.15 Q \quad \dots(iv)$$

$$\text{From (i) } \therefore Q_1 = Q - Q_2 = Q - 0.15 Q = 0.85 Q \quad \dots(v)$$

**2nd Case.** When the pipes are connected in series.

Total loss = Sum of head losses in the two pipes

$$\therefore H = \frac{4f \cdot L \cdot V_1^2}{d_1 \times 2g} + \frac{4f \cdot L \cdot V_2^2}{d_2 \times 2g}$$

$$\text{where } V_1 = \frac{Q}{\frac{\pi D^2}{4}} = \frac{4Q}{\pi D^2}, V_2 = \frac{Q}{\frac{\pi d^2}{4}} = \frac{4Q}{\pi d^2}$$

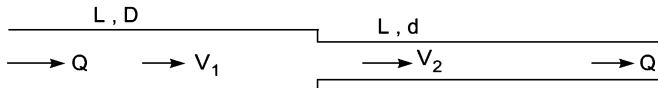


Fig. 11.22

$$\therefore H = \frac{4f \cdot L \times \left(\frac{4Q}{\pi D^2}\right)^2}{D \times 2g} + \frac{4fL \left(\frac{4Q}{\pi d^2}\right)^2}{d \times 2g}$$

$$\text{or } H = \frac{32 f L Q^2}{D^5 \pi^2 \times g} + \frac{32 f L Q^2}{d^5 \pi^2 \times g} \quad \dots(vi)$$

$$\text{From equation (ii), } \frac{32 f L}{\pi^2 D^5 \times g} = \frac{h}{Q_1^2}$$

$$\text{and from equation (iii), } \frac{32 f L}{\pi^2 d^5 \times g} = \frac{h}{Q_2^2}$$

Substituting these values in equation (vi), we have

$$H = Q^2 \times \frac{h}{Q_1^2} + Q^2 \times \frac{h}{Q_2^2} = \frac{Q^2}{Q_1^2} h + \frac{Q^2}{Q_2^2} h = h \left[ \frac{Q^2}{Q_1^2} + \frac{Q^2}{Q_2^2} \right]$$

$$\therefore \frac{H}{h} = \frac{Q^2}{Q_1^2} + \frac{Q^2}{Q_2^2}$$

But from equations (iv) and (v),  $Q_1 = .85 Q$  and  $Q_2 = 0.15 Q$

$$\therefore \frac{H}{h} = \frac{Q^2}{.85^2 Q^2} + \frac{Q^2}{.15^2 Q^2} = \frac{1}{.85^2} + \frac{1}{.15^2} = 1.384 + 44.444 = 45.828. \text{ Ans.}$$

**Problem 11.36 (A).** Three pipes of the same length  $L$ , diameter  $D$ , and friction factor  $f$  are connected in parallel. Determine the diameter of the pipe of length  $L$  and friction factor  $f$  which will carry the same discharge for the same head loss. Use the formula  $h_f = f \times L \times V^2/2g D$ .

**Solution.** Given :

$$\text{Length of each pipe} = L$$

$$\text{Diameter of each pipe} = D$$

$$\text{Friction factor of each pipe} = f$$

$$\text{Head loss, } h_f = f \times L \times V^2 / 2gD$$

When the three pipes are connected in parallel, then head loss in each pipe will be same. And total head loss will be equal to the head loss in each pipe.

$$\text{Let } h_f = \text{Total head loss,}$$

$$h_{f_1} = \text{Head loss in 1st pipe,}$$

$$h_{f_2} = \text{Head loss in 2nd pipe, and } h_{f_3} = \text{Head loss in 3rd pipe.}$$

$$\text{Then } h_f = h_{f_1} = h_{f_2} = h_{f_3} \text{ or } h_f = \frac{f \times L \times V^2}{2gD} \quad \dots(i)$$

Let  $Q_1$  = Discharge through 1st pipe,  $Q_2$  = Discharge through 2nd pipe,

$Q_3$  = Discharge through 3rd pipe, and  $Q$  = Total discharge.

When the three pipes are connected in parallel, then

$$Q = Q_1 + Q_2 + Q_3 = 3 \times Q_1 \quad (\because Q_1 = Q_2 = Q_3)$$

$$= 3 \times A_1 \times V_1$$

$$= 3 \times \frac{\pi}{4} D^2 \times V \left( \text{where } A_1 = \frac{\pi}{4} D^2 \text{ and } V_1 = V \right) \quad \dots(ii)$$

For a single pipe (or length  $L$ ; friction factor  $f$ ) which will carry same discharge as the three pipes in parallel

Let  $d$  = dia. of the single pipe

$v$  = velocity through single pipe

$$\text{Then discharge, } Q = \text{Area} \times \text{Velocity} = \left( \frac{\pi}{4} d^2 \right) \times v \quad \dots(iii)$$

Equating the two values of discharge, given by equations (ii) and (iii), we get

$$3 \times \frac{\pi}{4} D^2 \times V = \frac{\pi}{4} d^2 \times v \text{ or } 3 \times \frac{D^2}{d^2} = \frac{v}{V} \quad \dots(iv)$$

The head loss for the single pipe is also equal to the total head loss for three pipes when they are in parallel.

But head loss for the single pipe of length  $L$ , dia.  $d$ , friction factor  $f$  and velocity  $v$  is given by

$$h_f = \frac{f \times L \times v^2}{d \times 2g} \quad \dots(v)$$

Equating the two values of  $h_f$  given by equations (i) and (v), we get

$$\frac{f \times L \times V^2}{D \times 2g} = \frac{f \times L \times v^2}{d \times 2g} \text{ or } \frac{V^2}{D} = \frac{v^2}{d}$$

$$\text{or } \frac{d}{D} = \frac{v^2}{V^2} \text{ or } \left( \frac{d}{D} \right)^{1/2} = \frac{v}{V}$$

Substituting the value of  $v/V$  in equation (iv), we get

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$$3 \times \frac{D^2}{d^2} = \left(\frac{d}{D}\right)^{1/2} \text{ or } 3 = \left(\frac{d}{D}\right)^{1/2} \times \left(\frac{d}{D}\right)^2 = \left(\frac{d}{D}\right)^{5/2}$$

or  $\frac{d}{D} = 3^{2/5} = 3^{0.4} = 1.55$

$\therefore d = 1.55 D.$  Ans.

Hence dia. of single pipe should be 1.55 times the dia. of the three pipes connected in parallel.

**Problem 11.37** For a town water supply, a main pipe line of diameter 0.4 m is required. As pipes more than 0.35 m diameter are not readily available, two parallel pipes of the same diameter were used for water supply. If the total discharge in the parallel pipes is same as in the single main pipe, find the diameter of the parallel pipe. Assume the co-efficient of friction same for all pipes.

**Solution.** Given :

Dia. of single main pipe line,  $d = 0.4 \text{ m}$

Let the length of single pipe line  $= L$

Co-efficient of friction  $= f$

$$\text{Loss of head due to friction in single pipe} = \frac{4fLV^2}{d \times 2g} = \frac{4fLV^2}{0.4 \times 2 \times g} \quad \dots(i)$$

where  $V$  = Velocity of flow in the single pipe.

In case of parallel pipe, as the diameters and lengths of the two pipes are same. Hence discharge in each pipe will be half the discharge of single main pipe. As discharge in each parallel pipe is same, hence velocity will also be same.

Let  $V_*$  = Velocity in each parallel pipe

$d_*$  = Dia. of each parallel pipe

$$\text{Then loss of head due to friction in parallel pipes} = \frac{4f \times L \times V_*^2}{d_* \times 2g} \quad \dots(ii)$$

Equating the two losses given by equations (i) and (ii), we have

$$\frac{4f \cdot L \cdot V^2}{0.4 \times 2g} = \frac{4f \times L \times V_*^2}{d_* \times 2g}$$

$$\text{Cancelling } \frac{4fL}{2g}, \quad \frac{V^2}{0.4} = \frac{V_*^2}{d_*^2} \text{ or } \frac{V^2}{V_*^2} = \frac{0.4}{d_*^2} \quad \dots(iii)$$

From continuity

Total flow in single main = sum of flow in two parallel pipes

or Velocity of main  $\times$  Area = 2  $\times$  Velocity in each parallel pipe  $\times$  Area

$$V \times \frac{\pi}{4} (0.4)^2 = 2 \times V_* \times \frac{\pi}{4} d_*^2 \text{ or } \frac{V}{V_*} = \frac{2 \times \frac{\pi}{4} d_*^2}{\frac{\pi}{4} (0.4)^2} = \frac{2d_*^2}{0.16}$$

$$\text{Squaring both sides, } \frac{V^2}{V_*^2} = \frac{4d_*^4}{0.0256} \quad \dots(iv)$$

Comparing equations (iii) and (iv), we get

$$\frac{0.4}{d_*} = \frac{4d_*^4}{.0256} \quad \text{or} \quad d_*^5 = \frac{0.4 \times .0256}{4} = .00256$$

$$\therefore d_* = (.00256)^{1/5} = 0.303 \text{ m} = 30.3 \text{ cm. Ans.}$$

$\therefore$  Use two pipes of 30.3 cm diameter.

**Problem 11.38** An old water supply distribution pipe of 250 mm diameter of a city is to be replaced by two parallel pipes of smaller equal diameter having equal lengths and identical friction factor values. Find out the new diameter required.

**Solution.** Given :

Dia. of old pipe,  $D = 250 \text{ mm} = 0.25 \text{ m}$

Let  $d$  = Dia. of each of parallel pipes

$Q$  = Discharge in old pipe

$Q_1$  = Discharge in first parallel pipe

$Q_2$  = Discharge in second parallel pipe

$f$  = Friction factor.

When a single pipe is replaced by two parallel pipes, the head loss will be same in the single pipe and in each of the parallel pipes. Also the discharge in single pipe will be equal to the total discharge in two parallel pipes i.e.,

$$h_f = h_{f_1} = h_2 \quad \dots(i)$$

and

$$Q = Q_1 + Q_2 \quad \dots(ii)$$

As the dia. of each parallel pipe is same and also length of each parallel pipe is equal, hence

$$Q_1 = Q_2 \text{ or } Q_1 = Q_2 = Q/2$$

Now  $h_f$  = Head loss in single pipe

$$= \frac{f \times L \times V^2}{D \times 2g}, \quad \text{where } f = \text{Friction factor}$$

$$= \frac{f \times L \times \left( \frac{Q}{\frac{\pi}{4} \times 0.25^2} \right)^2}{0.25 \times 2 \times 9.81} \quad \left( \because V = \frac{Q}{\text{Area}} = \frac{Q}{\frac{\pi}{4} D^2} \right) \quad \dots(iii)$$

$$= \frac{f \times L \times (4Q)^2}{0.25 \times 2 \times 9.81 \times (\pi \times 0.25^2)^2}$$

$h_{f_1}$  = Head loss in 1st parallel pipe

$$= \frac{f \times L \times (V_1)^2}{d \times 2g} \quad (\because \text{Dia. of parallel pipe} = d \text{ and } V_1 \text{ is the velocity in 1st parallel pipe})$$

$$= \frac{f \times L \times \left( \frac{Q}{2 \times \frac{\pi}{4} d^2} \right)^2}{d \times 2g} \quad \left( \because V_1 = \frac{Q_1}{A_1} = \frac{Q_1}{\frac{\pi}{4} d^2} \text{ as } Q_1 = \frac{Q}{2} \right)$$

$$= \frac{f \times L \times (4Q)^2}{d \times 2 \times 9.81 \times (2 \times \pi \times d^2)^2} \quad \dots(iv)$$

But

$$h_f = h_{f_i}$$

$$\text{or } \frac{f \times L \times (4Q)^2}{0.25 \times 2 \times 9.81 \times (\pi \times 0.25^2)^2} = \frac{f \times L \times (4Q)^2}{d \times 2 \times 9.81 \times (2 \times \pi \times d^2)^2}$$

$$\text{or } d \times (2\pi d^2)^2 = 0.25 \times (\pi \times 0.25^2)^2$$

$$\text{or } d \times 4 \times d^4 = 0.25 \times 0.25^4$$

$$\text{or } d^5 = \frac{0.25^5}{4} \text{ or } d = \frac{0.25}{(4)^{1/5}} = \frac{0.25}{1.3195} = 0.1894 \text{ m} \approx 0.19 \text{ m. Ans.}$$

**Problem 11.39** A pipe of diameter 0.4 m and of length 2000 m is connected to a reservoir at one end. The other end of the pipe is connected to a junction from which two pipes of lengths 1000 m and diameter 300 mm run in parallel. These parallel pipes are connected to another reservoir, which is having level of water 10 m below the water level of the above reservoir. Determine the total discharge if  $f = 0.015$ . Neglect minor losses.

**Solution.** Given :

$$\text{Dia. of pipe, } d = 0.4 \text{ m}$$

$$\text{Length of pipe, } L = 2000 \text{ m}$$

$$\text{Dia. of parallel pipes, } d_1 = d_2 = 300 \text{ mm} = 0.30 \text{ m}$$

$$\text{Length of parallel pipes, } L_1 = L_2 = 1000 \text{ m}$$

$$\text{Difference of water level in two reservoir, } H = 10 \text{ m, } f = 0.015$$

Applying Bernoulli's equation to points E and F. Taking flow through ABC.

$$\begin{aligned} 10 &= \frac{4fLV^2}{d \times 2g} + \frac{4f \times L_1 \times V_1^2}{d_1 \times 2g} \\ &= \frac{4 \times 0.015 \times 2000 \times V^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.015 \times 1000 \times V_1^2}{0.3 \times 2 \times 9.81} \\ &= 15.29 V^2 + 10.19 V_1^2 \quad \dots(i) \end{aligned}$$

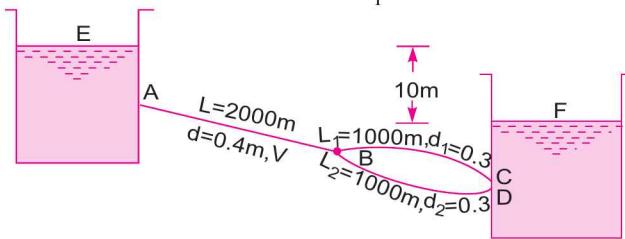


Fig. 11.23

From continuity equation

Discharge through AB = discharge through BC + discharge through BD

or  $\frac{\pi}{4} d^2 \times V = \frac{\pi}{4} d_1^2 \times V_1 + \frac{\pi}{4} d_1^2 V_2$

But  $d_1 = d_2$  and also the lengths of pipes  $BC$  and  $BD$  are equal and hence discharge through  $BC$  and  $BD$  will be same. This means  $V_1 = V_2$  also

$$\begin{aligned}\therefore \frac{\pi}{4} d^2 V &= \frac{\pi}{4} d_1^2 \times V_1 + \frac{\pi}{4} d_1^2 \times V_1 \\ &= 2 \times \frac{\pi}{4} d_1^2 \times V_1 \text{ or } d^2 V = 2d_1^2 V_1\end{aligned}$$

or  $(0.4)^2 \times V = 2 \times (0.3)^2 V_1$  or  $.16V = 0.18 V_1$

$$\therefore V_1 = \frac{0.16}{0.18} V = 0.888 V$$

Substituting this value of  $V_1$  in equation (i), we get

$$10 = 15.29 V^2 + (10.19)(.888)^2 V^2 = 15.29 V^2 + 8.035 V^2 = 23.325 V^2$$

$$\therefore V = \sqrt{\frac{10}{23.325}} = 0.654 \text{ m/s}$$

$\therefore$  Discharge  $= V \times \text{Area}$

$$= 0.654 \times \frac{\pi}{4} d^2 = 0.654 \times \frac{\pi}{4} (0.4)^2 = .0822 \text{ m}^3/\text{s. Ans.}$$

**Problem 11.40** Two sharp ended pipes of diameters 50 mm and 100 mm respectively, each of length 100 m are connected in parallel between two reservoirs which have a difference of level of 10 m. If the co-efficient of friction for each pipe is (4f) 0.32, calculate the rate of flow for each pipe and also the diameter of a single pipe 100 m long which would give the same discharge, if it were substituted for the original two pipes.

**Solution.** Given :

Dia. of 1st pipe,  $d_1 = 50 \text{ mm} = 0.05 \text{ m}$

Length of 1st pipe,  $L_1 = 100 \text{ m}$

Dia. of 2nd pipe,  $d_2 = 100 \text{ mm} = 0.10 \text{ m}$

Length of 2nd pipe,  $L_2 = 100 \text{ m}$

Difference in level in reservoirs,  $H = 10 \text{ m}$

Co-efficient of friction  $4f = 0.32$

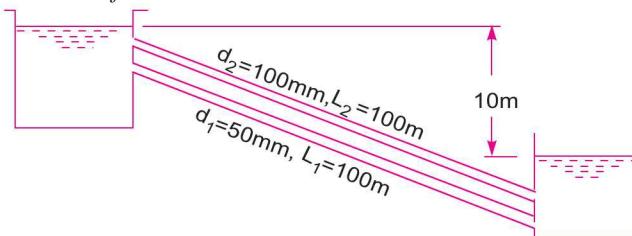


Fig. 11.24

Let  $V_1$  = velocity of flow in pipe 1, and  
 $V_2$  = velocity of flow in pipe 2.

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When the pipes are connected in parallel, the loss of head will be same in both the pipes.

For the first pipe, loss of head is given as

$$H = \frac{4f \times L_1 \times V_1^2}{d_1 \times 2g} = \frac{0.32 \times 100 \times V_1^2}{0.05 \times 2 \times 9.81} \quad (\because 4f = .32)$$

or

$$10 = 32.619 V_1^2$$

$$\therefore V_1 = \sqrt{\frac{10}{32.619}} = 0.5535 \text{ m/s}$$

$$\begin{aligned} \therefore \text{Rate of flow in 1st pipe, } Q_1 &= V_1 \times A_1 = 0.5536 \times \frac{\pi}{4} (d_1)^2 \\ &= .5536 \times \frac{\pi}{4} (0.05)^2 = .001087 \text{ m}^3/\text{s} = \mathbf{1.087 \text{ litres/s. Ans.}} \end{aligned}$$

For the 2nd pipe, loss of head is given by,

$$10 = H = \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} = \frac{0.32 \times 100 \times V_2^2}{0.10 \times 2 \times 9.81}$$

$$\therefore V_2 = \sqrt{\frac{10 \times 0.10 \times 2 \times 9.81}{.32 \times 100}} = 0.783 \text{ m/s}$$

$$\begin{aligned} \therefore \text{Rate of flow in 2nd pipe, } Q_2 &= A_2 \times V_2 = \frac{\pi}{4} d_2^2 \times V_2 \\ &= \frac{\pi}{4} (.1)^2 \times .783 = 0.00615 \text{ m}^3/\text{s} = \mathbf{6.15 \text{ litres/s. Ans.}} \end{aligned}$$

Let

$D$  = diameter of a single pipe which is substituted for the two original pipes

$L$  = length of single pipe = 100 m

$V$  = velocity through pipe

The discharge through single pipe,

$$Q = Q_1 + Q_2 = 1.087 + 6.15 = 7.237 \text{ litres/s} = .007237 \text{ m}^3/\text{s}$$

$$\therefore V = \frac{Q}{\text{Area}} = \frac{.007237}{\frac{\pi}{4} D^2} = \frac{4 \times .007237}{\pi D^2} = \frac{.009214}{D^2} \text{ m/s}$$

Loss of head through single pipe is

$$H = \frac{4f \times L \times V^2}{D \times 2g} = \frac{0.32 \times 100 \times \left(\frac{.009214}{D^2}\right)^2}{D \times 2 \times 9.81}$$

$$\text{or} \quad 10.0 = \frac{.32 \times 100 \times .009214^2}{2 \times 9.81 \times D^5} = \frac{.0001384}{D^5}$$

$$\text{or} \quad D^5 = \frac{.0001384}{10} = .00001384$$

$$\therefore D = (.00001384)^{1/5} = 0.1067 \text{ m} = \mathbf{106.7 \text{ mm. Ans.}}$$

**Problem 11.41** Two reservoirs are connected by a pipe line of diameter 600 mm and length 4000 m. The difference of water level in the reservoirs is 20 m. At a distance of 1000 m from the upper reservoir, a small pipe is connected to the pipe line. The water can be taken from the small pipe. Find the discharge to the lower reservoir, if

- (i) No water is taken from the small pipe, and
- (ii) 100 litres/s of water is taken from small pipe.

Take  $f = .005$  and neglect minor losses.

**Solution.** Given :

$$\text{Dia. of pipe, } d = 600 \text{ mm} = 0.60 \text{ m}$$

$$\text{Length of pipe, } L = 4000 \text{ m}$$

$$\text{Difference of water level, } H = 20 \text{ m}, f = .005$$

- (i) No water is taken from small pipe

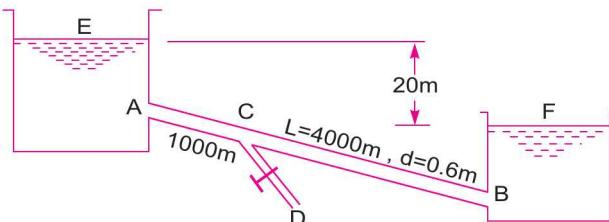


Fig. 11.25

$$\text{The head loss due to friction in pipe } AB = \frac{4f \times L \times V^2}{d \times 2g} \text{ or } 20 = \frac{4 \times .005 \times 4000 \times V^2}{0.6 \times 2 \times 9.81}$$

$$\therefore V = \sqrt{\frac{20 \times 0.6 \times 2 \times 9.81}{4 \times .005 \times 4000}} = \sqrt{2.943} = 1.715 \text{ m/s}$$

$$\therefore \text{Discharge, } Q = \text{Area} \times V = \frac{\pi}{4} (0.6)^2 \times 1.715 = 0.485 \text{ m}^3/\text{s. Ans.}$$

- (ii) 100 litres of water is taken from small pipe

Let

$$Q_1 = \text{discharge through pipe } AC$$

$$Q_2 = \text{discharge through pipe } CB$$

Then for parallel pipes

$$Q_1 = Q_2 + 100 \text{ litres/s} = Q_2 + 0.1 \text{ m}^3/\text{s}$$

$$\therefore Q_2 = (Q_1 - 0.1) \text{ m}^3/\text{s}$$

... (i)

$$\text{Length of pipe } AC, \quad L_1 = 1000 \text{ m}$$

$$\text{Length of pipe } CB, \quad L_2 = 4000 - 1000 = 3000 \text{ m}$$

Applying Bernoulli's equation to points E and F and taking flow through ABC, we have

$$20 = \frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} \quad \dots (ii)$$

$$\text{where } V_1 = \text{velocity through pipe } AC = \frac{Q_1}{\frac{\pi}{4}(0.6)^2} = \frac{4Q_1}{\pi \times .36}$$

$$d_1 = \text{dia. of pipe } AC = 0.6$$

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$$V_2 = \text{velocity through pipe } CB = \frac{Q_2}{\frac{\pi}{4}(0.6)^2} = \frac{4Q_2}{\pi \times .36}$$

$d_2$  = dia. of pipe  $CB = 0.6$

Substituting these values in equation (ii), we get

$$\begin{aligned} 20 &= \frac{4 \times .005 \times 1000}{0.6 \times 2 \times 9.81} \times \left( \frac{4Q_1}{\pi \times .36} \right)^2 + \frac{4 \times .005 \times 3000}{0.6 \times 2 \times 9.81} \times \left( \frac{4Q_2}{\pi \times .36} \right)^2 \\ 20 &= 21.25 Q_1^2 + 63.75 Q_2^2 \end{aligned} \quad \dots(iii)$$

But from (i),

$$Q_2 = Q_1 - 0.1 \text{ or } Q_1 = Q_2 + 0.1$$

Substituting the value of  $Q_1$  in equation (iii), we get

$$\begin{aligned} 20 &= 21.25 (Q_2 + 0.1)^2 + 63.75 Q_2^2 \\ &= 21.55 [Q_2^2 + .01 + 0.2 Q_2] + 63.75 Q_2^2 \\ &= 21.25 Q_2^2 + 0.2125 + 4.250 Q_2 + 63.75 Q_2^2 \\ &= 85 Q_2^2 + 4.25 Q_2 + .2125 \end{aligned}$$

$$\text{or } 85 Q_2^2 + 4.25 Q_2 - 19.7875 = 0$$

This is a quadratic equation in  $Q_2$

$$\begin{aligned} \therefore Q_2 &= \frac{-4.25 \pm \sqrt{4.25^2 + 4 \times 85 \times 19.7875}}{2 \times 85} \\ &= \frac{-4.25 \pm \sqrt{18.0625 + 6727.75}}{170} = \frac{-44.25 \pm 82.13}{170} = \frac{82.13 - 4.25}{170} \\ &= 0.458 \text{ m}^3/\text{s} \quad (\text{Neglecting negative root}) \end{aligned}$$

$\therefore$  Discharge to lower reservoir =  $Q_2 = 0.458 \text{ m}^3/\text{s}$ . Ans.

### ► 11.10 FLOW THROUGH BRANCHED PIPES

When three or more reservoirs are connected by means of pipes, having one or more junctions, the system is called a branching pipe system. Fig. 11.26 shows three reservoirs at different levels connected to a single junction, by means of pipes which are called branched pipes. The lengths, diameters and co-efficient of friction of each pipes is given. It is required to find the discharge and direction of flow in each pipe. The basic equations used for solving such problems are :

1. **Continuity equation** which means the inflow of fluid at the junction should be equal to the outflow of fluid.

2. **Bernoulli's equation, and**

3. **Darcy-Weisbach equation**

Also it is assumed that reservoirs are very large and the water surface levels in the reservoirs are constant so that steady conditions exist in the pipes. Also minor losses are assumed very small. The flow from reservoir  $A$  takes place to junction  $D$ . The flow from junction  $D$  is towards reservoirs  $C$ . Now the flow from junction  $D$  towards reservoir  $B$  will take place only when piezometric head at  $D$  (which is equal to  $\frac{p_D}{\rho g} + Z_D$ ) is more than the piezometric head at  $B$  (i.e.,  $Z_B$ ). Let us consider that flow is from  $D$  to reservoir  $B$ .

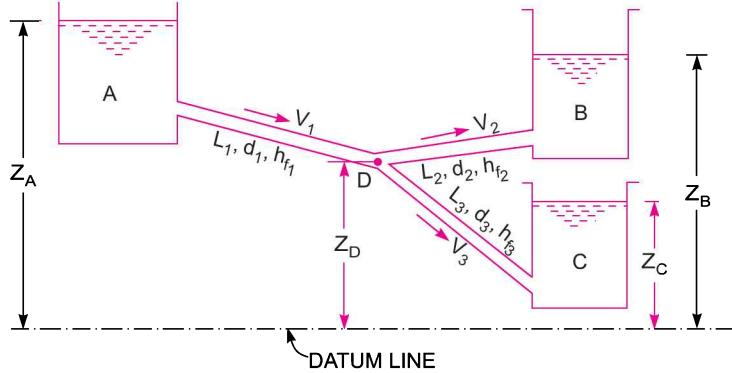


Fig. 11.26

For flow from *A* to *D* from Bernoulli's equation

$$Z_A = Z_D + \frac{p_D}{\rho g} + h_{f1} \quad \dots(i)$$

For flow from *D* to *B* from Bernoulli's equation

$$Z_D + \frac{p_D}{\rho g} = Z_B + h_{f2} \quad \dots(ii)$$

For flow from *D* to *C* from Bernoulli's equation

$$Z_D + \frac{p_D}{\rho g} = Z_C + h_{f3} \quad \dots(iii)$$

From continuity equation,

Discharge through *AD* = Discharge through *DB* + Discharge through *DC*

$$\therefore \frac{\pi}{4} d_1^2 V_1 = \frac{\pi}{4} d_2^2 \times V_2 + \frac{\pi}{4} d_3^2 V_3$$

$$\text{or } d_1^2 V_1 = d_2^2 V_2 + d_3^2 V_3 \quad \dots(iv)$$

There are four unknowns i.e.,  $V_1$ ,  $V_2$ ,  $V_3$  and  $\frac{p_D}{\rho g}$  and there are four equations (i), (ii), (iii) and (iv).

Hence unknown can be calculated.

**Problem 11.42** Three reservoirs *A*, *B* and *C* are connected by a pipe system shown in Fig. 11.27. Find the discharge into or from the reservoirs *B* and *C* if the rate of flow from reservoirs *A* is 60 litres/s. Find the height of water level in the reservoir *C*. Take  $f = .006$  for all pipes.

**Solution.** Given :

Length of pipe *AD*,  $L_1 = 1200 \text{ m}$

Dia. of pipe *AD*,  $d_1 = 30 \text{ cm} = 0.30 \text{ m}$

Discharge through *AD*,  $Q_1 = 60 \text{ litres/s} = 0.06 \text{ m}^3/\text{s}$

Height of water level in *A* from reference line,  $Z_A = 40 \text{ m}$

For pipe *DB*, length  $L_2 = 600 \text{ m}$ , dia.,  $d_2 = 20 \text{ cm} = 0.20 \text{ m}$ ,  $Z_B = 38.0$

For pipe *DC*, length  $L_3 = 800 \text{ m}$ , dia.,  $d_3 = 30 \text{ cm} = 0.30 \text{ m}$

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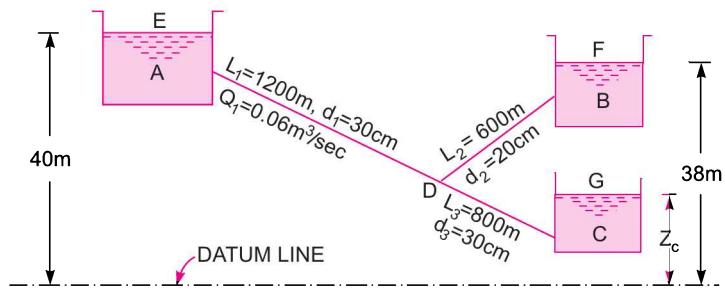


Fig. 11.27

Applying Bernoulli's equations to points E and D,  $Z_A = Z_D + \frac{p_D}{\rho g} + h_{f_1}$

where  $h_{f_1} = \frac{4 \cdot f \cdot L_1 \cdot V_1^2}{d_1 \times 2g}$ , where  $V_1 = \frac{Q_1}{\text{Area}} = \frac{0.06}{\frac{\pi}{4}(0.3)^2} = 0.848 \text{ m/sec}$

$$h_{f_1} = \frac{4 \times 0.006 \times 1200 \times 0.848^2}{0.3 \times 2 \times 9.81} = 3.518 \text{ m}$$

$$\therefore Z_A = Z_D + \frac{p_D}{\rho g} + 3.518 \text{ or } 40.0 = Z_D + \frac{p_D}{\rho g} + 3.518$$

$$\therefore \left( Z_D + \frac{p_D}{\rho g} \right) = 40.0 - 3.518 = 36.482 \text{ m}$$

Hence piezometric head at D = 36.482. But  $Z_B = 38 \text{ m}$ . Hence water flows from B to D.

Applying Bernoulli's equation to points B and D

$$Z_B = \left( Z_D + \frac{p_D}{\rho g} \right) + h_{f_2} \text{ or } 38 = 36.482 + h_{f_2}$$

$$\therefore h_{f_2} = 38 - 36.482 = 1.518 \text{ m}$$

$$\text{But } h_{f_2} = \frac{4 \cdot f \cdot L_2 \cdot V_2^2}{d_2 \times 2g} = \frac{4 \times 0.006 \times 600 \times V_2^2}{0.2 \times 2 \times 9.81}$$

$$\therefore 1.518 = \frac{4 \times 0.006 \times 600 \times V_2^2}{0.2 \times 2 \times 9.81}$$

$$\therefore V_2 = \sqrt{\frac{1.518 \times 0.2 \times 2 \times 9.81}{4 \times 0.006 \times 600}} = 0.643 \text{ m/s.}$$

$$\therefore \text{Discharge, } Q_2 = V_2 \times \frac{\pi}{4} (d_2)^2 = 0.643 \times \frac{\pi}{2} \times (.2)^2 \\ = 0.0202 \text{ m}^3/\text{s} = \mathbf{20.2 \text{ litres/s. Ans.}}$$

Applying Bernoulli's equation to points D and C

$$Z_D + \frac{p_D}{\rho g} = Z_C + h_{f_3}$$

or  $36.482 = Z_C + \frac{4f \cdot L_3 \cdot V_3^2}{d_3 \times 2g}$ , where  $V_3 = \frac{Q_3}{\frac{\pi}{4} d_3^2}$

But from continuity  $Q_1 + Q_2 = Q_3$

$$\therefore Q_3 = Q_1 + Q_2 = 0.06 + 0.0202 = 0.0802 \text{ m}^3/\text{s}$$

$$\therefore V_3 = \frac{Q_3}{\frac{\pi}{4} (0.3)^2} = \frac{0.0802}{\frac{\pi}{4} (0.09)} = 1.134 \text{ m/s}$$

$$\therefore 36.482 = Z_C + \frac{4 \times 0.005 \times 800 \times 1.134^2}{0.3 \times 2 \times 9.81} = Z_C + 4.194$$

$$\therefore Z_C = 36.482 - 4.194 = 32.288 \text{ m. Ans.}$$

**Problem 11.43** Three reservoirs, A, B and C are connected by a pipe system shown in Fig. 11.28. The lengths and diameters of pipes 1, 2 and 3 are 800 m, 1000 m, 800 mm, 200 mm and 150 mm respectively. Determine the piezometric head at junction D. Take  $f = .005$ .

**Solution.** Given :

The length of pipe 1,  $L_1 = 800 \text{ m}$  and its dia.,  $d_1 = 300 \text{ mm} = 0.3 \text{ m}$

The length of pipe 2,  $L_2 = 1000 \text{ m}$  and its dia.,  $d_2 = 200 \text{ mm} = 0.2 \text{ m}$

The length of pipe 3,  $L_3 = 800 \text{ m}$  and its dia.,  $d_3 = 150 \text{ mm} = 0.15 \text{ m}$

Height of reservoir, A from datum line,  $Z_A = 60 \text{ m}$

Similarly,  $Z_B = 40 \text{ m}$  and  $Z_C = 30 \text{ m}$ .

The direction of flow in pipes are shown (given) in Fig. 11.28. Applying Bernoulli's equation to points A and D

$$Z_A = \left( Z_D + \frac{p_D}{\rho g} \right) + h_{f_1}$$

or  $\left[ Z_A - \left( Z_D + \frac{p_D}{\rho g} \right) \right] = h_{f_1} = \frac{4 \times f \times L_1 \times V_1^2}{d_1 \times 2g} = \frac{4 \times 0.005 \times 800 \times V_1^2}{0.3 \times 2 \times 9.81}$

or  $60 - \left( Z_D + \frac{p_D}{\rho g} \right) = 2.718 V_1^2 \quad \dots(i)$

Applying Bernoulli's equation to points D and B

$$\begin{aligned} \left( Z_D + \frac{p_D}{\rho g} \right) &= Z_B + h_{f_2} = 40 + \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} \\ &= 40 + \frac{4 \times 0.005 \times 1000 \times V_2^2}{0.2 \times 2 \times 9.81} = 40.0 + 5.09 V_2^2 \end{aligned}$$

or  $\left( Z_D + \frac{p_D}{\rho g} \right) - 40.0 = 5.09 V_2^2 \quad \dots(ii)$

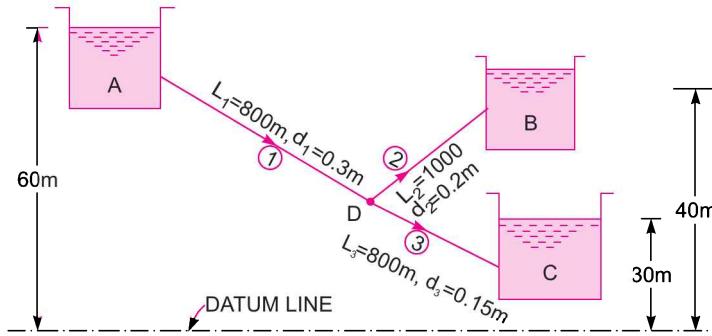


Fig. 11.28

Applying Bernoulli's equation to points D and C

$$\left( Z_D + \frac{P_D}{\rho g} \right) = Z_C + h_{f_3} = 30 + \frac{4f \times L_3 \times V_3^2}{d_3 \times 2g} = 30 + \frac{4 \times .005 \times 800 \times V_3^2}{0.15 \times 2 \times 9.81}$$

$$\text{or } \left( Z_D + \frac{P_D}{\rho g} \right) = 30.0 + 5.436 V_3^2 \quad \dots(iii)$$

Adding (i) and (ii), we have  $60 - 40 = 2.718 V_1^2 + 5.09 V_2^2$

$$\text{or } 20 = 2.718 V_1^2 + 5.09 V_2^2 \quad \dots(iv)$$

Adding (i) and (iii), we have  $60 = 2.718 V_1^2 + 30.0 + 5.436 V_3^2$

$$\text{or } 60 - 30 = 30 = 2.718 V_1^2 + 5.436 V_3^2 \quad \dots(v)$$

Also from continuity equation, we have

$$Q_1 = Q_2 + Q_3$$

$$\text{or } \frac{\pi}{4} d_1^2 \times V_1 = \frac{\pi}{4} d_2^2 V_2 + \frac{\pi}{4} d_3^2 V_3 \text{ or } d_1^2 V_1 = d_2^2 V_2 + d_3^2 V_3$$

$$\text{or } 0.3^2 V_1 = 0.2^2 V_2 + 0.15^2 \times V_3 \text{ or } .09 V_1 = .04 V_2 + .0225 V_3 \quad \dots(vi)$$

$$\text{Now from (iv), } V_2 = \sqrt{\frac{20 - 2.718 V_1^2}{5.09}} \quad \dots(vii)$$

$$\text{And from (v), } V_3 = \sqrt{\frac{30 - 2.718 V_1^2}{5.436}} \quad \dots(viii)$$

Substituting the value of  $V_2$  and  $V_3$  in (vi), we get

$$0.09 V_1 = .04 \sqrt{\frac{20 - 2.718 V_1^2}{5.09}} + .0225 \sqrt{\frac{30 - 2.718 V_1^2}{5.436}}$$

Squaring both sides, we get

$$\begin{aligned} (0.09 V_1)^2 &= (.04)^2 \times \left( \frac{20 - 2.718 V_1^2}{5.09} \right) + (0.0225)^2 \times \frac{30 - 2.718 V_1^2}{5.436} + 2 \times .04 \\ &\quad \times .0225 \times \sqrt{\frac{20 - 2.718 V_1^2}{5.09}} \times \sqrt{\frac{30 - 2.718 V_1^2}{5.436}} \end{aligned}$$

$$\text{or } .0081 V_1^2 = .00628 - .000854 V_1^2 + .00279 - .000253 V_1^2 + .0018$$

$$\text{or } .0081 V_1^2 + .000854 V_1^2 + .000253 V_1^2 = .00628 + .00279 + .0018 = .01087$$

$$\text{or } .009207 V_1^2 = .01087$$

$$\therefore V_1 = \sqrt{\frac{.01087}{.009207}} = 1.086 \text{ m/s}$$

Substituting this value of  $V_1$  in (vii) and (viii)

$$V_2 = \sqrt{\frac{20 - 2.718 \times V_1^2}{5.09}} = \sqrt{\frac{20 - 2.718 \times 1.086^2}{5.09}} = 1.816 \text{ m/s}$$

$$\therefore V_3 = \sqrt{\frac{30 - 2.718 \times 1.086^2}{5.436}} = 2.22 \text{ m/s}$$

$$\begin{aligned} \text{Piezometric head at } D &= Z_D + \frac{p_D}{\rho g} = 30.0 + 5.436 \times V_3^2 \\ &= 30.0 + 5.436 \times (2.22)^2 = 56.79 \text{ m. Ans.} \end{aligned}$$

**Problem 11.44** A pipe line 60 cm diameter bifurcates at a Y-junction into two branches 40 cm and 30 cm in diameter. If the rate of flow in the main pipe is 1.5 m<sup>3</sup>/s and mean velocity of flow in 30 cm diameter pipe is 7.5 m/s, determine the rate of flow in the 40 cm diameter pipe.

**Solution.** Given :

$$\text{Dia. of main pipe, } D = 60 \text{ cm} = 0.6 \text{ m}$$

$$\text{Dia. of branch pipe 1, } D_1 = 40 \text{ cm} = 0.4 \text{ m}$$

$$\text{Dia. of branch pipe 2, } D_2 = 30 \text{ cm} = 0.3 \text{ m}$$

$$\text{Velocity in branch pipe 2, } V_2 = 7.5 \text{ m/s}$$

$$\text{Rate of flow in main pipe, } Q = 1.5 \text{ m}^3/\text{s}$$

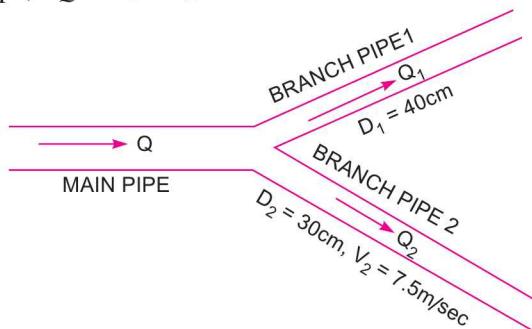


Fig. 11.29

Let  $Q_1$  = Rate of flow in branch pipe 1,

$Q_2$  = Rate of flow in branch pipe 2,

$Q$  = Rate of flow in main pipe,

Now rate of flow in main pipe is equal to the sum of rate of flow in branch pipes.

$$\therefore Q = Q_1 + Q_2 \quad \dots(i)$$

But  $Q_2 = \text{Area of branch pipe 2} \times \text{Velocity in branch pipe 2}$

$$= A_2 \times V_2 = \frac{\pi}{4} D_2^2 \times V_2 = \frac{\pi}{4} (0.3)^2 \times 7.5 = 0.53 \text{ m}^3/\text{s}$$

Substituting the values of  $Q$  and  $Q_2$  in equation (i), we get

$$\begin{aligned} 1.5 &= Q_1 + 0.53 \\ \therefore Q_1 &= 1.5 - 0.53 = 0.97 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

### ► 11.11 POWER TRANSMISSION THROUGH PIPES

Power is transmitted through pipes by flowing water or other liquids flowing through them. The power transmitted depends upon (i) the weight of liquid flowing through the pipe and (ii) the total head available at the end of the pipe. Consider a pipe  $AB$  connected to a tank as shown in Fig. 11.30. The power available at the end  $B$  of the pipe and the condition for maximum transmission of power will be obtained as mentioned below :

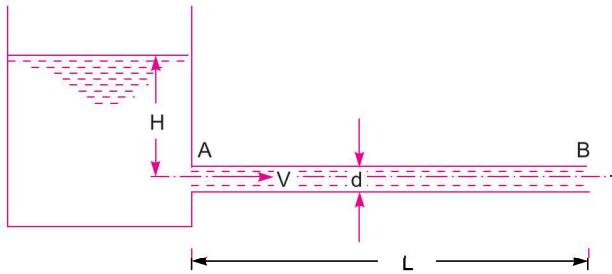


Fig. 11.30 Power transmission through pipe.

Let

$L$  = length of the pipe,

$d$  = diameter of the pipe,

$H$  = total head available at the inlet of pipe,

$V$  = velocity of flow in pipe,

$h_f$  = loss of head due to friction, and  $f$  = co-efficient of friction.

The head available at the outlet of the pipe, if minor losses are neglected

$$= \text{Total head at inlet} - \text{loss of head due to friction}$$

$$= H - h_f = H - \frac{4f \times L \times V^2}{d \times 2g} \quad \left( \because h_f = \frac{4f \times L \times V^2}{d \times 2g} \right)$$

Weight of water flowing through pipe per sec,

$$W = \rho g \times \text{volume of water per sec} = \rho g \times \text{Area} \times \text{Velocity}$$

$$= \rho g \times \frac{\pi}{4} d^2 \times V$$

∴ The power transmitted at the outlet of the pipe

$$= \text{weight of water per sec} \times \text{head at outlet}$$

$$= \left( \rho g \times \frac{\pi}{4} d^2 \times V \right) \times \left( H - \frac{4f \times L \times V^2}{d \times 2g} \right) \text{ Watts}$$

∴ Power transmitted at outlet of the pipe,

$$P = \frac{\rho g}{1000} \times \frac{\pi}{4} d^2 \times V \left( H - \frac{4fL V^2}{d \times 2g} \right) \text{ kW} \quad \dots(11.21)$$

Efficiency of power transmission,

$$\begin{aligned}\eta &= \frac{\text{Power available at outlet of the pipe}}{\text{Power supplied at the inlet of the pipe}} \\ &= \frac{\text{Weight of water per sec} \times \text{Head available at outlet}}{\text{Weight of water per sec} \times \text{Head at inlet}} \\ &= \frac{W \times (H - h_f)}{W \times H} = \frac{H - h_f}{H}. \quad \dots(11.22)\end{aligned}$$

**11.11.1 Condition for Maximum Transmission of Power.** The condition for maximum transmission of power is obtained by differentiating equation (11.21) with respect to  $V$  and equating the same to zero.

Thus  $\frac{d}{dV}(P) = 0$

or  $\frac{d}{dV} \left[ \frac{\rho g}{1000} \times \frac{\pi}{4} d^2 \left( HV - \frac{4fLV^3}{d \times 2g} \right) \right] = 0$

or  $\frac{\rho g}{1000} \times \frac{\pi}{4} d^2 \left( H - \frac{4 \times 3 \times f \times L \times V^2}{d \times 2g} \right) = 0$

or  $H - 3 \times \frac{4fLV^2}{d \times 2g} = 0 \quad \text{or} \quad H - 3 \times h_f = 0 \quad \left( \because \frac{4fLV^2}{d \times 2g} = h_f \right)$

$\therefore H = 3h_f \quad \text{or} \quad h_f = \frac{H}{3} \quad \dots(11.23)$

Equating (11.23) is the condition for maximum transmission of power. It states that power transmitted through a pipe is maximum when the loss of head due to friction is one-third of the total head at inlet.

**11.11.2 Maximum Efficiency of Transmission of Power.** Efficiency of power transmission through pipe is given by equation (11.22) as

$$\eta = \frac{H - h_f}{H}$$

For maximum power transmission through pipe the condition is given by equation (11.23) as

$$h_f = \frac{H}{3}$$

Substituting the value of  $h_f$  in efficiency, we get maximum  $\eta$ ,

$$\eta_{\max} = \frac{H - H/3}{H} = 1 - \frac{1}{3} = \frac{2}{3} \text{ or } 66.7\%. \quad \dots(11.24)$$

**Problem 11.45** A pipe of diameter 300 mm and length 3500 m is used for the transmission of power by water. The total head at the inlet of the pipe is 500 m. Find the maximum power available at the outlet of the pipe, if the value of  $f = .006$ .

**Solution.** Given :

Diameter of the pipe,  $d = 300 \text{ mm} = 0.30 \text{ m}$

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Length of the pipe,  $L = 3500 \text{ m}$

Total head at inlet,  $H = 500 \text{ m}$

Co-efficient of friction,  $f = .006$

For maximum power transmission, using equation (11.23)

$$h_f = \frac{H}{3} = \frac{500}{3} = 166.7 \text{ m}$$

Now  $h_f = \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .006 \times 3500 \times V^2}{0.3 \times 2 \times 9.81} = 14.27 V^2$

Equating the two values of  $h_f$ , we get

$$166.7 = 14.27 V^2 \text{ or } V = \sqrt{\frac{166.7}{14.27}} = 3.417 \text{ m/s}$$

$\therefore$  Discharge,  $Q = V \times \text{Area}$

$$= 3.417 \times \frac{\pi}{4} (d)^2 = 3.417 \times \frac{\pi}{4} (.3)^2 = 0.2415 \text{ m}^3/\text{s}$$

Head available at the end of the pipe

$$= H - h_f = H - \frac{H}{3} = \frac{2H}{3} = \frac{2 \times 500}{3} = 333.33 \text{ m}$$

$$\begin{aligned} \therefore \text{Maximum power available} &= \frac{\rho g \times Q \times \text{head at the end of pipe}}{1000} \text{ kW} \\ &= \frac{1000 \times 9.81 \times 0.2415 \times 333.33}{1000} \text{ kW} = 689.7 \text{ kW. Ans.} \end{aligned}$$

**Problem 11.46** A pipe line of length 2000 m is used for power transmission. If 110.3625 kW power is to be transmitted through the pipe in which water having a pressure of  $490.5 \text{ N/cm}^2$  at inlet is flowing. Find the diameter of the pipe and efficiency of transmission if the pressure drop over the length of pipe is  $98.1 \text{ N/cm}^2$ . Take  $f = .0065$ .

**Solution.** Given :

Length of pipe,  $L = 2000 \text{ m}$

Power transmitted  $= 110.3625 \text{ kW}$

Pressure at inlet,  $p = 490.5 \text{ N/cm}^2 = 490.5 \times 10^4 \text{ N/m}^2$

$$\therefore \text{Pressure head at inlet, } H = \frac{p}{\rho g} = \frac{490.5 \times 10^4}{1000 \times 9.81} = 500 \text{ m} \quad [\because \rho = 1000]$$

Pressure drop  $= 98.1 \text{ N/cm}^2 = 98.1 \times 10^4 \text{ N/m}^2$

$$\therefore \text{Loss of head, } h_f = \frac{98.1 \times 10^4}{\rho g} = \frac{98.1 \times 10^4}{1000 \times 9.81} = 100 \text{ m}$$

Co-efficient of friction,  $f = .0065$

Head available at the end of the pipe  $= H - h_f = 500 - 100 = 400 \text{ m}$

Let the diameter of the pipe  $= d$

Now power transmitted is given by,  $P = \frac{\rho g \times Q \times (H - h_f)}{1000}$  kW

or  $110.3625 = \frac{1000 \times 9.81 \times Q \times 400}{1000}$

$$\therefore Q = \frac{110.3625 \times 1000}{1000 \times 9.81 \times 400} = 0.02812$$

But discharge,  $Q = \text{Area} \times \text{Velocity} = \frac{\pi}{4} d^2 \times V$

$$\therefore \frac{\pi}{4} d^2 \times V = .02812$$

$$\therefore V = \frac{.02812 \times 4}{\pi d^2} = \frac{0.0358}{d^2} \quad \dots(i)$$

The head lost due to friction,  $h_f = \frac{4f \times L \times V^2}{d \times 2g}$

But  $h_f = 100 \text{ m}$

$$\begin{aligned} \therefore 100 &= \frac{4 \times f \times L \times V^2}{d \times 2g} = \frac{4 \times .0065 \times 2000 \times V^2}{d \times 2 \times 9.81} \\ &= \frac{2.65 \times V^2}{d} = \frac{2.65}{d} \times \left( \frac{.0358}{d^2} \right)^2 = \frac{.003396}{d^5} \end{aligned}$$

$\therefore$  From equation (i),  $V = \frac{.0358}{d^2}$

$$\therefore 100 = \frac{.003396}{d^5}$$

or  $d = \left( \frac{.003396}{100} \right)^{1/5} = 0.1277 \text{ m} = 127.7 \text{ mm. Ans.}$

Efficiency of power transmission is given by equation (11.22),

$$\eta = \frac{H - h_f}{H} = \frac{500 - 100}{500} = 0.80 = 80\%. \text{ Ans.}$$

**Problem 11.47** For Problem 11.46, find : (i) the diameter of the pipe corresponding to maximum efficiency of transmission, (ii) diameter of the pipe corresponding to 90% efficiency of transmission.

**Solution.** (i) Diameter of pipe corresponding to maximum efficiency.

Let the dia. of pipe for  $\eta_{\max} = d$

But from equation (11.24),  $\eta_{\max} = 66.67\% = \frac{2}{3}$

or  $\frac{H - h_f}{H} = \frac{2}{3} \text{ or } \frac{500 - h_f}{500} = \frac{2}{3}$

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or  $h_f = 500 - 500 \times \frac{2}{3} = \frac{1500 - 1000}{3} = \frac{500}{3} = 166.7 \text{ m}$

The other data given from Problem 11.46,

Power transmitted = 110.3625

Length of pipe,  $L = 2000 \text{ m}$

Co-efficient of friction,  $f = .0065$

Power transmitted is given by the relation,

$$P = \frac{\rho g \times Q \times (H - h_f)}{1000}$$

or  $110.3625 = \frac{1000 \times 9.81 \times Q \times (500 - 166.7)}{1000}$

or  $Q = \frac{110.3625 \times 1000}{1000 \times 9.81 \times (500 - 166.7)} = 0.03375 \text{ m}^3/\text{s}$

But

$Q = \text{area of pipe} \times \text{velocity of flow}$

$$= \frac{\pi}{4} d^2 \times V \{ \text{where } V = \text{velocity of flow} \}$$

$\therefore 0.03375 = \frac{\pi}{4} d^2 \times V$

$\therefore V = \frac{0.03375 \times 4}{\pi \times d^2} = \frac{0.04297}{d^2} \quad \dots(i)$

Now the head lost due to friction,  $h_f = \frac{4fLV^2}{d \times 2g}$

But  $h_f = 166.7 \text{ m}$

$\therefore 166.7 = \frac{4 \times .0065 \times 2000 \times V^2}{d \times 2 \times 9.81}$

$$= \frac{2.65 V^2}{d} = \frac{2.65}{d} \times \left( \frac{.04297}{d^2} \right)^2 = \frac{.00489}{d^5} \quad \left( \because V = \frac{.04297}{d^2} \right)$$

$\therefore d^5 = \frac{.00489}{166.7} = .00002933$

$\therefore d = (.00002933)^{1/5} = 0.1240 \text{ m} = 124 \text{ mm. Ans.}$

(ii) Let the diameter of pipe, when efficiency of transmission is 90% =  $d$

$$\eta = 90\% = 0.9$$

But  $\eta$  is given by equation (11.22) as,  $\eta = \frac{H - h_f}{H} = 0.9$

But  $H = 500 \text{ m}$

$$\therefore \frac{500 - h_f}{500} = 0.9 \text{ or } 500 - 500 \times 0.9 = h_f \text{ or } 500 - 450 = h_f$$

$$\therefore h_f = 500 - 450 = 50 \text{ m}$$

The other given data is,  $P = 110.3625, L = 2000, f = .0065$

$$\text{Using relation for power transmission, } P = \frac{\rho g \times Q \times (H - h_f)}{1000}$$

$$\text{or } 110.3625 = \frac{1000 \times 9.81 \times Q \times (500 - 50)}{1000}$$

$$Q = \frac{110.3625 \times 1000}{1000 \times 9.81 \times (500 - 50)} = .025 \text{ m}^3/\text{s}$$

But

$$Q = \frac{\pi}{4} d^2 \times V$$

$$\therefore \frac{\pi}{4} d^2 \times V = .025 \text{ or } V = \frac{.025 \times 4}{\pi d^2} = \frac{0.03183}{d^2} \quad \dots(i)$$

$$\text{Now the head lost due to friction, } h_f = \frac{4fLV^2}{d \times 2g}$$

$$\text{or } 50 = \frac{4 \times .0065 \times 2000 \times}{d \times 2g} \times \left( \frac{.03183}{d^2} \right)^2 = \frac{.002685}{d^5}$$

$$\therefore d^5 = \frac{.002685}{50} = .0000537$$

$$d = (.0000537)^{1/5} = .1399 \text{ m} \approx \mathbf{140 \text{ mm. Ans.}}$$

## ► 11.12 FLOW THROUGH NOZZLES

Fig. 11.31 shows a nozzle fitted at the end of a long pipe. The total energy at the end of the pipe consists of pressure energy and kinetic energy. By fitting the nozzle at the end of the pipe, the total energy is converted into kinetic energy. Thus nozzles are used, where higher velocities of flow are required. The examples are :

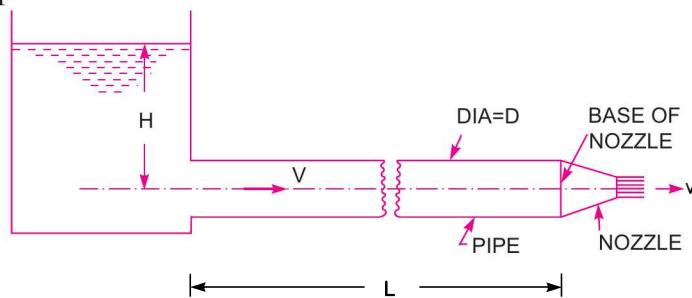


Fig. 11.31 Nozzle fitted to a pipe.

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1. In case of Pelton turbine, the nozzle is fitted at the end of the pipe (called penstock) to increase velocity.

2. In case of the extinguishing fire, a nozzle is fitted at the end of the hose pipe to increase velocity.

Let  $D$  = diameter of the pipe,  $L$  = length of the pipe,

$$A = \text{area of the pipe} = \frac{\pi}{4} D^2,$$

$V$  = velocity of flow in pipe,

$H$  = total head at the inlet of the pipe,

$d$  = diameter of nozzle at outlet,

$v$  = velocity of flow at outlet of nozzle,

$$a = \text{area of the nozzle at outlet} = \frac{\pi}{4} d^2,$$

$f$  = co-efficient of friction for pipe.

$$\text{Loss of head due to friction in pipe, } h_f = \frac{4fLV^2}{2g \times D}$$

$\therefore$  Head available at the end of the pipe or at the base of nozzle

= Head at inlet of pipe – head lost due to friction

$$= H - h_f = \left( H - \frac{4fLV^2}{2g \times D} \right)$$

Neglecting minor losses and also assuming losses in the nozzle negligible, we have

Total head at inlet of pipe = total head (energy) at the outlet of nozzle + losses

$$\text{But total head at outlet of nozzle} = \text{kinetic head} = \frac{v^2}{2g}$$

$$\therefore H = \frac{v^2}{2g} + h_f = \frac{v^2}{2g} + \frac{4fLV^2}{2gD} \quad \left( \because h_f = \frac{4fLV^2}{2gD} \right) \dots(i)$$

From continuity equation in the pipe and outlet of nozzle,

$$AV = av$$

$$\therefore V = \frac{av}{A}$$

Substituting this value in equation (i), we get

$$H = \frac{v^2}{2g} + \frac{4fL}{2gD} \times \left( \frac{av}{A} \right)^2 = \frac{v^2}{2g} + \frac{4fLa^2v^2}{2g \times D \times A^2} = \frac{v^2}{2g} \left( 1 + \frac{4fLa^2}{DA^2} \right)$$

$$\therefore v = \sqrt{\left( 1 + \frac{4fL}{D} \times \frac{a^2}{A^2} \right)} \quad \dots(11.25)$$

$\therefore$  Discharge through nozzle =  $a \times v$ .

**11.12.1 Power Transmitted Through Nozzle.** The kinetic energy of the jet at the outlet of nozzle =  $\frac{1}{2} mv^2$

Now mass of liquid at the outlet of nozzle per second =  $\rho av$

$$\therefore \text{Kinetic energy of the jet at the outlet per sec.} = \frac{1}{2} \rho av \times v^2 = \frac{1}{2} \rho av^3$$

$$\therefore \text{Power in kW at the outlet of nozzle} = (\text{K.E./sec}) \times \frac{1}{1000} = \frac{\frac{1}{2} \rho av^3}{1000}$$

$\therefore$  Efficiency of power transmission through nozzle,

$$\begin{aligned} \eta &= \frac{\text{Power at outlet of nozzle}}{\text{Power at the inlet of pipe}} = \frac{\frac{1}{2} \rho av^3}{\frac{1000}{\rho g \cdot Q \cdot H}} \\ &= \frac{\frac{1}{2} \rho av \cdot v^2}{\rho g \cdot Q \cdot H} = \frac{\frac{1}{2} \rho av \cdot v^2}{\rho g \cdot av \cdot H} \quad \{ \because Q = av \} \\ &= \frac{v^2}{2gH} = \left[ \frac{1}{1 + \frac{4fL}{D} \times \frac{a^2}{A^2}} \right] \quad \dots(11.26) \end{aligned}$$

$$\left( \because \text{From equation (11.25), } \frac{v^2}{2gH} = \left[ \frac{1}{1 + \frac{4fL}{D} \frac{a^2}{A^2}} \right] \right)$$

**11.12.2 Condition for Maximum Power Transmitted Through Nozzle.** We know that, the total head at inlet of pipe = total head at the outlet of the nozzle + losses

$$\begin{aligned} i.e., \quad H &= \frac{v^2}{2g} + h_f & \left[ \because \text{total head at outlet of nozzle} = \frac{v^2}{2g} \text{ and} \right. \\ & \quad \left. h_f = \frac{4 \cdot f \cdot L \cdot V^2}{D \times 2g} = \text{loss of liquid in pipe} \right] \end{aligned}$$

$$= \frac{v^2}{2g} + \frac{4 \cdot f \cdot L \cdot V^2}{D \times 2g}$$

$$\therefore \frac{v^2}{2g} = \left( H - \frac{4 \cdot f \cdot L \cdot V^2}{D \times 2g} \right)$$

$$\text{But power transmitted through nozzle} = \frac{\frac{1}{2} \rho av^3}{1000} = \frac{\frac{1}{2} \rho av}{1000} \times v^2 = \frac{\frac{1}{2} \rho av}{1000} \left[ 2g \left( H - \frac{4 \cdot f \cdot L \cdot V^2}{D \times 2g} \right) \right]$$

$$= \frac{\rho gav}{1000} \left[ H - \frac{4fLV^2}{D \times 2g} \right] \quad \dots(11.27)$$

Now from continuity equation,  $AV = av$

$$\therefore V = \frac{av}{A}$$

Substituting the value of  $V$  in equation (11.27), we get

$$\text{Power transmitted through nozzle} = \frac{\rho gav}{1000} \left[ H - \frac{4fLa^2}{D \times 2g} \frac{v^2}{A^2} \right]$$

The power ( $P$ ) will be maximum, when  $\frac{d(P)}{dv} = 0$

$$\text{or } \frac{d}{dv} \left[ \frac{\rho gav}{1000} \left( H - \frac{4fL}{D \times 2g} \frac{a^2 v^2}{A^2} \right) \right] = 0$$

$$\text{or } \frac{d}{dv} \left[ \frac{\rho ga}{1000} \left( Hv - \frac{4fL}{D \times 2g} \frac{a^2 v^3}{A^2} \right) \right] = 0$$

$$\text{or } \left[ \frac{\rho ga}{1000} \left( H - 3 \frac{4fL}{D \times 2g} \frac{a^2 v^2}{A^2} \right) \right] = 0 \text{ or } H - 3 \times \frac{4fL}{D \times 2g} \times V^2 = 0 \quad \left( \because V = \frac{av}{A} \right)$$

$$\text{or } H - 3h_f = 0 \quad \left( \because \frac{4fLV^2}{D \times 2g} = h_f = \text{head loss in pipe} \right)$$

$$\text{or } h_f = \frac{H}{3} \quad \dots(11.28)$$

Equation (11.28) gives the condition for maximum power transmitted through nozzle. It states that power transmitted through nozzle is maximum when the head lost due to friction in pipe is one-third the total head supplied at the inlet of pipe.

**11.12.3 Diameter of Nozzle for Maximum Transmission of Power Through Nozzle.** For maximum transmission of power, the condition is given by equation (11.28) as,  $h_f = \frac{H}{3}$

$$\text{But } h_f = \frac{4f \cdot L \cdot V^2}{D \times 2g}$$

$$\therefore \frac{4fLV^2}{D \times 2g} = \frac{H}{3} \text{ or } H = 3 \times \frac{4fLV^2}{D \times 2g}$$

But  $H$  is also = total head at outlet of nozzle + losses

$$= \frac{v^2}{2g} + h_f = \frac{v^2}{2g} + \frac{4fLV^2}{D \times 2g}$$

Equating the two values of  $H$ , we get

$$3 \times \frac{4fLV^2}{D \times 2g} = \frac{v^2}{2g} + \frac{4fLV^2}{D \times 2g} \text{ or } \frac{12fLV^2}{D \times 2g} - \frac{4fLV^2}{D \times 2g} = \frac{v^2}{2g}$$

or  $\frac{8fLV^2}{D \times 2g} = \frac{v^2}{2g}$  ... (i)

But from continuity,  $AV = av$  or  $V = \frac{av}{A}$ .

Substituting this value of  $V$  in equation (i), we get

$$\frac{8fL}{D \times 2g} \times \frac{a^2 v^2}{A^2} = \frac{v^2}{2g} \text{ or } \frac{8fL}{D} \times \frac{a^2}{A^2} = 1 \quad \left( \text{Divide by } \frac{v^2}{2g} \right) \dots (\text{ii})$$

or  $\frac{8fL}{D} \times \frac{\left(\frac{\pi d^2}{4}\right)^2}{\left(\frac{\pi D^2}{4}\right)^2} = 1 \text{ or } \frac{8fL}{D} \times \frac{d^4}{D^4} = 1 \text{ or } d^4 = \frac{D^5}{8fL}$

$\therefore d = \left( \frac{D^5}{8fL} \right)^{1/4}$  ... (11.29)

$$\text{From equation (ii), } \frac{8fL}{D} = \frac{A^2}{a^2}$$

$$\therefore \frac{A}{a} = \sqrt{\frac{8fL}{D}} \quad \dots (11.30)$$

Equation (11.30) gives the ratio of the area of the supply pipe to the area of the nozzle and hence from this equation, the diameter of the nozzle can be obtained.

**Problem 11.48** A nozzle is fitted at the end of a pipe of length 300 m and of diameter 100 mm. For the maximum transmission of power through the nozzle, find the diameter of nozzle. Take  $f = .009$ .

**Solution.** Given :

$$\text{Length of pipe, } L = 300 \text{ m}$$

$$\text{Diameter of pipe, } D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Co-efficient of friction, } f = .009$$

$$\text{Let the diameter of nozzle } = d$$

For maximum transmission of power, the diameter of nozzle is given by relation (11.29) as

$$d = \left( \frac{D^5}{8fL} \right)^{1/4} = \left( \frac{0.1^5}{8 \times .009 \times 300} \right)^{1/4} = 0.02608 \text{ m} = \mathbf{26.08 \text{ mm. Ans.}}$$

**Problem 11.49** The head of water at the inlet of a pipe 2000 m long and 500 mm diameter is 60 m. A nozzle of diameter 100 mm at its outlet is fitted to the pipe. Find the velocity of water at the outlet of the nozzle if  $f = .01$  for the pipe.

**Solution.** Given :

$$\text{Head of water at inlet of pipe, } H = 60 \text{ m}$$

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Length of pipe,  $L = 2000 \text{ m}$   
 Dia. of pipe,  $D = 500 \text{ mm} = 0.50 \text{ m}$   
 Dia. of nozzle at outlet,  $d = 100 \text{ mm} = 0.1 \text{ m}$   
 Co-efficient of friction,  $f = .01$

The velocity at outlet of nozzle is given by equation (11.25) as

$$v = \sqrt{\frac{2gH}{1 + \frac{4fL}{D} \times \frac{a^2}{A^2}}} = \sqrt{\frac{2 \times 9.81 \times 60}{1 + \frac{4 \times .01 \times 2000}{0.5} \left( \frac{\pi d^2}{\frac{\pi D^2}{4}} \right)^2}}$$

$$= \sqrt{\frac{2 \times 9.81 \times 60}{1 + \frac{4 \times .01 \times 2000}{0.5} \times \left( \frac{0.1 \times .1}{0.5 \times .5} \right)^2}} = 30.61 \text{ m/s. Ans.}$$

**Problem 11.50** Find the maximum power transmitted by a jet of water discharging freely out of nozzle fitted to a pipe = 300 m long and 100 mm diameter with co-efficient of friction as 0.01. The available head at the nozzle is 90 m.

**Solution.** Given :

Length of pipe,  $L = 300 \text{ m}$   
 Dia. of pipe,  $D = 100 \text{ mm} = 0.1 \text{ m}$   
 Co-efficient of friction,  $f = .01$   
 Head available at nozzle,  $= 90 \text{ m}$

For maximum power transmission through the nozzle, the diameter at the outlet of nozzle is given by equation (11.29) as

$$d = \left( \frac{D^5}{8fL} \right)^{1/4} = \left[ \frac{(0.1)^5}{8 \times .01 \times 300} \right]^{1/4} = .0254 \text{ m}$$

$$\therefore \text{Area at the nozzle, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.0254)^2 = .0005067 \text{ m}^2.$$

The nozzle at the outlet, discharges water into atmosphere and hence the total head available at the nozzle is converted into kinetic head.

$$\therefore \text{Head available at outlet} = v^2/2g \text{ or } 90 = v^2/2g$$

$$\therefore v = \sqrt{2 \times 9.81 \times 90} = 42.02 \text{ m/s}$$

$$\text{Discharge through nozzle, } Q = a \times v = .0005067 \times 42.02 = 0.02129 \text{ m}^3/\text{s}$$

$$\therefore \text{Maximum power transmitted} = \frac{\rho g \times Q \times \text{Head at outlet of nozzle}}{1000}$$

$$= \frac{1000 \times 9.81 \times 0.02129 \times 90}{1000} = 18.796 \text{ kW. Ans.}$$

**Problem 11.51** The rate of flow of water through a pipe of length 2000 m and diameter 1 m is  $2 \text{ m}^3/\text{s}$ . At the end of the pipe a nozzle of outside diameter 300 mm is fitted. Find the power transmitted

through the nozzle if the head of water at inlet of the pipe is 200 m and co-efficient of friction for pipe is 0.01.

**Solution.** Given :

Length of pipe,	$L = 2000 \text{ m}$
Dia. of pipe,	$D = 1 \text{ m}$
Discharge,	$Q = 2 \text{ m}^3/\text{s}$
Dia. of nozzle,	$d = 300 \text{ mm} = 0.3 \text{ m}$
Head at inlet of pipe,	$H = 200 \text{ m}$
Co-efficient of friction,	$f = .01$

$$\text{Now area of pipe, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 1^2 = 0.7854 \text{ m}^2$$

$$\text{Velocity of water through pipe, } V = \frac{Q}{A} = \frac{2.0}{0.7854} = 2.546 \text{ m/s}$$

Power transmitted through nozzle is given by equation (11.27) as

$$\begin{aligned} P &= \frac{\rho g \cdot a \cdot v}{1000} \left[ H - \frac{4fLV^2}{D \times 2g} \right] \\ &= \frac{1000 \times 9.81 \times 2.0}{1000} \left[ 200 - \frac{4 \times .01 \times 2000 \times (2.546)^2}{1 \times 2 \times 9.81} \right] (\because av = Q) \\ &= 3405.43 \text{ kW. Ans.} \end{aligned}$$

### ► 11.13 WATER HAMMER IN PIPES

Consider a long pipe  $AB$  as shown in Fig. 11.32 connected at one end to a tank containing water at a height of  $H$  from the centre of the pipe. At the other end of the pipe, a valve to regulate the flow of water is provided. When the valve is completely open, the water is flowing with a velocity,  $V$  in the pipe. If now the valve is suddenly closed, the momentum of the flowing water will be destroyed and consequently a wave of high pressure will be set up. This wave of high pressure will be transmitted along the pipe with a velocity equal to the velocity of sound wave and may create noise called knocking. Also this wave of high pressure has the effect of hammering action on the walls of the pipe and hence it is also known as water hammer.

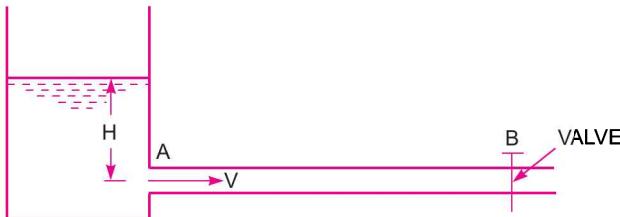


Fig. 11.32 Water hammer.

The pressure rise due to water hammer depends upon : (i) the velocity of flow of water in pipe, (ii) the length of pipe, (iii) time taken to close the valve, (iv) elastic properties of the material of the pipe. The following cases of water hammer in pipes will be considered :

1. Gradual closure of valve,
2. Sudden closure of valve and considering pipe rigid, and

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3. Sudden closure of valve and considering pipe elastic.

**11.13.1 Gradual Closure of Valve.** Let the water is flowing through the pipe  $AB$  shown in Fig. 11.32, and the valve provided at the end of the pipe is closed gradually.

Let       $A$  = area of cross-section of the pipe  $AB$ ,  
 $L$  = length of pipe,  
 $V$  = velocity of flow of water through pipe,  
 $T$  = time in second required to close the valve, and  
 $p$  = intensity of pressure wave produced.

$$\text{Mass of water in pipe } AB = \rho \times \text{volume of water} = \rho \times A \times L$$

The valve is closed gradually in time ' $T$ ' seconds and hence the water is brought from initial velocity  $V$  to zero velocity in time seconds.

$$\therefore \text{Retardation of water} = \frac{\text{Change of velocity}}{\text{Time}} = \frac{V - 0}{T} = \frac{V}{T}$$

$$\therefore \text{Retarding force} = \text{Mass} \times \text{Retardation} = \rho AL \times \frac{V}{T} \quad \dots(i)$$

If  $p$  is the intensity of pressure wave produced due to closure of the valve, the force due to pressure wave,

$$= p \times \text{area of pipe} = p \times A \quad \dots(ii)$$

Equating the two forces, given by equations (i) and (ii),

$$\begin{aligned} \rho AL \times \frac{V}{T} &= p \times A \\ \therefore p &= \frac{\rho LV}{T} \end{aligned} \quad \dots(11.31)$$

$$\text{Head of pressure, } H = \frac{p}{\rho g} = \frac{\rho LV}{\rho g \times T} = \frac{\rho LV}{\rho \times g \times T} \text{ or } H = \frac{LV}{gT} \quad \dots(11.32)$$

$$(i) \text{ The valve closure is said to be gradual if } T > \frac{2L}{C} \quad \dots(11.33)$$

where  $t$  = time in sec,  $C$  = velocity of pressure wave

$$(ii) \text{ The valve closure is said to be sudden if } T < \frac{2L}{C} \quad \dots(11.34)$$

where  $C$  = velocity of pressure wave.

**11.13.2 Sudden Closure of Valve and Pipe is Rigid.** Equation (11.31) gives the relation between increase of pressure due to water hammer in pipe and the time required to close the valve. If  $t = 0$ , the increase in pressure will be infinite. But from experiments, it is observed that the increase in pressure due to water hammer is finite, even for a very rapid closure of valve. Thus equation (11.31) is valid only for (i) incompressible fluids and (ii) when pipe is rigid. But when a wave of high pressure is created, the liquids get compressed to some extent and also pipe material gets stretched. For a sudden closure of valve [the value of  $t$  is small and hence a wave of high pressure is created] the following two cases will be considered :

- (i) Sudden closure of valve and pipe is rigid, and
- (ii) Sudden closure of valve and pipe is elastic.

Consider a pipe  $AB$  in which water is flowing as shown in Fig. 11.32. Let the pipe is rigid and valve fitted at the end  $B$  is closed suddenly.

Let       $A$  = Area of cross-section of pipe  $AB$ ,  
 $L$  = Length of pipe,  
 $V$  = Velocity of flow of water through pipe,  
 $p$  = Intensity of pressure wave produced,  
 $K$  = Bulk modulus of water.

When the valve is closed suddenly, the kinetic energy of the flowing water is converted into strain energy of water if the effect of friction is neglected and pipe wall is assumed perfectly rigid.

$$\therefore \text{Loss of kinetic energy} = \frac{1}{2} \times \text{mass of water in pipe} \times V^2 \\ = \frac{1}{2} \times \rho AL \times V^2 \quad (\because \text{mass} = \rho \times \text{volume} = \rho \times A \times L)$$

$$\text{Gain of strain energy} = \frac{1}{2} \left( \frac{p^2}{K} \right) \times \text{volume} = \frac{1}{2} \frac{p^2}{K} \times AL$$

Equating loss of kinetic energy to gain of strain energy

$$\therefore \frac{1}{2} \rho AL \times V^2 = \frac{1}{2} \frac{p^2}{K} \times AL$$

$$\text{or} \quad p^2 = \frac{1}{2} \rho AL \times V^2 \times \frac{2K}{AL} = \rho KV^2$$

$$\therefore p = \sqrt{\rho KV^2} = V \sqrt{K\rho} = V \sqrt{\frac{K\rho^2}{\rho}} \quad \dots(11.35)$$

$$= \rho V \times C \quad (\because \sqrt{K/\rho} = C) \quad \dots(11.36)$$

where  $C$  = velocity\* of pressure wave.

**11.13.3 Sudden Closure of Valve and Pipe is Elastic.** Consider the pipe  $AB$  in which water is flowing as shown in Fig. 11.32. Let the thickness ' $t$ ' of the pipe wall is small compared to the diameter  $D$  of the pipe and also let the pipe is elastic.

Let       $E$  = Modulus of Elasticity of the pipe material,

$$\frac{1}{m} = \text{Poisson's ratio for pipe material,}$$

$p$  = Increase of pressure due to water hammer,

$t$  = Thickness of the pipe wall,

$D$  = Diameter of the pipe.

When the valve is closed suddenly, a wave of high pressure of intensity  $p$  will be produced in the water. Due to this high pressure  $p$ , circumferential and longitudinal stresses in the pipe wall will be produced.

Let       $f_l$  = Longitudinal stress in pipe

$f_c$  = Circumferential stress in pipe,

$$\text{The magnitude of these stresses are given as } f_l = \frac{pD}{4t} \text{ and } f_c = \frac{pD}{2t}$$

Now from the knowledge of strength of material we know, strain energy stored in pipe material per unit volume

\* For derivation of velocity of pressure wave, please refer to chapter 15.

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$$\begin{aligned}
 &= \frac{1}{2E} \left[ f_l^2 + f_c^2 - \frac{2f_l \times f_c}{m} \right] \\
 &= \frac{1}{2E} \left[ \left( \frac{pD}{4t} \right)^2 + \left( \frac{pD}{2t} \right)^2 - \frac{2 \times \frac{pD}{4t} \times \frac{pD}{2t}}{m} \right] \\
 &= \frac{1}{2E} \left[ \frac{p^2 D^2}{16t^2} + \frac{p^2 D^2}{4t^2} - \frac{p^2 D^2}{4mt^2} \right]
 \end{aligned}$$

Taking  $\frac{1}{m} = \frac{1}{4}$   $\left( i.e., \text{Poisson ratio} = \frac{1}{4} \right)$

$\therefore$  Strain energy stored in pipe material per unit volume

$$= \frac{1}{2E} \left[ \frac{p^2 D^2}{16t^2} + \frac{p^2 D^2}{4t^2} - \frac{p^2 D^2}{4t^2 \times 4} \right] = \frac{1}{2E} \times \frac{p^2 D^2}{4t^2} = \frac{p^2 D^2}{8Et^2}$$

Total volume of pipe material =  $\pi D \times t \times L$ .

$\therefore$  Total strain energy stored in pipe material

$$\begin{aligned}
 &= \text{Strain energy per unit volume} \times \text{total volume} \\
 &= \frac{p^2 D^2}{8Et^2} \times \pi D \times t \times L = \frac{p^2 \pi D^3 L}{8Et} \\
 &= \frac{p^2 \times \pi D^2 \times DL}{8Et} = \frac{p^2 A \times DL}{2Et} \quad \left( \because \frac{\pi D^2}{4} = \text{Area of pipe} = A \right)
 \end{aligned}$$

$$\text{Now loss of kinetic energy of water} = \frac{1}{2} m V^2 = \frac{1}{2} \rho A L \times V^2 \quad (\because m = \rho A L)$$

$$\text{Gain of strain energy in water} = \frac{1}{2} \left( \frac{p^2}{K} \right) \times \text{volume} = \frac{1}{2} \frac{p^2}{K} \times A L$$

Then, loss of kinetic energy of water = Gain of strain energy in water + Strain energy stored in pipe material.

$$\therefore \frac{1}{2} \rho A L \times V^2 = \frac{1}{2} \left( \frac{p^2}{K} \right) \times A L + \frac{p^2 A \times D L}{2 E t}$$

$$\text{Divide by } A L, \quad \frac{\rho V^2}{2} = \frac{1}{2} \frac{p^2}{K} + \frac{p^2 D}{2 E t} = \frac{p^2}{2} \left[ \frac{1}{K} + \frac{D}{E t} \right] \text{ or } \rho V^2 = p^2 \left[ \frac{1}{K} + \frac{D}{E t} \right]$$

$$\therefore p^2 = \frac{\rho V^2}{\frac{1}{K} + \frac{D}{E t}} \text{ or } p = \sqrt{\frac{\rho V^2}{\frac{1}{K} + \frac{D}{E t}}} = V \times \sqrt{\frac{\rho}{\frac{1}{K} + \frac{D}{E t}}} \quad \dots(11.37)$$

#### 11.13.4 Time Taken by Pressure Wave to Travel from the Valve to the Tank and from Tank to the Valve

Let  $T$  = The required time taken by pressure wave

$L$  = Length of the pipe

$C$  = Velocity of pressure wave

Then total distance =  $L + L = 2L$

$$\therefore \text{Time, } T = \frac{\text{Distance}}{\text{Velocity of pressure wave}} = \frac{2L}{C}. \quad \dots(11.38)$$

**Problem 11.52** The water is flowing with a velocity of 1.5 m/s in a pipe of length 2500 m and of diameter 500 mm. At the end of the pipe, a valve is provided. Find the rise in pressure if the valve is closed in 25 seconds. Take the value of  $C = 1460$  m/s.

**Solution.** Given :

Velocity of water,  $V = 1.5$  m/s

Length of pipe,  $L = 2500$  m

Diameter of pipe,  $D = 500$  mm = 0.5 m

Time to close the valve,  $T = 25$  seconds

Value of,  $C = 1460$  m/s

Let the rise in pressure =  $p$

$$\text{The ratio, } \frac{2L}{C} = \frac{2 \times 2500}{1460} = 3.42$$

From equation (11.33), we have if  $T > \frac{2L}{C}$ , the closure of valve is said to be gradual.

$$\text{Here } T = 25 \text{ sec and } \frac{2L}{C} = 3.42$$

$\therefore T > \frac{2L}{C}$  and hence valve is closed gradually.

For gradually closure of valve, the rise in pressure is given by equation (11.31) as

$$p = \frac{\rho VL}{T} = 1000 \times 2500 \times \frac{1.5}{25} = 150000 \text{ N/m}^2$$

$$= \frac{150000}{10^4} \frac{\text{N}}{\text{cm}^2} = 15.0 \frac{\text{N}}{\text{cm}^2}. \text{ Ans.}$$

**Problem 11.53** If in Problem 11.52, the valve is closed in 2 sec, find the rise in pressure behind the valve. Assume the pipe to be rigid one and take Bulk modulus of water. i.e.,  $K = 19.62 \times 10^4$  N/cm<sup>2</sup>.

**Solution.** Given :

$V = 1.5$  m/s,  $L = 2500$  m

$D = 500$  mm = 0.5 m

Time to close the valve,  $T = 2$  sec

Bulk modulus of water,  $K = 19.62 \times 10^4$  N/cm<sup>2</sup>

$$= 19.62 \times 10^4 \times 10^4 \text{ N/m}^2 = 19.62 \times 10^8 \text{ N/m}^2$$

Velocity of pressure wave is given by,

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$$C = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{19.62 \times 10^8}{1000}} = 1400 \text{ m/s} \quad (\because \rho = 1000)$$

The ratio,  $\frac{2L}{C} = \frac{2 \times 2500}{1400} = 3.57 \quad \therefore T < \frac{2L}{C}$ .

$\therefore$  From equation (11.34), if  $T < \frac{2L}{C}$ , valve is closed suddenly. For sudden closure of valve, when pipe is rigid, the rise in pressure is given by equation (11.35) or (11.36) as

$$p = V \sqrt{K\rho} = 1.5 \sqrt{19.62 \times 10^8 \times 1000} \quad (\because \rho = 1000) \\ = 210.1 \times 10^4 \text{ N/m}^2 = 210.1 \text{ N/cm}^2. \text{ Ans.}$$

**Problem 11.54** If in Problem 11.52, the thickness of the pipe is 10 mm and the valve is suddenly closed at the end of the pipe, find the rise in pressure if the pipe is considered to be elastic. Take  $E = 19.62 \times 10^{10} \text{ N/m}^2$  for pipe material and  $K = 19.62 \times 10^4 \text{ N/cm}^2$  for water. Calculate the circumferential stress and longitudinal stress developed in the pipe wall.

**Solution.** Given :

$$V = 1.5 \text{ m/s}, L = 2500 \text{ m}, D = 0.5 \text{ m}$$

$$\text{Thickness of pipe, } t = 10 \text{ mm} = .01 \text{ m}$$

$$\text{Modulus of elasticity, } E = 19.62 \times 10^{10} \text{ N/m}^2$$

$$\text{Bulk modulus, } K = 19.62 \times 10^4 \text{ N/cm}^2 = 19.62 \times 10^8 \text{ N/m}^2$$

For sudden closure of the valve for an elastic pipe, the rise in pressure is given by equation (11.37) as

$$p = V \times \sqrt{\left(\frac{\rho}{K} + \frac{D}{Et}\right)} = 1.5 \times \sqrt{\left(\frac{1000}{\frac{1}{19.62 \times 10^8} + \frac{0.5}{19.62 \times 10^{10} \times .01}}\right)} \\ = 1.5 \times \sqrt{\frac{1000}{(5.09 \times 10^{-10} + 2.54 \times 10^{-10})}} \\ = 1715510 \text{ N/m}^2 = 171.55 \text{ N/cm}^2. \text{ Ans.}$$

Circumferential stress ( $f_c$ ) is given by

$$= \frac{p \times D}{2t} = \frac{171.55 \times 0.5}{2 \times .01} = 4286.9 \text{ N/m}^2$$

$$\text{Longitudinal stress is given by, } f_l = \frac{p \times D}{4t} = \frac{171.55 \times 0.5}{4 \times .01} = 2143.45 \text{ N/m}^2. \text{ Ans.}$$

**Problem 11.55** A valve is provided at the end of a cast iron pipe of diameter 150 mm and of thickness 10 mm. The water is flowing through the pipe, which is suddenly stopped by closing the valve. Find the maximum velocity of water, when the rise of pressure due to sudden closure of valve is 196.2 N/cm<sup>2</sup>. Take  $K$  for water as  $19.62 \times 10^4 \text{ N/cm}^2$  and  $E$  for cast iron pipe as  $11.772 \times 10^6 \text{ N/cm}^2$ .

**Solution.** Given :

$$\text{Diameter of pipe, } D = 150 \text{ mm} = 0.15 \text{ m}$$

$$\text{Thickness of pipe, } t = 10 \text{ mm} = .01 \text{ m}$$

$$\text{Rise of pressure, } p = 196.2 \text{ N/cm}^2 = 196.2 \times 10^4 \text{ N/m}^2$$

$$\text{Bulk modulus, } K = 19.62 \times 10^4 \text{ N/cm}^2 = 19.62 \times 10^8 \text{ N/m}^2$$

$$\text{Modulus of elasticity, } E = 11.772 \times 10^6 \text{ N/cm}^2 = 11.772 \times 10^{10} \text{ N/m}^2$$

For sudden closure of valve and when pipe is elastic, the pressure rise is given by equation (11.37) as

$$p = V \times \sqrt{\frac{\rho}{\left(\frac{1}{K} + \frac{D}{Et}\right)}} = V \times \sqrt{\frac{1000}{\left(\frac{1}{19.62 \times 10^8} + \frac{0.15}{11.772 \times 10^{10} \times .01}\right)}}$$

or

$$196.2 \times 10^4 = V \times \sqrt{\frac{1000}{5.09 \times 10^{-10} + 1.274 \times 10^{-10}}} \\ = V \times \sqrt{\frac{1000}{6.364 \times 10^{-10}}} = V \times 125.27 \times 10^4$$

$$\therefore V = \frac{196.2 \times 10^4}{125.27 \times 10^4} = 1.566 \text{ m/s}$$

$$\therefore \text{Maximum velocity} = \mathbf{1.566 \text{ m/s. Ans.}}$$

#### ► 11.14 PIPE NETWORK

A pipe network is an interconnected system of pipes forming several loops or circuits. The pipe network is shown in Fig. 11.33. The examples of such networks of pipes are the municipal water distribution systems in cities and laboratory supply system. In such system, it is required to determine the distribution of flow through the various pipes of the network. The following are the necessary conditions for any network of pipes :

(i) The flow into each junction must be equal to the flow out of the junction. This is due to continuity equation.

(ii) The algebraic sum of head losses round each loop must be zero. This means that in each loop, the loss of head due to flow in clockwise direction must be equal to the loss of head due to flow in anticlockwise direction.

(iii) The head loss in each pipe is expressed as  $h_f = rQ^n$ . The value of  $r$  depends upon the length of pipe, diameter of pipe and co-efficient of friction of pipe. The value of  $n$  for turbulent flow is 2. We know that,

$$h_f = \frac{4 \times f \times L \times V^2}{D \times 2g} = \frac{4fL \times \left(\frac{Q}{A}\right)^2}{D \times 2g} \quad \left( \because V = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4} D^2} \right)$$

$$= \frac{4fL \times Q^2}{D \times 2g \times \left(\frac{\pi}{4} D^2\right)^2} = \frac{4fL \times Q^2}{D \times 2g \times \left(\frac{\pi}{4}\right)^2 \times D^4}$$

$$= \frac{4f \times L \times Q^2}{2g \times \left(\frac{\pi}{4}\right)^2 \times D^5}$$

$$= rQ^2 \quad \dots(11.39) \left( \text{where } \frac{4f \times L}{2g \times \left(\frac{\pi}{4}\right)^2 \times D^5} = r \right)$$

This head loss will be positive, when the pipe is a part of loop and the flow in the pipe is clockwise.

Generally, the pipe network problems are difficult to solve analytically. Hence the methods of successive approximations are used. '**Hardy Cross Method**' is one such method which is commonly used.

#### 11.14.1 Hardy Cross Method.

The procedure for Hardy Cross Method is as follows :

- In this method a trial distribution of discharges is made arbitrary but in such a way that continuity equation is satisfied at each junction (or node).

2. With the assumed values of  $Q$ , the head loss in each pipe is calculated according to equation (11.39).

3. Now consider any loop (or circuits). The algebraic sum of head losses round each loop must be zero. This means that in each loop, the loss of head due to flow in clockwise direction must be equal to the loss of head due to flow in anticlockwise direction.

4. Now calculate the net head loss around each loop considering the head loss to be positive in clockwise flow and to be negative in anticlockwise flow.

If the net head loss due to assumed values of  $Q$  round the loop is zero, then the assumed values of  $Q$  in that loop is correct. But if the net head loss due to assumed values of  $Q$  is not zero, then the assumed values of  $Q$  are corrected by introducing a correction  $\Delta Q$  for the flows, till the circuit is balanced.

The correction factor  $\Delta Q^*$  is obtained by

$$\Delta Q = \frac{-\sum r Q_0^n}{\sum r n Q_0^{n-1}} \quad \dots(11.40)$$

For turbulent flow, the value of  $n = 2$  and hence above correction factor becomes as

$$\Delta Q = \frac{-\sum r Q_0^2}{\sum 2r Q_0} \quad \dots(11.41)$$

5. If the value of  $\Delta Q$  comes out to be positive, then it should be added to the flows in the clockwise direction ( $\because$  the flows in clockwise direction in a loops are considered positive) and subtracted from the flows in the anticlockwise direction.

6. Some pipes may be common to two circuits (or two loops), then the two corrections are applied to these pipes.

\* Let for any pipe  $Q_0$  = assumed discharge and  $Q$  = correct discharge, then

$$Q = Q_0 + \Delta Q$$

$$\therefore \text{Head loss for the pipe, } h_f = rQ^2 = r(Q_0 + \Delta Q)^2.$$

$$\begin{aligned} \text{For complete circuit, the net head loss, } \Sigma h_f &= \Sigma (rQ^2) = \Sigma r (Q_0 + \Delta Q)^2 = \Sigma r (Q_0^2 + 2Q_0 \Delta Q + \Delta Q^2) \\ &= \Sigma r (Q_0^2 + 2Q_0 \Delta Q) \text{ As } \Delta Q \text{ is small compared with } Q_0 \text{ and hence } \Delta Q^2 \text{ can be neglected.} \end{aligned}$$

$$\therefore \Sigma rQ^2 = \Sigma rQ_0^2 + \Sigma r \times 2Q_0 \Delta Q$$

$$\text{For the correct distribution, the net head loss for a circuit should be zero (i.e., } \Sigma h_f = \Sigma (rQ^2) = 0)$$

$$\therefore \Sigma rQ_0^2 + \Sigma r \times 2Q_0 \Delta Q = 0$$

or  $\Sigma rQ_0^2 + \Delta Q \Sigma r \times 2Q_0 = 0$  [As  $\Delta Q$  is same for one circuit, hence it can be taken out of the summation]

$$\therefore \Delta Q = \frac{-\sum r Q_0^2}{\sum 2r Q_0}.$$

7. After the corrections have been applied to each pipe in a loop and to all loops, a second trial calculation is made for all loops. The procedure is repeated till  $\Delta Q$  becomes negligible.

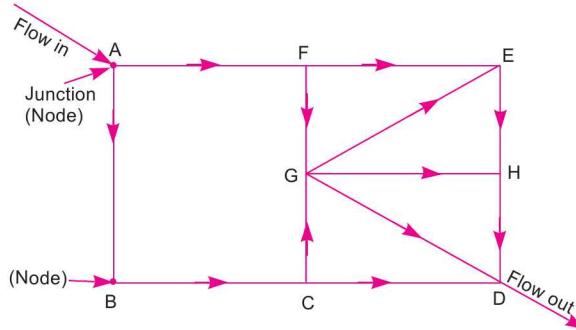


Fig. 11.33 Pipe network.

[Loops are : ABCGFA, FEGF, GEHG, GHDG and GCDG]

**Problem 11.56** Calculate the discharge in each pipe of the network shown in Fig. 11.34. The pipe network consists of 5 pipes. The head loss  $h_f$  in a pipe is given by  $h_f = rQ^2$ . The values of  $r$  for various pipes and also the inflow or outflows at nodes are shown in the figure.

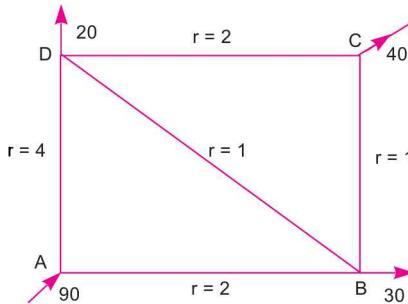


Fig. 11.34

**Solution.** Given :

Inflow at node  $A = 90$ , outflow at  $B = 30$ , at  $C = 40$  and at  $D = 20$ .

Values of  $r$  for  $AB = 2$ , for  $BC = 1$ , for  $CD = 2$ , for  $AD = 4$  and for  $BD = 1$ .

For the first trial, the discharges are assumed as shown in Fig. 11.34 (a) so that continuity is satisfied at each node (*i.e.*, flow into a node = flow out of the node). For this distribution of discharge, the corrections  $\Delta Q$  for the loops  $ABD$  and  $BCD$  are calculated.

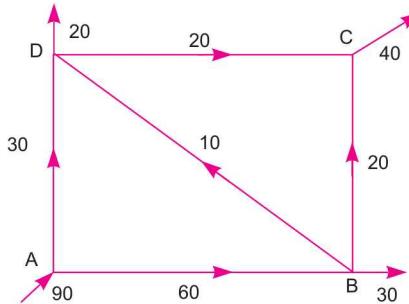


Fig. 11.34(a)

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### First Trial

Pipe	r	Loop ADB		Loop DCB							
		$Q_0$	$h_f = rQ_0^2$	$2rQ_0$	$Q_0$						
AD	4	30	$4 \times 30^2 = 3600$	$2 \times 4 \times 30 = 240$	DC	2	20	$2 \times 20^2 = 800$	$2 \times 2 \times 20 = 80$		
DB	1	10	$-1 \times 10^2 = -100$	$2 \times 1 \times 10 = 20$	CB	1	20	$-1 \times 20^2 = -400$	$2 \times 1 \times 20 = 40$		
AB	2	60	$-2 \times 60^2 = -7200$	$2 \times 2 \times 60 = 240$	BD	1	10	$1 \times 10^2 = 100$	$2 \times 1 \times 10 = 20$		
			$\Sigma rQ_0^2 = -3700$ , $\Sigma 2rQ_0 = 500$ ,					$\Sigma 2rQ_0 = 500$	$\Sigma 2rQ_0 = 140$		
..			$\Delta Q = \frac{-\Sigma rQ_0^2}{\Sigma 2rQ_0} = \frac{-(3700)}{500} = 7.4$					$\Delta Q = \frac{-\Sigma rQ_0^2}{\Sigma 2rQ_0} = \frac{-500}{140} = -3.57 \approx -3.6$			
In the loop ADB, the head loss $h_f$ is negative in pipes DB and AB as the direction of discharges in these pipes is anticlockwise.						The head loss in pipe BC for loop DCB is negative as the direction of discharge in pipe BC is anticlockwise.					
As $\Delta Q$ is positive for loop ADB, hence it should be added to the flow in the clockwise direction and subtracted from the flow in the anticlockwise direction. Hence the corrected flow for second trial for loop ADB will be as follows :						As $\Delta Q$ is negative for loop DCB, hence it should be subtracted from the flow in the clockwise direction and added to the flow in the anticlockwise direction. Hence corrected flow for second trial for loop DCB will be as follows :					
Pipe AD = $30 + 7.4 = 37.4$ (flow is clockwise)						Pipe DC = $20 - 3.6 = 16.4$					
Pipe AB = $60 - 7.4 = 52.6$ (flow is anticlockwise)						Pipe BC = $20 + 3.6 = 23.6$					
Pipe BD = $10 - 7.4 = 2.6$ (flow is anticlockwise)						Pipe BD* = $2.6 - 3.6 = -1$					

**Note.** The pipe BD is common to two loops (*i.e.*, loop ADB and loop DCB). Hence this pipe will get two corrections. After the two corrections, the resultant flow in pipe BD is negative in loop DCB. Hence the direction of flow will be anticlockwise in pipe BD for loop DCB.

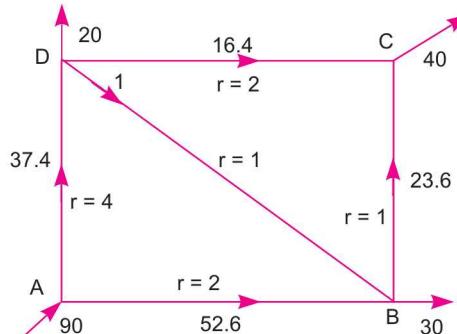


Fig. 11.34 (b)

The distribution of discharges in various pipes for second trial is shown in Fig. 11.34 (b). For second trial the correction  $\Delta Q$  for loops ADB and DCB are calculated as follows :

Loop ADB				Loop DCB			
Pipe	r	$Q_0$	$h_f = rQ_0^2$	Pipe	r	$Q_0$	$h_f = rQ_0^2$
AD	4	37.4	$4 \times 37.4^2 = 5595$	DC	2	16.4	$2 \times 16.4^2 = 537.9$
DB	1	1	$1 \times 1^2 = 1$	CB	1	23.6	$-1 \times 23.6^2 = -556.9$
AB	2	52.6	$-2 \times 52.6^2 = -5533.5$	BD	1	1	$2 \times 1 \times 1 = 2$
			$\Sigma rQ_0^2 = 62.54, \Sigma 2rQ_0 = 511.6$				$\Sigma rQ_0^2 = -20, \Sigma 2rQ_0 = 114.8$
			$\therefore \Delta Q = \frac{-\sum rQ_0^2}{\sum 2rQ_0} = \frac{62.54}{-511.6}$				$\therefore \Delta Q = \frac{-\sum rQ_0^2}{\sum 2rQ_0} = \frac{-(-20)}{114.8}$
			$= -0.122 \approx -0.1$				$= \frac{20}{114.8} = 0.174$
							$\approx 0.2$
As $\Delta Q$ is negative, hence it should be subtracted from the flow in the clockwise direction and added to the flow in the anticlockwise direction							
As the correction ( $\Delta Q$ ) is small (i.e., $\Delta Q = -0.1$ ), this correction is applied and further trials are discontinued.							
Hence corrected flow for loop ADB will be as follows :							
For pipe AD, $Q_0 = 37.4 - 0.1 = 37.3$ (as flow is clockwise)							
For pipe DB, $Q_0 = 1 - 0.1 = 0.9$ (as flow is clockwise)							
For pipe AB, $Q_0 = 52.6 + 0.1 = 52.7$ (as flow is anti-clockwise)							
As $\Delta Q$ is positive, hence it should be added to the flow in the clockwise direction and subtracted from the flow in the anticlockwise direction.							
As the correction ( $\Delta Q$ ) is small (i.e., $\Delta Q = 0.2$ ), this correction is applied and further trials are discontinued.							
Hence corrected flow for loop DCB will be as follows :							
For pipe DC, $Q_0 = 16.4 + 0.2 = 16.6$ (clockwise flow)							
For pipe CB, $Q_0 = 23.6 - 0.2 = 23.4$ (anticlockwise flow)							
For pipe BD, $Q_0 = 0.9 - 0.2 = 0.7$ (anticlockwise flow)							

The final distribution of discharges in each pipe is as follows :

Discharge in pipe AD = 37.3 from A to D

AB = 52.7 from A to B

DB = 0.7 from D to B

DC = 16.6 from D to C

BC = 23.4 from B to C

The final discharge in each pipe is shown in Fig. 11.34 (c)

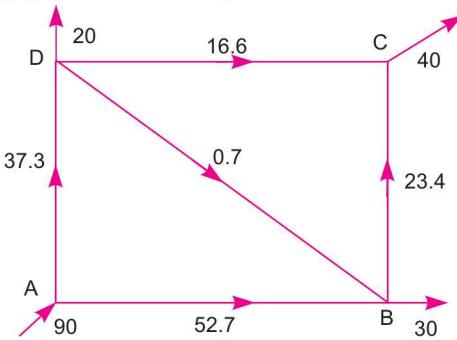


Fig. 11.34 (c)

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**Note.** The pipe  $DB$  is common to two loop (*i.e.*, loops  $ADB$  and loop  $DBC$ ). Hence this pipe will get two corrections. For loop  $ADB$ , the correction  $\Delta Q = -0.1$  and hence the corrected flow in pipe  $DB$  is  $1 - 0.1 = 0.9$ . Now again, the correction is applied to pipe  $DB$  when we consider loop  $DBC$ . For loop  $DBC$ , the correction  $\Delta Q = 0.2$  but flow is anticlockwise and hence the final correct flow in pipe  $DB$  will be  $0.9 - 0.2 = 0.7$ .

### HIGHLIGHTS

1. The energy loss in pipe is classified as major energy loss and minor energy losses. Major energy loss is due to friction while minor energy losses are due to sudden expansion of pipe, sudden contraction of pipe, bend in pipe and an obstruction in pipe.
2. Energy loss due to friction is given by Darcy Formula,  $h_f = \frac{4fLV^2}{d \times 2g}$ .
3. The head loss due to friction in pipe can also be calculated by Chezy's formula.

$$V = C\sqrt{mi} \text{ Chezy's formula}$$

where  $C$  = Chezy's Constant

$$m = \text{Hydraulic mean depth} = \frac{d}{4} \text{ (for pipe running full)}$$

$V$  = Velocity of flow

$$i = \text{Loss of head per unit length} = \frac{h_f}{L}$$

$$\therefore h_f = L \times i, \text{ where } i \text{ is obtained from Chezy's formula.}$$

4. Loss of head due to sudden expansion of pipe,  $h_c = \frac{(V_1 - V_2)^2}{2g}$

where  $V_1$  = Velocity in small pipe,  $V_2$  = Velocity in large pipe.

5. Loss of head due to sudden contraction of pipe,  $h_c = \left( \frac{1}{C_c} - 1 \right) \frac{V_2^2}{2g}$

$$\text{where } C_c = \text{co-efficient of contraction} = 0.375 \frac{V_2^2}{2g} \quad \dots(\text{For } C_c = 0.62)$$

$$= 0.5 \frac{V_2^2}{2g} \quad \dots(\text{if value of } C_c \text{ is not given})$$

6. Loss of head at the entrance of a pipe,  $h_i = 0.5 \frac{V^2}{2g}$ .

7. Loss of head at the exit of pipe,  $h_o = \frac{V^2}{2g}$ .

8. The line representing the sum of pressure head and datum head with respect to some reference line is called hydraulic gradient line (H.G.L.) while the line representing the sum of pressure head, datum head and velocity head with respect to some reference line is known as total energy line (T.E.L.).

9. Syphon is a long bent pipe used to transfer liquids from a reservoir at a higher level to another reservoir at a lower level, when the two reservoirs are separated by a high level ground.

10. The maximum vacuum created at the summit of syphon is only 7.4 m of water.

11. When pipes of different lengths and different diameters are connected end to end, pipes are called in series or compound pipes. The rate of flow through each pipe connected in series is same.

12. A single pipe of uniform diameter, having same discharge and same loss of head as compound pipe consisting of several pipes of different lengths and diameters, is known as equivalent pipe. The diameter of equivalent pipe is called equivalent size of the pipe.

13. The equivalent size of the pipe is obtained from

$$\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

where  $L$  = equivalent length of pipe  $= L_1 + L_2 + L_3$

$d_1, d_2, d_3$  = diameters of pipes connected in series

$d$  = equivalent size of the pipes.

14. When the pipes are connected in parallel, the loss of head in each pipe is same. The rate of flow in main pipe is equal to sum of the rate of flow in each pipe, connected in parallel.

15. For solving problems for branched pipes, the three basic, equations i.e., continuity, Bernoulli's and Darcy's equations are used.

16. Power transmitted in kW through pipe is given by  $P = \frac{\rho g \times Q \times (H - h_f)}{1000}$

where  $Q$  = discharge through pipe = area  $\times$  velocity  $= \frac{\pi}{4} d^2 \times V$

$H$  = total head at inlet of pipe

$h_f$  = head lost due to friction

$$= \frac{4fLV^2}{d \times 2g}, \text{ where } L = \text{Length of pipe}$$

In S.I. units, power transmitted is given by, Power  $= \frac{\rho g \times Q \times (H - h_f)}{1000}$  kW.

17. Efficiency of power transmission through pipes,  $\eta = \frac{H - h_f}{H}$ .

18. Condition for maximum transmission of power through pipe,  $h_f = \frac{H}{3}$  and maximum efficiency  $= 66.7\%$ .

19. The velocity of water at the outlet of the nozzle is  $v = \sqrt{\frac{2gH}{\left(1 + \frac{4fL}{D} \times \frac{a^2}{A^2}\right)}}$

where  $H$  = head at the inlet of the pipe,  $L$  = length of the pipe,

$D$  = diameter of the pipe,  $a$  = area of the nozzle at outlet,

$A$  = area of the pipe.

20. The power transmitted through nozzle,  $P = \frac{\rho g \times Q}{1000} \left[ H - \frac{4fLV^2}{D \times 2g} \right]$

and the efficiency of power transmission through nozzle,  $\eta = \frac{1}{\left[1 + \frac{4fL}{D} \times \frac{a^2}{A^2}\right]}$

21. Condition for maximum power transmission through nozzle,  $h_f = \frac{H}{3}$ .

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22. Diameter of nozzle for maximum power transmission through nozzle is,  $d = \left( \frac{D^5}{8fL} \right)^{1/4}$

where  $d$  = diameter of the nozzle at outlet,  $D$  = diameter of the pipe,  
 $L$  = length of the pipe,  $f$  = co-efficient of friction for pipe.

23. When a liquid is flowing through a long pipe fitted with a valve at the end of the pipe and the valve is closed suddenly, a pressure wave of high intensity is produced behind the valve. This pressure wave of high intensity is having the effect of hammering action on the walls of the pipe. This phenomenon is known as water hammer.

24. The intensity of pressure rise due to water hammer is given by

$$p = \frac{\rho LV}{T} \quad \dots \text{when valve is closed gradually.}$$
$$= V\sqrt{K\rho} \quad \dots \text{when valve is closed suddenly and pipe is assumed rigid}$$

$$= V \times \sqrt{\frac{\rho}{\frac{1}{K} + \frac{D}{Et}}} \quad \dots \text{when valve is closed suddenly and pipe is elastic.}$$

where  $L$  = Length of pipe,  
 $T$  = Time required to close the valve,  
 $D$  = Diameter of the pipe,  
 $t$  = Thickness of the pipe wall,

$V$  = Velocity of flow,  
 $K$  = Bulk modulus of water,  
 $E$  = Modulus of elasticity for pipe material.

25. If the time required to close the valve :

$$T > \frac{2L}{C} \quad \dots \text{the valve closure is said to be gradual,}$$
$$T < \frac{2L}{C} \quad \dots \text{the valve closure is said to be sudden}$$

where  $L$  = length of pipe,

$$C = \text{velocity of pressure wave produced due to water hammer} = \sqrt{\frac{K}{\rho}}.$$

## EXERCISE

### (A) THEORETICAL PROBLEMS

- How will you determine the loss of head due to friction in pipes by using (i) Darcy Formula and (ii) Chezy's formula ?
- (a) What do you understand by the terms : Major energy loss and minor energy losses in pipes ?  
(b) What do you understand by total energy line, hydraulic gradient line, pipes in series, pipes in parallel and equivalent pipe ?
- (a) Derive an expression for the loss of head due to : (i) Sudden enlargement and (ii) Sudden contraction of a pipe.  
(b) Obtain expression for head loss in a sudden expansion in the pipe. List all the assumptions made in the derivation.
- Define and explain the terms : (i) Hydraulic gradient line and (ii) Total energy line.

5. Show that the loss of head due to sudden expansion in pipe line is a function of velocity head.
6. What is a syphon ? On what principle it works ?
7. What is a compound pipe ? What will be loss of head when pipes are connected in series ?
8. Explain the terms : (i) Pipes in parallel (ii) Equivalent pipe and (iii) Equivalent size of the pipe.
9. Find an expression for the power transmission through pipes. What is the condition for maximum transmission of power and corresponding efficiency of transmission ?
10. Prove that the head loss due to friction is equal to one-third of the total head at inlet for maximum power transmission through pipes or nozzles.

11. Prove that the velocity through nozzle is given by  $v = \sqrt{\frac{2gH}{1 + \frac{4fL}{D} \times \frac{a^2}{A^2}}}$

where  $a$  = Area of nozzle at outlet,  $A$  = Area of the pipe.

12. Show that the diameter of the nozzle for maximum transmission of power is given by  $d = \left( \frac{D^5}{8fL} \right)^{1/4}$

where  $D$  = Diameter of pipe,  $L$  = Length of pipe.

13. Find an expression for the ratio of the outlet area of the nozzle to the area of the pipe for maximum transmission of power.
14. Explain the phenomenon of Water Hammer. Obtain an expression for the rise of pressure when the flowing water in a pipe is brought to rest by closing the valve gradually.
15. Show that the pressure rise due to sudden closure of a valve at the end of a pipe, through which water is

flowing is given by  $p = V \sqrt{\frac{d}{\frac{1}{K} + \frac{D}{Et}}}$

where  $V$  = Velocity of flow,  $D$  = Diameter of pipe,  $E$  = Young's Modulus,  $K$  = Bulk Modulus and  $t$  = Thickness of pipe.

16. Three pipes of different diameters and different lengths are connected in series to make a compound pipe. The ends of this compound pipe are connected with two tanks whose difference of water level is  $H$ . If co-efficient of friction for these pipes is same, then derive the formula for the total head loss, neglecting first the minor losses and then including them.
17. For the two cases of flow in a sudden contraction in a pipeline and flow in a sudden expansion in a pipe line, draw the flow pattern, piezometric grade line and total energy line.
18. What do you mean by "equivalent pipe" and "flow through parallel pipes" ?
19. (a) Define and explain the terms : (i) Hydraulic gradient line and (ii) total energy line.  
 (b) What do you mean by equivalent pipe ? Obtain an expression for equivalent pipe.

*(Delhi University, December 2002)*

### (B) NUMERICAL PROBLEMS

1. Find the head loss due to friction in a pipe of diameter 250 mm and length 60 m, through which water is flowing at a velocity of 3.0 m/s using (i) Darcy formula and (ii) Chezy's Formula for which  $C = 55$ . Take  $\nu$  for water = .01 stoke. [Ans. (i) 1.182, (ii) 2.856]
2. Find the diameter of a pipe of length 2500 m when the rate of flow of water through the pipe is 0.25 m<sup>3</sup>/s and head loss due to friction is 5 m. Take  $C = 50$  in Chezy's formula. [Ans. 605 mm]

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3. An oil of Kinematic Viscosity 0.5 stoke is flowing through a pipe of diameter 300 mm at the rate of 320 litres per sec. Find the head lost due to friction for a length of 60 m of the pipe. [Ans. 5.14 m]

4. Calculate the rate of flow of water through a pipe of diameter 300 mm, when the difference of pressure head between the two ends of a pipe 400 m apart is 5 m of water. Take the value of  $f = .009$  in the formula

$$h_f = \frac{4fLV^2}{d \times 2g} \quad [\text{Ans. } 0.101 \text{ m}^3/\text{s}]$$

5. The discharge through a pipe is 200 litres/s. Find the loss of head when the pipe is suddenly enlarged from 150 mm to 300 mm diameter. [Ans. 3.672 m]

6. The rate of flow of water through a horizontal pipe is  $0.3 \text{ m}^3/\text{s}$ . The diameter of the pipe is suddenly enlarged from 250 mm to 500 mm. The pressure intensity in the smaller pipe is  $13.734 \text{ N/cm}^2$ . Determine : (i) loss of head due to sudden enlargement, (ii) pressure intensity in the large pipe and (iii) power lost due to enlargement. [Ans. (i) 1.07 m, (ii)  $14.43 \text{ N/cm}^2$ , (iii) 3.15 kW]

7. A horizontal pipe of diameter 400 mm is suddenly contracted to a diameter of 200 mm. The pressure intensities in the large and smaller pipe is given as  $14.715 \text{ N/cm}^2$  and  $12.753 \text{ N/cm}^2$  respectively. If  $C_c = 0.62$ , find the loss of head due to contraction. Also determine the rate of flow of water.

[Ans. (i) 0.571 m, (ii) 171. 7 litres/s]

8. Water is flowing through a horizontal pipe of diameter 300 mm at a velocity of 4 m/s. A circular solid plate of diameter 200 mm is placed in the pipe to obstruct the flow. If  $C_c = 0.62$ , find the loss of head due to obstruction in the pipe. [Ans. 2.953 m]

9. Determine the rate of flow of water through a pipe of diameter 10 cm and length 60 cm when one end of the pipe is connected to a tank and other end of the pipe is open to the atmosphere. The height of water in the tank from the centre of the pipe is 5 cm. Pipe is given as horizontal and value of  $f = .01$ . Consider minor losses. [Ans. 15.4 litres/s]

10. A horizontal pipe-line 50 m long is connected to a water tank at one end and discharges freely into the atmosphere at the other end. For the first 30 m of its length from the tank, the pipe is 200 mm diameter and its diameter is suddenly enlarged to 400 mm. The height of water level in the tank is 10 m above the centre of the pipe. Considering all minor losses, determine the rate of flow. Take  $f = .01$  for both sections of the pipe. [Ans. 164.13 litres/s]

11. Determine the difference in the elevations between the water surfaces in the two tanks which are connected by a horizontal pipe of diameter 400 mm and length 500 m. The rate of flow of water through the pipe is 200 litres/s. Consider all losses and take the value of  $f = .009$ . [Ans. 11.79 m]

12. For the problems 9, 10 and 11 draw the hydraulic gradient lines (H.G.L.) and total energy lines (T.E.L.)

13. A syphon of diameter 150 mm connects two reservoirs having a difference in elevation of 15 m. The length of the syphon is 400 m and summit is 4.0 m above the water level in the upper reservoir. The length of the pipe from upper reservoir to the summit is 80 m. Determine the discharge through the syphon and also pressure at the summit. Neglect minor losses. The co-efficient of friction,  $f = .005$ .

[Ans. 41.52 litres/s, - 7.281 m of water]

14. A syphon of diameter 200 mm connects two reservoirs having a difference in elevation as 20 m. The total length of the syphon is 800 m and the summit is 5 m above the water level in the upper reservoir. If the separation takes place at 2.8 m of water absolute find the maximum length of syphon from upper reservoir to the summit. Take  $f = .004$  and atmospheric pressure = 10.3 m of water. [Ans. 87.52 m]

15. Three pipes of lengths 800 m, 600 m and 300 m and of diameters 400 mm, 300 mm and 200 mm respectively are connected in series. The ends of the compound pipe is connected to two tanks, whose water surface levels are maintained at a difference of 15 m. Determine the rate of flow of water through the pipes if  $f = .005$ . What will be diameter of a single pipe of length 1700 m and  $f = .005$ , which replaces the three pipes ? [Ans.  $0.0848 \text{ m}^3/\text{s}$ , 266.5 mm]

16. Two pipes of lengths 2500 m each and diameters 80 cm and 60 cm respectively, are connected in parallel. The co-efficient of friction for each pipe is 0.006. The total flow is equal to 250 litres/s. Find the rate of flow in each pipe. [Ans.  $0.1683 \text{ m}^3/\text{s}$ ,  $0.0817 \text{ m}^3/\text{s}$ ]
17. A pipe of diameter 300 mm and length 1000 m connects two reservoirs, having difference of water levels as 15 m. Determine the discharge through the pipe. If an additional pipe of diameter 300 mm and length 600 m is attached to the last 600 m length of the existing pipe, find the increase in the discharge. Take  $f = .02$  and neglect minor losses. [Ans.  $0.0742 \text{ m}^3/\text{s}$ ,  $0.0258 \text{ m}^3/\text{s}$ ]
18. Two sharp ended pipes of diameters 60 mm and 100 mm respectively, each of length 150 m are connected in parallel between two reservoirs which have a difference of level of 15 m. If co-efficient of friction for each pipe is 0.08, calculate the rate of flow for each pipe and also the diameter of a single pipe 150 m long which would give the same discharge if it were substituted for the original two pipes. [Ans.  $0.0017$ ,  $.00615$ ,  $110 \text{ mm}$ ]
19. Three reservoirs A, B and C are connected by a pipe system having length 700 m, 1200 m and 500 m and diameters 400 mm, 300 mm and 200 mm respectively. The water levels in reservoir A and B from a datum line are 50 m and 45 m respectively. The level of water in reservoir C is below the level of water in reservoir B. Find the discharge into or from the reservoirs B and C if the rate of flow from reservoir A is 150 litres per sec. Find the height of water level in the reservoir C. Take  $f = .005$  for all pipes. [Ans.  $.005 \text{ m}^3/\text{s}$ ,  $.095 \text{ m}^3/\text{s}$ ,  $24.16 \text{ m}$ ]
20. A pipe of diameter 300 mm and length 3000 m is used for the transmission of power by water. The total head at the inlet of the pipe is 400 m. Find the maximum power available at the outlet of the pipe. Take  $f = .005$ . [Ans.  $667.07 \text{ kW}$ ]
21. A pipe line of length 2100 m is used for transmitting 103 kW. The pressure at the inlet of the pipe is  $392.4 \text{ N/cm}^2$ . If the efficiency of transmission is 80%, find the diameter of the pipe. Take  $f = .005$ . [Ans.  $136 \text{ mm}$ ]
22. A nozzle is fitted at the end of a pipe of length 400 m and of diameter 150 mm. For the maximum transmission of power through the nozzle, find the diameter of the nozzle. Take  $f = .008$ . [Ans.  $41.5 \text{ mm}$ ]
23. The head of water at the inlet of a pipe of length 1500 m and of diameter 400 mm is 50 m. A nozzle of diameter 80 mm at the outlet, is fitted to the pipe. Find the velocity of water at the outlet of the nozzle if  $f = .01$  for the pipe. [Ans.  $28.12 \text{ m/s}$ ]
24. The rate of flow of water through a pipe of length 1500 m and diameter 800 mm is  $2 \text{ m}^3/\text{s}$ . At the end of the pipe a nozzle of outside diameter 200 mm is fitted. Find the power transmitted through the nozzle if the head of water at the inlet of the pipe is 180 m and  $f = .01$  for pipe. [Ans.  $2344.7 \text{ kW}$ ]
25. The water is flowing with a velocity of  $2 \text{ m/s}$  in a pipe of length 2000 m and of diameter 600 mm. At the end of the pipe, a valve is provided. Find the rise in pressure if the valve is closed in 20 seconds. Take the value of  $C = 1420 \text{ m/s}$ . [Ans.  $20 \text{ N/cm}^2$ ]
26. If the valve in the problem 25 is closed in 1.5 sec, find the rise in pressure. Take bulk modulus of water  $= 19.62 \times 10^4 \text{ N/cm}^2$  and consider pipe as rigid one. [Ans.  $186.75 \text{ N/cm}^2$ ]
27. If in the problem 25, the thickness of the pipe is 10 mm and the valve is closed suddenly. Find the rise in pressure if the pipe is considered to be elastic. Take value of  $E = 19.62 \times 10^6 \text{ N/cm}^2$  for pipe material and  $K = 19.62 \times 10^4 \text{ N/cm}^2$  for water. Calculate the circumferential stress and longitudinal stress developed in the pipe wall. [Ans.  $p = 221.47 \text{ N/cm}^2$ ,  $f_c = 6644.1 \text{ N/cm}^2$ ,  $f_l = 3322 \text{ N/cm}^2$ ]
28. The difference in water surface levels in two tanks, which are connected by two pipes in series of lengths 600 m and 400 m and of diameters 30 cm and 20 cm respectively, is 15 m. Determine the rate of flow of water if the co-efficient of friction is 0.005 for both the pipes. Neglect minor losses.
29. Water is flowing vertically downwards through a 10 cm diameter pipe at the rate of 50 l.p.s. At a particular location the pipe suddenly enlarges to 20 cm diameter. A point P is located 50 cm above the section of enlargement and another point Q is located 50 cm below it in the enlarged portion. A pressure gauge connected at P gives a reading of  $19.62 \text{ N/cm}^2$ . Calculate the pressure at location Q neglecting friction loss between P and Q but considering the loss due to sudden enlargement. What

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will be the pressure at  $Q$  if the same discharge flows upwards assuming that the pressure  $P$  remains the same ? Consider the loss due to contraction with  $C_c = 0.60$  but neglect friction loss between  $P$  and  $Q$ .

[Ans.  $21.36 \text{ N/cm}^2$ ,  $23.4 \text{ N/cm}^2$ ]

30. Two tanks are connected with the help of two pipes in series. The lengths of the pipes are 1000 m and 800 m whereas the diameters are 400 mm and 200 mm respectively. The co-efficient of friction for both the pipes is 0.008. The difference of water level in the two tanks is 15 m. Find the rate of flow of water through the pipes, considering all losses. Also draw the total energy line and hydraulic gradient lines for the system.

[Ans.  $0.0464 \text{ m}^3/\text{s}$ ]

[Hint.  $L_1 = 1000 \text{ m}$ ;  $L_2 = 800 \text{ m}$ ;  $d_1 = 400 \text{ mm} = 0.4 \text{ m}$ ;  $d_2 = 200 \text{ mm} = 0.2 \text{ m}$ ,  $f = 0.008$ ;  $H = 15 \text{ m}$ .

Now

$$H = h_i + hf_1 + h_c + hf_2 + h_o,$$

$$H = \frac{0.5 V_1^2}{2g} + \frac{4f \times L_1 \times V_1^2}{d \times 2g} + \frac{0.5 V_2^2}{2g} + \frac{4f \times L_2 \times V_2^2}{d_2 \times 2g} + \frac{V_2^2}{2g}$$

$$\text{or } 15 = \frac{0.5 V_1^2}{2 \times 9.81} + \frac{4 \times 0.008 \times 1000 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{0.5 V_2^2}{2 \times 9.81} + \frac{4 \times 0.008 \times 800 \times V_2^2}{0.2 \times 2 \times 9.81} + \frac{V_2^2}{2g}$$

Also

$$A_1 V_1 = A_2 V_2 \text{ or } V_2 = 4V_1$$

∴

$$\begin{aligned} 15 &= 0.02548 V_1^2 + 4.0775 V_1^2 + 0.02548 V_2^2 + 6.524 V_2^2 + 0.05097 V_2^2 \\ &= 4.103 V_1^2 + 6.6 V_2^2 = 4.103 V_1^2 + 6.6 \times (4V_1)^2 \\ &= 4.103 V_1^2 + 105.607 V_1^2 = 109.71 V_1^2 \end{aligned}$$

∴

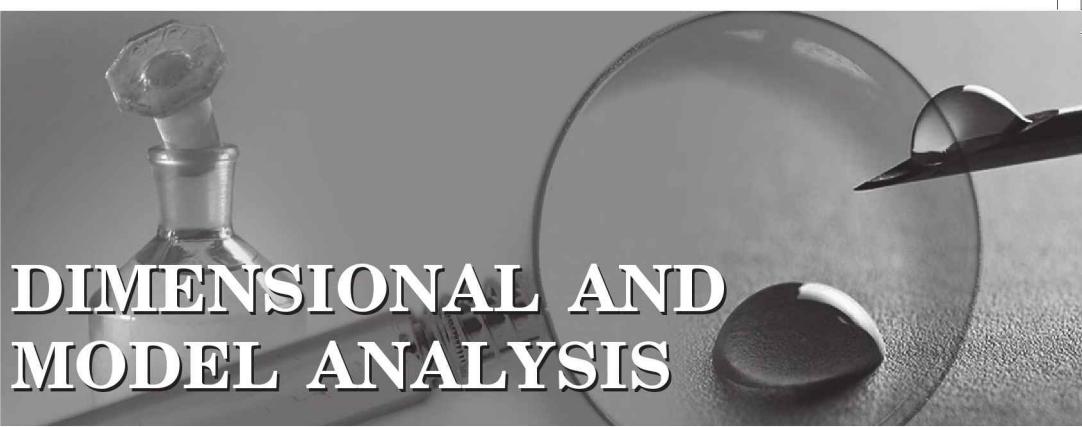
$$V_1 = \sqrt{\frac{15}{109.71}} = 0.3697 \text{ m/s} \therefore Q = A_1 V_1 = \frac{\pi}{4} (.4)^2 \times 0.3697 = 0.0464 \text{ m}^3/\text{s}]$$

31. A pipe of diameter 25 cm and length 2000 m connects two reservoirs, having difference of water level 25 m. Determine the discharge through the pipe. If an additional pipe of diameter 25 cm and length 1000 m is attached to the last 1000 m length of the existing pipe, find the increase in discharge. Take  $f = 0.015$ . Neglect minor losses. (Delhi University, December 2002) [Ans. (i) 49.62 l/s, (ii) 13.14 l/s]

# 12

## CHAPTER

# DIMENSIONAL AND MODEL ANALYSIS



### ► 12.1 INTRODUCTION

Dimensional analysis is a method of dimensions. It is a mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon. All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length  $L$ , mass  $M$  and time  $T$  are three fixed dimensions which are of importance in Fluid Mechanics. If in any problem of fluid mechanics, heat is involved then temperature is also taken as fixed dimension. These fixed dimensions are called fundamental dimensions or fundamental quantity.

### ► 12.2 SECONDARY OR DERIVED QUANTITIES

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is denoted by distance per unit time ( $L/T$ ), density by mass per unit volume  $\left(\frac{M}{L^3}\right)$  and acceleration by distance per second square ( $L/T^2$ ). Then velocity, density and acceleration become as secondary or derived quantities. The expressions  $(L/T)$ ,  $\left(\frac{M}{L^3}\right)$  and  $\left(\frac{L}{T^2}\right)$  are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanics are given in Table 12.1.

Table 12.1

S. No.	Physical Quantity	Symbol	Dimensions
(a) Fundamental			
1.	Length	$L$	$L$
2.	Mass	$M$	$M$
3.	Time	$T$	$T$

Contd...

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S.No.	Physical Quantity	Symbol	Dimensions
	(b) Geometric		
4.	Area	A	$L^2$
5.	Volume	V	$L^3$
	(c) Kinematic Quantities		
6.	Velocity	v	$LT^{-1}$
7.	Angular Velocity	$\omega$	$T^{-1}$
8.	Acceleration	a	$LT^{-2}$
9.	Angular Acceleration	$\alpha$	$T^{-2}$
10.	Discharge	Q	$L^3T^{-1}$
11.	Acceleration due to Gravity	g	$LT^{-2}$
12.	Kinematic Viscosity	$\nu$	$L^2T^{-1}$
	(d) Dynamic Quantities		
13.	Force	F	$MLT^{-2}$
14.	Weight	W	$MLT^{-2}$
15.	Density	$\rho$	$ML^{-3}$
16.	Specific Weight	w	$ML^{-2}T^{-2}$
17.	Dynamic Viscosity	$\mu$	$ML^{-1}T^{-1}$
18.	Pressure Intensity	p	$ML^{-1}T^{-2}$
19.	Modulus of Elasticity	$\{E\}$	$ML^{-1}T^{-2}$
20.	Surface Tension	$\sigma$	$MT^{-2}$
21.	Shear Stress	$\tau$	$ML^{-1}T^{-2}$
22.	Work, Energy	W or E	$ML^2T^{-2}$
23.	Power	P	$ML^2T^{-3}$
24.	Torque	T	$ML^2T^{-2}$
25.	Momentum	M	$MLT^{-1}$

**Problem 12.1** Determine the dimensions of the quantities given below : (i) Angular velocity, (ii) Angular acceleration, (iii) Discharge, (iv) Kinematic viscosity, (v) Force, (vi) Specific weight, and (vii) Dynamic viscosity.

**Solution.** (i) Angular velocity =  $\frac{\text{Angle covered in radians}}{\text{Time}} = \frac{1}{T} = T^{-1}$ .

(ii) Angular acceleration =  $\text{rad/sec}^2 = \frac{\text{rad}}{T^2} = \frac{1}{T^2} = T^{-2}$ .

(iii) Discharge = Area  $\times$  Velocity =  $L^2 \times \frac{L}{T} = \frac{L^3}{T} = L^3T^{-1}$ .

(iv) Kinematic viscosity ( $\nu$ ) =  $\frac{\mu}{\rho}$ , where  $\mu$  is given by  $\tau = \mu \frac{\partial u}{\partial y}$

$$\therefore \mu = \frac{\tau}{\frac{\partial u}{\partial y}} = \frac{\text{Shear Stress}}{\frac{L}{T} \times \frac{1}{L}} = \frac{\text{Force}}{\frac{1}{T} \text{Area}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Area} \times \text{Time}} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}} = \frac{ML}{L^2 T^2 \times \frac{1}{T}} = \frac{M}{LT} = ML^{-1} T^{-1}$$

and

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{L^3} = ML^{-3}$$

$$\therefore \text{Kinematic viscosity } (\nu) = \frac{\mu}{\rho} = \frac{ML^{-1} T^{-1}}{ML^{-3}} = L^2 T^{-1}.$$

$$(v) \text{ Force} = \text{Mass} \times \text{Acceleration} = M \times \frac{\text{Length}}{(\text{Time})^2} = \frac{ML}{T^2} = ML T^{-2}.$$

$$(vi) \text{ Specific weight} = \frac{\text{Weight}}{\text{Volume}} = \frac{\text{Force}}{\text{Volume}} = \frac{ML T^{-2}}{L^3} = ML^{-2} T^{-2}.$$

(vii) Dynamic viscosity,  $\mu$  is derived in (iv) as  $\mu = ML^{-1} T^{-1}$ .

### ► 12.3 DIMENSIONAL HOMOGENEITY

Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions (*i.e.*,  $L$ ,  $M$ ,  $T$ ) on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.

Let us consider the equation,  $V = \sqrt{2gH}$

$$\text{Dimension of L.H.S.} = V = \frac{L}{T} = LT^{-1}$$

$$\text{Dimension of R.H.S.} = \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$$

$$\text{Dimension of L.H.S.} = \text{Dimension of R.H.S.} = LT^{-1}$$

$\therefore$  Equation  $V = \sqrt{2gH}$  is dimensionally homogeneous. So it can be used in any system of units.

### ► 12.4 METHODS OF DIMENSIONAL ANALYSIS

If the number of variable involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods :

1. Rayleigh's method, and
2. Buckingham's  $\pi$ -theorem.

**12.4.1 Rayleigh's Method.** This method is used for determining the expression for a variable which depends upon maximum three or four variables only. If the number of independent variables becomes more than four, then it is very difficult to find the expression for the dependent variable.

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Let  $X$  is a variable, which depends on  $X_1$ ,  $X_2$  and  $X_3$  variables. Then according to Rayleigh's method,  $X$  is function of  $X_1$ ,  $X_2$  and  $X_3$  and mathematically it is written as  $X = f[X_1, X_2, X_3]$ .

This can also be written as  $X = KX_1^a \cdot X_2^b \cdot X_3^c$   
where  $K$  is constant and  $a, b$  and  $c$  are arbitrary powers.

The values of  $a$ ,  $b$  and  $c$  are obtained by comparing the powers of the fundamental dimension on both sides. Thus the expression is obtained for dependent variable.

**Problem 12.2** The time period ( $t$ ) of a pendulum depends upon the length ( $L$ ) of the pendulum and acceleration due to gravity ( $g$ ). Derive an expression for the time period.

**Solution.** Time period  $t$  is a function of (i)  $L$  and (ii)  $g$

$$t = KL^a \cdot g^b, \text{ where } K \text{ is a constant} \quad \dots(i)$$

Substituting the dimensions on both sides  $T^1 = KL^a \cdot (LT^{-2})^b$

Equating the powers of  $M$ ,  $L$  and  $T$  on both sides, we have

$$\text{Power of } T, \quad 1 = -2b \quad \therefore \quad b = -\frac{1}{2}$$

$$\text{Power of } L, \quad 0 = a + b \quad \therefore \quad a = -b = -\left(-\frac{1}{2}\right) = \frac{1}{2}$$

Substituting the values of  $a$  and  $b$  in equation (i),

$$t = KL^{1/2} \cdot g^{-1/2} = K \sqrt{\frac{L}{g}}$$

The value of  $K$  is determined from experiments which is given as

$$K = 2\pi$$

$$\therefore t = 2\pi \sqrt{\frac{L}{g}}. \text{ Ans.}$$

**Problem 12.3** Find an expression for the drag force on smooth sphere of diameter  $D$ , moving with a uniform velocity  $V$  in a fluid of density  $\rho$  and dynamic viscosity  $\mu$ .

**Solution.** Drag force  $F$  is a function of



$$F = K D^a \cdot V^b \cdot \rho^c \cdot \mu^d \quad \dots(i)$$

where  $K$  is non-dimensional factor.

Substituting the dimensions on both sides,

$$MLT^{-2} = KL^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d$$

Equating the powers of  $M$ ,  $L$  and  $T$  on both sides,

Power of  $M$ :  $1 = c + d$

Power of  $L_1$   $\equiv a + b - 3c = d$

Power of  $\Sigma$ ,  $\text{Power of } T = 2 \equiv h - d$

There are four unknowns ( $a, b, c, d$ ) but equations are three. Hence it is not possible to find the values of  $a, b, c$  and  $d$ . But three of them can be expressed in terms of fourth variable which is most important. Here viscosity is having a vital role and hence  $a, b, c$  are expressed in terms of  $d$  which is the power to viscosity.

$$\therefore \begin{aligned} c &= 1 - d \\ b &= 2 - d \\ a &= 1 - b + 3c + d = 1 - 2 + d + 3(1 - d) + d \end{aligned}$$

$$= 1 - 2 + d + 3 - 3d + d = 2 - d$$

Substituting these values of  $a$ ,  $b$  and  $c$  in (i), we get

$$F = KD^{2-d} \cdot V^{2-d} \cdot \rho^{1-d} \cdot \mu^d$$

$$= KD^2V^2\rho(D^{-d} \cdot V^{-d} \cdot \rho^{-d} \cdot \mu^d) = K\rho D^2V^2 \left( \frac{\mu}{\rho VD} \right)^d$$

$$= K\rho D^2V^2 \phi \left( \frac{\mu}{\rho VD} \right). \text{Ans.}$$

**Problem 12.4** Find the expression for the power  $P$ , developed by a pump when  $P$  depends upon the head  $H$ , the discharge  $Q$  and specific weight  $w$  of the fluid.

**Solution.** Power  $P$  is a function of

$$(i) \text{ Head, } H \quad (ii) \text{ Discharge, } Q$$

$$(iii) \text{ Specific weight, } w$$

$$\therefore P = KH^a \cdot Q^b \cdot w^c \quad \dots(i)$$

where  $K$  = Non-dimensional constant.

Substituting the dimensions on both sides of equation (i)

$$ML^2T^{-3} = KL^a \cdot (L^3T^{-1})^b \cdot (ML^{-2}T^{-2})^c$$

Equating the powers of  $M$ ,  $L$  and  $T$  on both sides,

$$\text{Power of } M, \quad 1 = c, \quad \therefore \quad c = 1$$

$$\text{Power of } L, \quad 2 = a + 3b - 2c, \quad a = 2 - 3b + 2c = 2 - 3 + 2 = 1$$

$$\text{Power of } T, \quad -3 = -b - 2c \quad \therefore \quad b = 3 - 2c = 3 - 2 = 1$$

Substituting the values of  $a$ ,  $b$  and  $c$  in (i)

$$P = KH^1 \cdot Q^1 \cdot w^1 = KHQw. \text{ Ans.}$$

**Problem 12.5** The efficiency  $\eta$  of a fan depends on the density  $\rho$ , the dynamic viscosity  $\mu$  of the fluid, the angular velocity  $\omega$ , diameter  $D$  of the rotor and the discharge  $Q$ . Express  $\eta$  in terms of dimensionless parameters.

**Solution.** The efficiency  $\eta$  depends on

$$(i) \text{ density, } \rho \quad (ii) \text{ viscosity, } \mu$$

$$(iii) \text{ angular velocity, } \omega \quad (iv) \text{ diameter, } D$$

$$(v) \text{ discharge, } Q$$

$$\therefore \eta = K\rho^a \cdot \mu^b \cdot \omega^c \cdot D^d \cdot Q^e \quad \dots(i)$$

where  $K$  = Non-dimensional constant.

Substituting the dimensions on both sides of equation (i)

$$M^0L^0T^0 = K(ML^{-3})^a \cdot (ML^{-1}T^{-1})^b \cdot (T^{-1})^c \cdot (L)^d \cdot (L^3T^{-1})^e$$

Equating powers of  $M$ ,  $L$ ,  $T$  on both sides,

$$\text{Power of } M, \quad 0 = a + b$$

$$\text{Power of } L, \quad 0 = -3a - b + d + 3e$$

$$\text{Power of } T, \quad 0 = -b - c - e.$$

There are five unknowns but equations are three. Express the three unknowns in terms of the other two unknowns which are more important. Viscosity and discharge are more important in this problem. Hence expressing  $a$ ,  $c$  and  $d$  in terms of  $b$  and  $e$ , we get

$$a = -b$$

$$c = -(b + e)$$

$$d = +3a + b - 3e = -3b + b - 3e = -2b - 3e.$$

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Substituting these values in equation (i), we get

$$\begin{aligned}\eta &= K\rho^{-b} \cdot \mu^b \cdot \omega^{-(b+e)} \cdot D^{-2b-3e} \cdot Q^e \\ &= K\rho^{-b} \cdot \mu^b \cdot \omega^{-b} \cdot \omega^{-e} \cdot D^{-2b} \cdot D^{-3e} \cdot Q^e \\ &= K \left( \frac{\mu}{\rho \omega D^2} \right)^b \cdot \left( \frac{Q}{\omega D^3} \right)^e = \phi \left[ \left( \frac{\mu}{\rho \omega D^2} \right) \cdot \left( \frac{Q}{\omega D^3} \right) \right]. \text{Ans.}\end{aligned}$$

**Problem 12.6** The resisting force  $R$  of a supersonic plane during flight can be considered as dependent upon the length of the aircraft  $l$ , velocity  $V$ , air viscosity  $\mu$ , air density  $\rho$  and bulk modulus of air  $K$ . Express the functional relationship between these variables and the resisting force.

**Solution.** The resisting force  $R$  depends upon

- (i) density,  $l$ ,
- (ii) velocity,  $V$ ,
- (iii) viscosity,  $\mu$ ,
- (iv) density,  $\rho$ ,
- (v) Bulk modulus,  $K$ .

$$\therefore R = Al^a \cdot V^b \cdot \mu^c \cdot \rho^d \cdot K^e \quad \dots(i)$$

where  $A$  is the non-dimensional constant.

Substituting the dimensions on both sides of the equation (i),

$$MLT^{-2} = AL^a \cdot (LT^{-1})^b \cdot (ML^{-1}T^{-1})^c \cdot (ML^{-3})^d \cdot (ML^{-1}T^{-2})^e$$

Equating the powers of  $M$ ,  $L$ ,  $T$  on both sides,

$$\begin{aligned}\text{Power of } M, \quad 1 &= c + d + e \\ \text{Power of } L, \quad 1 &= a + b - c - 3d - e \\ \text{Power of } T, \quad -2 &= -b - c - 2e.\end{aligned}$$

There are five unknowns but equations are only three. Expressing the three unknowns in terms of two unknowns ( $\mu$  and  $K$ ).

$\therefore$  Express the values of  $a$ ,  $b$  and  $d$  in terms of  $c$  and  $e$ .

Solving,

$$\begin{aligned}d &= 1 - c - e \\ b &= 2 - c - 2e \\ a &= 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e \\ &= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.\end{aligned}$$

Substituting these values in (i), we get

$$\begin{aligned}R &= A l^{2-c} \cdot V^{2-c-2e} \cdot \mu^c \cdot \rho^{1-c-e} \cdot K^e \\ &= A l^2 \cdot V^2 \cdot \rho(l^{-c} V^{-c} \mu^c \rho^{-c}) \cdot (V^{-2e} \cdot \rho^{-e} \cdot K^e) \\ &= A l^2 V^2 \rho \left( \frac{\mu}{\rho V L} \right)^c \cdot \left( \frac{K}{\rho V^2} \right)^e \\ &= A \rho l^2 V^2 \phi \left[ \left( \frac{\mu}{\rho V L} \right) \cdot \left( \frac{K}{\rho V^2} \right) \right]. \text{Ans.}\end{aligned}$$

**Problem 12.7** A partially sub-merged body is towed in water. The resistance  $R$  to its motion depends on the density  $\rho$ , the viscosity  $\mu$  of water, length  $l$  of the body, velocity  $v$  of the body and the acceleration due to gravity  $g$ . Show that the resistance to the motion can be expressed in the form

$$R = \rho L^2 V^2 \phi \left[ \left( \frac{\mu}{\rho V L} \right) \cdot \left( \frac{l g}{V^2} \right) \right].$$

**Solution.** The resistance  $R$  depends on

- (i) density,  $\rho$ ,
- (ii) viscosity,  $\mu$ ,
- (iii) length,  $l$ ,
- (iv) velocity,  $V$ ,
- (v) acceleration,  $g$ .

$$\therefore R = K\rho^a \cdot \mu^b \cdot l^c \cdot V^d \cdot g^e \quad \dots(i)$$

where  $K$  = Non-dimensional constant.

Substituting the dimensions on both sides of the equation (i),

$$MLT^{-2} = K(ML^{-3})^a \cdot (ML^{-1}T^{-1})^b \cdot L^c \cdot (LT^{-1})^d \cdot (LT^{-2})^e$$

Equating the powers of  $M, L, T$  on both sides

Power of $M$ ,	$1 = a + b$
Power of $L$ ,	$1 = -3a - b + c + d + e$
Power of $T$ ,	$-2 = -b - d - 2e$

There are five unknowns and equations are only three. Expressing the three unknowns in terms of two unknowns ( $\mu$  and  $g$ ). Hence express  $a, c$  and  $d$  in terms of  $b$  and  $e$ . Solving, we get

$$\begin{aligned} a &= 1 - b \\ d &= 2 - b - 2e \\ c &= 1 + 3a + b - d - e = 1 + 3(1 - b) + b - (2 - b - 2e) - e \\ &= 1 + 3 - 3b + b - 2 + b + 2e - e = 2 - b + e. \end{aligned}$$

Substituting these values in equation (i), we get

$$\begin{aligned} R &= K\rho^{1-b} \cdot \mu^b \cdot l^{2-b+e} \cdot V^{2-b-2e} \cdot g^e \\ &= K\rho l^2 \cdot V^2 \cdot (\rho^{-b} \mu^b l^{-b} V^{-b}) \cdot (l^e \cdot V^{-2e} \cdot g^e) \\ &= K\rho l^2 V^2 \cdot \left(\frac{\mu}{\rho V l}\right)^b \cdot \left(\frac{l g}{V^2}\right)^e = K\rho l^2 V^2 \phi \left[ \left(\frac{\mu}{\rho V l}\right) \cdot \left(\frac{l g}{V^2}\right) \right]. \text{ Ans.} \end{aligned}$$

**12.4.2 Buckingham's  $\pi$ -Theorem.** The Rayleigh's method of dimensional analysis becomes more laborious if the variables are more than the number of fundamental dimensions ( $M, L, T$ ). This difficulty is overcome by using Buckingham's  $\pi$ -theorem, which states, "If there are  $n$  variables (independent and dependent variables) in a physical phenomenon and if these variables contain  $m$  fundamental dimensions ( $M, L, T$ ), then the variables are arranged into  $(n - m)$  dimensionless terms. Each term is called  $\pi$ -term".

Let  $X_1, X_2, X_3, \dots, X_n$  are the variables involved in a physical problem. Let  $X_1$  be the dependent variable and  $X_2, X_3, \dots, X_n$  are the independent variables on which  $X_1$  depends. Then  $X_1$  is a function of  $X_2, X_3, \dots, X_n$  and mathematically it is expressed as

$$X_1 = f(X_2, X_3, \dots, X_n) \quad \dots(12.1)$$

Equation (12.1) can also be written as

$$f_1(X_1, X_2, X_3, \dots, X_n) = 0. \quad \dots(12.2)$$

Equation (12.2) is a dimensionally homogeneous equation. It contains  $n$  variables. If there are  $m$  fundamental dimensions then according to Buckingham's  $\pi$ -theorem, equation (12.2) can be written in terms of number of dimensionless groups or  $\pi$ -terms in which number of  $\pi$ -terms is equal to  $(n - m)$ . Hence equation (12.2) becomes as

$$f(\pi_1, \pi_2, \dots, \pi_{n-m}) = 0. \quad \dots(12.3)$$

Each of  $\pi$ -terms is dimensionless and is independent of the system. Division or multiplication by a constant does not change the character of the  $\pi$ -term. Each  $\pi$ -term contains  $m + 1$  variables, where  $m$  is the number of fundamental dimensions and is also called repeating variables. Let in the above case  $X_2, X_3$  and  $X_4$  are repeating variables if the fundamental dimension  $m$  ( $M, L, T$ ) = 3. Then each  $\pi$ -term is written as

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$$\left. \begin{aligned} \pi_1 &= X_2^{a_1} \cdot X_3^{b_1} \cdot X_4^{c_1} \cdot X_1 \\ \pi_2 &= X_2^{a_2} \cdot X_3^{b_2} \cdot X_4^{c_2} \cdot X_5 \\ &\vdots \\ \pi_{n-m} &= X_2^{a_{n-m}} \cdot X_3^{b_{n-m}} \cdot X_4^{c_{n-m}} \cdot X_n \end{aligned} \right\} \quad \dots(12.4)$$

Each equation is solved by the principle of dimensional homogeneity and values of  $a_1, b_1, c_1$  etc., are obtained. These values are substituted in equation (12.4) and values of  $\pi_1, \pi_2, \dots, \pi_{n-m}$  are obtained. These values of  $\pi$ 's are substituted in equation (12.3). The final equation for the phenomenon is obtained by expressing any one of the  $\pi$ -terms as a function of others as

$$\begin{aligned} \pi_1 &= \phi [\pi_2, \pi_3, \dots, \pi_{n-m}] \\ \text{or} \quad \pi_2 &= \phi_1 [\pi_1, \pi_3, \dots, \pi_{n-m}] \end{aligned} \quad \dots(12.5)$$

**12.4.3 Method of Selecting Repeating Variables.** The number of repeating variables are equal to the number of fundamental dimensions of the problem. The choice of repeating variables is governed by the following considerations :

1. As far as possible, the dependent variable should not be selected as repeating variable.
2. The repeating variables should be chosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Variables with Geometric Property are

$$(i) \text{ Length, } l \qquad (ii) \text{ } d \qquad (iii) \text{ Height, } H \text{ etc.}$$

Variables with flow property are

$$(i) \text{ Velocity, } V \qquad (ii) \text{ Acceleration etc.}$$

Variables with fluid property : (i)  $\mu$ , (ii)  $\rho$ , (iii)  $\omega$  etc.

3. The repeating variables selected should not form a dimensionless group.
4. The repeating variables together must have the same number of fundamental dimensions.
5. No two repeating variables should have the same dimensions.

In most of fluid mechanics problems, the choice of repeating variables may be (i)  $d, v, \rho$  (ii)  $l, v, \rho$  or (iii)  $l, v, \mu$  or (iv)  $d, v, \mu$ .

**12.4.4 Procedure for Solving Problems by Buckingham's  $\pi$ -theorem.** The procedure for solving problems by Buckingham's  $\pi$ -theorem is explained by considering the problem 12.6 which is also solved by the Rayleigh's method. The problem is :

The resisting force  $R$  of a supersonic plane during flight can be considered as dependent upon the length of the aircraft  $l$ , velocity  $V$ , air viscosity  $\mu$ , air density  $\rho$  and bulk modulus of air  $K$ . Express the functional relationship between these variables and the resisting force.

**Solution.** Step 1. The resisting force  $R$  depends upon (i)  $l$ , (ii)  $V$ , (iii)  $\mu$ , (iv)  $\rho$  and (v)  $K$ . Hence  $R$  is a function of  $l, V, \mu, \rho$  and  $K$ . Mathematically,

$$R = f(l, V, \mu, \rho, K) \quad \dots(i)$$

or it can be written as  $f_1(R, l, V, \mu, \rho, K) = 0$   $\dots(ii)$

$\therefore$  Total number of variables,  $n = 6$ .

Number of fundamental dimensions,  $m = 3$ .

[ $m$  is obtained by writing dimensions of each variables as  $R = MLT^{-2}$ ,  $V = LT^{-1}$ ,  $\mu = ML^{-1}T^{-1}$ ,  $\rho = ML^{-3}$ ,  $K = ML^{-1}T^{-2}$ . Thus as fundamental dimensions in the problem are  $M, L, T$  and hence  $m = 3$ .]

Number of dimensionless  $\pi$ -terms =  $n - m = 6 - 3 = 3$ .

Thus three  $\pi$ -terms say  $\pi_1, \pi_2$  and  $\pi_3$  are formed. Hence equation (ii) is written as

$$f_1(\pi_1, \pi_2, \pi_3) = 0. \quad \dots(iii)$$

**Step 2.** Each  $\pi$  term =  $m + 1$  variables, where  $m$  is equal to 3 and also called repeating variables. Out of six variables  $R, l, V, \mu, \rho$  and  $K$ , three variables are to be selected as repeating variable.  $R$  is a dependent variable and should not be selected as a repeating variable. Out of the five remaining

variables, one variable should have geometric property, the second variable should have flow property and third one fluid property. These requirements are fulfilled by selecting  $l$ ,  $V$  and  $\rho$  as repeating variables. The repeating variables themselves should not form a dimensionless term and should have themselves fundamental dimensions equal to  $m$ , i.e., 3 here. Dimensions of  $l$ ,  $V$  and  $\rho$  are  $L$ ,  $LT^{-1}$ ,  $ML^{-3}$  and hence the three fundamental dimensions exist in  $l$ ,  $V$  and  $\rho$  and they themselves do not form dimensionless group.

**Step 3.** Each  $\pi$ -term is written as according to equation (12.4)

$$\left. \begin{aligned} \pi_1 &= l^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot R \\ \pi_2 &= l^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot \mu \\ \pi_3 &= l^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot K \end{aligned} \right\} \dots(iv)$$

**Step 4.** Each  $\pi$ -term is solved by the principle of dimensional homogeneity. For the first  $\pi$ -term, we have

$$\pi_1 = M^0 L^0 T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot MLT^{-2}.$$

Equating the powers of  $M$ ,  $L$ ,  $T$  on both sides, we get

$$\begin{aligned} \text{Power of } M, \quad 0 &= c_1 + 1 & \therefore c_1 &= -1 \\ \text{Power of } L, \quad 0 &= a_1 + b_1 - 3c_1 + 1, & \\ \therefore \quad a_1 &= -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2 \\ \text{Power of } T, \quad 0 &= -b_1 - 2 & \therefore b_1 &= -2 \end{aligned}$$

Substituting the values of  $a_1$ ,  $b_1$  and  $c_1$  in equation (iv),

$$\pi_1 = l^{-2} \cdot V^{-2} \cdot \rho^{-1} \cdot R$$

$$\text{or} \quad \pi_1 = \frac{R}{l^2 V^2 \rho} = \frac{R}{\rho l^2 V^2} \dots(v)$$

Similarly for the 2nd  $\pi$ -term, we get  $\pi_2 = M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot ML^{-1} T^{-1}$ .

Equating the powers of  $M$ ,  $L$ ,  $T$  on both sides

$$\begin{aligned} \text{Power of } M, \quad 0 &= c_2 + 1, & \therefore c_2 &= -1 \\ \text{Power of } L, \quad 0 &= a_2 + b_2 - 3c_2 - 1, & \\ a_2 &= -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1 \\ \text{Power of } T, \quad 0 &= -b_2 - 1, & \therefore b_2 &= -1 \end{aligned}$$

Substituting the values of  $a_2$ ,  $b_2$  and  $c_2$  in  $\pi_2$  of (iv)

$$\pi_2 = l^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{l V \rho}.$$

### 3rd $\pi$ -term

$$\begin{aligned} \pi_3 &= l^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot K \\ \text{or} \quad M^0 L^0 T^0 &= l^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1} T^{-2} \end{aligned}$$

Equating the powers of  $M$ ,  $L$ ,  $T$  on both sides, we have

$$\begin{aligned} \text{Power of } M, \quad 0 &= c_3 + 1, & \therefore c_3 &= -1 \\ \text{Power of } L, \quad 0 &= a_3 + b_3 - 3c_3 - 1, & \therefore a_3 &= -b_3 + 3c_3 + 1 = 2 - 3 + 1 = 0 \\ \text{Power of } T, \quad 0 &= -b_3 - 2, & \therefore b_3 &= -2 \end{aligned}$$

Substituting the values of  $a_3$ ,  $b_3$  and  $c_3$  in  $\pi_3$  term

$$\pi_3 = l^0 \cdot V^{-2} \cdot \rho^{-1} \cdot K = \frac{K}{V^2 \rho}.$$

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**Step 5.** Substituting the values of  $\pi_1$ ,  $\pi_2$  and  $\pi_3$  in equation (iii), we get

$$f_1 \left( \frac{R}{\rho l^2 V^2}, \frac{\mu}{lV\rho}, \frac{K}{V^2 \rho} \right) = 0 \quad \text{or} \quad \frac{R}{\rho l^2 V^2} = \phi \left[ \frac{\mu}{lV\rho}, \frac{K}{V^2 \rho} \right]$$

or

$$R = \rho l^2 V^2 \phi \left[ \frac{\mu}{lV\rho}, \frac{K}{V^2 \rho} \right]. \text{ Ans.}$$

**Problem 12.8** (a) State Buckingham's  $\pi$ -theorem.

(b) The efficiency  $\eta$  of a fan depends on density  $\rho$ , dynamic viscosity  $\mu$  of the fluid, angular velocity  $\omega$ , diameter  $D$  of the rotor and the discharge  $Q$ . Express  $\eta$  in terms of dimensionless parameters.

**Solution.** (a) Statement of Buckingham's  $\pi$ -theorem is given in Article 12.4.2.

(b) Given :  $\eta$  is a function of  $\rho$ ,  $\mu$ ,  $\omega$ ,  $D$  and  $Q$

$$\therefore \eta = f(\rho, \mu, \omega, D, Q) \quad \text{or} \quad f_1(\eta, \rho, \mu, \omega, D, Q) = 0 \quad \dots(i)$$

Hence total number of variables,  $n = 6$ .

The value of  $m$ , i.e., number of fundamental dimensions for the problem is obtained by writing dimensions of each variable. Dimensions of each variable are

$$\eta = \text{Dimensionless}, \rho = ML^{-3}, \mu = ML^{-1}T^{-1}, \omega = T^{-1}, D = L \text{ and } Q = L^3T^{-1}$$

$$\therefore m = 3$$

$$\text{Number of } \pi\text{-terms} = n - m = 6 - 3 = 3$$

$$\text{Equation (i) is written as } f_1(\pi_1, \pi_2, \pi_3) = 0 \quad \dots(ii)$$

Each  $\pi$ -term contains  $m + 1$  variables, where  $m$  is equal to three and is also repeating variable.

Choosing  $D$ ,  $\omega$  and  $\rho$  as repeating variables, we have

$$\pi_1 = D^{a_1} \cdot \omega^{b_1} \cdot \rho^{c_1} \cdot \eta$$

$$\pi_2 = D^{a_2} \cdot \omega^{b_2} \cdot \rho^{c_2} \cdot \mu$$

$$\pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$$

**First  $\pi$ -term**

$$\pi_1 = D^{a_1} \cdot \omega^{b_1} \cdot \rho^{c_1} \cdot \eta$$

Substituting dimensions on both sides of  $\pi_1$ ,

$$M^0 L^0 T^0 = L^{a_1} \cdot (T^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot M^0 L^0 T^0$$

Equating the powers of  $M$ ,  $L$ ,  $T$  on both sides

$$\text{Power of } M, \quad 0 = c_1 + 0, \quad \therefore \quad c_1 = 0$$

$$\text{Power of } L, \quad 0 = a_1 + 0, \quad \therefore \quad a_1 = 0$$

$$\text{Power of } T, \quad 0 = -b_1 + 0, \quad \therefore \quad b_1 = 0$$

Substituting the values of  $a_1$ ,  $b_1$  and  $c_1$  in  $\pi_1$ , we get

$$\pi_1 = D^0 \omega^0 \rho^0 \cdot \eta = \eta$$

[If a variable is dimensionless, it itself is a  $\pi$ -term. Here the variable  $\eta$  is a dimensionless and hence  $\eta$  is a  $\pi$ -term. As it exists in first  $\pi$ -term and hence  $\pi_1 = \eta$ . Then there is no need of equating the powers. Directly the value can be obtained.]

$$\text{Second } \pi\text{-term} \quad \pi_2 = D^{a_2} \cdot \omega^{b_2} \cdot \rho^{c_2} \cdot \mu$$

Substituting the dimensions on both sides

$$M^0 L^0 T^0 = L^{a_2} \cdot (T^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot ML^{-1} T^{-1}$$

Equating the powers of  $M$ ,  $L$ ,  $T$  on both sides

$$\begin{aligned}
 \text{Power of } M, \quad & 0 = c_2 + 1, \quad \therefore \quad c_2 = -1 \\
 \text{Power of } L, \quad & 0 = a_2 - 3c_2 - 1, \quad \therefore \quad a_2 = 3c_2 + 1 = -3 + 1 = -2 \\
 \text{Power of } T, \quad & 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1 \\
 \text{Substituting the values of } a_2, b_2 \text{ and } c_2 \text{ in } \pi_2, \quad &
 \end{aligned}$$

$$\pi_2 = D^{-2} \cdot \omega^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{D^2 \omega \rho}$$

**Third  $\pi$ -term**  $\pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$

Substituting the dimensions on both sides

$$M^0 L^0 T^0 = L^{a_3} \cdot (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot L^3 T^{-1}$$

Equating the powers of  $M, L$  and  $T$  on both sides

$$\begin{aligned}
 \text{Power of } M, \quad & 0 = c_3, \quad \therefore \quad c_3 = 0 \\
 \text{Power of } L, \quad & 0 = a_3 - 3c_3 + 3, \quad \therefore \quad a_3 = 3c_3 - 3 = -3 \\
 \text{Power of } T, \quad & 0 = -b_3 - 1, \quad \therefore \quad b_3 = -1
 \end{aligned}$$

Substituting the values of  $a_3, b_3$  and  $c_3$  in  $\pi_3$ ,

$$\pi_3 = D^{-3} \cdot \omega^{-1} \cdot \rho^0 \cdot Q = \frac{Q}{D^2 \omega}$$

Substituting the values of  $\pi_1, \pi_2$  and  $\pi_3$  in equation (ii)

$$f_1 \left( \eta, \frac{\mu}{D^2 \omega \rho}, \frac{Q}{D^2 \omega} \right) = 0 \text{ or } \eta = \phi \left[ \frac{\mu}{D^2 \omega \rho}, \frac{Q}{D^2 \omega} \right]. \text{ Ans.}$$

**Problem 12.9** Using Buckingham's  $\pi$ -theorem, show that the velocity through a circular orifice is given by  $V = \sqrt{2gH} \phi \left[ \frac{D}{H}, \frac{\mu}{\rho VH} \right]$ , where  $H$  is the head causing flow,  $D$  is the diameter of the orifice,  $\mu$  is co-efficient of viscosity,  $\rho$  is the mass density and  $g$  is the acceleration due to gravity.

**Solution.** Given :

$V$  is a function of  $H, D, \mu, \rho$  and  $g$

$$\therefore V = f(H, D, \mu, \rho, g) \text{ or } f_1(V, H, D, \mu, \rho, g) = 0$$

$$\therefore \text{Total number of variable, } n = 6 \quad \dots(i)$$

Writing dimension of each variable, we have

$$V = LT^{-1}, H = L, D = L, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, g = LT^{-2}.$$

Thus number of fundamental dimensions,  $m = 3$

$$\therefore \text{Number of } \pi\text{-terms} = n - m = 6 - 3 = 3.$$

$$\text{Equation (i) can be written as } f_1(\pi_1, \pi_2, \pi_3) = 0 \quad \dots(ii)$$

Each  $\pi$ -term contains  $m + 1$  variables, where  $m = 3$  and is also equal to repeating variables. Here  $V$  is a dependent variable and hence should not be selected as repeating variable. Choosing  $H, g, \rho$  as repeating variable, we get three  $\pi$ -terms as

$$\pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$$

$$\pi_2 = H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D$$

$$\pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$$

**First  $\pi$ -term**

$$\pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$$

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Substituting dimensions on both sides

$$M^0 L^0 T^0 = L^{a_1} \cdot (LT^{-2})^{b_1} \cdot (MT^{-3})^{c_1} \cdot (LT^{-1})$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_1, \quad \therefore c_1 = 0$$

$$\text{Power of } L, \quad 0 = a_1 + b_1 - 3c_1 + 1, \quad \therefore a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\text{Power of } T, \quad 0 = -2b_1 - 1, \quad \therefore b_1 = -\frac{1}{2}$$

Substituting the values of  $a_1, b_1$  and  $c_1$  in  $\pi_1$ ,

$$\pi_1 = H^{-\frac{1}{2}} \cdot g^{-\frac{1}{2}} \cdot \rho^0 \cdot V = \frac{V}{\sqrt{gH}}.$$

**Second  $\pi$ -term**  $\pi_2 = H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D$

Substituting the dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-2})^{b_2} \cdot (ML^{-3})^{c_2} \cdot L$$

Equating the powers of  $M, L, T$ ,

$$\text{Power of } M, \quad 0 = c_2, \quad \therefore c_2 = 0$$

$$\text{Power of } L, \quad 0 = a_2 + b_2 - 3c_2 + 1, \quad a_2 = -b_2 + 3c_2 - 1 = -1$$

$$\text{Power of } T, \quad 0 = -2b_2, \quad \therefore b_2 = 0$$

Substituting the values of  $a_2, b_2, c_2$  in  $\pi_2$ ,

$$\pi_2 = H^{-1} \cdot g^0 \rho^0 \cdot D = \frac{D}{H}.$$

**Third  $\pi$ -term**  $\pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$

Substituting the dimensions on both sides

$$M^0 L^0 T^0 = L^{a_3} \cdot (LT^{-2})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1} T^{-1}$$

Equating the powers of  $M, L, T$  on both sides

$$\text{Power of } M, \quad 0 = c_3 + 1, \quad \therefore c_3 = -1$$

$$\text{Power of } L, \quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2}$$

$$\text{Power of } T, \quad 0 = -2b_3 - 1, \quad \therefore b_3 = -\frac{1}{2}$$

Substituting the values of  $a_3, b_3$  and  $c_3$  in  $\pi_3$ ,

$$\pi_3 = H^{-3/2} \cdot g^{-1/2} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{H^{3/2} \rho \sqrt{g}}$$

$$= \frac{\mu}{H \rho \sqrt{gH}} = \frac{\mu V}{H \rho V \sqrt{gH}} \quad [\text{Multiply and Divide by } V]$$

$$= \frac{\mu}{H \rho V} \cdot \pi_1 \quad \left\{ \because \frac{V}{\sqrt{gH}} = \pi_1 \right\}$$

Substituting the values of  $\pi_1, \pi_2$  and  $\pi_3$  in equation (ii),

$$f_1 \left( \frac{V}{\sqrt{gH}}, \frac{D}{H}, \pi_1 \frac{\mu}{H \rho V} \right) = 0 \text{ or } \frac{V}{\sqrt{gH}} = \phi \left[ \frac{D}{H}, \pi_1 \frac{\mu}{H \rho V} \right]$$

or  $V = \sqrt{2gH} \phi \left[ \frac{D}{H}, \frac{\mu}{\rho VH} \right]. \text{ Ans.}$

Multiplying by a constant does not change the character of  $\pi$ -terms.

**Problem 12.10** The pressure difference  $\Delta p$  in a pipe of diameter  $D$  and length  $l$  due to turbulent flow depends on the velocity  $V$ , viscosity  $\mu$ , density  $\rho$  and roughness  $k$ . Using Buckingham's  $\pi$ -theorem, obtain an expression for  $\Delta p$ .

**Solution.** Given :

$\Delta p$  is a function of  $D, l, V, \mu, \rho, k$

$$\therefore \Delta p = f(D, l, V, \mu, \rho, k) \text{ or } f_1(\Delta p, D, l, V, \mu, \rho, k) = 0 \quad \dots(i)$$

$\therefore$  Total number of variables,  $n = 7$ .

Writing dimensions of each variable,

Dimension of  $\Delta p$  = Dimension of pressure =  $ML^{-1}T^{-2}$

$$D = L, l = L, V = LT^{-1}, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, k = L$$

$\therefore$  Number of fundamental dimensions,  $m = 3$

Number of  $\pi$ -terms  $= n - m = 7 - 3 = 4$ .

Now equation (i) can be grouped in 4  $\pi$ -terms as

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \quad \dots(ii)$$

Each  $\pi$ -term contains  $m + 1$  or  $3 + 1 = 4$  variables. Out of four variables, three are repeating variables. Choosing  $D, V, \rho$  as the repeating variables, we have the four  $\pi$ -terms as

$$\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot \Delta p$$

$$\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot l$$

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$$

$$\pi_4 = D^{a_4} \cdot V^{b_4} \cdot \rho^{c_4} \cdot k$$

**First  $\pi$ -term**  $\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot \Delta p$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot ML^{-1} T^{-2}$$

Equating the powers of  $M, L, T$  on both sides,

Power of  $M$ ,  $0 = c_1 + 1 \quad \therefore c_1 = -1$

Power of  $L$ ,  $0 = a_1 + b_1 - 3c_1 - 1, \quad \therefore a_1 = -b_1 + 3c_1 + 1 = 2 - 3 + 1 = 0$

Power of  $T$ ,  $0 = -b_1 - 2, \quad \therefore b_1 = -2$

Substituting the values of  $a_1, b_1$  and  $c_1$  in  $\pi_1$ ,

$$\pi_1 = D^0 \cdot V^{-2} \cdot \rho^{-1} \cdot \Delta p = \frac{\Delta p}{\rho V^2}$$

**Second  $\pi$ -term**  $\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot l$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot L$$

Equating powers of  $M, L, T$  on both sides,

Power of  $M$ ,  $0 = c_2, \quad \therefore c_2 = 0$

Power of  $L$ ,  $0 = a_2 - b_2 - 3c_2 + 1, \quad \therefore a_2 = b_2 + 3c_2 - 1 = -1$

Power of  $T$ ,  $0 = -b_2, \quad \therefore b_2 = 0$

Substituting the values of  $a_2, b_2, c_2$  in  $\pi_2$ ,

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$$\pi_2 = D^{-1} \cdot V^0 \cdot \rho^0 \cdot l = \frac{l}{D}$$

**Third  $\pi$ -term**

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1} T^{-1}$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_3 + 1, \quad \therefore c_3 = -1$$

$$\text{Power of } L, \quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore a_3 = -b_3 + 3c_3 + 1 = 1 - 3 + 1 = -1$$

$$\text{Power of } T, \quad 0 = -b_3 - 1, \quad \therefore b_3 = -1$$

Substituting the values of  $a_3, b_3$  and  $c_3$  in  $\pi_3$ ,

$$\pi_3 = D^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{DV\rho}.$$

**Fourth  $\pi$ -term**

$$\pi_4 = D^{a_4} \cdot V^{b_4} \cdot \rho^{c_4} \cdot k$$

$$\text{or } M^0 L^0 T^0 = L^{a_4} \cdot (LT^{-1})^{b_4} \cdot (ML^{-3})^{c_4} \cdot L \quad \{\text{Dimension of } k = L\}$$

Equating the power of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_4, \quad \therefore c_4 = 0$$

$$\text{Power of } L, \quad 0 = a_4 - b_4 - 3c_4 + 1, \quad \therefore a_4 = b_4 + 3c_4 - 1 = -1$$

$$\text{Power of } T, \quad 0 = -b_4, \quad \therefore b_4 = 0$$

Substituting the values of  $a_4, b_4, c_4$  in  $\pi_4$ ,

$$\pi_4 = D^{-1} \cdot V^0 \cdot \rho^0 \cdot k = \frac{k}{D}$$

Substituting the values of  $\pi_1, \pi_2, \pi_3$  and  $\pi_4$  in (ii), we get

$$f_1 \left( \frac{\Delta p}{\rho V^2}, \frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D} \right) = 0 \quad \text{or} \quad \frac{\Delta p}{\rho V^2} = \phi \left[ \frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D} \right]. \text{ Ans.}$$

**Expression for  $h_f$**  (Difference of pressure-head). From experiments, it was observed that pressure difference,  $\Delta p$  is a linear function of  $\frac{l}{D}$  and hence it is taken out of function

$$\therefore \frac{\Delta p}{\rho V^2} = \frac{l}{D} \phi \left[ \frac{\mu}{DV\rho}, \frac{k}{D} \right]$$

$$\therefore \frac{\Delta p}{\rho} = V^2 \cdot \frac{l}{D} \phi \left[ \frac{\mu}{DV\rho}, \frac{k}{D} \right]$$

$$\text{Dividing by } g \text{ to both sides, we have } \frac{\Delta p}{\rho g} = \frac{V^2 \cdot l}{g \cdot D} \phi \left[ \frac{\mu}{DV\rho}, \frac{k}{D} \right].$$

Now  $\phi \left[ \frac{\mu}{DV\rho}, \frac{k}{D} \right]$  contains two terms. First one is  $\frac{\mu}{DV\rho}$  which is  $\frac{1}{\text{Reynolds number}}$  or  $\frac{1}{R_e}$  and

second one is  $\frac{k}{D}$  which is called roughness factor. Now  $\phi \left[ \frac{1}{R_e}, \frac{k}{D} \right]$  is put equal to  $f$ , where  $f$  is the co-efficient of friction which is a function of Reynolds number and roughness factor.

$$\therefore \frac{\Delta p}{\rho g} = \frac{4f}{2} \cdot \frac{V^2 l}{gD} \quad \left\{ \because f = \phi \left( \frac{\mu}{DV\rho}, \frac{K}{D} \right) \right\}$$

Multiplying or dividing by any constant does not change the character of  $\pi$ -terms.

$$\therefore \frac{\Delta p}{\rho g} = h_f = \frac{4f \cdot LV^2}{D \times 2g} . \text{Ans.}$$

**Problem 12.11** The pressure difference  $\Delta p$  in a pipe of diameter  $D$  and length  $l$  due to viscous flow depends on the velocity  $V$ , viscosity  $\mu$  and density  $\rho$ . Using Buckingham's  $\pi$ -theorem, obtain an expression for  $\Delta p$ .

**Solution.** This problem is similar to problem 12.10. The only difference is that  $\Delta p$  is to be calculated for viscous flow. Then in the repeating variable instead of  $\rho$ , the fluid property  $\mu$  is to be chosen.

Now  $\Delta p$  is a function of  $D, l, V, \mu, \rho$  or  $\Delta p = f(D, l, V, \mu, \rho)$   
or  $f_1(\Delta p, D, l, V, \mu, \rho) = 0$  ... (i)

Total number of variables,  $n = 6$

Number of fundamental dimensions,  $m = 3$

Number of  $\pi$ -terms  $= n - 3 = 6 - 3 = 3$

Hence equation (i) is written as  $f_1(\pi_1, \pi_2, \pi_3) = 0$  ... (ii)

Each  $\pi$ -term contains  $m + 1$  variables, i.e.,  $3 + 1 = 4$  variables. Out of four variables, three are repeating variables.

Choosing  $D, V, \mu$  as repeating variables, we have  $\pi$ -terms as

$$\begin{aligned}\pi_1 &= D^{a_1} \cdot V^{b_1} \cdot \mu^{c_1} \cdot \Delta p \\ \pi_2 &= D^{a_2} \cdot V^{b_2} \cdot \mu^{c_2} \cdot l \\ \pi_3 &= D^{a_3} \cdot V^{b_3} \cdot \mu^{c_3} \cdot \rho\end{aligned}$$

**First  $\pi$ -term**  $\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \mu^{c_1} \cdot \Delta p$

Substituting the dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-1}T^{-1})^{c_1} \cdot ML^{-1}T^{-2}$$

Equating the powers of  $M, L, T$  on both sides,

$$\begin{aligned}\text{Power of } M, \quad 0 &= c_1 + 1, \quad \therefore c_1 = -1 \\ \text{Power of } L, \quad 0 &= a_1 + b_1 - c_1 - 1, \quad \therefore a_1 = -b_1 + c_1 + 1 = 1 - 1 + 1 = 1 \\ \text{Power of } T, \quad 0 &= -b_1 - c_1 - 2, \quad \therefore b_1 = -c_1 - 2 = 1 - 2 = -1\end{aligned}$$

Substituting the values of  $a_1, b_1$  and  $c_1$  in  $\pi_1$ ,

$$\pi_1 = D^1 \cdot V^{-1} \cdot \mu^{-1} \cdot \Delta p = \frac{D\Delta p}{\mu V}.$$

**Second  $\pi$ -term**  $\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \mu^{c_2} \cdot l$

Substituting the dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-1}T^{-1})^{c_2} \cdot L$$

Equating the powers of  $M, L, T$  on both sides

$$\begin{aligned}\text{Power of } M, \quad 0 &= c_2, \quad \therefore c_2 = 0 \\ \text{Power of } L, \quad 0 &= a_2 + b_2 - c_2 + 1, \quad \therefore a_2 = -b_2 + c_2 - 1 = -1 \\ \text{Power of } T, \quad 0 &= -b_2 - c_2, \quad \therefore b_2 = -c_2 = 0\end{aligned}$$

Substituting the values of  $a_2, b_2$  and  $c_2$  in  $\pi_2$ ,

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$$\pi_2 = D^{-1} \cdot V^0 \cdot \mu^0 \cdot l = \frac{l}{D}$$

**Third  $\pi$ -term**

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \mu^{c_3} \cdot \rho$$

Substituting the dimension on both sides,

$$M^0 L^0 T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-1}T^{-1})^{c_3} \cdot ML^{-3}.$$

Equating the powers of  $M, L, T$  on both sides

$$\text{Power of } M, \quad 0 = c_3 + 1, \quad \therefore c_3 = -1$$

$$\text{Power of } L, \quad 0 = a_3 + b_3 - c_3 - 3, \quad \therefore a_3 = -b_3 + c_3 + 3 = -1 - 1 + 3 = 1$$

$$\text{Power of } T, \quad 0 = -b_3 - c_3, \quad \therefore b_3 = -c_3 = -(-1) = 1$$

Substituting the values of  $a_3, b_3$  and  $c_3$  in  $\pi_3$ ,

$$\pi_3 = D^1 \cdot V^1 \cdot \mu^{-1} \cdot \rho = \frac{\rho DV}{\mu}.$$

Substituting the values of  $\pi_1, \pi_2$  and  $\pi_3$  in equation (ii),

$$f_1 \left( \frac{D\Delta p}{\mu V}, \frac{l}{D}, \frac{\rho DV}{\mu} \right) = 0 \quad \text{or} \quad \frac{D\Delta p}{\mu V} = \phi \left[ \frac{l}{D}, \frac{\rho DV}{\mu} \right] \quad \text{or} \quad \Delta p = \frac{\mu V}{D} \phi \left[ \frac{l}{D}, \frac{\rho DV}{\mu} \right]$$

Experiments show that the pressure difference  $\Delta p$  is a linear function  $\frac{l}{D}$ . Hence  $\frac{l}{D}$  can be taken out of the functional as

$$\Delta p = \frac{\mu V}{D} \times \frac{l}{D} \phi \left[ \frac{\rho DV}{\mu} \right]. \text{ Ans.}$$

Expression for difference of pressure head for viscous flow

$$h_f = \frac{\Delta p}{\rho g} = \frac{\mu V}{D} \times \frac{l}{D} \times \frac{1}{\rho g} \phi [R_e] \quad \left\{ \because \frac{\rho DV}{\mu} = R_e \right\}$$

$$= \frac{\mu V l}{w D^2} \phi [R_e]. \text{ Ans.}$$

**Problem 12.12** Derive on the basis of dimensional analysis suitable parameters to present the thrust developed by a propeller. Assume that the thrust  $P$  depends upon the angular velocity  $\omega$ , speed of advance  $V$ , diameter  $D$ , dynamic viscosity  $\mu$ , mass density  $\rho$ , elasticity of the fluid medium which can be denoted by the speed of sound in the medium  $C$ .

**Solution.** Thrust  $P$  is a function of  $\omega, V, D, \mu, \rho, C$

$$\text{or} \quad P = f(\omega, V, D, \mu, \rho, C)$$

$$\text{or} \quad f_1 = (P, \omega, V, D, \mu, \rho, C) = 0 \quad \dots(i)$$

$\therefore$  Total number of variables,  $n = 7$

Writing dimensions of each variable, we have

$$P = MLT^{-2}, \omega = T^{-1}, V = LT^{-1}, D = L,$$

$$\mu = ML^{-1} T^{-1}, \rho = ML^{-3}, C = LT^{-1}$$

$\therefore$  Number of fundamental dimensions,  $m = 3$

$\therefore$  Number of  $\pi$ -terms  $= n - m = 7 - 3 = 4$

Hence, equation (i) can be written as  $f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0$

$\dots(ii)$

Each  $\pi$ -term contains  $m + 1$ , i.e.,  $3 + 1 = 4$  variables. Out of four variables, three are repeating variables.

Choosing  $D, V, \rho$  as repeating variables, we get  $\pi$ -terms as

$$\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot P$$

$$\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot \omega$$

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$$

$$\pi_4 = D^{a_4} \cdot V^{b_4} \cdot \rho^{c_4} \cdot C$$

**First  $\pi$ -term**

$$\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot P$$

Writing dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot MLT^{-2}$$

Equating powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_1 + 1, \quad \therefore c_1 = -1$$

$$\text{Power of } L, \quad 0 = a_1 + b_1 - 3c_1 + 1,$$

$$a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$$

$$\text{Power of } T, \quad 0 = -b_1 - 2, \quad \therefore b_1 = -2$$

Substituting the values of  $a_1, b_1$  and  $c_1$  in  $\pi_1$ ,

$$\pi_1 = D^{-2} \cdot V^{-2} \cdot \rho^{-1} P = \frac{P}{D^2 V^2 \rho}.$$

**Second  $\pi$ -term**

$$\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \Delta^{c_2} \cdot \omega$$

Writing dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot T^{-1}$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_2, \quad \therefore c_2 = 0$$

$$\text{Power of } L, \quad 0 = a_2 + b_2 - 3c_2, \quad \therefore a_2 = -b_2 + 3c_2 = 1 + 0 = 1$$

$$\text{Power of } T, \quad 0 = -b_2 - 1, \quad \therefore b_2 = -1$$

Substituting the values of  $a_2, b_2, c_2$  in  $\pi_2$ ,

$$\pi_2 = D^1 \cdot V^{-1} \cdot \rho^0 \cdot \omega = \frac{D\omega}{V}.$$

**Third  $\pi$ -term**

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$$

Writing dimensions on both sides,

$$M^0 L^0 T^0 = D^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1} T^{-1}$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_3 + 1, \quad \therefore c_3 = -1$$

$$\text{Power of } L, \quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore a_3 = -b_3 + 3c_3 + 1 = 1 - 3 + 1 = -1$$

$$\text{Power of } T, \quad 0 = -b_3 - 1 \quad \therefore b_3 = -1$$

Substituting the values of  $a_3, b_3$  and  $c_3$  in  $\pi_3$ ,

$$\pi_3 = D^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{DV\rho}.$$

**Fourth  $\pi$ -term**

$$\pi_4 = D^{a_4} \cdot V^{b_4} \cdot \rho^{c_4} \cdot C$$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_4} \cdot (LT^{-1})^{b_4} \cdot (ML^{-3})^{c_4} \cdot LT^{-1}$$

Equating the powers of  $M, L, T$  on both sides,

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$$\begin{aligned} \text{Power of } M, \quad & 0 = c_4, \quad \therefore c_4 = 0 \\ \text{Power of } L, \quad & 0 = a_4 + b_4 - 3c_4 + 1, \quad \therefore a_4 = -b_4 + 3c_4 - 1 = 1 + 0 - 1 = 0 \\ \text{Power of } T, \quad & 0 = -a_4 - 1, \quad \therefore b_4 = -1 \end{aligned}$$

Substituting the values of  $a_4$ ,  $b_4$  and  $c_4$  in  $\pi_4$ ,

$$\pi_4 = D^0 \cdot V^{-1} \cdot \rho^0 \cdot C = \frac{C}{V}.$$

Substituting the values of  $\pi_1$ ,  $\pi_2$ ,  $\pi_3$  and  $\pi_4$  in equation (ii),

$$f_1 \left( \frac{P}{D^2 V^2 \rho}, \frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V} \right) = 0 \quad \text{or} \quad \frac{P}{D^2 V^2 \rho} = \phi \left( \frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V} \right)$$

$$\text{or} \quad P = D^2 V^2 \rho \phi \left( \frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V} \right). \text{ Ans.}$$

**Problem 12.13** The frictional torque  $T$  of a disc of diameter  $D$  rotating at a speed  $N$  in a fluid of viscosity  $\mu$  and density  $\rho$  in a turbulent flow is given by  $T = D^5 N^2 \rho \phi \left[ \frac{\mu}{D^2 N \rho} \right]$ .

Prove this by the method of dimensions.

**Solution.** Given :  $T = f(D, N, \mu, \rho)$  or  $f_1(T, D, N, \mu, \rho) = 0$  ... (i)

$\therefore$  Total number of variables,  $n = 5$

Dimensions of each variable are expressed as

$$T = ML^2 T^{-2}, D = L, N = T^{-1}, \mu = ML^{-1} T^{-1}, \rho = ML^{-3}$$

$\therefore$  Number of fundamental dimensions,  $m = 3$

Number of  $\pi$ -terms  $= n - m = 5 - 3 = 2$

Hence equation (i) can be written as  $f_1(\pi_1, \pi_2) = 0$  ... (ii)

Each  $\pi$ -term contains  $m + 1$  variable, i.e.,  $3 + 1 = 4$  variables. Three variables are repeating variables.

Choosing  $D, N, \rho$  as repeating variables, the  $\pi$ -terms are

$$\begin{aligned} \pi_1 &= D^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot T \\ \pi_2 &= D^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu \end{aligned}$$

**Dimensional Analysis of  $\pi_1$**

$$\pi_1 = D^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot T$$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_1} \cdot (T^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot ML^2 T^{-2}.$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_1 + 1, \quad \therefore c_1 = -1$$

$$\text{Power of } L, \quad 0 = a_1 - 3c_1 + 2, \quad \therefore a_1 = 3c_1 - 2 = -3 - 2 = -5$$

$$\text{Power of } T, \quad 0 = -b_1 - 2, \quad \therefore b_1 = -2$$

Substituting the values of  $a_1, b_1, c_1$  in  $\pi_1$ ,

$$\pi_1 = D^{-5} \cdot N^{-2} \cdot \rho^{-1} \cdot T = \frac{T}{D^5 N^2 \rho}.$$

**Dimensional Analysis of  $\pi_2$**

$$\pi_2 = D^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu$$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (T^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot ML^{-1} T^{-1}.$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_2 + 1, \quad \therefore c_2 = -1$$

$$\text{Power of } L, \quad 0 = a_2 - 3c_2 - 1, \quad \therefore a_2 = 3c_2 + 1 = -3 + 1 = -2$$

$$\text{Power of } T, \quad 0 = -b_2 - 1, \quad \therefore b_2 = -1$$

Substituting the values of  $a_2, b_2$  and  $c_2$  in  $\pi_2$ ,

$$\pi_2 = D^{-2} N^{-1} \rho^{-1} \cdot \mu = \frac{\mu}{D^2 N \rho}.$$

Substituting the values of  $\pi_1$  and  $\pi_2$  in equation (ii),

$$f_1 \left( \frac{T}{D^5 N^2 \rho}, \frac{\mu}{D^2 N \rho} \right) = 0 \quad \text{or} \quad \frac{T}{D^5 N^2 \rho} = \phi \left( \frac{\mu}{D^2 N \rho} \right)$$

or

$$T = D^5 N^2 \rho \phi \left[ \frac{\mu}{D^2 N \rho} \right]. \text{ Ans.}$$

**Problem 12.14** Using Buckingham's  $\pi$ -theorem, show that the discharge  $Q$  consumed by an oil ring is given by

$$Q = N d^3 \phi \left[ \frac{\mu}{\rho N d^2}, \frac{\sigma}{\rho N^2 d^3}, \frac{w}{\rho N^2 d} \right]$$

where  $d$  is the internal diameter of the ring,  $N$  is rotational speed,  $\rho$  is density,  $\mu$  is viscosity,  $\sigma$  is surface tension and  $w$  is the specific weight of oil.

**Solution.** Given :  $Q = f(d, N, \rho, \mu, \sigma, w)$  or  $f_1(Q, d, N, \rho, \mu, \sigma, w) = 0$  ... (i)

$\therefore$  Total number of variables,  $n = 7$

Dimensions of each variables are

$$Q = L^3 T^{-1}, d = L, N = T^{-1}, \rho = ML^{-3}, \mu = ML^{-1} T^{-1}, \sigma = MT^{-2}$$

and

$$w = ML^{-2} T^{-2}$$

$\therefore$  Total number of fundamental dimensions,  $m = 3$

$\therefore$  Total number of  $\pi$ -terms  $= n - m = 7 - 3 = 4$

$\therefore$  Equation (i) becomes as  $f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0$  ... (ii)

Choosing  $d, N, \rho$  as repeating variables, the  $\pi$ -terms are

$$\pi_1 = d^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot Q$$

$$\pi_2 = d^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu$$

$$\pi_3 = d^{a_3} \cdot N^{b_3} \cdot \rho^{c_3} \cdot \sigma$$

$$\pi_4 = d^{a_4} \cdot N^{b_4} \cdot \rho^{c_4} \cdot w.$$

**First  $\pi$ -term**  $\pi_1 = d^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot Q.$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_1} \cdot (T^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot L^3 T^{-1}.$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_1, \quad \therefore c_1 = 0$$

$$\text{Power of } L, \quad 0 = a_1 - 3c_1 + 3, \quad \therefore a_1 = 3c_1 - 3 = 0 - 3 = -3$$

$$\text{Power of } T, \quad 0 = -b_1 - 1, \quad \therefore b_1 = -1$$

$$\text{Substituting } a_1, b_1, c_1 \text{ in } \pi_1, \pi_1 = d^{-3} \cdot N^{-1} \cdot \rho^0 \cdot Q = \frac{Q}{d^3 N}$$

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**Second  $\pi$ -term**  $\pi_2 = d^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu.$

Substituting the dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_2} \cdot (T^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot ML^{-1} T^{-1}$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_2 + 1, \quad \therefore \quad c_2 = -1$$

$$\text{Power of } L, \quad 0 = a_2 - 3c_2 - 1, \quad 0 = a_2 - 3(-1) - 1, \quad a_2 = 2$$

$$\therefore \quad a_2 = 3c_2 + 1 = -3 + 1 = -2$$

$$\text{Power of } T, \quad 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$$

Substituting the values of  $a_2, b_2, c_2$  in  $\pi_2$ ,

$$\pi_2 = d^{-2} \cdot N^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{d^2 N \rho} \quad \text{or} \quad \frac{\mu}{\rho N d^2}.$$

**Third  $\pi$ -term**  $\pi_3 = d^{a_3} \cdot N^{b_3} \cdot \rho^{c_3} \cdot \sigma.$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_3} \cdot (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot MT^{-2}$$

Equating the powers of  $M, L, T$  on the sides,

$$\text{Power of } M, \quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$$

$$\text{Power of } L, \quad 0 = a_3 - 3c_3, \quad \therefore \quad a_3 = 3c_3 = -3$$

$$\text{Power of } T, \quad 0 = -b_3 - 2, \quad \therefore \quad b_3 = -2$$

Substituting the values of  $a_3, b_3, c_3$  in  $\pi_3$ ,

$$\pi_3 = d^{-3} \cdot N^{-2} \cdot \rho^{-1} \cdot \sigma = \frac{\sigma}{d^3 N^2 \rho}.$$

**Fourth  $\pi$ -term**  $\pi_4 = d^{a_4} \cdot N^{b_4} \cdot \rho^{c_4} \cdot w$

Substituting dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_4} \cdot (T^{-1})^{b_4} \cdot (ML^{-3})^{c_4} \cdot ML^{-2} T^{-2}$$

Equating the powers of  $M, L, T$  on both sides,

$$\text{Power of } M, \quad 0 = c_4 + 1, \quad \therefore \quad c_4 = -1$$

$$\text{Power of } L, \quad 0 = a_4 - 3c_4 - 2, \quad \therefore \quad a_4 = 3c_4 + 2 = -3 + 2 = -1$$

$$\text{Power of } T, \quad 0 = -b_4 - 2, \quad \therefore \quad b_4 = -2$$

Substituting the values of  $a_4, b_4$  and  $c_4$  in  $\pi_4$ ,

$$\pi_4 = d^{-1} \cdot N^{-2} \cdot \rho^{-1} \cdot w = \frac{w}{d N^2 \rho}.$$

Now substituting the values of  $\pi_1, \pi_2, \pi_3, \pi_4$  in (ii),

$$f\left(\frac{Q}{d^3 N}, \frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 \rho}, \frac{w}{d N^2 \rho}\right) = 0 \quad \text{or} \quad \frac{Q}{d^3 N} = f_1\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 \rho}, \frac{w}{d N^2 \rho}\right]$$

or

$$Q = d^3 N \phi \left[ \frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 \rho}, \frac{w}{d N^2 \rho} \right]. \text{ Ans.}$$

### ► 12.5 MODEL ANALYSIS

For predicting the performance of the hydraulic structures (such as dams, spillways etc.) or hydraulic machines (such as turbines, pumps etc.), before actually constructing or manufacturing,

models of the structures or machines are made and tests are performed on them to obtain the desired information.

The **model** is the small scale replica of the actual structure or machine. The actual structure or machine is called **Prototype**. It is not necessary that the models should be smaller than the prototypes (though in most of cases it is), they may be larger than the prototype. The study of models of actual machines is called **Model analysis**. Model analysis is actually an experimental method of finding solutions of complex flow problems. Exact analytical solutions are possible only for a limited number of flow problems. The followings are the advantages of the dimensional and model analysis :

1. The performance of the hydraulic structure or hydraulic machine can be easily predicted, in advance, from its model.
2. With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests on the model.
3. The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be, finally, adopted.
4. The tests performed on the models can be utilized for obtaining, in advance, useful information about the performance of the prototypes only if a complete similarity exists between the model and the prototype.

## ► 12.6 SIMILITUDE-TYPES OF SIMILARITIES

Similitude is defined as the similarity between the model and its prototype in every respect, which means that the model and prototype have similar properties or model and prototype are completely similar. Three types of similarities must exist between the model and prototype. They are

1. Geometric Similarity, 2. Kinematic Similarity, and 3. Dynamic Similarity.

**1. Geometric Similarity.** The geometric similarity is said to exist between the model and the prototype. The ratio of all corresponding linear dimension in the model and prototype are equal.

Let

$L_m$  = Length of model,  $b_m$  = Breadth of model,

$D_m$  = Diameter of model,  $A_m$  = Area of model,

$\nabla_m$  = Volume of model,

and  $L_p, b_p, D_p, A_p, \nabla_p$  = Corresponding values of the prototype.

For geometric similarity between model and prototype, we must have the relation,

$$\frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{D_p}{D_m} = L_r \quad \dots(12.6)$$

where  $L_r$  is called the scale ratio.

For area's ratio and volume's ratio the relation should be as given below :

$$\frac{A_p}{A_m} = \frac{L_p \times b_p}{L_m \times b_m} = L_r \times L_r = L_r^2 \quad \dots(12.7)$$

and

$$\frac{\nabla_p}{\nabla_m} = \left(\frac{L_p}{L_m}\right)^3 = \left(\frac{b_p}{b_m}\right)^3 = \left(\frac{D_p}{D_m}\right)^3 \quad \dots(12.8)$$

**2. Kinematic Similarity.** Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios of the velocity and acceleration at the corresponding points in the model and at the corresponding

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points in the prototype are the same. Since velocity and acceleration are vector quantities, hence not only the ratio of magnitude of velocity and acceleration at the corresponding points in model and prototype should be same ; but the directions of velocity and accelerations at the corresponding points in the model and prototype also should be parallel.

Let  $V_{P_1}$  = Velocity of fluid at point 1 in prototype,  
 $V_{P_2}$  = Velocity of fluid at point 2 in prototype,  
 $a_{P_1}$  = Acceleration of fluid at point 1 in prototype,  
 $a_{P_2}$  = Acceleration of fluid at point 2 in prototype, and  
 $V_{m_1}, V_{m_2}, a_{m_1}, a_{m_2}$  = Corresponding values at the corresponding points of fluid velocity and acceleration in the model.

For kinematic similarity, we must have

$$\frac{V_{P_1}}{V_{m_1}} = \frac{V_{P_2}}{V_{m_2}} = V_r \quad \dots(12.9)$$

where  $V_r$  is the velocity ratio.

$$\text{For acceleration, we must have } \frac{a_{P_1}}{a_{m_1}} = \frac{a_{P_2}}{a_{m_2}} = a_r \quad \dots(12.10)$$

where  $a_r$  is the acceleration ratio.

Also the directions of the velocities in the model and prototype should be same.

**3. Dynamic Similarity.** Dynamic similarity means the similarity of forces between the model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points are equal. Also the directions of the corresponding forces at the corresponding points should be same.

Let  $(F_i)_P$  = Inertia force at a point in prototype,  
 $(F_v)_P$  = Viscous force at the point in prototype,  
 $(F_g)_P$  = Gravity force at the point in prototype,

and  $(F_i)_m, (F_v)_m, (F_g)_m$  = Corresponding values of forces at the corresponding point in model.

Then for dynamic similarity, we have

$$\frac{(F_i)_P}{(F_i)_m} = \frac{(F_v)_P}{(F_v)_m} = \frac{(F_g)_P}{(F_g)_m} \dots = F_r, \text{ where } F_r \text{ is the force ratio.}$$

Also the directions of the corresponding forces at the corresponding points in the model and prototype should be same.

### ► 12.7 TYPES OF FORCES ACTING IN MOVING FLUID

For the fluid flow problems, the forces acting on a fluid mass may be any one, or a combination of the several of the following forces :

1. Inertia force,  $F_i$ .
2. Viscous force,  $F_v$ .
3. Gravity force,  $F_g$ .
4. Pressure force,  $F_p$ .
5. Surface tension force,  $F_s$ .
6. Elastic force,  $F_e$ .

1. **Inertia Force ( $F_i$ )**. It is equal to the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration. It is always existing in the fluid flow problems.

2. **Viscous Force ( $F_v$ )**. It is equal to the product of shear stress ( $\tau$ ) due to viscosity and surface area of the flow. It is present in fluid flow problems where viscosity is having an important role to play.

3. **Gravity Force ( $F_g$ )**. It is equal to the product of mass and acceleration due to gravity of the flowing fluid. It is present in case of open surface flow.

4. **Pressure Force ( $F_p$ )**. It is equal to the product of pressure intensity and cross-sectional area of the flowing fluid. It is present in case of pipe-flow.

5. **Surface Tension Force ( $F_s$ )**. It is equal to the product of surface tension and length of surface of the flowing fluid.

6. **Elastic Force ( $F_e$ )**. It is equal to the product of elastic stress and area of the flowing fluid.

For a flowing fluid, the above-mentioned forces may not always be present. And also the forces, which are present in a fluid flow problem, are not of equal magnitude. There are always one or two forces which dominate the other forces. These dominating forces govern the flow of fluid.

## ► 12.8 DIMENSIONLESS NUMBERS

Dimensionless numbers are those numbers which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force. As this is a ratio of one force to the other force, it will be a dimensionless number. These dimensionless numbers are also called non-dimensional parameters. The followings are the important dimensionless numbers :

- |                      |                     |
|----------------------|---------------------|
| 1. Reynold's number, | 2. Froude's number, |
| 3. Euler's number,   | 4. Weber's number,  |
| 5. Mach's number.    |                     |

**12.8.1 Reynold's Number ( $R_e$ )**. It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for Reynold's number is obtained as

$$\begin{aligned} \text{Inertia force } (F_i) &= \text{Mass} \times \text{Acceleration of flowing fluid} \\ &= \rho \times \text{Volume} \times \frac{\text{Velocity}}{\text{Time}} = \rho \times \frac{\text{Volume}}{\text{Time}} \times \text{Velocity} \\ &= \rho \times A V \times V \quad \{ \because \text{Volume per sec} = \text{Area} \times \text{Velocity} = A \times V \} \\ &= \rho A V^2 \end{aligned} \quad \dots(12.11)$$

$$\begin{aligned} \text{Viscous force } (F_v) &= \text{Shear stress} \times \text{Area} \quad \left\{ \because \tau = \mu \frac{du}{dy} \therefore \text{Force} = \tau \times \text{Area} \right\} \\ &= \tau \times A \\ &= \left( \mu \frac{du}{dy} \right) \times A = \mu \cdot \frac{V}{L} \times A \quad \left\{ \because \frac{du}{dy} = \frac{V}{L} \right\} \end{aligned}$$

By definition, Reynold's number,

$$\begin{aligned} R_e &= \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \cdot \frac{V}{L} \times A} = \frac{\rho V L}{\mu} \\ &= \frac{V \times L}{(\mu / \rho)} = \frac{V \times L}{\nu} \quad \left\{ \because \frac{\mu}{\rho} = \nu = \text{Kinematic viscosity} \right\} \end{aligned}$$

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In case of pipe flow, the linear dimension  $L$  is taken as diameter,  $d$ . Hence Reynold's number for pipe flow,

$$R_e = \frac{V \times d}{\nu} \quad \text{or} \quad \frac{\rho V d}{\mu}. \quad \dots(12.12)$$

**12.8.2 Froude's Number ( $F_e$ ).** The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as

$$F_e = \sqrt{\frac{F_i}{F_g}}$$

where  $F_i$  from equation (12.11)  $= \rho A V^2$

and  $F_g$  = Force due to gravity

= Mass  $\times$  Acceleration due to gravity

=  $\rho \times \text{Volume} \times g = \rho \times L^3 \times g$

=  $\rho \times L^2 \times L \times g = \rho \times A \times L \times g$

$\{\because \text{Volume} = L^3\}$

$\{\because L^2 = A = \text{Area}\}$

$$\therefore F_e = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \sqrt{\frac{V^2}{L g}} = \frac{V}{\sqrt{L g}} \quad \dots(12.13)$$

**12.8.3 Euler's Number ( $E_u$ ).** It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as

$$E_u = \sqrt{\frac{F_i}{F_p}}$$

where  $F_p$  = Intensity of pressure  $\times$  Area  $= p \times A$

and  $F_i = \rho A V^2$

$$\therefore E_u = \sqrt{\frac{\rho A V^2}{p \times A}} = \sqrt{\frac{V^2}{p / \rho}} = \frac{V}{\sqrt{p / \rho}} \quad \dots(12.14)$$

**12.8.4 Weber's Number ( $W_e$ ).** It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

$$\text{Weber's Number, } W_e = \sqrt{\frac{F_i}{F_s}}$$

where  $F_i$  = Inertia force  $= \rho A V^2$

and  $F_s$  = Surface tension force

= Surface tension per unit length  $\times$  Length  $= \sigma \times L$

$$\begin{aligned} \therefore W_e &= \sqrt{\frac{\rho A V^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 \times V^2}{\sigma \times L}} && \{\because A = L^2\} \\ &= \sqrt{\frac{\rho L \times V^2}{\sigma}} = \sqrt{\frac{V^2}{\sigma / \rho L}} = \frac{V}{\sqrt{\sigma / \rho L}}. && \dots(12.15) \end{aligned}$$

**12.8.5 Mach's Number ( $M$ ).** Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. Mathematically, it is defined as

$$M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{F_i}{F_e}}$$

where  $F_i = \rho A V^2$

and  $F_e = \text{Elastic force} = \text{Elastic stress} \times \text{Area}$   
 $= K \times A = K \times L^2$

$\{\because K = \text{Elastic stress}\}$

$$\therefore M = \sqrt{\frac{\rho A V^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 \times V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}$$

But  $\sqrt{\frac{K}{\rho}} = C = \text{Velocity of sound in the fluid}$

$$\therefore M = \frac{V}{C}. \quad \dots(12.16)$$

## ► 12.9 MODEL LAWS OR SIMILARITY LAWS

For the dynamic similarity between the model and the prototype, the ratio of the corresponding forces acting at the corresponding points in the model and prototype should be equal. The ratio of the forces are dimensionless numbers. It means for dynamic similarity between the model and prototype, the dimensionless numbers should be same for model and the prototype. But it is quite difficult to satisfy the condition that all the dimensionless numbers (*i.e.*,  $R_e$ ,  $F_e$ ,  $W_e$ ,  $E_u$  and  $M$ ) are the same for the model and prototype. Hence models are designed on the basis of ratio of the force, which is dominating in the phenomenon. The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The followings are the model laws :

- 1. Reynold's model law,
- 2. Froude model law,
- 3. Euler model law,
- 4. Weber model law,
- 5. Mach model law.

**12.9.1 Reynold's Model Law.** Reynold's model law is the law in which models are based on Reynold's number. Models based on Reynold's number includes :

- (i) Pipe flow
- (ii) Resistance experienced by sub-marines, airplanes, fully immersed bodies etc.

As defined earlier that Reynold number is the ratio of inertia force and viscous force, and hence fluid flow problems where viscous forces alone are predominant, the models are designed for dynamic similarity on Reynolds law, which states that the Reynold number for the model must be equal to the Reynold number for the prototype.

Let  $V_m$  = Velocity of fluid in model,  
 $\rho_m$  = Density of fluid in model,  
 $L_m$  = Length or linear dimension of the model,  
 $\mu_m$  = Viscosity or fluid in model,

and  $V_p$ ,  $\rho_p$ ,  $L_p$  and  $\mu_p$  are the corresponding values of velocity, density, linear dimension and viscosity of fluid in prototype. Then according to Reynold's model law,

$$[R_e]_m = [R_e]_P \text{ or } \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \quad \dots(12.17)$$

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or  $\frac{\rho_P \cdot V_P \cdot L_P}{\rho_m \cdot V_m \cdot L_m} \times \frac{1}{\frac{\mu_P}{\mu_m}} = 1 \quad \text{or} \quad \frac{\rho_r \cdot V_r \cdot L_r}{\mu_r} = 1$

$$\left\{ \text{where } \rho_r = \frac{\rho_P}{\rho_m}, V_r = \frac{V_P}{V_m} \text{ and } L_r = \frac{L_P}{L_m}, \frac{\mu_P}{\mu_m} = \mu_r \right\}$$

And also  $\rho_r$ ,  $V_r$ ,  $L_r$  and  $\mu_r$  are called the scale ratios for density, velocity, linear dimension and viscosity.

The scale ratios for time, acceleration, force and discharge for Reynold's model law are obtained as

$$t_r = \text{Time scale ratio} = \frac{L_r}{V_r} \quad \left\{ \because V = \frac{L}{t} \therefore t = \frac{L}{V} \right\}$$

$$a_r = \text{Acceleration scale ratio} = \frac{V_r}{t_r}$$

$$F_r = \text{Force scale ratio} = (\text{Mass} \times \text{Acceleration})_r \\ = m_r \times a_r = \rho_r A_r V_r \times a_r \\ = \rho_r L_r^2 V_r \times a_r \quad \{A_r = \text{Area ratio}\}$$

$$Q_r = \text{Discharge scale ratio} = (\rho A V)_r \\ = \rho_r A_r V_r = \rho_r \cdot L_r^2 \cdot V_r.$$

**Problem 12.15** A pipe of diameter 1.5 m is required to transport an oil of sp. gr. 0.90 and viscosity  $3 \times 10^{-2}$  poise at the rate of 3000 litre/s. Tests were conducted on a 15 cm diameter pipe using water at 20°C. Find the velocity and rate of flow in the model. Viscosity of water at 20°C = 0.01 poise.

**Solution.** Given :

Dia. of prototype,

$$D_P = 1.5 \text{ m}$$

Viscosity of fluid,

$$\mu_P = 3 \times 10^{-2} \text{ poise}$$

$Q$  for prototype,

$$Q_P = 3000 \text{ lit/s} = 3.0 \text{ m}^3/\text{s}$$

Sp. gr. of oil,

$$S_P = 0.9$$

∴ Density of oil,

$$\rho_P = S_P \times 1000 = 0.9 \times 1000 = 900 \text{ kg/m}^3$$

Dia. of the model,

$$D_m = 15 \text{ cm} = 0.15 \text{ m}$$

Viscosity of water at 20°C

$$= 0.01 \text{ poise} = 1 \times 10^{-2} \text{ poise or } \mu_m = 1 \times 10^{-2} \text{ poise}$$

Density of water or

$$\rho_m = 1000 \text{ kg/m}^3.$$

For pipe flow, the dynamic similarity will be obtained if the Reynold's number in the model and prototype are equal

Hence using equation (12.17),  $\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_P V_P D_P}{\mu_P}$  {For pipe, linear dimension is  $D$ }

$$\therefore \frac{V_m}{V_P} = \frac{\rho_P}{\rho_m} \cdot \frac{D_P}{D_m} \cdot \frac{\mu_m}{\mu_P}$$

$$= \frac{900}{1000} \times \frac{1.5}{0.15} \times \frac{1 \times 10^{-2}}{3 \times 10^{-2}} = \frac{900}{1000} \times 10 \times \frac{1}{3} = 3.0$$

But

$$V_p = \frac{\text{Rate of flow in prototype}}{\text{Area of prototype}} = \frac{3.0}{\frac{\pi}{4}(D_p)^2} = \frac{3.0}{\frac{\pi}{4}(1.5)^2}$$

$$= \frac{3.0 \times 4}{\pi \times 2.25} = 1.697 \text{ m/s}$$

∴

$$V_m = 3.0 \times V_p = 3.0 \times 1.697 = 5.091 \text{ m/s. Ans.}$$

$$\text{Rate of flow through model, } Q_m = A_m \times V_m = \frac{\pi}{4} (D_m)^2 \times V_m = \frac{\pi}{4} (0.15)^2 \times 5.091 \text{ m}^3/\text{s}$$

$$= 0.0899 \text{ m}^3/\text{s} = 0.0899 \times 1000 \text{ lit/s} = 89.9 \text{ lit/s. Ans.}$$

**Problem 12.16** Water is flowing through a pipe of diameter 30 cm at a velocity of 4 m/s. Find the velocity of oil flowing in another pipe of diameter 10 cm, if the condition of dynamic similarity is satisfied between the two pipes. The viscosity of water and oil is given as 0.01 poise and .025 poise. The sp. gr. of oil = 0.8.

**Solution.** Given :

Two pipes having different liquids.

Let for pipe 1, Liquid = Water

Dia. of pipe,  $d_1 = 30 \text{ cm} = 0.30 \text{ m}$

Velocity of flow,  $V_1 = 4 \text{ m/s}$

Viscosity,  $\mu_1 = 0.01 \text{ poise} = \frac{0.01}{10} \text{ (S.I. Units)}$

Density,  $\rho_1 = 1000 \text{ kg/m}^3$

For pipe 2, Liquid = Oil

Dia. of pipe,  $d_2 = 10 \text{ cm} = 0.1 \text{ m}$

Velocity of flow,  $V_2 = ?$

Viscosity,  $\mu_2 = 0.025 \text{ poise} = \frac{0.025}{10} \text{ (S.I. Units)}$

Sp. gr. of oil  $= 0.8$

∴ Density,  $\rho_2 = 0.8 \times 1000 = 800 \text{ kg/m}^3$

If the pipes are dynamically similar, the Reynold's number for both the pipes should be same.

$$\therefore \frac{\rho_1 V_1 d_1}{\mu_1} = \frac{\rho_2 V_2 d_2}{\mu_2} \quad \text{or} \quad V_2 = \frac{\rho_1}{\rho_2} \cdot \frac{d_1}{d_2} \cdot \frac{\mu_2}{\mu_1} V_1$$

$$= \frac{1000}{800} \times \frac{0.30}{0.10} \times \frac{\frac{0.025}{0.01}}{\frac{10}{10}} \times 4.0 = \frac{1000}{800} \times 3 \times \frac{0.025}{0.01} \times 4.0$$

$$= 37.5 \text{ m/s. Ans.}$$

**Problem 12.17** The ratio of lengths of a sub-marine and its model is 30 : 1. The speed of sub-marine (prototype) is 10 m/s. The model is to be tested in a wind tunnel. Find the speed of air in wind tunnel. Also determine the ratio of the drag (resistance) between the model and its prototype. Take the value of kinematic viscosities for sea water and air as .012 stokes and .016 stokes respectively. The density for sea-water and air is given as  $1030 \text{ kg/m}^3$  and  $1.24 \text{ kg/m}^3$  respectively.

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**Solution.** Given :

Prototype (sub-marine) and its model.

For prototype, Speed  $V_p = 10 \text{ m/s}$

Fluid = Sea - water

$$\begin{aligned} \text{Kinematic viscosity, } v_p &= 0.012 \text{ stokes} = .012 \text{ cm}^2/\text{s} \\ &= .012 \times 10^{-4} \text{ m}^2/\text{s} \end{aligned} \quad \{ \because \text{ Stoke} = \text{cm}^2/\text{s} \}$$

$$\text{Density, } \rho_p = 1030 \text{ kg/m}^3$$

For model Fluid = Air

$$\text{Kinematic viscosity, } v_m = 0.016 \text{ stokes} = 0.016 \text{ cm}^2/\text{s} = .016 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Density, } \rho_m = 1.24 \text{ kg/m}^3$$

$$\text{Also } \frac{\text{Length of prototype}}{\text{Length of model}} = \frac{L_p}{L_m} = 30.0$$

Let the velocity of air in model =  $V_m$ .

For dynamic similarity between model and sub-marine, the viscous resistance is to be overcome and hence for fully submerged sub-marine, the Reynold's number for model and prototype should be same.

$$\therefore \frac{\rho_p V_p D_p}{\mu_p} = \frac{\rho_m V_m D_m}{\mu_m} \quad \text{or} \quad \frac{V_p D_p}{(\mu / \rho)_p} = \frac{V_m D_m}{(\mu / \rho)_m}; \frac{V_p D_p}{v_p} = \frac{V_m D_m}{v_m}$$

$$\begin{aligned} \therefore V_m &= \frac{v_m}{v_p} \times \frac{D_p}{D_m} \times V_p \\ &= \frac{0.016 \times 10^{-4}}{.012 \times 10^{-4}} \times 30 \times 10 \text{ m/s} \quad \left\{ \because \frac{D_p}{D_m} = \frac{L_p}{L_m} = 30 \right\} \\ &= \frac{0.016}{.012} \times 30 \times 10 = 400 \text{ m/s. Ans.} \end{aligned}$$

Ratio of drag force (resistance) :

Drag force = Mass × Acceleration

$$= \rho L^3 \times \frac{V}{t} = \rho \cdot L^2 \cdot \frac{L}{t} \times V = \rho L^2 V^2 \quad \left\{ \because \frac{L}{t} = V \right\}$$

Let  $F_p$  and  $F_m$  denote the drag force for the prototype and for the model respectively then,

$$\begin{aligned} \frac{F_p}{F_m} &= \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_p}{\rho_m} \times \left( \frac{L_p}{L_m} \right)^2 \times \left( \frac{V_p}{V_m} \right)^2 \\ &= \frac{1030}{1.24} \times 30^2 \times \left( \frac{10}{400} \right)^2 = 467.22. \text{ Ans.} \end{aligned}$$

**Problem 12.18** A ship 300 m long moves in sea-water, whose density is  $1030 \text{ kg/m}^3$ , A 1 : 100 model of this ship is to be tested in a wind tunnel. The velocity of air in the wind tunnel around the model is 30 m/s and the resistance of the model is 60 N. Determine the velocity of ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as  $1.24 \text{ kg/m}^3$ . Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively.

**Solution.** Given :

For Prototype,

Length,

$$L_P = 300 \text{ m}$$

Density of water

Fluid = Sea-water

$$= 1030 \text{ kg/m}^3$$

Kinematic viscosity,

$$\nu_P = 0.012 \text{ stokes} = 0.012 \times 10^{-4} \text{ m}^2/\text{s}$$

Let velocity of ship

$$= V_P$$

Resistance

$$= F_P$$

**For model**

Length,

$$L_m = \frac{1}{100} \times 300 = 3 \text{ m}$$

Velocity,

$$V_m = 30 \text{ m/s}$$

Resistance,

$$F_m = 60 \text{ N}$$

Density of air,

$$\rho_m = 1.24 \text{ kg/m}^3$$

Kinematic viscosity of air,  $\nu_m = 0.018 \text{ stokes} = .018 \times 10^{-4} \text{ m}^2/\text{s}$ .

For dynamic similarity between the prototype and its model, the Reynolds number for both of them should be equal.

$$\therefore \frac{V_p \times L_p}{\nu_p} = \frac{V_m \times L_m}{\nu_m} \quad \text{or} \quad V_p = \frac{\nu_p}{\nu_m} \times \frac{L_m}{L_p} \times V_m$$

$$= \frac{.012 \times 10^{-4}}{.018 \times 10^{-4}} \times \frac{3}{300} \times 30 = \frac{1}{1.5} \times \frac{1}{100} \times 30 = \mathbf{0.2 \text{ m/s. Ans.}}$$

Resistance

= Mass  $\times$  Acceleration

$$= \rho L^3 \times \frac{V}{t} = \rho L^2 \times \frac{V}{1} \times \frac{L}{t} = \rho L^2 V^2$$

Then

$$\frac{F_p}{F_m} = \frac{(\rho L^2 V^2)_p}{(\rho L^2 V^2)_m} = \frac{\rho_p}{\rho_m} \times \left( \frac{L_p}{L_m} \right)^2 \times \left( \frac{V_p}{V_m} \right)^2$$

But

$$\frac{\rho_p}{\rho_m} = \frac{1030}{1.24}$$

$$\therefore \frac{F_p}{F_m} = \frac{1030}{1.24} \times \left( \frac{300}{3} \right)^2 \times \left( \frac{0.2}{30} \right)^2 = 369.17$$

$$\therefore F_p = 369.17 \times F_m = 369.17 \times 60 = \mathbf{22150.2 \text{ N. Ans.}}$$

**12.9.2 Froude Model Law.** Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal. Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems :

1. Free surface flows such as flow over spillways, weirs, sluices, channels etc.,
2. Flow of jet from an orifice or nozzle,
3. Where waves are likely to be formed on surface,
4. Where fluids of different densities flow over one another.

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Let

$V_m$  = Velocity of fluid in model,

$L_m$  = Linear dimension or length of model,

$g_m$  = Acceleration due to gravity at a place where model is tested.

and  $V_p$ ,  $L_p$  and  $g_p$  are the corresponding values of the velocity, length and acceleration due to gravity for the prototype. Then according to Froude model law,

$$(F_e)_{model} = (F_e)_{prototype} \text{ or } \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}} \quad \dots(12.18)$$

If the tests on the model are performed on the same place where prototype is to operate, then  $g_m = g_p$  and equation (12.18) becomes as

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}} \quad \dots(12.19)$$

or

$$\frac{V_m}{V_p} \times \frac{1}{\sqrt{\frac{L_m}{L_p}}} = 1$$

$$\frac{V_p}{V_m} = \sqrt{\frac{L_p}{L_m}} = \sqrt{L_r} \quad \left\{ \because \frac{L_p}{L_m} = L_r \right\}$$

where  $L_r$  = Scale ratio for length

$$\frac{V_p}{V_m} = V_r = \text{Scale ratio for velocity.}$$

$$\therefore \frac{V_p}{V_m} = V_r = \sqrt{L_r}. \quad \dots(12.20)$$

**Scale ratios** for various physical quantities based on Froude model law are :

(a) **Scale ratio for time**

$$\text{As} \quad \text{time} = \frac{\text{Length}}{\text{Velocity}},$$

then ratio of time for prototype and model is

$$T_r = \frac{T_p}{T_m} = \frac{\left(\frac{L}{V}\right)_p}{\left(\frac{L}{V}\right)_m} = \frac{\frac{L_p}{V_p}}{\frac{L_m}{V_m}} = \frac{L_p}{L_m} \times \frac{V_m}{V_p} = L_r \times \frac{1}{\sqrt{L_r}} \quad \left\{ \because \frac{V_p}{V_m} = \sqrt{L_r} \right\}$$

$$= \sqrt{L_r}. \quad \dots(12.21)$$

(b) **Scale ratio for acceleration**

$$\text{Acceleration} = \frac{V}{T}$$

$$\therefore a_r = \frac{a_p}{a_m} = \frac{\left(\frac{V}{T}\right)_p}{\left(\frac{V}{T}\right)_m} = \frac{V_p}{T_p} \times \frac{T_m}{V_m} = \frac{V_p}{V_m} \times \frac{T_m}{T_p}$$

$$= \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} = 1. \quad \left\{ \because \frac{V_p}{V_m} = \sqrt{L_r}, \frac{T_p}{T_m} = \sqrt{L_r} \right\} \quad \dots(12.22)$$

(c) Scale ratio for discharge

$$Q = A \times V = L^2 \times \frac{L}{T} = \frac{L^3}{T}$$

$$\therefore Q_r = \frac{Q_p}{Q_m} = \frac{\left(\frac{L^3}{T}\right)_p}{\left(\frac{L^3}{T}\right)_m} = \left(\frac{L_p}{L_m}\right)^3 \times \left(\frac{T_m}{T_p}\right) = L_r^3 \times \frac{1}{\sqrt{L_r}} = L_r^{2.5} \quad \dots(12.23)$$

(d) Scale ratio for force

$$\text{As Force} = \text{Mass} \times \text{Acceleration} = \rho L^3 \times \frac{V}{T} = \rho L^2 \cdot \frac{L}{T} \cdot V = \rho L^2 V^2$$

$$\therefore \text{Ratio for force, } F_r = \frac{F_p}{F_m} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2.$$

If the fluid used in model and prototype is same, then

$$\frac{\rho_p}{\rho_m} = 1 \quad \text{or} \quad \rho_p = \rho_m$$

$$\text{and hence } F_r = \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{V_p}{V_m}\right)^2 = L_r^2 \times (\sqrt{L_r})^2 = L_r^2 \cdot L_r = L_r^3. \quad \dots(12.24)$$

(e) Scale ratio for pressure intensity

$$\text{As } p = \frac{\text{Force}}{\text{Area}} = \frac{\rho L^2 V^2}{L^2} = \rho V^2$$

$$\therefore \text{Pressure ratio, } p_r = \frac{p_p}{p_m} = \frac{\rho_p V_p^2}{\rho_m V_m^2}$$

If fluid is same, then  $\rho_p = \rho_m$

$$\therefore p_r = \frac{V_p^2}{V_m^2} = \left(\frac{V_p}{V_m}\right)^2 = L_r^2. \quad \dots(12.25)$$

(f) Scale ratio for work, energy, torque, moment etc.

$$\text{Torque} = \text{Force} \times \text{Distance} = F \times L$$

$$\therefore \text{Torque ratio, } T_r^* = \frac{T_p^*}{T_m^*} = \frac{(F \times L)_p}{(F \times L)_m} = F_r \times L_r = L_r^3 \times L_r = L_r^4. \quad \dots(12.26)$$

(g) Scale ratio for power

$$\text{As Power} = \text{Work per unit time}$$

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$$\begin{aligned}
 &= \frac{F \times L}{T} \\
 \therefore \text{Power ratio, } P_r &= \frac{P_p}{P_m} = \frac{\frac{F_p \times L_p}{T_p}}{\frac{F_m \times L_m}{T_m}} = \frac{F_p}{F_m} \times \frac{L_p}{L_m} \times \frac{1}{\frac{T_p}{T_m}} \\
 &= F_r \cdot L_r \cdot \frac{1}{T_r} = L_r^3 \cdot L_r \cdot \frac{1}{\sqrt{L_r}} = L^{3.5}. \quad \dots(12.27)
 \end{aligned}$$

**Problem 12.19** In 1 in 40 model of a spillway, the velocity and discharge are 2 m/s and 2.5 m<sup>3</sup>/s. Find the corresponding velocity and discharge in the prototype.

**Solution.** Given :

$$\begin{aligned}
 \text{Scale ratio of length, } L_r &= 40 \\
 \text{Velocity in model, } V_m &= 2 \text{ m/s} \\
 \text{Discharge in model, } Q_m &= 2.4 \text{ m}^3/\text{s}
 \end{aligned}$$

Let  $V_p$  and  $Q_p$  are the velocity and discharge in prototype.

$$\text{Using equation (12.20) for velocity ratio, } \frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{40}$$

$$\therefore V_p = V_m \times \sqrt{40} = 2 \times \sqrt{40} = 12.65 \text{ m/s. Ans.}$$

Using equation (12.23) for discharge ratio,

$$\begin{aligned}
 \frac{Q_p}{Q_m} &= L_r^{2.5} = (40)^{2.5} \\
 \therefore Q_p &= Q_m \times 40^{2.5} = 2.5 \times 40^{2.5} = 25298.2 \text{ m}^3/\text{s. Ans.}
 \end{aligned}$$

**Problem 12.20** A ship model of scale  $\frac{1}{50}$  is towed through sea water at a speed of 1 m/s. A force of 2 N is required to tow the model. Determine the speed of ship and the propulsive force on the ship, if prototype is subjected to wave resistance only.

**Solution.** Given :

$$\begin{aligned}
 \text{Scale ratio of length, } L_r &= 50 \\
 \text{Speed of model, } V_m &= 1 \text{ m/s} \\
 \text{Force required for model, } F_m &= 2 \text{ N}
 \end{aligned}$$

Let the speed of ship  $= V_p$

and the propulsive force for ship  $= F_p$ .

As prototype is subjected to wave resistance only for dynamic similarity, the Froude number should be same for model and prototype. Hence for velocity ratio, for Froude model law using equation (12.20), we have

$$\therefore \frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{50}$$

$$\therefore V_p = \sqrt{50} \times V_m = \sqrt{50} \times 1 = 7.071 \text{ m/s. Ans.}$$

Force scale ratio is given by equation (12.24),

$$\therefore F_r = \frac{F_p}{F_m} = L_r^3$$

$$\therefore F_p = F_m \times L_r^3 = 2 \times (50)^3 = 250000 \text{ N. Ans.}$$

**Problem 12.21** In the model test of a spillway the discharge and velocity of flow over the model were  $2 \text{ m}^3/\text{s}$  and  $1.5 \text{ m/s}$  respectively. Calculate the velocity and discharge over the prototype which is 36 times the model size.

**Solution.** Given :

$$\text{Discharge over model, } Q_m = 2 \text{ m}^3/\text{s}$$

$$\text{Velocity over model, } V_m = 1.5 \text{ m/s}$$

$$\text{Linear scale ratio, } L_r = 36.$$

For dynamic similarity, Froude model law is used. Using equation (12.20), we have

$$\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{36} = 6.0$$

$$\therefore V_p = \text{Velocity over prototype} = V_m \times 6.0 = 1.5 \times 6.0 = 9 \text{ m/s. Ans.}$$

For discharge, using equation (12.23), we get

$$\frac{Q_p}{Q_m} = L_r^{2.5} = (36)^{2.5}.$$

$$\therefore Q_p = Q_m \times (36)^{2.5} = 2 \times 36^{2.5} = 15552 \text{ m}^3/\text{s. Ans.}$$

**Problem 12.22** In a geometrically similar model of spillway the discharge per metre length is

$\frac{1}{6} \text{ m}^3/\text{s}$ . If the scale of the model is  $\frac{1}{36}$ , find the discharge per metre length of the prototype.

**Solution.** Given :

$$\text{Discharge per metre length for model, } q_m = \frac{1}{6} \text{ m}^3/\text{s}$$

$$\text{Linear scale ratio, } L_r = 36$$

$$\text{Discharge per metre length for prototype, } q_p = ?$$

The discharge ratio for spillway is given by equation (12.23),  $\frac{Q_p}{Q_m} = L_r^{2.5}$ .

But discharge ratio per metre length is given as

$$\frac{q_p}{q_m} = \frac{Q_p / L_p}{Q_m / L_m} = \frac{Q_p}{Q_m} \times \frac{L_m}{L_p} = L_r^{2.5} \times \frac{1}{L_r} = L_r^{1.5}$$

$$\therefore q_p = q_m \times L_r^{1.5} = \frac{1}{6} \times (36)^{1.5} = \frac{1}{6} \times 6^{2 \times 1.5} \\ = 6^{3-1} = 6^2 = 36 \text{ m}^3/\text{s per metre length. Ans.}$$

**Problem 12.23** A spillway model is to be built to a geometrically similar scale of  $\frac{1}{50}$  across a

flume of  $600 \text{ mm}$  width. The prototype is  $15 \text{ m}$  high and maximum head on it is expected to be  $1.5 \text{ m}$ .

(i) What height of model and what head on the model should be used? (ii) If the flow over the model at a particular head is  $12 \text{ litres per second}$ , what flow per metre length of the prototype is expected?

(iii) If the negative pressure in the model is  $200 \text{ mm}$ , what is the negative pressure in prototype? Is it practicable?

**Solution.** Given :

$$\text{Scale ratio for length, } L_r = 50$$

$$\text{Width of model, } B_m = 600 \text{ mm} = 0.6 \text{ m}$$

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Flow over model,  $Q_m = 12 \text{ litres/s}$   
 Pressure in model,  $h_m = -200 \text{ mm of water} = -0.2 \text{ m}$

Height of prototype,  $H_p = 15 \text{ m}$   
 Head on prototype,  $H_p^* = 1.5 \text{ m}$

(i) Let the height of model  
 and head on model  $= H_m$   
 $= H_m^*$

Linear scale ratio,  $L_r = \frac{H_p}{H_m} = \frac{H_p^*}{H_m^*} = 50$

$\therefore$  Height of model,  $H_m = \frac{H_p}{50} = \frac{15}{50} = 0.3 \text{ m. Ans.}$

And head on model,  $H_m^* = \frac{H_p^*}{50} = \frac{1.50}{50} = 0.03 \text{ m. Ans.}$

Width of prototype,  $B_p = L_r \times B_m = 50 \times 0.6 = 30 \text{ m.}$

(ii) Discharge ratio is given by equation (12.23) as

$$\frac{Q_p}{Q_m} = L_r^{2.5} = (50)^{2.5} = 17677.67$$

$$\therefore Q_p = Q_m \times 17677.67 = 12 \times 17677.67 = 212132.04 \text{ lit/s}$$

$$\begin{aligned} \text{Discharge per metre length of prototype} &= \frac{Q_p}{\text{Length of prototype}} = \frac{212132.04}{\text{Width of prototype}} \\ &= \frac{212132.04}{30} = 7071.078 \text{ litres/s. Ans.} \end{aligned}$$

(iii) Negative pressure head in prototype,

$$h_p = L_r \times h_m = 50 \times (-0.2) = -10.0 \text{ m. Ans.}$$

This negative pressure is not practicable. Maximum practicable negative pressure head is  $-7.50 \text{ m}$  of water.

**Problem 12.24** In a 1 in 20 model of stilling basin, the height of the hydraulic jump in the model is observed to be 0.20 metre. What is the height of the hydraulic jump in the prototype? If the energy

dissipated in the model is  $\frac{1}{10} \text{ kW}$ , what is the corresponding value in prototype?

**Solution.** Given :

Linear scale ratio,  $L_r = 20$

Height of hydraulic jump in model,  $h_m = 0.20 \text{ m}$

Energy dissipated in model,  $P_m = \frac{1}{10} \text{ kW}$

(i) Let the height of hydraulic jump in the prototype  $= h_p$

Then  $\frac{h_p}{h_m} = L_r = 20$

$$\therefore h_p = h_m \times 20 = 0.20 \times 20 = 4 \text{ m. Ans.}$$

(ii) Let the energy dissipated in prototype  $= F_p$

Using equation (12.27) for power ratio,  $\frac{P_p}{P_m} = L_r^{3.5} = 20^{3.5} = 35777.088$

$$\therefore P_p = P_m \times 35777.088 = \frac{1}{10} \times 35777.088 = 3577.708 \text{ kW. Ans.}$$

**Problem 12.25** The characteristics of the spillway are to be studied by means of a geometrically similar model constructed to the scale ratio of 1 : 10.

- (i) If the maximum rate of flow in the prototype is 28.3 cumecs, what will be the corresponding flow in model ?
- (ii) If the measured velocity in the model at a point on the spillway is 2.4 m/s, what will be the corresponding velocity in prototype ?
- (iii) If the hydraulic jump at the foot of the model is 50 mm high, what will be the height of jump in prototype ?
- (iv) If the energy dissipated per second in the model is 3.5 Nm, what energy is dissipated per second in the prototype ?

**Solution.** Given :

$$\frac{\text{Linear dimension of model}}{\text{Linear dimension of prototype}} = \frac{1}{10}$$

$$\therefore \text{Scale ratio, } L_r = 10.$$

$$(i) \text{ Discharge in prototype, } Q_p = 28.3 \text{ m}^3/\text{s}$$

$$\text{Let } Q_m = \text{Discharge in model}$$

For discharge using equation (12.23), we get

$$\frac{Q_p}{Q_m} = L_r^{2.5}$$

$$\therefore Q_m = \frac{Q_p}{L_r^{2.5}} = \frac{28.3}{10^{2.5}} = 0.0895 \text{ m}^3/\text{s. Ans.}$$

$$(ii) \text{ Velocity in the model, } V_m = 2.4 \text{ m/s}$$

$$\text{Let } V_p = \text{Velocity in the prototype}$$

For velocity using equation (12.20), we get

$$\frac{V_p}{V_m} = \sqrt{L_r}$$

$$\therefore V_p = V_m \times \sqrt{L_r} = 2.4 \times \sqrt{10} = 7.589 \text{ m/s. Ans.}$$

$$(iii) \text{ Hydraulic jump in model, } H_m = 50 \text{ mm}$$

$$\text{Let } H_p = \text{Hydraulic jump in prototype}$$

$$\text{Now scale ratio } = \frac{H_p}{H_m}$$

$$\therefore H_p = H_m \times \text{Scale ratio} = 50 \times 10 = 500 \text{ mm. Ans.}$$

$$(iv) \text{ Energy dissipated/s in model, } E_m = 3.5 \text{ N m/s}$$

$$\text{Let } E_p = \text{Energy dissipated/s in prototype}$$

$$\text{Now using equation (12.27), we get } \frac{E_p}{E_m} = L_r^{3.5}$$

$$\therefore E_p = E_m \times L_r^{3.5} = 3.5 \times 10^{3.5} = 11067.9 \text{ N m/s. Ans.}$$

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**Problem 12.26** A 1 : 64 model is constructed of an open channel in concrete which has Manning's  $N = 0.014$ . Find the value of  $N$  for the model.

**Solution.** Given :

$$\text{Linear scale ratio, } L_r = 64$$

$$\text{Value of } N \text{ for prototype, } N_p = 0.014$$

$$\text{Let } N_m = \text{Value of } N \text{ for model.}$$

$$\text{The Manning's formula* is given by, } V = \frac{1}{N} m^{3/2} \cdot i^{1/2}$$

in which  $m$  = Hydraulic mean depth in  $m$

$i$  = Slope of the bed of the channel

Now for the model, the Manning's formula becomes as

$$V_m = \frac{1}{N_m} \cdot (m_m)^{2/3} \cdot (i_m)^{1/2} \quad \dots(i)$$

and for the prototype, the Manning's formula is written as

$$V_p = \frac{1}{N_p} \cdot (m_p)^{2/3} \cdot (i_p)^{1/2} \quad \dots(ii)$$

Dividing equation (ii) by equation (i), we get

$$\frac{V_p}{V_m} = \frac{\frac{1}{N_p} \cdot (m_p)^{2/3} \cdot (i_p)^{1/2}}{\frac{1}{N_m} \cdot (m_m)^{2/3} \cdot (i_m)^{1/2}} = \frac{N_m}{N_p} \cdot \left(\frac{m_p}{m_m}\right)^{2/3} \cdot \left(\frac{i_p}{i_m}\right)^{1/2} \quad \dots(iii)$$

For dynamic similarity, Froude model law is used. Using equation (12.20), we have

$$\frac{V_p}{V_m} = \sqrt{L_r} = \sqrt{64} = 8$$

But  $\frac{m_p}{m_m} = L_r$  and  $\frac{i_p}{i_m} = 1$  as  $i_p$  and  $i_m$  are dimensionless.

Substituting these values in equation (iii), we get

$$8 = \frac{N_m}{N_p} \times (L_r)^{2/3} \times 1 = \frac{N_m}{0.014} \times (64)^{2/3} \quad (\because N_p = 0.014)$$

$$\therefore N_m = \frac{8 \times 0.014}{64^{2/3}} = \frac{8 \times 0.014}{16} = \mathbf{0.007. \text{ Ans.}}$$

**Problem 12.27** A 7.2 m height and 15 m long spillway discharges 94  $\text{m}^3/\text{s}$  discharge under a head of 2.0 m. If a 1 : 9 scale model of this spillway is to be constructed, determine model dimensions, head over spillway model and the model discharge. If model experiences a force of 7500 N (764.53 kgf), determine force on the prototype.

**Solution.** Given :

$$\text{For prototype : Height } h_p = 7.2 \text{ m}$$

$$\text{Length, } L_p = 15 \text{ m}$$

$$\text{Discharge, } Q_p = 94 \text{ m}^3/\text{s}$$

\*See chapter 16 where Manning's formula for velocity through an open channel flow is given.

Head,  $H_P = 2.0 \text{ m}$

Size of model =  $\frac{1}{9}$  of the size of prototype.

$\therefore$  Linear scale ratio,  $L_r = 9$

Force experienced by model,  $F_P = 7500 \text{ N}$

Find : (i) Model dimensions i.e., height and length of model ( $h_m$  and  $L_m$ )

(ii) Head over model i.e.,  $H_m$

(iii) Discharge through model i.e.,  $Q_m$

(iv) Force on prototype (i.e.,  $F_p$ )

(i) *Model dimensions ( $h_m$  and  $L_m$ )*

$$\frac{h_P}{h_m} = \frac{L_P}{L_m} = L_r = 9$$

$$\therefore h_m = \frac{h_P}{9} = \frac{7.2}{9} = 0.8 \text{ m. Ans.}$$

$$\text{And } L_m = \frac{L_P}{9} = \frac{15}{9} = 1.67 \text{ m. Ans.}$$

(ii) *Head over model ( $H_m$ )*

$$\frac{H_P}{H_m} = L_r = 9$$

$$\therefore H_m = \frac{H_P}{9} = \frac{2}{9} = 0.222 \text{ m. Ans.}$$

(iii) *Discharge through model ( $Q_m$ )*

Using equation (12.23), we get  $\frac{Q_P}{Q_m} = L_r^{2.5}$

$$\therefore Q_m = \frac{Q_P}{L_r^{2.5}} = \frac{94}{9^{2.5}} = \frac{94}{243} = 0.387 \text{ m}^3/\text{s. Ans.}$$

(iv) *Force on the Prototype ( $F_p$ )*

Using equation (12.24), we get  $F_r = \frac{F_P}{F_m} = L_r^3$

$$\therefore F_p = F_m \times L_r^3 = 7500 \times 9^3 = 5467500 \text{ N. Ans.}$$

**12.9.3 Euler's Model Law.** Euler's model law is the law in which the models are designed on Euler's number which means for dynamic similarity between the model and prototype, the Euler number for model and prototype should be equal. Euler's model law is applicable when the pressure forces are alone predominant in addition to the inertia force. According to this law :

$$(E_u)_{\text{model}} = (E_u)_{\text{prototype}} \quad \dots(12.28)$$

If

$V_m$  = Velocity of fluid in model,

$p_m$  = Pressure of fluid in model,

$\rho_m$  = Density of fluid in model,

and

$V_p, p_p, \rho_p$  = Corresponding values in prototype, then

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Substituting these values in equation (12.28), we get

$$\frac{V_m}{\sqrt{p_m/\rho_m}} = \frac{V_p}{\sqrt{p_p/\rho_p}} \quad \dots(12.29)$$

If fluid is same in model and prototype, then equation (12.29) becomes as

$$\frac{V_m}{\sqrt{p_m}} = \frac{V_p}{\sqrt{p_p}} \quad \dots(12.30)$$

Euler's model law is applied for fluid flow problems where flow is taking place in a closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension force is absent. This law is also used where the phenomenon of cavitation takes place.

**12.9.4 Weber Model Law.** Weber model law is the law in which models are based on Weber's number, which is the ratio of the square root of inertia force to surface tension force. Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between the model and prototype is obtained by equating the Weber number of the model and its prototype. Hence according to this law :

$$(W_e)_{\text{model}} = (W_e)_{\text{prototype}}, \quad \text{where } W_e \text{ is Weber number and } = \frac{V}{\sqrt{\sigma / \rho L}}$$

If

$V_m$  = Velocity of fluid in model,  
 $\sigma_m$  = Surface tensile force in model,  
 $\rho_m$  = Density of fluid in model,  
 $L_m$  = Length of surface in model,

and  $V_p, \sigma_p, \rho_p, L_p$  = Corresponding values of fluid in prototype.

Then according to Weber law, we have

$$\frac{V_m}{\sqrt{\sigma_m / \rho_m L_m}} = \frac{V}{\sqrt{\sigma_p / \rho_p L_p}} \quad \dots(12.31)$$

Weber model law is applied in following cases :

1. Capillary rise in narrow passages,
2. Capillary movement of water in soil,
3. Capillary waves in channels,
4. Flow over weirs for small heads.

**12.9.5 Mach Model Law.** Mach model law is the law in which models are designed on Mach number, which is the ratio of the square root of inertia force to elastic force of a fluid. Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the Mach number of the model and its prototype. Hence according to this law :

$$(M)_{\text{model}} = (M)_{\text{prototype}}$$

$$\text{where } M = \text{Mach number} = \frac{V}{\sqrt{K / \rho}}$$

If

$V_m$  = Velocity of fluid in model,  
 $K_m$  = Elastic stress for model,  
 $\rho_m$  = Density of fluid in model,

and  $V_p, K_p$  and  $\rho_p$  = Corresponding values for prototype. Then according to Mach law,

$$= \frac{V_m}{\sqrt{K_m / \rho_m}} = \frac{V}{\sqrt{K_p / \rho_p}} \quad \dots(12.32)$$

Mach model law is applied in the following cases :

1. Flow of aeroplane and projectile through air at supersonic speed, i.e., at a velocity more than the velocity of sound,
2. Aerodynamic testing,
3. Under water testing of torpedoes,
4. Water-hammer problems.

**Problem 12.28** The pressure drop in an aeroplane model of size  $\frac{1}{10}$  of its prototype is  $80 \text{ N/cm}^2$ .

The model is tested in water. Find the corresponding pressure drop in the prototype. Take density of air =  $1.24 \text{ kg/m}^3$ . The viscosity of water is 0.01 poise while the viscosity of air is 0.00018 poise.

**Solution.** Given :

$$\text{Pressure drop in model, } p_m = 80 \text{ N/cm}^2 = 80 \times 10^4 \text{ N/m}^2$$

$$\text{Linear scale ratio, } L_r = 40$$

$$\text{Fluid in model} = \text{Water, while in prototype} = \text{Air}$$

$$\text{Viscosity of water, } \mu_m = 0.01 \text{ poise}$$

$$\text{Density of water, } \rho_m = 1000 \text{ kg/m}^3$$

$$\text{Viscosity of air, } \mu_P = .00018 \text{ poise}$$

$$\text{Density of air, } \rho_P = 1.24 \text{ kg/m}^3$$

Let the corresponding pressure drop in prototype =  $p_P$ .

As the problem involves pressure force and viscous force and hence for dynamic similarity between the model and prototype, Euler's number and Reynold's number should be considered. Making first of all, Reynold's number equal, we get from equation (12.17)

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_P V_P L_P}{\mu_P} \quad \text{or} \quad \frac{V_m}{V_P} = \frac{\rho_P}{\rho_m} \times \frac{L_P}{L_m} \times \frac{\mu_m}{\mu_P}$$

But

$$\frac{\rho_P}{\rho_m} = \frac{1.24}{1000}$$

$$\frac{L_P}{L_m} = L_r = 40, \quad \frac{\mu_m}{\mu_P} = \frac{0.01}{.00018}$$

$$\therefore \frac{V_m}{V_P} = \frac{1.24}{1000} \times 40 \times \frac{.01}{.00018} = 2.755.$$

Now making Euler's number equal, we get from equation (12.29) as

$$\frac{V_m}{\sqrt{\rho_m}} = \frac{V_P}{\sqrt{\rho_P}} \quad \text{or} \quad \frac{V_m}{V_P} = \frac{\sqrt{p_m/\rho_m}}{\sqrt{p_P/\rho_P}} = \sqrt{\frac{p_m}{p_P}} \times \sqrt{\frac{\rho_P}{\rho_m}}$$

But

$$\frac{V_m}{V_P} = 2.755 \text{ and } \frac{\rho_P}{\rho_m} = \frac{1.24}{1000}$$

$$\therefore 2.755 = \sqrt{\frac{p_m}{p_P}} \times \sqrt{\frac{1.24}{1000}} = \sqrt{\frac{p_m}{p_P}} \times .0352$$

$$\therefore \sqrt{\frac{p_m}{p_P}} = \frac{2.755}{.0352} = 78.267$$

$$\therefore \frac{p_m}{p_P} = (78.267)^2 \text{ or } p_P = \frac{p_m}{(78.267)^2} = \frac{80}{(78.267)^2}$$

$$= \mathbf{0.01306 \text{ N/cm}^2. \text{ Ans.}}$$

## ► 12.10 MODEL TESTING OF PARTIALLY SUB-MERGED BODIES

Let us consider the testing of a ship model (ship is a partially sub-merged body) in a water-tunnel in order to find the drag force  $F$  or resistance experienced by a ship. The drag experienced by a ship consists of :

1. The wave resistance, which is the resistance offered by the waves on the free sea-surface, and
2. The frictional or viscous resistance, which is offered by the water on the surface of contact of the ship with water.

Thus in this case three forces namely inertia, gravity and viscous forces are present. Then for dynamic similarity between the model and its prototype, the Reynold's number (which is ratio of inertia force to viscous force) and the Froude number (which is the ratio of inertia force to gravity force) should be taken into account. This means that in this case, the Reynold model law and Froude model law should be applied.

But for Reynold model law, the condition is

Reynold number of model = Reynold number of prototype

or

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

If fluid is same for the model and prototype, then  $\rho_m = \rho_p$  and  $\mu_m = \mu_p$

$\therefore$

$$V_m L_m = V_p L_p$$

$$V_m = \frac{V_p L_p}{L_m} = L_r V_p \quad \left\{ \because \frac{L_p}{L_m} = L_r \right\} \dots(12.33)$$

For Froude model law, have from equation (12.18) as  $\frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$

If fluid is same for model and prototype and test is conducted at the same place where prototype is to operate, then  $g_m = g_p$

$\therefore$

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}}$$

$\therefore$

$$V_m = \sqrt{\frac{L_m}{L_p}} \times V_p = V_p \times \frac{1}{\sqrt{\frac{L_p}{L_m}}} = V_p \times \frac{1}{\sqrt{L_r}} \quad \left\{ \because \frac{L_p}{L_m} = L_r \right\} \dots(12.34)$$

From equations (12.33) and (12.34), we observe that the velocity of fluid in model for Reynold model law and Froude model law is different. Thus it is quite impossible to satisfy both the laws together, which means the dynamic similarity between the model and its prototype will not exist. To overcome this difficulty, the method suggested by William Froude is adopted for testing the ship model (or partially sub-merged bodies) as :

**Step 1.** The total resistance experienced by a ship is equal to the wave resistance plus frictional or viscous resistance.

Let

$(R)_P$  = Total resistance experienced by prototype,

$(R_w)_P$  = Wave resistance experienced by prototype,

$(R_f)_P$  = Frictional resistance experienced by prototype, and

$(R)_m, (R_w)_m, (R_f)_m$  = Corresponding values for model.

Then, we have for prototype,  $(R)_P = (R_w)_P + (R_f)_P$  ... (12.35)

and for model,  $(R)_m = (R_w)_m + (R_f)_m$  ... (12.36)

**Step 2.** The frictional resistances for the model and the ship [i.e.,  $(R_f)_m$  and  $(R_f)_P$ ] are calculated from the expressions given below :

$$(R_f)_P = f_P A_P V_P^n \quad \dots(12.37)$$

and  $(R_f)_m = f_m A_m V_m^n$  ... (12.38)

where  $f_P$  = Frictional resistance per unit area per unit velocity of prototype,

$A_P$  = Wetted surface area of the prototype,

$V_P$  = Velocity of prototype,

$n$  = Constant, and

$f_m, A_m, V_m$  = Corresponding values of frictional resistance, wetted area and velocity of model.

The values of  $f_P$  and  $f_m$  are determined from experiments.

**Step 3.** The model is tested by towing it in water contained in a towing tank such that the dynamic similarity for Froude number is satisfied i.e.,  $(F_e)_m = (F_e)_P$ . The total resistance of the model  $(R_m)$  is measured for this condition.

**Step 4.** The total resistance  $(R_m)$  for the model is known from step 3 and frictional resistance of the model  $(R_f)_m$  is calculated from equation (12.37). Then the wave resistance for the model is known from equation (12.36) as

$$(R_w)_m = R_m - (R_f)_m \quad \dots(12.39)$$

**Step 5.** The resistance experienced by a ship of length  $L$ , flowing with velocity  $V$  in fluid of viscosity  $\mu$ , density  $\rho$  depends upon  $g$ , the acceleration due to gravity. By dimensional analysis, the expression for resistance is given by

$$\frac{R}{\rho L^2 V^2} = \phi \left[ \frac{\rho VL}{\mu}, \frac{V^2}{gL} \right] = \phi [R_e, F_e^2]$$

Thus resistance is a function of Reynold number  $(R_e)$  and Froude number  $(F_e)$ .

For dynamic similarity for model and prototype for wave resistance only, we have

$$\frac{(R_w)_P}{\rho_P L_P^2 V_P^2} = \frac{(R_w)_m}{\rho_m L_m^2 V_m^2}$$

or wave resistance for prototype is given as

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \frac{L_P^2}{L_m^2} \times \frac{V_P^2}{V_m^2} \times (R_w)_m \quad \dots(12.40)$$

But from Step 3,  $(F_e)_m = (F_e)_P$  or  $\frac{V_m}{\sqrt{L_m g_m}} = \frac{V_P}{\sqrt{L_P g_P}}$

If the model and ship are at the same place,  $g_m = g_P$

$$\therefore \frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}} \quad \text{or} \quad V_m = \sqrt{\frac{L_m}{L_P}} \cdot V_P$$

Substituting the value of  $V_m$  in equation (12.40), we have

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \frac{L_P^2}{L_m^2} \times \frac{V_P^2}{V_P^2 \times \frac{L_m}{L_P}} \times (R_w)_m$$

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$$= \frac{\rho_p}{\rho_m} \times \frac{L_p^3}{L_m^3} \times (R_w)_m. \quad \dots(12.41)$$

**Step 6.** The total resistance of the ship is given by adding  $(R_w)_p$  from equation (12.41) to  $(R_f)_p$  given by equation (12.37) as

$$R_p = \frac{\rho_p}{\rho_m} \times \left( \frac{L_p}{L_m} \right)^3 \times (R_w)_m + f_p A_p V_p^2. \quad \dots(12.42)$$

**Problem 12.29** A 1 in 20 model of a naval ship having a sub-merged surface area of  $5 \text{ m}^2$  and length 8 m has a total drag of 20 N when towed through water at a velocity of 1.5 m/s. Calculate the total drag on the prototype when moving at the corresponding speed. Use the relation  $F_f = \frac{1}{2} C_f \rho A V^2$

for calculating the skin (frictional) resistance. The value of  $C_f$  is given by  $C_f = \frac{0.0735}{(R_e)^{1.5}}$ .

Take kinematic viscosity of water (or sea-water) as 0.01 stoke and density of water (or sea-water) as  $1000 \text{ kg/m}^3$ .

**Solution.** Given :

Linear scale ratio,	$L_r = 20$
Sub-merged area of model,	$A_m = 5.0 \text{ m}^2$
Length of model,	$L_m = 8.0 \text{ m}$
Total drag of model,	$R_m = 20 \text{ N}$
Velocity of model,	$V_m = 1.5 \text{ m/s}$

Let  $A_p, L_p, R_p, V_p$  = Corresponding values for prototype.

Fluid in model is the same as in prototype and is sea-water.

Kinematic viscosity of sea-water,  $v_m = v_p = 0.01 \text{ stokes} = .01 \text{ cm}^2/\text{s} = .01 \times 10^{-4} \text{ m}^2/\text{s}$

Density of water,  $\rho_m = 1000 \text{ kg/m}^3$

The skin (frictional) resistance of model is given by

$$(F_f)_m = \frac{1}{2} C_{f_m} \rho_m A_m V_m^2 \quad \dots(i)$$

where  $C_{f_m} = \frac{0.0735}{[(R_e)_m]^{1/5}} \quad \dots(ii)$

where  $(R_e)_m$  = Reynold's number for model

$$\begin{aligned} &= \frac{\rho_m V_m L_m}{\mu_m} \text{ or } \frac{V_m L_m}{v_m} \quad \left\{ \because v = \frac{\mu}{\rho} \right\} \\ &= \frac{1.5 \times 8.0}{.01 \times 10^{-4}} = 1.2 \times 10^7. \end{aligned}$$

Substituting this value in equation (ii), we get

$$C_{f_m} = \frac{0.0735}{(1.2 \times 10^7)^{1/5}} = \frac{.0735}{26.0517} = 2.82 \times 10^{-3} \quad \dots(iii)$$

Substituting the value of  $C_{f_m}$  in equation (i), we get

$$(F_f)_m = \frac{1}{2} \times 2.82 \times 10^{-3} \times 1000 \times 5.0 \times (1.5)^2 = 15.8617 = 15.862 \text{ N}$$

Using equation (12.36), we get  $R_m = (R_w)_m + (R_f)_m$

where  $(R_f)_m = (F_f)_m = 15.862$  or  $20 = (R_w)_m + 15.862$

$\therefore$  Wave resistance for model,  $(R_w)_m = 20 - 15.862 = 4.138 \text{ N}$  ... (iv)

The wave resistance experienced by the ship is given by equation (12.41) as

$$\begin{aligned} (R_w)_P &= \frac{\rho_P}{\rho_m} \times \left( \frac{L_P}{L_m} \right)^3 \times (R_w)_m \\ &= 1 \times L_r^3 \times 4.138 \text{ N} \quad \left\{ \because \frac{\rho_P}{\rho_m} = 1 \text{ for same fluid} \right\} \\ &= 1 \times 20^3 \times 4.138 = 33104 \text{ N} \end{aligned}$$

and skin (frictional) resistance of prototype is given by

$$(R_f)_P = (F_f)_P = \frac{1}{2} C_{f_p} \times \rho_P \times A_P \times V_P^2 \quad \dots(v)$$

where  $V_P$  is the velocity of prototype and is given by Froude model law,

$$i.e., \quad (F_e)_m = (F_e)_P \quad \text{or} \quad \frac{V_m}{\sqrt{L_m g}} = \frac{V}{\sqrt{L_P g}} \quad \text{or} \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}}$$

$$\begin{aligned} \therefore \quad V_P &= \sqrt{\frac{L_P}{L_m}} \times V_m = \sqrt{L_r} \times V_m \quad \left\{ \because \frac{L_P}{L_m} = L_r \right\} \\ &= \sqrt{20} \times 1.5 = 6.708 \text{ m/s} \end{aligned}$$

$$\text{Now} \quad \frac{A_P}{A_m} = L_r^2 = 20^2$$

$$\therefore \quad A_P = A_m \times 20^2 = 5 \times 400 = 2000 \text{ m}^2$$

$$\text{and} \quad L_P = L_m \times L_r = 8 \times 20 = 160 \text{ m}$$

$$\text{In equation (v), the value of } C_{f_p} \text{ is given by } C_{f_p} = \frac{0.0735}{[(R_e)_P]^{1/5}}$$

where  $(R_e)_P$  = Reynolds number for prototype

$$= \frac{V_P \times L_P}{\nu_P} = \frac{6.708 \times 160}{.01 \times 10^{-4}} = 1.073 \times 10^9$$

$$\therefore \quad C_{f_p} = \frac{0.0735}{(1.073 \times 10^9)^{1/5}} = \frac{0.0735}{63.99} = 1.1486 \times 10^{-3}$$

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Substituting this value of  $C_{f_p}$  in equation (v), we get

$$(R_f)_P = (F_f)_P = \frac{1}{2} \times 1.1486 \times 10^{-3} \times 1000 \times 2000 \times (6.708)^2 = 51683.8 \text{ N}$$

$\therefore$  Total drag on prototype is obtained by using equation (12.35).

$$\therefore R_P = (R_w)_P + (R_f)_P = 33104 + 51683.8 = 84787.8 \text{ N. Ans.}$$

**Problem 12.30** A 1 : 15 model of a flying boat is towed through water. The prototype is moving in sea-water of density  $1024 \text{ kg/m}^3$  at a velocity of 20 m/s. Find the corresponding speed of the model. Also determine the resistance due to waves on model if the resistance due to waves of prototype is 600 N.

**Solution.** Given :

$$\text{Linear scale ratio, } L_r = 15$$

$$\text{Velocity of prototype, } V_p = 20 \text{ m/s}$$

Fluid in prototype is sea-water while in model it is water

$$\text{Density of sea-water, } \rho_p = 1024 \text{ kg/m}^3$$

$$\text{Density of water, } \rho_m = 1000 \text{ kg/m}^3$$

Resistance due to waves for prototype is,  $(R_w)_P = 600 \text{ N.}$

Find  $V_m$  and  $(R_w)_m$ .

(i) The velocity,  $V_m$  from model is given by Froude model law,

$$\therefore \frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p \times g}}$$

$$\therefore V_m = \sqrt{\frac{L_m}{L_p}} \times V_p = \frac{V_p}{\sqrt{L_p / L_m}} = \frac{20}{\sqrt{15}} \quad \left\{ \because \frac{L_p}{L_m} = L_r = 15 \right\}$$

$$= \frac{20}{3.872} = 5.165 \text{ m/s. Ans.}$$

(ii) For dynamic similarity between model and its prototype for wave resistance only, we have equation (12.41) as

$$(R_w)_P = \frac{\rho_p}{\rho_m} \times \left( \frac{L_p}{L_m} \right)^3 \times (R_w)_m$$

$$\text{Substituting the known values, } 600 = \frac{1024}{1000} \times 15^3 \times (R_w)_m = \frac{1024}{1000} \times 15^3 \times (R_w)_m$$

$$\therefore (R_w)_m = \frac{600 \times 1000}{1024 \times 15^3} = 0.1736 \text{ N. Ans.}$$

**Problem 12.31** A 1 : 40 model of an ocean tanker is dragged through fresh water at 2 m/s with a total measured drag of 12 N. The skin (frictional) drag co-efficient 'f' for model and prototype are 0.03 and 0.002 respectively in the equation  $R_f = f \cdot A V^2$ . The wetted surface area of the model is  $25 \text{ m}^2$ . Determine the total drag on the prototype and the power required to drive the prototype.

Take  $\rho_p = 1030 \text{ kg/m}^3$  and  $\rho_m = 1000 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Linear scale ratio, } L_r = 40$$

Velocity of model,  $V_m = 2 \text{ m/s}$

Total drag of model,  $R_m = 12 \text{ N}$

Wetted area of model,  $A_m = 25 \text{ m}^2$

Co-efficient of friction for model,  $f_m = .03$

for prototype,  $f_p = .002$ .

Let the total drag on prototype  $= R_p$

And power required to drive the prototype  $= P$

Frictional drag on model,  $(R_f)_m = f_m A_m V_m^2 = .03 \times 25 \times 2^2 = 3 \text{ N}$

$\therefore$  Wave drag on model,  $(R_w)_m = R_m - (R_f)_m = 12 - 3 = 9 \text{ N}$ .

The waves drag on prototype is obtained from equation (12.41) as

$$(R_w)_p = \frac{\rho_p}{\rho_m} \times \left( \frac{L_p}{L_m} \right)^3 \times (R_w)_m = \frac{1030}{1000} \times L_r^3 \times 9 \quad \left\{ \because \frac{L_p}{L_m} = L_r = 40 \right\}$$

$$= \frac{1030}{1000} \times 40^3 \times 9 = 593291.8 \text{ N} \quad \dots(i)$$

The frictional drag on prototype is given by

$$(R_f)_p = f_p \times A_p \times V_p^2 \quad \dots(ii)$$

where the velocity of prototype  $V_p$  is obtained from Froude model law as

$$\frac{V_m}{\sqrt{L_m \times g}} = \frac{V_p}{\sqrt{L_p \times g}} \quad \text{or} \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_p}{\sqrt{L_p}}$$

$$\therefore V_p = \sqrt{\frac{L_p}{L_m}} \times V_m = \sqrt{L_r} \times V_m = \sqrt{40} \times 2 = 12.65 \text{ m/s}$$

and

$$\begin{aligned} \frac{A_p}{A_m} &= L_r^2 = 40 \times 40 \text{ or } A_p = 40 \times 40 \times A_m \\ &= 40 \times 40 \times 25 = 40000 \text{ m}^2. \end{aligned}$$

Substituting these values in (ii), we get

$$(R_f)_p = .002 \times 40000 \times (12.65)^2 = 12801.8 \text{ N} \quad \dots(iii)$$

Total drag on the prototype is obtained by adding equations (i) and (ii) as

$$\begin{aligned} R_p &= (R_w)_p + (R_f)_p \\ &= 593291.8 + 12801.8 = \mathbf{606093.6 \text{ N. Ans.}} \end{aligned}$$

Power required to drive the prototype,

$$\begin{aligned} P &= \frac{(\text{Total drag on prototype}) \times \text{Velocity of prototype}}{1000} \\ &= \frac{606093.6 \times 12.65}{1000} = \mathbf{7667 \text{ kW. Ans.}} \end{aligned}$$

**Problem 12.32** Resistance  $R$ , to the motion of a completely submerged body is given by

$$R = \rho V^2 l^2 \phi \left( \frac{Vl}{v} \right),$$

where  $\rho$  and  $v$  are density and kinematic viscosity of the fluid while  $l$  is the length of the body and  $V$  is the velocity of flow. If the resistance of a one-eighth scale air-ship model when tested in water at

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12 m/s is 22 N, what will be the resistance in air of the air-ship at the corresponding speed? Kinematic viscosity of air is 13 times that of water and density of water is 810 times of air.

**Solution.** Given :

$$\text{Linear scale ratio, } L_r = 8$$

$$\text{Velocity of model, } V_m = 12 \text{ m/s}$$

$$\text{Resistance to model, } R_m = 22 \text{ N}$$

The fluid for model is water and for prototype the fluid is air.

$$\text{Kinematic viscosity of air} = 13 \times \text{Kinematic viscosity of water}$$

$$\therefore V_p = 13 \times V_m$$

$$\text{Density of water} = 810 \times \text{Density of air}$$

$$\therefore \rho_m = 810 \times \rho_p$$

Let  $V_p$  = Velocity of the air-ship (Prototype)

$R_p$  = Resistance of the air-ship

$$\text{The resistance, } R, \text{ is given by } R = \rho V^2 l^2 \phi\left(\frac{Vl}{v}\right)$$

$\therefore$  The non-dimensional terms  $\frac{R}{\rho V^2 l^2}$  and  $\frac{Vl}{v}$  should be same for the prototype and its model.

$$\therefore \left(\frac{Vl}{v}\right)_{\text{prototype}} = \left(\frac{Vl}{v}\right)_{\text{model}} \quad \text{or} \quad \frac{V_p l_p}{v_p} = \frac{V_m l_m}{v_m}$$

$$\therefore V_p = V_m \frac{l_m}{l_p} \times \frac{v_p}{v_m} = 12 \times \frac{1}{L_r} \times 13 \quad \left\{ \because \frac{l_p}{l_m} = L_r \right\}$$

$$= 12 \times \frac{1}{8} \times 13 = 19.5 \text{ m/s}$$

$$\text{Also } \left(\frac{R}{\rho V^2 l^2}\right)_{\text{prototype}} = \left(\frac{R}{\rho V^2 l^2}\right)_{\text{model}} \quad \text{or} \quad \frac{R_p}{\rho_p V_p^2 l_p^2} = \frac{R_m}{\rho_m V_m^2 l_m^2}$$

$$\therefore R_p = R_m \times \frac{\rho_p}{\rho_m} \times \frac{V_p^2}{V_m^2} \times \frac{l_p^2}{l_m^2}$$

$$= 22 \times \frac{1}{810} \times \frac{(19.5)^2}{12^2} \times 8^2 \quad \left( \because \frac{l_p}{l_m} = L_r = 8.0 \right)$$

$$= 4.59 \text{ N. Ans.}$$

### ► 12.11 CLASSIFICATION OF MODELS

The hydraulic models are classified as :

1. Undistorted models, and
2. Distorted models.

**12.11.1 Undistorted Models.** Undistorted models are those models which are geometrically similar to their prototypes or in other words if the scale ratio for the linear dimensions of the model and its prototype is same, the model is called undistorted model. The behaviour of the prototype can be easily predicted from the results of undistorted model.

**12.11.2 Distorted Models.** A model is said to be distorted if it is not geometrically similar to its prototype. For a distorted model different scale ratios for the linear dimensions are adopted. For example, in case of rivers, harbours, reservoirs etc., two different scale ratios, one for horizontal dimensions and other for vertical dimensions are taken. Thus the models of rivers, harbours and reservoirs will become as distorted models. If for the river, the horizontal and vertical scale ratios are taken to be same so that the model is undistorted, then the depth of water in the model of the river will be very-very small which may not be measured accurately. The following are the advantage of distorted models :

1. The vertical dimensions of the model can be measured accurately.
2. The cost of the model can be reduced.
3. Turbulent flow in the model can be maintained.

Though there are some advantages of the distorted model, yet the results of the distorted model cannot be directly transferred to its prototype. But sometimes from the distorted models very useful information can be obtained.

**12.11.3 Scale Ratios for Distorted Models.** As mentioned above, two different scale ratios, one for horizontal dimensions and other for vertical dimensions, are taken for distorted models.

Let

$(L_r)_H$  = Scale ratio for horizontal dimension

$$= \frac{L_p}{L_m} = \frac{B_p}{B_m} = \frac{\text{Linear horizontal dimension of prototype}}{\text{Linear horizontal dimension of model}}$$

$(L_r)_V$  = Scale ratio for vertical dimension

$$= \frac{\text{Linear vertical dimension of prototype}}{\text{Linear vertical dimension of model}} = \frac{h_p}{h_m}$$

Then the scale ratios of velocity, area of flow, discharge etc., in terms of  $(L_r)_H$  and  $(L_r)_V$  can be obtained for distorted models as given below :

#### 1. Scale ratio for velocity

Let

$V_p$  = Velocity in prototype

$V_m$  = Velocity in model.

Then

$$\frac{V_p}{V_m} = \frac{\sqrt{2gh_p}}{\sqrt{2gh_m}} = \sqrt{\frac{h_p}{h_m}} = \sqrt{(L_r)_V} \quad \left( \because \frac{h_p}{h_m} = (L_r)_V \right)$$

#### 2. Scale ratio for area of flow

Let

$A_p$  = Area of flow in prototype =  $B_p \times h_p$

$A_m$  = Area of flow in model =  $B_m \times h_m$

∴

$$\frac{A_p}{A_m} = \frac{B_p \times h_p}{B_m \times h_m} = \frac{B_p}{B_m} \times \frac{h_p}{h_m} = (L_r)_H \times (L_r)_V$$

#### 3. Scale ratio for discharge

Let

$Q_p$  = Discharge through prototype =  $A_p \times V_p$

$Q_m$  = Discharge through model =  $A_m \times V_m$

∴

$$\frac{Q_p}{Q_m} = \frac{A_p \times V_p}{A_m \times V_m} = (L_r)_H \times (L_r)_V \times \sqrt{(L_r)_V} = (L_r)_H \times [(L_r)_V]^{3/2} \dots (12.43)$$

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**Problem 12.33** The discharge through a weir is  $1.5 \text{ m}^3/\text{s}$ . Find the discharge through the model of the weir if the horizontal dimension of the model =  $\frac{1}{50}$  the horizontal dimension of the prototype and vertical dimension of the model =  $\frac{1}{10}$  the vertical dimension of the prototype.

**Solution.** Given :

Discharge through weir (prototype),  $Q_P = 1.5 \text{ m}^3/\text{s}$

$$\text{Horizontal dimension of model} = \frac{1}{50} \times \text{Horizontal dimension of prototype}$$

$$\therefore \frac{\text{Horizontal dimension of prototype}}{\text{Horizontal dimension of model}} = 50 \text{ or } (L_r)_H = 50$$

$$\text{Vertical dimension of model} = \frac{1}{10} \times \text{Vertical dimension of prototype}$$

$$\therefore \frac{\text{Vertical dimension of prototype}}{\text{Vertical dimension of model}} = 10$$

$$\therefore (L_r)_V = 10.$$

$$\text{Using equation (12.43), we get } \frac{Q_P}{Q_m} = (L_r)_H \times [(L_r)_V]^{3/2} = 50 \times 10^{3/2} = 1581.14$$

$$\begin{aligned} Q_m &= \frac{Q_P}{1581.14} = \frac{1.50}{1581.14} = .000948 \text{ m}^3/\text{s} \\ &= \mathbf{0.948 \text{ litres/s. Ans.}} \end{aligned}$$

### HIGHLIGHTS

1. Dimensional analysis is the method of dimensions, in which fundamental dimensions are  $M$ ,  $L$  and  $T$ .
2. Dimensional analysis is performed by two methods namely Rayleigh's Method and Buckingham's  $\pi$ -theorem.
3. Rayleigh's method is used for finding an expression for a variable which depends on maximum three or four variables while there is no restriction on the number of variables for Buckingham's  $\pi$ -theorem.
4. Model analysis is an experimental method of finding solutions of complex flow problems. A model is a small scale replica of the actual machine or structure. The actual machine or structure is called prototype.
5. Three types of similarities must exist between the model and prototype. They are : (i) Geometric Similarity, (ii) Kinematic Similarity, and (iii) Dynamic Similarity.
6. For geometric similarity, the ratio of all linear dimensions of the model and of the prototype should be equal.
7. Kinematic similarity means the similarity of motion between model and prototype.
8. Dynamic similarity means the similarity of forces between the model and prototype.
9. Reynold's number is defined as the ratio of inertia force and viscous force of a flowing fluid. It is given by,

$$R_e = \frac{\rho VL}{\mu} = \frac{VL}{v} = \frac{V \times d}{v} \text{ for pipe flow}$$

where  $V$  = Velocity of flow,  $d$  = Diameter of pipe and  
 $\nu$  = Kinematic viscosity of fluid.

10. Froude's Number is the ratio of the square root of inertia force and gravity force and is given by

$$F_e = \sqrt{\frac{F_i}{F_g}} = \frac{V}{\sqrt{Lg}}.$$

11. Euler's number is the ratio of the square root of inertia force and pressure force and is given by,

$$E_u = \sqrt{\frac{F_i}{F_p}} = \frac{V}{\sqrt{p/\rho}}.$$

12. Mach number is the ratio of the square root of inertia force and elastic force and is given by

$$M = \sqrt{\frac{F_i}{F_e}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}.$$

13. The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The model laws are (i) Reynold's Model Law, (ii) Froude Model Law, (iii) Euler Model Law, (iv) Weber Model Law, (v) Mach Model Law.  
 14. The drag experienced by a ship model (or partially sub-merged body) is obtained by Froude's method.  
 15. Hydraulic models are classified as (i) undistorted models and (ii) distorted models.  
 16. If the models are geometrically similar to its prototype, the models are known as undistorted model. And if the models are having different scale ratio for horizontal and vertical dimensions, the models are known as distorted model.

## **EXERCISE**

### **(A) THEORETICAL PROBLEMS**

1. Define the terms dimensional analysis and model analysis.
2. What do you mean by fundamental units and derived units ? Give examples.
3. Explain the term, 'dimensionally homogeneous equation'.
4. What are the methods of dimensional analysis ? Describe the Rayleigh's method for dimensional analysis.
5. State Buckingham's  $\pi$ -theorem. Why this theorem is considered superior over the Rayleigh's method for dimensional analysis ?
6. What do you mean by repeating variables ? How are the repeating variables selected for dimensional analysis ?
7. Define the terms : model, prototype, model analysis, hydraulic similitude.
8. Explain the different types of hydraulic similarities that must exist between a prototype and its model.
9. What do you mean by dimensionless numbers ? Name any four dimensionless numbers.  
 Define and explain Reynold's number, Froude's number's and Mach number. Derive expressions for any above two numbers.
10. What is meant by geometric, kinematic and dynamic similarities ? Are these similarities truly attainable ? If not why ?
11. Define the following non-dimensional numbers : Reynold's number, Froude's number and Mach's number. What are their significances for fluid flow problems ?
12. What are the different laws on which models are designed for dynamic similarity ? Where are they used ?
13. How will you determine the total drag of a ship or partially sub-merged bodies ?
14. Explain the terms : distorted models and undistorted models. What is the use of distorted models ?

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15. Prove that the scale ratio for discharge for a distorted model is given as

$$\frac{Q_p}{Q_m} = (L_r)_H \times (L_r)_V^{3/2}$$

where  $Q_p$  = Discharge through prototype  
 $Q_m$  = Discharge through model  
 $(L_r)_H$  = Horizontal scale ratio  
 $(L_r)_V$  = Vertical scale ratio.

16. Show that ratio of inertia force to viscous force gives the Reynolds number.  
 17. State Buckingham's  $\pi$ -theorem. What do you mean by repeating variables ? How are the repeating variables selected in dimensional analysis ?  
 18. What is the significance of the non-dimensional numbers : Reynolds number, Froude number and Mach number in the theory of similarity ? What is the dimensional analysis ? How is this analysis related to the theory of similarity ?  
 (Delhi University, December 2001)  
 19. Define and explain : (i) Froude's number, (ii) Mach number, (iii) Hydraulic similarities (iv) Distorted and undistorted models.  
 (Delhi University, December 2002)

### (B) NUMERICAL PROBLEMS

1. Give the dimensions of : (i) Force (ii) Viscosity (iii) Power and (iv) Kinematic viscosity.  
 [Ans.  $MLT^{-2}$ ,  $ML^{-1}T^{-1}$ ,  $ML^{+2}T^{-3}$ ,  $L^2T^{-1}$ ]  
 2. The variables controlling the motion of a floating vessel through water are the drag force  $F$ , the speed  $V$ , the length  $L$ , the density  $\rho$  and dynamic viscosity  $\mu$  of water and acceleration due to gravity  $g$ . Derive an expression for  $F$  by dimensional analysis. [Ans.  $F = \rho L^2 V^2 \phi \left[ \frac{\mu}{\rho VL}, \frac{Lg}{V^2} \right]$   
 3. The resistance  $R$ , to the motion of a completely sub-merged body depends upon the length of the body  $L$ , velocity of flow  $V$ , mass density of fluid  $\rho$  and kinematic viscosity of fluid  $\nu$ . By dimensional analysis prove that

$$R = \rho V^2 L^2 \phi \left( \frac{VL}{\nu} \right).$$

4. A pipe of diameter 1.8 m is required to transport an oil of sp. gr. 0.8 and viscosity .04 poise at the rate of  $4 \text{ m}^3/\text{s}$ . Tests were conducted on a 20 cm diameter pipe using water at  $20^\circ\text{C}$ . Find the velocity and rate of flow in the model. Viscosity of water at  $20^\circ\text{C} = .01$  poise. [Ans. 2.829 m/s, 88.8 litres/s]  
 5. A model of a sub-marine of scale  $\frac{1}{40}$  is tested in a wind tunnel. Find the speed of air in wind tunnel if the speed of sub-marine in sea-water is 15 m/s. Also find the ratio of the resistance between the model and its prototype. Take the values of kinematic viscosities for sea-water and air as .012 stokes and 0.016 stokes respectively. The density of sea-water and of air are given as  $1030 \text{ kg/m}^3$  and  $1.24 \text{ kg/m}^3$  respectively.

$$\left[ \text{Ans. } 800 \text{ m/s, } \frac{F_m}{F_P} = 0.00214 \right]$$

6. A ship 250 m long moves in sea-water, whose density is  $1030 \text{ kg/m}^3$ . A  $1 : 125$  model of this ship is to be tested in wind tunnel. The velocity of air in the wind tunnel around the model is 20 m/s and the resistance of the model is 50 N. Determine the velocity of ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as  $1.24 \text{ kg/m}^3$ . Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively. [Ans. 0.106 m/s, 18228.7 N]

7. In 1 : 30 model of a spillway, the velocity and discharge are 1.5 m/s and 2.0 m<sup>3</sup>/s. Find the corresponding velocity and discharge in the prototype.  
 [Ans. 8.216 m/s, 9859 m<sup>3</sup>/s]
8. A ship-model of scale  $\frac{1}{60}$  is towed through sea-water at a speed of 0.5 m/s. A force of 1.5 N is required to tow the model. Determine the speed of the ship and propulsive force on the ship, if prototype is subjected to wave resistance only.  
 [Ans. 3.873 m/s, 324000 N]
9. A spillway model is to be built to a geometrically similar scale of  $\frac{1}{40}$  across a flume, of 50 cm width. The prototype is 20 m high and maximum head on it is expected to be 2 m. (i) What height of model and what head on the model should be used ? (ii) If the flow over the model at a particular head is 10 litres/s, what flow per metre length of the prototype is expected ? (iii) If the negative pressure in the model is 150 mm, what is the negative pressure in the prototype ? Is it practicable ?  
 [Ans. (i) 0.5 m, 0.05 m, (ii) 5059.64 litres/s (iii) 6 m. Yes]
10. The pressure drop in an aeroplane model of size  $\frac{1}{50}$  of its prototype is 4 N/cm<sup>2</sup>. The model is tested in water. Find the corresponding pressure drop in the prototype. Take density of air = 1.24 kg/m<sup>3</sup>. The viscosity of water is 0.01 poise while the viscosity of air is 0.00018 poise.  
 [Ans. .00042 N/cm<sup>2</sup>]
11. A 1 : 20 model of a flying boat is towed through water. The prototype is moving in sea-water of density 1024 kg/m<sup>3</sup> at a velocity of 15 m/s. Find the corresponding speed of the model. Also determine the resistance due to waves on model, if the resistance due to waves of prototype is 500 N.  
 [Ans. 3.354 m/s, .061 N]
12. A 1 : 50 model of an ocean tanker is dragged through fresh water at 1.5 m/s with a total measured drag of 10 N. The frictional drag co-efficient 'f' for model and prototype are 0.03 and 0.02 respectively in the equation,  

$$R_f = f \cdot A \cdot V^2 ;$$
  
 the wetted surface area of the model is 20 m<sup>2</sup>. Determine the total drag on the prototype and the power required to derive the prototype. Take density of sea-water and of fresh water as 1024 kg/m<sup>3</sup> and 1000 kg/m<sup>3</sup> respectively.  
 [Ans. 1118436 N, 158072 h.p.]
13. If model prototype ratio is 1 : 75, show that the ratio of discharges per unit width of spillway is given by  

$$\left(\frac{1}{75}\right)^{3/2}.$$
14. A fluid of density  $\rho$  and viscosity  $\mu$ , flows at an average velocity  $V$  through a circular pipe of diameter  $D$ . Show by dimensional analysis, that the shear stress at the pipe wall is given as  

$$\tau_0 = \rho V^2 \phi \left[ \frac{\rho V D}{\mu} \right].$$
15. The drag force exerted by a flowing fluid on a solid body depends upon the length of the body,  $L$ , velocity of flow  $V$ , density of fluid  $\rho$ , and viscosity  $\mu$ . Find an expression for drag force using Buckingham's theorem.
16. The efficiency  $\eta$  of geometrically similar fans depends upon the mass density of air  $\rho$ , its viscosity  $\mu$ , speed of fan  $N$  (revolutions per sec), diameter of blades  $D$  and discharge  $Q$ . Perform dimensional analysis.  
**[Hint.** Take  $D, N, \rho$  as repeating variables. Then three  $\pi$ -terms will be  

$$\pi_1 = D^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot \eta = \eta ;$$
  

$$\pi_2 = D^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu = \frac{\mu}{D^2 N P} \text{ and } \pi_3 = D^{a_3} \cdot N^{b_3} \cdot \rho^{c_3} \cdot Q = \frac{Q}{D^3 N}$$
  

$$\therefore \eta = \phi(\pi_2, \pi_3) = \phi\left(\frac{\mu}{D^2 N P}, \frac{Q}{D^3 N}\right)$$

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17. The discharge through an orifice depends on the diameter  $D$  of the orifice, head  $H$  over the orifice, density  $\rho$  of liquid, viscosity  $\mu$  of the liquid and acceleration  $g$  due to gravity. Using dimensional analysis, find an expression for the discharge. Hence find the dimensionless parameters on which the discharge co-efficient of an orifice meter depend.

[ Hint.  $Q = f(D, H, \rho, \mu, g)$ . Hence  $N = 6$ ,  $m = 3$  and number of  $\pi$ -terms = 3. Take  $\rho, D, g$  as repeating variables. Then

$$\pi_1 = \rho^{a_1} D^{b_1} g^{c_1} Q, \pi_2 = \rho^{a_2} D^{b_2} g^{c_2} \mu \text{ and } \pi_3 = \rho^{a_3} D^{b_3} g^{c_3} H.$$

Find  $\pi_1, \pi_2$  and  $\pi_3$ . They will be  $\pi_1 = \frac{Q}{D^{2.5} \times g^{1/2}}$ ,  $\pi_2 = \frac{\mu}{\rho \cdot D^{3/2} \cdot g^{1/2}}$  and  $\pi_3 = \frac{H}{D}$ .

Hence

$$Q = D^{2.5} \cdot g^{1/2} \cdot \left( \frac{H}{D}, \frac{\mu}{\rho \cdot D^{3/2} \cdot g^{1/2}} \right) = D^2 \cdot g^{1/2} \cdot D^{1/2} \left( \frac{H}{D}, \frac{\mu}{\rho \cdot D^{3/2} \cdot g^{1/2}} \right).$$

Also

$$Q = C_d \times A \times \sqrt{2gh}.$$

Comparing the two values of  $Q$ .

We get  $C_d = \phi \left( \frac{H}{D}, \frac{\mu}{\rho \cdot D^{3/2} \cdot g^{1/2}} \right)$ . Hence  $C_d$  depends upon  $\frac{H}{D}$  and  $\frac{\mu}{\rho \cdot D^{3/2} \cdot g^{1/2}}$

18. The force exerted by a flowing fluid on a stationary body depends upon the length ( $L$ ) of the body, velocity ( $V$ ) of the fluid, density ( $\rho$ ) of fluid, viscosity ( $\mu$ ) of the fluid and acceleration ( $g$ ) due to gravity. Find an expression for the force using dimensional analysis.

$$\boxed{\text{Ans. } F = \rho L^2 V^2 \phi \left( \frac{\mu}{\rho V L}, \frac{L \times g}{V^2} \right)}$$

19. The pressure difference  $\Delta p$  in a pipe of diameter  $D$  and length  $L$  due to turbulent flow depends upon the velocity  $V$ , viscosity  $\mu$ , density  $\rho$  and roughness  $k$ . Using Buckingham's  $\pi$ -theorem or otherwise obtain an expression for  $\Delta p$ .

(Delhi University, December 2002)

# 13

## CHAPTER



### ► 13.1 INTRODUCTION

When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the boundary will be same as that of the boundary. If the boundary is stationary, the velocity of fluid at the boundary will be zero. Farther away from the boundary, the velocity will be higher and as a result of this variation of velocity, the velocity gradient  $\frac{du}{dy}$  will exist. The velocity of fluid increases from zero velocity on the stationary boundary to free-stream velocity ( $U$ ) of the fluid in the direction normal to the boundary. This variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called boundary layer theory.

According to boundary layer theory, the flow of fluid in the neighbourhood of the solid boundary may be divided into two regions as shown in Fig. 13.1.

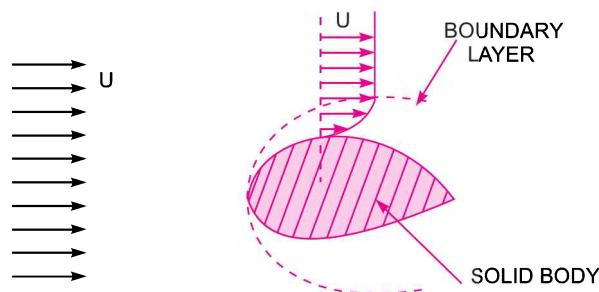


Fig. 13.1 *Flow over solid body.*

1. A very thin layer of the fluid, called the boundary layer, in the immediate neighbourhood of the solid boundary, where the variation of velocity from zero at the solid boundary to free-stream velocity in the direction normal to the boundary takes place. In this region, the velocity gradient  $\frac{du}{dy}$  exists and hence the fluid exerts a shear stress on the wall in the direction of motion. The value of shear stress is given by

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$$\tau = \mu \frac{du}{dy}$$

2. The remaining fluid, which is outside the boundary layer. The velocity outside the boundary layer is constant and equal to free-stream velocity. As there is no variation of velocity in this region, the velocity gradient  $\frac{du}{dy}$  becomes zero. As a result of this the shear stress is zero.

### ► 13.2 DEFINITIONS

**13.2.1 Laminar Boundary Layer.** For defining the boundary layer (*i.e.*, laminar boundary layer or turbulent boundary layer) consider the flow of a fluid, having free-stream velocity ( $U$ ), over a smooth thin plate which is flat and placed parallel to the direction of free stream of fluid as shown in Fig. 13.2. Let us consider the flow with zero pressure gradient on one side of the plate, which is stationary.

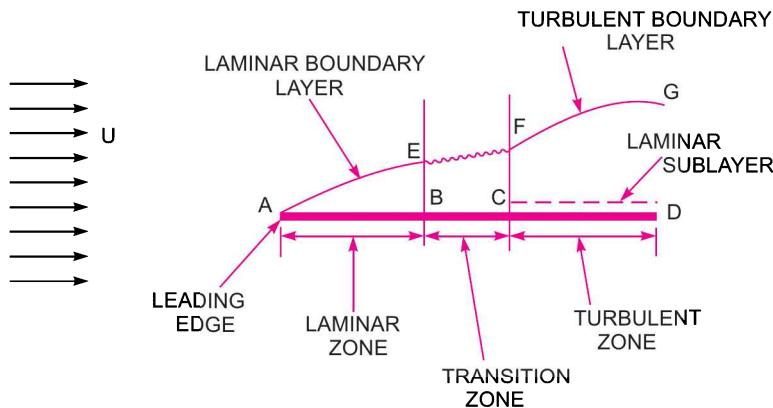


Fig. 13.2 Flow over a plate.

The velocity of fluid on the surface of the plate should be equal to the velocity of the plate. But plate is stationary and hence velocity of fluid on the surface of the plate is zero. But at a distance away from the plate, the fluid is having certain velocity. Thus a velocity gradient is set up in the fluid near the surface of the plate. This velocity gradient develops shear resistance, which retards the fluid. Thus the fluid with a uniform free stream velocity ( $U$ ) is retarded in the vicinity of the solid surface of the plate and the boundary layer region begins at the sharp leading edge. At subsequent points downstream the leading edge, the boundary layer region increases because the retarded fluid is further retarded. This is also referred as the growth of boundary layer. Near the leading edge of the surface of the plate, where the thickness is small, the flow in the boundary layer is laminar though the main flow is turbulent. This layer of the fluid is said to be laminar boundary layer. This is shown by AE in Fig. 13.2. The length of the plate from the leading edge, upto which laminar boundary layer exists, is called laminar zone. This is shown by distance AB. The distance of B from leading edge is obtained from Reynold number equal to  $5 \times 10^5$  for a plate. Because upto this Reynold number the boundary layer is laminar. The Reynold number is given by  $(R_e)_x = \frac{U \times x}{v}$

where  $x$  = Distance from leading edge,  
 $U$  = Free-stream velocity of fluid,  
 $v$  = Kinematic viscosity of fluid,

Hence for laminar boundary layer, we have  $5 \times 10^5 = \frac{U \times x}{v}$  ... (13.1)

If the values of  $U$  and  $v$  are known,  $x$  or the distance from the leading edge upto which laminar boundary layer exists can be calculated.

**13.2.2 Turbulent Boundary Layer.** If the length of the plate is more than the distance  $x$ , calculated from equation (13.1), the thickness of boundary layer will go on increasing in the downstream direction. Then the laminar boundary layer becomes unstable and motion of fluid within it, is disturbed and irregular which leads to a transition from laminar to turbulent boundary layer. This short length over which the boundary layer flow changes from laminar to turbulent is called transition zone. This is shown by distance  $BC$  in Fig. 13.2. Further downstream the transition zone, the boundary layer is turbulent and continues to grow in thickness. This layer of boundary is called turbulent boundary layer, which is shown by the portion  $FG$  in Fig. 13.2.

**13.2.3 Laminar Sub-layer.** This is the region in the turbulent boundary layer zone, adjacent to the solid surface of the plate as shown in Fig. 13.2. In this zone, the velocity variation is influenced only by viscous effects. Though the velocity distribution would be a parabolic curve in the laminar sub-layer zone, but in view of the very small thickness we can reasonably assume that velocity variation is linear and so the velocity gradient can be considered constant. Therefore, the shear stress in the laminar sub-layer would be constant and equal to the boundary shear stress  $\tau_0$ . Thus the shear stress in the sub-layer is

$$\tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \mu \frac{u}{y} \quad \left\{ \because \text{For linear variation, } \frac{\partial u}{\partial y} = \frac{u}{y} \right\}$$

**13.2.4 Boundary Layer Thickness ( $\delta$ ).** It is defined as the distance from the boundary of the solid body measured in the  $y$ -direction to the point, where the velocity of the fluid is approximately equal to 0.99 times the free stream velocity ( $U$ ) of the fluid. It is denoted by the symbol  $\delta$ . For laminar and turbulent zone it is denoted as :

1.  $\delta_{lam}$  = Thickness of laminar boundary layer,
2.  $\delta_{tur}$  = Thickness of turbulent boundary layer, and
3.  $\delta'$  = Thickness of laminar sub-layer.

**13.2.5 Displacement Thickness ( $\delta^*$ ).** It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate on account of boundary layer formation. It is denoted by  $\delta^*$ . It is also defined as :

"The distance perpendicular to the boundary, by which the free-stream is displaced due to the formation of boundary layer".

**Expression for  $\delta^*$ .**

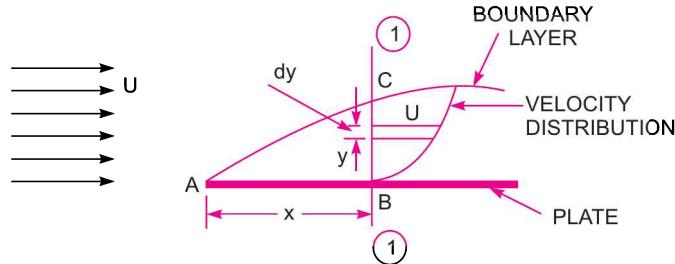


Fig. 13.3 Displacement thickness.

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Consider the flow of a fluid having free-stream velocity equal to  $U$  over a thin smooth plate as shown in Fig. 13.3. At a distance  $x$  from the leading edge consider a section 1-1. The velocity of fluid at  $B$  is zero and at  $C$ , which lies on the boundary layer, is  $U$ . Thus velocity varies from zero at  $B$  to  $U$  at  $C$ , where  $BC$  is equal to the thickness of boundary layer i.e.,

$$\text{Distance } BC = \delta$$

At the section 1-1, consider an elemental strip.

Let  $y$  = distance of elemental strip from the plate,

$dy$  = thickness of the elemental strip,

$u$  = velocity of fluid at the elemental strip,

$b$  = width of plate.

Then area of elemental strip,  $dA = b \times dy$

Mass of fluid per second flowing through elemental strip

$$\begin{aligned} &= \rho \times \text{Velocity} \times \text{Area of elemental strip} \\ &= \rho u \times dA = \rho u \times b \times dy \end{aligned} \quad \dots(i)$$

If there had been no plate, then the fluid would have been flowing with a constant velocity equal to free-stream velocity ( $U$ ) at the section 1-1. Then mass of fluid per second flowing through elemental strip would have been

$$= \rho \times \text{Velocity} \times \text{Area} = \rho \times U \times b \times dy \quad \dots(ii)$$

As  $U$  is more than  $u$ , hence due to the presence of the plate and consequently due to the formation of the boundary layer, there will be a reduction in mass flowing per second through the elemental strip.

This reduction in mass/sec flowing through elemental strip

$$\begin{aligned} &= \text{mass/sec given by equation (ii)} - \text{mass/sec given by equation (i)} \\ &= \rho U b dy - \rho u b dy = \rho b(U - u)dy \end{aligned}$$

∴ Total reduction in mass of fluid/s flowing through  $BC$  due to plate

$$= \int_0^\delta \rho b(U - u)dy = \rho b \int_0^\delta (U - u)dy \quad \dots(iii)$$

{if fluid is incompressible}

Let the plate is displaced by a distance  $\delta^*$  and velocity of flow for the distance  $\delta^*$  is equal to the free-stream velocity (i.e.,  $U$ ). Loss of the mass of the fluid/sec flowing through the distance  $\delta^*$

$$\begin{aligned} &= \rho \times \text{Velocity} \times \text{Area} \\ &= \rho \times U \times \delta^* \times b \quad \{\because \text{Area} = \delta^* \times b\} \quad \dots(iv) \end{aligned}$$

Equating equation (iii) and (iv), we get

$$\rho b \int_0^\delta (U - u)dy = \rho \times U \times \delta^* b$$

Cancelling  $\rho b$  from both sides, we have

$$\int_0^\delta (U - u)dy = U \times \delta^*$$

or 
$$\delta^* = \frac{1}{U} \int_0^\delta (U - u)dy = \int_0^\delta \frac{(U - u)dy}{U} \quad \left\{ \begin{array}{l} \because U \text{ is constant and can} \\ \text{be taken inside the integral} \end{array} \right\}$$

$$\therefore \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy. \quad \dots(13.2)$$

**13.2.6 Momentum Thickness ( $\theta$ ).** Momentum thickness is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in **momentum** of the flowing fluid on account of boundary layer formation. It is denoted by  $\theta$ .

Consider the flow over a plate as shown in Fig. 13.3. Consider the section 1-1 at a distance  $x$  from leading edge. Take an elemental strip at a distance  $y$  from the plate having thickness ( $dy$ ). The mass of fluid flowing per second through this elemental strip is given by equation (i) and is equal to  $\rho b dy$ .

$$\text{Momentum of this fluid} = \text{Mass} \times \text{Velocity} = (\rho b dy) u$$

$$\text{Momentum of this fluid in the absence of boundary layer} = (\rho b dy) U$$

$$\therefore \text{Loss of momentum through elemental strip} = (\rho b dy) U - (\rho b dy) \times u = \rho b u (U - u) dy$$

$$\therefore \text{Total loss of momentum/sec through } BC = \int_0^\delta \rho b u (U - u) dy \quad \dots(13.3)$$

Let  $\theta$  = distance by which plate is displaced when the fluid is flowing with a constant velocity  $U$

$\therefore$  Loss of momentum/sec of fluid flowing through distance  $\theta$  with a velocity  $U$

$$= \text{Mass of fluid through } \theta \times \text{velocity}$$

$$= (\rho \times \text{area} \times \text{velocity}) \times \text{velocity}$$

$$= [\rho \times \theta \times b \times U] \times U$$

$$= \rho \theta b U^2$$

$$\{\because \text{Area} = \theta \times b\} \quad \dots(13.4)$$

Equating equations (13.4) and (13.3), we have

$$\rho \theta b U^2 = \int_0^\delta \rho b u (U - u) dy = \rho b \int_0^\delta u (U - u) dy \quad \{\text{If fluid is assumed incompressible}\}$$

$$\text{or} \quad \theta U^2 = \int_0^\delta u (U - u) dy \quad \{\text{cancelling } \rho b \text{ from both sides}\}$$

$$\text{or} \quad \theta = \frac{1}{U^2} \int_0^\delta u (U - u) dy = \int_0^\delta \frac{u (U - u) dy}{U^2}$$

$$\therefore \theta = \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u}{U} \right] dy. \quad \dots(13.5)$$

**13.2.7 Energy Thickness ( $\delta^{**}$ ).** It is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is denoted by  $\delta^{**}$ .

Consider the flow over the plate as shown in Fig. 13.3 having section 1-1 at a distance  $x$  from leading edge. The mass of fluid flowing per second through the elemental strip of thickness 'dy' at a distance  $y$  from the plate as given by equation (i) =  $\rho b dy$

$$\text{Kinetic energy of this fluid} = \frac{1}{2} m \times \text{velocity}^2 = \frac{1}{2} (\rho b dy) u^2$$

Kinetic energy of this fluid in the absence of boundary layer

$$= \frac{1}{2} (\rho b dy) U^2$$

$\therefore$  Loss of K.E. through elemental strip

$$= \frac{1}{2} (\rho b dy) U^2 - \frac{1}{2} (\rho b dy) u^2 = \frac{1}{2} \rho b [U^2 - u^2] dy$$

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∴ Total loss of K.E. of fluid passing through BC

$$= \int_0^\delta \frac{1}{2} \rho u b [U^2 - u^2] dy = \frac{1}{2} \rho b \int_0^\delta u (U^2 - u^2) dy$$

{If fluid is considered incompressible}

Let  $\delta^{**}$  = distance by which the plate is displaced to compensate for the reduction in K.E.

∴ Loss of K.E. through  $\delta^{**}$  of fluid flowing with velocity  $U$

$$\begin{aligned} &= \frac{1}{2} (\text{mass}) \times \text{velocity}^2 = \frac{1}{2} (\rho \times \text{area} \times \text{velocity}) \times \text{velocity}^2 \\ &= \frac{1}{2} (\rho \times b \times \delta^{**} \times U) U^2 \quad \{\because \text{Area} = b \times \delta^{**}\} \\ &= \frac{1}{2} \rho b \delta^{**} U^3 \end{aligned}$$

Equating the two losses of K.E., we get

$$\begin{aligned} \frac{1}{2} \rho b \delta^{**} U^3 &= \frac{1}{2} \rho b \int_0^\delta u (U^2 - u^2) dy \\ \text{or} \quad \delta^{**} &= \frac{1}{U^3} \int_0^\delta u (U^2 - u^2) dy \\ \therefore \quad \delta^{**} &= \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u^2}{U^2} \right] dy. \end{aligned} \quad \dots(13.6)$$

**Problem 13.1** Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by  $\frac{u}{U} = \frac{y}{\delta}$ , where  $u$  is the velocity at a distance  $y$  from the plate and  $u = U$  at  $y = \delta$ , where  $\delta$  = boundary layer thickness. Also calculate the value of  $\delta^*/\theta$ .

**Solution.** Given :

$$\text{Velocity distribution} \quad \frac{u}{U} = \frac{y}{\delta}$$

(i) Displacement thickness  $\delta^*$  is given by equation (13.2),

$$\begin{aligned} \delta^* &= \int_0^\delta \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \left( 1 - \frac{y}{\delta} \right) dy \quad \left\{ \because \frac{u}{U} = \frac{y}{\delta} \right\} \\ &= \left[ y - \frac{y^2}{2\delta} \right]_0^\delta \quad \{\delta \text{ is constant across a section}\} \\ &= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \frac{\delta}{2}. \quad \text{Ans.} \end{aligned}$$

(ii) Momentum thickness,  $\theta$  is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy$$

Substituting the value of  $\frac{u}{U} = \frac{y}{\delta}$ ,

$$\begin{aligned}\theta &= \int_0^\delta \frac{y}{\delta} \left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy \\ &= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6} = \frac{\delta}{6}. \text{ Ans.}\end{aligned}$$

(iii) Energy thickness  $\delta^{**}$  is given by equation (13.6), as

$$\begin{aligned}\delta^{**} &= \int_0^\delta \frac{u}{U} \left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \frac{y}{\delta} \left[1 - \frac{y^2}{\delta^2}\right] dy \quad \left\{ \because \frac{u}{U} = \frac{y}{\delta} \right\} \\ &= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3} \\ &= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \frac{\delta}{4}. \text{ Ans.}\end{aligned}$$

$$(iv) \quad \frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta} = 3. \text{ Ans.}$$

**Problem 13.2** Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by  $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ .

**Solution.** Given :

$$\text{Velocity distribution} \quad \frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

(i) Displacement thickness  $\delta^*$  is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of  $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ , we have

$$\begin{aligned}\delta^* &= \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy \\ &= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta \\ &= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{3}. \text{ Ans.}\end{aligned}$$

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(ii) Momentum thickness  $\theta$ , is given by equation (13.5),

$$\begin{aligned}
 \theta &= \int_0^\delta \frac{u}{U} \left\{ 1 - \frac{u}{U} \right\} dy = \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left[ 1 - \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \right] dy \\
 &= \int_0^\delta \left[ \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right] \left[ 1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2} \right] dy \\
 &= \int_0^\delta \left[ \frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4} \right] dy \\
 &= \int_0^\delta \left[ \frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4} \right] dy = \left[ \frac{2y^2}{2\delta} - \frac{5y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4} \right]_0^\delta \\
 &= \left[ \frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4} \right] = \delta - \frac{5\delta}{3} + \delta - \frac{\delta}{5} \\
 &= \frac{15\delta - 25\delta + 15\delta - 3\delta}{15} = \frac{30\delta - 28\delta}{15} = \frac{2\delta}{15}. \quad \text{Ans.}
 \end{aligned}$$

(iii) Energy thickness  $\delta^{**}$  is given by equation (13.6),

$$\begin{aligned}
 \delta^{**} &= \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u^2}{U^2} \right] dy = \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left( 1 - \left[ \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right]^2 \right) dy \\
 &= \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left( 1 - \left[ \frac{4y^2}{\delta^2} + \frac{y^4}{\delta^4} - \frac{4y^3}{\delta^3} \right] \right) dy \\
 &= \int_0^\delta \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \left( 1 - \frac{4y^2}{\delta^2} - \frac{y^4}{\delta^4} + \frac{4y^3}{\delta^3} \right) dy \\
 &= \int_0^\delta \left( \frac{2y}{\delta} - \frac{8y^3}{\delta^3} - \frac{2y^5}{\delta^5} + \frac{8y^4}{\delta^4} - \frac{y^2}{\delta^2} + \frac{4y^4}{\delta^4} + \frac{y^6}{\delta^6} - \frac{4y^5}{\delta^5} \right) dy \\
 &= \int_0^\delta \left[ \frac{2y}{\delta} - \frac{y^2}{\delta^2} - \frac{8y^3}{\delta^3} + \frac{12y^4}{\delta^4} - \frac{6y^5}{\delta^5} + \frac{y^6}{\delta^6} \right] dy \\
 &= \left[ \frac{2y^2}{2\delta} - \frac{y^3}{3\delta^2} - \frac{8y^4}{4\delta^3} + \frac{12y^5}{5\delta^4} - \frac{6y^6}{6\delta^5} + \frac{y^7}{7\delta^6} \right]_0^\delta \\
 &= \frac{\delta^2}{\delta} - \frac{\delta^3}{3\delta^2} - \frac{2\delta^4}{\delta^3} + \frac{12\delta^5}{5\delta^4} - \frac{\delta^6}{\delta^5} + \frac{\delta^7}{7\delta^6} = \delta - \frac{\delta}{3} - 2\delta + \frac{12}{5}\delta - \delta + \frac{\delta}{7} \\
 &= -2\delta - \frac{\delta}{3} + \frac{12}{5}\delta + \frac{\delta}{7} = \frac{-210\delta - 35\delta + 252\delta + 15\delta}{105} \\
 &= \frac{-245\delta + 267\delta}{105} = \frac{22\delta}{105}. \quad \text{Ans.}
 \end{aligned}$$

### ► 13.3 DRAG FORCE ON A FLAT PLATE DUE TO BOUNDARY LAYER

Consider the flow of a fluid having free-stream velocity equal to  $U$ , over a thin plate as shown in Fig. 13.4. The drag force on the plate can be determined if the velocity profile near the plate is known. Consider a small length  $\Delta x$  of the plate at a distance of  $x$  from the leading edge as shown in Fig. 13.4 (a). The enlarged view of the small length of the plate is shown in Fig. 13.4 (b).

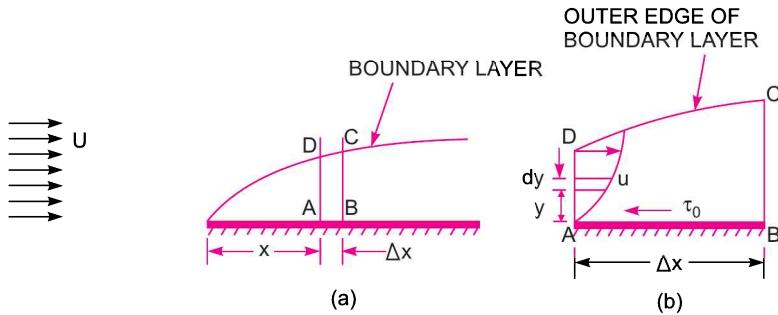


Fig. 13.4 Drag force on a plate due to boundary layer.

The shear stress  $\tau_0$  is given by  $\tau_0 = \mu \left( \frac{du}{dy} \right)_{y=0}$ , where  $\left( \frac{du}{dy} \right)_{y=0}$  is the velocity distribution near the plate at  $y = 0$ .

Then drag force or shear force on a small distance  $\Delta x$  is given by

$$\begin{aligned} \Delta F_D &= \text{shear stress} \times \text{area} \\ &= \tau_0 \times \Delta x \times b \quad \dots(13.7) \quad \{\text{Taking width of plate} = b\} \end{aligned}$$

where  $\Delta F_D$  = drag force on distance  $\Delta x$

The drag force  $\Delta F_D$  must also be equal to the rate of change of momentum over the distance  $\Delta x$ .

Consider the flow over the small distance  $\Delta x$ . Let  $ABCD$  is the control volume of the fluid over the distance  $\Delta x$  as shown in Fig. 13.4 (b). The edge  $DC$  represents the outer edge of the boundary layer.

Let  $u$  = velocity at any point within the boundary layer

$b$  = width of plate

Then mass rate of flow entering through the side  $AD$

$$\begin{aligned} &= \int_0^\delta \rho \times \text{velocity} \times \text{area of strip of thickness } dy \\ &= \int_0^\delta \rho \times u \times b \times dy \quad \{\because \text{Area of strip} = b \times dy\} \\ &= \int_0^\delta \rho ubdy \end{aligned}$$

Mass rate of flow leaving the side  $BC$

$$\begin{aligned} &= \text{mass through } AD + \frac{\partial}{\partial x} (\text{mass through } AD) \times \Delta x \\ &= \int_0^\delta \rho ubdy \frac{\partial}{\partial x} \left[ \int_0^\delta (\rho ubdy) \right] \times \Delta x \end{aligned}$$

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From continuity equation for a steady incompressible fluid flow, we have

Mass rate of flow entering  $AD$  + mass rate of flow entering  $DC$

= mass rate of flow leaving  $BC$

$\therefore$  Mass rate of flow entering  $DC$  = mass rate of flow through  $BC$  – mass rate of flow through  $AD$

$$\begin{aligned} &= \int_0^\delta \rho u b dy + \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u b dy \right] \times \Delta x - \int_0^\delta \rho u b dy \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u b dy \right] \times \Delta x \end{aligned}$$

The fluid is entering through side  $DC$  with a uniform velocity  $U$ .

Now let us calculate momentum flux through control volume.

Momentum flux entering through  $AD$

$$\begin{aligned} &= \int_0^\delta \text{momentum flux through strip of thickness } dy \\ &= \int_0^\delta \text{mass through strip} \times \text{velocity} = \int_0^\delta (\rho u b dy) \times u = \int_0^\delta \rho u^2 b dy \end{aligned}$$

$$\text{Momentum flux leaving the side } BC = \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u^2 b dy \right] \times \Delta x$$

Momentum flux entering the side  $DC$  = mass rate through  $DC$   $\times$  velocity

$$\begin{aligned} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u b dy \right] \times \Delta x \times U \quad (\because \text{Velocity} = U) \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u U b dy \right] \times \Delta x \end{aligned}$$

As  $U$  is constant and so it can be taken inside the differential and integral.

$\therefore$  Rate of change of momentum of the control volume

$$\begin{aligned} &= \text{Momentum flux through } BC - \text{Momentum flux through } AD \\ &\quad - \text{momentum flux through DC} \\ &= \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u^2 b dy \right] \times \Delta x - \int_0^\delta \rho u^2 b dy - \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u U b dy \right] \times \Delta x \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u^2 b dy \right] \times \Delta x - \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u U b dy \right] \times \Delta x \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \rho u^2 b dy - \int_0^\delta \rho u U b dy \right] \times \Delta x \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta (\rho u^2 b - \rho u U b) dy \right] \times \Delta x \\ &= \frac{\partial}{\partial x} \left[ \rho b \int_0^\delta (u^2 - u U) dy \right] \times \Delta x \end{aligned}$$

{For incompressible fluid  $\rho$  is constant}

$$= \rho b \frac{\partial}{\partial x} \left[ \int_0^\delta (u^2 - uU) dy \right] \times \Delta x \quad \dots(13.8)$$

Now the rate of change of momentum on the control volume  $ABCD$  must be equal to the total force on the control volume in the same direction according to the momentum principle. But for a flat plate  $\frac{\partial p}{\partial x} = 0$ , which means there is no external pressure force on the control volume. Also the force on the side  $DC$  is negligible as the velocity is constant and velocity gradient is zero approximately. The only external force acting on the control volume is the shear force acting on the side  $AB$  in the direction from  $B$  to  $A$  as shown in Fig. 13.4 (b). The value of this force is given by equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

$\therefore$  Total external force in the direction of rate of change of momentum

$$= -\tau_0 \times \Delta x \times b \quad \dots(13.9)$$

According to momentum principle, the two values given by equations (13.9) and (13.8) should be the same.

$$\therefore -\tau_0 \times \Delta x \times b = \rho b \frac{\partial}{\partial x} \left[ \int_0^\delta (u^2 - uU) dy \right] \times \Delta x$$

Cancelling  $\Delta x \times b$ , to both sides, we have

$$\begin{aligned} -\tau_0 &= \rho \frac{\partial}{\partial x} \left[ \int_0^\delta (u^2 - uU) dy \right] \\ \text{or } \tau_0 &= -\rho \frac{\partial}{\partial x} \left[ \int_0^\delta (u^2 - uU) dy \right] = \rho \frac{\partial}{\partial x} \left[ \int_0^\delta (uU - u^2) dy \right] \\ &= \rho \frac{\partial}{\partial x} \left[ \int_0^\delta U^2 \left( \frac{u}{U} - \frac{u^2}{U^2} \right) dy \right] = \rho U^2 \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u}{U} \right] dy \right] \\ \text{or } \frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u}{U} \right] dy \right] \end{aligned} \quad \dots(13.10)$$

In equation (13.10), the expression  $\int_0^\delta \frac{u}{U} \left[ 1 - \frac{u}{U} \right] dy$  is equal to momentum thickness  $\theta$ . Hence equation (13.10) is also written as

$$\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x} \quad \dots(13.11)$$

Equation (13.11) is known as **Von Karman momentum integral equation** for boundary layer flows.

This is applied to :

1. Laminar boundary layers,
2. Transition boundary layers, and
3. Turbulent boundary layer flows.

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For a given velocity profile in laminar zone, transition zone or turbulent zone of a boundary layer, the shear stress  $\tau_0$  is obtained from equation (13.10) or (13.11). Then drag force on a small distance  $\Delta x$  of the plate is obtained from equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

Then total drag on the plate of length  $L$  on one side is

$$F_D = \int_0^L \Delta F_D = \int_0^L \tau_0 \times b \times dx \quad \{\text{change } \Delta x = dx\}. \quad \dots(13.12)$$

**13.3.1 Local Co-efficient of Drag [ $C_D^*$ ].** It is defined as the ratio of the shear stress  $\tau_0$  to the quantity  $\frac{1}{2} \rho U^2$ . It is denoted by  $C_D^*$

Hence

$$C_D^* = \frac{\tau_0}{\frac{1}{2} \rho U^2}. \quad \dots(13.13)$$

**13.3.2 Average Co-efficient of Drag [ $C_D$ ].** It is defined as the ratio of the total drag force to the quantity  $\frac{1}{2} \rho A U^2$ . It is also called co-efficient of drag and is denoted by  $C_D$ .

Hence

$$C_D = \frac{F_D}{\frac{1}{2} \rho A U^2} \quad \dots(13.14)$$

where  $A$  = Area of the surface (or plate)

$U$  = Free-stream velocity

$\rho$  = Mass density of fluid.

**13.3.3 Boundary Conditions for the Velocity Profiles.** The followings are the boundary conditions which must be satisfied by any velocity profile, whether it is in laminar boundary layer zone, or in turbulent boundary layer zone :

1. At  $y = 0$ ,  $u = 0$  and  $\frac{du}{dy}$  has some finite value
2. At  $y = \delta$ ,  $u = U$
3. At  $y = \delta$ ,  $\frac{du}{dy} = 0$ .

**Problem 13.3** For the velocity profile for laminar boundary layer flows given as

$$\frac{u}{U} = 2(y/\delta) - (y/\delta)^2$$

find an expression for boundary layer thickness ( $\delta$ ), shear stress ( $\tau_0$ ) and co-efficient of drag ( $C_D$ ) in terms of Reynold number.

**Solution.** Given :

$$(i) \text{ The velocity distribution } \frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \quad \dots(i)$$

Substituting this value of  $\frac{u}{U}$  in equation (13.10), we get

$$\begin{aligned}
 \frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \right] = \frac{\partial}{\partial x} \left[ \int_0^\delta \left[ \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right] \left[ 1 - \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right) \right] dy \right] \\
 &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left[ \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right] \left[ 1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2} \right] dy \right] \\
 &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left[ \frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4} \right] dy \right] \\
 &= \frac{\partial}{\partial x} \int_0^\delta \left[ \frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4} \right] dy = \frac{\partial}{\partial x} \left[ \frac{2y^2}{2\delta} - \frac{5 \times y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4} \right]_0^\delta \\
 &= \frac{\partial}{\partial x} \left[ \frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4} \right] = \frac{\partial}{\partial x} \left[ \delta - \frac{5}{3}\delta + \delta - \frac{\delta}{5} \right] \\
 &= \frac{\partial}{\partial x} \left[ \frac{15\delta - 25\delta + 15\delta - 3\delta}{15} \right] = \frac{\partial}{\partial x} \left[ \frac{30\delta - 28\delta}{15} \right] = \frac{\partial}{\partial x} \left[ \frac{2\delta}{15} \right] = \frac{2}{15} \frac{\partial}{\partial x} [\delta] \\
 \therefore \quad \tau_0 &= \rho U^2 \times \frac{2}{15} \frac{\partial}{\partial x} [\delta] = \frac{2}{15} \rho U^2 \frac{\partial [\delta]}{\partial x} \quad \dots(13.15)
 \end{aligned}$$

The shear stress at the boundary in laminar flow is also given by Newton's law of viscosity as

$$\tau_0 = \mu \left( \frac{du}{dy} \right)_{y=0} \quad \dots(ii)$$

$$\begin{aligned}
 \text{But from equation (i),} \quad u &= U \left[ \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right] \\
 \therefore \quad \frac{du}{dy} &= U \left[ \frac{2}{\delta} - \frac{2y}{\delta^2} \right] \quad \{ \because U \text{ is constant} \} \\
 \therefore \quad \left( \frac{du}{dy} \right)_{y=0} &= U \left[ \frac{2}{\delta} - \frac{2 \times (0)}{\delta^2} \right] = \frac{2U}{\delta}
 \end{aligned}$$

Substituting this value in (ii), we get

$$\tau_0 = \mu \times \frac{2U}{\delta} = \frac{2\mu U}{\delta} \quad \dots(iii)$$

Equating the two values of  $\tau_0$  given by equation (13.15) and (iii)

$$\begin{aligned}
 \frac{2}{15} \rho U^2 \frac{\partial}{\partial x} [\delta] &= \frac{2\mu U}{\delta} \\
 \text{or} \quad \frac{\delta \partial}{\partial x} [\delta] &= \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U} \quad \text{or} \quad \delta \partial [\delta] = \frac{15\mu}{\rho U} \partial x
 \end{aligned}$$

As the boundary layer thickness ( $\delta$ ) is a function of  $x$  only.

Hence partial derivative can be changed to total derivative

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$$\therefore \delta d[\delta] = \frac{15\mu}{\rho U} dx$$

On integration, we get  $\frac{\delta^2}{2} = \frac{15\mu}{\rho U} x + C$   $\left\{ \frac{\mu}{\rho U} \text{ is constant} \right\}$   
 $x = 0, \delta = 0$  and hence  $C = 0$

$$\therefore \frac{\delta^2}{2} = \frac{15\mu x}{\rho U}$$

$$\therefore \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = \sqrt{\frac{30\mu x}{\rho U}} = 5.48 \sqrt{\frac{\mu x}{\rho U}} \quad \dots(13.16)$$

$$= 5.48 \sqrt{\frac{\mu x \times x}{\rho U \times x}} = 5.48 \sqrt{\frac{x^2}{R_{e_x}}} \quad \left\{ \because R_{e_x} = \frac{\rho U x}{\mu} \right\}$$

$$= 5.48 \frac{x}{\sqrt{R_{e_x}}} \quad \dots(13.17)$$

In equation (13.16),  $\mu$ ,  $\rho$  and  $U$  are constant and hence it is clear from this equation that thickness of laminar boundary layer is proportional to the square root of the distance from the leading edge. Equation (13.17) gives the thickness of laminar boundary layer in terms of Reynolds number.

### (ii) Shear stress ( $\tau_0$ ) in terms of Reynolds number

$$\text{From equation (iii), we have } \tau_0 = \frac{2\mu U}{\delta}$$

Substituting the value of  $\delta$  from equation (13.17), in the above equation, we get

$$\tau_0 = \frac{2\mu U}{5.48 \frac{x}{\sqrt{R_{e_x}}}} = \frac{2\mu U \sqrt{R_{e_x}}}{5.48 x} = 0.365 \frac{\mu U}{x} \sqrt{R_{e_x}}$$

### (iii) Co-efficient of Drag ( $C_D$ )

$$\text{From equation (13.14), we have } C_D = \frac{F_D}{\frac{1}{2} \rho A U^2}$$

where  $F_D$  is given by equation (13.12) as

$$\begin{aligned} F_D &= \int_0^L \tau_0 \times b \times dx = \int_0^L 0.365 \frac{\mu U}{x} \sqrt{R_{e_x}} \times b \times dx \\ &= 0.365 \int_0^L \frac{\mu U}{x} \sqrt{\frac{\rho U x}{\mu}} \times b \times dx \quad \left\{ \because R_{e_x} = \frac{\rho U x}{\mu} \right\} \\ &= 0.365 \int_0^L \mu U \sqrt{\frac{\rho U}{\mu}} \times \frac{1}{\sqrt{x}} \times b \times dx \end{aligned}$$

$$\begin{aligned}
 &= 0.365 \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2} dx = 0.365 \mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \left[ \frac{x^{1/2}}{\frac{1}{2}} \right]_0^L \\
 &= 0.365 \times 2\mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \sqrt{L} \\
 &= 0.73 b \mu U \sqrt{\frac{\rho UL}{\mu}} \\
 \therefore C_D &= \frac{0.73 b \mu U \sqrt{\frac{\rho UL}{\mu}}}{\frac{1}{2} \rho A U^2} \quad \dots(13.18)
 \end{aligned}$$

where  $A$  = Area of plate = Length of plate  $\times$  width =  $L \times b$

$$\begin{aligned}
 \therefore C_D &= \frac{0.73 b \mu U}{\frac{1}{2} \rho \times L \times b \times U^2} \sqrt{\frac{\rho UL}{\mu}} = \frac{1.46 \mu}{\rho L U} \sqrt{\frac{\rho UL}{\mu}} \\
 &= \frac{1.46 \sqrt{\mu}}{\sqrt{\rho UL}} = 1.46 \sqrt{\frac{\mu}{\rho UL}} = \frac{1.46}{\sqrt{R_{e_L}}} \quad \left\{ \because \sqrt{\frac{\mu}{\rho UL}} = \frac{1}{\sqrt{R_{e_L}}} \right\} \quad \dots(13.19)
 \end{aligned}$$

**Problem 13.4** For the velocity profile given in problem 13.3, find the thickness of boundary layer at the end of the plate and the drag force on one side of a plate 1 m long and 0.8 m wide when placed in water flowing with a velocity of 150 mm per second. Calculate the value of co-efficient of drag also. Take  $\mu$  for water = 0.01 poise.

**Solution.** Given :

Length of plate,

$$L = 1 \text{ m}$$

Width of plate,

$$b = 0.8 \text{ m}$$

Velocity of fluid (water),

$$U = 150 \text{ mm/s} = 0.15 \text{ m/s}$$

$\mu$  for water

$$= 0.01 \text{ poise} = \frac{0.01 \text{ Ns}}{10 \text{ m}^2} = 0.001 \frac{\text{Ns}}{\text{m}^2}$$

Reynold number at the end of the plate i.e., at a distance of 1 m from leading edge is given by

$$\begin{aligned}
 R_{e_L} &= \frac{\rho U L}{\mu} = 1000 \times \frac{0.15 \times 1.0}{0.001} \quad (\because \rho = 1000) \\
 &= \frac{1000 \times 0.15 \times 1.0}{0.001} = 150000
 \end{aligned}$$

(i) As laminar boundary layer exists upto Reynold number =  $2 \times 10^5$ . Hence this is the case of laminar boundary layer. Thickness of boundary layer at  $x = 1.0 \text{ m}$  is given by equation (13.17) as

$$\delta = 5.48 \frac{x}{\sqrt{R_{e_x}}} = \frac{5.48 \times 1.0}{\sqrt{150000}} = 0.01415 \text{ m} = 14.15 \text{ mm. Ans.}$$

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(ii) Drag force on one side of the plate is given by equation (13.18)

$$F_D = 0.73 b\mu U \sqrt{\frac{\rho UL}{\mu}}$$

$$= 0.73 \times 0.8 \times 0.001 \times 0.15 \times \sqrt{150000} \quad \left\{ \because \frac{\rho UL}{\mu} = R_{e_L} \right\}$$

$$= 0.0338 \text{ N. Ans.}$$

(iii) Co-efficient of drag,  $C_D$  is given by equation (13.19) as

$$C_D = \frac{1.46}{\sqrt{R_{e_L}}} = \frac{1.46}{\sqrt{150000}} = .00376. \text{ Ans.}$$

**Problem 13.5** For the velocity profile for laminar boundary layer  $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ .

Determine the boundary layer thickness, shear stress, drag force and co-efficient of drag in terms of Reynold number.

**Solution.** Given :

$$\text{Velocity distribution, } \frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\text{Using equation (13.10), we have } \frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \right]$$

Substituting the value of  $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$  in the above equation

$$\begin{aligned} \frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left[ \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right] \left[ 1 - \left\{ \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right\} \right] dy \right] \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{3y}{2\delta} - \frac{y^3}{2\delta^3} \right) \left( 1 - \frac{3y}{2\delta} + \frac{y^3}{2\delta^3} \right) dy \right] \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{3y}{2\delta} - \frac{9y^2}{4\delta^2} + \frac{3y^4}{4\delta^4} - \frac{y^3}{2\delta^3} + \frac{3y^4}{4\delta^4} - \frac{y^6}{4\delta^6} \right) dy \right] \\ &= \frac{\partial}{\partial x} \left[ \frac{3y^2}{2 \times 2\delta} - \frac{9y^3}{3 \times 4\delta^2} + \frac{3y^5}{5 \times 4\delta^4} - \frac{y^4}{4 \times 2\delta^3} + \frac{3y^5}{5 \times 4\delta^4} - \frac{y^7}{7 \times 4\delta^6} \right]_0^\delta \\ &= \frac{\partial}{\partial x} \left[ \frac{3\delta^2}{4\delta} - \frac{3\delta^3}{4\delta^2} + \frac{3}{20} \frac{\delta^5}{\delta^4} - \frac{1}{8} \frac{\delta^4}{\delta^3} + \frac{3}{20} \frac{\delta^5}{\delta^4} - \frac{1}{28} \frac{\delta^7}{\delta^6} \right] \\ &= \frac{\partial}{\partial x} \left[ \frac{3}{4} \delta - \frac{3}{4} \delta + \frac{3}{20} \delta - \frac{1}{8} \delta + \frac{3}{20} \delta - \frac{1}{28} \delta \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial x} \left[ \frac{6}{20} \delta - \frac{1}{8} \delta - \frac{1}{28} \delta \right] = \frac{\partial \delta}{\partial x} \left[ \frac{84 - 35 - 10}{280} \right] = \frac{39}{280} \frac{\partial \delta}{\partial x} \\
 \tau_0 &= \rho U^2 \times \frac{39}{280} \frac{\partial \delta}{\partial x} = \frac{39}{280} \rho U^2 \frac{\partial \delta}{\partial x} \quad \dots(13.20)
 \end{aligned}$$

Also the shear stress  $\tau_0$  is given by  $\tau_0 = \mu \left( \frac{du}{dy} \right)_{y=0}$ , where  $u = U \left[ \frac{3}{2} \frac{y}{\delta} - \frac{y^3}{2\delta^3} \right]$

$$\therefore \frac{du}{dy} = U \left[ \frac{3}{2\delta} - \frac{3y^2}{2\delta^3} \right]$$

$$\text{Hence } \left( \frac{du}{dy} \right)_{y=0} = U \left[ \frac{3}{2\delta} - \frac{3}{2\delta^3} \times 0 \right] = \frac{3U}{2\delta}$$

$$\therefore \tau_0 = \mu \left( \frac{du}{dy} \right)_{y=0} = \mu \frac{3U}{2\delta} = \frac{3}{2} \frac{\mu U}{\delta} \quad \dots(13.21)$$

Equating the two values of  $\tau_0$  given by equations (13.20) and (13.21)

$$\begin{aligned}
 \frac{39}{280} \rho U^2 \frac{\partial \delta}{\partial x} &= \frac{3}{2} \frac{\mu U}{\delta} \\
 \therefore \delta \partial \delta &= \frac{3}{2} \mu U \times \frac{280}{39} \times \frac{1}{\rho U^2} \partial x = \frac{420}{39} \frac{\mu}{\rho U} \partial x
 \end{aligned}$$

$$\text{Integrating, we get } \frac{\delta^2}{2} = \frac{420}{39} \frac{\mu}{\rho U} x + C$$

where  $x = 0, \delta = 0, \therefore C = 0$

$$\begin{aligned}
 \therefore \frac{\delta^2}{2} &= \frac{420}{39} \cdot \frac{\mu}{\rho U} x \\
 \text{or } \delta &= \sqrt{\frac{420 \times 2}{39} \frac{\mu}{\rho U} x} = 4.64 \sqrt{\frac{\mu x}{\rho U}} = 4.64 \sqrt{\frac{\mu x \times x}{\rho U x}} \\
 &= 4.64 \sqrt{\frac{\mu}{\rho U x}} x = \frac{4.64 x}{\sqrt{R_{e_x}}} \quad \dots(13.22)
 \end{aligned}$$

(i) **Shear Stress  $\tau_0$** . Substituting the value of  $\delta$  from equation (13.22) into equation (13.21), we get

$$\tau_0 = \frac{3}{2} \frac{\mu U}{\frac{4.64 x}{\sqrt{R_{e_x}}}} = \frac{3}{9.28} \frac{\mu U \sqrt{R_{e_x}}}{x} = 0.323 \frac{\mu U}{x} \sqrt{R_{e_x}}$$

(ii) **Drag force ( $F_D$ )**

Using equation (13.12), we get the drag force as

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.323 \frac{\mu U}{x} \sqrt{R_{e_x}} \times b \times dx$$

$$\begin{aligned}
&= 0.323 \int_0^L \frac{\mu U}{x} \sqrt{\frac{\rho U x}{\mu}} \times b \times dx = 0.323 \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L \frac{1}{\sqrt{x}} dx \\
&= 0.323 \mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2} dx \\
&= 0.323 \mu U \sqrt{\frac{\rho U}{\mu}} \times b \left[ \frac{x^{1/2}}{\frac{1}{2}} \right]_0^L = 0.323 \times 2 \mu U \sqrt{\frac{\rho U}{\mu}} \times b [\sqrt{L}] \\
&= 0.646 \mu U \sqrt{\frac{\rho U L}{\mu}} \times b
\end{aligned} \tag{13.23}$$

(iii) **Drag Co-efficient ( $C_D$ )**. Using equation (13.14), we get the value of  $C_D$  as

$$\begin{aligned}
C_D &= \frac{F_D}{\frac{1}{2} \rho A U^2}, \text{ where } A = b \times L \\
&= \frac{0.646 \mu U \sqrt{\frac{\rho U L}{\mu}} \times b}{\frac{1}{2} \rho \times b \times L \times U^2} = 0.646 \times 2 \times \frac{\mu}{\rho U L} \times \sqrt{\frac{\rho U L}{\mu}} = \frac{1.292}{\sqrt{\frac{\rho U L}{\mu}}} \\
&= \frac{1.292}{\sqrt{R_{e_L}}}. \quad \left\{ \because \sqrt{\frac{\rho U L}{\mu}} = \sqrt{R_{e_L}} \right\}
\end{aligned} \tag{13.24}$$

**Problem 13.6** For the velocity profile for laminar boundary layer

$$\frac{u}{U} = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4$$

obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in term of Reynold number.

**Solution.** Given :

$$(i) \text{ The velocity profile, } \frac{u}{U} = \frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4}$$

Using equation (13.10), we have

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \right]$$

Substituting the given velocity profile in the above equation

$$\begin{aligned}
\frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4} \right) \left( 1 - \left\{ \frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4} \right\} \right) dy \right] \\
&= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4} \right) \left( 1 - \frac{2y}{\delta} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4} \right) dy \right]
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{4y^4}{\delta^4} - \frac{2y^5}{\delta^5} - \frac{2y^3}{\delta^3} + \frac{4y^4}{\delta^4} - \frac{4y^6}{\delta^6} + \frac{2y^7}{\delta^7} + \frac{y^4}{\delta^4} - \frac{2y^5}{\delta^5} + \frac{2y^7}{\delta^7} - \frac{y^8}{\delta^8} \right) dy \right] \\
 &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{2y}{\delta} - \frac{4y^2}{\delta^2} - \frac{2y^3}{\delta^3} + \frac{9y^4}{\delta^4} - \frac{4y^5}{\delta^5} - \frac{4y^6}{\delta^6} + \frac{4y^7}{\delta^7} - \frac{y^8}{\delta^8} \right) dy \right] \\
 &= \frac{\partial}{\partial x} \left[ \frac{2y^2}{2\delta} - \frac{4y^3}{3\delta^2} - \frac{2y^4}{4\delta^3} + \frac{9y^5}{5\delta^4} - \frac{4y^6}{6\delta^5} - \frac{4y^7}{7\delta^6} + \frac{4y^8}{8\delta^7} - \frac{y^9}{9\delta^8} \right]_0 \\
 &= \frac{\partial}{\partial x} \left[ \delta - \frac{4}{3}\delta - \frac{1}{2}\delta + \frac{9}{5}\delta - \frac{2}{3}\delta - \frac{4}{7}\delta + \frac{1}{2}\delta - \frac{1}{9}\delta \right] \\
 &= \frac{\partial}{\partial x} \left[ \frac{315 - 420 + 63 \times 9 - 210 - 45 \times 4 - 35}{315} \right] \delta \\
 &= \frac{\partial}{\partial x} \left[ \frac{315 - 420 + 567 - 210 - 180 - 35}{315} \right] \delta \\
 &= \frac{\partial}{\partial x} \left[ \frac{882 - 845}{815} \right] \delta = \frac{\partial}{\partial x} \left[ \frac{37}{315} \right] \delta = \frac{37}{315} \frac{\partial \delta}{\partial x} \\
 \therefore \tau_0 &= \frac{37}{315} \rho U^2 \frac{\partial \delta}{\partial x} \quad \dots(13.25)
 \end{aligned}$$

Also shear stress is given by Newton's law of viscosity as

$$\tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}$$

$$\text{where } u = U \left[ \frac{2y}{\delta} - \frac{2y^2}{\delta^2} + \frac{y^4}{\delta^4} \right]$$

$$\begin{aligned}
 \therefore \left( \frac{\partial u}{\partial y} \right) &= U \left[ \frac{2}{\delta} - \frac{4y}{\delta^2} - \frac{4y^3}{\delta^4} \right] \\
 \therefore \left( \frac{\partial u}{\partial y} \right)_{y=0} &= U \left[ \frac{2}{\delta} - \frac{4}{\delta^2}(0) - \frac{4}{\delta^4}(0) \right] = \frac{2U}{\delta} \\
 \therefore \tau_0 &= \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \mu \times \frac{2U}{\delta} = \frac{2U\mu}{\delta} \quad \dots(13.26)
 \end{aligned}$$

Equating the two values of  $\tau_0$  given by equations (13.25) and (13.26)

$$\frac{37}{315} \rho U^2 \frac{\partial \delta}{\partial x} = \frac{2U\mu}{\delta} \quad \text{or} \quad \delta \frac{\partial \delta}{\partial x} = \frac{315}{37} \times \frac{2U\mu}{\rho U^2} \partial x = \frac{630}{37} \frac{\mu}{\rho U} \partial x$$

On integration, we get  $\frac{\delta^2}{2} = \frac{630}{37} \frac{\mu}{\rho U} x + C$ , where  $C$  = Constant of integration

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At  $x = 0$ ,  $\delta = 0$  and hence  $C = 0$

$$\begin{aligned}\therefore \frac{\delta^2}{2} &= \frac{630}{37} \frac{\mu}{\rho U} x \\ \therefore \delta &= \sqrt{\frac{630 \times 2}{37} \frac{\mu}{\rho U}} x = 5.84 \sqrt{\frac{\mu x}{\rho U}} \\ &= 5.84 \sqrt{\frac{\mu x \times x}{\rho U x}} = 5.84 \sqrt{\frac{\mu}{\rho U x}} \times x = \frac{5.84 x}{\sqrt{R_{e_x}}} \quad \dots(13.27)\end{aligned}$$

(ii) **Shear Stress ( $\tau_0$ )**. Substituting the value of  $\delta$  from (13.27) into (13.26)

$$\tau_0 = \frac{2U\mu}{\delta} = \frac{2U\mu}{\frac{5.84x}{\sqrt{R_{e_x}}}} = \frac{2U\mu}{5.84x} \sqrt{R_{e_x}} = 0.34 \frac{U\mu}{x} \sqrt{R_{e_x}}.$$

(iii) **Drag Force ( $F_D$ )** on one side of the plate :

Using equation (13.12), we get

$$\begin{aligned}F_D &= \int_0^L \tau_0 \times b \times dx = \int_0^L 0.34 \frac{U\mu}{x} \sqrt{R_{e_x}} \times b \times dx = \int_0^L 0.34 \frac{U\mu}{\mu} \frac{\sqrt{\rho U \mu}}{\mu} b dx \\ &= 0.34 U\mu \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2} dx = 0.34 U\mu \sqrt{\frac{\rho U}{\mu}} \times b \times \left[ \frac{x^{1/2}}{1/2} \right]_0^L \\ &= 0.34 \times 2U\mu \sqrt{\frac{\rho U}{\mu}} b \sqrt{L} = 0.68 b\mu U \sqrt{\frac{\rho UL}{\mu}} \quad \dots(13.28)\end{aligned}$$

(iv) **Drag Co-efficient ( $C_D$ )**

Using equation (13.14),  $C_D = \frac{F_D}{\frac{1}{2} \rho A U^2}$ , where  $A = b \times L$

$$\begin{aligned}&= \frac{0.68 \times b \times \mu U \times \sqrt{\frac{\rho UL}{\mu}}}{\frac{1}{2} \rho \times b \times L \times U^2} = 0.68 \times 2 \frac{\mu}{\rho UL} \times \sqrt{\frac{\rho UL}{\mu}} = 1.36 \times \frac{1}{\sqrt{\frac{\rho UL}{\mu}}} \\ &= 1.36 \times \frac{1}{\sqrt{R_{e_L}}}. \quad \dots(13.29)\end{aligned}$$

**Problem 13.7** For the velocity profile for laminar boundary flow  $\frac{u}{U} = \sin\left(\frac{\pi}{2} \frac{y}{\delta}\right)$ .

Obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in terms of Reynold number.

**Solution.** (i) The velocity profile is  $\frac{u}{U} = \sin\left(\frac{\pi}{2} \frac{y}{\delta}\right)$ .

Substituting this value in equation (13.10), we have

$$\begin{aligned}
 \frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \right] = \frac{\partial}{\partial x} \left[ \int_0^\delta \sin \left( \frac{\pi}{2} \frac{y}{\delta} \right) \left[ 1 - \sin \left( \frac{\pi}{2} \cdot \frac{y}{\delta} \right) \right] dy \right] \\
 &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left[ \sin \left( \frac{\pi}{2} \frac{y}{\delta} \right) - \sin^2 \left( \frac{\pi}{2} \frac{y}{\delta} \right) \right] dy \right] \\
 &= \frac{\partial}{\partial x} \left[ \left[ \frac{-\cos \frac{\pi y}{2\delta}}{\frac{\pi}{2\delta}} \right] - \left[ \frac{\frac{\pi y}{2\delta} \times \frac{1}{2}}{\frac{\pi}{2\delta}} - \frac{\sin 2 \left( \frac{\pi}{2} \frac{y}{\delta} \right)}{4 \times \frac{\pi}{2\delta}} \right] \right]_0^\delta \\
 &\quad \left\{ \because \int \sin^2 \left( \frac{\pi}{2} \frac{y}{\delta} \right) dy = \frac{\frac{\pi y}{2\delta} \times \frac{1}{2}}{\frac{\pi}{2\delta}} - \frac{\sin 2 \left( \frac{\pi}{2} \frac{y}{\delta} \right)}{4 \times \frac{\pi}{2\delta}} \right\} \\
 \therefore \frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \left( \frac{-\cos \frac{\pi}{2} \frac{\delta}{\delta}}{\frac{\pi}{2\delta}} + \frac{\cos \frac{\pi}{2} \times 0}{\frac{\pi}{2\delta}} \right) - \left[ \frac{\frac{\pi}{2} \frac{\delta}{\delta} \times \frac{1}{2}}{\frac{\pi}{2\delta}} - 0 \right] \right] \\
 &= \frac{\partial}{\partial x} \left[ \left( 0 + \frac{1}{\frac{\pi}{2\delta}} \right) - \frac{\left( \frac{\pi}{4} \right)}{\frac{\pi}{2\delta}} \right] = \frac{\partial}{\partial x} \left[ \frac{2\delta}{\pi} - \frac{\pi}{4} \times \frac{2\delta}{\pi} \right] \\
 &= \frac{\partial}{\partial x} \left[ \frac{2\delta}{\pi} - \frac{\delta}{2} \right] = \frac{\partial}{\partial x} \left[ \frac{4 - \pi}{2\pi} \right] \delta = \left( \frac{4 - \pi}{2\pi} \right) \frac{\partial \delta}{\partial x} \\
 \therefore \tau_0 &= \left( \frac{4 - \pi}{2\pi} \right) \rho U^2 \frac{\partial \delta}{\partial x} \quad \dots(13.30)
 \end{aligned}$$

$$\tau_0 \text{ is also equal} = \mu \left( \frac{du}{dy} \right)_{\text{at } y=0}$$

But

$$u = U \sin \left( \frac{\pi}{2} \frac{y}{\delta} \right)$$

$\therefore$

$$\left( \frac{du}{dy} \right) = U \cos \left( \frac{\pi}{2} \frac{y}{\delta} \right) \times \frac{\pi}{2\delta}$$

$$\left( \frac{du}{dy} \right)_{y=0} = U \times \frac{\pi}{2\delta} \cos \left( \frac{\pi}{2} \times \frac{0}{\delta} \right) = \frac{U\pi}{2\delta}$$

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$$\therefore \tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \frac{\mu U \pi}{2\delta} \quad \dots(13.31)$$

Equating the two values  $\tau_0$  given by equations (13.30) and (13.31)

$$\begin{aligned} \left( \frac{4-\pi}{2\pi} \right) \rho U^2 \frac{\partial \delta}{\partial x} &= \frac{\mu U \pi}{2\delta} \quad \text{or} \quad \delta \partial \delta = \frac{\mu U \pi}{2} \times \frac{2\pi}{4-\pi} \times \frac{1}{\rho U^2} \partial x \\ \therefore \delta \partial \delta &= \frac{\pi^2}{(4-\pi)} \frac{\mu U}{\rho U^2} \cdot \partial x = 11.4975 \frac{\mu}{\rho U} \partial x \end{aligned}$$

Integrating, we get  $\frac{\delta^2}{2} = 11.4975 \frac{\mu}{\rho U} x + C$

At  $x = 0, \delta = 0$  and hence  $C = 0$

$$\begin{aligned} \therefore \frac{\delta^2}{2} &= 11.4975 \frac{\mu}{\rho U} x \\ \therefore \delta &= \sqrt{2 \times 11.4975 \frac{\mu}{\rho U} x} = 4.795 \sqrt{\frac{\mu}{\rho U}} x \\ &= 4.795 \sqrt{\frac{\mu}{\rho U x}} = 4.795 \sqrt{\frac{\mu}{\rho U x}} \times x \\ &= \frac{4.795 x}{\sqrt{R_{e_x}}} \quad \dots(13.32) \end{aligned}$$

### (ii) Shear Stress ( $\tau_0$ )

From equation (13.31), 
$$\begin{aligned} \tau_0 &= \frac{\mu U \pi}{2\delta} = \frac{\mu U \pi}{2 \times 4.795 x} = \frac{\mu U \pi \sqrt{R_{e_x}}}{2 \times 4.795 x} \\ &= \frac{\pi}{2 \times 4.795} \frac{\mu U}{x} \sqrt{R_{e_x}} = 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}}. \end{aligned}$$

### (iii) Drag force ( $F_D$ ) on one side of the plate is given by equation (13.12)

$$\begin{aligned} F_D &= \int_0^L \tau_0 \times b \times dx = \int_0^L 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}} \times b \times dx = 0.327 \mu U \times b \int_0^L \frac{1}{x} \sqrt{\frac{\rho U x}{\mu}} dx \\ &= 0.327 \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \int_0^L x^{-1/2} dx = 0.327 \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \left[ \frac{x^{1/2}}{\frac{1}{2}} \right]_0^L \\ &= 0.327 \times 2 \times \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \times \sqrt{L} \\ &= 0.655 \times \mu U \times b \times \sqrt{\frac{\rho U L}{\mu}} \quad \dots(13.33) \end{aligned}$$

(iv) Co-efficient of drag,  $C_D$  is given by equation (13.14),

$$C_D = \frac{F_D}{\frac{1}{2} \rho A U^2}, \text{ where } A = b \times L$$

$$\begin{aligned} \therefore C_D &= \frac{0.655 \times \mu U \times b \times \sqrt{\frac{\rho UL}{\mu}}}{\frac{1}{2} \rho U^2 \times b \times L} = 0.655 \times 2 \times \frac{\mu}{\rho UL} \times \sqrt{\frac{\rho UL}{\mu}} \\ &= 1.31 \times \frac{1}{\sqrt{\frac{\rho UL}{\mu}}} = \frac{1.31}{\sqrt{R_{e_L}}} \end{aligned} \quad \dots(13.34)$$

Note.  $\int \sin^2 x ds = \left( \frac{x}{2} - \frac{\sin 2x}{4} \right)$  is used.

Table 13.1 shows the values of boundary layer thickness and co-efficients of drag in terms of Reynold number for various velocity distributions

**Table 13.1**

Velocity Distribution	$\delta$	$C_D$
1. $\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2$	$5.48 x/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
2. $\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3$	$4.64 x/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
3. $\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - 2 \left( \frac{y}{\delta} \right)^3 + \left( \frac{y}{\delta} \right)^4$	$5.84 x/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
4. $\frac{u}{U} = \sin \left( \frac{\pi}{2} \frac{y}{\delta} \right)$	$4.79 x/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$
5. Blasius's Solution	$4.91 x/\sqrt{R_{e_x}}$	$1.328/\sqrt{R_{e_L}}$

**Problem 13.8** For the velocity profile in laminar boundary layer as,

$$\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3$$

find the thickness of the boundary layer and the shear stress 1.5 m from the leading edge of a plate. The plate is 2 m long and 1.4 m wide and is placed in water which is moving with a velocity of 200 mm per second. Find the total drag force on the plate if  $\mu$  for water = .01 poise.

**Solution.** Given :

Velocity profile is  $\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3$

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Distance of  $x$  from leading edge,  $x = 1.5$  m

Length of plate,  $L = 2$  m

Width of plate,  $b = 1.4$  m

Velocity of plate,  $U = 200$  mm/s = 0.2 m/s

Viscosity of water,  $\mu = 0.01$  poise =  $\frac{0.01}{10} = 0.001$  Ns/m<sup>2</sup>

For the given velocity profile, thickness of boundary layer is given by equation (13.22) as

$$\delta = \frac{4.64 x}{\sqrt{R_{e_x}}}$$

$$\left[ \text{Here } R_{e_x} = \frac{\rho U x}{\mu} = 1000 \times \frac{0.2 \times 1.5}{0.001} = 300000 \right]$$

$$\delta = \frac{4.64 \times 1.5}{\sqrt{300000}} = 0.0127 \text{ m} = 12.7 \text{ mm. Ans.}$$

Shear stress ( $\tau_0$ ) is given by  $\tau_0 = 0.323 \frac{\mu U}{x} \sqrt{R_{e_x}}$

$$= 0.323 \times 0.001 \times \frac{0.2}{1.5} \times \sqrt{300000} = 0.0235 \text{ N/m}^2. \text{ Ans.}$$

Drag Force ( $F_D$ ) on one side of the plate is given by (13.23) as

$$\begin{aligned} F_D &= 0.646 \mu U \sqrt{\frac{\rho U L}{\mu}} \times b \\ &= 0.646 \times 0.001 \times 0.2 \times \sqrt{1000 \times \frac{0.2 \times 2.0}{0.001}} \times 1.4 \\ &= .646 \times 0.001 \times 0.2 \times \sqrt{400000} \times 1.4 = 0.1138 \text{ N} \\ \therefore \text{Total drag force} &= \text{Drag force on both sides of the plate} \\ &= 2 \times 0.1138 = 0.2276 \text{ N. Ans.} \end{aligned}$$

**Problem 13.9** Air is flowing over a smooth plate with a velocity of 10 m/s. The length of the plate is 1.2 m and width 0.8 m. If laminar boundary layer exists up to a value of  $R_e = 2 \times 10^5$ , find the maximum distance from the leading edge upto which laminar boundary layer exists. Find the maximum thickness of laminar boundary layer if the velocity profile is given by

$$\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2$$

Take kinematic viscosity for air = 0.15 stokes.

**Solution.** Given :

Velocity of air,  $U = 10$  m/s

Length of plate,  $L = 1.2$  m

Width of plate,  $b = 0.8$  m

Reynold number upto which laminar boundary exists =  $2 \times 10^5$

$\nu$  for air = 0.15 stokes =  $0.15 \times 10^{-4}$  m<sup>2</sup>/s

Reynold number  $R_{e_x} = \frac{\rho U x}{\mu} = \frac{U x}{v}$

If  $R_{e_x} = 2 \times 10^5$ , then  $x$  denotes the distance from leading edge upto which laminar boundary layer exists

$$\therefore 2 \times 10^5 = \frac{10 \times x}{0.15 \times 10^{-4}}$$

$$\therefore x = \frac{2 \times 10^5 \times 0.15 \times 10^{-4}}{10} = 0.30 \text{ m} = 300 \text{ mm. Ans.}$$

Maximum thickness of the laminar boundary for the velocity profile,  $\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2$  is given by equation (13.17) as

$$\delta = \frac{5.48 \times x}{\sqrt{R_{e_x}}} = \frac{5.48 \times 0.30}{\sqrt{2 \times 10^5}} = 0.00367 \text{ m} = 3.67 \text{ mm. Ans.}$$

**Problem 13.10** Air is flowing over a flat plate 500 mm long and 600 mm wide with a velocity of 4 m/s. The kinematic viscosity of air is given as  $0.15 \times 10^{-4} \text{ m}^2/\text{s}$ . Find (i) the boundary layer thickness at the end of the plate, (ii) Shear stress at 200 mm from the leading edge and (iii) drag force on one side of the plate. Take the velocity profile over the plate as  $\frac{u}{U} = \sin \left( \frac{\pi}{2} \cdot \frac{y}{\delta} \right)$  and density of air  $1.24 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Length of plate, } L = 500 \text{ mm} = 0.5 \text{ m}$$

$$\text{Width of plate, } b = 600 \text{ mm} = 0.6 \text{ m}$$

$$\text{Velocity of air, } U = 4 \text{ m/s}$$

$$\text{Kinematic viscosity, } v = 0.15 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\therefore \text{Mass density, } \rho = 1.24 \text{ kg/m}^3$$

For the velocity profile  $\frac{u}{U} = \sin \left( \frac{\pi}{2} \cdot \frac{y}{\delta} \right)$ , we have

(i) Boundary layer thickness at the end of the plate means value of  $\delta$  at  $x = 0.5 \text{ m}$ . First find Reynold number.

$$R_{e_x} = \frac{\rho U x}{\mu} = \frac{U x}{v} = \frac{4 \times 0.5}{0.15 \times 10^{-4}} = 1.33 \times 10^5.$$

Hence boundary layer is laminar over the entire length of the plate as Reynold number at the end of the plate is  $1.33 \times 10^5$ .

$\therefore \delta$  at  $x = 0.5 \text{ m}$  for the given velocity profile is given by equation (13.32) as

$$\delta = \frac{4.795x}{\sqrt{R_{e_x}}} = \frac{4.795 \times 0.5}{\sqrt{1.33 \times 10^5}} = 0.00656 \text{ m} = 6.56 \text{ mm. Ans.}$$

(ii) Shear stress at any distance from leading edge is given by  $\tau_0 = 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}}$

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$$\text{At } x = 200 \text{ mm} = 0.2 \text{ m}, R_{e_x} = \frac{U \times x}{v} = \frac{4 \times 0.2}{0.15 \times 10^{-4}} = 53333$$

$$\therefore \tau_0 = \frac{0.327 \times \mu \times 4 \times \sqrt{53333}}{0.2}$$

$$\begin{aligned} \text{But } \mu &= v \times \rho & \left\{ \because v = \frac{u}{\rho}, \therefore \mu = v \times \rho \right\} \\ &= 0.15 \times 10^{-4} \times 1.24 = 0.186 \times 10^{-4} \\ \therefore \tau_0 &= \frac{0.327 \times 0.186 \times 10^{-4} \times 4 \times \sqrt{53333}}{0.2} = \mathbf{0.02805 \text{ N/m}^2. \text{ Ans.}} \end{aligned}$$

(iii) Drag force on one side of the plate is given by equation (13.33)

$$\begin{aligned} F_D &= 0.655 \times \mu U \times b \times \sqrt{\frac{\rho UL}{\mu}} \\ &= 0.655 \times 0.186 \times 10^{-4} \times 4.0 \times 0.6 \times \sqrt{\frac{UL}{v}} & \left\{ \because v = \frac{\mu}{\rho} \right\} \\ &= 0.29234 \times 10^{-4} \times \sqrt{\frac{4 \times 0.5}{1.5 \times 10^{-4}}} = \mathbf{0.01086 \text{ N. Ans.}} \end{aligned}$$

**Problem 13.11** A thin plate is moving in still atmospheric air at a velocity of 5 m/s. The length of the plate is 0.6 m and width 0.5 m. Calculate (i) the thickness of the boundary layer at the end of the plate, and (ii) drag force on one side of the plate. Take density of air as 1.24 kg/m<sup>3</sup> and kinematic viscosity 0.15 stokes.

**Solution.** Given :

Velocity of plate,	$U = 5 \text{ m/s}$
Length of plate,	$L = 0.6 \text{ m}$
Width of plate,	$b = 0.5 \text{ m}$
Density of air,	$\rho = 1.24 \text{ kg/m}^3$
Kinematic viscosity,	$v = 0.15 \text{ stokes} = 0.15 \times 10^{-4} \text{ m}^2/\text{s}$
Reynold number,	$R_e = \frac{UL}{v} = \frac{5 \times 0.6}{0.15 \times 10^{-4}} = 200000$

As  $R_e$  is less than  $5 \times 10^5$ , hence boundary layer is laminar over the entire length of the plate.

(i) Thickness of boundary layer at the end of the plate by Blasius's solution is

$$\delta = \frac{4.91x}{\sqrt{R_{e_x}}} = \frac{4.91 L}{\sqrt{R_{e_1}}} = \frac{4.91 \times 0.6}{\sqrt{200000}} = .00658 \text{ m} = \mathbf{6.58 \text{ mm. Ans.}}$$

(ii) Drag force on one side of the plate is given by equation (13.14) as

$$C_D = \frac{F_D}{\frac{1}{2} \rho A U^2}$$

$$\therefore F_D = \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D$  from Blasius's solution,  $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{200000}} = 0.002969 \approx .00297$

$$\therefore F_D = \frac{1}{2} \times 1.24 \times 0.6 \times 0.5 \times 5^2 \times .002970 \quad \{ \because A = L \times b = 0.6 \times 0.5 \}$$

$$= \mathbf{0.01373 \text{ N. Ans.}}$$

**Note.** If no velocity profile is given in the numerical problem but boundary layer is laminar, then Blasius's solution is used.

**Problem 13.12** A plate of 600 mm length and 400 mm wide is immersed in a fluid of sp. gr. 0.9 and kinematic viscosity ( $\nu$ ) =  $10^{-4} \text{ m}^2/\text{s}$ . The fluid is moving with a velocity of 6 m/s. Determine (i) boundary layer thickness, (ii) shear stress at the end of the plate, and (iii) drag force on one side of the plate.

**Solution.** As no velocity profile is given in the above problem, hence Blasius's solution will be used.

Given : length of plate,	$L = 600 \text{ mm} = 0.60 \text{ m}$
Width of plate,	$b = 400 \text{ mm} = 0.40 \text{ m}$
Sp. gr. of fluid,	$S = 0.9$
∴ Density,	$\rho = 0.9 \times 1000 = 900 \text{ kg/m}^3$
Velocity of fluid,	$U = 6 \text{ m/s}$
Kinematic viscosity,	$\nu = 10^{-4} \text{ m}^2/\text{s}$
Reynold number,	$R_{e_L} = \frac{U \times L}{\nu} = \frac{6 \times 0.6}{10^{-4}} = 3.6 \times 10^4$

As  $R_{e_L}$  is less than  $5 \times 10^5$ , hence boundary layer is laminar over the entire length of the plate.

(i) Thickness of boundary layer at the end of the plate from Blasius's solution is

$$\delta = \frac{4.91 x}{\sqrt{R_{e_x}}}, \text{ where } x = 0.6 \text{ m and } R_{e_x} = 3.6 \times 10^4$$

$$= \frac{4.91 \times 0.6}{\sqrt{3.6 \times 10^4}} = 0.0155 \text{ m} = \mathbf{15.5 \text{ mm. Ans.}}$$

(ii) Shear stress at the end of the plate is

$$\tau_0 = 0.332 \frac{\rho U^2}{\sqrt{R_{e_L}}} = \frac{0.332 \times 900 \times 6^2}{\sqrt{3.6 \times 10^4}} = \mathbf{56.6 \text{ N/m}^2. \text{ Ans.}}$$

(iii) Drag force ( $F_D$ ) on one side of the plate is given by

$$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D$  from Blasius's solution is  $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{3.6 \times 10^4}} = 0.00699$

$$\therefore F_D = \frac{1}{2} \rho A U^2 \times C_D$$

$$= \frac{1}{2} \times 900 \times 0.6 \times 0.4 \times 6^2 \times .00699 \quad \{ \because A = L \times b = 0.6 \times .4 \}$$

$$= \mathbf{26.78 \text{ N. Ans.}}$$

### ► 13.4 TURBULENT BOUNDARY LAYER ON A FLAT PLATE

The thickness of the boundary layer, drag force on one side of the plate and co-efficient of drag due to turbulent boundary layer on a smooth plate at zero pressure gradient are determined as in case of laminar boundary layer provided the velocity profile is known. Blasius on the basis of experiments give the following velocity profile for turbulent boundary layer

$$\frac{u}{U} = \left( \frac{y}{\delta} \right)^n \quad \dots(13.35)$$

where  $n = \frac{1}{7}$  for  $R_e < 10^7$  but more than  $5 \times 10^5$

$$\therefore \frac{u}{U} = \left( \frac{y}{\delta} \right)^{1/7} \quad \dots(13.36)$$

Equation (13.36) is not applicable very near the boundary, where the thin laminar sub-layer of thickness  $\delta'$  exists. Here velocity distribution is influenced only by viscous effects.

$$\text{The value of } \tau_0 \text{ for flat plate is taken as } \tau_0 = 0.0225 \rho U^2 \left( \frac{\mu}{\rho \delta U} \right)^{1/4} \quad \dots(13.37)$$

**Problem 13.13** For the velocity profile for turbulent boundary layer  $\frac{u}{U} = \left( \frac{y}{\delta} \right)^{1/7}$ , obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in terms of Reynold number. Given the shear stress ( $\tau_0$ ) for turbulent boundary layer as

$$\tau_0 = 0.0225 \rho U^2 \left( \frac{\mu}{\rho U \delta} \right)^{1/4}.$$

**Solution.** Given :  $\frac{u}{U} = \left( \frac{y}{\delta} \right)^{1/7}$

(i) Substituting this value in Von Karman momentum integral equation (13.10),

$$\begin{aligned} \frac{\tau_0}{\rho U^2} &= \frac{\partial}{\partial x} \left[ \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \right] \\ &= \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{y}{\delta} \right)^{1/7} \left[ 1 - \left( \frac{y}{\delta} \right)^{1/7} \right] dy \right] = \frac{\partial}{\partial x} \left[ \int_0^\delta \left( \frac{y^{1/7}}{\delta^{1/7}} - \frac{y^{2/7}}{\delta^{2/7}} \right) dy \right] \\ &= \frac{\partial}{\partial x} \left[ \frac{y^{1/7+1}}{\left( \frac{1}{7} + 1 \right) \delta^{1/7}} - \frac{y^{2/7+1}}{\left( \frac{2}{7} + 1 \right) \delta^{2/7}} \right]_0^\delta \\ &= \frac{\partial}{\partial x} \left[ \frac{7}{8} \frac{y^{8/7}}{\delta^{1/7}} - \frac{7}{9} \frac{y^{9/7}}{\delta^{2/7}} \right]_0^\delta = \frac{\partial}{\partial x} \left[ \frac{7}{8} \frac{\delta^{8/7}}{\delta^{1/7}} - \frac{7}{9} \frac{\delta^{9/7}}{\delta^{2/7}} \right] \end{aligned}$$

$$= \frac{\partial}{\partial x} \left[ \frac{7}{8} \delta - \frac{7}{9} \delta \right] = \frac{\partial}{\partial x} \left[ \frac{63 - 56}{72} \right] \delta = \frac{\partial}{\partial x} \left[ \frac{7}{72} \right] \delta = \frac{7}{72} \frac{\partial \delta}{\partial x}$$

In the above expression, the integration limits should be from  $\delta'$  to  $\delta$ . But as the laminar sub-layer is very thin that is  $\delta'$  is very small. Hence the limits of integration are taken from 0 to  $\delta$ .

$$\text{Now } \tau_0 = \frac{7}{72} \rho U^2 \frac{\partial \delta}{\partial x} \quad \dots(13.38)$$

But the value of  $\tau_0$  for turbulent boundary layer is given,

$$\tau_0 = 0.0225 \rho U^2 \left( \frac{\mu}{\rho U \delta} \right)^{1/4} \quad \dots(13.39)$$

Equating the two values of  $\tau_0$  given by equations (13.38) and (13.39), we have

$$\frac{7}{72} \rho U^2 \frac{\partial \delta}{\partial x} = 0.0225 \rho U^2 \left( \frac{\mu}{\rho U \delta} \right)^{1/4}$$

$$\text{or } \frac{7}{72} \frac{\partial \delta}{\partial x} = 0.0225 \left( \frac{\mu}{\rho U} \right)^{1/4} \times \frac{1}{\delta^{1/4}} \quad \{ \text{cancelling } \rho U^2 \}$$

$$\text{or } \delta^{1/4} \partial \delta = 0.0225 \times \frac{72}{7} \times \left( \frac{\mu}{\rho U} \right)^{1/4} \partial x = 0.2314 \left( \frac{\mu}{\rho U} \right)^{1/4} \partial x.$$

$$\text{Integrating, we get } \frac{\delta^{1/4+1}}{\left( \frac{1}{4} + 1 \right)} = 0.2314 \left( \frac{\mu}{\rho U} \right)^{1/4} x + C$$

$$\text{or } \frac{4}{5} \times \delta^{5/4} = 0.2314 \left( \frac{\mu}{\rho U} \right)^{1/4} x + C$$

where  $C$  is constant of integration.

To determine the value of  $C$ , assume turbulent boundary layer starts from the leading edge, though in actual practice the turbulent boundary layer starts after the transition from laminar boundary layer. The laminar layer exists for a very short distance and hence this assumption will not affect the subsequent analysis.

Hence at  $x = 0, \delta = 0$  and so  $C = 0$

$$\therefore \frac{4}{5} \delta^{5/4} = 0.2314 \left( \frac{\mu}{\rho U} \right)^{1/4} x \text{ or } \delta^{5/4} = \frac{0.2314 \times 5}{4} \left( \frac{\mu}{\rho U} \right)^{1/4} x$$

$$\begin{aligned} \text{or } \delta &= \left[ \frac{0.2314 \times 5}{4} \left( \frac{\mu}{\rho U} \right)^{1/4} x \right]^{4/5} = \left( \frac{0.2314 \times 5}{4} \right)^{4/5} \left( \frac{\mu}{\rho U} \right)^{1/5} x^{4/5} \\ &= 0.37 \left( \frac{\mu}{\rho U} \right)^{1/5} x^{4/5} \end{aligned} \quad \dots(13.40)$$

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$$= 0.37 \left( \frac{\mu}{\rho U x} \right)^{1/5} x^{1/5} \times x^{4/5} = 0.37 \left( \frac{1}{R_{e_x}} \right)^{1/5} \times x = \frac{0.37 x}{\left( R_{e_x} \right)^{1/5}} \quad \dots(13.41)$$

From equation (13.40), it is clear that  $\delta$  varies as  $x^{4/5}$  in turbulent boundary layer while in case of laminar boundary layer  $\delta$  varies as  $\sqrt{x}$ .

(ii) **Shear Stress ( $\tau_0$ )** at any point from leading edge is given by equation (13.40) as

$$\tau_0 = 0.225 \rho U^2 \left( \frac{\mu}{\rho U \delta} \right)^{1/4}$$

Substituting the value of  $\delta$  from equation (13.40), we have

$$\begin{aligned} \tau_0 &= 0.0225 \rho U^2 \left( \frac{\mu}{\rho U \times 0.37 \times \left( \frac{\mu}{\rho U} \right)^{1/5} \times x^{4/5}} \right)^{1/4} \\ &= \frac{.0225 \times 2}{2} \rho U^2 \left( \frac{\mu^{4/5}}{0.37 \times (\rho U)^{4/5} \times x^{4/5}} \right)^{1/4} \\ &= .0225 \times 2 \times \frac{\rho U^2}{2} \times \frac{1}{(0.37)^{1/4}} \left( \frac{\mu}{\rho U x} \right)^{1/5} \\ &= 0.0577 \times \frac{\rho U^2}{2} \left( \frac{\mu}{\rho U x} \right)^{1/5} \end{aligned} \quad \dots(13.42)$$

(iii) **Drag force ( $F_D$ )** on one side of the plate is

$$\begin{aligned} F_D &= \int_0^L \tau_0 \times b \times dx = \int_0^L 0.0577 \times \frac{\rho U^2}{2} \times \left( \frac{\mu}{\rho U} \right)^{1/5} \frac{1}{x^{1/5}} \times b \times dx \\ &= 0.0577 \times \frac{\rho U^2}{2} \times \left( \frac{\mu}{\rho U} \right)^{1/5} \times b \int_0^L x^{-1/5} dx \\ &= .0577 \times \frac{\rho U^2}{2} \times \left( \frac{\mu}{\rho U} \right)^{1/5} \times b \times \left[ \frac{x^{4/5}}{4/5} \right]_0^L \\ &= .0577 \times \frac{5}{4} \times \frac{\rho U^2}{2} \left( \frac{\mu}{\rho U} \right)^{1/5} \times b \times L^{4/5} \\ &= 0.072 \times \frac{\rho U^2}{2} \times \left( \frac{\mu}{\rho U} \right)^{1/5} \times b \times L^{4/5} \end{aligned}$$

(iv) **Drag co-efficient,  $C_D$**  is given by

$$C_D = \frac{F_D}{\frac{1}{2} \rho A U^2}, \text{ where } A = L \times b$$

$$\begin{aligned}
 &= \frac{.072 \times \frac{\rho U^2}{2} \times \left( \frac{\mu}{\rho U} \right)^{1/5} \times b \times L^{4/5}}{\frac{\rho U^2}{2} \times b \times L} \\
 &= 0.072 \times \left( \frac{\mu}{\rho U} \right)^{1/5} \cdot \frac{1}{L^{1/5}} = 0.072 \left( \frac{\mu}{\rho U L} \right)^{1/5} \\
 &= \frac{.072}{R_{e_L}^{1/5}}
 \end{aligned} \quad \dots(13.43) \quad \left\{ \because R_{e_L} = \frac{\rho U L}{\mu} \right\}$$

This is valid for  $R_{e_L} > 5 \times 10^5$  but less than  $10^7$ .

### ► 13.5 ANALYSIS OF TURBULENT BOUNDARY LAYER

(a) If Reynold number is more than  $5 \times 10^5$  and less than  $10^7$  the thickness of boundary layer and drag co-efficient are given as :

$$\delta = \frac{0.37x}{(R_{e_x})^{1/5}} \text{ and } C_D = \frac{0.072}{(R_{e_L})^{1/5}} \quad \dots(13.44)$$

where  $x$  = Distance from the leading edge

$R_{e_x}$  = Reynold number for length  $x$

$R_{e_L}$  = Reynold number at the end of the plate.

(b) If Reynold number is more than  $10^7$  but less than  $10^9$ , Schlichting gave the empirical equation as

$$C_D = \frac{0.455}{(\log_{10} R_{e_L})^{2.58}} \quad \dots(13.44A)$$

### ► 13.6 TOTAL DRAG ON A FLAT PLATE DUE TO LAMINAR AND TURBULENT BOUNDARY LAYER

Consider the flow over a flat plate as shown in Fig. 13.5.

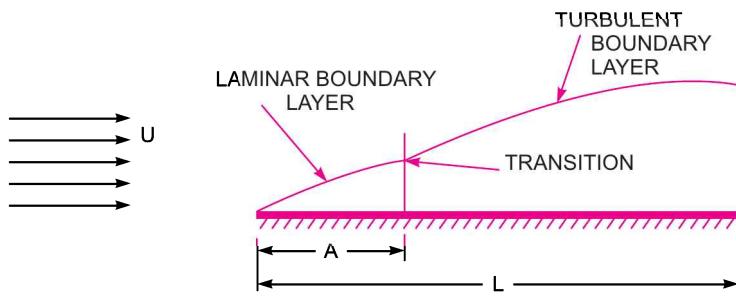


Fig. 13.5 Drag due to laminar and turbulent boundary layer.

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Let

$L$  = Total length of the plate,  $b$  = Width of plate,

$A$  = Length of laminar boundary layer

If the length of transition region is assumed negligible, then

$L - A$  = Length of turbulent boundary layer.

We have obtained the drag on a flat plate for the laminar as well as turbulent boundary layer on the assumption that turbulent boundary layer starts from the leading edge. This assumption is valid only when the length of laminar boundary layer is negligible. But if the length of laminar boundary layer is not negligible, then the total drag on the plate due to laminar and turbulent boundary layer is calculated as :

(1) Find the length from the leading edge upto which laminar boundary layer exists. This is done by equating  $5 \times 10^5 = \frac{Ux}{v}$ . The value of  $x$  gives the length of laminar boundary layer. Let this length is equal to  $A$ .

(2) Find drag using Blasius solution for laminar boundary layer for length  $A$ .

(3) Find drag due to turbulent boundary layer for the whole length of the plate.

(4) Find the drag due to turbulent boundary layer for a length  $A$  only

Then total drag on the plate

$$\begin{aligned} &= \text{Drag given by (2)} + \text{Drag given by (3)} - \text{Drag given by (4)} \\ &= \text{Drag due to laminar boundary layer for length } A \\ &\quad + \text{Drag due to turbulent boundary layer for length } L \\ &\quad - \text{Drag due to turbulent boundary layer for length } A. \end{aligned} \quad \dots(13.45)$$

**Problem 13.14 (S.I. Units).** Determine the thickness of the boundary layer at the trailing edge of smooth plate of length 4 m and of width 1.5 m, when the plate is moving with a velocity of 4 m/s in stationary air. Take kinematic viscosity of air as  $1.5 \times 10^{-5} \text{ m}^2/\text{s}$ .

**Solution.** Given :

Length of plate,  $L = 4 \text{ m}$

Width of plate,  $b = 1.5 \text{ m}$

Velocity of plate,  $U = 4 \text{ m/s}$

Kinematic viscosity,  $v = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$

$$\text{Reynold number, } R_{e_L} = \frac{U \times L}{v} = \frac{4.0 \times 4.0}{1.5 \times 10^{-5}} = 10.66 \times 10^5$$

As the Reynold number is more than  $5 \times 10^5$  and hence the boundary layer at the trailing edge is turbulent.

The boundary layer thickness for turbulent boundary layer is given by equation (13.44) as

$$\delta = \frac{0.37x}{\left(R_{e_x}\right)^{1/5}} \quad | \text{ Here } x = L \text{ and } R_{e_x} = R_{e_L}$$

$$= \frac{0.37 \times 4.0}{\left(10.66 \times 10^5\right)^{1/5}} = 0.0921 \text{ m} = 92.1 \text{ mm. Ans.}$$

**Problem 13.15** In Problem 13.14, determine the total drag on one side of the plate assuming that (i) the boundary layer is laminar over the entire length of the plate and (ii) the boundary layer is turbulent from the very beginning. Take  $\rho$  for air =  $1.226 \text{ kg/m}^3$ .

**Solution.** The data of problem 13.14,

$$L = 4 \text{ m}, b = 1.5 \text{ m}, U = 4 \text{ m/s},$$

$$\nu = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$$

$$R_{e_L} = 10.66 \times 10^5 \text{ and } \rho = 1.226 \text{ kg/m}^3.$$

(i) When the boundary layer is laminar over the entire length, the value of  $C_D$  is given by Blasius's solution as

$$C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{10.66 \times 10^5}} = .001286$$

∴ Drag force ( $F_D$ ) on one side of the plate is

$$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

$$\text{where } A = b \times L = 1.5 \times 4 = 6.0 \text{ m}^2$$

$$= \frac{1}{2} \times 1.226 \times 6.0 \times 4^2 \times .001286 = \mathbf{0.0757 \text{ N. Ans.}}$$

(ii) When the boundary layer is turbulent from the very beginning, the value of co-efficient of drag,  $C_D$  is given by equation (13.43) as

$$C_D = \frac{0.072}{(R_{e_L})^{1/5}} = \frac{0.072}{(10.66 \times 10^5)^{1/5}} = .00448$$

$$\begin{aligned} \therefore \text{Drag force, } F_D &= \frac{1}{2} \rho A U^2 \times C_D \\ &= \frac{1}{2} \times 1.226 \times 6.0 \times 4^2 \times .00448 \quad \{ \because A = b \times L = 1.5 \times 4 = 6 \text{ m}^2 \} \\ &= \mathbf{0.2637 \text{ N. Ans.}} \end{aligned}$$

**Problem 13.16** Water is flowing over a thin smooth plate of length 4 m and width 2 m at a velocity of 1.0 m/s. If the boundary layer flow changes from laminar to turbulent at a Reynold number  $5 \times 10^5$ , find (i) the distance from leading edge upto which boundary layer is laminar, (ii) the thickness of the boundary layer at the transition point, and (iii) the drag force on one side of the plate. Take viscosity of water  $\mu = 9.81 \times 10^{-4} \text{ Ns/m}^2$ .

**Solution.** Given :

$$\text{Length of plate, } L = 4 \text{ m}$$

$$\text{Width of plate, } b = 2 \text{ m}$$

$$\text{Velocity of flow, } U = 1.0 \text{ m/s}$$

$$\text{Reynold number for laminar boundary layer} = 5 \times 10^5$$

$$\text{Viscosity of water, } \mu = 9.81 \times 10^{-4} \frac{\text{Ns}}{\text{m}^2}$$

(i) Let the distance from leading edge upto which laminar boundary layer exists =  $x$

$$\therefore 5 \times 10^5 = \frac{\rho U x}{\mu} = 1000 \times \frac{1.0 \times x}{9.81 \times 10^{-4}} \quad (\because \rho = 1000)$$

$$\therefore x = \frac{5 \times 10^5 \times 9.81 \times 10^{-4}}{1000} = 0.4900 \text{ m} = \mathbf{490 \text{ mm. Ans.}}$$

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(ii) Thickness of boundary layer at the point where the boundary layer changes from laminar to turbulent i.e., at Reynold number =  $5 \times 10^5$ , is given by Blasius's solution as

$$\delta = \frac{4.91 \times x}{\sqrt{R_{e_x}}} \quad | \text{ Here } x = 49 \text{ cm} = 0.49 \text{ m}, R_{e_x} = 5 \times 10^5$$

$$\delta = \frac{4.91 \times 0.49}{\sqrt{5 \times 10^5}} = 0.0034 \text{ m} = 3.4 \text{ mm. Ans.}$$

(iii) Drag force on the plate on one side

= Drag due to laminar boundary layer + Drag due to turbulent boundary.

(a) Drag due to laminar boundary layer (i.e., from E to F)

$$F_{EF} = \frac{1}{2} \rho A U^2 \times C_D \quad \dots(i)$$

where  $C_D$  is given by Blasius solution for laminar boundary layer as

$$C_D = \frac{1.328}{\sqrt{R_{e_x}}}, \text{ where for } EF, R_{e_x} = 5 \times 10^5$$

$$= \frac{1.328}{\sqrt{5 \times 10^5}} = 0.001878$$

$$A = \text{Area of plate upto laminar boundary layer} \\ = 0.49 \times b = 0.49 \times 2 = 0.98 \text{ m}^2$$

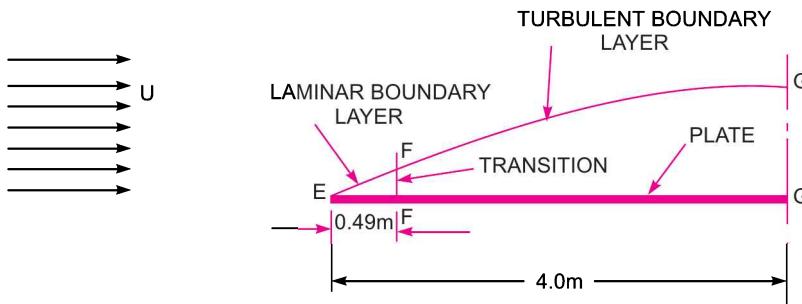


Fig. 13.6 (a)

Substituting the value of  $C_D$  and  $A$  in equation (i), we get

$$F_{EF} = \frac{1}{2} \times 1000 \times 0.98 \times 1.0^2 \times .001878 = 0.92 \text{ N.} \quad \dots(ii)$$

(b) Drag force due to turbulent boundary layer from F to G

$$\begin{aligned} &= \text{Drag force due to turbulent boundary layer from } E \text{ to } G \\ &\quad - \text{Drag force due to turbulent flow from } E \text{ to } F \\ &= (F_{EG})_{\text{turb.}} - (F_{EF})_{\text{turb.}} \end{aligned}$$

Now

$$(F_{FG})_{\text{turb.}} = \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D$  from equation (13.44) is  $C_D = \frac{0.072}{(R_{e_L})^{1/5}}$

But

$$R_{eL} = \frac{\rho UL}{\mu} = 1000 \times \frac{1.0 \times 4.0}{9.81 \times 10^{-4}} = 40.77 \times 10^5$$

∴

$$C_D = \frac{0.072}{(40.77 \times 10^5)^{1/5}} = 0.00343$$

∴

$$(F_{EG})_{turb.} = \frac{1}{2} \rho A U^2 \times C_D = \frac{1}{2} \times 1000 \times (4 \times 2) \times 1^2 \times .00343 = 13.72 \text{ N}$$

Also

$$(F_{EF})_{turb.} = \frac{1}{2} \rho A_{EF} \times U^2 \times C_D$$

where  $A_{EF}$  = Area of plate upto  $EF = EF \times b = 0.49 \times 2 = 0.98 \text{ m}^2$

and

$$C_D = \frac{0.072}{(R_{EF})^{1/5}} = \frac{0.072}{(5 \times 10^5)^{1/5}} = .00522$$

$$(F_{EF})_{turb.} = \frac{1}{2} \times 1000 \times 0.98 \times 1^2 \times .00522 = 2.557 \text{ N}$$

∴ Drag force due to turbulent boundary layer from  $F$  to  $G$

$$= (F_{EG})_{turb.} - (F_{EF})_{turb.} = 13.72 - 2.557 = 11.163 \text{ N}$$

∴ Drag force on the plate on one side

$$\begin{aligned} &= \text{Drag force due to laminar boundary layer upto } F \\ &\quad + \text{Drag due to turbulent boundary layer from } F \text{ to } G \\ &= 0.92 + 11.163 = 12.083 \text{ N. Ans.} \end{aligned}$$

**Problem 13.16 (A)** Air flows at 10 m/s past a smooth rectangular flat plate 0.3 m wide and 3 m long. Assuming that the turbulence level in the oncoming stream is low and that transition occurs at  $R_e = 5 \times 10^5$ , calculate ratio of total drag when the flow is parallel to the length of the plate to the value when the flow is parallel to the width.  
(R.G.P.V., Bhopal S 2001)

**Solution.** Given :

$$U = 10 \text{ m/s} ; b = 0.3 \text{ m} ; L = 3 \text{ m} ;$$

Reynolds number for laminar B.L. =  $5 \times 10^5$ .

The kinematic viscosity of air and density of air may be assumed as their values are not given in the question. Take  $\rho = 1.24 \text{ kg/m}^3$  and  $v = 0.15 \text{ stoke}$

$$\therefore \rho = 1.24 \text{ kg/m}^3 \text{ and } v = 0.15 \text{ stoke} = 0.15 \times 10^{-4} \text{ m}^2/\text{s.}$$

(i) **Drag when flow is parallel to the length of the plate**

Let  $x$  = the distance from leading edge upto which laminar boundary exists

$$\therefore 5 \times 10^5 = \frac{\rho \times U \times x}{\mu} = \frac{U \times x}{v} = \frac{10 \times x}{0.15 \times 10^{-4}}$$

$$\therefore x = \frac{5 \times 10^5 \times 0.15 \times 10^{-4}}{10} = 0.75 \text{ m}$$

Now the drag force on the plate on one side

$$= \text{Drag due to laminar boundary layer} + \text{Drag due to turbulent boundary layer} \dots (i)$$

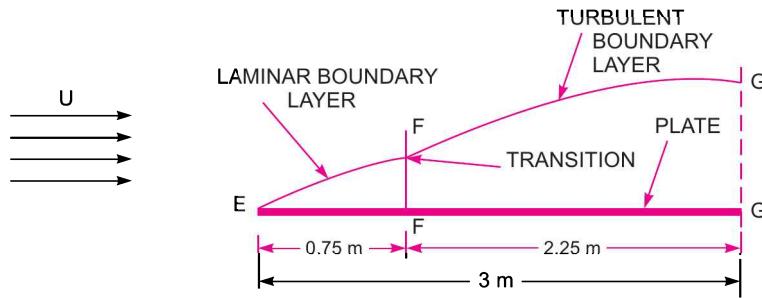


Fig. 13.6 (b)

(a) Drag due to laminar boundary layer (i.e., from E to F)

$$F_{EF} = \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D$  is given by Blasius solution for laminar boundary layer as

$$\begin{aligned} C_D &= \frac{1.328}{\sqrt{R_{ex}}}, \text{ where } R_{ex} = 5 \times 10^5 \\ &= \frac{1.328}{\sqrt{5 \times 10^5}} = 0.001878 \end{aligned}$$

$$\begin{aligned} A &= \text{Area of plate upto laminar boundary layer} \\ &= 0.75 \times b = 0.75 \times 0.3 = 0.225 \text{ m}^2 \\ \rho &= 1.24 \text{ kg/m}^3 \end{aligned}$$

$$\therefore F_{EF} = \frac{1}{2} \times 1.24 \times 0.225 \times 10^2 \times 0.001878 = 0.0262 \text{ N}$$

(b) Drag force due to turbulent boundary layer from F to G

$$\begin{aligned} &= \text{Drag force due to turbulent boundary layer from } E \text{ to } G \\ &\quad - \text{Drag force due to turbulent B.L. from } E \text{ to } F \\ &= (F_{EG})_{\text{turb.}} - (F_{EF})_{\text{turb.}} \end{aligned}$$

$$\text{Now } (F_{EG})_{\text{turb.}} = \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D$  for turbulent boundary layer is given by equation (13.44) as

$$C_D = \frac{0.072}{(R_{e_L})^{1/5}}$$

$$\text{But } R_{e_L} = \frac{U \times L}{v} = \frac{10 \times 3}{0.15 \times 10^{-4}} = 20 \times 10^5$$

$$\therefore C_D = \frac{0.072}{(20 \times 10^5)^{1/5}} = 0.00395$$

$$\begin{aligned} \therefore (F_{EG})_{\text{turb.}} &= \frac{1}{2} \rho A U^2 \times C_D = \frac{1}{2} \times 1.24 \times (3 \times 0.3) \times 10^2 \times 0.00395 \\ &= \mathbf{0.2204 \text{ N}} \end{aligned}$$

Now

$$(F_{EF})_{\text{turb.}} = \frac{1}{2} \rho \times A_{EF} \times U^2 \times C_D$$

where  $A_{EF}$  = Area of plate upto  $EF = EF \times b = 0.75 \times 0.3 = 0.225 \text{ m}^2$

and  $C_D = \frac{0.072}{[(R_e)_{EF}]^{1/5}} = \frac{0.072}{(5 \times 10^5)^{1/5}} = 0.00522$

$$\therefore (F_{EF})_{\text{turb.}} = \frac{1}{2} \times 1.24 \times 0.225 \times 10^2 \times 0.00522 = 0.0728 \text{ N}$$

$\therefore$  Drag force due to turbulent boundary layer from  $F$  to  $G$

$$= (F_{EG})_{\text{turb.}} - (F_{EF})_{\text{turb.}} = 0.2204 - 0.0728 = 0.1476 \text{ N}$$

$\therefore$  Total drag force when flow is parallel to the length of the plate

$$\begin{aligned} &= \text{Drag due to laminar boundary layer upto } F \\ &\quad + \text{Drag due to turbulent boundary layer from } F \text{ to } G \\ &= 0.0262 + 0.1476 = 0.1738 \text{ N} \end{aligned} \quad \dots(A)$$

### (ii) Drag when flow is parallel to the width of the plate

We have already calculated that upto the length of 0.75 m from the leading edge, the boundary layer is laminar. As the width of the plate is only 0.3 m, hence when flow is parallel to the width of the plate, only laminar boundary layer will be formed.

$\therefore$  Drag force on the plate

$$= \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D$  from Blasius solution for laminar boundary layer is given as

$$C_D = \frac{1.328}{\sqrt{R_{e_x}}}, \text{ here } x = \text{width of plate} = 0.3 \text{ m hence}$$

$$R_{e_x} = \frac{U \times x}{v} = \frac{10 \times 0.3}{0.15 \times 10^{-4}} = 2 \times 10^5$$

$$= \frac{1.328}{\sqrt{2 \times 10^5}} = 0.00297$$

$A$  = Area of plate upto width (0.3 m) =  $3 \times 0.3 = 0.9$

$$\rho = 1.24 \text{ kg/m}^3$$

$\therefore$  Total drag on the plate

$$\begin{aligned} &= \frac{1}{2} \times 1.24 \times 0.9 \times 10^2 \times 0.00297 \\ &= 0.1657 \text{ N} \end{aligned} \quad \dots(B)$$

$\therefore$  Ratio of two total drags given by equations (A) and (B) becomes as

$$\frac{\text{Total drag when flow is parallel to the length of the plate}}{\text{Total drag when flow is parallel to the width of the plate}} = \frac{\text{Equation (A)}}{\text{Equation (B)}} = \frac{0.1738}{0.1657} = 1.05. \text{ Ans.}$$

**Problem 13.17** Oil with a free-stream velocity of 2 m/s flows over a thin plate 2 m wide and 2 m long. Calculate the boundary layer thickness and the shear stress at the trailing end point and determine the total surface resistance of the plate. Take specific gravity as 0.86 and kinematic viscosity as  $10^{-5} \text{ m}^2/\text{s}$ .

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**Solution.** Given :

Free-stream velocity of oil,  $U = 2 \text{ m/s}$

Width of plate,  $b = 2 \text{ m}$

Length of plate,  $L = 2 \text{ m}$

$\therefore$  Area of plate,  $A = b \times L = 2 \times 2 = 4 \text{ m}^2$

Specific gravity of oil,  $S = 0.86$

$\therefore$  Density of oil,  $\rho = 0.86 \times 1000 = 860 \text{ kg/m}^3$

Kinematic viscosity,  $v = 10^{-5} \text{ m}^2/\text{s}$

Now the Reynold number at the trailing end,

$$R_{e_L} = \frac{UL}{v} = \frac{2 \times 2}{10^{-5}} = 4 \times 10^5.$$

Since  $R_{e_L}$  is less than  $5 \times 10^5$ , the boundary layer is laminar over the entire length of the plate.

$\therefore$  Thickness of boundary layer at the end of the plate from Blasius's solution is,

$$\delta = \frac{4.91 \times L}{\sqrt{R_{e_L}}} = \frac{4.91 \times 2.0}{\sqrt{4 \times 10^5}} = 0.0155 \text{ m} = 15.5 \text{ mm. Ans.}$$

Shear stress at the end of the plate is,  $\tau_0 = 0.332 \times = 1.805 \text{ N/m}^2$ . Ans.

Surface resistance on one side of the plate is given by

$$F_D = \frac{1}{2} \rho A U^2 \times C_D$$

where  $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{4 \times 10^5}} = 0.0021$

$$\therefore F_D = \frac{1}{2} \times 860 \times 4.0 \times 2^2 \times .0021 = 14.44 \text{ N}$$

$$\therefore \text{Total resistance} = 2 \times F_D = 2 \times 14.44 = 28.88 \text{ N. Ans.}$$

### ► 13.7 SEPARATION OF BOUNDARY LAYER

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free-stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking to the solid body if it cannot provide kinetic energy to overcome the resistance offered by the solid body. In other words, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

**13.7.1 Effect of Pressure Gradient on Boundary Layer Separation.** The effect of pressure gradient  $\left(\frac{dp}{dx}\right)$  on boundary layer separation can be explained by considering the flow over a

curved surface  $ABCSD$  as shown in Fig. 13.7. In the region  $ABC$  of the curved surface, the area of flow decreases and hence velocity increases. This means that flow gets accelerated in this region. Due to the increase of the velocity, the pressure decreases in the direction of the flow and hence pressure gradient  $\frac{dp}{dx}$  is negative in this region. As long as  $\frac{dp}{dx} < 0$ , the entire boundary layer moves forward as shown in Fig. 13.7.

**Region  $CSD$  of the curved surface.** The pressure is minimum at the point  $C$ . Along the region  $CSD$  of the curved surface, the area of flow increases and hence velocity of flow along the direction of fluid decreases. Due to decrease of velocity, the pressure increases in the direction of flow and hence pressure gradient  $\frac{dp}{dx}$  is positive or  $\frac{dp}{dx} > 0$ . Thus in the region  $CSD$ , the pressure gradient is positive and velocity of fluid layer along the direction of flow decreases. As explained in the Art. 13.7, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Thus the combined effect of positive pressure gradient and surface resistance reduce the momentum of the fluid is unable to the surface. A stage comes, when the momentum of the fluid is unable to overcome the surface resistance and the boundary layer starts separating from the surface at the point  $S$ . Downstream the point  $S$ , the flow is taking place in reverse direction and the velocity gradient becomes negative.

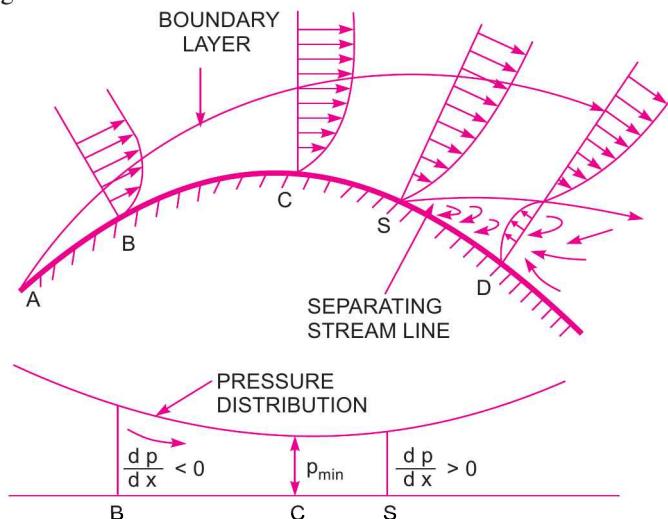


Fig. 13.7 Effect of pressure gradient on boundary layer separation.

Thus the positive pressure gradient helps in separating the boundary layer.

**13.7.2 Location of Separation Point.** The separation point  $S$  is determined from the condition,

$$\left( \frac{\partial u}{\partial y} \right)_{y=0} = 0 \quad \dots(13.46)$$

For a given velocity profile, it can be determined whether the boundary layer has separated, or on the verge of separation or will not separate from the following conditions :

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1. If  $\left(\frac{\partial u}{\partial y}\right)_{y=0}$  is negative ... the flow has separated.
2. If  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$  ... the flow is on the verge of separation.
3. If  $\left(\frac{\partial u}{\partial y}\right)_{y=0}$  is positive ... the flow will not separate or flow will remain attached with the surface.

**Problem 13.18** For the following velocity profiles, determine whether the flow has separated or on the verge of separation or will attach with the surface :

$$(i) \frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3, \quad (ii) \frac{u}{U} = 2 \left( \frac{y}{\delta} \right)^2 - \left( \frac{y}{\delta} \right)^3,$$

$$(iii) \frac{u}{U} = -2 \left( \frac{y}{\delta} \right) + \left( \frac{y}{\delta} \right)^2.$$

**Solution.** Given :

### 1st Velocity Profile

$$\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3 \quad \text{or} \quad u = \frac{3U}{2} \left( \frac{y}{\delta} \right) - \frac{U}{2} \left( \frac{y}{\delta} \right)^3$$

Differentiating w.r.t.  $y$ , the above equation becomes,

$$\frac{\partial u}{\partial y} = \frac{3U}{2} \times \frac{1}{\delta} - \frac{U}{2} \times 3 \left( \frac{y}{\delta} \right)^2 \times \frac{1}{\delta}$$

$$\text{At } y = 0, \left( \frac{\partial u}{\partial y} \right)_{y=0} = \frac{3U}{2\delta} - \frac{3U}{2} \left( \frac{0}{\delta} \right)^2 \times \frac{1}{\delta} = \frac{3U}{2\delta}.$$

As  $\left( \frac{\partial u}{\partial y} \right)_{y=0}$  is positive. Hence flow will not separate or flow will remain attached with the surface.

### 2nd Velocity Profile

$$\frac{u}{U} = 2 \left( \frac{y}{\delta} \right)^2 - \left( \frac{y}{\delta} \right)^3$$

$$\therefore u = 2U \left( \frac{y}{\delta} \right)^2 - U \left( \frac{y}{\delta} \right)^3$$

$$\therefore \frac{\partial u}{\partial y} = 2U \times 2 \left( \frac{y}{\delta} \right) \times \frac{1}{\delta} - U \times 3 \left( \frac{y}{\delta} \right)^2 \times \frac{1}{\delta}$$

$$\text{at } y = 0, \left( \frac{\partial u}{\partial y} \right)_{y=0} = 2U \times 2 \left( \frac{0}{\delta} \right) \times \frac{1}{\delta} - U \times 3 \left( \frac{0}{\delta} \right)^2 \times \frac{1}{\delta} = 0$$

As  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ , the flow is on the verge of separation. **Ans.**

### 3rd Velocity Profile

$$\begin{aligned}\frac{u}{U} &= -2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2 \\ \therefore u &= -2U\left(\frac{y}{\delta}\right) + U\left(\frac{y}{\delta}\right)^2 \\ \therefore \frac{\partial u}{\partial y} &= -2U\left(\frac{1}{\delta}\right) + 2U\left(\frac{y}{\delta}\right) \times \frac{1}{\delta}\end{aligned}$$

at  $y = 0, \left(\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{2U}{\delta} + 2U\left(\frac{0}{\delta}\right) \times \frac{1}{\delta} = -\frac{2U}{\delta}$

As  $\left(\frac{\partial u}{\partial y}\right)_{y=0}$  is negative the flow has separated. **Ans.**

**13.7.3 Methods of Preventing the Separation of Boundary Layer.** When the boundary layer separates from the surface as shown in Fig. 13.7 at point *S*, a certain portion adjacent to the surface has a back flow and eddies are continuously formed in this region and hence continuous loss of energy takes place. Thus separation of boundary layer is undesirable and attempts should be made to avoid separation by various methods. The following are the methods for preventing the separation of boundary layer :

1. Suction of the slow moving fluid by a suction slot.
2. Supplying additional energy from a blower.
3. Providing a bypass in the slotted wing.
4. Rotating boundary in the direction of flow.
5. Providing small divergence in a diffuser.
6. Providing guide-blades in a bend.
7. Providing a trip-wire ring in the laminar region for the flow over a sphere.

### HIGHLIGHTS

1. When a solid body is immersed in a flowing fluid, there is a narrow region of the fluid in the neighbourhood of the solid body, where the velocity of fluid varies from zero to free-stream velocity. This narrow region of fluid is called boundary layer.
2. The boundary layer is called laminar boundary layer if the Reynold number of the flow defined as

$$R_e = \frac{U \times x}{v} \text{ is less than } 5 \times 10^5$$

where  $U$  = Free-stream velocity of flow,  $x$  = Distance from leading edge,  
and  $v$  = Kinematic viscosity of fluid.

3. If the Reynold number is more than  $5 \times 10^5$ , the boundary layer is called turbulent boundary layer.

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4. The distance from the surface of the solid body in the direction perpendicular to flow, where the velocity of fluid is approximately equal to 0.99 times the free-stream velocity is called boundary layer thickness and is denoted by  $\delta$ . For different zones,  $\delta$  is represented as

$\delta_{\text{lam.}}$  = Thickness of laminar boundary layer

$\delta_{\text{tur.}}$  = Thickness of turbulent boundary layer

$\delta'$  = Thickness of laminar sub-layer.

5. Displacement thickness ( $\delta^*$ ) is given by  $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ .

6. Momentum thickness ( $\theta$ ) is given by  $\theta = \int_0^\delta \frac{u}{U} \left[1 - \frac{u}{U}\right] dy$ .

7. Energy thickness ( $\delta^{**}$ ) is given by  $\delta^{**} = \int_0^\delta \frac{u}{U} \left[1 - \frac{u^2}{U^2}\right] dy$ .

8. Von Karman momentum integral equation is given as  $\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x}$

where  $\theta = \int_0^\delta \frac{u}{U} \left[1 - \frac{u}{U}\right] dy$ ,  $\tau_0$  = shear stress at surface.

This equation is applicable to laminar, transition and turbulent boundary layer flows.

9. Thickness of laminar boundary layer and co-efficient of drag from Blasius's solution is given as

$$\delta = \frac{4.91 x}{\sqrt{R_{e_x}}}$$

where  $R_{e_x}$  = Reynold number,  $C_D = \frac{1.328}{\sqrt{R_{e_L}}}$

10. Velocity profile for turbulent boundary layer is  $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$

This equation is not valid very near the boundary, where laminar sub-layer exists.

11. The shear stress at the boundary for turbulent boundary layer over a flat plate is given as

$$\tau_0 = 0.0225 \rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}$$

12. Total drag on a flat plate due to laminar and turbulent boundary layer flows = Drag due to laminar boundary layer upto distance  $x$  + Drag due to turbulent boundary layer for length  $L$

– Drag due to turbulent boundary layer for length  $x$ .

$$\left[ \text{where } x \text{ is given by } 5 \times 10^5 = \frac{Ux}{v} \right]$$

13. If the pressure gradient is positive, the boundary layer separates from the surface and back flow and eddies formation take place due to which a great loss of energy occur.

14. The conditions for separation, attached flow and detached flow are :

$$(i) \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \dots \text{condition for separation} \quad (ii) \left(\frac{\partial u}{\partial y}\right)_{y=0} = \text{positive} \dots \text{condition for attached flow}$$

$$(iii) \left(\frac{\partial u}{\partial y}\right)_{y=0} = \text{negative} \dots \text{condition for detached flow.}$$

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. What do you understand by the terms boundary layer, and boundary layer theory ?
  2. Define : laminar boundary layer, turbulent boundary layer, laminar sub-layer and boundary layer thickness.
  3. Define displacement thickness. Derive an expression for the displacement thickness.
  4. Prove that the momentum thickness and energy thickness for boundary layer flows are given by
- $$\theta = \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u}{U} \right] dy \quad \text{and} \quad \delta^{**} = \int_0^\delta \frac{u}{U} \left[ 1 - \frac{u^2}{U} \right] dy.$$
5. Obtain an expression for the boundary shear stress in terms of momentum thickness.
  6. Obtain Von Karman momentum integral equation.
  7. What are the boundary conditions that must be satisfied by a given velocity profile in laminar boundary layer flows ?
  8. How will you find the drag on a flat plate due to laminar and turbulent boundary layers ?
  9. What do you mean by separation of boundary layer ? What is the effect of pressure gradient on boundary layer separation ?
  10. How will you determine whether a boundary layer flow is attached flow, detached flow or on the verge of separation ?
  11. What are the different methods of preventing the separation of boundary layers ?
  12. What is meant by boundary layer ? Why does it increase with distance from the upstream edge ?
  13. Define the terms : boundary layer, boundary layer thickness, drag, lift and momentum thickness.
  14. What do you mean by boundary layer separation ? What is the effect of pressure gradient on boundary layer separation ?
- (R.G.P.V., Bhopal S, 2001)

### (B) NUMERICAL PROBLEMS

1. (a) Find the ratios of displacement thickness to momentum thickness and momentum thickness to energy thickness for the velocity distribution in the boundary layer given by

$$\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2$$

where  $u$  = Velocity in boundary layer at a distance  $y$

$U$  = Free-stream velocity

$\delta$  = Boundary layer thickness

[Ans. 2.5, 7/11]

- (b) Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by

$$\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2.$$

(Delhi University, December, 2002)

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2. For the velocity profile in laminar boundary layer given as  $\frac{u}{U} = \frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3$ , find the thickness of the boundary layer and shear stress 1.8 m from the leading edge of a plate. The plate is 2.5 m long and 1.5 m wide and is placed in water which is moving with a velocity of 15 cm per second. Find the drag on one side of the plate if the viscosity of water = 0.01 poise. [Ans. 1.6 cm, 0.0014 N/m<sup>2</sup>, 0.0889 N]
3. Air is flowing over a smooth plate with a velocity of 8 m/s. The length of the plate is 1.5 m and width 1 m. If the laminar boundary exists upto a value of Reynold number =  $5 \times 10^5$ , find the maximum distance from the leading edge upto which laminar boundary layer exists. Find the maximum thickness of laminar boundary layer if the velocity profile is given by

$$\frac{u}{U} = (y/\delta) - (y/\delta)^2. \text{ Take } v \text{ for air} = 0.15 \text{ stokes. [Ans. } 0.9375 \text{ m, } 7.26 \text{ mm}]$$

4. If in Problem 3, the velocity profile over the plate is given as  $\frac{u}{U} = \sin\left(\frac{\pi}{2} \times \frac{y}{\delta}\right)$  and density of air as 1.24 kg/m<sup>3</sup>, find : (i) maximum thickness of the laminar boundary layer, (ii) shear stress at 20 cm from the leading edge and (iii) drag force on one side of the plate assuming the laminar boundary layer over the entire length of the plate. [Ans. (i) 0.635 cm, (ii) 0.099 N/m<sup>2</sup>, (iii) 0.0871 N]
5. A thin plate is moving in still atmospheric air at a velocity of 4 m/s. The length of the plate is 0.5 m and width 0.4 m. Calculate the (i) thickness of the boundary layer at the end of the plate and (ii) drag force on one side of the plate. Take density of air as 1.25 kg/m<sup>3</sup> and kinematic viscosity 0.15 stokes. [Ans. (i) 0.672 cm (ii) 0.00728 N]
6. Find the frictional drag on one side of the plate 200 mm wide and 500 mm long placed longitudinally in a stream of crude oil (specific gravity = 0.925, kinematic viscosity = 0.9 stoke) flowing with undisturbed velocity of 5 m/s. Also find the thickness of boundary layer and the shear stress at the trailing edge of the plate. [Ans. 9.34 N, 14.75 mm]
7. A smooth flat plate of length 5 m and width 2 m is moving with a velocity of 4 m/s in stationary air of density as 1.25 kg/m<sup>3</sup> and kinematic viscosity  $1.5 \times 10^{-5}$  m<sup>2</sup>/s. Determine thickness of the boundary layer at the trailing edge of the smooth plate. Find the total drag on one side of the plate assuming that the boundary layer is turbulent from the very beginning. [Ans. 110 mm, 0.43 N]
8. Water is flowing over a thin smooth plate of length 4.5 m and width 2.5 m at a velocity of 0.9 m/s. If the boundary layer flow changes from laminar to turbulent at a Reynold number  $5 \times 10^5$ , find (i) the distance from leading edge upto, which boundary layer is laminar, (ii) thickness of the boundary layer at the transition point, and (iii) the drag force on-one side of the plate. Take viscosity of water as 0.01 poise. [Ans. (i) 555 mm (ii) 3.85 mm (iii) 13.75 N]
9. For the velocity profile given below, state whether the boundary layer has separated or on the verge of separation or will remain attached with the surface :

$$(i) \frac{u}{U} = 2(y/\delta) - (y/\delta)^2$$

$$(ii) \frac{u}{U} = -2(y/\delta) + \frac{1}{2}(y/\delta)^3 \text{ and}$$

$$(iii) \frac{u}{U} = \frac{3}{2}(y/\delta)^2 + \frac{1}{2}(y/\delta)^3.$$

[Ans. (i) Remain attached (ii) has separated (iii) on the verge of separation]

10. Oil with a free-stream velocity of 1.5 m/s flow over a thin plate 1.4 m wide and 2.2 m long. Calculate the boundary layer thickness and the shear stress at the trailing end point and determine the total surface resistance of the plate. Take specific gravity of oil as 0.80 and kinematic viscosity as 0.1 stoke.

[Ans. 1.88 cm, 1.04 N/cm<sup>2</sup>, 12.8 N]

11. (a) For the velocity profile  $\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^2$ . Calculate the co-efficient of drag in terms of Reynolds number.  
(b) A thin smooth plate of 0.3 m width and 1.0 m length moves at 4 m/s viscosity in still atmospheric air of density  $1.20 \text{ kg/m}^3$  and kinematic viscosity of  $1.49 \times 10^{-5} \text{ m}^2/\text{s}$ . Calculate the drag force on the plate.  
[Ans. 0.00716 N]
12. Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by,

$$\frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2 \text{ where } \delta = \text{boundary layer thickness.} \quad \left[ \text{Ans. } \frac{\delta}{3}; \frac{2}{15}\delta; \frac{22}{105}\delta \right]$$



# 14

## CHAPTER

# FORCES ON SUB-MERGED BODIES



### ► 14.1 INTRODUCTION

When a fluid is flowing over a stationary body, a force is exerted by the fluid on the body. Similarly, when a body is moving in a stationary fluid, a force is exerted by the fluid on the body. Also when the body and fluid both are moving at different velocities, a force is exerted by the fluid on the body. Some of the examples of the fluids flowing over stationary bodies or bodies moving in a stationary fluid are :

1. Flow of air over buildings,
2. Flow of water over bridges,
3. Submarines, ships, airplanes and automobiles moving through water or air.

### ► 14.2 FORCE EXERTED BY A FLOWING FLUID ON A STATIONARY BODY

Consider a body held stationary in a real fluid, which is flowing at a uniform velocity  $U$  as shown in Fig. 14.1.

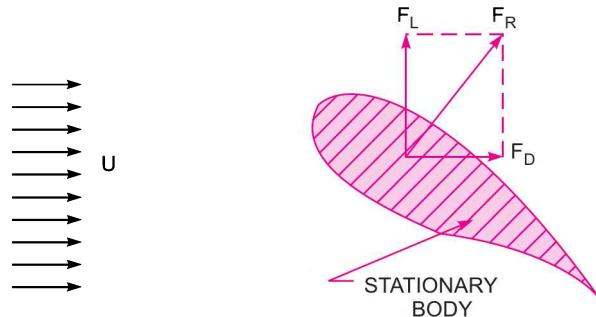


Fig. 14.1 Force on a stationary body.

The fluid will exert a force on the stationary body. The total force ( $F_R$ ) exerted by the fluid on the body is perpendicular to the surface of the body. Thus the total force is inclined to the direction of motion. The total force can be resolved in two components, one in the direction of motion and other perpendicular to the direction of motion.

**14.2.1 Drag.** The component of the total force ( $F_R$ ) in the direction of motion is called 'drag'. This component is denoted by  $F_D$ . Thus drag is the force exerted by the fluid in the direction of motion.

**14.2.2 Lift.** The component of the total force ( $F_R$ ) in the direction perpendicular to the direction of motion is known as 'lift'. This is denoted by  $F_L$ . Thus lift is the force exerted by the fluid in the direction perpendicular to the direction of motion. Lift force occurs only when the axis of the body is inclined to the direction of fluid flow. If the axis of the body is parallel to the direction of fluid flow, lift force is zero. In that case only drag force acts.

If the fluid is assumed ideal and the body is symmetrical such as a sphere or cylinder, both the drag and lift will be zero.

### ► 14.3 EXPRESSION FOR DRAG AND LIFT

Consider an arbitrary shaped solid body placed in a real fluid, which is flowing with a uniform velocity  $U$  in a horizontal direction as shown in Fig. 14.2. Consider a small elemental area  $dA$  on the surface of the body. The forces acting on the surface area  $dA$  are :

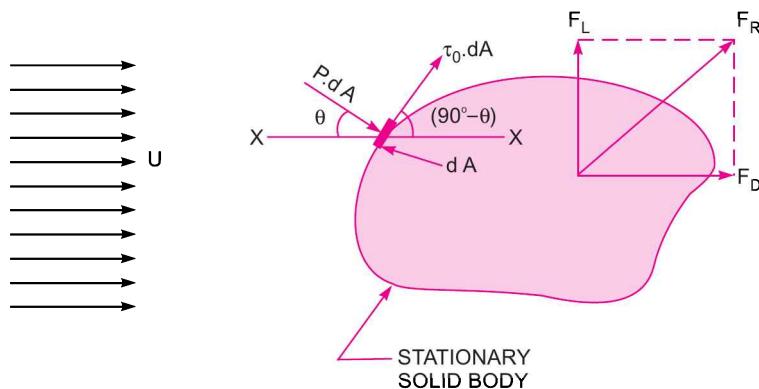


Fig. 14.2 Drag and lift.

1. Pressure force equal to  $p \times dA$ , acting perpendicular to the surface and
2. Shear force equal to  $\tau_0 \times dA$ , acting along the tangential direction to the surface.

Let  $\theta$  = Angle made by pressure force with horizontal direction.

(a) **Drag Force ( $F_D$ )**. The drag force on elemental area

$$\begin{aligned} &= \text{Force due to pressure in the direction of fluid motion} \\ &\quad + \text{Force due to shear stress in the direction of fluid motion.} \end{aligned}$$

$$= pdA \cos \theta + \tau_0 dA \cos (90^\circ - \theta) = pdA \cos \theta + \tau_0 dA \sin \theta$$

∴ Total drag,

$$\begin{aligned} F_D &= \text{Summation of } pdA \cos \theta + \text{Summation of } \tau_0 dA \sin \theta \\ &= \int p \cos \theta dA + \int \tau_0 \sin \theta dA. \end{aligned} \quad \dots(14.1)$$

The term  $\int p \cos \theta dA$  is called the pressure drag or form drag while the term  $\int \tau_0 \sin \theta dA$  is called the friction drag or skin drag or shear drag.

(b) **Lift Force ( $F_L$ )**. The lift force on elemental area

$$\begin{aligned} &= \text{Force due to pressure in the direction perpendicular to the direction of motion} \\ &\quad + \text{Force due to shear stress in the direction perpendicular to the direction of motion.} \end{aligned}$$

$$= -pdA \sin \theta + \tau_0 dA \sin (90^\circ - \theta) = -pdA \sin \theta + \tau_0 dA \cos \theta$$

Negative sign is taken with pressure force as it is acting in the downward direction while shear force is acting vertically up.

$$\therefore \text{Total lift, } F_L = \int \tau_0 dA \cos \theta - \int pdA \sin \theta \\ = \int \tau_0 \cos \theta dA - \int p \sin \theta dA \quad \dots(14.2)$$

The drag and lift for a body moving in a fluid of density  $\rho$ , at a uniform velocity  $U$  are calculated mathematically, as

$$F_D = C_D A \frac{\rho U^2}{2} \quad \dots(14.3)$$

$$F_L = C_L A \frac{\rho U^2}{2} \quad \dots(14.4)$$

where  $C_D$  = Co-efficient of drag,

$C_L$  = Co-efficient of lift,

$A$  = Area of the body which is the projected area of the body perpendicular to the direction of flow

Or

= Largest projected area of the immersed body.

$$\text{Then resultant force on the body, } F_R = \sqrt{F_D^2 + F_L^2} \quad \dots(14.5)$$

The equations (14.3) and (14.4) which give the mathematical expression for drag and lift are derived by the method of dimensional analysis.

**14.3.1 Dimensional Analysis of Drag and Lift.** In the chapter of Dimensional and Model Analysis it is shown in problem 12.6 that the force exerted by a fluid on a supersonic plane is given by :

$$F = \rho L^2 U^2 \phi \left[ \frac{\mu}{\rho UL} \cdot \frac{K}{\rho U^2} \right] \quad \dots(i)$$

Also in problem 12.7, it is shown that the force exerted by a fluid on a partially sub-merged body is given by :

$$F = \rho L^2 U^2 \phi \left[ \frac{\mu}{\rho UL} \cdot \frac{Lg}{U^2} \right] \quad \dots(ii)$$

Thus the general expression for the force exerted by a fluid (air or water) on a body (completely sub-merged or partially sub-merged) is given as

$$F = \rho L^2 U^2 \phi \left[ \frac{\mu}{\rho UL} \cdot \frac{Lg}{U^2} \cdot \frac{K}{\rho U^2} \right] \quad \dots(14.6)$$

where  $L$  = Length of body,  $U$  = Velocity of body,

$\mu$  = Viscosity of fluid,  $\rho$  = Density of fluid

$F$  = Force exerted,  $k$  = Bulk modulus of fluid,

$g$  = Acceleration due to gravity.

If the body is completely sub-merged in the fluid, the force exerted by the fluid on the body due to gravitational effect is negligible. Hence the non-dimensional term containing 'g' in equation (14.6),

i.e.,  $\frac{Lg}{U^2}$  is neglected. If the velocity of the body is comparable with velocity of sound, the effect due to

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compressibility is to be considered. But if the ratio of the velocity of the body to the velocity of the sound is less than 0.3, the force exerted by the fluid on the body due to compressibility is negligible. Hence the non-dimensional term in equation (14.6) containing  $K$  can be neglected. Then the force exerted by fluid on the body is given as

$$F = \rho L^2 U^2 \phi \left[ \frac{\mu}{\rho UL} \right] = \rho L^2 U^2 \phi \left[ \frac{\rho UL}{\mu} \right]$$

where  $\frac{\rho UL}{\mu}$  = Reynolds number =  $R_e$

$$F = \rho L^2 U^2 \phi [R_e]. \quad \dots(14.7)$$

Now  $F$  is the total force exerted by the fluid on the body. The total force is having two components, one in the direction of motion called drag force and other component in the direction perpendicular to the direction of motion, called lift force.

The two components of  $F$  are expressed as

$$F_D = \frac{\rho L^2 U^2}{2} \times C_D$$

where  $C_D$  is a function of  $R_e$  and is called co-efficient of drag

$$= C_D A \frac{\rho U^2}{2} \quad \{ \because L^2 = \text{Area} = A \}$$

And

$$F_L = \frac{\rho L^2 U^2}{2} \times C_L$$

where  $C_L$  is a function of  $R_e$  and is called co-efficient of lift

$$= C_L \cdot A \frac{\rho U^2}{2}.$$

**Problem 14.1** A flat plate  $1.5 \text{ m} \times 1.5 \text{ m}$  moves at  $50 \text{ km/hour}$  in stationary air of density  $1.15 \text{ kg/m}^3$ . If the co-efficients of drag and lift are 0.15 and 0.75 respectively, determine :

- (i) The lift force,
- (ii) The drag force,
- (iii) The resultant force, and
- (iv) The power required to keep the plate in motion.

**Solution.** Given :

Area of the plate,  $A = 1.5 \times 1.5 = 2.25 \text{ m}^2$

Velocity of the plate,  $U = 50 \text{ km/hr} = \frac{50 \times 1000}{60 \times 60} \text{ m/s} = 13.89 \text{ m/s}$

Density of air  $\rho = 1.15 \text{ kg/m}^3$

Co-efficient of drag,  $C_D = 0.15$

Co-efficient of lift,  $C_L = 0.75$

(i) **Lift Force ( $F_L$ )**. Using equation (14.4),

$$F_L = C_L A \times \frac{\rho U^2}{2} = 0.75 \times 2.25 \times \frac{1.15 \times 13.89^2}{2} \text{ N} = \mathbf{187.20 \text{ N. Ans.}}$$

(ii) **Drag Force ( $F_D$ )**. Using equation (14.3),

$$F_D = C_D \times A \times \frac{\rho U^2}{2} = 0.15 \times 2.25 \times \frac{1.15 \times 13.89^2}{2} \text{ N} = \mathbf{37.44 \text{ N. Ans.}}$$

(iii) **Resultant Force ( $F_R$ )**. Using equation (14.5),

$$\begin{aligned} F_R &= \sqrt{F_D^2 + F_L^2} = \sqrt{37.44^2 + 187.20^2} \text{ N} \\ &= \sqrt{1400 + 35025} = \mathbf{190.85 \text{ N. Ans.}} \end{aligned}$$

(iv) **Power Required to keep the Plate in Motion**

$$\begin{aligned} P &= \frac{\text{Force in the direction of motion} \times \text{Velocity}}{1000} \text{ kW} \\ &= \frac{F_D \times U}{1000} = \frac{37.425 \times 13.89}{1000} \text{ kW} = \mathbf{0.519 \text{ kW. Ans.}} \end{aligned}$$

**Problem 14.2** Experiments were conducted in a wind tunnel with a wind speed of 50 km/hour on a flat plate of size 2 m long and 1 m wide. The density of air is 1.15 kg/m<sup>3</sup>. The co-efficients of lift and drag are 0.75 and 0.15 respectively. Determine :

- |   |   |
|---|---|
| (i) the lift force,<br>(iii) the resultant force,<br>(v) power exerted by air on the plate. | (ii) the drag force,<br>(iv) direction of resultant force and |
|---|---|

**Solution.** Given :

Area of plate,

$$A = 2 \times 1 = 2 \text{ m}^2$$

Velocity of air,

$$U = 50 \text{ km/hr} = \frac{50 \times 1000}{60 \times 60} \text{ m/s} = 13.89 \text{ m/s}$$

Density of air,

$$\rho = 1.15 \text{ kg/m}^3$$

Value of

$$C_D = 0.15 \text{ and } C_L = 0.75$$

(i) **Lift force ( $F_L$ )**

Using equation (14.4),

$$\begin{aligned} F_L &= C_L \times A \times \rho \times U^2/2 \\ &= 0.75 \times 2 \times 1.15 \times 13.89^2/2 = \mathbf{166.404 \text{ N. Ans.}} \end{aligned}$$

(ii) **Drag force ( $F_D$ )**

Using equation (14.3),

$$\begin{aligned} F_D &= C_D \times A \times \rho \times U^2/2 \\ &= 0.15 \times 2 \times 1.15 \times 13.89^2/2 = \mathbf{33.28 \text{ N. Ans.}} \end{aligned}$$

(iii) **Resultant force ( $F_R$ )**

$$F_R = \sqrt{F_D^2 + F_L^2} = \sqrt{33.28^2 + 166.404^2} = \mathbf{169.67 \text{ N. Ans.}}$$

(iv) **The direction of resultant force ( $\theta$ )**

The direction of resultant force is given by,

$$\tan \theta = \frac{F_L}{F_D} = \frac{166.38}{33.275} = 5$$

∴

$$\theta = \tan^{-1} 5 = \mathbf{78.69^\circ. Ans.}$$

(v) **Power exerted by air on the plate**

Power

$$\begin{aligned} &= \text{Force in the direction of motion} \times \text{Velocity} \\ &= F_D \times U \text{ N m/s} = 33.280 \times 13.89 \text{ W} \quad (\because \text{Watt} = \text{N m/s}) \\ &= \mathbf{462.26 \text{ W. Ans.}} \end{aligned}$$

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**Problem 14.3** Find the difference in drag force exerted on a flat plate of size  $2 \text{ m} \times 2 \text{ m}$  when the plate is moving at a speed of  $4 \text{ m/s}$  normal to its plane in : (i) water, (ii) air of density  $1.24 \text{ kg/m}^3$ . Co-efficient of drag is given as 1.15.

**Solution.** Given :

$$\text{Area of plate, } A = 2 \times 2 = 4 \text{ m}^2$$

$$\text{Velocity of plate, } U = 4 \text{ m/s}$$

$$\text{Co-efficient of drag, } C_D = 1.15$$

(i) Drag force when the plate is moving in water.

$$\text{Using equation (14.3), } F_D = C_D \times A \times \frac{\rho U^2}{2}, \quad \text{where } \rho \text{ for water} = 1000$$

$$= 1.15 \times 4 \times 1000 \times \frac{4^2}{2} \text{ N} = 36800 \text{ N.} \quad \dots(i)$$

(ii) Drag force when the plate is moving in air,

$$F_D = C_D \times A \times \frac{\rho U^2}{2}, \quad \text{where } \rho \text{ for air} = 1.24$$

$$\therefore F_D = 1.15 \times 4.0 \times 1.24 \times \frac{4.0^2}{2.0} \text{ N} = 45.6 \text{ N} \quad \dots(ii)$$

$$\begin{aligned} \therefore \text{Difference in drag force} &= (i) - (ii) \\ &= 36800 - 45.6 = 36754.4 \text{ N. Ans.} \end{aligned}$$

**Problem 14.4** A truck having a projected area of  $6.5 \text{ square metres}$  travelling at  $70 \text{ km/hour}$  has a total resistance of  $2000 \text{ N}$ . Of this 20 per cent is due to rolling friction and 10 per cent is due to surface friction. The rest is due to form drag. Calculate the co-efficient of form drag. Take density of air =  $1.25 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Area of truck, } A = 6.5 \text{ m}^2$$

$$\text{Speed of truck, } U = 70 \text{ km/hr} = \frac{70 \times 100}{60 \times 60} = 19.44 \text{ m/s}$$

$$\text{Total resistance, } F_T = 2000 \text{ N}$$

$$\text{Rolling friction resistance, } F_C = 20\% \text{ of total resistance} = \frac{20}{100} \times 2000 = 400 \text{ N}$$

$$\text{Surface friction resistance, } F_S = 10\% \text{ of total resistance} = \frac{10}{100} \times 2000 = 200 \text{ N}$$

$$\therefore \text{Form drag, } F_D = 2000 - F_C - F_S = 2000 - 400 - 200 = 1400 \text{ N}$$

$$\text{Using equation (14.3), } F_D = C_D \times A \times \frac{\rho U^2}{2}$$

where if  $F_D$  = Form drag then  $C_D$  = Co-efficient of form drag

$$\therefore 1400 = C_D \times 6.5 \times 1.25 \times \frac{19.44^2}{2} \quad (\rho = \text{Density of air} = 1.25 \text{ kg/m}^3)$$

$$\therefore C_D = \frac{1400 \times 2}{6.5 \times 1.25 \times 19.44 \times 19.44} = 0.912. \text{ Ans.}$$

**Problem 14.5** A circular disc 3 m in diameter is held normal to a 26.4 m/s wind of density 0.0012 gm/cc. What force is required to hold it at rest? Assume co-efficient of drag of disc = 1.1.

**Solution.** Given :

$$\text{Diameter of disc} = 3 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} \times (3)^2 = 7.0685 \text{ m}^2$$

$$\text{Velocity of wind, } U = 26.4 \text{ m/s}$$

$$\begin{aligned} \text{Density of wind, } \rho &= 0.0012 \text{ gm/cm}^3 = \frac{0.0012}{1000} \text{ kg/cm}^3 \\ &= \frac{0.0012}{1000} \times 10^6 \frac{\text{kg}}{\text{m}^3} = 1.2 \text{ kg/m}^3 \end{aligned}$$

$$\text{Co-efficient of drag, } C_D = 1.1$$

The force required to hold the disc at rest is equal to the drag exerted by wind on the disc.

Drag ( $F_D$ ) is given by equation (14.3) as

$$F_D = C_D \times A \times \frac{\rho U^2}{2} = \frac{1.1 \times 7.0685 \times 1.2 \times 26.4^2}{2.0} = 3251.4 \text{ N. Ans.}$$

**Problem 14.6** A man weighing 90 kgf descends to the ground from an aeroplane with the help of a parachute against the resistance of air. The velocity with which the parachute, which is hemispherical in shape, comes down is 20 m/s. Find the diameter of the parachute. Assume  $C_D = 0.5$  and density of air = 1.25 kg/m<sup>3</sup>.

**Solution.** Given :

$$\text{Weight of man, } W = 90 \text{ kgf} = 90 \times 9.81 \text{ N} = 882.9 \text{ N} \quad (\because 1 \text{ kgf} = 9.81 \text{ N})$$

$$\text{Velocity of parachute, } U = 20 \text{ m/s}$$

$$\text{Co-efficient of drag, } C_D = 0.5$$

$$\text{Density of air, } \rho = 1.25 \text{ kg/m}^3$$

$$\text{Let the diameter of parachute} = D$$

$$\therefore \text{Area, } A = \frac{\pi}{4} D^2 \text{ m}^2.$$

When the parachute with the man comes down with a uniform velocity,  $U = 20 \text{ m/s}$ , the drag resistance will be equal to the weight of man, neglecting the weight of parachute. And projected area of

the hemispherical parachute will be equal to  $\frac{\pi}{4} D^2$ .

$$\therefore \text{Drag, } F_D = 90 \text{ kgf} = 90 \times 9.81 = 882.9 \text{ N}$$

$$\text{Using equation (14.3), } F_D = C_D \times A \times \frac{\rho U^2}{2}$$

$$\therefore 882.9 = 0.5 \times \frac{\pi}{4} D^2 \times \frac{1.25 \times 20^2}{2}$$

$$\therefore D^2 = \frac{882.9 \times 4 \times 2.0}{0.5 \times \pi \times 1.25 \times 20 \times 20} = 8.9946 \text{ m}^2$$

$$\text{or } D = \sqrt{8.9946} = 2.999 \text{ m. Ans.}$$



Fig. 14.3

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**Problem 14.7** A man weighing 981 N descends to the ground from an aeroplane with the help of a parachute against the resistance of air. The shape of the parachute is hemispherical of 2 m diameter. Find the velocity of the parachute with which it comes down. Assume  $C_d = 0.5$  and  $\rho$  for air = 0.00125 gm/cc and  $v = 0.015$  stoke.

**Solution.** Given :

$$\text{Weight of the man, } W = 981 \text{ N}$$

$$\therefore \text{Drag force, } F_D = W = 981 \text{ N}$$

$$\text{Diameter of the parachute, } D = 2 \text{ m}$$

$$\therefore \text{Projected area, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 2^2 = \pi \text{ m}^2$$

$$\text{Co-efficient of drag, } C_d = 0.5$$

$$\text{Density for air, } \rho = 0.00125 \text{ gm/cm}^3 = \frac{0.00125}{1000} \text{ kg/cm}^3$$

$$= \frac{0.00125}{1000} \times 10^6 \frac{\text{kg}}{\text{m}^3} = 1.25 \text{ kg/m}^3$$

$$\text{Let the velocity of parachute } = U$$

$$\text{Using equation (14.3), } F_D = C_D \times A \times \frac{\rho U^2}{2} \quad \text{or} \quad 981 = 0.5 \times \pi \times \frac{1.25 \times U^2}{2.0}$$

$$\therefore U = \sqrt{\frac{981 \times 2.0}{0.5 \times \pi \times 1.25}} = 31.61 \text{ m/s. Ans.}$$

**Problem 14.8** A man descends to the ground from an aeroplane with the help of a parachute which is hemispherical having a diameter of 4 m against the resistance of air with a uniform velocity of 25 m/s. Find the weight of the man if the weight of parachute is 9.81 N. Take  $C_D = 0.6$  and density of air = 1.25 kg/m<sup>3</sup>.

**Solution.** Given :

$$\text{Diameter of parachute, } D = 4 \text{ m}$$

$$\therefore \text{Projected area, } A = \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times 4^2 = 4\pi \text{ m}^2$$

$$\text{Velocity of parachute, } U = 25 \text{ m/s}$$

$$\text{Weight of parachute, } W_1 = 9.81 \text{ N}$$

$$\text{Co-efficient of drag, } C_D = 0.6$$

$$\text{Density of air, } \rho = 1.25 \text{ kg/m}^3$$

$$\text{Let the weight of man } = W_2$$

$$\text{Then weight of man + Weight of parachute } = W_2 + W_1 = (W_2 + 9.81)$$

Hence drag force will be equal to the weight of man plus weight of parachute.

$$\therefore \text{Drag force, } F_D = (W_2 + 9.81)$$

$$\text{Using equation (14.3), we have } F_D = C_D \times A \times \frac{\rho U^2}{2}$$

$$\text{or } (W_2 + 9.81) = 0.6 \times 4\pi \times \frac{1.25 \times 25^2}{2.0} = 2945.24 \text{ N}$$

$$\therefore W_2 = 2945.24 - 9.81 = 2935.43 \text{ N. Ans.}$$

**Problem 14.9** Calculate the diameter of a parachute to be used for dropping an object of mass 100 kg so that the maximum terminal velocity of dropping is 5 m/s. The drag co-efficient for the parachute, which may be treated as hemispherical is 1.3. The density of air is  $1.216 \text{ kg/m}^3$ .

**Solution.** Given :

$$\begin{aligned}\text{Mass of object, } & M = 100 \text{ kg} \\ \text{Weight of object, } & W = 100 \times 9.81 = 981 \text{ N} \\ \therefore \text{ Drag force, } & F_D = 981 \text{ N} \\ \text{Velocity of object, } & U = 5 \text{ m/s} \\ \text{Drag co-efficient, } & C_D = 1.3 \\ \text{Density of air, } & \rho = 1.216 \text{ kg/m}^3 \\ \text{Let the diameter of parachute } & = D \text{ m}\end{aligned}$$

$$\therefore \text{Projected area, } A = \frac{\pi}{4} D^2 \text{ m}^2$$

$$\text{Using equation (14.3), } F_D = C_D \times A \times \frac{\rho U^2}{2}$$

$$\text{or } 981 = 1.3 \times \frac{\pi}{4} D^2 \times \frac{1.216 \times 5^2}{2}$$

$$\text{or } D^2 = \frac{981 \times 4 \times 2}{1.3 \times \pi \times 1.216 \times 5 \times 5} = 63.21$$

$$\therefore D = \sqrt{63.21} = 7.95 \text{ m. Ans.}$$

**Problem 14.10** A kite  $0.8 \text{ m} \times 0.8 \text{ m}$  weighing 0.4 kgf (3.924 N) assumes an angle of  $12^\circ$  to the horizontal. The string attached to the kite makes an angle of  $45^\circ$  to the horizontal. The pull on the string is 2.5 kgf (24.525 N) when the wind is flowing at a speed of 30 km/hour. Find the corresponding co-efficient of drag and lift. Density of air is given as  $1.25 \text{ kg/m}^3$ .

**Solution.** Given :

$$\begin{aligned}\text{Projected area of kite, } & A = 0.8 \times 0.8 = 0.64 \text{ m}^2 \\ \text{Weight of kite, } & W = 0.4 \text{ kgf} = 0.4 \times 9.81 = 3.924 \text{ N} \\ \text{Angle made by kite with horizontal, } & \theta_1 = 12^\circ \\ \text{Angle made by string with horizontal, } & \theta_2 = 45^\circ \\ \text{Pull on the string, } & P = 2.5 \text{ kgf} = 2.5 \times 9.81 = 24.525 \text{ N} \\ \text{Speed of wind, } & U = 30 \text{ km/hr} = \frac{30 \times 1000}{60 \times 60} \text{ m/s} = 8.333 \text{ m/s} \\ \text{Density of air, } & \rho = 1.25 \text{ kg/m}^3 \\ \text{Drag force, } & F_D = \text{Force exerted by wind in the direction of motion} \\ & \quad (\text{i.e., in the } X\text{-}X \text{ direction}) \\ & = \text{Component of pull, } P \text{ along } X\text{-}X \\ & = P \cos 45^\circ = 24.525 \cos 45^\circ = 17.34 \text{ N} \\ \text{And lift force, } & F_L = \text{Force exerted by wind on the kite perpendicular to the} \\ & \quad \text{direction of motion (i.e., along } Y\text{-}Y \text{ direction)} \\ & = \text{Component of } P \text{ in vertically downward direction} \\ & + \text{Weight of kite (W)}$$

$$= P \sin 45^\circ + W = 24.525 \sin 45^\circ + 3.924 \text{ N} \\ = 17.34 + 3.924 = 21.264 \text{ N.}$$

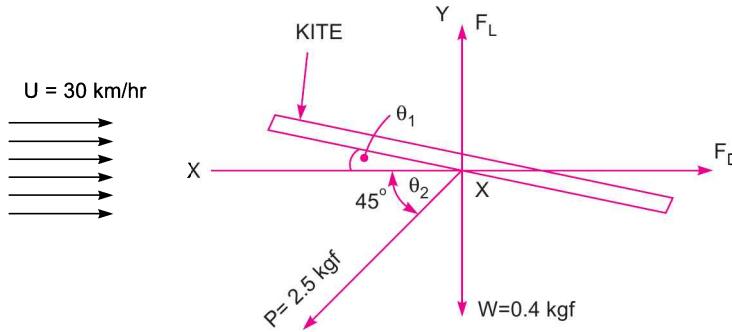


Fig. 14.4

(i) **Drag Co-efficient ( $C_D$ )**. Using equation (14.3), we have

$$F_D = C_D \times A \times \frac{\rho U^2}{2}$$

or  $C_D = \frac{2 \times F_D}{A \rho U^2} = \frac{2 \times 17.34}{0.64 \times 1.25 \times 8.333^2} = 0.624. \text{ Ans.}$

(ii) **Lift Co-efficient ( $C_L$ )**. Using equation (14.4), we have

$$F_L = C_L \times A \times \frac{\rho U^2}{2}$$

or  $C_L = \frac{2 \times F_L}{A \times \rho \times U^2} = \frac{2 \times 21.264}{0.64 \times 1.25 \times 8.333^2} = 0.765. \text{ Ans.}$

**Problem 14.11** A kite weighing 0.8 kgf (7.848 N) has an effective area of  $0.8 \text{ m}^2$ . It is maintained in air at an angle of  $10^\circ$  to the horizontal. The string attached to the kite makes an angle of  $45^\circ$  to the horizontal and at this position the value of co-efficient of drag and lift are 0.6 and 0.8 respectively. Find the speed of the wind and the tension in the string. Take the density of air as  $1.25 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Weight of kite, } W = 0.8 \text{ kgf} = 0.8 \times 9.81 = 7.848 \text{ N}$$

$$\text{Effective area, } A = 0.8 \text{ m}^2$$

$$\text{Angle made by kite with horizontal} = 10^\circ$$

$$\text{Angle made by string with horizontal} = 45^\circ$$

$$C_D = 0.6$$

$$C_L = 0.8$$

$$\therefore \text{Density of air, } \rho = 1.25 \text{ kg/m}^3$$

$$\text{Let the speed of the wind } = U \text{ m/s}$$

$$\text{and tension in string } = T$$

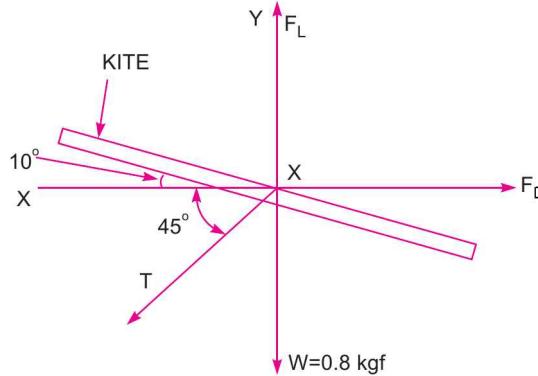


Fig. 14.5

The free body diagram for the kite is shown in Fig. 14.5.

$$\text{Drag force, } F_D = \text{Component of } T \text{ along } X-X = T \cos 45^\circ$$

$$\text{But drag force } F_D \text{ is also } = C_D \times A \times \frac{\rho U^2}{2} = \frac{0.6 \times 0.8 \times 1.25 \times U^2}{2} = 0.3 U^2$$

$$\text{Equating the two values of } F_D, T \cos 45^\circ = 0.3 U^2 \quad \dots(i)$$

$$\begin{aligned} \text{Now lift force from Fig. 14.5, } F_L &= \text{Component of } T \text{ vertically downward} + \text{Weight of kite} \\ &= T \sin 45^\circ + 7.848 \end{aligned}$$

$$\text{Also lift force, } F_L = \frac{C_L \times A \times \rho U^2}{2} = \frac{0.8 \times 0.8 \times 1.25 \times U^2}{2.0} = 0.4 U^2$$

$$\text{Equating the two values of } F_L, T \sin 45^\circ + 7.848 = 0.4 U^2$$

$$\text{or } T \sin 45^\circ = .04076 U^2 - 7.848 \quad \dots(ii)$$

From equations (i) and (ii), as  $T \cos 45^\circ = T \sin 45^\circ$ ,

$$0.3 U^2 = 0.4 U^2 - 7.848$$

$$\text{or } 7.848 = 0.4 U^2 - 0.3 U^2 = 0.1 U^2$$

$$\therefore U^2 = \frac{7.848}{0.1} = 78.48 \quad \dots(iii)$$

$$\therefore U = \sqrt{78.48} = 8.86 \text{ m/s} = \frac{8.86 \times 60 \times 60}{1000} \text{ km/hr} = 31.89 \text{ km/hr. Ans.}$$

Substituting the value of  $U^2 = 78.48$  given by equation (iii) into equation (i), we get

$$T \cos 45^\circ = 0.3 \times 78.48 = 23.544$$

$$\therefore T = \frac{23.544}{\cos 45^\circ} = 33.3 \text{ N. Ans.}$$

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**Problem 14.12** The air is flowing over a cylinder of diameter 50 mm and infinite length with a velocity of 0.1 m/s. Find the total drag, shear drag and pressure drag on 1 m length of the cylinder if the total drag co-efficient is equal to 1.5 and shear drag co-efficient equal to 0.2. Take density of air = 1.25 kg/cm<sup>3</sup>.

**Solution.** Given :

Diameter of cylinder,	$D = 50 \text{ mm} = 0.05 \text{ m}$
Length of cylinder,	$L = 1.0 \text{ m}$
. . . Projected Area,	$A = L \times D = 1 \times 0.05 = 0.05 \text{ m}^2$
Velocity of air,	$U = 0.1 \text{ m/s}$
Total drag co-efficient,	$C_{DT} = 1.5$
Shear drag co-efficient,	$C_{DB} = 0.2$
Density of air,	$\rho = 1.25 \text{ kg/m}^3$

Total drag is given by, 
$$F_{DT} = C_{DT} \times A \times \frac{\rho U^2}{2}$$

$$= 1.5 \times 0.05 \times \frac{1.25 \times (0.1)^2}{2} = \mathbf{0.000468 \text{ N. Ans.}}$$

Shear drag is given by, 
$$F_{DS} = C_{DS} \times A \times \frac{\rho U^2}{2}$$

$$= 0.2 \times 0.05 \times \frac{1.25 \times (0.1)^2}{2} = \mathbf{0.0000625 \text{ N. Ans.}}$$

From equation (14.1), Total drag,  $F_{DT}$  = Pressure drag + Shear drag

. . . Pressure drag,  $= \text{Total drag} - \text{Shear drag}$   
 $= 0.000468 - 0.0000625 = \mathbf{0.0004055 \text{ N. Ans.}}$

**Problem 14.13** A body of length 2.0 m has a projected area 1.5 m<sup>2</sup> normal to the direction of its motion. The body is moving through water, which is having viscosity = 0.01 poise. Find the drag on the body if it has a drag co-efficient 0.5 for a Reynold number of  $8 \times 10^6$ .

**Solution.** Given :

Length of body,	$L = 2.0 \text{ m}$
Projected Area,	$A = 1.5 \text{ m}^2$
Viscosity of water,	$\mu = 0.01 \text{ poise} = \frac{0.01}{10} = 0.001 \frac{\text{Ns}}{\text{m}^2}$
Drag co-efficient,	$C_d = 0.5$
Reynold number,	$R_e = 8 \times 10^6$
Let the drag force on body	$= F_D$

First find the velocity with which body is moving in water. It is calculated from the given Reynold number.

$$R_e = \frac{\rho UL}{\mu}, \text{ where } \rho \text{ for water} = 1000$$

or

$$8 \times 10^6 = \frac{1000 \times U \times 2.0}{0.001} = 2 \times 10^6 U$$

$$\therefore U = \frac{8 \times 10^6}{2 \times 10^6} = 4.0 \text{ m/s}$$

Using equation (14.3),  $F_D = C_d \times A \times \frac{\rho U^2}{2} = 0.5 \times 1.5 \times 1000 \times \frac{4.0^2}{2} = 6049 \text{ N. Ans.}$

**Problem 14.14** A sub-marine which may be supposed to approximate a cylinder 4 m in diameter and 20 m long travels sub-merged at 1.3 m/s in sea-water. Find the drag exerted on it, if the drag coefficient for Reynold number greater than  $10^5$  may be taken as 0.75. The density of sea-water is given as  $1035 \text{ kg/m}^3$  and kinematic viscosity as .015 stokes.

**Solution.** Given :

Dia. of cylinder,

$$D = 4 \text{ m}$$

Length of cylinder,

$$L = 20 \text{ m}$$

Velocity of cylinder,

$$U = 1.3 \text{ m/s}$$

Density of sea-water,

$$\rho = 1035 \text{ kg/m}^3$$

Kinematic viscosity

$$\nu = 0.015 \text{ stokes} = 0.015 \text{ cm}^2/\text{s} = 0.015 \times 10^{-4} \text{ m}^2/\text{s}$$

Let the drag force

$$= F_D$$

Reynold number,

$$R_e = \frac{U \times D}{\nu} = \frac{1.3 \times 4.0}{0.015 \times 10^{-4}} = 3.466 \times 10^6$$

Since  $R_e > 10^5$ , hence  $C_D = 0.75$

Drag force is given by equation (14.3) as

$$F_D = C_D \times A \times \frac{\rho U^2}{2}$$

where  $A = \text{projected area of cylinder} = L \times D = 20 \times 4.0 = 80.0 \text{ m}^2$

$$\therefore F_D = 0.75 \times 80 \times \frac{1035 \times 1.3^2}{2.0} \text{ kgf} = 52472.2 \text{ N. Ans.}$$

**Problem 14.15** A jet plane which weighs 29.43 kN and having a wing area of  $20 \text{ m}^2$  flies at a velocity of 950 km/hour, when the engine delivers 7357.5 kW power. 65% of the power is used to overcome the drag resistance of the wing. Calculate the co-efficients of lift and drag for the wing. The density of the atmospheric air is  $1.21 \text{ kg/m}^3$ .

**Solution.** Given :

Weight of plane,

$$W = 29.43 \text{ kN} = 29.43 \times 1000 \text{ N} = 29430 \text{ N}$$

Wing area,

$$A = 20 \text{ m}^2$$

Speed of plane,

$$U = 950 \text{ km/hr} = \frac{950 \times 1000}{60 \times 60} = 263.88 \text{ m/s}$$

Engine power,

$$P = 7357.5 \text{ kW}$$

$$\text{Power used to overcome drag resistance} = 65\% \text{ of } 7357.5 = \frac{65}{100} \times 7357.5 = 4782.375 \text{ kW}$$

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∴ Density of air,  $\rho = 1.21 \text{ kg/m}^3$

Let  $C_D$  = Co-efficient of drag and  $C_L$  = Co-efficient of lift.

Now power used in kW to overcome drag resistance =  $\frac{F_D \times U}{1000}$  or  $4782.375 = \frac{F_D \times 263.88}{1000}$

$$\therefore F_D = \frac{4782.375 \times 1000}{263.88}$$

But from equation (14.3), we have  $F_D = C_D \cdot A \cdot \frac{\rho U^2}{2}$

$$\therefore \frac{4782.375 \times 1000}{263.88} = C_D \times 20 \times 1.21 \times \frac{263.88^2}{2}$$

$$\therefore C_D = \frac{4782.375 \times 1000 \times 2}{20 \times 1.21 \times 263.88^3} = 0.0215. \text{ Ans.}$$

The lift force should be equal to weight of the plane

$$\therefore F_L = W = 29430 \text{ N}$$

But  $F_L = C_L \cdot A \cdot \frac{\rho U^2}{2}$  or  $29430 = C_L \times 20 \times 1.21 \times \frac{263.88^2}{2}$

$$\therefore C_L = \frac{29430 \times 2}{20 \times 1.21 \times 263.88^2} = 0.0349. \text{ Ans.}$$

**14.3.2 Pressure Drag and Friction Drag.** The total drag on a body is given by equation (14.1) as

$$\text{Total drag, } F_D = \int p \cos \theta dA + \int \tau_0 \sin \theta dA \quad \dots(i)$$

where  $\int p \cos \theta dA$  = Pressure drag or form drag, and

$\int \tau_0 \sin \theta dA$  = Friction drag or skin drag or shear drag

The relative contribution of the pressure drag and friction drag to the total drag depends on :

- (i) Shape of the immersed body,
- (ii) Position of the body immersed in the fluid, and
- (iii) Fluid characteristics.

Consider the flow of a fluid over a flat plate when the plate is placed parallel to the direction of the flow as shown in Fig. 14.6. In this  $\cos \theta$ , which is the angle made by pressure with the direction of motion, will be  $90^\circ$ . Thus the term  $\int p \cos \theta dA$  will be zero and hence total drag will be equal to friction drag (or shear drag). If the plate is placed perpendicular to the flow as shown in Fig. 14.7, the angle  $\theta$ , made by the pressure with the direction of motion will be zero. Hence the term  $\int \tau_0 \sin \theta dA$  will become equal to zero and hence total drag will be due to the pressure difference between the upstream and downstream side of the plate. If the plate is held at an angle with the direction of flow, both the terms  $\int p \cos \theta dA$  and  $\int \tau_0 \sin \theta dA$  will exist and total drag will be equal to the sum of pressure drag and friction drag.

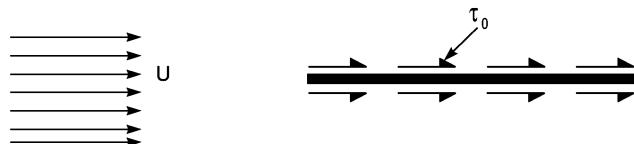


Fig. 14.6 Flat plate parallel to flow.

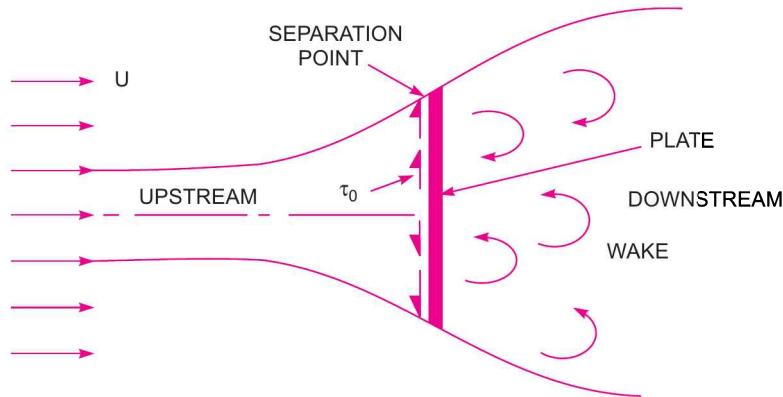


Fig. 14.7 Flat plate perpendicular to flow.

**14.3.3 Stream-lined Body.** A stream-lined body is defined as that body whose surface coincides with the stream-lines, when the body is placed in a flow. In that case the separation of flow will take place only at the trailing edge (or rearmost part of the body). Though the boundary layer will start at the leading edge, will become turbulent from laminar, yet it does not separate upto the rearmost part of the body in the case of stream-lined body. Thus behind a stream-lined body, wake formation zone will be very small and consequently the pressure drag will be very small. Then the total drag on the stream-lined body will be due to friction (shear) only. A body may be stream-lined :

1. at low velocities but may not be so at higher velocities.
2. when placed in a particular position in the flow but may not be so when placed in another position.

**14.3.4 Bluff Body.** A bluff body is defined as that body whose surface does not coincide with the streamlines, when placed in a flow. Then the flow is separated from the surface of the body much ahead of its trailing edge with the result of a very large wake formation zone. Then the drag due to pressure will be very large as compared to the drag due to friction on the body. Thus the bodies of such a shape in which the pressure drag is very large as compared to friction drag are called bluff bodies.

#### ► 14.4 DRAG ON A SPHERE

Consider the flow of a real fluid past a sphere.

Let  $U$  = Velocity of the flow of fluid over sphere,

$D$  = Diameter of sphere,

$\rho$  = Mass density of fluid, and

$\mu$  = Viscosity of fluid.

If the Reynolds number of the flow is very small less than 0.2 (*i.e.*,  $R_e = \frac{UD\rho}{\mu} < 0.2$ ), the viscous

forces are much more important than the inertial forces as in this case viscous forces are much more predominate than the inertial forces, which may be assumed negligible. G.G. Stokes, developed a mathematical equation for the total drag on a sphere immersed in a flowing fluid for which Reynolds number is upto 0.2, so that inertia forces may be assumed negligible. According to his solution, total drag is

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$$F_D = 3\pi\mu DU. \quad \dots(14.8)$$

He further observed that out of the total drag given by equation (14.8), two-third is contributed by skin friction and the remaining one-third by pressure difference. Thus

$$\text{Skin friction drag, } F_{D_f} = \frac{2}{3} F_D = \frac{2}{3} \times 3\pi\mu DU = 2\pi\mu DU$$

$$\text{and pressure drag, } F_{D_p} = \frac{1}{3} F_D = \frac{1}{3} \times 3\pi\mu DU = \pi\mu DU.$$

(i) **Expression of  $C_d$  for Sphere when Reynolds Number is less than 0.2.** From equation (14.3), the total drag is given by

$$F_D = C_D \times A \times \frac{\rho U^2}{2}$$

For sphere,

$$F_D = 3\pi\mu DU$$

$$A = \text{Projected area of the sphere} = \frac{\pi}{4} D^2$$

$$\therefore 3\pi\mu DU = C_D \times \frac{\pi}{4} D^2 \frac{\rho U^2}{2}$$

$$C_D = \frac{3\pi\mu DU}{\frac{\pi}{4} D^2 \times \frac{\rho U^2}{2}} = \frac{24\mu}{\rho UD} = \frac{24}{R_e} \quad \dots(14.9) \quad \left( \because \frac{\mu}{\rho DU} = R_e \right)$$

Equation (14.9) is called ‘Stoke’s law’.

(ii) **Value of  $C_D$  for Sphere when  $R_e$  is between 0.2 and 5.** With the increase of Reynolds number, the inertia forces increase and must be taken into account. When  $R_e$  lies between 0.2 and 5, Oseen, a Swedish physicist, improved Stoke’s law as

$$C_D = \frac{24}{R_e} \left[ 1 + \frac{3}{16R_e} \right] \quad \dots(14.10)$$

Equation (14.10) is called Oseen formulae and is valid for  $R_e$  between 0.2 and 5.

(iii) **Value of  $C_D$  for  $R_e$  from 5.0 to 1000.** The drag co-efficient for the Reynolds number from 5 to 1000 is equal to 0.4.

(iv) **Value of  $C_D$  for  $R_e$  from 1000 to 100,000.** In this range,  $C_D$  is independent of the Reynolds number and its value is approximately equal to 0.5.

(v) **Value of  $C_D$  for  $R_e$  more than  $10^5$ .** The value of  $C_D$  is approximately equal to 0.2 for the Reynolds number more than  $10^5$ .

**Problem 14.16** Calculate the weight of a ball of diameter 80 mm which is just supported in a vertical air stream which is flowing at a velocity of 7 m/s. The density of air is given as  $1.25 \text{ kg/m}^3$ . The kinematic viscosity of air = 1.5 stokes.

**Solution.** Given :

$$\text{Dia. of ball, } D = 80 \text{ mm} = 0.08 \text{ m}$$

$$\text{Velocity of air, } U = 7 \text{ m/s}$$

$$\text{Density of air, } \rho = 1.25 \text{ kg/m}^3$$

$$\text{Kinematic viscosity, } v = 1.5 \text{ stokes} = 1.5 \times 10^{-4} \text{ m}^2/\text{s}$$

$$\text{Reynold number, } R_e = \frac{U \times D}{v} = \frac{7 \times 0.08}{1.5 \times 10^{-4}} = 0.373 \times 10^4 = 3730.$$

Thus the value of  $R_e$  lies between 1000 and 100,000 and hence  $C_D = 0.5$ .

When the ball is supported in a vertical air stream, the weight of ball is equal to the drag force as shown in Fig. 14.8.

$$\text{But drag force, } F_D = C_D \times A \times \frac{\rho U^2}{2}$$

where  $A$  = projected area of ball

$$= \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.08)^2 = 0.005026 \text{ m}^2$$

$$\therefore \text{Drag force, } F_D = 0.5 \times 0.005026 \times \frac{1.25 \times 7^2}{2} \text{ N} \\ = 0.07696 \text{ N}$$

$$\therefore \text{Weight of ball} = F_D = \mathbf{0.07696 \text{ N. Ans.}}$$

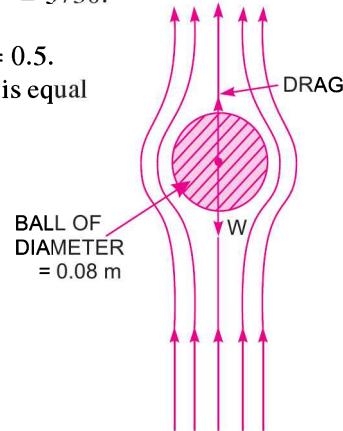


Fig. 14.8

## ► 14.5 TERMINAL VELOCITY OF A BODY

Terminal velocity is defined as the maximum constant velocity of a falling body (such as sphere or a composite body such as parachute together with man) with which the body will be travelling. When the body is allowed to fall from rest in the atmosphere, the velocity of the body increases due to acceleration of gravity. With the increase of the velocity, the drag force, opposing the motion of body also increases. A stage is reached when the upward drag force acting on the body will be equal to the weight of the body. Then the net external force acting on the body will be zero and the body will be travelling at constant speed. This constant speed is called terminal velocity of the falling body.

If the body drops in a fluid, at the instant it has acquired terminal velocity, the net force acting on the body will be zero. The forces acting on the body at this state will be :

1. Weight of body ( $W$ ), acting downward,
2. Drag force ( $F_D$ ), acting vertically upward, and
3. Buoyant force ( $F_B$ ), acting vertically up.

The net force on the body should be zero, i.e.,  $W = F_D + F_B$  ... (14.11)

**Problem 14.17** A metallic sphere of sp. gr. 7.0 falls in an oil of density  $800 \text{ kg/m}^3$ . The diameter of the sphere is 8 mm and it attains a terminal velocity of 40 mm/s. Find the viscosity of the oil in poise.

**Solution.** Given :

$$\text{Sp. gr. of metallic sphere} = 7.0$$

$$\therefore \text{Density of metallic sphere, } \rho_s = 7 \times 1000 = 7000 \text{ kg/m}^3$$

$$\text{Density of oil, } \rho_o = 800 \text{ kg/m}^3$$

$$\text{Dia. of sphere, } D = 8 \text{ mm} = 8 \times 10^{-3} \text{ m}$$

$$\text{Terminal velocity, } U = 40 \text{ mm/s} = .04 \text{ m/s}$$

$$\text{Let the viscosity of oil} = \mu$$

$$\text{Weight of sphere, } W = \rho_s \times g \times \text{Volume of sphere}$$

$$= 7000 \times 9.81 \times \frac{\pi}{6} D^3$$

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$$= 7000 \times 9.81 \times \frac{\pi}{6} \times (8 \times 10^{-3})^3 = 18.4 \times 10^{-3} = 0.0184 \text{ N}$$

Buoyant force on sphere,  $F_B = \text{Density of oil} \times g \times \text{Volume of sphere}$

$$= 800 \times 9.81 \times \frac{\pi}{6} (8 \times 10^{-3})^3 \text{ N} = 0.002103 \text{ N.}$$

Drag force,  $F_D$  on the sphere is given by equation (14.8) as

$$F_D = 3\pi\mu DU = 3\pi\mu \times 8 \times 10^{-3} \times .04 = .003015 \mu$$

Using equation (14.11),  $W = F_D + F_B$

$$\text{or } 0.0184 = .003015 \mu + 0.002103$$

$$\text{or } .003015 \mu = 0.0184 - 0.002103 = 0.016297$$

$$\text{or } \mu = \frac{0.016297}{.003015} = \frac{5.4 \text{ Ns}}{\text{m}^2} = 5.4 \times 10 = \mathbf{54.0 \text{ poise. Ans.}}$$

The expression for drag given by equation (14.11) is valid only upto Reynolds number less than 0.2. Hence it is necessary to calculate Reynold number for the flow.

$$\therefore \text{Reynold number, } R_e = \frac{\rho UD}{\mu}, \quad \text{where } \rho \text{ for oil} = 800 \text{ kg/m}^3$$

$$\therefore R_e = 800 \times \frac{.04 \times 8 \times 10^{-3}}{5.4} = 0.0474$$

Hence  $R_e < 0.2$  and so the expression  $F_D = 3\pi\mu DU$  is valid.

**Problem 14.18** A spherical steel ball of diameter 40 mm and of density  $8500 \text{ kg/m}^3$  is dropped in large mass of water. The co-efficient of drag of the ball in water is given as 0.45. Find the terminal velocity of the ball in water. If the ball is dropped in air, find the increase in terminal velocity of ball. Take the density of air =  $1.25 \text{ kg/m}^3$  and  $C_D = 0.1$ .

**Solution.** Given :

$$\text{Diameter of steel ball, } D = 40 \text{ mm} = 0.04 \text{ m}$$

$$\text{Density of ball, } \rho_s = 8500 \text{ kg/m}^3$$

$$C_D \text{ for ball in water} = 0.45$$

Let the terminal velocity in water =  $U_1$

The forces acting on the spherical ball are :

1. Weight,  $W = \text{Density of ball} \times g \times \text{Volume of spherical ball}$

$$= \rho_s \times g \times \frac{\pi}{6} D^3 = 8500 \times 9.81 \times \frac{\pi}{6} (.04)^3 = \mathbf{2.794 \text{ N.}}$$

2. Buoyant force,  $F_B = \text{Density of water} \times g \times \text{Volume of ball}$

$$= 1000 \times 9.81 \times \frac{\pi}{6} (0.04)^3 = 0.3286 \text{ N.}$$

3. Drag force,  $F_D = C_D \times A \times \frac{\rho U^2}{2}$

where  $A = \text{projected area} = \frac{\pi}{4} D^2 = \frac{\pi}{4} (.04)^2$ ,  $\rho = 1000$  for water

$$F_D = 0.45 \times \frac{\pi}{4} \times (.04)^2 \times 1000 \times \frac{U_1^2}{2} = 0.2825 U_1^2$$

Using equation (14.11), we get  $W = F_D + F_B$   
 or  $2.794 = 0.2825 U_1^2 + 0.3286$

or  $U_1^2 = \frac{2.794 - 0.3286}{0.2825} = 8.725$

$\therefore U_1 = \sqrt{8.725} = 2.953 \text{ m/s. Ans.}$

**When ball is dropped in air.** Let the terminal velocity =  $U_2$

Weight,  $W = 2.794$

Buoyant force,  $F_B = \text{Density of air} \times g \times \text{Volume of ball}$

$$= 1.25 \times 9.81 \times \frac{\pi}{6} (.04)^3 = 0.000411 \text{ N}$$

Drag force,  $F_D = C_D \times A \times \frac{\rho U^2}{2}$ , where  $\rho$  for air = 1.25

$$F_D = 0.1 \times \frac{\pi}{4} (.04)^2 \times 1.25 \frac{U_2^2}{2} = 0.0000785 U_2^2.$$

The buoyant force in air is 0.000411, while weight of the ball is 2.794 N. Hence buoyant force is negligible.

$\therefore$  For equilibrium of the ball in air,  $F_D = \text{Weight of ball}$

or  $0.0000785 U_2^2 = 2.794$  or  $U_2 = \sqrt{\frac{2.794}{0.0000785}} = 188.67 \text{ m/s}$

$\therefore$  Increase in terminal velocity in air =  $U_2 - U_1 = 188.67 - 2.9533 = 185.717 \text{ m/s. Ans.}$

**Problem 14.19** A metallic ball of diameter  $2 \times 10^{-3} \text{ m}$  drops in a fluid of sp. gr. 0.95 and viscosity 15 poise. The density of the metallic ball is  $12000 \text{ kg/m}^3$ . Find :

- (i) The drag force exerted by fluid on metallic ball,
- (ii) The pressure drag and skin friction drag,
- (iii) The terminal velocity of ball in fluid.

**Solution.** Given :

Diameter of metallic ball,  $D = 2 \times 10^{-3} \text{ m}$

Sp. gr. of fluid,  $S_0 = 0.95$

$\therefore$  Density of fluid,  $\rho_0 = 0.95 \times 1000 = 950 \text{ kg/m}^3$

Viscosity of fluid,  $\mu = 15 \text{ poise} = \frac{15}{10} = 1.5 \frac{\text{Ns}}{\text{m}^2}$

Density of ball,  $\rho_s = 12000 \text{ kg/m}^3$

The forces acting on the ball are :

Weight of ball,  $W = \text{Density of ball} \times g \times \text{Volume of ball}$

$$= 12000 \times 9.81 \times \frac{\pi}{6} D^3$$

$$= 12000 \times 9.81 \times \frac{\pi}{6} \times (2 \times 10^{-3})^3 \text{ N} = 0.000493 \text{ N}$$

Buoyant force,  $F_B = \text{Density of fluid} \times g \times \text{Volume of ball}$

$$= 950 \times 9.81 \times \frac{\pi}{6} (2 \times 10^{-3})^3 \text{ N} = 0.000039 \text{ N}$$

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When the metallic ball reaches the terminal velocity, equation (14.11) is applicable.

$$\therefore W = F_D + F_B \quad \text{or} \quad F_D = W - F_B = 0.000493 - 0.000039 \\ = 0.000454 \text{ N. Ans.}$$

$$(i) \text{ Drag force, } F_D = 0.000454 \text{ N}$$

$$(ii) \text{ Pressure drag} \quad = \frac{1}{3} F_D = \frac{1}{3} \times 0.000454 = 0.0001513 \text{ N. Ans.}$$

$$\text{Skin friction drag} \quad = \frac{2}{3} \times F_D = \frac{2}{3} \times 0.000454 = 0.0003028 \text{ N. Ans.}$$

$$(iii) \text{ Let the terminal velocity } = U$$

Then drag force ( $F_D$ ) is given by equation (14.8) as  $F_D = 3\pi\mu DU$

$$\text{But } F_D = 0.000454 \text{ N}$$

Equating the two values of  $F_D$ , we have

$$3\pi\mu DU = 0.000454 \quad \text{or} \quad 3\pi \times \frac{15}{10} \times 2 \times 10^{-3} \times U = 0.000454$$

$$\text{or} \quad U = \frac{10 \times 0.000454}{3\pi \times 15 \times 2 \times 10^{-3}} = 0.016 \text{ m/s. Ans.}$$

Let us check for Reynold number,  $R_e$

$$R_e = \frac{\rho UD}{\mu}, \quad \text{where } \rho = 950 \text{ kg/m}^3$$

$$= 950 \times \frac{0.016 \times 2 \times 10^{-3}}{1.5} = 0.02$$

Hence the Reynolds number is less than 0.2 and so the expression  $F_D = 3\pi\mu DU$  for calculating terminal velocity is valid.

**Problem 14.20** Determine the velocity of fall of rain drops of a  $30 \times 10^{-3}$  cm diameter, density  $0.0012 \text{ gm/cm}^3$  and kinematic viscosity  $0.15 \text{ cm}^2/\text{s}$ .

**Solution.** Given :

$$\text{Diameter of rain drops, } D = 30 \times 10^{-3} \text{ cm}$$

$$\text{Density of rain drops, } \rho = 0.0012 \text{ gm/cm}^3$$

$$\text{Kinematic viscosity, } v = 0.15 \text{ cm}^2/\text{s}$$

$$\text{Using the relation, } v = \frac{\mu}{\rho} \quad \text{or} \quad 0.15 = \frac{\mu}{0.0012}$$

$$\therefore \mu = 0.15 \times 0.0012 = 0.00018 \frac{\text{gm}}{\text{cm sec}}$$

$$\text{Now weight of rain drop} \quad = \rho \times g \times \text{Volume of rain drop}$$

$$= \rho \times g \times \frac{\pi}{6} D^3 \quad (\because \text{Rain drop is a sphere})$$

Drag force,  $F_D$ , on rain drop is given by equation (14.8) as  $F_D = 3\pi\mu DU$

When rain drop is falling with a uniform velocity  $U$ , the drag force must be equal to the weight of rain drop. Hence equating these two values, we get

$$\text{Weight of rain drop} \quad = \text{Drag force}$$

$$\text{or} \quad \rho \times g \times \frac{\pi}{6} D^3 = 3\pi\mu DU \quad \text{or} \quad U = \frac{\rho \times g \times \frac{\pi}{6} \times D^3}{3\pi\mu D} = \frac{\rho g D^2}{18\mu}$$

$$= \frac{0.0012 \times 981 \times (30 \times 10^{-3})^2}{18 \times .00018} = 0.327 \text{ cm/s. Ans.}$$

Let us check for Reynolds number,  $R_e$

$$R_e = \frac{\rho UD}{\mu} = \frac{UD}{v} = \frac{0.327 \times 30 \times 10^{-3}}{0.15} = 0.0654$$

As the Reynolds number is less than 0.2, the expression  $F_D = 3\pi\mu DU$  is valid.

#### ► 14.6 DRAG ON A CYLINDER

Consider a real fluid flowing over a circular cylinder of diameter  $D$  and length  $L$ , when the cylinder is placed in the fluid such that its length is perpendicular to the direction of flow. If the Reynolds number of the flow is less than 0.2 (*i.e.*,  $\frac{U \times d}{v} < 0.2$ ), the inertia force is negligibly small as compared to viscous force and hence the flow pattern about the cylinder will be symmetrical. As the Reynolds number is increased, inertia forces increase and hence they must be taken into consideration for analysis of flow over cylinder. With the increase of the Reynolds number, the flow pattern becomes unsymmetrical with respect to an axis perpendicular to the direction of flow. The drag force, *i.e.*, the force exerted by the flowing fluid on the cylinder in the direction of flow depends upon the Reynolds number of the flow. From experiments, it has been observed that :

- (i) When Reynolds number ( $R_e$ )  $< 1$ , the drag force is directly proportional to velocity and hence the drag co-efficient ( $C_D$ ) is inversely proportional to Reynolds number.
- (ii) With the increase of the Reynolds number from 1 to 2000, the drag co-efficient decreases and reaches a minimum value of 0.95 at  $R_e = 2000$ .
- (iii) With the further increase of the Reynolds number from 2000 to  $3 \times 10^4$ , the co-efficient of drag increases and attains maximum value of 1.2 at  $R_e = 3 \times 10^4$ .
- (iv) The value of co-efficient of drag decreases if the Reynolds number is increased from  $3 \times 10^4$  to  $3 \times 10^5$ . At  $R_e = 3 \times 10^5$ , the value of  $C_D = 0.3$ .
- (v) If the Reynolds number is increased beyond  $3 \times 10^6$ , the value of  $C_D$  increases and it becomes equal to 0.7 in the end.

#### ► 14.7 DEVELOPMENT OF LIFT ON A CIRCULAR CYLINDER

When a body is placed in a fluid in such a way that its axis is parallel to the direction of fluid flow and body is symmetrical, the resultant force acting on the body is in the direction of flow. There is no force component on the body perpendicular to the direction of flow. But the component to the force on the body perpendicular to the direction of flow, is known as '*Lift*'. Hence in this case lift will be zero.

The lift will be acting on the body when the axis of the symmetrical body is inclined to the direction of flow or body is unsymmetrical. In the case of circular cylinder, the body is symmetrical and the axis is parallel to the direction of flow when cylinder is stationary. Hence the lift will be zero. But if the cylinder is rotated, the axis of the cylinder is no longer parallel to the direction of flow and hence lift will be acting on the rotating cylinder. This is explained by considering the following cases :

**14.7.1 Flow of Ideal Fluid over Stationary Cylinder.** Consider the flow of an ideal fluid over a cylinder, which is stationary as shown in Fig. 14.9.

Let

$U$  = Free stream velocity of fluid

$R$  = Radius of the cylinder

$\theta$  = Angle made by any point say  $C$  on the circumference of the cylinder with the direction of flow.

The flow pattern will be symmetrical and the velocity at any point say  $C$  on the surface of the cylinder is given by  $u_\theta = 2U \sin \theta$

...(14.12)

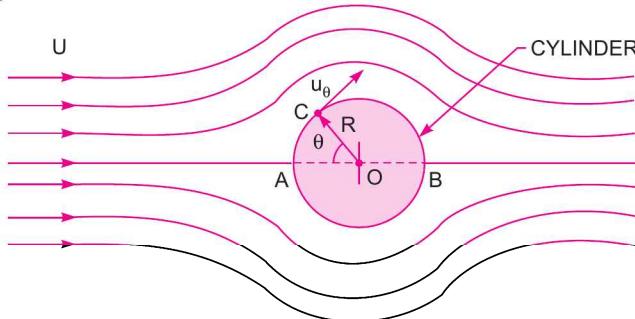


Fig. 14.9 Flow of ideal fluid over stationary cylinder.

The velocity distribution over the upper half and lower half of the cylinder from the axis  $AB$  of the cylinder are identical and hence the pressure distributions will also be same. Hence the lift acting on the cylinder will be zero.

**14.7.2 Flow Pattern Around the Cylinder when a Constant Circulation  $\Gamma$  is Imparted to the Cylinder.** Circulation is defined as the flow along a closed curve. Mathematically, the circulation is obtained if the product of the velocity component along the curve at any point and the length of the small element containing that point is integrated around the curve.

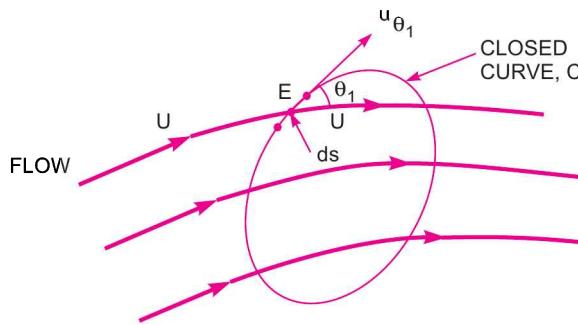


Fig. 14.10 Circulation.

Consider a fluid flowing with a free stream velocity equal to  $U$ . Within the fluid consider a closed curve as shown in Fig. 14.10. Let  $E$  is any point on the closed curve and ' $ds$ ' is a small length of the closed curve containing point  $E$ .

Let  $\theta_1$  = Angle made by the tangent at  $E$  with the direction of flow,

$$\begin{aligned} u_{\theta_1} &= \text{Component of free stream velocity along the tangent at } E \text{ and is given as} \\ &= U \cos \theta_1 \end{aligned}$$

∴ By definition, circulation along the closed curve is

$$\begin{aligned}\Gamma &= \oint \text{velocity component along curve} \times \text{Length of element} \\ &= \oint U \cos \theta_1 \times dS\end{aligned}\quad \dots(14.13)$$

where  $\oint$  = Integral for the complete closed curve.

**Circulation for the Flow-field in a Free-Vortex.** The equation for the free vortex flow is given by

$$u_{\theta_1} \times r = \text{Constant say } = k \quad \dots(i)$$

where  $u_{\theta_1}$  = velocity of fluid in a free-vortex flow

$$r = \text{Radius, where velocity is } u_{\theta_1}.$$

The flow-pattern for a free-vortex flow consists of streamlines which are series of concentric circles as shown in Fig. 14.11 (a).

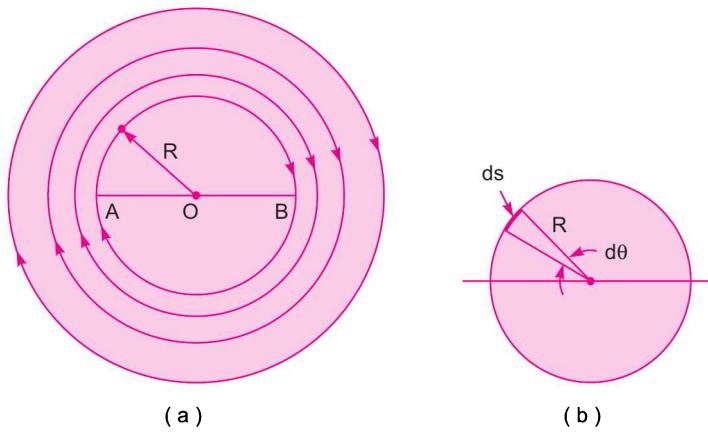


Fig. 14.11 Stream-lines for free vortex.

In case of free-vortex flow, the stream velocity at any point on a circle of radius  $R$  is equal to the tangential velocity at that point. This means that angle between the stream-lines and tangent on the stream is zero. Also from Fig. 14.11 (b), the length of the element 'ds' is given as  $ds = Rd\theta$

∴ For a free-vortex flow,  $U = u_{\theta_1}; \cos \theta_1 = 1; ds = Rd\theta$

Substituting these values in equation (14.13), we get the circulation for a free vortex as

$$\Gamma = \oint u_{\theta_1} \times 1 \times Rd\theta$$

But from equation (i), for a radius  $R$ , we have

$$u_{\theta_1} \times R = K$$

$$\begin{aligned}\therefore \Gamma &= \oint Kd\theta = 2\pi K && (\because \oint d\theta = 2\pi) \\ &= 2\pi u_{\theta_1} \times R && (\because K = u_{\theta_1} \times R) \\ \therefore u_{\theta_1} &= \frac{\Gamma}{2\pi R}.\end{aligned}\quad \dots(14.14)$$

**Flow over Cylinder due to Constant Circulation.** The flow pattern over a cylinder to which a constant circulation ( $\Gamma$ ) is imparted is obtained by combining the flow patterns shown in Figs. 14.9 and Fig. 14.11 (a). The resultant flow pattern is shown in Fig. 14.12. The velocity at any point on the surface of the cylinder is obtained by adding equations (14.12) and (14.14) as

$$u = u_\theta + u_{\theta_1} = 2U \sin \theta + \frac{\Gamma}{2\pi R} \quad \dots(14.15)$$

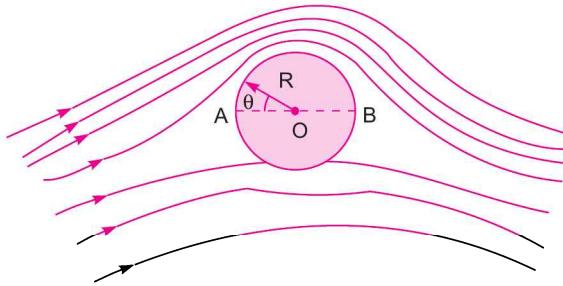


Fig. 14.12 Flow pattern over a rotating cylinder.

For the upper half portion of the cylinder,  $\theta$  varies from  $0^\circ$  to  $180^\circ$  and hence component of velocity,  $2U \sin \theta$  is positive. But for the lower half portion of the cylinder,  $\theta$  varies from  $180^\circ$  to  $360^\circ$ . As  $\sin \theta$  for the values of  $\theta$  more than  $180^\circ$  and less than  $360^\circ$  is negative and hence component of velocity  $2U \sin \theta$  will be negative. This means, the velocity on the upper half portion of the cylinder will be more than the velocity on the lower half portion of the cylinder. But from Bernoulli's theorem we know that at a surface where velocity is less, pressure will be more there and *vice-versa*. Hence on the lower half portion of cylinder, where velocity is less, pressure will be more than the pressure on the upper half portion of the cylinder. Due to this difference of pressure on the two portions of the cylinder, a force will be acting on the cylinder in a direction perpendicular to the direction of flow. This force is nothing but a lift force. Thus by rotating a cylinder at constant velocity in a uniform flow field, a lift force can be developed.

**14.7.3 Expression for Lift Force Acting on Rotating Cylinder.** Let a cylinder is rotating in a uniform flow field. The resultant flow pattern will be as shown in Fig. 14.12. Consider a small length of the element on the surface of the cylinder.

Let  $p_s$  = Pressure on the surface of the element on cylinder

$ds$  = Length of element

$R$  = Radius of cylinder

$d\theta$  = Angle made by the length  $ds$  at the centre of the cylinder as shown in Fig. 14.13.

$p$  = Pressure of the fluid far away from the cylinder

$U$  = Velocity of fluid far away from the cylinder

$u_s$  = Velocity of fluid on the surface of the cylinder.

Applying Bernoulli's equation to a point far away from cylinder and to a point lying on the surface of cylinder such that both the points are on the same horizontal line, we have

$$\begin{aligned}\frac{p}{\rho g} + \frac{U^2}{2g} &= \frac{p_s}{\rho g} + \frac{u_s^2}{2g} \\ \therefore \frac{p_s}{\rho g} &= \frac{p}{\rho g} + \frac{U^2}{2g} - \frac{u_s^2}{2g} \\ &= \frac{p}{\rho g} + \frac{U^2}{2g} \left[ 1 - \frac{u_s^2}{U^2} \right] \quad \dots(i)\end{aligned}$$

But the velocity on the surface of the cylinder is given by equation (14.15). Hence

$$u_s = u = 2U \sin \theta + \frac{\Gamma}{2\pi R}$$

Substituting this value of  $u_s$  in (i), we get

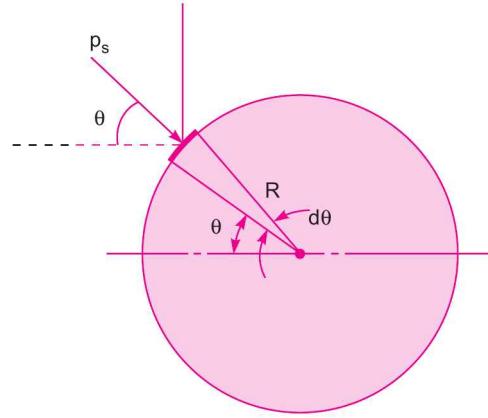


Fig. 14.13 Lift on a rotating cylinder.

$$\begin{aligned}\frac{p_s}{\rho g} &= \frac{p}{\rho g} + \frac{U^2}{2g} \left[ 1 - \frac{\left( 2U \sin \theta + \frac{\Gamma}{2\pi R} \right)^2}{U^2} \right] \\ &= \frac{p}{\rho g} + \frac{U^2}{2g} \left[ 1 - \frac{\left( 4U^2 \sin^2 \theta + \frac{\Gamma^2}{4\pi^2 R^2} + 4U \sin \theta \frac{\Gamma}{2\pi R} \right)}{U^2} \right] \\ &= \frac{p}{\rho g} + \frac{U^2}{2g} \left[ 1 - \left( 4 \sin^2 \theta + \frac{\Gamma^2}{4\pi^2 R^2 U^2} + \frac{4 \sin \theta \Gamma}{U \times 2\pi R} \right) \right]\end{aligned}$$

or

$$p_s = p + \frac{\rho g U^2}{2g} \left[ 1 - 4 \sin^2 \theta - \frac{\Gamma^2}{4\pi^2 R^2 U^2} - \frac{4 \sin \theta \Gamma}{U \times 2\pi R} \right] \quad \dots(ii)$$

From Fig. 14.13, we have the lift force acting on the small length  $ds$  on the element, due to pressure  $p_s$  as

$$\begin{aligned}&= \text{Component of } p_s \text{ in the direction perpendicular to flow} \times \text{Area of the element} \\ &= (-p_s \sin \theta) \times (ds \times L)\end{aligned}$$

Negative sign is taken, as the component of  $p_s$  perpendicular to the flow is acting in the downward direction.

Now

$$L = \text{length of the cylinder}$$

$$ds = R \times d\theta$$

$$\therefore \text{Lift force on the element} = -p_s \sin \theta \times R d\theta \times L \quad (\because ds = R d\theta) \quad \dots(iii)$$

The total force is obtained by integrating equation (iii) over the centre surface of the cylinder.

$$\therefore \text{Total lift, } F_L = \int_0^{2\pi} -p_s \sin \theta \times R d\theta \times L = \int_0^{2\pi} -p_s \times R \times L \times \sin \theta d\theta$$

Substituting the value of  $p_s$  from equation (ii), we get

$$F_L = \int_0^{2\pi} - \left[ p + \frac{\rho g U^2}{2g} \left( 1 - 4 \sin^2 \theta - \frac{\Gamma^2}{4\pi^2 R^2 U^2} - \frac{4 \sin \theta \Gamma}{U \times 2\pi R} \right) \right] RL \times \sin \theta d\theta$$

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$$= -RL \int_0^{2\pi} \left[ p \sin \theta + \frac{\rho g U^2}{2g} \left( \sin \theta - 4 \sin^3 \theta - \frac{\Gamma \sin \theta}{4\pi^2 R^2 U^2} - \frac{4 \sin^2 \theta \Gamma}{U \times 2\pi R} \right) \right] d\theta$$

But  $\int_0^{2\pi} \sin \theta d\theta = \int_0^{2\pi} \sin^3 \theta d\theta = 0$

$$\begin{aligned} \therefore F_L &= -R \times L \int_0^{2\pi} \frac{\rho g U^2}{2g} \left( -\frac{4 \sin^2 \theta \Gamma}{U \times 2\pi R} \right) d\theta \\ &= R \times L \times \frac{\rho g U^2}{2g} \times \frac{4 \Gamma}{U \times 2\pi R} \int_0^{2\pi} \sin^2 \theta d\theta = \frac{L}{g} \frac{\rho g U \Gamma}{\pi} \int_0^{2\pi} \sin^2 \theta d\theta \end{aligned}$$

But  $\int_0^{2\pi} \sin^2 \theta d\theta = \left[ \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi} = \left( \frac{2\pi}{2} - \frac{\sin 4\pi}{4} \right) = \pi$

$$\therefore F_L = \frac{L}{g} \frac{\rho g}{\pi} U \Gamma \times \pi = \frac{L}{g} \rho g U \Gamma = \frac{\rho g}{g} L U \Gamma = \rho L U \Gamma \quad \dots(14.16)$$

Equation (14.16) is known as Kutta-Joukowski equation.

**14.7.4 Drag Force Acting on a Rotating Cylinder.** The resultant flow pattern for a rotating cylinder in a uniform flow field is shown in Fig. 14.12. The resultant flow pattern is symmetrical about the vertical axis of the cylinder. Hence the velocity distribution and also pressure distribution is symmetrical about the vertical axis and as such there will be no drag on the cylinder.

**14.7.5 Expression for Lift Co-efficient for Rotating Cylinder.** The lift co-efficient is defined by the equation (14.4) as

$$F_L = C_L A \frac{\rho U^2}{2} \quad \dots(i)$$

where  $C_L$  = Lift co-efficient,  $A$  = Projected area

$U$  = Free stream velocity or uniform velocity of flow.

For a rotating cylinder, the lift force is given by equation (14.16)

$$\begin{aligned} F_L &= \rho L U \Gamma \\ A &= \text{Projected area of cylinder} = 2RL \end{aligned}$$

$\therefore$  Substituting these values in equation (i), we get

$$\rho L U \Gamma = C_L \times 2RL \times \frac{\rho U^2}{2} \quad \text{or} \quad C_L = \frac{\rho L U \Gamma}{R L \rho U^2} = \frac{\Gamma}{R U} \quad \dots(14.17)$$

From equation (14.14), we have  $u_{\theta_1} = \frac{\Gamma}{2\pi R}$  or  $\frac{\Gamma}{R} = 2\pi u_{\theta_1}$

Substituting this value of  $\frac{\Gamma}{R}$  in equation (14.17),  $C_L$  is also expressed

$$C_L = \frac{2\pi u_{\theta_1}}{U} \quad \dots(14.18)$$

where  $u_{\theta_1}$  = Velocity of rotation of the cylinder in the tangential direction.

#### 14.7.6 Location of Stagnation Points for a Rotating Cylinder in a Uniform Flow-field.

Stagnation points are those points on the surface of the cylinder, where velocity is zero. For a rotating cylinder as shown in Fig. 14.12, the resultant velocity is given by equation (14.15) as

$$u = 2U \sin \theta + \frac{\Gamma}{2\pi R}.$$

For stagnation point,  $u = 0$

$$\therefore 2U \sin \theta + \frac{\Gamma}{2\pi R} = 0 \text{ or } 2U \sin \theta = -\frac{\Gamma}{2\pi R}$$

$$\text{or } \sin \theta = -\frac{\Gamma}{4\pi UR}. \quad \dots(14.19)$$

The solution of equation (14.19) gives the location of stagnation points on the surface of the cylinder. There are two values of  $\theta$ , which satisfy equation (14.19). As  $\sin \theta$  is negative in equation (14.19), it means  $\theta$  is more than  $180^\circ$  but less than  $360^\circ$ . The two values of  $\theta$  are such that one value is between  $180^\circ$  and  $270^\circ$  and other value is between  $270^\circ$  and  $360^\circ$ .

For a single stagnation point,  $\theta = 270^\circ$  and then equation (14.19) becomes as

$$\sin 270^\circ = -\frac{\Gamma}{4\pi UR} \text{ or } -1 = -\frac{\Gamma}{4\pi UR} \quad (\because \sin 270^\circ = -1)$$

$$\therefore \Gamma = 4\pi UR. \quad \dots(14.20)$$

**14.7.7 Magnus Effect.** When a cylinder is rotated in a uniform flow, a lift force is produced on the cylinder. This phenomenon of the lift force produced by a rotating cylinder in a uniform flow is known as Magnus Effect. This fact was investigated by a German physicist H.G. Magnus and hence the name is given as Magnus Effect.

**Problem 14.21** A cylinder rotates at 150 r.p.m. with its axis perpendicular in an air stream which is having uniform velocity of 25 m/s. The cylinder is 1.5 m in diameter and 10 m long. Assuming ideal fluid theory, find (i) the circulation, (ii) lift force, and (iii) position of stagnation points. Take density of air as  $1.25 \text{ kg/m}^3$ .

**Solution.** Given :

$$\text{Speed of cylinder, } N = 150 \text{ r.p.m.}$$

$$\text{Velocity of air, } U = 25 \text{ m/s}$$

$$\text{Diameter of cylinder, } D = 1.5 \text{ m}$$

$$\therefore \text{Radius of cylinder, } R = \frac{D}{2} = \frac{1.5}{2} = 0.75 \text{ m}$$

$$\text{Length of cylinder, } L = 10 \text{ m}$$

$$\text{Density of air, } \rho = 1.25 \text{ kg/m}^3$$

$$\text{Tangential velocity of cylinder is given as } u_\theta = \frac{\pi DN}{60} = \frac{\pi \times 1.5 \times 150}{60} = 11.78 \text{ m/s.}$$

$$(i) \text{ Circulation } (\Gamma) \text{ is obtained from equation (14.14), as } u_\theta = \frac{\Gamma}{2\pi R}$$

$$\therefore \Gamma = 2\pi R \times u_\theta = 2\pi \times 0.75 \times 11.78 = 55.51 \text{ m}^2/\text{s. Ans.}$$

$$(ii) \text{ Lift force, } F_L \text{ is given by equation (14.16) as}$$

$$F_L = \rho L U \Gamma = 1.25 \times 10 \times 25 \times 55.51$$

$$= 17344 \text{ N. Ans.}$$

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(iii) Position of stagnation points are given by equation (14.19) as

$$\begin{aligned}\sin \theta &= -\frac{\Gamma}{4\pi UR} = -\frac{55.51}{4\pi \times 25 \times 0.75} = 0.2356 \\&= -\sin (13.62^\circ) \\&= \sin [180^\circ + 13.62^\circ] \text{ and } \sin [360^\circ - 13.62^\circ] \\&\therefore \theta = (180^\circ + 13.62^\circ) \text{ and } (360^\circ - 13.62^\circ) \\&= \mathbf{193.62^\circ \text{ and } 346.38^\circ. \quad Ans.}\end{aligned}$$

The location of stagnation points are shown in Fig. 14.14.

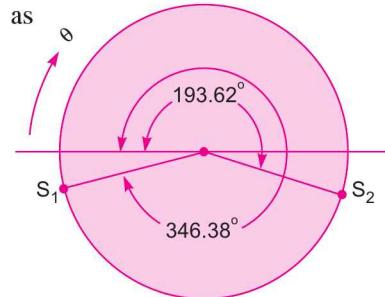


Fig. 14.14

**Problem 14.22** A cylinder whose axis is perpendicular to the stream of air having a velocity of 20 m/s, rotates at 300 r.p.m. The cylinder is 2 m in diameter and 10 m long. (a) Find : (i) the circulation, (ii) theoretical lift force per unit length, (iii) position of stagnation points, and (iv) the actual lift, drag and resultant force. Take density of air 1.24 kg/m<sup>3</sup>. For actual drag and lift, take C<sub>L</sub> = 3.4, C<sub>D</sub> = 0.65 and  $\frac{u_\theta}{U} = 1.57$ . (b) Find the speed of rotation of the cylinder which will give only a single stagnation point.

**Solution.** Given :

$$\begin{array}{ll} \text{Velocity of air,} & U = 20 \text{ m/s} \\ \text{Speed of rotation,} & N = 300 \text{ r.p.m.} \\ \text{Diameter of cylinder,} & D = 2 \text{ m} \\ \text{Length of cylinder,} & L = 10 \text{ m} \\ \text{Density of air,} & \rho = 1.24 \text{ kg/m}^3 \end{array}$$

$$\text{Tangential velocity of cylinder is given as } u_\theta = \frac{\pi DN}{60} = \frac{\pi}{60} \times 2.0 \times 300 = 31.42 \text{ m/s.}$$

(a) (i) Now the circulation ( $\Gamma$ ) is given by equation (14.14) as  $u_\theta = \frac{\Gamma}{2\pi R}$

$$\begin{aligned}\therefore \Gamma &= 2\pi R u_\theta = 2\pi \times \frac{D}{2} \times 31.42 \\&= 2\pi \times \frac{3}{2} \times 31.42 = \mathbf{197.41 \text{ m}^2/\text{s. Ans.}}\end{aligned}$$

(ii) The theoretical lift ( $F_L$ ) is given by equation (14.16) as

$$F_L = \rho U L \Gamma = 1.24 \times 20 \times 10 \times 197.41 = 48957.7 \text{ N}$$

$$\therefore \text{Theoretical lift per unit length} = \frac{F_L}{L} = \frac{48957.7}{10} = \mathbf{4895.77 \text{ N/m. Ans.}}$$

(iii) Position of stagnation points are obtained from equation (14.19) as

$$\begin{aligned}\sin \theta &= -\frac{\Gamma}{4\pi UR} = -\frac{197.41}{4\pi \times 20 \times D/2} = \frac{197.41}{4\pi \times 20 \times 1} = -0.7854 \\&= \sin [180^\circ + 51.75^\circ] \text{ and } \sin [360^\circ - 51.75^\circ] \\&= \sin [231.75^\circ] \text{ and } \sin [308.25^\circ] \\&\therefore \theta = \mathbf{231.75^\circ \text{ and } 308.25^\circ. \quad Ans.}\end{aligned}$$

Stagnation points will be at an angle of 231.75° and 308.25°.

(iv) Actual Lift, Drag and Direction of Resultant Force

For actual lift and drag, given  $\frac{u_\theta}{U} = 1.57$ ,  $C_L = 3.4$  and  $C_D = 0.65$ .

The ratio of  $\frac{u_\theta}{U}$  from theoretical consideration is given as  $\frac{u_\theta}{U} = \frac{31.42}{20} = 1.57$

$$\begin{aligned}\text{Now actual lift force is given by } F_L &= \frac{1}{2} \rho A U^2 \times C_L \\ &= \frac{1}{2} \times 1.24 \times (2 \times 10) \times 20^2 \times 3.4 = 16864 \text{ N}\end{aligned}$$

where  $A$  = projected area of cylinder =  $2 \times 10 \text{ m}^2$

$$\therefore F_L = 16864 \text{ N. Ans.}$$

$$\text{Actual drag force, } F_D = \frac{1}{2} \rho A U^2 \times C_D = \frac{1}{2} \times 1.24 \times (2 \times 10) \times 20^2 \times 0.65 = 3224 \text{ N. Ans.}$$

$$\begin{aligned}\therefore \text{Resultant force} &= \sqrt{F_L^2 + F_D^2} = \sqrt{16864^2 + 3224^2} \\ &= \sqrt{284394496 + 10394176} = 17169.4 \text{ N. Ans.}\end{aligned}$$

The direction of the resultant force with the horizontal is given by

$$\tan \theta = \frac{F_L}{F_D} = \frac{16864}{3224} = 5.23$$

$$\therefore \theta = \tan^{-1} 5.23 = 79.1^\circ. \text{ Ans.}$$

(b) Speed of rotation of the cylinder for single stagnation point.

For a single point stagnation, the equation (14.20) is used.

$$\therefore \Gamma = 4\pi UR = 4\pi \times 20 \times 1 = 251.32 \text{ m}^2/\text{s} \quad \left( \because R = \frac{D}{2} = \frac{2}{2} = 1 \text{ m} \right)$$

The speed of rotation, corresponding to the circulation = 251.32 is given by equation (14.14) as

$$u_\theta = \frac{\Gamma}{2\pi R} = \frac{251.32}{2\pi \times 1} = 40.0$$

$$\text{But } u_\theta = \frac{\pi DN}{60}$$

$$\therefore N = \frac{60 \times u_\theta}{\pi \times D} = \frac{60 \times 40}{\pi \times 2.0} = 381.97 \text{ r.p.m.} \approx 382.0 \text{ (say) r.p.m. Ans.}$$

**Problem 14.23** The air having a velocity of 40 m/s is flowing over a cylinder of diameter 1.5 m and length 10 m, when the axis of the cylinder is perpendicular to the air stream. The cylinder is rotated about its axis and a lift of 6867 N per metre length of the cylinder is developed. Find the speed of rotation and location of the stagnation points. The density of air is given as  $1.25 \text{ kg/m}^3$ .

**Solution.** Given :

Velocity of air,  $U = 40 \text{ m/s}$

Diameter of cylinder,  $D = 1.5 \text{ m}$

Length of the cylinder,  $L = 10 \text{ m}$

$$\text{Lift/metre length, } \frac{F_L}{L} = 6867 \text{ N}$$

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Density of air,  $\rho = 1.25 \text{ kg/m}^2$

From equation (14.16), we have  $F_L = \rho L U \Gamma$  or  $\frac{F_L}{L} = \rho U \Gamma$

$$\therefore 6867 = 1.25 \times 40 \times \Gamma$$

$$\therefore \Gamma = \frac{6867}{1.25 \times 40} = 137.36 \text{ m}^2/\text{s.}$$

Let the speed of rotation corresponding to circulation  $137.36 = u_\theta$ . Using equation (14.14),

$$u_\theta = \frac{\Gamma}{2\pi R} = \frac{137.36}{2\pi \times \frac{D}{2}} = \frac{137.36 \times 2}{2\pi \times 1.5} = 29.15 = \frac{\pi D N}{60}$$

$$\therefore N = \frac{60 \times 29.15}{\pi \times D} = \frac{60 \times 29.15}{\pi \times 1.5} = 371.15 \text{ r.p.m. Ans.}$$

Position of stagnation points are given by equation (14.19)

$$\begin{aligned} \sin \theta &= -\frac{\Gamma}{4\pi U R} = -\frac{137.36}{4\pi \times 40 \times \frac{D}{2}} = -\frac{137.36 \times 2}{4\pi \times 53 \times 1.5} \\ &= -0.3643 = -\sin 21.36^\circ \\ &= \sin (180^\circ + 21.36^\circ) \text{ and } \sin [360^\circ - 21.36^\circ] \\ &= \sin 201.36^\circ \text{ and } \sin 338.64^\circ \\ \therefore \theta &= 201.36^\circ \text{ and } 338.64^\circ. \text{ Ans.} \end{aligned}$$

### ► 14.8 DEVELOPMENT OF LIFT ON AN AIRFOIL

Fig. 14.15 shows the two shapes of the airfoils, which are stream-lines bodies which may be symmetrical or unsymmetrical in shapes. The airfoil is characterized by its chord length  $C$ , angle of attack  $\alpha$  (which is the angle between the direction of the fluid flowing and chord line) and span  $L$  of the airfoil. The lift on the airfoil is due to negative pressure created on the upper part of the airfoil. The drag force on the airfoil is always small due to the design of the shape of the body, which is stream-lined.

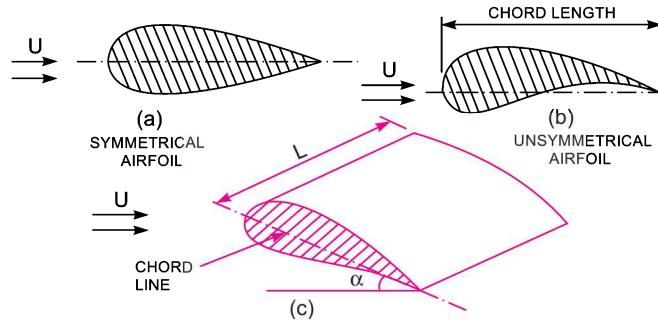


Fig. 14.15 *Shapes of airfoils.*

From the theoretical analysis, the circulation  $\Gamma$  developed on the airfoil so that the stream-line at the trailing edge of the airfoil is tangential to the airfoil, is given as

$$\Gamma = \pi C U \sin \alpha \quad \dots(14.21)$$

where  $C$  = Chord length,  $U$  = Free stream velocity of airfoil,  $\alpha$  = Angle of attack

Lift force  $F_L$  is given by equation (14.16) as

$$\begin{aligned} F_L &= \rho U L \Gamma = \rho U L \times \pi C U \sin \alpha \\ &= \pi \rho C U^2 L \sin \alpha \end{aligned} \quad (\because \Gamma = \pi C U \sin \alpha) \quad \dots(14.22)$$

The lift force is also given by equation (14.4), as

$$F_L = C_L \times A \times \frac{\rho U^2}{2}$$

where  $C_L$  = Co-efficient of lift

$A$  = Projected area =  $C \times L$  for airfoil

$$\therefore F_L = C_L \times C \times L \times \frac{\rho U^2}{2} \quad \dots(14.23)$$

Equating the two value of lift force given by equations (14.22) and (14.23), we get

$$\begin{aligned} \pi \rho C U^2 L \sin \alpha &= C_L \times C \times L \times \frac{\rho U^2}{2} \\ \therefore C_L &= \frac{2 \pi \rho C U^2 L \sin \alpha}{C \times L \times \rho U^2} = 2 \pi \sin \alpha \end{aligned} \quad \dots(14.24)$$

Thus it is clear from equation (14.23), that co-efficient of lift depends upon the angle of attack.

**14.8.1 Steady-state of a Flying Object.** When a flying object for example airplane is in a steady-state, the weight of the airplane is equal to the lift force and thrust developed by the engine is equal to the drag force. Hence

$$W = \text{Lift force} = C_L \frac{\rho A U^2}{2} \quad \dots(14.25)$$

where  $W$  = Weight of the airplane and  $C_L \frac{\rho A U^2}{2}$  = Lift force.

**Problem 14.24** An airfoil of chord length 2 m and of span 15 m has an angle of attack as  $6^\circ$ . The airfoil is moving with a velocity of 80 m/s in air whose density is  $1.25 \text{ kg/m}^3$ . Find the weight of the airfoil and the power required to drive it. The values of co-efficient of drag and lift corresponding to angle of attack are given as 0.03 and 0.5 respectively.

**Solution.** Given :

Chord length,	$C = 2 \text{ m}$
Span of airfoil,	$L = 15 \text{ m}$
Angle of attack,	$\alpha = 6^\circ$
Velocity of airfoil,	$U = 80 \text{ m/s}$
Density of air,	$\rho = 1.25 \text{ kg/m}^3$
Co-efficient of drag,	$C_D = 0.03$
Co-efficient of lift,	$C_L = 0.50$

From equation (14.25), we know that

$$\begin{aligned} \text{Weight of airfoil} &= \text{Lift force} = C_L \frac{\rho A U^2}{2} = 0.50 \times 1.25 \times (C \times L) \times \frac{80^2}{2} \end{aligned}$$

$$= 0.50 \times 1.25 \times (2 \times 15) \times \frac{80^2}{2} = 60000 \text{ N. Ans.}$$

Now drag force,  $F_D = C_D \times \rho \times \frac{AU^2}{2}$

$$= 0.03 \times 1.25 \times \frac{(2 \times 15) \times 80^2}{2} = 3600 \text{ N. Ans.}$$

$$\therefore \text{Power required in kW} = \frac{F_D \times U}{1000} = \frac{3600 \times 80}{1000} = 288 \text{ kW. Ans.}$$

**Problem 14.25** A jet plane which weighs 29430 N and has a wing area of  $20 \text{ m}^2$  flies at a velocity of 250 km/hr. When the engine delivers 7357.5 kW. 65% of the power is used to overcome the drag resistance of the wing. Calculate the co-efficient of lift and co-efficient of drag for the wing. Take density of air equal to  $1.21 \text{ kg/m}^3$ .

**Solution.** Given :

Weight of plane,  $W = 29430 \text{ N}$

Wing area,  $A = 20 \text{ m}^2$

Velocity of plane,  $U = 250 \text{ km/hr} = \frac{250 \times 1000}{60 \times 60} \text{ m/s} = 69.44 \text{ m/s}$

Power delivered by engine = 7357.5 kW

Power required to overcome drag resistance  
= 65% of 7357.5 =  $0.65 \times 7357.5 = 4782.375 \text{ kW.}$

Density of air,  $\rho = 1.21 \text{ kg/m}^3$

Now, weight of plane = Lift force =  $C_L \times A \times \frac{\rho U^2}{2}$

$\therefore 29430 = C_L \times 20 \times 1.21 \times \frac{69.44^2}{2}$

$\therefore C_L = \frac{29430 \times 2}{20 \times 1.21 \times 69.44^2} = 0.5046. \text{ Ans.}$

Let  $F_D = \text{Drag force}$

Power required to overcome drag resistance =  $\frac{F_D \times U}{1000} \text{ kW}$

$\therefore 4782.375 = \frac{F_D \times 69.44}{1000}$

$\therefore F_D = \frac{4782.375 \times 1000}{69.44} = 68870.6 \text{ N}$

Now drag force,  $F_D = C_D \times A \times \frac{\rho U^2}{2}$

$\therefore 68870.6 = C_D \times 20 \times \frac{1.21 \times 69.44^2}{2}$

$\therefore C_D = \frac{68870.6 \times 2}{20 \times 1.21 \times 69.44^2} = 1.18. \text{ Ans.}$

## HIGHLIGHTS

1. The force exerted by a fluid on a solid body immersed in the fluid in the direction of motion is called drag force while the force perpendicular to the direction of motion, on the body is known as lift force.
2. The mathematical expression for the drag and lift force are,

$$F_L = C_D A \frac{\rho U^2}{2}; F_L = C_L A \frac{\rho U^2}{2}$$

where  $C_D$  = Co-efficient of drag,  $C_L$  = Co-efficient of lift,  
 $A$  = Projected area of the body,  $\rho$  = Density of fluid,  
 $U$  = Free-stream velocity of fluid.

3. The resultant force exerted by fluid on solid body is  $F_R = \sqrt{F_D^2 + F_L^2}$ .
4. Total drag on a body is the sum of pressure drag and friction drag.
5. A body whose surface coincides with the stream-lines, when the body is placed in a flow is called streamlined body. If the surface of the body does not coincide with the stream-lines, the body is called bluff body.
6. The drag on a sphere for Reynolds number less than 0.2 is given by  $F_D = 3\pi\mu DU$

Out of this total drag, Skin friction drag  $= \frac{2}{3} \times 3\pi\mu DU = 2\pi\mu DU$

and pressure drag  $= \frac{1}{3} \times 3\pi\mu DU = \pi\mu DU$ .

7. Values of  $C_D$  for sphere for different Reynold number is

$$\begin{aligned} C_D &= \frac{24}{R_e} \dots \text{when } R_e < 0.2 \\ &= \frac{24}{R_e} \left[ 1 + \frac{3}{16R_e} \right] \dots \text{when } 0.2 < R_e < 5.0 \\ &= 0.4 \dots \text{when } R_e \text{ lies between 5 and 1000} \\ &= 0.5 \dots \text{when } R_e \text{ lies between 100 and 100000} \\ &= 0.2 \dots \text{when } R_e > 10^5. \end{aligned}$$

8. Terminal velocity is defined as the maximum constant velocity of a falling body with which it will travel. At the terminal velocity, the weight of the body is equal to the drag force plus the buoyant force. Hence  $W = F_D + F_B$ .

9. The velocity of ideal fluid at any point on the surface of the cylinder is given by

$$u_\theta = 2U \sin \theta$$

where  $u_\theta$  = Tangential velocity on the surface of the cylinder

$U$  = Uniform velocity or free stream velocity

$\theta$  = Angle made by any point on the surface of the cylinder with the direction of flow.

10. Circulation is the flow along a closed curve and is obtained when the product of velocity along the closed curve and length of the small element is integrated around the curve. Circulation for free vortex at any radius  $R$  is given by

$$\Gamma = 2\pi R \times u_\theta.$$

11. The resultant velocity on a circular cylinder which is rotated at constant speed in uniform flow-field is

$$u = 2U \sin \theta + \frac{\Gamma}{2\pi R}.$$

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12. When a circular cylinder is rotated in a uniform flow, a lift force is produced on the cylinder. The magnitude of the lift force  $F_L$  is given by

$$F_L = \rho L U \Gamma$$

where  $L$  = Length of cylinder,  $U$  = Free stream velocity,  $\Gamma$  = Circulation.

13. The expression for lift co-efficients for a rotating cylinder in a uniform flow is given by

$$C_L = \frac{\Gamma}{R U} \quad \dots(\text{in term of circulation})$$

$$= \frac{2\pi u_\theta}{U} \quad \dots(\text{in term of tangential speed})$$

14. The location of stagnation points is given by  $\sin \theta = - \frac{\Gamma}{4\pi U R}$

where  $R$  = Radius of cylinder,  $U$  = Free stream velocity.

15. For a single stagnation point, the condition is

$$\Gamma = 4\pi U R \quad \dots(\text{in terms of circulation})$$

or  $u_\theta = 2U \quad \dots(\text{in terms of tangential velocity})$

16. Circulation developed on the airfoil is given by

$$\Gamma = \pi C U \sin \alpha$$

where  $C$  = Chord length,  $U$  = Velocity of airfoil,  $\alpha$  = Angle of attack.

17. The expression for co-efficient of lift for an airfoil is  $C_L = 2\pi \sin \alpha$ .

18. When an airplane is in steady-state,

Weight of plane = Lift force

Thrust by engine = Drag force.

## EXERCISE

### (A) THEORETICAL PROBLEMS

- Define the terms : drag and lift.
- What do you understand by : Total drag on a body, resultant force on a body, co-efficient of drag and co-efficient of lift.
- Differentiate between (i) stream-lines body and bluff body, (ii) Friction drag and pressure drag.
- (a) What is the expression for the drag on a sphere, when the Reynolds number of the flow is upto 0.2 ? Hence prove that the co-efficient of drag for sphere for this range of the Reynolds number is given by

$$C_D = \frac{24}{R_e}, \text{ where } R_e = \text{Reynolds number.}$$

(b) Draw  $C_D$  versus  $R_e$  diagram for a sphere and explain why  $C_D$  suddenly drops at  $R_e = 3 \times 10^3$ .

(c) Draw pressure distribution diagrams in dimensionless form for flow past sphere when fluid has no viscosity, when  $R_e = 10^4$  and when  $R_e = 10^6$ .

- What do you mean by 'Terminal velocity of a body' ? What is the relation between the weight of the body, drag force on the body and buoyant force when the body has acquired terminal velocity ?
- What is circulation ? Find an expression for circulation for a free-vortex of radius  $R$ .
- Obtain an expression for the lift produced on a rotating cylinder placed in a uniform flow field such that the axis of the cylinder is perpendicular to the direction of flow.
- What is Magnus effect ? Why is it known as Magnus effect ?

9. Prove that the co-efficient of lift for a rotating cylinder placed in a uniform flow is given by

$$C_L = \frac{\Gamma}{RU}$$

where  $\Gamma$  = Circulation,  $R$  = Radius of cylinder,  $U$  = Free-stream velocity.

10. Define stagnation points. How the position of the stagnation points for a rotating cylinder in a uniform flow is determined ? What is the condition for single stagnation point ?
11. Define the terms : Airfoil, chord length, angle of attack, span of an airfoil.
12. If the circulation developed on an airfoil is equal to  $\pi CU \sin \alpha$ , then prove that co-efficient of lift for airfoil is given by  $C_L = 2\pi \sin \alpha$ , where  $\alpha$  = angle of attack.
13. Explain the terms : (i) Friction drag, (ii) Pressure drag and profile drag.
14. (a) How are drag and lift forces caused on a body immersed in a moving fluid ?  
 (b) What is the drag force on a sphere in the stoke range ?

### (B) NUMERICAL PROBLEMS

1. A flat plate  $2 \text{ m} \times 2 \text{ m}$  moves at 40 km/hour in stationary air of density  $1.25 \text{ kg/m}^3$ . If the co-efficient of drag and lift are 0.2 and 0.8 respectively, find : (i) the lift force, (ii) the drag force, (iii) the resultant force, and (iv) the power required to keep the plate in motion.  
 [Ans. (i) 246.86 N, (ii) 61.715 N, (iii) 254.4 N, (iv) 0.684 kW]
2. Find the drag force difference on a flat plate of size  $1.5 \text{ m} \times 1.5 \text{ m}$  when the plate is moving at a speed of 5 m/s normal to its plate first in water and second in air of density  $1.24 \text{ kg/m}^3$ . Co-efficient of drag is given as 1.10.  
 [Ans. 30899 N]
3. A truck having a projected area of 12 square metres travelling at 60 km/hr has a total resistance of 2943 N. Of this 25% is due to rolling friction and 15% is due to surface friction. The rest is due to form drag. Calculate the coefficient of form drag if the density of air =  $1.25 \text{ kg/m}^3$ .  
 [Ans. 0.847]
4. A circular disc 4 m in diameter is held normal to 30 m/s wind of density  $1.25 \text{ kg/m}^3$ . If the co-efficient of drag of disc = 1.1, what force is required to hold the disc at rest ?  
 [Ans. 7775.4 N]
5. Find the diameter of a parachute with which a man of mass 80 kg descends to the ground from an aeroplane against the resistance of air, with a velocity of 25 m/s. Take  $C_d = 0.5$  and density of air =  $1.25 \text{ kg/m}^3$ .  
 [Ans. 2.26 m]
6. A man descends to the ground with the help of a parachute from an aeroplane against the resistance of air with a uniform velocity of 10 m/s. The parachute is hemispherical in shape and is having diameter of 5 m. Find the weight of man if  $C_d = 0.5$  and density of air =  $1.25 \text{ kg/m}^3$ .  
 [Ans. 613.52 N]
7. A kite  $60 \text{ cm} \times 60 \text{ cm}$  weighing 2.943 N assumes an angle of  $10^\circ$  to the horizontal. The string attached to the kite makes an angle of  $45^\circ$  to the horizontal. If the pull on the string is 29.43 N when the wind is flowing at a speed of 40 km/hr. Find the corresponding co-efficient to drag and lift. Density of air is given as  $1.25 \text{ kg/m}^3$ .  
 [Ans.  $C_D = 0.7489$ ,  $C_L = .8548$ ]
8. The air is flowing over a cylinder of diameter 100 mm and of infinite length with a velocity of 150 mm/s. Find the total drag, shear drag and pressure drag on 1 m length of the cylinder if the total drag co-efficient = 1.5 m and shear drag co-efficient = 0.25. The density of air is given as =  $1.25 \text{ kg/m}^3$ .  
 [Ans. 0.00211 N, 0.000351 N, 0.001756 N]
9. A body of length 2.5 m has a projected area  $1.8 \text{ m}^2$  normal to the direction of its motion. The body is moving through water with a velocity such that the Reynold number =  $6 \times 10^6$  and the drag co-efficient = 0.5. Find the drag on the body. Take viscosity of water = 0.01 poise.  
 [Ans. 2592 N]
10. Calculate the weight of a ball of diameter 50 mm which is just supported in a vertical air stream which is flowing at a velocity of 10 m/s. The density of air =  $1.25 \text{ kg/m}^3$  and kinematic viscosity = 1.5 stokes.  
 [Ans. 0.0613 N]

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11. A metallic sphere of sp. gr. 8.0 falls in an oil of density  $800 \text{ kg/m}^3$ . The diameter of the sphere is 10 mm and it attains a terminal velocity of 50 mm/s. Find the viscosity of the oil in poise. [Ans. 78.48 poise]
12. A metallic ball of diameter 5 mm drops in a fluid of sp. gr. 0.8 and viscosity 30 poise. The specific gravity of the metallic ball, is 9.0. Find : (i) the drag force exerted by fluid on metallic ball, (ii) the pressure drag and skin friction drag, and (iii) terminal velocity of ball in fluid.  
[Ans. (i) 0.005264 N, (ii) 0.001754 N, 0.003527 N, (iii) 3.7 cm/s]
13. A cylinder rotates at 200 r.p.m. with its axis perpendicular in an air stream which is having uniform velocity of 20 m/s. The cylinder is 2 m in diameter and 8 m long. Assuming ideal fluid theory, find (i) the circulation, (ii) lift force, and (iii) position of stagnation points. Take density of air as  $1.25 \text{ kg/m}^3$ .  
[Ans. (i)  $131.57 \text{ m}^2/\text{s}$ , (ii) 26309.4 N, (iii)  $\theta = 211.56^\circ$  and  $328.44^\circ$ ]
14. For the problem 13, find the speed of rotation of the cylinder which will give only a single stagnation point.  
[Ans. 381.97 r.p.m.]
15. The air having a velocity of 30 m/s is flowing over a cylinder of diameter 1.4 m and length 10 m, when the axis of the cylinder is perpendicular to the air stream. The cylinder is rotated about its axis and a total lift of 58860 N is produced. Find the speed of rotation and location of the stagnation points. The density of air is given as  $1.25 \text{ kg/m}^3$ .  
[Ans.  $N = 486.87 \text{ r.p.m.}$ ,  $\theta = 216.5^\circ$  and  $323.5^\circ$ ]
16. A jet plane which weighs 19620 N has a wing area of  $25 \text{ m}^2$ . It is flying at a speed of 200 km per hour. When the engine develops 588.6 kW, 70% of this power is used to overcome the drag resistance of the wing. Calculate the co-efficient of lift and co-efficient of drag for the wing. Taken density of air as  $1.25 \text{ kg/m}^3$ .  
[Ans.  $C_L = .407$ ,  $C_D = .114$ ]
17. Experiments were conducted in a wind tunnel with a wind speed of 50 km/hour on a flat plate of size 2 m long and 1 m wide. The density of air is  $1.15 \text{ kg/m}^3$ . The plate is kept at such an angle that co-efficients of lift and drag are 0.75 and 0.15 respectively. Determine : (i) lift force, (ii), drag force, (iii) resultant force, (iv) its direction, and (v) power exerted by the air stream on the plate.  
[Ans. (i) 166.4 N, (ii) 33.28 N, (iii) 169.64 N, (iv)  $78.7^\circ$ , (v) 0.461 kW]

# 15

## CHAPTER



### ► 15.1 INTRODUCTION

Compressible flow is defined as that flow in which the density of the fluid does not remain constant during flow. This means that the density changes from point to point in compressible flow. But in case of incompressible flow, the density of the fluid is assumed to be constant. In the previous chapters, the fluid was assumed incompressible, and the basic equations such as equation of continuity, Bernoulli's equation and impulse momentum equations were derived on the assumption that fluid is incompressible. This assumption is true for flow of liquids, which are incompressible fluids. But in case of flow of fluids, such as

- (i) flow of gases through orifices and nozzles,
- (ii) flow of gases in machines such as compressors, and
- (iii) projectiles and airplanes flying at high altitude with high velocities, the density of the fluid changes during the flow. The change in density of a fluid is accompanied by the changes in pressure and temperature and hence the thermodynamic behaviour of the fluids will have to be taken into account.

### ► 15.2 THERMODYNAMIC RELATIONS

The thermodynamic relations have been discussed in Chapter 1, which are as follows :

**15.2.1 Equation of State.** Equation of state is defined as the equation which gives the relationship between the pressure, temperature and specific volume of a gas. For a perfect gas the equation of state is

$$p \nabla = RT \quad \dots(15.1)$$

where  $p$  = Absolute pressure in kgf/m<sup>2</sup> or N/m<sup>2</sup>

$\nabla$  = Specific volume or volume per unit mass

$T$  = Absolute temperature =  $273 + t^\circ$  (centigrade)

$R$  = Gas constant in kgf-m/kg °K or (J/kg K)

= 29.2 kgf-m/kg °K or 287 J/kg K for air.

In equation (15.1),  $\nabla$  is the specific volume which is the reciprocal of density or

$$\nabla = \frac{1}{\rho}.$$

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Substituting this value of  $\nabla$  in equation (15.1), we get

$$\frac{p}{\rho} = RT \quad \dots(15.2)$$

**Note.** In the equation of state given by equation (15.2), the dimensions of  $p$ ,  $\rho$  and  $R$  should be used with care. The following points must be remembered :

1. If the value of  $R$  is given as  $29.2 \text{ kgf-m/kg } ^\circ\text{K}$  for air, the corresponding value of  $p$  and  $\rho$  should be taken in  $\text{kgf/m}^2$  and  $\text{kg/m}^3$ . The mass rate of flow of the gas will be in  $\text{kg/sec}$ .
2. If the value of  $R$  is given as  $287 \text{ J/kg K}$ , the corresponding value of  $p$  and  $\rho$  should be in  $\text{N/m}^2$  and  $\text{kg/m}^3$ . The mass rate of flow will be in  $\text{kg/sec}$ .

**Value of  $\frac{p}{\rho}$  in Bernoulli's Equation\*.** (i) If the value of  $p$  is taken in  $\text{N/m}^2$ , the corresponding value of  $\rho$  is in  $\text{kg/m}^3$ . And as mentioned above (point number 2), the value of  $R$  should be  $287 \text{ J/kg K}$ .

(ii) If the value of  $p$  is taken in  $\text{kgf/m}^2$  in Bernoulli's equation, the corresponding value of  $\rho$  should be in  $\text{msl/m}^3$ . But as mentioned in point number 1, if the value of  $R$  is taken  $29.2$ , the corresponding values of  $p$  and  $\rho$  are in  $\text{kgf/m}^2$  and  $\text{kg/m}^3$ . Hence the mass density in equation of state is in  $\text{kg/m}^3$  while in Bernoulli's equation it is in  $\text{msl/m}^3$ . The density calculated from equation of state must be converted into  $\text{msl/m}^3$ .

**Note.** It is better to use pressure in  $\text{N/m}^2$ , density in  $\text{kg/m}^3$  and value of  $R = 287 \text{ J/kg K}$ . The value of density calculated from equation of state will be in the same dimensions as used in Bernoulli's equation.

**15.2.2 Expansion and Compression of Perfect Gas.** When the expansion or compression of a perfect gas takes place, the pressure, temperature and density are changed. The change in pressure, temperature and density of a gas is brought about by the two processes which are known as

1. Isothermal process, and
2. Adiabatic process.

**1. Isothermal Process.** This is the process in which a gas is compressed or expanded while the temperature is kept constant. The gas obeys Boyle's law, according to which we have

$$p\nabla = \text{Constant, where } \nabla = \text{Specific volume}$$

or

$$\frac{p}{\rho} = \text{Constant} \quad \left( \because \nabla = \frac{1}{\rho} \right) \dots(15.3)$$

**2. Adiabatic Process.** If the compression or expansion of a gas takes place in such a way that the gas neither gives heat, nor takes heat from its surrounding, then the process is said to be adiabatic. According to this process,

$$p\nabla^k = \text{Constant}$$

where  $k$  = Ratio of the specific heat at constant pressure to the specific heat at constant volume

$$= \frac{C_p}{C_v} = 1.4 \text{ for air.}$$

The above relation is also written as  $\frac{p}{\rho^k} = \text{Constant.}$   $\dots(15.4)$

If the adiabatic process is reversible (or frictionless), it is known as isentropic process. And if the pressure and density are related in such a way that  $k$  is not equal to  $\frac{C_p}{C_v}$  but equal to some positive value then the process is known as polytropic. According to which

$$\frac{p}{\rho^n} = \text{Constant} \quad \dots(15.5)$$

where  $n \neq k$  but equal to some positive constant.

\* Please, refer to equations (15.10) and (15.11).

## ► 15.3 BASIC EQUATIONS OF COMPRESSIBLE FLOW

The basic equations of the compressible flows are

1. Continuity Equation,
2. Bernoulli's Equation or Energy Equation,
3. Momentum Equation,
4. Equation of state.

**15.3.1 Continuity Equation.** This is based on law of conservation of mass which states that matter cannot be created nor destroyed. Or in other words, the matter or mass is constant. For one-dimensional steady flow, the mass per second =  $\rho AV$

where  $\rho$  = Mass density,  $A$  = Area of cross-section,  $V$  = Velocity

As mass or mass per second is constant according to law of conservation of mass. Hence

$$\rho AV = \text{Constant.} \quad \dots(15.6)$$

Differentiating equation (15.6),  $d(\rho AV) = 0$  or  $\rho d(AV) + AVd\rho = 0$   
or  $\rho [AdV + VdA] + AVd\rho = 0$  or  $\rho AdV + \rho VdA + AVd\rho = 0$

Dividing by  $\rho AV$ , we get  $\frac{dV}{V} + \frac{dA}{A} + \frac{dp}{\rho} = 0.$   $\dots(15.7)$

Equation (15.7) is also known as continuity equation in differential form.

**15.3.2 Bernoulli's Equation.** Bernoulli's equation has been derived for incompressible fluids in Chapter 6. The same procedure is followed. The flow of a fluid particle along a stream-line in the direction of  $S$  is considered. The resultant force on the fluid particle in the direction of  $S$  is equated to the mass of the fluid particle and its acceleration. As the flow of compressible fluid is steady, the same Euler's equation as given by equation (6.3) is obtained as

$$\frac{dp}{\rho} + VdV + gdZ = 0 \quad \dots(15.8)$$

Integrating the above equation, we get

$$\int \frac{dp}{\rho} + \int VdV + \int gdZ = \text{Constant}$$

or  $\int \frac{dp}{\rho} + \frac{V^2}{2} + gZ = \text{Constant} \quad \dots(15.9)$

In case of incompressible flow, the density  $\rho$  is constant and hence integration of  $\frac{dp}{\rho}$  is equal to  $\frac{p}{\rho}$ .

But in case of compressible flow, the density  $\rho$  is not constant. Hence  $\rho$  cannot be taken outside the integration sign. With the change of  $\rho$ , the pressure  $p$  also changes for compressible fluids. This change of  $\rho$  and  $p$  takes place according to equations (15.3) or (15.4) depending upon the type of process during compressible flow. The value of  $\rho$  from these equations in terms of  $p$  is obtained and is substituted in  $\int \frac{dp}{\rho}$  and then the integration is done. The Bernoulli's equation will be different for isothermal process and for adiabatic process.

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**(A) Bernoulli's Equation for Isothermal Process.** For isothermal process, the relation between pressure ( $p$ ) and density ( $\rho$ ) is given by equation (15.3) as

$$\frac{p}{\rho} = \text{Constant} = C_1 \text{ (say)} \quad \dots(i)$$

$$\therefore \rho = \frac{p}{C_1}$$

$$\begin{aligned} \text{Hence} \quad \int \frac{dp}{\rho} &= \int \frac{dp}{p/C_1} = \int \frac{C_1 dp}{p} = C_1 \int \frac{dp}{p} \quad (\because C_1 \text{ is constant}) \\ &= C_1 \log_e p = \frac{p}{\rho} \log_e p \quad \left( \because C_1 = \frac{p}{\rho} \text{ from equation (i)} \right) \end{aligned}$$

Substituting the value  $\int \frac{dp}{\rho}$  in equation (15.9), we get

$$\frac{p}{\rho} \log_e p + \frac{V^2}{2} + gZ = \text{Constant}$$

$$\text{Dividing by 'g', } \frac{p}{\rho g} \log_e p + \frac{V^2}{2g} + Z = \text{Constant.} \quad \dots(15.10)$$

Equation (15.10) is the Bernoulli's equation for compressible flow undergoing isothermal process. For the two points 1 and 2, this equation is written as

$$\frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g} + Z_2 \quad \dots(15.11)$$

**(B) Bernoulli's Equation for Adiabatic Process.** For the adiabatic process, the relation between pressure ( $p$ ) and density ( $\rho$ ) is given by equation (15.4) as

$$\frac{p}{\rho^k} = \text{Constant} = \text{say } C_2 \quad \dots(ii)$$

$$\therefore \rho^k = \frac{p}{C_2} \quad \text{or} \quad \rho = \left( \frac{p}{C_2} \right)^{1/k}$$

$$\begin{aligned} \text{Hence} \quad \int \frac{dp}{\rho} &= \int \frac{dp}{\left( \frac{p}{C_2} \right)^{1/k}} = \int \frac{C_2^{1/k}}{p^{1/k}} dp = C_2^{1/k} \int \frac{1}{p^{1/k}} dp \\ &= C_2^{1/k} \int p^{-1/k} dp = C_2^{1/k} \frac{p^{\left( -\frac{1}{k} + 1 \right)}}{\left( -\frac{1}{k} + 1 \right)} \\ &= \frac{C_2^{1/k} p^{\left( \frac{k-1}{k} \right)}}{\left( \frac{k-1}{k} \right)} = \left( \frac{k}{k-1} \right) C_2^{1/k} p^{\left( \frac{k-1}{k} \right)} \end{aligned}$$

$$= \left( \frac{k}{k-1} \right) \left( \frac{p}{\rho^k} \right)^{1/k} p^{\left( \frac{1-k}{k} \right)} \quad \left( \because C_2^{1/k} = \frac{p}{\rho^k} \text{ from (ii)} \right)$$

$$= \left( \frac{k}{k-1} \right) \frac{p^{1/k}}{\rho^{k \times 1/k}} p^{\left( \frac{k-1}{k} \right)} = \left( \frac{k}{k-1} \right) \frac{p^{\frac{1}{k} + \frac{k-1}{k}}}{\rho} = \left( \frac{k}{k-1} \right) \frac{p}{\rho}$$

Substituting the value of  $\int \frac{dp}{\rho} = \left( \frac{k}{k-1} \right) \frac{p}{\rho}$  in equation (15.9), we get

$$\left( \frac{k}{k-1} \right) \frac{p}{\rho} + \frac{V^2}{2} + gZ = \text{Constant}$$

$$\text{Dividing by } 'g' \quad \left( \frac{k}{k-1} \right) \frac{p}{\rho g} + \frac{V^2}{2g} + Z = \text{Constant.} \quad \dots(15.12)$$

Equation (15.12) is the Bernoulli's equation for compressible flow undergoing adiabatic process. For the two points 1 and 2, this equation is written as

$$\left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left( \frac{k}{k-1} \right) \frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2 \quad \dots(15.13)$$

**Problem 15.1** A gas is flowing through a horizontal pipe at a temperature of  $4^\circ\text{C}$ . The diameter of the pipe is 8 cm and at a section 1-1 in this pipe, the pressure is  $30.3 \text{ N/cm}^2$  (gauge). The diameter of the pipe changes from 8 cm to 4 cm at the section 2-2, where pressure is  $20.3 \text{ N/cm}^2$  (gauge). Find the velocities of the gas at these sections assuming an isothermal process. Take  $R = 287.14 \text{ Nm/kg K}$ , and atmospheric pressure =  $10 \text{ N/cm}^2$ .

**Solution.** Given :

For the section 1-1,

Temperature,  $t_1 = 4^\circ\text{C}$

$\therefore$  Absolute temperature,  $T_1 = 4 + 273 = 277^\circ\text{K}$

Diameter pipe,  $D_1 = 8 \text{ cm} = 0.08 \text{ m}$

$\therefore$  Area of pipe,  $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (0.08)^2 = .005026 \text{ m}^2$

Pressure,  $p_1 = 30.3 \text{ N/cm}^2$  (gauge)  
 $= 30.3 + 10 = 40.3 \text{ N/cm}^2$  (absolute)  $= 40.3 \times 10^4 \text{ N/m}^2$  (abs.)

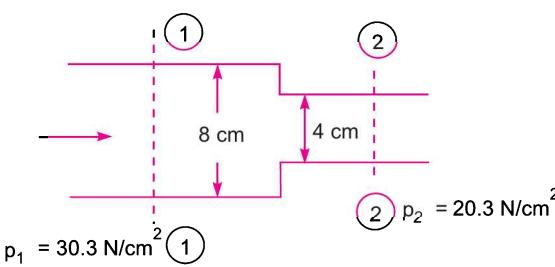


Fig. 15.1

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For the section 2-2,

$$\text{Diameter of pipe, } D_2 = 4 \text{ cm} = .04 \text{ m}$$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (.04)^2 = .0012565 \text{ m}^2$$

$$\text{Pressure, } p_2 = 20.3 + 10 = 30.3 \text{ N/cm}^2 (\text{abs.}) = 30.3 \times 10^4 \text{ N/m}^2 (\text{abs.})$$

$$\text{Gas constant, } R = 287.14 \text{ N-m/kg}^\circ\text{K}$$

$$\text{Ratio of specific heat, } k = 1.4.$$

Applying continuity equation at sections (1) and (2), we get

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$\text{or } \frac{V_2}{V_1} = \frac{\rho_1 A_1}{\rho_2 A_2} = \frac{\rho_1 \times .005026}{\rho_2 \times .0012565} = 4 \times \frac{\rho_1}{\rho_2} \quad \dots(i)$$

For isothermal process using equation (15.3),

$$\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2} \text{ or } \frac{\rho_1}{\rho_2} = \frac{p_1}{p_2} = \frac{40.3 \times 10^4}{30.3 \times 10^4} = 1.33$$

Substituting the value of  $\frac{\rho_1}{\rho_2} = 1.33$  in equation (i), we get

$$\frac{V_2}{V_1} = 4 \times 1.33 = 5.32$$

$$\therefore V_2 = 5.32 V_1 \quad \dots(ii)$$

Applying Bernoulli's equation at sections 1-1 and 2-2 for isothermal process which is given by equation (15.11), we get

$$\frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g} + Z_2$$

$$\text{For horizontal pipe, } Z_1 = Z_2$$

$$\therefore \frac{p_1}{\rho_1 g} \log_e p_1 + \frac{V_1^2}{2g} = \frac{p_2}{\rho_2 g} \log_e p_2 + \frac{V_2^2}{2g}$$

$$\text{or } \frac{p_1}{\rho_1 g} \log_e p_1 - \frac{p_2}{\rho_2 g} \log_e p_2 = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$\text{But for isothermal process, } \frac{p_1}{\rho_1} = \frac{p_2}{\rho_2}$$

$$\therefore \frac{p_1}{\rho_1 g} \log_e p_1 - \frac{p_1}{\rho_1 g} \log_e p_2 = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$\text{or } \frac{p_1}{\rho_1 g} \left[ \log_e \frac{p_1}{p_2} \right] = \frac{(5.32 V_1)^2}{2g} - \frac{V_1^2}{2g} \quad (\because \text{ From (ii), } V_2 = 5.32 V_1)$$

$$\text{or } \frac{p_1}{\rho_1 g} \log_e \left( \frac{40.3 \times 10^4}{30.3 \times 10^4} \right) = \frac{V_1^2}{2g} (5.32^2 - 1) = 27.30 \frac{V_1^2}{2g}$$

$$\text{or } \frac{p_1}{\rho_1 g} \log_e 1.33 = 27.30 \frac{V_1^2}{2g}$$

$$\text{or } \frac{p_1}{\rho_1 g} \times 0.285 = 27.30 \frac{V_1^2}{2g}$$

$$\text{or } \frac{p_1}{\rho_1} = \frac{27.30}{2 \times 0.285} V_1^2 = 47.894 V_1^2 \quad \dots(iii)$$

Now from equation of state, i.e., from equation (15.2), we have

$$\frac{p}{\rho} = RT \text{ or at section 1, } \frac{p_1}{\rho_1} = RT_1$$

$$\text{or } \frac{p_1}{\rho_1} = RT_1 = 287.14 \times 277 = 79537.4$$

Substituting this value of  $\frac{p_1}{\rho_1} = 79537.4$  in equation (iii), we get  $79537.4 = 47.894 V_1^2$

$$\therefore V_1 = \sqrt{\frac{79537.4}{47.894}} = 40.75 \text{ m/s. Ans.}$$

From equation (ii),  $V_2 = 5.32 \times V_1 = 5.32 \times 40.75 = 216.79 \text{ m/s. Ans.}$

**Problem 15.2** A gas is flowing through a horizontal pipe which is having area of cross-section as  $40 \text{ cm}^2$ , where pressure is  $40 \text{ N/cm}^2$  (gauge) and temperature is  $15^\circ\text{C}$ . At another section the area of cross-section is  $20 \text{ cm}^2$  and pressure is  $30 \text{ N/cm}^2$  (gauge). If the mass rate of flow of gas through the pipe is  $0.5 \text{ kg/s}$ , find the velocities of the gas at these sections, assuming an isothermal change. Take  $R = 292 \text{ N-m/kg}^\circ\text{K}$ , and atmospheric pressure =  $10 \text{ N/cm}^2$ .

**Solution.** Given :

	Section 1	Section 2
Area,	$A_1 = 40 \text{ cm}^2 = 40 \times 10^{-4} \text{ m}^2$	Area, $A_2 = 20 \text{ cm}^2 = 20 \times 10^{-4} \text{ m}^2$
Pressure,	$p_1 = 40 \text{ N/cm}^2$ (gauge) $= 40 + 10 = 50 \text{ N/cm}^2$ (abs.) $= 50 \times 10^4 \text{ N/m}^2$	$p_2 = 30 \text{ N/cm}^2$ (gauge) $= 30 + 10 = 40 \text{ N/cm}^2$ (abs.) $= 40 \times 10^4 \text{ N/m}^2$
Temperature,	$t_1 = 15^\circ\text{C}$ $\therefore T_1 = 15 + 273 = 288^\circ\text{K}$	
Mass rate of flow	$= 0.5 \text{ kg/s.}$	
Gas constant,	$R = 292 \text{ N-m/kg}^\circ\text{K}$	

From equation of state, i.e., equation (15.2),  $\frac{p_1}{\rho_1} = RT_1$

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$$\therefore \rho_1 = \frac{p_1}{RT_1} = \frac{50 \times 10^4}{292 \times 288} \frac{\text{kg}}{\text{m}^3} = 5.945 \frac{\text{kg}}{\text{m}^3}$$

Mass rate of flow is given by  $= \rho_1 A_1 V_1$  or  $0.5 = 5.945 \times 40 \times 10^{-4} \times V_1$

$$\therefore V_1 = \frac{0.5}{5.945 \times 40 \times 10^{-4}} = 21.02 \text{ m/s}$$

For isothermal process, temperature is constant and hence temperature at section 2 is also  $288^\circ\text{K}$ .

$$\therefore T_2 = 288^\circ\text{K}$$

Using equation (15.2), we get  $\frac{p_2}{\rho_2} = RT_2$

$$\therefore \rho_2 = \frac{p_2}{RT_2} = \frac{40 \times 10^4}{292 \times 288} = 4.756 \text{ kg/m}^3$$

Now mass rate of flow  $= \rho_2 A_2 V_2$

$$\therefore 0.5 = 4.756 \times 20 \times 10^{-4} \times V_2$$

$$\therefore V_2 = \frac{0.5}{4.756 \times 20 \times 10^{-4}} = 52.565 \text{ m/s. Ans.}$$

**Problem 15.3** A gas with a velocity of  $300 \text{ m/s}$  is flowing through a horizontal pipe at a section where pressure is  $6 \times 10^4 \text{ N/m}^2$  (absolute) and temperature  $40^\circ\text{C}$ . The pipe changes in diameter and at this section the pressure is  $9 \times 10^4 \text{ N/m}^2$ . Find the velocity of the gas at this section if the flow of the gas is adiabatic.

Take  $R = 287 \text{ J/kg}^\circ\text{K}$  and  $k = 1.4$ .

**Solution.** Given :

<b>Section 1</b> $V_1 = 300 \text{ m/s}$ $p_1 = 6 \times 10^4 \text{ N/m}^2$ $t_1 = 40^\circ\text{C}$ $\therefore T_1 = 273 + 40 = 313^\circ\text{K}$	<b>Section 2</b> $p_2 = 9 \times 10^4 \text{ N/m}^2$ $V_2 = \text{velocity at section 2}$ $R = 287 \text{ J/kg}^\circ\text{K}$
---	---

Adiabatic flow,  $k = 1.4$

Applying Bernoulli's equation at sections 1 and 2, given by equation (15.13), we get

$$\left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left( \frac{k}{k-1} \right) \frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} \quad (\because Z_1 = Z_2)$$

or 
$$\left( \frac{k}{k-1} \right) \left[ \frac{p_1}{\rho_1 g} - \frac{p_2}{\rho_2 g} \right] = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

Dividing by 'g', we get 
$$\left( \frac{k}{k-1} \right) \left[ \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

$$\text{or } \left( \frac{1.4}{1.4 - 1.0} \right) \frac{p_1}{\rho_1} \left[ 1 - \frac{p_2}{\rho_2} \times \frac{\rho_1}{p_1} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2} \quad \dots(i)$$

For adiabatic flow, using equation (15.4), we get

$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} \quad \text{or} \quad \frac{p_1}{p_2} = \left( \frac{\rho_1}{\rho_2} \right)^k \quad \text{or} \quad \left( \frac{\rho_1}{\rho_2} \right) = \left( \frac{p_1}{p_2} \right)^{1/k}$$

Substituting the value of  $\frac{\rho_1}{\rho_2}$  in equation (i), we get

$$\frac{1.4}{0.4} \frac{p_1}{\rho_1} \left[ 1 - \frac{p_2}{p_1} \times \left( \frac{p_1}{p_2} \right)^{1/k} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

$$\text{or } 3.5 \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right) \times \left( \frac{p_2}{p_1} \right)^{-1/k} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

$$\text{or } 3.5 \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{1 - \frac{1}{k}} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2} \quad \text{or } 3.5 \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

Substituting the value of  $p_2$  and  $p_1$ , we get

$$3.5 \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{9 \times 10^4}{6 \times 10^4} \right)^{\frac{1.4 - 1}{1.4}} \right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

$$\text{or } 3.5 \frac{p_1}{\rho_1} [1 - 1.5^{2/7}] = \frac{V_2^2}{2} - \frac{V_1^2}{2} \quad \text{or } 3.5 \frac{p_1}{\rho_1} [1 - 1.5^{2.857}] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

$$\text{or } -0.4298 \frac{p_1}{\rho_1} = \frac{V_2^2}{2} - \frac{V_1^2}{2} = \frac{V_2^2}{2} - \frac{300^2}{2}$$

$$\text{or } -0.4298 \frac{p_1}{\rho_1} = \frac{V_2^2}{2} - 45000 \quad \dots(ii)$$

From equation of state, we have  $\frac{p_1}{\rho_1} = RT_1 = 287 \times 313 = 89831$

Substituting the value of  $\frac{p_1}{\rho_1} = 89831$  in equation (ii),

$$-0.4298 \times 89831 = \frac{V_2^2}{2} - 45000$$

$$\text{or } \frac{V_2^2}{2} = 45000 - 0.4298 \times 89831 = 6390.6$$

$$\therefore V_2 = \sqrt{2 \times 6390.6} = 113.0 \text{ m/s. Ans.}$$

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**15.3.3 Momentum Equations.** The momentum per second of a flowing fluid (or momentum flux) is equal to the product of mass per second and the velocity of the flow. Mathematically, the momentum per second of a flowing fluid (compressible or incompressible) is

$$= \rho A V \times V, \text{ where } \rho A V = \text{Mass per second.}$$

The term  $\rho A V$  is constant at every section of flow due to continuity equation. This means the momentum per second at any section is equal to the product of a constant quantity and the velocity. This also implies that momentum per second is independent of compressible effect. Hence the momentum equation for incompressible and compressible fluid is the same. The momentum equation for compressible fluid for any direction may be expressed as,

$$\begin{aligned} \text{Net force in the direction of } S &= \text{Rate of change of momentum in the direction of } S \\ &= \text{Mass per second [change of velocity]} \\ &= \rho A V [V_2 - V_1] \end{aligned} \quad \dots(15.14)$$

where  $V_2$  = Final velocity in the direction of  $S$ ,

$V_1$  = Initial velocity in the direction of  $S$ .

## ►15.4 VELOCITY OF SOUND OR PRESSURE WAVE IN A FLUID

The disturbance in a solid, liquid or gas is transmitted from one point to the other. The velocity with which the disturbance is transmitted depends upon the distance between the molecules of the medium. In case of solids, molecules are closely packed and hence the disturbance is transmitted instantaneously. In case of liquids and gases (or fluids) the molecules are relatively apart. The disturbance will be transmitted from one molecule to the next molecule. But in case of fluids, there is some distance between two adjacent molecules. Hence each molecule will have to travel a certain distance before it can transmit the disturbance. Thus the velocity of disturbance in case of fluids will be less than the velocity of the disturbance in solids.

The distance between the molecules is related with the density, which in turn depends upon pressure in case of fluids. Hence the velocity of disturbance depends upon the changes of pressure and density of the fluid.

**15.4.1 Expression for Velocity of Sound Wave in a Fluid.** The disturbance creates the pressure waves in a fluid. These pressure waves travel with a velocity of sound waves in all directions. But for the sake of simplicity, one-dimensional case will be considered.

Fig. 15.2 shows the model for one-dimensional propagation of the pressure waves. It is a right long pipe of uniform cross-sectional area, fitted with a piston. Let the pipe is filled with a compressible fluid, which is at rest initially. The piston is moved towards right and a disturbance is created in the fluid. This disturbance is in the form of pressure wave, which travels in the fluid with a velocity of sound wave.

Let  $A$  = Cross-sectional area of the pipe

$V$  = Velocity of piston

$p$  = Pressure of the fluid in pipe before the movement of the piston

$\rho$  = Density of fluid before the movement of the piston

$dt$  = A small interval of time with which piston is moved

$C$  = Velocity of pressure wave or sound wave travelling in the fluid

Distance travelled by the piston in time  $dt$

$$= \text{Velocity of piston} \times dt = V \times dt$$

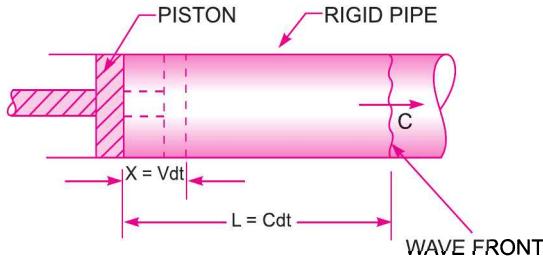


Fig. 15.2 Propagation of pressure wave.

Distance travelled by the pressure wave in time  $dt$

$$= \text{Velocity of pressure wave} \times dt = C \times dt$$

As the value of  $C$  will be very large, hence  $C \times dt$  will be more than  $V \times dt$ . For the time interval ' $dt$ ', the pressure wave has travelled a distance  $L$  and piston has moved through  $x$ . Thus in the length of the tube equal to  $(L - x)$ , the fluid will be compressed. Due to compression of the fluid, the pressure and density of the fluid will change.

Let  $p + dp$  = Pressure after compression

$\rho + d\rho$  = Density after compression or the density of fluid in the length  $(L - x)$

Now mass of fluid for a length ' $L$ ' before compression

$$\begin{aligned} &= \rho \times \text{Volume of fluid upto length } L \\ &= \rho \times A \times L = \rho \times A \times C \times dt \quad (\because L = Cdt) \dots(i) \end{aligned}$$

Mass of fluid after compression for length  $(L - x)$

$$\begin{aligned} &= \text{Density after compression} \times \text{Area} \times \text{Length} \\ &= (\rho + d\rho) A \times (L - x) \\ &= (\rho + d\rho) A \times (Cdt - Vdt) \quad (\because L = Cdt, x = Vdt) \dots(ii) \end{aligned}$$

From the continuity equation, we have

Mass of fluid before compression

$$= \text{Mass of fluid after compression}$$

$$\therefore \rho A C dt = (\rho + d\rho) A \times (Cdt - Vdt) \text{ or } \rho A C dt = (\rho + d\rho) A \times dt (C - V)$$

$$\text{Dividing by } A \times dt, \quad \rho C = (\rho + d\rho) (C - V) = \rho C - \rho V + C d\rho - V d\rho$$

$$\therefore C d\rho = \rho C - \rho C + \rho V + V d\rho = \rho V + V d\rho \dots(iii)$$

But the velocity of the piston,  $V$ , is very small as compared to the velocity of the pressure wave  $C$ . Also the value of  $d\rho$  is very small. Hence the term  $(V \times d\rho)$  will be very-very small and can be neglected. Hence equation (iii) becomes,

$$C d\rho = \rho V \dots(iv)$$

Now when the piston is moved with a velocity  $V$  for time  $dt$ , the fluid which is at rest initially will move with a velocity equal to the velocity of the piston. Also the pressure of the fluid will increase from  $p$  to  $p + dp$  due to the movement of the piston. Hence applying the impulse momentum equation, we get

$$\begin{aligned} \text{Net force on the fluid} &= \text{Rate of change of momentum} \\ \text{or} \quad (p + dp) A - p \times A &= \text{Mass per second} \end{aligned}$$

[change of velocity of fluid]

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or

$$\begin{aligned}
 dp \times A &= \frac{\text{Total mass}}{\text{Time}} [V - 0] = \frac{\rho A L}{dt} [V - 0] \\
 &= \frac{\rho A C dt}{dt} [V - 0] \quad (\because L = C dt) \\
 &= \rho A C [V - 0] = \rho A C V \quad \text{or} \quad dp = \frac{\rho A C V}{A} = \rho C V
 \end{aligned}$$

or

$$C = \frac{dp}{\rho V} \quad \dots(v)$$

Multiplying equations (iv) and (v), we get

$$\begin{aligned}
 C^2 d\rho &= \rho V \times \frac{dp}{\rho V} = dp \\
 C^2 &= \frac{dp}{d\rho} \\
 \therefore C &= \sqrt{\frac{dp}{d\rho}} \quad \dots(15.15)
 \end{aligned}$$

Hence equation (15.15) gives the velocity of sound wave which is the square root of the ratio of change of pressure to the change of density of a fluid due to disturbance.

**15.4.2 Velocity of Sound in Terms of Bulk Modulus.** Bulk modulus  $K$  is defined as

$$K = \frac{\text{Increase in pressure}}{\frac{\text{Decrease in volume}}{\text{Original volume}}} = \frac{dp}{-\left(\frac{dV}{V}\right)} \quad \dots(vi)$$

where  $dV$  = Decrease in volume,  $V$  = Original volume.

Negative sign is taken, as with the increase of pressure, volume decreases.

Now we know mass of a fluid is constant. Hence

$$\rho \times \text{Volume} = \text{Constant} \quad (\because \text{Mass} = \rho \times \text{Volume})$$

or

$$\rho \times V = \text{Constant}$$

Differentiating the above equation ( $\rho$  and  $V$  are variables),

$$\rho dV + V d\rho = 0 \quad \text{or} \quad \rho dV = -V d\rho \quad \text{or} \quad -\frac{dV}{V} = \frac{d\rho}{\rho}$$

Substituting the value  $\left(-\frac{dV}{V}\right)$  in equation (vi), we get

$$K = \frac{dp}{d\rho} = \rho \frac{dp}{d\rho} \quad \text{or} \quad \frac{dp}{d\rho} = \frac{K}{\rho}$$

From equation (15.15), the velocity of sound wave is

$$C = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{K}{\rho}} \quad \dots(15.16)$$

Equation (15.16) gives the velocity of sound wave in terms of bulk modulus and density. This equation is applicable for liquids and gases.

**15.4.3 Velocity of Sound for Isothermal Process.** Isothermal process is given by equation (15.3), as

$$\frac{p}{\rho} = \text{Constant or } p\rho^{-1} = \text{Constant}$$

Differentiating the above equation ( $p$  and  $\rho$  both are variable),

$$p(-1)\rho^{-2}d\rho + \rho^{-1}dp = 0$$

Dividing by  $\rho^{-1}$ , we get  $-p\rho^{-1}d\rho + dp = 0$  or  $\frac{-p}{\rho}d\rho + dp = 0$

$$\therefore dp = \frac{p}{\rho}d\rho \text{ or } \frac{dp}{d\rho} = \frac{p}{\rho} = RT \quad \left( \because \text{From equation of state } \frac{p}{\rho} = RT \right)$$

Substituting the value of  $\frac{dp}{d\rho}$  in equation (15.15),  $C = \sqrt{\frac{p}{\rho}} = \sqrt{RT}$  ... (15.17)

**15.4.4 Velocity of Sound for Adiabatic Process.** Adiabatic process is given by equation (15.4), as

$$\frac{p}{\rho^k} = \text{Constant or } p\rho^{-k} = \text{Constant}$$

Differentiating the above equation, we get

$$p(-k)\rho^{-k-1}d\rho + \rho^{-k}dp = 0$$

Dividing by  $\rho^{-k}$ , we get  $-pk\rho^{-1}d\rho + dp = 0$  or  $dp = \frac{pk}{\rho}d\rho$

$$\therefore \frac{dp}{d\rho} = \frac{p}{\rho}k = RTk \quad \left( \because \frac{p}{\rho} = RT \right)$$

$$= kRT$$

Substituting the value of  $\frac{dp}{d\rho}$  in equation (15.15), we get  $C = \sqrt{kRT}$ . ... (15.18)

**Note 1.** For the propagation of the minor disturbances through air, the process is assumed to be adiabatic. The velocity of the disturbances (pressure waves) through air is very high and hence there is no time for any appreciable heat transfer.

**2.** Isothermal process is considered for the calculation of the velocity of the sound waves (or pressure waves) only when it is given in the numerical problem that process is isothermal. If no process is mentioned, it is assumed to be adiabatic.

## ►15.5 MACH NUMBER

In Chapter 12, Art. 12.8.5, Mach number has been defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force.

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$$\begin{aligned}
 \therefore \text{ Mach number} &= M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{\rho A V^2}{K A}} \\
 &= \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C} \quad \left[ \because \sqrt{\frac{K}{\rho}} = C \text{ from equation (15.16)} \right] \\
 \text{Thus Mach number} &= M \\
 &= \frac{\text{Velocity of fluid or body moving in fluid}}{\text{Velocity of sound in the fluid}} \\
 &= \frac{V}{C}. \quad \dots(15.19)
 \end{aligned}$$

For the compressible fluid flow, Mach number is an important non-dimensional parameter. On the basis of the Mach number, the flow is defined as :

1. Sub-sonic flow,    2. Sonic flow, and    3. Super-sonic flow.

**1. Sub-sonic Flow.** A flow is said sub-sonic flow if the Mach number is less than 1.0 (or  $M < 1$ ) which means the velocity of flow is less than the velocity of sound wave (or  $V < C$ ).

**2. Sonic Flow.** A flow is said to be sonic flow if the Mach number ( $M$ ) is equal to 1.0. This means that when the velocity of flow  $V$  is equal to the velocity of sound  $C$ , the flow is said to be sonic flow.

**3. Super-sonic Flow.** A flow is said to be super-sonic flow if the Mach number is greater than 1.0 (or  $M > 1$ ). This means that when velocity of flow  $V$  is greater than the velocity of sound wave, the flow is said to be super-sonic flow.

**Problem 15.4** Find the sonic velocity for the following fluids :

- (i) Crude oil of sp. gr. 0.8 and bulk modulus  $153036 \text{ N/cm}^2$ .
- (ii) Mercury having a bulk modulus of  $2648700 \text{ N/cm}^2$ .

**Solution.** Given :

$$\begin{aligned}
 (i) \text{ For Crude oil,} \quad \text{sp. gr.} &= 0.8 \\
 \therefore \text{ Density,} \quad \rho &= 0.8 \times 1000 = 800 \text{ kg/m}^3 \\
 \text{Bulk modulus,} \quad K &= 153036 \text{ N/cm}^2 = 153036 \times 10^4 \text{ N/m}^2
 \end{aligned}$$

Using equation (15.16) for sonic velocity, as

$$C = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{153036 \times 10^4}{800}} = 1383.09 \approx \mathbf{1383 \text{ m/s. Ans.}}$$

$$\begin{aligned}
 (ii) \text{ For Mercury,} \quad \text{sp. gr.} &= 13.6 \\
 \therefore \text{ Density of Mercury,} \quad \rho &= 13.6 \times 1000 \text{ kg/m}^3 \\
 \text{Bulk modulus,} \quad K &= 2648700 \text{ N/cm}^2 = 2648700 \times 10^4 \text{ N/m}^2
 \end{aligned}$$

$$\text{The sonic velocity, } C = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{2648700 \times 10^4}{13.6 \times 1000}} = \mathbf{1395.55 \text{ m/s. Ans.}}$$

**Problem 15.5** Find the speed of the sound wave in air at sea-level where the pressure and temperature are  $10.1043 \text{ N/cm}^2$  (abs.) and  $15^\circ\text{C}$  respectively. Take  $R = 287 \text{ J/kg}^\circ\text{K}$  and  $k = 1.4$ .

**Solution.** Given :

$$\text{Pressure, } p = 10.1043 \text{ N/cm}^2 = 10.1043 \times 10^4 \text{ N/m}^2$$

$$\text{Temperature, } t = 15^\circ\text{C}$$

$$\therefore T = 273 + 15 = 288 \text{ K}, R = 287 \text{ J/kg}^\circ\text{K}, k = 1.4$$

For adiabatic process, the velocity of sound is given by equation (15.18), as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 288} = 340.17 \text{ m/s. Ans.}$$

**Problem 15.6** Calculate the Mach number at a point on a jet propelled aircraft, which is flying at  $1100 \text{ km/hour}$  at sea-level where air temperature is  $20^\circ\text{C}$ . Take  $k = 1.4$  and  $R = 287 \text{ J/kg}^\circ\text{K}$ .

**Solution.** Given :

$$\text{Speed of aircraft, } V = 1100 \text{ km/hour} = \frac{1100 \times 1000}{60 \times 60} = 305.55 \text{ m/s}$$

$$\text{Temperature, } t = 20^\circ\text{C}$$

$$\therefore T = 273 + 20 = 293^\circ\text{K}, k = 1.4, R = 287 \text{ J/kg}^\circ\text{K}$$

Using equation (15.18), the velocity of sound is

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 293} = 343.11 \text{ m/s}$$

Mach number is given by equation (15.19) as

$$M = \frac{V}{C} = \frac{305.55}{343.11} = 0.89. \text{ Ans.}$$

**Problem 15.7** An aeroplane is flying at an height of  $15 \text{ km}$  where the temperature is  $-50^\circ\text{C}$ . The speed of the plane is corresponding to  $M = 2.0$ . Assuming  $k = 1.4$  and  $R = 287 \text{ J/kg}^\circ\text{K}$ , find the speed of the plane.

**Solution.** Given :

$$\text{Height of the plane, } Z = 15 \text{ km (Extra Data)}$$

$$\text{Temperature, } t = -50^\circ\text{C}$$

$$\therefore T = -50 + 273 = 223^\circ\text{K}$$

$$\text{Mach number, } M = 2.0, k = 1.4, R = 287 \text{ J/kg}^\circ\text{K}.$$

Using equation (15.18), we get the velocity of sound as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 223} = 299.33 \text{ m/s}$$

$$\text{Using equation (15.19), we have } M = \frac{V}{C} \text{ or } 2.0 = \frac{V}{299.33}$$

$$\therefore V = 2.0 \times 299.33 = 598.66 \text{ m/s}$$

$$= \frac{598.66 \times 60 \times 60}{1000} = 2155.17 \text{ km/hour. Ans.}$$

### ► 15.6 PROPAGATION OF PRESSURE WAVES (OR DISTURBANCES) IN A COMPRESSIBLE FLUID

Whenever any disturbance is produced in a compressible fluid, the disturbance is propagated in all directions with a velocity of sound (*i.e.*, equal to  $C$ ). The nature of propagation of the disturbance depends upon the Mach number. Let us consider a small projectile moving from left to right in a straight line in a stationary fluid. Due to the movement of the projectile, the disturbances will be created in the fluid. This disturbance will be moving in all directions with a velocity  $C$ .

Hence let  $V$  = Velocity of the projectile,

$C$  = Velocity of pressure wave or disturbance created in the fluid.

Let us find the nature of propagation of the disturbance for different Mach numbers.

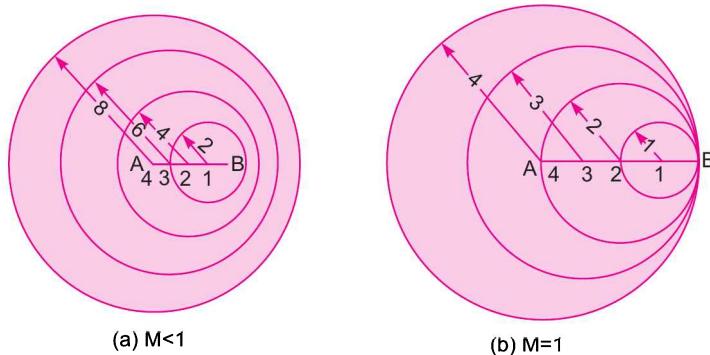
**1st Case : When  $M < 1$ .** When Mach number is less than 1.0, the flow is called sub-sonic flow. For  $M < 1$  means  $\frac{V}{C} < 1$  or  $V < C$ . To find the nature of propagation for this case, let  $V = 1$  unit and

$C = 2$  unit, so that  $\frac{V}{C} = \frac{1}{2}$  which is less than 1.0. Let the projectile is at  $A$  and is moving towards right.

Let in 4 seconds the projectile reaches to the position  $B$ . At  $A$ , the point 4 is also marked. The position of the projectile after 1 sec, 2 sec, 3 sec and 4 sec along the lines are shown by the points 3, 2, 1 and  $B$  respectively. The projectile moves from  $A$  to  $B$  in 4 seconds and hence the distance  $AB = 4 \times V = 4 \times 1 = 4$  units. The disturbance created at  $A$  in 4 seconds will move a distance  $= 4C = 4 \times 2 = 8$  units in all directions. Hence taking  $A$  as centre and radius equal to 8 units, a circle is drawn. This circle gives the position of disturbance after 4 seconds. When the projectile is at point 3, it will reach  $B$  in three seconds and distance  $3B = 3 \times V = 3 \times 1 = 3$  units. But the disturbance created at point 3 in three seconds will move a distance having a radius  $= 3 \times C = 3 \times 2 = 6$  units. Similarly at point 2, the disturbance will have a radius  $= 2 \times C = 4$  units and at point 1, the disturbance will have a radius  $= 1 \times C = 1 \times 2 = 2$  units. This is shown in Fig. 15.3 (a).

As in this case  $V < C$ , the pressure wave is always ahead of the projectile and point  $B$  is inside the sphere of radius 8 units.

**2nd Case : When  $M = 1$ .** When  $M = 1$ , the flow is known as sonic flow. In this case, the disturbance always travels with the projectile as shown in Fig. 15.3 (b). Let  $V = 1$  unit, and  $C = 1$  unit so that  $M = \frac{V}{C} = \frac{1}{1} = 1.0$ . Let the projectile moves from  $A$  to  $B$  in 4 seconds. The disturbance created at  $A$  in 4 seconds will move a distance having radius  $= 4 \times C = 4 \times 1 = 4$  units in all directions. The projectile



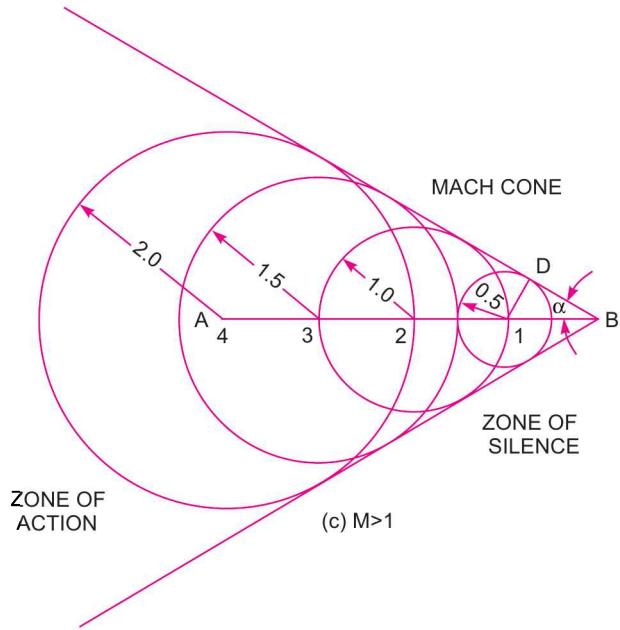


Fig. 15.3 Propagation of disturbance for different Mach numbers.

from point 3 will move to position *B* in three seconds. The disturbance created at point 3, will move a distance having radius =  $3 \times C = 3 \times 1 = 3$  units in all directions in three seconds. Similarly at the point 2 and point 1, the disturbance created at these points will move a distance having radius 2 and 1 in all directions respectively.

**3rd Case : When  $M > 1$ .** When  $M > 1$ , the flow is known as supersonic flow. Let  $V = 1$  unit and  $C = 0.5$  unit so that  $M = \frac{V}{C} = \frac{1}{0.5} = 2.0$ , which is greater than unity. Let the projectile moves from *A* to *B* in 4 seconds. The distance travelled by the projectile in 4 seconds =  $4 \times V = 4 \times 1 = 4$  units. Hence, take  $AB = 4$  units. The disturbance created at *A* will move in all directions and in 4 seconds, the radius of disturbance will be equal to  $4 \times C = 4 \times 0.5 = 2$  units. Hence taking *A* as centre, draw a circle with radius equal to 2 units. After one second from *A*, the projectile will be at point 3 and distance  $A3 = V \times 1 = 1 \times 1 = 1$  unit. The projectile from point 3 will reach point *B* in three seconds. Hence the disturbance created at point 3 will move in all directions and in three seconds, the radius of disturbance from point 3 will be equal to  $3 \times C = 3 \times 0.5 = 1.5$  units. Similarly the radius of disturbance at point 2 and 1 will be  $2 \times C = 2 \times 0.5 = 1$  unit and  $1 \times C = 1 \times 0.5 = 0.5$  unit respectively as shown in Fig. 15.3 (c). In this case the sphere of propagation of disturbance always lags behind the projectile. If we draw a tangent to the different circles which represent the propagated spherical waves on both sides, we shall get a cone with vertex at *B*. This cone is known as **Mach Cone**.

**15.6.1 Mach Angle.** This is defined as the half of the angle of the Mach cone. In Fig. 15.3 (c), angle  $\alpha$  is known as Mach angle. In the  $\Delta 1BD$  of Fig. 15.3 (c), the distance  $1B$  = Velocity of projectile =  $V$ , the distance  $1D$  = Velocity of sound wave =  $C$ . Hence we have

$$\sin \alpha = \frac{1D}{1B} = \frac{C}{V} = \frac{1}{V/C} = \frac{1}{M} \quad \dots(15.20)$$

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**15.6.2 Zone of Action.** When  $M > 1$ , the effect of the disturbance is felt only in the region inside the Mach cone. This region is called the zone of action.

**15.6.3 Zone of Silence.** When  $M > 1$ , there is no effect of disturbance in the region outside the Mach cone. The region which is outside the Mach cone is called zone of silence.

**Problem 15.8** A projectile is travelling in air having pressure and temperature as  $8.829 \text{ N/cm}^2$  and  $-2^\circ\text{C}$ . If the Mach angle is  $40^\circ$ , find the velocity of the projectile. Take  $k = 1.4$  and  $R = 287 \text{ J/kg}^\circ\text{K}$ .

**Solution.** Given :

$$\text{Pressure of air, } p = 8.829 \text{ N/cm}^2 = 8.829 \times 10^4 \text{ N/m}^2$$

$$\text{Temperature of air, } t = -2^\circ\text{C}$$

$$\therefore T = -2 + 273 = 271^\circ\text{K}$$

$$\text{Mach angle, } \alpha = 40^\circ, k = 1.4, R = 287 \text{ J/kg}^\circ\text{K}$$

$$\text{Let the velocity of projectile } = V$$

$$\text{Using equation (15.20), we have } \sin \alpha = \frac{C}{V} \text{ or } \sin 40^\circ = 0.6427 = \frac{C}{V}$$

The velocity of sound,  $C$  is given by equation (15.18) as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 271} = 329.98 \text{ m/s} \approx 330 \text{ m/s}$$

$$\therefore \sin 40^\circ = 0.6427 = \frac{C}{V} = \frac{330}{V}$$

$$\therefore V = \frac{330}{0.6427} = 513 \text{ m/s. Ans.}$$

**Problem 15.9** A projectile travels in air of pressure  $10.1043 \text{ N/cm}^2$  at  $10^\circ\text{C}$  at a speed of  $1500 \text{ km/hour}$ . Find the Mach number and the Mach angle. Take  $k = 1.4$  and  $R = 287 \text{ J/kg}^\circ\text{K}$ .

**Solution.** Given :

$$\text{Pressure, } p = 10.1043 \text{ N/cm}^2 = 10.1043 \times 10^4 \text{ N/cm}^2$$

$$\text{Temperature, } t = 10^\circ\text{C}$$

$$\therefore T = 10 + 273 = 283^\circ\text{K}$$

$$\text{Speed of projectile, } V = 1500 \text{ km/hour} = \frac{1500 \times 1000}{60 \times 60} \text{ m/s} = 416.67 \text{ m/s}$$

$$k = 1.4, R = 287 \text{ J/kg}^\circ\text{K}$$

For adiabatic process, the velocity of sound is given by

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 283} = 337.20 \text{ m/s}$$

$$\therefore \text{Mach number, } M = \frac{V}{C} = \frac{416.67}{337.20} = 1.235. \text{ Ans.}$$

∴ Mach angle is obtained from equation (15.20) as

$$\sin \alpha = \frac{C}{V} = \frac{1}{M} = \frac{1}{1.235} = 0.8097$$

$$\therefore \text{Mach angle, } \alpha = \sin^{-1} 0.8097 = 54.06^\circ. \text{ Ans.}$$

**Problem 15.10** Find the velocity of bullet fired in standard air if the Mach angle is  $30^\circ$ . Take  $R = 287.14 \text{ J/kg}^\circ\text{K}$  and  $k = 1.4$  for air. Assume temperature as  $15^\circ\text{C}$ .

**Solution.** Given :

Mach angle

$$\alpha = 30^\circ$$

$$R = 287.14 \text{ J/kg}^\circ\text{K}$$

$$k = 1.4$$

Temperature,

$$t = 15^\circ\text{C}$$

∴

$$T = 15 + 273 = 288^\circ\text{K}$$

Velocity of sound is given by equation (15.18) as

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287.14 \times 288} = 340.25 \text{ m/s}$$

Using the relation,

$$\sin \alpha = \frac{C}{V} \text{ given by equation (15.20)}$$

$$\sin 30^\circ = \frac{340.25}{V}$$

∴

$$V = \frac{340.25}{\sin 30^\circ} = 680.50 \text{ m/s. Ans.}$$

## ► 15.7 STAGNATION PROPERTIES

When a fluid is flowing past an immersed body, and at a point on the body if the resultant velocity becomes zero, the values of pressure, temperature and density at that point are called stagnation properties. The point is called the **stagnation point**. The values of pressure, density and temperature are called stagnation pressure, stagnation density and stagnation temperature respectively. They are denoted by  $p_s$ ,  $\rho_s$  and  $T_s$  respectively.

**15.7.1 Expression for Stagnation Pressure ( $p_s$ ).** Consider a compressible fluid flowing past an immersed body under frictionless adiabatic conditions as in Fig. 15.4. Consider two points 1 and 2 on a stream-line as shown in Fig. 15.4.

Let  $p_1$  = Pressure of compressible fluid at point 1,

$V_1$  = Velocity of fluid at 1, and

$\rho_1$  = Density of fluid at 1,

$p_2$ ,  $V_2$ ,  $\rho_2$  = Corresponding values of pressure, velocity and density at point 2.

Applying Bernoulli's equation for adiabatic flow given by equation (15.13) at point 1 and 2, we get

$$\left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left( \frac{k}{k-1} \right) \frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2$$

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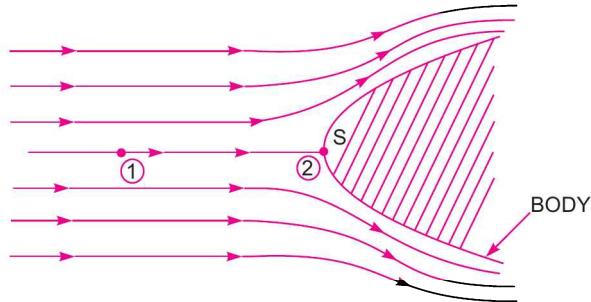


Fig. 15.4 Stagnation properties.

But

$$Z_1 = Z_2$$

$$\therefore \left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left( \frac{k}{k-1} \right) \frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g}$$

$$\text{or } \left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \left( \frac{k}{k-1} \right) \frac{p_2}{\rho_2} + \frac{V_2^2}{2} \quad \left( \text{Cancelling } \frac{1}{g} \right)$$

Point 2 is a stagnation point. Hence velocity will become zero at stagnation point and pressure and density will be denoted by  $p_s$  and  $\rho_s$ .

$$\therefore V_2 = 0, p_2 = p_s \text{ and } \rho_2 = \rho_s$$

Substituting these values in the above Bernoulli's equation,

$$\left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \left( \frac{k}{k-1} \right) \frac{p_s}{\rho_s} + 0$$

$$\text{or } \left( \frac{k}{k-1} \right) \left[ \frac{p_1}{\rho_1} - \frac{p_s}{\rho_s} \right] = -\frac{V_1^2}{2} \text{ or } \left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} \left[ 1 - \frac{p_s}{\rho_s} \times \frac{\rho_1}{p_1} \right] = -\frac{V_1^2}{2}$$

$$\text{or } \left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} \left[ 1 - \frac{p_s}{p_1} \times \frac{\rho_1}{\rho_s} \right] = -\frac{V_1^2}{2} \quad \dots(i)$$

But for adiabatic process from equation (15.4), we have

$$\frac{p}{\rho^k} = \text{constant} \text{ or } \frac{p_1}{\rho_1^k} = \frac{p_s}{\rho_s^k} \text{ or } \frac{p_1}{p_s} = \frac{\rho_1^k}{\rho_s^k} \text{ or } \left( \frac{\rho_1}{\rho_s} \right) = \left( \frac{p_1}{p_s} \right)^{\frac{1}{k}} \quad \dots(ii)$$

Substituting the value of  $\frac{\rho_1}{\rho_s}$  in equation (i), we get

$$\left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} \left[ 1 - \frac{p_s}{p_1} \times \left( \frac{p_1}{p_s} \right)^{\frac{1}{k}} \right] = -\frac{V_1^2}{2}$$

$$\text{or } \left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} \left[ 1 - \frac{p_s}{p_1} \times \left( \frac{p_s}{p_1} \right)^{-\frac{1}{k}} \right] = -\frac{V_1^2}{2}$$

$$\text{or } \left( \frac{k}{k-1} \right) \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_s}{p_1} \right)^{1-\frac{1}{k}} \right] = -\frac{V_1^2}{2}$$

$$\text{or } \left[ 1 - \left( \frac{p_s}{p_1} \right)^{\frac{k-1}{k}} \right] = -\frac{V_1^2}{2} \times \left( \frac{k-1}{k} \right) \frac{\rho_1}{p_1}$$

$$\text{or } 1 + \frac{V_1^2}{2} \left( \frac{k-1}{k} \right) \frac{\rho_1}{p_1} = \left( \frac{p_s}{p_1} \right)^{\frac{k-1}{k}} \quad \dots(iii)$$

Now for adiabatic process, the velocity of sound is given by equation (15.18) as

$$C = \sqrt{kRT} = \sqrt{k \frac{p}{\rho}} \quad \left( \because \frac{p}{\rho} = RT \right)$$

$$\text{For the point 1, } C_1 = \sqrt{k \frac{p_1}{\rho_1}} \text{ or } C_1^2 = k \frac{p_1}{\rho_1}$$

Substituting the value of  $\frac{k p_1}{\rho_1} = C_1^2$  in equation (iii)

$$1 + \frac{V_1^2}{2} (k-1) \times \frac{1}{C_1^2} = \left( \frac{p_s}{p_1} \right)^{\left( \frac{k-1}{k} \right)} \text{ or } 1 + \frac{V_1^2}{2C_1^2} \times (k-1) = \left( \frac{p_s}{p_1} \right)^{\left( \frac{k-1}{k} \right)}$$

$$\text{or } 1 + \frac{M_1^2}{2} (k-1) = \left( \frac{p_s}{p_1} \right)^{\left( \frac{k-1}{k} \right)} \quad \left( \because \frac{V_1^2}{C_1^2} = M_1^2 \right)$$

$$\text{or } \left( \frac{p_s}{p_1} \right)^{\frac{k-1}{k}} = \left[ 1 + \frac{(k-1)}{2} M_1^2 \right] \text{ or } \frac{p_s}{p_1} = \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\left( \frac{k}{k-1} \right)} \quad \dots(iv)$$

$$\therefore p_s = p_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\left( \frac{k}{k-1} \right)} \quad \dots(15.21)$$

Equation (15.21) gives the value of stagnation pressure.

In equation (15.21), for  $M < 1$ , the term  $\frac{k-1}{2} M_1^2$  will be less than 1 and hence the R.H.S. of this equation can be expressed by Binomial theorem as

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$$\begin{aligned}
 p_s &= p_1 \left[ 1 + \left( \frac{k}{k-1} \right) \times \left( \frac{k-1}{2} \cdot M_1^2 \right) + \left( \frac{k}{k-1} \right) \left( \frac{k}{k-1} - 1 \right) \left( \frac{k-1}{2} \cdot M_1^2 \right)^2 / 2! \right. \\
 &\quad \left. + \left( \frac{k}{k-1} \right) \left( \frac{k}{k-1} - 1 \right) \left( \frac{k}{k-1} - 2 \right) \left( \frac{k-1}{2} \cdot M_1^2 \right)^3 / 3! + \dots \right] \\
 &= p_1 \left[ 1 + \frac{k}{2} M_1^2 + \frac{k}{8} M_1^4 + \frac{k(2-k)}{48} M_1^6 + \dots \right] \\
 &= p_1 + p_1 \left[ \frac{k}{2} M_1^2 + \frac{k}{8} M_1^4 + \frac{k(2-k)}{48} M_1^6 + \dots \right] \\
 \text{or } \frac{p_s - p_1}{p_1} &= \left[ \frac{k}{2} M_1^2 + \frac{k}{8} M_1^4 + \frac{k(2-k)}{48} M_1^6 + \dots \right] \\
 &= \frac{k}{2} M_1^2 \left[ 1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots \right] \\
 &= \frac{k}{2} \frac{V_1^2}{C_1^2} \left[ 1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots \right] \\
 &= \frac{k}{2} \frac{V_1^2}{\left( k \frac{p_1}{\rho_1} \right)} \left[ 1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots \right] \quad \left( \because C_1 = \sqrt{\frac{kp_1}{\rho_1}} \right) \\
 &= \frac{1}{2} \cdot \frac{\rho_1 V_1^2}{p_1} \left[ 1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots \right]
 \end{aligned}$$

or       $(p_s - p_1) = \frac{1}{2} \cdot \rho_1 \cdot V_1^2 \left[ 1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots \right]$  ... (15.21A)

From the above equation, it is clear that when the approaching velocity  $V_1$  is small compared with the velocity of sound wave  $C_1$ , then  $M_1$  will be very small and the term  $1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots$  will be nearly equal to 1.

Hence equation (15.21A) becomes as

$$\therefore p_s - p_1 = \frac{1}{2} \cdot \rho_1 \times V_1^2 \text{ or } p_s = p_1 + \frac{1}{2} \rho_1 V_1^2$$

But when approaching velocity becomes high then  $M_1$  is not small and equation (15.21A) is expressed as

$$\frac{1}{2} \times \rho_1 \times V_1^2 = 1 + \frac{1}{4} M_1^2 + \frac{(2-k)}{24} M_1^4 + \dots \quad \dots (15.21B)$$

### 15.7.2 Expression for Stagnation Density ( $\rho_s$ ).

From equation (ii), we have

$$\left(\frac{\rho_1}{\rho_s}\right) = \left(\frac{p_1}{p_s}\right)^{\frac{1}{k}} \text{ or } \left(\frac{\rho_s}{\rho_1}\right) = \left(\frac{p_s}{p_1}\right)^{\frac{1}{k}} \quad (\text{Taking reciprocal})$$

$$\therefore \rho_s = \rho_1 \left[ \frac{p_s}{p_1} \right]^{\frac{1}{k}}$$

Substituting the value of  $\left(\frac{p_s}{p_1}\right)$  from equation (iv),

$$\rho_s = \rho_1 \left[ \left( 1 + \frac{k-1}{2} M_1^2 \right)^{\frac{k}{k-1}} \right]^{\frac{1}{k}} \text{ or } \rho_s = \rho_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{1}{k-1}} \quad \dots(15.22)$$

### 15.7.3 Expression for Stagnation Temperature ( $T_s$ ).

Equation of state is given by equation (15.2) as  $\frac{p}{\rho} = RT$

For the stagnation point, we have equation of state as  $\frac{p_s}{\rho_s} = RT_s$   $\dots(15.22A)$

$$\therefore T_s = \frac{1}{R} \frac{p_s}{\rho_s}$$

Substituting the value of  $p_s$  and  $\rho_s$  from equations (15.21) and (15.22), we have

$$\begin{aligned} T_s &= \frac{1}{R} \frac{p_1 \left[ 1 + \left( \frac{k-1}{2} \right) M_1^2 \right]^{\frac{k}{k-1}}}{\rho_1 \left[ 1 + \left( \frac{k-1}{2} \right) M_1^2 \right]^{\frac{1}{k-1}}} \\ &= \frac{1}{R} \frac{p_1}{\rho_1} \left[ 1 + \left( \frac{k-1}{2} \right) M_1^2 \right]^{\frac{k}{k-1} - \frac{1}{k-1}} \\ &= \frac{1}{R} \frac{p_1}{\rho_1} \left[ 1 + \left( \frac{k-1}{2} \right) M_1^2 \right]^{\frac{(k-1)}{k-1}} = \frac{p_1}{\rho_1 R} \left[ 1 + \left( \frac{k-1}{2} \right) M_1^2 \right] \\ &= T_1 \left[ 1 + \left( \frac{k-1}{2} \right) M_1^2 \right] \quad \left( \because \frac{p_1}{\rho_1} = RT_1 \right) \quad \dots(15.23) \end{aligned}$$

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**Problem 15.11** Find the Mach number when an aeroplane is flying at 1100 km/hour through still air having a pressure of 7 N/cm<sup>2</sup> and temperature -5°C. Wind velocity may be taken as zero. Take R = 287.14 J/kg K. Calculate the pressure, temperature and density of air at stagnation point on the nose of the plane. Take k = 1.4.

**Solution.** Given :

Speed of aeroplane,	$V = 1100 \text{ km/hr} = \frac{1100 \times 1000}{60 \times 60} = 305.55 \text{ m/s}$
Pressure of air,	$p_1 = 7 \text{ N/cm}^2 = 7 \times 10^4 \text{ N/m}^2$
Temperature,	$t_1 = -5^\circ\text{C}$
..	$T_1 = -5 + 273 = 268^\circ\text{K}$
	$R = 287.14 \text{ J/kg K}$
	$k = 1.4$

Using relation  $C = \sqrt{kRT}$  for velocity of sound for adiabatic process, we have

$$C_1 = \sqrt{1.4 \times 287.14 \times 268} = 328.2 \text{ m/s}$$

$$\therefore \text{Mach number, } M_1 = \frac{V_1}{C_1} = \frac{305.55}{328.20} = 0.9309 \approx \mathbf{0.931. \text{ Ans.}}$$

**Stagnation Pressure,  $p_s$ .** Using equation (15.21) for stagnation pressure,

$$\begin{aligned} p_s &= p_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{k}{k-1}} \\ &= 7.0 \times 10^4 \left[ 1 + \frac{1.4-1.0}{2.0} \times (.931)^2 \right]^{\frac{1.4}{1.4-1.0}} \\ &= 7.0 \times 10^4 [1 + .1733]^{\frac{1.4}{.4}} \\ &= 7.0 \times 10^4 [1.1733]^{3.5} = 12.24 \times 10^4 \text{ N/m}^2 = \mathbf{12.24 \text{ N/cm}^2. \text{ Ans.}} \end{aligned}$$

**Stagnation Temperature,  $T_s$ .** Using equation (15.23) for stagnation temperature,

$$\begin{aligned} T_s &= T_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right] \\ &= 268 \left[ 1 + \frac{1.4-1.0}{2.0} \times (.931)^2 \right] = 268 [1.1733] = 314.44^\circ\text{K} \\ \therefore t_s &= T_s - 273 = 314.43 - 273 = \mathbf{41.44^\circ\text{C. Ans.}} \end{aligned}$$

**Stagnation Density,  $\rho_s$ .** Using equation of state (15.22 A) for stagnation density,  $\frac{p_s}{\rho_s} = RT_s$

$$\therefore \rho_s = \frac{p_s}{RT_s} \quad \dots(i)$$

In equation (i) given above, if R is taken as 287.14 J/kg K, then pressure should be taken in N/m<sup>2</sup> so that the value of ρ is in kg/m<sup>3</sup>. Hence  $p_s = 12.24 \times 10^4 \text{ N/m}^2$  and  $T_s = 314.44^\circ\text{K}$ .

$$\therefore \rho_s = \frac{12.24 \times 10^4}{287.14 \times 314.44} = \mathbf{1.355 \text{ kg/m}^3. \text{ Ans.}}$$

**Problem 15.12** Calculate the stagnation pressure, temperature and density at the stagnation point on the nose of a plane, which is flying at 800 km/hour through still air having a pressure 8.0 N/cm<sup>2</sup> (abs.) and temperature – 10°C. Take R = 287 J/kg K and k = 1.4.

**Solution.** Given :

$$\text{Speed of plane, } V = 800 \text{ km/hour} = \frac{800 \times 1000}{60 \times 60} = 222.22 \text{ m/s}$$

$$\text{Pressure of air, } p_1 = 8.0 \text{ N/cm}^2 = 8.0 \times 10^4 \text{ N/m}^2$$

$$\text{Temperature, } t_1 = -10^\circ\text{C}$$

$$\therefore T_1 = -10 + 273 = 263^\circ\text{K}$$

$$R = 287 \text{ J/kg}^\circ\text{K}$$

$$k = 1.4$$

For adiabatic flow, the velocity of sound is given by

$$C = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 263} = 325.07 \text{ m/s}$$

$$\therefore \text{Mach number, } M = \frac{V}{C} = \frac{222.22}{325.07} = 0.683.$$

This Mach number is the local Mach number and hence equal to  $M_1$ .

$$\therefore M_1 = 0.683$$

Using equation (15.21) for stagnation pressure,

$$p_s = p_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{k}{k-1}} = 8.0 \times 10^4 \left[ 1 + \frac{1.4-1.0}{2.0} \times (0.683)^2 \right]^{\frac{1.4}{1.4-1.0}} \\ = 8.0 \times 10^4 [1.0933]^{3.5} = 10.93 \times 10^4 \text{ N/m}^2 = 10.93 \text{ N/cm}^2. \text{ Ans.}$$

Using equation (15.23) for stagnation temperature,

$$T_s = T_1 \left( 1 + \frac{k-1}{2} M_1^2 \right) = 263 \left( 1 + \frac{1.4-1.0}{2.0} \times (0.683)^2 \right) \\ = 263 [1.0933] = 287.5 \text{ K}$$

$$\therefore t_s = T_s - 273 = 287.5 - 273 = 14.5^\circ\text{C. Ans.}$$

$$\text{Using equation (15.2), } \frac{p}{\rho} = RT$$

$$\text{For stagnation point, } \frac{p_s}{\rho_s} = RT_s \quad \therefore \rho_s = \frac{p_s}{RT_s}$$

As R = 287 J/kg K, the value of  $p_s$  should be taken in N/m<sup>2</sup> so that the value of  $\rho_s$  is obtained in kg/m<sup>3</sup>.

$$\therefore p_s = 10.93 \times 10^4 \text{ N/m}^2$$

$$\therefore \text{Stagnation density, } \rho_s = \frac{10.93 \times 10^4}{287 \times 287.5} = 1.324 \text{ kg/m}^3. \text{ Ans.}$$

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### ►15.8 AREA VELOCITY RELATIONSHIP FOR COMPRESSIBLE FLOW

The area velocity relationship for incompressible fluid is given by the continuity equation as  
 $A \times V = \text{Constant}$ .

From the above equation, it is clear that with the increase of area, velocity decreases. But in case of compressible fluid, the continuity equation is given by,  $\rho AV = \text{Constant}$ . ... (i)

From this relation, it is clear that with the change of area, both the velocity and density are affected. Hence to find the relation between area and velocity for compressible fluid we proceed as given below.

Differentiating equation (i), we get

$$\rho d(AV) + AVdp = 0 \quad \text{or} \quad \rho [AdV + VdA] + AVdp = 0$$

or  $\rho AdV + \rho VdA + AVdp = 0$ .

$$\text{Dividing by } \rho AV, \text{ we get } \frac{dV}{V} + \frac{dA}{A} + \frac{dp}{\rho} = 0 \quad \dots(ii)$$

The Euler's equation for compressible fluid is given by equation (15.8), as

$$\frac{dp}{\rho} + VdV + gdz = 0$$

Neglecting the  $z$  term, the above equation is written as  $\frac{dp}{\rho} + VdV = 0$ .

This equation can also be written as  $\frac{dp}{\rho} \times \frac{dp}{dp} + VdV = 0 \quad (\text{Dividing and multiplying by } dp)$

$$\text{or } \frac{dp}{dp} \times \frac{dp}{\rho} + VdV = 0$$

$$\text{But } \frac{dp}{dp} = C^2 \text{ from equation (15.15)}$$

Hence above equation becomes as

$$C^2 \frac{dp}{\rho} + VdV = 0 \quad \text{or} \quad C^2 \frac{dp}{\rho} = -VdV \quad \text{or} \quad \frac{dp}{\rho} = -\frac{VdV}{C^2}$$

Substituting the value of  $\frac{dp}{\rho}$  in equation (ii), we get

$$\frac{dV}{V} + \frac{dA}{A} - \frac{VdV}{C^2} = 0 \quad \text{or} \quad \frac{dA}{A} = \frac{VdV}{C^2} - \frac{dV}{V} = \frac{dV}{V} \left[ \frac{V^2}{C^2} - 1 \right]$$

$$\therefore \frac{dA}{A} = \frac{dV}{V} [M^2 - 1] \quad \dots(15.24)$$

Equation (15.24) gives the relationship between change of area with change of velocity for different Mach numbers. The following are the important conclusions :

- (i) For  $M < 1$ , the flow is sub-sonic and the right-hand side of equation (15.24) is negative as  $(M^2 - 1)$  is negative for the values of  $M < 1$ . Hence  $\frac{dA}{A} > 0$ ,  $\frac{dV}{V} < 0$ . This means that with the increase of area, the velocity decreases and vice versa.

(ii) For  $M > 1$ , the flow is super-sonic. The value of  $(M^2 - 1)$  will be positive and hence right-hand side of equation (15.24) will be positive. Hence  $\frac{dA}{A} > 0$  and also  $\frac{dV}{V} > 0$ . This means that with the increase of area, velocity also increases.

(iii) For  $M = 1$ , the flow is called sonic flow. The value of  $(M^2 - 1)$  is zero. Hence right-hand side of equation (15.24) will be zero. Hence  $\frac{dA}{A} = 0$ . This means area is constant.

### ► 15.9 FLOW OF COMPRESSIBLE FLUID THROUGH ORIFICES AND NOZZLES FITTED TO A LARGE TANK

Consider a compressible fluid filled in a large reservoir or vessel to which a short nozzle is fitted as shown in Fig. 15.5. If the pressure drop of the compressible fluid, flowing through the nozzle from reservoir is small, the process is considered to be isothermal. But if the pressure drop is large, the process is considered to be adiabatic.

Consider two points 1 and 2 inside the tank and at the exit of the nozzle respectively.

Let  $V_1$  = Velocity of fluid in the tank,

$p_1$  = Pressure of fluid at point 1,

$\rho_1$  = Density of fluid at point 1,

$T_1$  = Temperature of fluid at point 1, and

$V_2, p_2, \rho_2, T_2$  = Corresponding values of velocity, pressure, density and temperature at point 2.

Considering the process to be adiabatic. Then from Bernoulli's equation for adiabatic flow from equation (15.13), we have

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + Z_1 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + Z_2.$$

But  $Z_1 = Z_2$  and also  $V_1 = \text{Velocity of fluid in tank} = 0$

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + 0 = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g}$$

or 
$$\left(\frac{k}{k-1}\right)\left[\frac{p_1}{\rho_1 g} - \frac{p_2}{\rho_2 g}\right] = \frac{V_2^2}{2g}$$

or 
$$\left(\frac{k}{k-1}\right)\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = \frac{V_2^2}{2} \quad \left(\text{Cancelling } \frac{1}{g}\right)$$

or 
$$V_2 = \sqrt{\frac{2k}{(k-1)}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]} = \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left(1 - \frac{p_2}{\rho_2} \times \frac{\rho_1}{p_1}\right)} \quad ... (i)$$

But for adiabatic flow from equation (15.4), we have

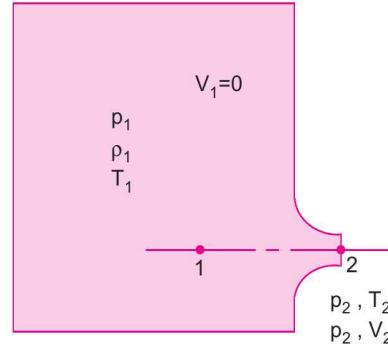


Fig. 15.5 Pressure tank fitted with a nozzle.

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$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} \text{ or } \frac{p_1}{p_2} = \left( \frac{\rho_1}{\rho_2} \right)^k$$

$$\therefore \frac{\rho_1}{\rho_2} = \left( \frac{p_1}{p_2} \right)^{\frac{1}{k}} \quad \dots(ii)$$

Substituting the value of  $\frac{\rho_1}{\rho_2}$  in equation (i), we get

$$V_2 = \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - \frac{p_2}{p_1} \times \left( \frac{p_1}{p_2} \right)^{\frac{1}{k}} \right]} = \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{1-\frac{1}{k}} \right]}$$

$$= \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right]} \quad \dots(15.25)$$

$$\text{Let } \frac{p_2}{p_1} = n. \text{ Then above equation becomes as } V_2 = \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - n^{\frac{k-1}{k}} \right]} \quad \dots(iii)$$

The mass rate of flow of the compressible fluid is given as

$$m = \rho_2 A_2 V_2, \text{ where } A_2 = \text{Area at the exit of nozzle}$$

$$= \rho_2 A_2 \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - n^{\frac{k-1}{k}} \right]} \quad [\text{Substitute } V_2 \text{ from (iii)}]$$

$$\Rightarrow = A_2 \sqrt{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \times \rho_2^2 \left[ 1 - n^{\frac{k-1}{k}} \right]} \quad (\text{taking } \rho_2 \text{ inside})$$

$$\text{But from equation (ii), we have } \rho_2 = \frac{\rho_1}{\left( \frac{p_1}{p_2} \right)^{1/k}} = \rho_1 \left( \frac{p_2}{p_1} \right)^{\frac{1}{k}} = \rho_1 n^{\frac{1}{k}} \quad \left( \because \frac{p_2}{p_1} = n \right)$$

$$\therefore \rho_2^2 = \rho_1^2 n^{\frac{2}{k}}$$

Substituting this value of  $\rho_2^2$  in the above equation, we get

$$m = A_2 \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \times \rho_1^2 n^{2/k} \left[ 1 - n^{\frac{k-1}{k}} \right]} = A_2 \sqrt{\frac{2k}{(k-1)} p_1 \rho_1 \left[ n^{\frac{2}{k}} - n^{\frac{k-1}{k} + \frac{2}{k}} \right]}$$

$$= A_2 \sqrt{\frac{2k}{k-1} p_1 \rho_1 \left[ n^{\frac{2}{k}} - n^{\frac{k+1}{k}} \right]} \quad \dots(15.26)$$

The mass rate of flow ( $m$ ) depends on the value of  $n$  for the given values of  $p_1$  and  $\rho_1$  at 1.

**15.9.1 Value of n or  $\frac{p_2}{p_1}$  for Maximum Value of Mass Rate of Flow.** For maximum value of  $m$ ,

we have

$$\frac{\partial m}{\partial n} = 0$$

or  $\frac{\partial}{\partial n} \left[ n^{\frac{2}{k}} - n^{\frac{k+1}{k}} \right] = 0 \quad \left( \because \frac{2k}{k-1} p_1 \rho_1 = \text{Constant} \right)$

or  $\frac{2}{k} n^{\frac{2}{k}-1} - \frac{k+1}{k} n^{\frac{k+1}{k}-1} = 0 \quad \text{or} \quad \frac{2}{k} n^{\frac{2-k}{k}} = \frac{k+1}{k} n^{\frac{k+1-k}{k}} = \frac{k+1}{k} n^{\frac{1}{k}}$

or  $n^{\frac{2-k}{k}} = \left( \frac{k+1}{k} \right) \times \frac{k}{2} \times n^{\frac{1}{k}} = \frac{k+1}{2} n^{\frac{1}{k}} \quad \text{or} \quad \frac{n^{\frac{2-k}{k}}}{n^{\frac{1}{k}}} = \frac{k+1}{2}$

$\therefore n^{\frac{2-k}{k}-\frac{1}{k}} = \frac{k+1}{2} \quad \text{or} \quad n^{\frac{1-k}{k}} = \frac{k+1}{2}$

$\therefore n^{\frac{-(k-1)}{k}} = \frac{k+1}{2} \quad \text{or} \quad \frac{1}{n^{\frac{k-1}{k}}} = \frac{k+1}{2} = \frac{1}{\left( \frac{2}{k+1} \right)}$

$\therefore n^{\frac{k-1}{k}} = \frac{2}{k+1} \quad \dots(15.27)$

Equation (15.27) is the condition for maximum value of mass rate of flow through the nozzle.

For  $k = 1.4$ , the value of  $n$  can be obtained from equation (15.27) as

$$n^{\frac{1.4-1.0}{1.4}} = \frac{2}{1.4+1} = \frac{2}{2.4} \quad \text{or} \quad n^{2/7} = \frac{2}{2.4}$$

$\therefore n = \left( \frac{2}{2.4} \right)^{7/2} = 0.528 \quad \text{or} \quad \frac{p_2}{p_1} = n = 0.528. \quad \dots(15.28)$

**15.9.2 Value of  $V_2$  for Maximum Rate of Flow of Fluid.** From equation (15.27), the value of  $n$  is given as

$$n = \left( \frac{2}{k+1} \right)^{\frac{k}{k-1}}$$

Substituting this value of  $n$  in equation (iii), we get

$$V_2 = \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left( 1 - n^{\frac{k-1}{k}} \right)}$$

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$$\begin{aligned}
 &= \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{2}{k+1} \right)^{\frac{k}{k-1} \times \frac{(k-1)}{k}} \right]} \\
 &= \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left( 1 - \frac{2}{k+1} \right)} = \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ \frac{k+1-2}{k+1} \right]} \\
 &= \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ \frac{k-1}{k+1} \right]} = \sqrt{\frac{2k}{k+1} \frac{p_1}{\rho_1}}
 \end{aligned} \quad \dots(15.29)$$

**15.9.3 Maximum Rate of Flow of Fluid Through Nozzle.** Mass rate of flow of fluid through nozzle is given by equation (15.26) as

$$m = A_2 \sqrt{\frac{2k}{(k-1)} p_1 \rho_1 \left[ n^{2/k} - n^{\frac{k+1}{k}} \right]}$$

From maximum rate of flow, from equation (15.27), we have

$$\begin{aligned}
 n^{\frac{k-1}{k}} &= \frac{2}{k+1} \\
 \therefore n &= \left( \frac{2}{k+1} \right)^{\frac{k}{k-1}}
 \end{aligned}$$

Substituting the value of  $n$  in the above equation, we have

$$\begin{aligned}
 m_{\max} &= A_2 \sqrt{\left( \frac{2k}{k-1} \right) p_1 \rho_1 \left[ \left( \frac{2}{k+1} \right)^{\frac{k}{k-1} \times \frac{2}{k}} - \left( \frac{2}{k+1} \right)^{\frac{k}{k-1} \times \frac{k+1}{k}} \right]} \\
 &= A_2 \sqrt{\frac{2k}{k-1} p_1 \rho_1 \left[ \left( \frac{2}{k+1} \right)^{\frac{2}{k-1}} - \left( \frac{2}{k+1} \right)^{\frac{k+1}{k-1}} \right]}
 \end{aligned}$$

For air,

$$k = 1.4,$$

$$\begin{aligned}
 \therefore m_{\max} &= A_2 \sqrt{\frac{2 \times 1.4}{1.4 - 1.0} p_1 \rho_1 \left[ \left( \frac{2}{1.4 + 1} \right)^{\frac{2}{1.4 - 1.0}} - \left( \frac{2}{1.4 + 1} \right)^{\frac{1.4 + 1.0}{1.4 - 1.0}} \right]} \\
 &= A_2 \sqrt{\frac{2.8}{0.4} p_1 \rho_1 \left[ \left( \frac{2}{2.4} \right)^{2/1.4} - \left( \frac{2}{2.4} \right)^{2.4/1.4} \right]} \\
 &= A_2 \sqrt{7 \times p_1 \rho_1 [ .4018 - .3348 ]} = A_2 \times 0.685 \times \sqrt{p_1 \rho_1} \\
 &= 0.685 A_2 \sqrt{p_1 \rho_1}.
 \end{aligned} \quad \dots(15.30)$$

#### 15.9.4 Variation of Mass Rate of Flow of Compressible Fluid with Pressure Ratio $\left(\frac{p_2}{p_1}\right)$ .

Fig. 15.6 shows the variation of mass rate of flow of compressible fluid with different pressure ratio  $\left(\frac{p_2}{p_1}\right)$ .

It is seen that when  $\frac{p_2}{p_1}$  is less than critical pressure ratio of 0.528, the mass rate of flow is constant and is equal to the mass rate of flow corresponding to pressure ratio = 0.528. But if the pressure ratio is more than 0.528, mass rate of flow decreases as shown by dotted line.

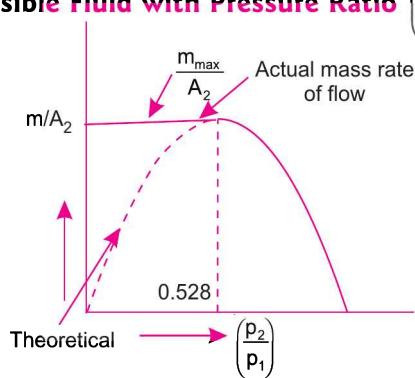


Fig. 15.6

#### 15.9.5 Velocity at Outlet of Nozzle for Maximum Rate of Flow is Equal to Sonic Velocity.

This is proved as given below.

The velocity at the outlet of nozzle for maximum rate of flow is given by equation (15.29) as

$$V_2 = \sqrt{\left(\frac{2k}{k+1}\right) \frac{p_1}{\rho_1}} \quad \dots(i)$$

$$\text{Now pressure ratio } \frac{p_2}{p_1} = n \quad \therefore \quad p_1 = \frac{p_2}{n}$$

$$\text{Also for adiabatic flow, } \frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} \text{ or } \frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^k$$

$$\text{or } \frac{p_1}{p_2} = \left(\frac{p_1}{p_2}\right)^{1/k} = \left(\frac{p_2}{p_1}\right)^{-1/k} = n^{-1/k} \quad \left( \because \frac{p_2}{p_1} = n \right)$$

$$\therefore \rho_1 = \rho_2 n^{-1/k}$$

Substituting the value of  $p_1$  and  $\rho_1$  in equation (i), we get

$$\begin{aligned} V_2 &= \sqrt{\left(\frac{2k}{k+1}\right) \frac{p_2}{\rho_2 n^{-1/k}}} = \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{1}{n} \times \frac{1}{n^{1/k}}} \\ &= \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{1}{n^{(1-1/k)}}} = \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{1}{n^{(k-1)/k}}} \end{aligned}$$

$$\text{But } n^{\frac{k-1}{k}} = \frac{2}{k+1} \text{ from equation (15.27)}$$

$$\begin{aligned} \therefore V_2 &= \sqrt{\frac{2k}{k+1} \times \frac{p_2}{\rho_2} \times \frac{(k+1)}{2}} = \sqrt{\frac{kp_2}{\rho_2}} \\ &= C_2. \quad \left( \because \sqrt{\frac{kp_2}{\rho_2}} = C_2 \right) \dots(15.31) \end{aligned}$$

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**Problem 15.13** Find the velocity of air flowing at the outlet of a nozzle, fitted to a large vessel which contains air at a pressure of  $294.3 \text{ N/cm}^2$  (abs.) and at a temperature of  $20^\circ\text{C}$ . The pressure at the outlet of the nozzle is  $206 \text{ N/cm}^2$  (abs). Take  $k = 1.4$  and  $R = 287 \text{ J/kg}^\circ\text{K}$ .

**Solution.** Given :

$$\text{Pressure inside vessel, } p_1 = 294.3 \text{ N/cm}^2 = 294.3 \times 10^4 \text{ N/m}^2$$

$$\text{Temperature inside vessel, } t_1 = 20^\circ\text{C}$$

$$\therefore T_1 = 20 + 273 = 293^\circ\text{K}$$

$$\text{Pressure at the nozzle, } p_2 = 206 \text{ N/cm}^2 = 206 \times 10^4 \text{ N/m}^2$$

$$R = 287 \text{ J/kg}^\circ\text{K}$$

$$k = 1.4$$

Using equation (15.25) for the velocity,

$$\begin{aligned} V_2 &= \sqrt{\left(\frac{2k}{k-1}\right) \frac{p_1}{\rho_1} \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]} \\ &= \sqrt{\left(\frac{2 \times 1.4}{1.4 \times 1.0}\right) \frac{294.3 \times 10^4}{\rho_1} \left[1 - \left(\frac{206 \times 10^4}{294.3 \times 10^4}\right)^{\frac{1.4-1.0}{1.4}}\right]} \\ &= \sqrt{\frac{2.8}{0.4} \times \frac{294.3 \times 10^4}{\rho_1} \left[1 - 0.7^{0.4/1.4}\right]} = \sqrt{\frac{7 \times 294.3 \times 10^4}{\rho_1} [1 - .903]} \quad \dots(i) \end{aligned}$$

The value of  $\rho_1$  is calculated from equation of state as

$$\frac{p_1}{\rho_1} = RT_1 \quad \therefore \rho_1 = \frac{p_1}{RT_1}$$

In this equation if  $R$  is taken in  $\text{J/kg K}$ ,  $p_1$  should be in  $\text{N/m}^2$ . Then  $\rho_1$  will be in  $\text{kg/m}^3$ .

$$\therefore p_1 = 294.3 \times 10^4 \text{ N/m}^2$$

$$\therefore \rho_1 = \frac{294.3 \times 10^4}{287 \times 293} = 34.99 \text{ kg/m}^3.$$

Substituting the value of  $\rho_1 = 34.99 \text{ kg/m}^3$  in equation (i),

$$V_2 = \sqrt{7 \times \frac{294.3 \times 10^4}{34.99} [1 - .903]} = 239.2 \text{ m/s. Ans.}$$

**Problem 15.14** A tank contains air at a temperature of  $30^\circ\text{C}$ . Air flows from the tank into atmosphere through a convergent nozzle. The diameter at the outlet of the nozzle is  $25 \text{ mm}$ . Assuming adiabatic flow, find the mass rate of flow of air through the nozzle when the pressure of air in tank is (i)  $3.924 \text{ N/cm}^2$  (gauge), (ii)  $33.354 \text{ N/cm}^2$  (gauge). Take  $k = 1.4$ ,  $R = 287 \text{ J/kg}^\circ\text{K}$  and atmospheric pressure =  $10.104 \text{ N/cm}^2$  (abs).

**Solution.** Given :

$$\text{Temperature in tank, } t_1 = 30^\circ\text{C}$$

$$\therefore T_1 = 30 + 273 = 303^\circ\text{K}$$

$$\text{Diameter at the nozzle, } D_2 = 25 \text{ mm} = 0.025 \text{ m.}$$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (0.025)^2 = 0.0004908 \text{ m}^2$$

$$R = 287 \text{ J/kg}^\circ\text{K}$$

$$k = 1.4.$$

(i) Mass rate of flow of air when pressure in tank is 3.924 N/cm<sup>2</sup> (gauge).

$$\begin{aligned} p_1 &= 3.924 \text{ N/cm}^2 \text{ (gauge)} \\ &= 3.924 + 10.104 = 14.028 \text{ N/cm}^2 \text{ (abs.)} \\ &= 14.028 \times 10^4 \text{ N/m}^2 \text{ (abs.)} \end{aligned}$$

$$\begin{aligned} \text{Pressure at the nozzle, } p_2 &= \text{Atmospheric pressure} \\ &= 10.104 \text{ N/cm}^2 = 10.104 \times 10^4 \text{ N/m}^2 \end{aligned}$$

$$\therefore \text{Pressure ratio, } n = \frac{p_2}{p_1} = \frac{10.104 \times 10^4}{14.028 \times 10^4} = 0.7203.$$

This pressure ratio is more than the pressure ratio 0.528 given by equation (15.28), hence mass rate of flow of air is given by equation (15.26), as

$$m = A_2 \sqrt{\frac{2k}{(k-1)} p_1 \rho_1 \left[ n^{\frac{2}{k}} - n^{-\frac{k+1}{k}} \right]} \quad \dots(i)$$

In this equation if  $p_1$  is taken into N/m<sup>2</sup>, then  $\rho_1$  will be in kg/m<sup>3</sup>. The value of  $\rho_1$  is obtained from equation of state as

$$\frac{p_1}{\rho_1} = RT_1 \text{ or } \rho_1 = \frac{p_1}{RT_1}, \quad \text{where } p_1 = 14.028 \text{ N/m}^2$$

$$\therefore \rho_1 = \frac{14.028 \times 10^4}{287 \times 303} = 1.613 \text{ kg/m}^3.$$

Substituting this value of  $\rho_1 = 1.613 \text{ kg/m}^3$  and  $p_1 = 14.028 \times 10^4 \text{ N/m}^2$  in equation (i), we get

$$\begin{aligned} m &= .0004908 \sqrt{\frac{2 \times 1.4}{1.4 - 1.0} \times 14.028 \times 10^4 \times 1.613 \left[ .7203^{\frac{2}{1.4}} - .7203^{\frac{1.4+1.0}{1.4}} \right]} \\ &= .0004908 \sqrt{1583935 \left[ .7203^{1.4285} - .7203^{1.7142} \right]} \\ &= .0004908 \sqrt{1583935 \left[ .6258 - .5698 \right]} = \mathbf{0.146 \text{ kg/s. Ans.}} \end{aligned}$$

(ii) Mass rate of flow of air when pressure in the tank is 33.354 N/cm<sup>2</sup> (gauge).

$$\therefore p_1 = 33.354 + 10.104 = 43.458 \text{ N/cm}^2 \text{ (abs.)} = 43.458 \times 10^4 \text{ N/m}^2 \text{ (abs.)}$$

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$$p_2 = \text{Atmospheric pressure} = 10.104 \times 10^4 \text{ N/m}^2$$

$$\therefore \text{Pressure ratio, } n = \frac{p_2}{p_1} = \frac{10.104 \times 10^4}{43.458 \times 10^4} = 0.2325.$$

This pressure ratio is less than the pressure ratio of 0.528. Hence as mentioned in Art. 15.9.4, the mass rate of flow will be corresponding to the pressure ratio of 0.528. Hence

$$m = A_2 \sqrt{\frac{2k}{k-1} p_1 \rho_1 \left[ n^{\frac{2}{k}} - n^{-\frac{(k+1)}{k}} \right]}$$

where  $n = 0.528$ ,  $p_1 = 43.458 \times 10^4 \text{ N/m}^2$ .

$$\text{The value of } \rho_1 = \frac{p_1}{RT_1} = \frac{43.458 \times 10^4}{287 \times 303} = 4.99 \text{ kg/m}^3$$

$$\therefore m = .0004908 \sqrt{\frac{2 \times 1.4}{(1.4 - 1.0)} \times 43.458 \times 10^4 \times 4.99 \left[ .528^{\frac{2}{1.4}} - .528^{-\frac{1.4+1.0}{1.4}} \right]} \\ = .004908 \sqrt{4906875 [4.015 - .3346]} = \mathbf{0.494 \text{ kg/s. Ans.}}$$

**Problem 15.15** A large tank contains air at  $28.449 \text{ N/cm}^2$  gauge pressure and  $24^\circ\text{C}$  temperature. The air flows from the tank to the atmosphere through an orifice. If the diameter of the orifice is 20 mm, find the maximum rate of flow of air. Tank  $R = 287 \text{ J/kg K}$ ,  $k = 1.4$ , atmospheric pressure =  $10.104 \text{ N/cm}^2$ .

**Solution.** Given :

$$\begin{aligned} \text{Pressure in tank, } p_1 &= 28.449 \text{ N/cm}^2 \text{ (gauge)} \\ &= 28.449 + 10.104 = 38.553 \text{ N/cm}^2 \text{ (abs.)} = 38.553 \times 10^4 \text{ N/m}^2 \end{aligned}$$

$$\begin{aligned} \text{Temperature in tank, } t_1 &= 24^\circ\text{C} \\ \therefore T_1 &= 273 + 24 = 297^\circ\text{K} \\ R &= 287 \text{ J/kg K} \\ k &= 1.4 \end{aligned}$$

$$\text{Diameter of orifice, } D = 20 \text{ mm} = 0.02 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} (0.02)^2 = .0003141 \text{ m}^2$$

$$\text{Using equation of state, we get } \frac{p_1}{\rho_1} = RT_1 \text{ or } \rho_1 = \frac{p_1}{RT_1} = \frac{38.553 \times 10^4}{287 \times 297} = 4.522 \text{ kg/m}^3$$

Maximum rate of flow of air is given by equation (15.30) as

$$\begin{aligned} m_{\max} &= 0.685 A_2 \sqrt{p_1 \rho_1} \quad (\text{Here } A_2 = A = 0.0003141) \\ &= 0.685 \times .0003141 \sqrt{38.553 \times 10^4 \times 4.522} = \mathbf{0.284 \text{ kg/s. Ans.}} \end{aligned}$$

► 15.10 MASS RATE OF FLOW OF COMPRESSIBLE FLUID THROUGH VENTURI METER

Consider a compressible fluid flowing through the horizontal venturimeter. Let the conditions of flow is represented by suffix 1 at the inlet of venturimeter and by suffix 2 at the throat of the venturimeter. Considering the flow as adiabatic, we have from Bernoulli's equation at sections 1 and 2 from equation (15.13), as

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} \quad (\because z_1 = z_2)$$

or  $\frac{k}{(k-1)}\frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2} + \frac{V_2^2}{2}$  (Cancelling  $\frac{1}{g}$ )

or  $\frac{k}{(k-1)}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right] = \frac{V_2^2}{2} - \frac{V_1^2}$   
 $\frac{k}{(k-1)}\frac{p_1}{\rho_1}\left[1 - \frac{p_2}{\rho_2} \times \frac{\rho_1}{p_1}\right] = \frac{V_2^2}{2} - \frac{V_1^2}$  ... (i)

For adiabatic flow,  $\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} \quad \therefore \quad \frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^k$

or  $\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{1/k} = \left(\frac{p_2}{p_1}\right)^{-1/k}$  ... (ii)

Substituting this value of  $\frac{\rho_1}{\rho_2}$  in equation (i), we get

$$\frac{k}{k-1}\frac{p_1}{\rho_1}\left[1 - \frac{p_2}{p_1} \times \left(\frac{p_2}{p_1}\right)^{-1/k}\right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

or  $\frac{k}{k-1}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{1-1/k}\right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$

or  $\frac{k}{k-1}\frac{p_1}{\rho_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{k-1/k}\right] = \frac{V_2^2}{2} - \frac{V_1^2}{2}$  ... (iii)

Applying continuity for sections 1 and 2, we have

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \therefore \quad V_1 = \frac{\rho_2 A_2 V_2}{\rho_1 A_1}$$

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Substituting the value of  $V_1$  in equation (iii), we get

$$\frac{k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{k-1/k} \right] = \frac{V_2^2}{2} - \left( \frac{\rho_2 A_2 V_2}{\rho_1 A_1} \right)^2 \times \frac{1}{2} = \frac{V_2^2}{2} \left[ 1 - \frac{\rho_2^2 A_2^2}{\rho_1^2 A_1^2} \right] \quad \dots(iv)$$

But from equation (ii), we have

$$\frac{\rho_1}{\rho_2} = \left( \frac{p_1}{p_2} \right)^{1/k} \quad \text{or} \quad \frac{\rho_2}{\rho_1} = \left( \frac{p_2}{p_1} \right)^{1/k} \quad \therefore \left( \frac{\rho_2}{\rho_1} \right)^2 = \left( \frac{p_2}{p_1} \right)^{2/k}$$

Substituting this value in equation (iv), we get

$$\begin{aligned} \frac{k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{k-1/k} \right] &= \frac{V_2^2}{2} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{2/k} \times \frac{A_2^2}{A_1^2} \right] \\ V_2^2 &= \frac{\frac{2k}{(k-1)} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{k-1/k} \right]}{\left[ 1 - \left( \frac{p_2}{p_1} \right)^{2/k} \times \frac{A_2^2}{A_1^2} \right]} \\ \therefore V_2 &= \sqrt{\frac{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right]}{\left[ 1 - \left( \frac{A_2^2}{A_1^2} \right) \left( \frac{p_2}{p_1} \right)^{\frac{2}{k}} \right]}} \end{aligned}$$

$\therefore$  Mass rate of flow through venturimeter,

$$m = \rho_2 A_2 V_2 = \rho_2 A_2 \sqrt{\frac{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{k-1/k} \right]}{\left[ 1 - \left( \frac{A_2^2}{A_1^2} \right) \left( \frac{p_2}{p_1} \right)^{2/k} \right]}} \quad \dots(15.32)$$

The only condition for equation (15.32) is that the pressure ratio  $\left( \frac{p_2}{p_1} \right)$  should be more than the pressure ratio 0.528

**Problem 15.16** Find the mass rate of flow of air though a venturimeter having inlet diameter as 300 mm and throat diameter 150 mm. The pressure at the inlet of venturimeter is 1.4 kgf/cm<sup>2</sup> (1.4 × 9.81 N/cm<sup>2</sup>) absolute and temperature of air at inlet is 15°C. The pressure at the throat is given as 1.3 kgf/cm<sup>2</sup> (1.3 × 9.81 N/cm<sup>2</sup>) absolute. Take R = 287 J/kg°K and k = 1.4.

**Solution.** Given :

Diameter at inlet,  $D_1 = 300 \text{ mm} = 0.30 \text{ m}$

$$\therefore \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.30)^2 = .07068 \text{ m}^2$$

$$\text{Diameter at throat, } D_2 = 150 \text{ mm} = 0.15 \text{ m}$$

$$\therefore \text{Area, } A_2 = \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2$$

$$\text{Pressure of air at inlet, } p_1 = 1.4 \text{ kgf/cm}^2 = 1.4 \times 10^4 \text{ kgf/m}^2 = 1.4 \times 10^4 \times 9.81 \text{ N/m}^2$$

$$\text{Throat pressure, } p_2 = 1.3 \text{ kgf/cm}^2 = 1.3 \times 10^4 \times 9.81 \text{ N/m}^2$$

$$R = 287 \text{ J/kg}^\circ\text{K}$$

$$k = 1.4$$

$$\text{Temperature at inlet, } t_1 = 15^\circ\text{C}$$

$$\text{The pressure ratio, } \frac{p_2}{p_1} = \frac{1.3 \times 10^4 \times 9.81}{1.4 \times 10^4 \times 9.81} = 0.9285$$

Density of gas at inlet is obtained from equation of state,

$$\frac{p_1}{\rho_1} = RT_1 \text{ or } \rho_1 = \frac{p_1}{RT_1} = \frac{1.4 \times 10^4 \times 9.81}{287 \times (273 + 15)}$$

$$\text{where } T_1 = t_1 + 273 = 15 + 273 = 288^\circ\text{K}$$

$$\therefore \rho_1 = \frac{1.4 \times 10^4 \times 9.81}{287 \times 288} = 1.66 \text{ kg/cm}^3$$

$$\text{For adiabatic process, we have } \frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} \text{ or } \left(\frac{\rho_2}{\rho_1}\right)^k = \frac{p_2}{p_1} \text{ or } \frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{1/k}$$

$$\therefore \rho_2 = \rho_1 \left(\frac{p_2}{p_1}\right)^{1/k} = 1.66 \times (0.9285)^{1/1.4} \quad \left( \because \frac{p_2}{p_1} = .9285 \right) \\ = 1.574 \text{ kg/m}^3.$$

Using equation (15.32) for the mass rate of flow through venturimeter, we get

$$m = \rho_2 A_2 \sqrt{\frac{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ 1 - \left(\frac{p_2}{p_1}\right)^{k-1/k} \right]}{\left[ 1 - \left(\frac{A_2^2}{A_1^2}\right) \left(\frac{p_2}{p_1}\right)^{2/k} \right]}} \\ = 1.574 \times 0.01767 \sqrt{\frac{\frac{2 \times 1.4}{1.4-1} \times \frac{1.4 \times 10^4 \times 9.81}{1.66} \left[ 1 - .9285^{\frac{1.4-1.0}{1.4}} \right]}{\left[ 1 - \left(\frac{0.01767^2}{0.07068^2}\right) (.9285^{2/1.4}) \right]}} \\ = .0278 \sqrt{\frac{579144 [1 - .9285^{2857}]}{1 - .0625 \times .899}} = .0278 \times \sqrt{\frac{12145.57}{.9438}} \\ = \mathbf{315 \text{ kg/s. Ans.}}$$

### ► 15.11 PITOT-STATIC TUBE IN A COMPRESSIBLE FLOW

The pitot-static tube, when used for determining the velocity at any point in a compressible fluid, gives only the difference between the stagnation head and static head. From this difference, the velocity of the incompressible fluid at that point is obtained from the relation

$$V = \sqrt{2gh}, \text{ where } h = \text{Difference in two heads.}$$

But when the pitot-static tube is used for finding velocity at any point in a compressible fluid, the actual pressure difference shown by the gauges of the pitot-tube should be multiplied by a factor, for obtaining correct velocity at that point. The value of the factor depends upon the Mach number of the flow. Let us find an expression for the correction factor for *sub-sonic* flow.

At a point in pitot-static tube, the pressure becomes stagnation pressure, denoted by  $p_s$ . The expression for stagnation pressure,  $p_s$  is given by equation (15.21), as

$$p_s = p_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{k}{k-1}} \quad \dots(i)$$

where  $p_1$  = Pressure of fluid far away from stagnation point,  
 $M_1$  = Mach number at point 1 far away from stagnation point.

For  $M < 1$ , the term  $\frac{k-1}{2} M_1^2$  will be less than 1 and hence the right-hand side of equation (i) can be expanded by Binomial theorem\* as

$$\begin{aligned} p_s &= \left[ 1 + \frac{k-1}{2} M_1^2 \times \frac{k}{k-1} \times \frac{\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)}{2!} \times \left(\frac{k-1}{2} M_1^2\right)^2 \right. \\ &\quad \left. + \frac{\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)\left(\frac{k}{k-1}-2\right)\left(\frac{k-1}{2} M_1^2\right)^3}{3!} + \dots \right] \\ &= p_1 \left[ 1 + \frac{k}{2} M_1^2 + \frac{k}{8} M_1^4 + \frac{k(2-k)}{48} M_1^6 + \dots \right] \\ &= p_1 + p_1 \left[ \frac{k}{2} M_1^2 + \frac{k}{8} M_1^4 + \frac{k(2-k)}{48} M_1^6 + \dots \right] \\ p_s - p_1 &= p_1 \times \frac{k}{2} M_1^2 \left[ 1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \dots \right] \quad \dots(ii) \end{aligned}$$

But  $M_1^2 = \frac{V_1^2}{C_1^2}$ , where  $C_1^2 = \frac{kp_1}{\rho_1}$

---

\* Binomial Theorem  $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!} \dots$

$$= \frac{V_1^2}{kp_1} = \frac{V_1^2 \rho_1}{kp_1}$$

Substituting the value of  $M_1^2$  in equation (ii), we get

$$\begin{aligned} p_s - p_1 &= p_1 \times \frac{k}{2} \times \frac{V_1^2 \rho_1}{kp_1} \left[ 1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \dots \right] \\ &= \frac{\rho_1 V_1^2}{2} \left[ 1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \dots \right] \end{aligned}$$

The term  $\left[ 1 + \frac{M_1^2}{4} + \frac{(2-k)}{24} M_1^4 + \dots \right]$  is known as **Compressibility Correction Factor**. And  $\frac{\rho_1 V_1^2}{2}$

is the reading of the pitot-static tube. Thus the reading of the pitot-tube must be multiplied by a correction factor given below for correct value of velocity measured by the pitot-tube.

$$\text{Compressibility Correction Factor, C.C.F.} = \left[ 1 + \frac{M_1^2}{4} + \frac{2-k}{24} M_1^4 + \dots \right] \quad \dots(15.33)$$

**Problem 15.17** Calculate the numerical factor by which the actual pressure difference shown by the gauge of a pitot-static tube must be multiplied to allow for compressibility when the value of the Mach number is 0.9. Take  $k = 1.4$ .

**Solution.** Given :

$$\begin{aligned} \text{Mach number, } M_1 &= 0.9 \\ k &= 1.4 \end{aligned}$$

Using equation (15.33), Compressibility Correction Factor is

$$\begin{aligned} \text{C.C.F.} &= \left[ 1 + \frac{M_1^2}{4} + \frac{2-k}{24} M_1^4 + \dots \right] = 1 + \frac{.9^2}{4} + \frac{2-1.4}{24} (.9)^4 + \dots \\ &= 1.0 + .2025 + .0164 + \dots = \mathbf{1.2189. Ans.} \end{aligned}$$

∴ Numerical factor by which the actual pressure difference is to be multiplied = 1.2189.

## HIGHLIGHTS

1. A flow in which density does not remain constant during flow, is called compressible flow.

2. Equation of state is given by,  $\frac{P}{\rho} = RT$

where  $P$  = Absolute pressure in kgf/m<sup>2</sup> or N/m<sup>2</sup>

$T$  = Absolute temperature =  $273 + t^\circ\text{C}$

$R$  = Gas constant in J/kg K or m/kg

$\rho$  = Density of gas.

If the value of  $P$  is taken in N/m<sup>2</sup>,  $R$  is J/kg °K and  $T$  in °K, the value of density is given in kg/m<sup>3</sup>

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3. The pressure density of a gas are related as

$$\frac{p}{\rho} = \text{Constant.... for isothermal process}$$

$$\frac{p}{\rho^k} = \text{Constant....for adiabatic process}$$

where  $k$  = Ratio of specific heats = 1.4 for air.

4. Continuity equation for compressible flow is given as  $\rho A V = \text{Constant}$ .

And in differential form, continuity equation is  $\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$ .

5. Bernoulli's equation for compressible fluids is given as

$$\frac{p}{\rho g} \log_e p + \frac{V^2}{2g} + Z = \text{Constant.....for isothermal process.}$$

$$\frac{k}{(k-1)} \frac{p}{\rho g} + \frac{V^2}{2g} + Z = \text{Constant.....for adiabatic process.}$$

6. Velocity of sound wave is given by

$$C = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{K}{\rho}} \quad \dots\dots \text{in term of Bulk modulus}$$

$$C = \sqrt{\frac{p}{\rho}} = \sqrt{RT} \quad \dots\dots \text{for isothermal process}$$

$$= \sqrt{\frac{kp}{\rho}} = \sqrt{kRT} \quad \dots\dots \text{for adiabatic process.}$$

7. Mach number,  $M$  is given as  $M = \frac{V}{C}$

If  $M < 1$  .....flow is sub-sonic flow,

$M > 1$  .....flow is super-sonic flow,

$M = 1$  .....flow is sonic flow.

8. In sub-sonic flow, the disturbance always moves ahead of the projectile. In sonic flow, the disturbance moves along the projectile while in super-sonic flow, the projectile always moves ahead of the disturbance .

9. Mach angle is given by  $\sin \alpha = \frac{C}{V} = \frac{1}{M}$ .

10. The pressure, temperature and density at a point where velocity is zero are called stagnation pressure, temperature and stagnation density. Their values are given as

$$p_s = p_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{k}{k-1}} ; \rho_s = \rho_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{1}{k-1}}$$

$$T_s = T_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right] \text{ also } = \frac{1}{R} \frac{p_s}{\rho_s}.$$

11. Area velocity relationship for compressible fluid is given as  $\frac{dA}{A} = \frac{dV}{V} [M^2 - 1]$

If  $M < 1$ ,  $\frac{dV}{V} < 0$  and  $\frac{dA}{A} > 0$  which means with the increase of area, velocity decreases.

If  $M > 1$ ,  $\frac{dV}{V} > 0$  and  $\frac{dA}{A} > 0$  which means with the increase of area, velocity also increases.

If  $M = 1$ ,  $\frac{dA}{A} = 0$  means area is constant.

12. Velocity through a nozzle or orifice fitted to a large tank is

$$V_2 = \sqrt{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{\rho_1} \right)^{\frac{k-1}{k}} \right]}$$

where  $p_2$  = Pressure at the outlet of nozzle or orifice,  $p_1$  = Pressure in the tank.

13. The mass rate of flow of compressible fluid through orifice or nozzle fitted to the tank is

$$m = A_2 \sqrt{\frac{2k}{k-1} p_1 \rho_1 \left[ n^{2/k} - n^{\frac{k+1}{k}} \right]}$$

where  $n$  = Pressure ratio =  $\frac{p_2}{p_1}$ .

14. For maximum flow through orifice or nozzle fitted to the tank, pressure ratio =  $\frac{p_2}{p_1} = n = 0.528$

$$\text{Also } n^{\frac{k-1}{k}} = \frac{2}{k+1}.$$

And velocity at the outlet of the orifice or nozzle is  $V_2 = \sqrt{\frac{2k}{k+1} \frac{p_1}{\rho_1}} = C_2$ .

And mass rate of flow of fluid is given by  $m_{\max} = 0.685 A_2 \sqrt{p_1 \rho_1}$ .

15. If the pressure ratio for the nozzle or orifice fitted to the large tank is less than 0.528, the mass rate of flow of the fluid is always corresponding to the pressure ratio of 0.528. But if the pressure ratio is more than 0.528, the mass rate of flow of fluid is corresponding to the given pressure ratio.

16. Mass rate of flow of compressible fluid through venturimeter is given by

$$m = \rho_2 A_2 \sqrt{\frac{\frac{2k}{k-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right]}{\left[ 1 - \left( \frac{A_2}{A_1} \right)^2 \left( \frac{p_2}{p_1} \right)^{2/k} \right]}}$$

where  $A_2$  = Area at the throat,  $A_1$  = Area at inlet.

17. The compressibility correction factor is given by C.C.F =  $\left[ 1 + \frac{M_1^2}{4} + \frac{2-k}{24} M_1^4 + \dots \right]$

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define compressible and incompressible flow.
2. What is the relation between pressure and density of a compressible fluid for  
(a) isothermal process and (b) adiabatic process ?
3. Derive the continuity equation for one-dimensional compressible flow in differential form.
4. State the Bernoulli's theorem for compressible flow. Derive an expression for Bernoulli's equation when the process is (i) isothermal and (ii) adiabatic.
5. Write an expression for momentum equation for compressible fluid.  
(a) isothermal process and (b) adiabatic process ?
6. (a) Obtain an expression for velocity of the sound wave in a compressible fluid in terms of change of pressure and change of density.  
(b) Show that the velocity of propagation of the pressure wave in a compressible fluid is given by

$$C = \sqrt{E/\rho}, \text{ where } E \text{ is volume modulus of elasticity of fluid.}$$

7. Prove that the velocity of sound wave in a compressible fluid is given by  $C = \sqrt{\frac{K}{\rho}}$   
where  $K$  = Bulk modulus of fluid,  $\rho$  = Density of fluid.
8. Derive an expression for the velocity of sound wave for compressible fluid when the process is assumed as (i) isothermal and (ii) adiabatic.
9. Define Mach number. What is the significance of Mach number in compressible fluid flows ?
10. Define the terms: Sub-sonic flow, super-sonic flow, sonic flow, Mach angle and Mach cone.
11. Show by means of diagrams the nature of propagation of disturbance in compressible flow when Mach number is less than one, is equal to one and is more than one. *(Delhi University, Dec. 2002)*
12. What do you understand by stagnation pressure? Obtain an expression for stagnation pressure of a compressible fluid in terms of approaching Mach number and pressure.
13. Prove that stagnation temperature and stagnation density are given as

$$T_s = T_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right] \text{ and } \rho_s = \rho_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{1}{k-1}}$$

14. Derive an expression for area velocity relationship for a compressible fluid in the form.

$$\frac{dA}{A} = \frac{dV}{V} [M^2 - 1] \quad \text{(Delhi University, Dec. 2002)}$$

15. Find an expression for mass rate of flow of compressible fluid through an orifice or nozzle fitted to a large tank. What is condition for maximum rate of flow?
16. What do you mean by compressibility correction factor ? Find an expression for compressibility factor.
17. Derive an expression for velocity of sound for an adiabatic process.
18. What do you mean by sub-sonic, sonic and super-sonic flows?
19. For frictionless adiabatic flow (*i.e.*, isentropic flow) show that the stagnation pressure at a given point is given by

$$\frac{p_s}{p_1} = 1 + \frac{1}{4} M_0^2 + \frac{(2-k)}{24} M_0^4 + \dots$$

where  $p_s$  = stagnation pressure,  $p_0$  = pressure in ambient flow and  $M_0 = U_0 / \sqrt{E/\rho}$ .

20. Differentiate between isentropic and adiabatic processes.
21. (a) Derive an expression for the velocity of sound waves moving in a compressible fluid.  
 (b) Define and explain the terms :  
 Mach number, Froude number, Reynolds number, Mach cone and Mach angle
22. State the Bernoulli's theorem for compressible flow. Derive an expression for Bernoulli's equation when the process is adiabatic.  
*(R.G.P.V., Summer, 2002)*

### (B) NUMERICAL PROBLEMS

- A gas is flowing through a horizontal pipe of cross-sectional area of  $30 \text{ cm}^2$ . At a point the pressure is  $30 \text{ N per cm}^2$  (gauge) and temperature  $20^\circ\text{C}$ . At another section the area of cross-section is  $15 \text{ cm}^2$  and pressure is  $25 \text{ N/cm}^2$  gauge. If the mass rate of flow of gas is  $0.15 \text{ kg/s}$ , find the velocities of the gas at these two sections, assuming an isothermal change. Take  $R = 287 \text{ J/kg K}$ , and atmospheric pressure  $10 \text{ N/cm}^2$ .  
**[Ans.  $V_1 = 10.71 \text{ m/s}$ ;  $V_2 = 24.5 \text{ m/s}$ ]**
- A gas with a velocity of  $350 \text{ m/s}$  is flowing through a horizontal pipe at a section where pressure is  $8 \text{ N/cm}^2$  (absolute) and temperature is  $30^\circ\text{C}$ . The pipe changes in diameter and at this section the pressure is  $12 \text{ N/cm}^2$  (absolute). Find the velocity of the gas at this section if the flow of the gas is adiabatic. Take  $R = 287 \text{ J/kg K}$  and  $k = 1.4$ .  
**[Ans.  $218.63 \text{ m/s}$ ]**
- Find the speed of the sound wave in air at sea-level where the pressure and temperature are  $9.81 \text{ N/cm}^2$  (abs.) and  $20^\circ\text{C}$  respectively. Take  $R = 287 \text{ J/kg K}$  and  $k = 1.4$ .  
**[Ans.  $343.11 \text{ m/s}$ ]**
- Calculate the Mach number at a point on a jet propelled aircraft which is flying at  $900 \text{ km/hour}$  at sea-level where air temperature is  $15^\circ\text{C}$ . Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ .  
**[Ans.  $0.735$ ]**
- An aeroplane is flying at an height of  $20 \text{ km}$ , where the temperature is  $-40^\circ\text{C}$ . The speed of the plane is corresponding to  $M = 1.8$ . Assuming  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ , find the speed of the plane.  
**[Ans.  $1982.66 \text{ m/hr}$ ]**
- A projectile is travelling in air having pressure and temperature as  $8.829 \text{ N/cm}^2$  and  $-5^\circ\text{C}$ . If the Mach angle is  $30^\circ$ , find the velocity of the projectile. Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ .  
**[Ans.  $656.30 \text{ m/s}$ ]**
- A projectile travels in air of pressure  $8.829 \text{ N/cm}^2$  at  $-10^\circ\text{C}$  at a speed of  $1200 \text{ km/hour}$ . Find the Mach number and the Mach angle. Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ .  
**[Ans.  $1.025, \theta = 77.2^\circ$ ]**
- Find the Mach number when an aeroplane is flying at  $900 \text{ km/hour}$  through still air having a pressure of  $8.0 \text{ N/cm}^2$  and temperature  $-15^\circ\text{C}$ . Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ . Calculate the pressure, temperature and density of air at the stagnation point on the nose of the plane.  
**[Ans.  $0.776, 11.9 \text{ N/cm}^2, 16.06^\circ\text{C}, 1.434 \text{ kg/m}^3$ ]**
- Find the velocity of air flowing at the outlet of a nozzle, fitted to a large vessel which contains air at a pressure of  $294.3 \text{ N/cm}^2$  (abs.) and at a temperature of  $30^\circ\text{C}$ . The pressure at the outlet of the nozzle is  $137.34 \text{ N/cm}^2$  (abs.) Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ .  
**[Ans.  $242.98 \text{ m/s}$ ]**
- A nozzle of diameter  $20 \text{ mm}$  is fitted to a large tank which contains air at  $20^\circ\text{C}$ . The air flows from the tank into atmosphere. For adiabatic flow, find the mass rate of flow of air through the nozzle when pressure of air in tank is (i)  $5.886 \text{ N/cm}^2$  (gauge) and (ii)  $29.43 \text{ N/cm}^2$  (gauge). Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$  and atmospheric pressure =  $9.81 \text{ N/cm}^2$ .  
**[Ans. (i)  $0.114 \text{ kg/s}$ , (ii)  $0.291 \text{ kg/s}$ ]**
- Find the mass rate of flow of air through a venturimeter having inlet diameter as  $400 \text{ mm}$  and throat diameter  $200 \text{ mm}$ . The pressure at the inlet of the venturimeter is  $27.468 \text{ N/cm}^2$  (abs.) and temperature of air at inlet is  $20^\circ\text{C}$ . The pressure at the throat is given as  $25.506 \text{ N/cm}^2$  absolute. Take  $k = 1.4$  and  $R = 287 \text{ J/kg K}$ .  
**[Ans.  $11.13 \text{ kg/s}$ ]**
- Calculate the numerical factor by which the actual pressure difference shown by the gauge of a pitot-tube should be multiplied to allow for compressibility when the value of the Mach number is  $0.7$ . Take  $k = 1.4$ .  
**[Ans.  $1.1285$ ]**

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13. Find the Mach number when an aeroplane is flying at 1000 km/hour through still air having pressure of  $7 \text{ N/cm}^2$  and temperature of  $-5^\circ\text{C}$ . Take  $R = 287.14 \text{ J/kg K}$ . Calculate the pressure and temperature of air at stagnation point. Take  $k = 1.4$ .  
*(Delhi University, Dec. 2002)*

[Hint.  $V = 1000 \text{ km/hour} = \frac{1000 \times 1000}{60 \times 60} = 277.77 \text{ m/s}$ ;  $p_1 = 7 \text{ N/m}^2 = 7 \times 10^4 \text{ N/m}^2$ ,  $t_1 = -5^\circ\text{C}$

$$\therefore T_1 = -5 + 273 = 268^\circ \text{K}; R = 287.14 \text{ J/kg K}, k = 1.4$$

$$\text{Now } C_1 = \sqrt{kRT_1} = \sqrt{1.4 \times 287.14 \times 268} = 328.2 \text{ m/s} \quad \therefore M = M_1 = \frac{V}{C} = \frac{V_1}{C_1} = \frac{277.77}{328.2}$$

$$\therefore M_1 = 0.846 \quad \therefore p_s = p_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right]^{\frac{k}{k-1}} = 7 \times 10^4 \left[ 1 + \frac{1.4-1}{2} \times (0.846)^2 \right]^{\frac{1.4}{1.4-1}}$$

$$= 7 \times 10^4 [1 + 0.2 \times 0.846^2]^{\frac{1.4}{0.4}} = 11.18 \times 10^4 \text{ N/m}^2 = \mathbf{11.18 \text{ N/cm}^2}.$$

$$T_s = T_1 \left[ 1 + \frac{k-1}{2} M_1^2 \right] = 268 \left[ 1 + \frac{1.4-1}{2} \times (0.846)^2 \right] = 306.38^\circ \text{K} = 306.38 - 273 = \mathbf{33.38^\circ\text{C}.}$$

# 16

## CHAPTER



# FLOW IN OPEN CHANNELS

### ► 16.1 INTRODUCTION

Flow in open channels is defined as the flow of a liquid with a free surface. A free surface is a surface having constant pressure such as atmospheric pressure. Thus a liquid flowing at atmospheric pressure through a passage is known as flow in open channels. In most of cases, the liquid is taken as water. Hence flow of water through a passage under atmospheric pressure is called flow in open channels. The flow of water through pipes at atmospheric pressure or when the level of water in the pipe is below the top of the pipe, is also classified as open channel flow.

In case of open channel flow, as the pressure is atmospheric, the flow takes place under the force of gravity which means the flow takes place due to the slope of the bed of the channel only. The hydraulic gradient line coincides with the free surface of water.

### ► 16.2 CLASSIFICATION OF FLOW IN CHANNELS

The flow in open channel is classified into the following types :

1. Steady flow and unsteady flow,
2. Uniform flow and non-uniform flow,
3. Laminar flow and turbulent flow, and
4. Sub-critical, critical and super critical flow.

**16.2.1 Steady Flow and Unsteady Flow.** If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow. Mathematically, steady flow is expressed as

$$\frac{\partial V}{\partial t} = 0, \frac{\partial Q}{\partial t} = 0 \quad \text{or} \quad \frac{\partial y}{\partial t} = 0 \quad \dots(16.1)$$

where  $V$  = velocity,  $Q$  = rate of flow and  $y$  = depth of flow.

If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady flow. Mathematically, unsteady flow means

$$\frac{\partial V}{\partial t} \neq 0 \quad \text{or} \quad \frac{\partial y}{\partial t} \neq 0 \quad \text{or} \quad \frac{\partial Q}{\partial t} \neq 0.$$

**16.2.2 Uniform Flow and Non-uniform Flow.** If for a given length of the channel, the velocity of flow, depth of flow, slope of the channel and cross-section remain constant, the flow is

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said to be uniform. On the other hand, if for a given length of the channel, the velocity of flow, depth of flow etc., do not remain constant, the flow is said to be non-uniform flow. Mathematically, uniform and non-uniform flow are written as :

$$\frac{\partial y}{\partial S} = 0, \frac{\partial V}{\partial S} = 0 \text{ for uniform flow}$$

and  $\frac{\partial y}{\partial S} \neq 0, \frac{\partial V}{\partial S} \neq 0$  for non-uniform flow.

Non-uniform flow in open channels is also called varied flow, which is classified in the following two types as :

- (i) Rapidly Varied Flow (R.V.F.), and
- (ii) Gradually Varied Flow (G.V.F.).

(i) **Rapidly Varied Flow (R.V.F.).** Rapidly varied flow is defined as that flow in which depth of flow changes abruptly over a small length of the channel. As shown in Fig. 16.1 when there is any obstruction in the path of flow of water, the level of water rises above the obstruction and then falls and again rises over a small length of channel. Thus the depth of flow changes rapidly over a short length of the channel. For this short length of the channel the flow is called rapidly varied flow (R.V.F.).

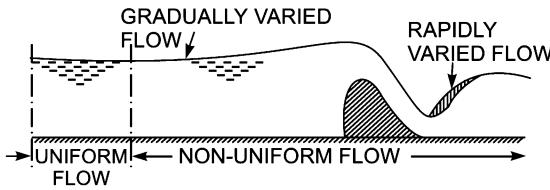


Fig. 16.1 Uniform and non-uniform flow.

(ii) **Gradually Varied Flow (G.V.F.).** If the depth of flow in a channel changes gradually over a long length of the channel, the flow is said to be gradually varied flow and is denoted by G.V.F.

**16.2.3 Laminar Flow and Turbulent Flow.** The flow in open channel is said to be laminar if the Reynold number ( $R_e$ ) is less than 500 or 600. Reynold number in case of open channels is defined as :

$$R_e = \frac{\rho V R}{\mu} \quad \dots(16.2)$$

where  $V$  = Mean velocity of flow of water

$R$  = Hydraulic radius or Hydraulic mean depth

$$= \frac{\text{Cross-section area of flow normal to the direction of flow}}{\text{Wetted perimeter}}$$

$\rho$  and  $\mu$  = Density and viscosity of water.

If the Reynold number is more than 2000, the flow is said to be turbulent in open channel flow. If  $R_e$  lies between 500 to 2000, the flow is considered to be in transition state.

**16.2.4 Sub-critical, Critical and Super Critical Flow.** The flow in open channel is said to be sub-critical if the Froude number ( $F_e$ ) is less than 1.0. The Froude number is defined as :

$$F_e = \frac{V}{\sqrt{gD}} \quad \dots(16.3)$$

where  $V$  = Mean velocity of flow

$D$  = Hydraulic depth of channel and is equal to the ratio of wetted area to the top width of channel

$$= \frac{A}{T}, \text{ where } T = \text{Top width of channel.}$$

Sub-critical flow is also called tranquil or streaming flow. For sub-critical flow,  $F_e < 1.0$ .

The flow is called critical if  $F_e = 1.0$ . And if  $F_e > 1.0$ , the flow is called super critical or shooting or rapid or torrential.

### ► 16.3 DISCHARGE THROUGH OPEN CHANNEL BY CHEZY'S FORMULA

Consider uniform flow of water in a channel as shown in Fig. 16.2. As the flow is uniform, it means the velocity, depth of flow and area of flow will be constant for a given length of the channel. Consider sections 1-1 and 2-2.

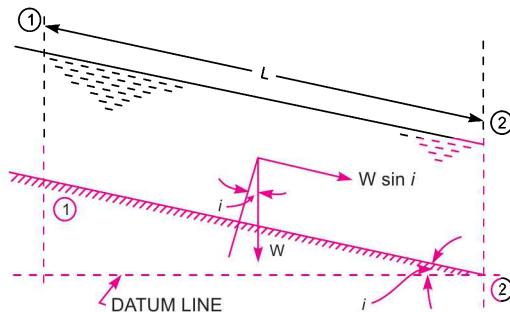


Fig. 16.2 Uniform flow in open channel.

Let

$L$  = Length of channel,

$A$  = Area of flow of water,

$i$  = Slope of the bed,

$V$  = Mean velocity of flow of water,

$P$  = Wetted perimeter of the cross-section,

$f$  = Frictional resistance per unit velocity per unit area.

The weight of water between sections 1-1 and 2-2.

$W$  = Specific weight of water  $\times$  volume of water

$$= w \times A \times L$$

Component of  $W$  along direction of flow =  $W \times \sin i = wAL \sin i$  ... (i)

Frictional resistance against motion of water =  $f \times \text{surface area} \times (\text{velocity})^n$

The value of  $n$  is found experimentally equal to 2 and surface area =  $P \times L$

$\therefore$  Frictional resistance against motion =  $f \times P \times L \times V^2$  ... (ii)

The forces acting on the water between sections 1-1 and 2-2 are:

1. Component of weight of water along the direction of flow,
2. Friction resistance against flow of water,
3. Pressure force at section 1-1,
4. Pressure force at section 2-2.

As the depths of water at the sections 1-1 and 2-2 are the same, the pressure forces on these two sections are same and acting in the opposite direction. Hence they cancel each other. In case of uniform flow, the velocity of flow is constant for the given length of the channel. Hence there is no acceleration acting on the water. Hence the resultant force acting in the direction of flow must be zero.

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∴ Resolving all forces in the direction of flow, we get

$$wAL \sin i - f \times P \times L \times V^2 = 0$$

or

$$wAL \sin i = f \times P \times L \times V^2$$

$$V^2 = \frac{wAL \sin i}{f \times P \times L} = \frac{w}{f} \times \frac{A}{P} \times \sin i$$

or

$$V = \sqrt{\frac{w}{f}} \times \sqrt{\frac{A}{P} \times \sin i} \quad \dots(iii)$$

But

$$\frac{A}{P} = m$$

= hydraulic mean depth or hydraulic radius,

$$\sqrt{\frac{w}{f}} = C = \text{Chezy's constant}$$

Substituting the values of  $\frac{A}{P}$  and  $\sqrt{\frac{w}{f}}$  in equation (iii),  $V = C \sqrt{m \sin i}$

$$\text{For small values of } i, \quad \sin i \approx \tan i \approx i \quad \therefore V = C \sqrt{mi} \quad \dots(16.4)$$

$$\therefore \text{Discharge,} \quad Q = \text{Area} \times \text{Velocity} = A \times V$$

$$= A \times C \sqrt{mi} \quad \dots(16.5)$$

**Problem 16.1** Find the velocity of flow and rate of flow of water through a rectangular channel of 6 m wide and 3 m deep, when it is running full. The channel is having bed slope as 1 in 2000. Take Chezy's constant  $C = 55$ .

**Solution.** Given :

$$\text{Width of rectangular channel, } b = 6 \text{ m}$$

$$\text{Depth of channel, } d = 3 \text{ m}$$

$$\therefore \text{Area, } A = 6 \times 3 = 18 \text{ m}^2$$

$$\text{Bed slope, } i = 1 \text{ in } 2000 = \frac{1}{2000}$$

$$\text{Chezy's constant, } C = 55$$

$$\text{Perimeter, } P = b + 2d = 6 + 2 \times 3 = 12 \text{ m}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{18}{12} = 1.5 \text{ m}$$

Velocity of flow is given by equation (16.4) as,

$$V = C \sqrt{mi} = 55 \sqrt{1.5 \times \frac{1}{2000}} = 1.506 \text{ m/s. Ans.}$$

$$\text{Rate of flow, } Q = V \times \text{Area} = V \times A = 1.506 \times 18 = 27.108 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.2** Find the slope of the bed of a rectangular channel of width 5 m when depth of water is 2 m and rate of flow is given as  $20 \text{ m}^3/\text{s}$ . Take Chezy's constant,  $C = 50$ .

**Solution.** Given :

$$\text{Width of channel, } b = 5 \text{ m}$$

$$\text{Depth of water, } d = 2 \text{ m}$$

$$\text{Rate of flow, } Q = 20 \text{ m}^3/\text{s}$$

$$\begin{array}{ll} \text{Chezy's constant} & C = 50 \\ \text{Let the bed slope} & = i \end{array}$$

Using equation (16.5), we have  $Q = AC\sqrt{mi}$   
where  $A = \text{Area} = b \times d = 5 \times 2 = 10 \text{ m}^2$

$$m = \frac{A}{P} = \frac{10}{b+2d} = \frac{10}{5+2 \times 2} = \frac{10}{5+4} = \frac{10}{9} \text{ m}$$

$$\therefore 20.0 = 10 \times 50 \times \sqrt{\frac{10}{9} \times i} \quad \text{or} \quad \sqrt{\frac{10}{9} i} = \frac{20.0}{500} = \frac{2}{50}$$

$$\text{Squaring both sides, we have } \frac{10}{9} i = \frac{4}{2500}$$

$$\therefore i = \frac{4}{2500} \times \frac{9}{10} = \frac{36}{25000} = \frac{1}{25000} = \frac{1}{694.44}. \text{ Ans.}$$

$\therefore$  Bed slope is 1 in 694.44.

**Problem 16.3** A flow of water of 100 litres per second flows down in a rectangular flume of width 600 mm and having adjustable bottom slope. If Chezy's constant C is 56, find the bottom slope necessary for uniform flow with a depth of flow of 300 mm. Also find the conveyance K of the flume.

**Solution.** Given :

$$\text{Discharge,} \quad Q = 100 \text{ litres/s} = \frac{100}{1000} = 0.10 \text{ m}^3/\text{s}$$

$$\text{Width of channel,} \quad b = 600 \text{ mm} = 0.60 \text{ m}$$

$$\text{Depth of flow,} \quad d = 300 \text{ mm} = 0.30 \text{ m}$$

$$\therefore \text{Area of flow,} \quad A = b \times d = 0.6 \times 0.3 = 0.18 \text{ m}^2$$

$$\text{Chezy's constant,} \quad C = 56$$

$$\text{Let the slope of bed} \quad = i$$

$$\text{Hydraulic mean depth,} \quad m = \frac{A}{P} = \frac{0.18}{b+2d} = \frac{0.18}{0.6+2 \times 0.30} = \frac{0.18}{1.2} = 0.15 \text{ m}$$

Using equation (16.5), we have  $Q = AC\sqrt{mi}$

$$\text{or} \quad 0.10 = 0.18 \times 56 \times \sqrt{0.15 \times i} \quad \text{or} \quad \sqrt{0.15i} = \frac{0.10}{0.18 \times 56}$$

$$\text{Squaring both sides, we have } 0.15i = \left( \frac{0.10}{0.18 \times 56} \right)^2 = .000098418$$

$$\therefore i = \frac{.000098418}{0.15} = .0006512 = \frac{1}{\frac{1}{.0006512}} = \frac{1}{1524} \cdot \text{Ans.}$$

$\therefore$  Slope of the bed is 1 in 1524.

#### Conveyance K of the channel

Equation (16.5) is given as  $Q = AC\sqrt{mi}$

which can be written as  $Q = K\sqrt{i}$

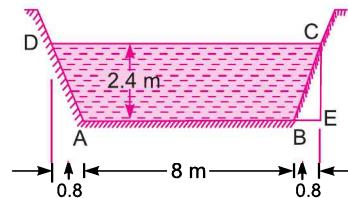
where  $K = AC\sqrt{m}$  and  $K$  is called conveyance of the channel section.

$$\therefore K = AC\sqrt{m} = 0.18 \times 56 \times \sqrt{0.15} = 3.9039 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.4** Find the discharge through a trapezoidal channel of width 8 m and side slope of 1 horizontal to 3 vertical. The depth of flow of water is 2.4 m and value of Chezy's constant,  $C = 50$ . The slope of the bed of the channel is given 1 in 4000.

**Solution.** Given :

Width,	$b = 8 \text{ m}$
Side slope	= 1 hor. to 3 vertical
Depth,	$d = 2.4 \text{ m}$
Chezy's constant,	$C = 50$
Bed slope,	$i = \frac{1}{4000}$



From Fig. 16.3 when depth ,  $CE = 2.4$ ,

Fig. 16.3

$$\text{the horizontal distance } BE = 2.4 \times \frac{1}{3} = 0.8 \text{ m}$$

$\therefore$  Top width of the channel,

$$CD = AB + 2 \times BE = 8.0 + 2 \times 0.8 = 9.6 \text{ m}$$

$\therefore$  Area of trapezoidal channel,  $ABCD$  is given as,

$$A = (AB + CD) \times \frac{CE}{2} = (8 + 9.6) \times \frac{2.4}{2} = 17.6 \times 1.2 = 21.12 \text{ m}^2$$

Wetted perimeter ,

$$P = AB + BC + AD = AB + 2BC \quad (\because BC = AD)$$

But

$$BC = \sqrt{BE^2 + CE^2} = \sqrt{(0.8)^2 + (2.4)^2} = 2.529 \text{ m}$$

$\therefore$

$$P = 8.0 + 2 \times 2.529 = 13.058 \text{ m}$$

Hydraulic mean depth,

$$m = \frac{A}{P} = \frac{21.12}{13.058} = 1.617 \text{ m}$$

The discharge,  $Q$  is given by equation (16.5) as

$$Q = AC\sqrt{mi} = 21.12 \times 50 \sqrt{1.617 \times \frac{1}{4000}} = 21.23 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.5** Find the bed slope of trapezoidal channel of bed width 6 m, depth of water 3 m and side slope of 3 horizontal to 4 vertical, when the discharge through the channel is  $30 \text{ m}^3/\text{s}$ . Take Chezy's constant,  $C = 70$ .

**Solution.** Given :

Bed width,	$b = 6.0 \text{ m}$
Depth of flow,	$d = 3.0 \text{ m}$
Side slope	= 3 horizontal to 4 vertical
Discharge,	$Q = 30 \text{ m}^3/\text{s}$
Chezy's constant,	$C = 70$

From Fig. 16.4, for depth of flow

$$= 3 \text{ m} = CE$$

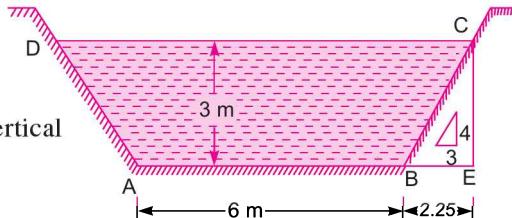


Fig. 16.4

$$\text{Distance, } BE = 3 \times \frac{3}{4} = \frac{9}{4} = 2.25 \text{ m}$$

$$\therefore \text{Top width, } CD = AB + 2 \times BE = 6.0 + 2 \times 2.25 = 10.50 \text{ m}$$

Wetted perimeter,  $P = AD + AB + BC = AP + 2BC$   $(\because BC = AD)$

$$= AB + 2\sqrt{BE^2 + CE^2} = 6.0 + 2\sqrt{(2.25)^2 + (3)^2} = 13.5 \text{ m}$$

Area of flow,  $A = \text{Area of trapezoidal } ABCD$

$$= \frac{(AB + CD) \times CE}{2} = \frac{(6 + 10.50)}{2} \times 3.0 = 24.75 \text{ m}^2$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{24.75}{13.50} = 1.833$$

Using equation (16.5),  $Q = AC\sqrt{mi}$

$$\text{or } 30.0 = 24.75 \times 70 \times \sqrt{1.833 \times i} = 2345.6\sqrt{i}$$

$$i = \left( \frac{30}{2345.6} \right)^2 = \frac{1}{\left( \frac{2345.6}{30} \right)^2} = \frac{1}{6133}. \text{ Ans.}$$

**Problem 16.6** Find the discharge of water through the channel shown in Fig. 16.5. Take the value of Chezy's constant = 60 and slope of the bed as 1 in 2000.

**Solution.** Given :

$$\text{Chezy's constant, } C = 60$$

$$\text{Bed slope, } i = \frac{1}{2000}$$

$$\text{From Fig. 16.5, Area, } A = \text{Area } ABCD + \text{Area } BEC$$

$$\begin{aligned} &= (1.2 \times 3.0) + \frac{\pi R^2}{2} \\ &= 3.6 + \frac{(1.5)^2}{2} = 7.134 \text{ m}^2 \end{aligned}$$

$$\text{Wetted perimeter, } P = AB + BEC + CD$$

$$= 1.2 + \pi R + 1.2 = 1.2 + \pi \times 1.5 + 1.2 = 7.1124 \text{ m}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{7.134}{7.1124} = 1.003$$

The discharge,  $Q$  is given by equation (16.5) as

$$\begin{aligned} Q &= AC\sqrt{mi} \\ &= 7.134 \times 60 \times \sqrt{1.003 \times \frac{1}{2000}} = 9.585 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

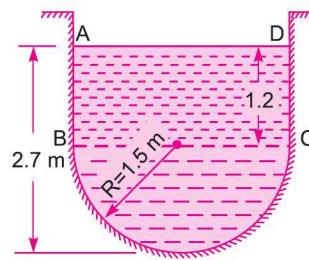


Fig. 16.5

**Problem 16.7** Find the rate of flow of water through a V-shaped channel as shown in Fig. 16.6. Take the value of  $C = 55$  and slope of the bed 1 in 2000.

**Solution.** Given :

$$\text{Chezy's constant, } C = 55$$

$$\text{Bed slope, } i = \frac{1}{1000}$$

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Depth of flow,  $d = 4.0 \text{ m}$

Angle made by each side with vertical,

i.e.,  $\angle ABD = \angle CBD = 30^\circ$

From Fig. 16.6, we have

Area,  $A = \text{Area of } ABC$

$$= 2 \times \text{Area } ABD = \frac{2 \times AD \times BD}{2} = AD \times BD$$

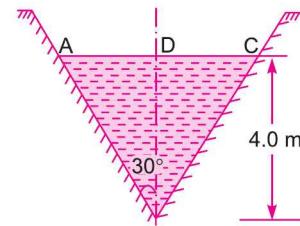


Fig. 16.6

$$= BD \tan 30^\circ \times BD \quad \left( \because \tan 30^\circ = \frac{AD}{BD}, AD = BD \tan 30^\circ \right)$$

$$= 4 \tan 30^\circ \times 4 = 9.2376 \text{ m}^2$$

Wetted perimeter,  $P = AB + BC = 2AB$

( $\because AB = BC$ )

$$= 2\sqrt{BD^2 + AD^2} = 2\sqrt{4.0^2 + (4 \tan 30^\circ)^2}$$

$$= 2\sqrt{16.0 + 5.333} = 9.2375 \text{ m.}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{9.2376}{9.2375} = 1.0 \text{ m}$$

Using equation (16.5) for discharge,

$$Q = AC\sqrt{mi} = 9.2376 \times 55 \times \sqrt{1 \times \frac{1}{1000}} = 16.066 \text{ m}^3/\text{s. Ans.}$$

### ► 16.4 EMPIRICAL FORMULAE FOR THE VALUE OF CHEZY'S CONSTANT

Equation (16.4) is known Chezy's formula after the name of a French Engineer, Antoine Chezy who developed this formula in 1975. In this equation  $C$  is known as Chezy's constant, which is not a dimensionless co-efficient. The dimension of  $C$  is

$$\begin{aligned} &= \frac{V}{\sqrt{mi}} = \frac{L/T}{\sqrt{\frac{A}{P} i}} = \frac{L/T}{\sqrt{\frac{L^2}{L} i}} = \frac{L}{T\sqrt{Li}} = \frac{\sqrt{L}}{T} \\ &= L^{1/2} T^{-1} \quad \{ i \text{ is dimensionless} \} \end{aligned}$$

Hence the value of  $C$  depends upon the system of units. The following are the empirical formulae, after the name of their inventors, used to determine the value of  $C$ :

$$1. \text{ Bazin formula (In MKS units)} : C = \frac{157.6}{1.81 + \frac{K}{\sqrt{m}}} \quad \dots(16.6)$$

where  $K$  = Bazin's constant and depends upon the roughness of the surface of channel, whose values are given in Table 16.1.

$m$  = Hydraulic mean depth or hydraulic radius.

**2. Ganguillet-Kutter Formula.** The value of  $C$  is given in MKS unit as

$$C = \frac{23 + \frac{0.00155}{i} + \frac{1}{N}}{1 + \left(23 + \frac{0.00155}{i}\right) \frac{N}{\sqrt{m}}} \quad \dots(16.7)$$

where  $N$  = Roughness co-efficient which is known as Kutter's constant, whose value for different surfaces are given in Table 16.2

$i$  = Slope of the bed

$m$  = Hydraulic mean depth.

**Table 16.1 Values of  $K$  in the Bazin's Formula**

S. No.	Nature of Channel inside surface	Value of $K$
1.	Smooth cemented or planned wood	0.11
2.	Brick or concrete or unplanned wood	0.21
3.	Rubble masonry or Ashlar or poor brick work	0.83
4.	Earthen channel of very good surface	1.54
5.	Earthen channel of ordinary surface	2.36
6.	Earthen channel of rough surface	3.17

**Table 16.2 Value of  $N$  in the Ganguillet-Kutter Formula**

S. No.	Nature of Channel inside surface	Value of $N$
1.	Very smooth surface of glass, plastic or brass	0.010
2.	Smooth surface of concrete	0.012
3.	Rubble masonry or poor brick work	0.017
4.	Earthen channels neatly excavated	0.018
5.	Earthen channels of ordinary surface	0.027
6.	Earthen channels of rough surface	0.030
7.	Natural streams, clean and straight	0.030
8.	Natural streams with weeds, dappools etc.	0.075 to .15

**3. Manning's Formula.** The value of  $C$  according to this formula is given as

$$C = \frac{1}{N} m^{1/6} \quad \dots(16.8)$$

where  $m$  = Hydraulic mean depth

$N$  = Manning's constant which is having same value as Kutter's constant for the normal range of slope and hydraulic mean depth. The values of  $N$  are given in Table 16.2.

**Problem 16.8** Find the discharge through a rectangular channel 2.5 m wide, having depth of water 1.5 m and bed slope as 1 in 2000. Take the value of  $k = 2.36$  in Bazin's formula.

**Solution.** Given :

Width of channel,  $b = 2.5$  m

Depth of flow,  $d = 1.5$  m

$$\therefore \text{Area}, \quad A = b \times d = 2.5 \times 1.5 = 3.75 \text{ m}^2$$

$$\text{Wetted Perimeter}, \quad P = d + b + d = 1.5 + 2.5 + 1.5 = 5.5 \text{ m}$$

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$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{3.75}{5.50} = 0.682$$

$$\text{Bed slope, } i = \frac{1}{2000}$$

$$\text{Bazin's constant, } K = 2.36$$

Using Bazin's formula given by equation (16.6), as

$$C = \frac{157.6}{1.81 + \frac{K}{\sqrt{m}}} = \frac{157.6}{1.81 + \frac{2.36}{\sqrt{0.682}}} = 33.76$$

Discharge,  $Q$  is given by equation (16.5), as

$$Q = AC\sqrt{mi}$$

$$= 3.75 \times 33.76 \times \sqrt{0.682 \times \frac{1}{2000}} = 2.337 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.9** Find the discharge through a rectangular channel 14 m wide, having depth of water 3 m and bed slope 1 in 1500. Take the value of  $N = 0.03$  in the Kutter's formula.

**Solution.** Given :

$$\text{Width of channel, } b = 4 \text{ m}$$

$$\text{Depth of water, } d = 3 \text{ m}$$

$$\text{Bed slope, } i = \frac{1}{1500} = 0.000667$$

$$\text{Kutter's constant, } N = 0.03$$

$$\text{Area of flow, } A = b \times d = 4 \times 3 = 12 \text{ m}^2$$

$$\text{Wetted perimeter, } P = d + b + d = 3 + 4 + 3 = 10 \text{ m}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{12}{10} = 1.2 \text{ m}$$

Using Kutter's formula given by equation (16.7), as

$$C = \frac{23 + \frac{.00155}{i} + \frac{1}{N}}{1 + \left( 23 + \frac{.00155}{i} \right) \times \frac{N}{\sqrt{m}}} = \frac{23 + \frac{.00155}{.000667} + \frac{1}{.03}}{1 + \left( 23 + \frac{.00155}{.000667} \right) \times \frac{.03}{\sqrt{1.20}}}$$

$$= \frac{23 + 2.3238 + 33.33}{1 + (23 + 2.3238) \times .03286} = \frac{58.633}{1.832} = 32.01$$

Discharge,  $Q$  is given by equation (16.5), as

$$Q = AC\sqrt{mi} = 12 \times 32.01 \times \sqrt{12 \times .000667} = 10.867 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.10** Find the discharge through a rectangular channel of width 2 m, having a bed slope of 4 in 8000. The depth of flow is 1.5 m and take the value of  $N$  in Manning's formula as 0.012.

**Solution.** Given :

$$\text{Width of the channel, } b = 2 \text{ m}$$

$$\text{Depth of the flow, } d = 1.5 \text{ m}$$

$$\therefore \text{Area of flow, } A = b \times d = 2 \times 1.5 = 3.0 \text{ m}^2$$

Wetted perimeter,  $P = b + d + d = 2 + 1.5 + 1.5 = 5.0 \text{ m}$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{3.0}{5.0} = 0.6$$

Bed slope,  $i = 4 \text{ in } 8000 = \frac{4}{8000} = \frac{1}{2000}$

Value of  $N$   $= 0.012$

Using Manning's formula, given by equation (16.8), as

$$C = \frac{1}{N} m^{1/6} = \frac{1}{0.012} \times 0.6^{1/6} = 76.54$$

Discharge,  $Q$  is given by equation (16.5), as

$$Q = AC\sqrt{mi}$$

$$= 3.0 \times 76.54 \sqrt{0.6 \times \frac{1}{2000}} \text{ m}^2/\text{s} = 3.977 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.11** Find the bed slope of trapezoidal channel of bed width 4 m, depth of water 3 m and side slope of 2 horizontal to 3 vertical, when the discharge through the channel is  $20 \text{ m}^3/\text{s}$ .

Take Manning's  $N = 0.03$  in Manning's formula  $C = \frac{1}{N} m^{1/6}$ .

**Solution.** Given :

Bed width,  $b = 4 \text{ cm}$

Depth of flow,  $d = 3 \text{ m}$

Side slope  $= 2 \text{ hor. to } 3 \text{ vert.}$

Discharge,  $Q = 20.0 \text{ m}^3/\text{s}$

Manning's,  $N = 0.03$

From Fig. 16.7, we have

Distance,  $BE = d \times \frac{2}{3} = 3 \times \frac{2}{3} = 2 \text{ m}$

$\therefore$  Top width,  $CD = AB + 2BE$   
 $= 4 + 2 \times 2 = 8.0 \text{ m}$

$\therefore$  Area of flow,  $A = \text{Area of trapezoidal section } ABCD$

$$= \frac{(AB + CD)}{2} \times d = \frac{(4 + 8)}{2} \times 3 = 18 \text{ m}^2$$

Wetted perimeter,  $P = AD + AB + BC = AB + 2BC \quad (\because AD = BC)$

$$= 4.0 + 2\sqrt{BE^2 + EC^2} = 4.0 + 2\sqrt{2^2 + 3^2} = 4.0 + 2 \times \sqrt{13} = 11.21 \text{ m}$$

$\therefore$  Hydraulic mean depth,  $m = \frac{A}{P} = \frac{18}{11.21} = 1.6057$

Using Manning's formula,  $C = \frac{1}{N} m^{1/6} = \frac{1}{0.03} \times (1.6057)^{1/6} = 36.07$

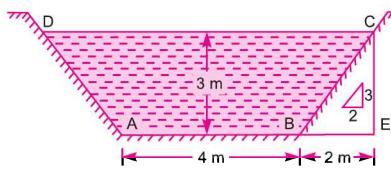


Fig. 16.7

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Using equation (16.5) for discharge,

$$Q = AC\sqrt{mi} = 18 \times 36.07 \times \sqrt{1.6057 \times i} \text{ or } 20.0 = 822.71\sqrt{i}$$

$$\therefore i = \left( \frac{20.0}{822.71} \right)^2 = 0.0005909 = \frac{1}{1692}. \text{ Ans.}$$

**Problem 16.12** Find the diameter of a circular sewer pipe which is laid at a slope of 1 in 8000 and carries a discharge of 800 litres/s when flowing half full. Take the value of Manning's  $N = 0.020$ .

**Solution.** Given :

$$\text{Slope of pipe, } i = \frac{1}{8000}$$

$$\text{Discharge, } Q = 800 \text{ litres/s} = 0.8 \text{ m}^3/\text{s}$$

$$\text{Manning's, } N = 0.020$$

$$\text{Let the dia. of sewer pipe, } D$$

$$\text{Depth of flow, } d = \frac{D}{2}$$

$$\therefore \text{Area of flow, } A = \frac{\pi}{4} D^2 \times \frac{1}{2} = \frac{\pi D^2}{8}$$

$$\text{Wetted perimeter, } P = \frac{\pi D}{2}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{\frac{\pi D^2}{8}}{\frac{\pi D}{2}} = \frac{D}{4}$$

Using Manning's formula given by equation (16.8),  $C = \frac{1}{N} m^{1/6}$

The discharge,  $Q$  through pipe is given by equation (16.6), as

$$Q = AC\sqrt{mi}$$

$$= \frac{\pi D^2}{8} \times \frac{1}{N} m^{1/6} \sqrt{mi}$$

$$\text{or } 0.80 = \frac{\pi}{8} D^2 \times \frac{1}{0.020} \times m^{1/6} \times m^{1/2} \times \sqrt{i}$$

$$= \frac{\pi}{8} D^2 \times \frac{1}{0.020} m^{(1/6 + 1/2)} \times \sqrt{\frac{1}{8000}} = \frac{\pi}{8} D^2 \times \frac{1}{0.020} \times m^{2/3} \times 0.01118$$

$$= 0.2195 \times D^2 \times \left( \frac{D}{4} \right)^{2/3} \quad \left( \because m = \frac{D}{4} \right)$$

$$= \frac{0.2195}{4^{2/3}} \times D^2 \times D^{2/3} = 0.0871 D^{8/3}$$

$$\text{or } D^{8/3} = \frac{0.80}{0.0871} = 9.1848$$

$$\therefore D = (9.1848)^{3/8} = (9.1848)^{0.375} = \mathbf{2.296 \text{ m. Ans.}}$$

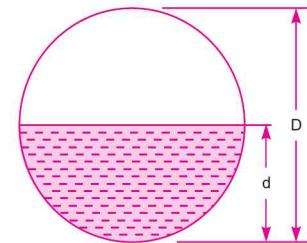


Fig. 16.8

## ► 16.5 MOST ECONOMICAL SECTION OF CHANNELS

A section of a channel is said to be most economical when the cost of construction of the channel is minimum. But the cost of construction of a channel depends upon the excavation and the lining. To keep the cost down or minimum, the wetted perimeter, for a given discharge, should be minimum. This condition is utilized for determining the dimensions of a economical sections of different form of channels.

Most economical section is also called the best section or most efficient section as the discharge, passing through a most economical section of channel for a given cross-sectional area ( $A$ ), slope of the bed ( $i$ ) and a resistance co-efficient, is maximum. But the discharge,  $Q$  is given by equation (16.5) as

$$Q = AC\sqrt{mi} = AC\sqrt{\frac{A \times i}{P}} \quad \left( \because m = \frac{A}{P} \right)$$

For a given  $A$ ,  $i$  and resistance co-efficient  $C$ , the above equation is written as

$$Q = K \frac{1}{\sqrt{P}}, \quad \text{where } K = AC\sqrt{Ai} = \text{constant}$$

Hence the discharge,  $Q$  will be maximum, when the wetted perimeter  $P$  is minimum. This condition will be used for determining the best section of a channel *i.e.*, best dimensions of a channel for a given area.

The conditions to be most economical for the following shapes of the channels will be considered :

1. Rectangular Channel,
2. Trapezoidal Channel, and
3. Circular Channel.

**16.5.1 Most Economical Rectangular Channel.** The condition for most economical section, is that for a given area, the perimeter should be minimum. Consider a rectangular channel as shown in Fig. 16.9

Let

$b$  = width of channel,

$d$  = depth of the flow,

$\therefore$  Area of flow,

$$A = b \times d \quad \dots(i)$$

Wetted perimeter,

$$P = d + b + d = b + 2d \quad \dots(ii)$$

From equation (i),

$$b = \frac{A}{d}$$

Substituting the value of  $b$  in (ii),

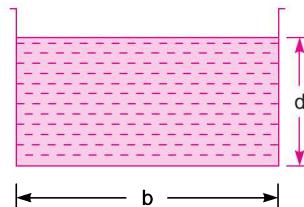


Fig. 16.9 Rectangular channel.

$$P = b + 2d = \frac{A}{d} + 2d \quad \dots(iii)$$

For most economical section,  $P$  should be minimum for a given area.

or

$$\frac{dP}{d(d)} = 0$$

Differentiating the equation (iii) with respect to  $d$  and equating the same to zero, we get

$$\frac{d}{d(d)} \left[ \frac{A}{d} + 2d \right] = 0 \quad \text{or} \quad -\frac{A}{d^2} + 2 = 0 \quad \text{or} \quad A = 2d^2$$

But

$$A = b \times d, \therefore b \times d = 2d^2 \text{ or } b = 2d$$

...(16.9)

$$\text{Now hydraulic mean depth, } m = \frac{A}{P} = \frac{b \times d}{b + 2d}$$

( $\because A = bd, P = b + 2d$ )

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$$\begin{aligned}
 &= \frac{2d \times d}{2d + 2d} && (\because b = 2d) \\
 &= \frac{2d^2}{4d} = \frac{d}{2} && \dots(16.10)
 \end{aligned}$$

From equations (16.9) and (16.10), it is clear that rectangular channel will be most economical when:

(i) Either  $b = 2d$  means width is two times depth of flow.

(ii) Or  $m = \frac{d}{2}$  means hydraulic depth is half the depth of flow.

**Problem 16.13** A rectangular channel of width, 4 m is having a bed slope of 1 in 1500. Find the maximum discharge through the channel. Take value of  $C = 50$ .

**Solution.** Given :

Width of channel,  $b = 4 \text{ m}$

Bed slope,  $i = \frac{1}{1500}$

Chezy's constant,  $C = 50$

Discharge will be maximum, when the channel is most economical. The conditions for most economical rectangular channel are :

$$(i) \quad b = 2d \quad \text{or} \quad d = \frac{b}{2} = \frac{4}{2} = 2.0 \text{ m}$$

$$(ii) \quad m = \frac{d}{2} = \frac{2}{2} = 1.0 \text{ m}$$

$\therefore$  Area of most economical rectangular channel,  $A = b \times d = 4.0 \times 2.0 = 8 \text{ m}^2$

Using equation (16.5) for discharge as

$$Q = AC\sqrt{mi} = 8.0 \times 50 \times \sqrt{1.0 \times \frac{1}{1500}} = 10.328 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.14** A rectangular channel carries water at the rate of 400 litres/s when bed slope is 1 in 2000. Find the most economical dimensions of the channel if  $C = 50$ .

**Solution.** Given :

Discharge,  $Q = 400 \text{ litres/s} = 0.4 \text{ m}^3/\text{s}$

Bed slope,  $i = \frac{1}{2000}$

Chezy's constant,  $C = 50$

For the rectangular channel to be most economical,

(i) Width,  $b = 2d$

(ii) Hydraulic mean depth,  $m = \frac{d}{2}$

$\therefore$  Area of flow,  $A = b \times d = 2d \times d = 2d^2$

Using equation (16.5) for discharge,

$$Q = AC\sqrt{mi}$$

or  $0.4 = 2d^2 \times 50 \times \sqrt{\frac{d}{2} \times \frac{1}{2000}} = 2 \times 50 \times \sqrt{\frac{1}{2 \times 2000}} d^{5/2}$   $d^{5/2} = 1.581$

$$\therefore d^{5/2} = \frac{0.4}{1.581} = 0.253$$

$$\therefore d = (0.253)^{2/5} = 0.577 \text{ m. Ans.}$$

$$b = 2d = 2 \times 0.577 = 1.154 \text{ m. Ans.}$$

**Problem 16.15** A rectangular channel 4 m wide has depth of water 1.5 m. The slope of the bed of the channel is 1 in 1000 and value of Chezy's constant  $C = 55$ . It is desired to increase the discharge to a maximum by changing the dimensions of the section for constant area of cross-section, slope of the bed and roughness of the channel. Find the new dimensions of the channel and increase in discharge.

**Solution.** Given :

Width of channel,

$$b = 4.0 \text{ m}$$

Depth of flow,

$$d = 1.5 \text{ m}$$

$\therefore$  Area of flow,

$$A = b \times d = 4 \times 1.5 = 6.0 \text{ m}^2$$

Slope of bed,

$$i = \frac{1}{1000}$$

Chezy's constant,

$$C = 55$$

Wetted perimeter,

$$P = d + b + d = 1.5 + 4 + 1.5 = 7.0 \text{ m}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{4.0}{7.0} = 0.857$$

The discharge,  $Q$  is given by  $Q = AC\sqrt{mi} = 6.0 \times 55 \sqrt{0.857 \times \frac{1}{1000}} = 9.66 \text{ m}^3/\text{s}$  ... (i)

For maximum discharge for a given area, slope of bed and roughness we proceed as :

Let  $b'$  = new width of channel

$d'$  = new depth of flow

Then, Area,

$A = b' \times d'$ , where  $A = \text{constant} = 6.0 \text{ m}^2$

$$\therefore b' \times d' = 6.0 \quad \dots(ii)$$

$$\text{Also for maximum discharge } b' = 2d' \quad \dots(iii)$$

Substituting the value of  $b'$  in equation (ii), we have

$$2d' \times d' = 6.0 \text{ or } d'^2 = \frac{6.0}{2} = 3.0$$

$$\therefore d' = \sqrt{3} = 1.732$$

Substituting the value of  $d'$  in (iii), we get

$$b' = 2 \times 1.732 = 3.464$$

$\therefore$  New dimensions of the channel are

Width,  $b' = 3.464 \text{ m. Ans.}$

Depth,  $d' = 1.732 \text{ m. Ans.}$

Wetted perimeter,  $P' = d' + b' + d' = 1.732 + 3.464 + 1.732 = 6.928$

$$\therefore \text{Hydraulic mean depth, } m' = \frac{A}{P'} = \frac{6.0}{6.928} = 0.866 \text{ m}$$

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(New hydraulic mean depth,  $m'$  corresponds to the condition of maximum discharge. And hence also equal to

$$\frac{d'}{2} = \frac{1.732}{2} = 0.866 \text{ m}$$

Max. discharge,  $Q'$ , is given by  $Q' = AC\sqrt{m'i} = 6.0 \times 55 \times \sqrt{0.866 \times \frac{1}{1000}} = 9.71 \text{ m}^3/\text{s}$  ... (iv)

$\therefore$  Increase in discharge  $= Q' - Q = 9.71 - 9.66 = 0.05 \text{ m}^3/\text{s. Ans.}$

**16.5.2 Most Economical Trapezoidal Channel.** The trapezoidal section of a channel will be most economical, when its wetted perimeter is minimum. Consider a trapezoidal section of channel as shown in Fig. 16.10.

Let

$b$  = width of channel at bottom,

$d$  = depth of flow,

$\theta$  = angle made by the sides with horizontal,

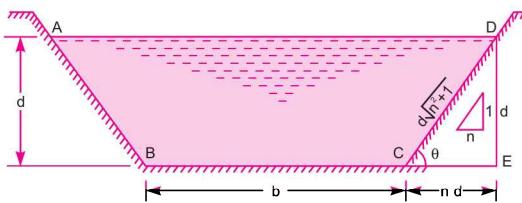


Fig. 16.10 Trapezoidal section.

(i) The side slope is given as 1 vertical to  $n$  horizontal.

$$\therefore \text{Area of flow, } A = \frac{(BC + AD)}{2} \times d = \frac{b + (b + 2nd)}{2} \times d \quad (\because AD = b + 2nd)$$

$$= \frac{2b + 2nd}{2} \times d = (b + nd) \times d \quad \dots(i)$$

$$\therefore \frac{A}{d} = b + nd$$

$$\therefore b = \frac{A}{d} - nd \quad \dots(ii)$$

$$\begin{aligned} \text{Now wetted perimeter, } P &= AB + BC + CD = BC + 2CD \quad (\because AB = CD) \\ &= b + 2\sqrt{CE^2 + DE^2} = b + 2\sqrt{n^2d^2 + d^2} = b + 2d\sqrt{n^2 + 1} \quad \dots(iia) \end{aligned}$$

Substituting the value of  $b$  from equation (ii), we get

$$P = \frac{A}{d} - nd + 2d\sqrt{n^2 + 1} \quad \dots(iii)$$

For most economical section,  $P$  should be minimum or  $\frac{dP}{d(d)} = 0$

$\therefore$  Differentiating equation (iii) with respect to  $d$  and equating it equal to zero, we get

$$\frac{d}{d(d)} \left[ \frac{A}{d} - nd + 2d\sqrt{n^2 + 1} \right] = 0$$

$$\text{or } -\frac{A}{d^2} - n + 2\sqrt{n^2 + 1} = 0 \quad (\because n \text{ is constant})$$

or

$$\frac{A}{d^2} + n = 2\sqrt{n^2 + 1}$$

Substituting the value of  $A$  from equation (i) in the above equation,

$$\frac{(b+nd)d}{d^2} + n = 2\sqrt{n^2 + 1} \quad \text{or} \quad \frac{b+nd}{d} + n = 2\sqrt{n^2 + 1}$$

or

$$\frac{b+nd+nd}{d} = \frac{b+2nd}{d} = 2\sqrt{n^2 + 1} \quad \text{or} \quad \frac{b+2nd}{2} = d\sqrt{n^2 + 1} \quad \dots(16.11)$$

But from Fig. 16.10,  $\frac{b+2nd}{2}$  = Half of top width

and  $d\sqrt{n^2 + 1} = CD$  = one of the sloping side

Equation (16.11) is the required condition for a trapezoidal section to be most economical, which can be expressed as half of the top width must be equal to one of the sloping sides of the channel.

### (ii) Hydraulic mean depth

$$\text{Hydraulic mean depth, } m = \frac{A}{P}$$

$$\text{Value of } A \text{ from (i), } A = (b+nd) \times d$$

$$\text{Value of } P \text{ from (ii), } P = b + 2d\sqrt{n^2 + 1} = b + (b+2nd) \quad (\because \text{ From equation (16.11) })$$

$$= 2b + 2nd = 2(b+nd)$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{(b+nd)d}{2(b+nd)} = \frac{d}{2} \quad \dots(16.12)$$

Hence for a trapezoidal section to be most economical hydraulic mean depth must be equal to half the depth of flow,

(iii) The three sides of the trapezoidal section of most economical section are tangential to the semi-circle described on the water line. This is proved as :

Let Fig. 16.11 shows the trapezoidal channel of most economical section.

Let  $\theta$  = angle made by the sloping side with horizontal, and

$O$  = the centre of the top width,  $AD$ .

Draw  $OF$  perpendicular to the sloping side  $AB$ .

$\triangle OAF$  is a right-angled triangle and angle  $OAF = \theta$

$$\therefore \sin \theta = \frac{OF}{OA} \quad \therefore OF = AO \sin \theta \quad \dots(iv)$$

$$\text{In } \triangle AEB, \quad \sin \theta = \frac{AE}{AB} = \frac{d}{\sqrt{d^2 + n^2 d^2}}$$

$$= \frac{d}{d\sqrt{1+n^2}} = \frac{1}{\sqrt{1+n^2}}$$

Substituting  $\sin \theta = \frac{1}{\sqrt{1+n^2}}$  in equation (iv), we get

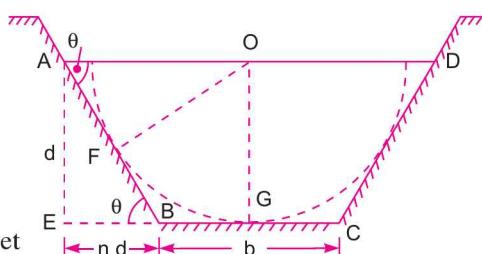


Fig. 16.11

$$OF = AO \times \frac{1}{\sqrt{1+n^2}} \quad \dots(v)$$

But

$AO$  = half of top width

$$= \frac{b + 2nd}{2} = d\sqrt{n^2 + 1} \text{ from equation (16.11)}$$

Substituting this value of  $AO$  in equation (v),

$$OF = \frac{d\sqrt{n^2 + 1}}{\sqrt{n^2 + 1}} = d \text{ depth of flow} \quad \dots(16.13)$$

Thus, if a semi-circle is drawn with  $O$  as centre and radius equal to the depth of flow  $d$ , the three sides of most economical trapezoidal section will be tangential to the semi-circle.

Hence the conditions for the most economical trapezoidal section are:

$$1. \quad \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

$$2. \quad m = \frac{d}{2}$$

3. A semi-circle drawn from  $O$  with radius equal to depth of flow will touch the three sides of the channel.

**Problem 16.16** A trapezoidal channel has side slopes of 1 horizontal to 2 vertical and the slope of the bed is 1 in 1500. The area of the section is  $40 \text{ m}^2$ . Find the dimensions of the section if it is most economical. Determine the discharge of the most economical section if  $C = 50$ .

**Solution.** Given :

$$\text{Side slope, } n = \frac{\text{Horizontal}}{\text{Vertical}} = \frac{1}{2}$$

$$\text{Bed slope, } i = \frac{1}{1500}$$

$$\text{Area of section, } A = 40 \text{ m}^2$$

$$\text{Chezy's constant, } C = 50$$

For the most economical section, using equation (16.11)

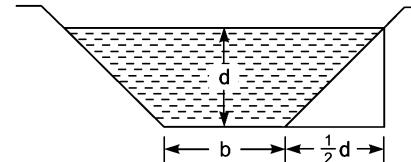


Fig. 16.12

$$\frac{b + 2nd}{2} = d\sqrt{n^2 + 1} \quad \text{or} \quad \frac{b + 2 \times \frac{1}{2} \times d}{2} = d\sqrt{\left(\frac{1}{2}\right)^2 + 1}$$

$$\text{or} \quad \frac{b + d}{2} = d\sqrt{\frac{1}{4} + 1} = 1.118 d$$

$$\text{or} \quad b = 2 \times 1.118d - d = 1.236 d \quad \dots(i)$$

$$\begin{aligned} \text{But area of trapezoidal section, } A &= \frac{b + (b + 2nd)}{2} \times d = (b + nd) d \\ &= (1.236 d + \frac{1}{2} d) d \quad (\because b = 1.236 d \text{ and } n = \frac{1}{2}) \\ &= 1.736 d^2 \end{aligned}$$

$$\begin{aligned} \text{But} \quad A &= 40 \text{ m}^2 \quad (\text{given}) \\ \therefore \quad 40 &= 1.736 d^2 \end{aligned}$$

$$\therefore d = \sqrt{\frac{40}{1.736}} = 4.80 \text{ m. Ans.}$$

Substituting the value of  $d$  in equation (i), we get

$$b = 1.236 \times 4.80 = 5.933 \text{ m. Ans.}$$

**Discharge for most economical section.** Hydraulic mean depth for most economical section is

$$m = \frac{d}{2} = \frac{4.80}{2} = 2.40 \text{ m}$$

$$\therefore \text{Discharge } Q = AC\sqrt{mi} = 40 \times 50 \times \sqrt{2.40 \times \frac{1}{1500}} \\ = 80 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.17** A trapezoidal channel has side slopes of 3 horizontal to 4 vertical and slope of its bed is 1 in 2000. Determine the optimum dimensions of the channel, if it is to carry water at  $0.5 \text{ m}^3/\text{s}$ . Take Chezy's constant as 80.

**Solution.** Given :

$$\text{Side slopes } n = \frac{\text{Horizontal}}{\text{Vertical}} = \frac{3}{4}$$

$$\text{Slope of bed, } i = \frac{1}{2000}$$

$$\text{Discharge, } Q = 0.5 \text{ m}^3/\text{s}$$

$$\text{Chezy's constant, } C = 80$$

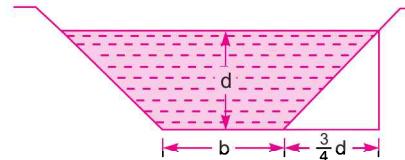


Fig. 16.13

For the most economical section, the condition is given by equation (16.11) as

$$\frac{b + 2nd}{2} = d\sqrt{n^2 + 1}, \text{ where } b = \text{width of section, } d = \text{depth of flow}$$

$$\text{or } \frac{b + 2 \times \frac{3}{4}d}{2} = d\sqrt{\left(\frac{3}{4}\right)^2 + 1} = d\sqrt{\frac{25}{16}} = \frac{5}{4}d \quad \text{or} \quad \frac{b + 1.5d}{2} = 1.25d$$

$$\text{or } b = 2 \times 1.25d - 1.5d = d \quad \dots(i)$$

For the discharge,  $Q$ , using equation (16.5) as

$$Q = AC\sqrt{mi} \quad \dots(ii)$$

$$\text{But for most economical section, hydraulic mean depth } m = \frac{d}{2}$$

Substituting the value of  $m$  and other known values in equation (ii)

$$0.50 = A \times 80 \times \sqrt{\frac{d}{2} \times \frac{1}{2000}} \quad \dots(iii)$$

But area of trapezoidal section is given as

$$\begin{aligned} A &= (b + nd) \times d = (d + \frac{3}{4}d) \times d \quad (\because \text{From (i) } b = d \text{ and } n = \frac{3}{4}) \\ &= \frac{7}{4}d^2 = 1.75d^2 \end{aligned}$$

Substituting the value of  $A$  in equation (iii), we get

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$$0.50 = 1.75 d^2 \times 80 \times \sqrt{\frac{d}{2} \times \frac{1}{2000}} = 2.2135 d^{5/2}$$

$$\therefore d = \left( \frac{0.50}{2.2135} \right)^{2/5} = 0.55 \text{ m. Ans.}$$

From equation (i),  $b = d = 0.55 \text{ m. Ans.}$

$\therefore$  Optimum dimensions of the channel are width = depth = 0.55 m.

**Problem 16.18** A trapezoidal channel with side slopes of 1 to 1 has to be designed to convey 10 m<sup>3</sup>/s at a velocity of 2 m/s so that the amount of concrete lining for the bed and sides is the minimum. Calculate the area of lining required for one metre length of canal.

**Solution.** Given :

$$\text{Side slope, } n = \frac{\text{Horizontal}}{\text{Vertical}} = 1$$

$$\text{Discharge } Q = 10 \text{ m}^3/\text{s}$$

$$\text{Velocity, } V = 2.0 \text{ m/s}$$

$$\therefore \text{Area of flow, } A = \frac{\text{Discharge}}{\text{Velocity}} = \frac{10.0}{2.0} = 5 \text{ m}^2 \quad \dots(i)$$

Let  $b$  = Width of the channel

$d$  = Depth of flow

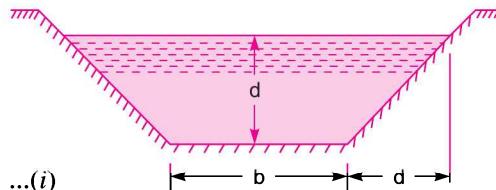


Fig. 16.14

For the amount of concrete lining for the bed and sides to be minimum the section should be most economical. But for the most economical trapezoidal section, the condition is from equation (16.11) as

Half of the top width = one of the sloping side

$$\text{i.e., } \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

For  $n = 1$ , the condition becomes

$$\frac{b + 2 \times 1d}{2} = d\sqrt{1^2 + 1} = 1.414 d$$

or

$$b = 2 \times 1.414d - 2d = 0.828 d \quad \dots(ii)$$

But area,

$$A = (b + nd)d = (0.828d + 1 \times d)d \quad (\because b = 0.828 d, n = 1)$$

$$= 1.828 d^2$$

Also from equation (i),  $A = 5 \text{ m}^2$

Equating the two values of  $A$ , we get

$$5 = 1.828 d^2 \quad \text{or} \quad d = \sqrt{\frac{5}{1.828}} = 1.6538 \approx 1.654 \text{ m}$$

From equation (ii),  $b = 0.828 d = 0.828 \times 1.654 = 1.369 \text{ m}$

Area of lining required for one metre length of canal

$$= \text{Wetted perimeter} \times \text{length of canal}$$

$$= P \times 1$$

$$\text{where } P = b + 2d\sqrt{n^2 + 1} = 1.369 + 2 \times 1.654\sqrt{1^2 + 1} = 6.047 \text{ m}$$

$$\therefore \text{Area of lining} = 6.047 \times 1 = 6.047 \text{ m}^2. \text{ Ans.}$$

**Problem 16.19** A trapezoidal channel has side slopes 1 to 1. It is required to discharge  $13.75 \text{ m}^3/\text{s}$  of water with a bed gradient of 1 in 1000. If unlined the value of Chezy's  $C$  is 44. If lined with concrete, its value is 60. The cost per  $\text{m}^3$  of excavation is four times the cost per  $\text{m}^2$  of lining. The channel is to be the most efficient one. Find whether the lined canal or the unlined canal will be cheaper. What will be the dimensions of that economical canal?

**Solution.** Given :

$$\text{Side slope, } n = \frac{1}{1} = 1$$

$$\text{Discharge, } Q = 13.75 \text{ m}^3/\text{s}$$

$$\text{Slope of bed, } i = \frac{1}{1000}$$

$$\text{For unlined, } C = 44$$

$$\text{For lined } C = 60$$

$$\text{Cost per } \text{m}^3 \text{ of excavation} = 4 \times \text{cost per } \text{m}^2 \text{ of lining}$$

$$\text{Let the cost per } \text{m}^2 \text{ of lining} = x$$

$$\text{Then cost per } \text{m}^3 \text{ of excavation} = 4x$$

As the channel is most efficient,

$$\therefore \text{Hydraulic mean depth, } m = \frac{d}{2}, \text{ where } d = \text{depth of channel}$$

$$\text{Let } b = \text{width of channel}$$

Also for the most efficient trapezoidal channel, from equation (16.11), we have

Half of top width = length of sloping side

$$\text{or } \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

$$\text{or } \frac{b + 2 \times 1 \times d}{2} = d\sqrt{1^2 + 1} = \sqrt{2d}$$

$$\therefore b = 2 \times \sqrt{2d} - 2d = 0.828d \quad \dots(i)$$

$$\begin{aligned} \text{Area, } A &= (b + nd) \times d = (0.828d + 1 \times d) \times d \\ &= 1.828d^2 \end{aligned} \quad \dots(ii)$$

### 1. For unlined channel

$$\text{Value of } C = 44$$

The discharge,  $Q$  is given by,  $Q = A \times V = A \times C\sqrt{mi}$

$$\begin{aligned} \text{or } 13.75 &= 1.828d^2 \times 44 \times \sqrt{\frac{d}{2} \times \frac{1}{1000}} \quad \left( \because A = 1.828d^2, m = \frac{d}{2} \right) \\ &= \frac{1.828 \times 44}{\sqrt{2000}} \times d^{5/2} \end{aligned}$$

$$d^{5/2} = \frac{13.75 \times \sqrt{2000}}{1.828 \times 44} = 7.6452$$

$$\therefore d = (7.6452)^{2/5} = 2.256 \text{ m}$$

Substituting this value in equation (i), we get

$$b = 0.828 \times 2.256 = 1.868 \text{ m.}$$

Now cost of excavation per running metre length of unlined channel

$$\begin{aligned} &= \text{Volume of channel} \times \text{cost per m}^3 \text{ of excavation} \\ &= (\text{Area of channel} \times 1) \times 4x = [(b + nd) \times d \times 1] \times 4x \\ &= (1.868 + 1 \times 2.256) \times 2.256 \times 1 \times 4x = 37.215x \quad \dots(iii) \end{aligned}$$

## 2. For lined channels

Value of  $C = 60$

The discharge is given by the equation,  $Q = A \times C \times \sqrt{mi}$

Substituting the value of  $A$  from equation (ii) and value of  $m = \frac{d}{2}$ , we get

$$\begin{aligned} 13.75 &= 1.828 d^2 \times 60 \times \sqrt{\frac{d}{2} \times \frac{1}{1000}} \quad (\because Q = 13.75) \\ &= 1.828 \times 60 \times \frac{1}{\sqrt{2000}} \times d^{5/2} \\ \therefore d^{5/2} &= \frac{13.75 \times \sqrt{2000}}{1.828 \times 60} = 5.606 \\ \therefore d &= (5.606)^{2/5} = 1.992 \text{ m} \end{aligned}$$

Substituting this value in equation (i), we get

$$b = 0.828 \times 1.992 = 1.649 \text{ m}$$

In case of lined channel, the cost of lining as well as cost of excavation is to be considered.

$$\begin{aligned} \text{Now cost of excavation} &= (\text{Volume of channel}) \times \text{cost per m}^3 \text{ of excavation} \\ &= (b + nd) \times d \times 1 \times 4x \\ &= (1.649 + 1 \times 1.992) \times 1.992 \times 1 \times 4x = 29.01x \end{aligned}$$

$$\begin{aligned} \text{Cost of lining} &= \text{Area of lining} \times \text{cost per m}^2 \text{ of lining} \\ &= (\text{Perimeter of lining} \times 1) \times x \\ &= (b + 2d\sqrt{1+n^2}) \times 1 \times x = (1.649 + 2 \times 1.992\sqrt{1+1^2}) \times 1 \times x \\ &= (1.649 + 2 \times 1.992 \times \sqrt{2})x = 7.283x \end{aligned}$$

$$\therefore \text{Total cost} = 29.01x + 7.283x = 36.293x$$

The total cost of lined channel is  $36.293x$  whereas the total cost of unlined channel is  $37.215x$ .

Hence lined channel will be cheaper. The dimensions are  $b = 1.649 \text{ m}$  and  $d = 1.992 \text{ m}$ . Ans.

**Problem 16.20** An open channel of most economical section, having the form of a half hexagon with horizontal bottom is required to give a maximum discharge of  $20.2 \text{ m}^3/\text{s}$  of water. The slope of the channel bottom is 1 in 2500. Taking Chezy's constant,  $C = 60$  in Chezy's equation, determine the dimensions of the cross-section.

**Solution.** Given :

$$\text{Maximum discharge, } Q = 20.2 \text{ m}^3/\text{s}$$

$$\text{Bed slope, } i = \frac{1}{2500}$$

$$\text{Chezy's constant, } C = 60$$

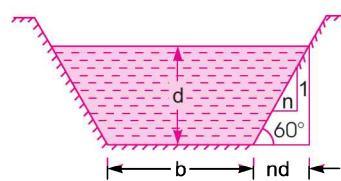


Fig. 16.15

Channel is the form of a half hexagon as shown in Fig. 16.15. This means that the angle made by the sloping side with horizontal will be  $60^\circ$ .

$$\therefore \tan \theta = \tan 60^\circ = \sqrt{3} = \frac{1}{n}$$

$$\therefore n = \frac{1}{\sqrt{3}}$$

Let  $b$  = width of the channel,  $d$  = depth of the flow.

As the channel given is of most economical section, hence the condition given by equations (16.11) and (16.12) should be satisfied i.e.,

Half of the top width = one of the sloping side

And hydraulic mean depth = half of depth of flow

$$\text{From equation (16.11), } \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

$$\text{For } n = \frac{1}{\sqrt{3}}, \quad \frac{b + 2 \times \frac{1d}{\sqrt{3}}}{2} = d\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} = \frac{2d}{\sqrt{3}}$$

$$\text{or } \frac{\sqrt{3}b + 2d}{2\sqrt{3}} = \frac{2d}{\sqrt{3}} \quad \text{or} \quad \frac{\sqrt{3}b + 2d}{2} = 2d$$

$$\therefore b = \frac{2 \times 2d - 2d}{\sqrt{3}} = \frac{2d}{\sqrt{3}} \quad \dots(i)$$

$$\begin{aligned} \text{Area of flow, } A &= (b + nd)d = \left( \frac{2}{\sqrt{3}}d + \frac{d}{\sqrt{3}} \right)d \\ &= \frac{3}{\sqrt{3}}d^2 = \sqrt{3}d^2 \end{aligned} \quad \left( \because n = \frac{1}{\sqrt{3}}, b = \frac{2d}{\sqrt{3}} \right)$$

$$\text{From equation (16.12)} \quad m = \frac{d}{2}$$

Using equation (16.5) for discharge  $Q$  as

$$Q = AC\sqrt{mi} \quad \text{or} \quad 20.2 = \sqrt{3} d^2 \times 60 \times \sqrt{\frac{d}{2} \times \frac{1}{2500}} = 1.4694 d^{5/2}$$

$$\therefore d^{5/2} = \frac{20.2}{1.4696} = 13.745$$

$$\therefore d = (13.745)^{2/5} = 2.852 \text{ m. Ans.}$$

Substituting this value in equation (i), we get

$$b = \frac{2d}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times 2.852 = 3.293 \text{ m. Ans.}$$

**Problem 16.21** A trapezoidal channel to carry  $142 \text{ m}^3/\text{minute}$  of water is designed to have a minimum cross-section. Find the bottom width and depth if the bed slope is 1 in 1200, the side slopes at  $45^\circ$  and Chezy's co-efficient = 55.

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**Solution.** Given : Discharge,  $Q = 142 \text{ m}^3/\text{min.} = \frac{142}{60} = 2.367 \text{ m}^3/\text{s}$

Bed slope,

$$i = 1 \text{ in } 1200 = \frac{1}{1200}$$

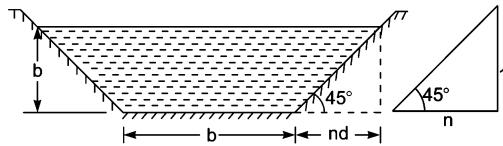


Fig. 16.16

Side slope,

$$\theta = 45^\circ$$

$$\therefore \tan \theta = \frac{1}{n} \quad \text{or} \quad \tan 45^\circ = \frac{1}{n}$$

$$\therefore 1 = \frac{1}{n} \quad \text{or} \quad n = 1$$

Chezy's constant,

$$C = 55$$

Let

$b$  = Width of the channel,  $d$  = Depth of the flow.

As the channel is to be designed for a minimum cross-section (*i.e.*, channel is of most economical section), the conditions given by equations (16.11) and (16.12) should be satisfied *i.e.*,

- (i) Half of top width = Length of sloping side
- (ii) Hydraulic mean depth = Half of depth of flow

$$\text{From equation (16.11), } \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

$$\text{or } \frac{b + 2 \times 1 \times d}{2} = d\sqrt{1^2 + 1} \quad (\because n = 1)$$

$$\text{or } b + 2d = 2\sqrt{2d} = 2 \times 1.414 d = 2.828 d$$

$$\therefore b = 2.828 d - 2d = 0.828 d \quad \dots(i)$$

Now using equation (16.5) for discharge  $Q$ , we get

$$Q = A \cdot C \cdot \sqrt{mi}$$

$$\text{or } 2.367 = (b + nd) d \times 55 \sqrt{\frac{d}{2} \times \frac{1}{1200}} \quad \left( \because A = (b + nd) \times d \text{ and } m = \frac{d}{2} \right)$$

$$= (0.828d \times 1 \times d) d \times 55 \sqrt{\frac{d}{2400}} \quad (\because b = 0.828d)$$

$$= (1.828d) \times d \times 55 \sqrt{\frac{d}{2400}} = 2.052 d^{5/2}$$

$$\therefore d = \left( \frac{2.367}{2.052} \right)^{2/5} = 1.058 \approx 1.06 \text{ m. Ans.}$$

Substituting this value in equation (i), we get

$$b = 0.828 \times 1.06 = 0.877 \text{ m. Ans.}$$

**Problem 16.22** A trapezoidal channel with side slopes of 3 horizontal to 2 vertical has to be designed to convey  $10 \text{ m}^3/\text{s}$  at a velocity of  $1.5 \text{ m/s}$ , so that the amount of concrete lining for the bed and sides is minimum. Find

(i) the wetted perimeter, and

(ii) slope of the bed if Manning's  $N = 0.014$  in the formula  $C = \frac{1}{N} \times m^{1/6}$

**Solution.** (i) Given :

$$\text{Side slope, } n = \frac{\text{Horizontal}}{\text{Vertical}} = \frac{3}{2} = 1.5$$

$$\begin{aligned} \text{Discharge, } Q &= 10 \text{ m}^3/\text{s} \\ \text{Velocity, } V &= 1.5 \text{ m/s} \\ \text{Manning's constant, } N &= .014 \end{aligned}$$

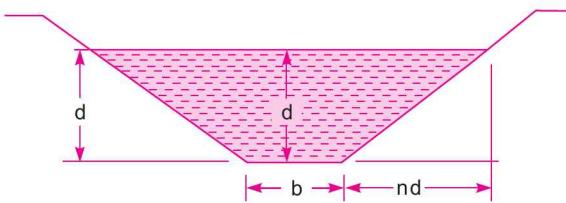


Fig. 16.17

Let  $b$  = width of the channel,  $d$  = depth of the flow.

The amount of concrete lining for the bed and sides will be minimum, when the section is most economical. For most economical trapezoidal section, the condition is given by equation (16.11) as

$$\frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

$$\text{For } n = 1.5, \quad \frac{b + 2 \times 1.5 \times d}{2} = d\sqrt{1.5^2 + 1} = \sqrt{3.25} d = 1.8 d \quad \text{or} \quad \frac{b + 3d}{2} = 1.8 d$$

$$\therefore b = 2 \times 1.8 - 3d = 0.6 d \quad \dots(i)$$

$$\begin{aligned} \text{But area of trapezoidal section, } A &= (b + nd)d = (0.6d + 1.5d)d \\ &= 2.1 d^2 \end{aligned} \quad (\because b = 0.6d, n = 1.5)$$

$$\text{Also area, } A = \frac{\text{Discharge}}{\text{Velocity}} = \frac{Q}{V} = \frac{10.0}{1.5} = 6.67 \text{ m}^2$$

Equating the two values of  $A$ , we have  $2.1 d^2 = 6.67$

$$\therefore d = \sqrt{\frac{6.67}{2.1}} = 1.78 \text{ m}$$

$$\text{From equation (i), } b = 0.6d = 0.6 \times 1.78 = 1.068 \approx 1.07 \text{ m}$$

$$\text{Hence wetted perimeter, } P = b + 2d\sqrt{n^2 + 1} = 1.07 + 2 \times 1.78\sqrt{1.5^2 + 1} = 7.48 \text{ m. Ans.}$$

$$(ii) \text{ Slope of the bed when } N = 0.014 \text{ in the formula, } C = \frac{1}{N} m^{1/6}$$

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For the most economical trapezoidal section, hydraulic mean depth  $m$ , is given by equation (16.12) as

$$m = \frac{d}{2} = \frac{1.78}{2} = 0.89 \text{ m}$$

$$C = \frac{1}{0.014} \times (0.89)^{1/6} = 66.09$$

Using equation (16.5),  $Q = AC\sqrt{mi}$

$$\text{or } 10.0 = 6.67 \times 66.09 \sqrt{0.89} \times i = 415.86\sqrt{i}$$

$$\therefore i = \left( \frac{10}{415.86} \right)^2 = \frac{1}{1729.4}. \text{ Ans.}$$

Hence slope of the bed is 1 in 1729.4.

### 16.5.3 Best Side Slope for Most Economical Trapezoidal Section.

Area of trapezoidal section,  $A = (b + nd)d$  ... (i)

where  $b$  = width of trapezoidal channel,  $d$  = depth of flow, and

$n$  = slope of the side of the channel

$$\text{From equation (i), } b = \frac{A}{d} - nd \quad \dots (\text{ii})$$

$$\text{Perimeter (wetted) of channel, } P = b + 2d\sqrt{n^2 + 1}$$

Substituting the value of  $b$  from equation (ii), perimeter becomes

$$P = \frac{A}{d} - nd + 2d\sqrt{n^2 + 1} \quad \dots (\text{iii})$$

For the most economical trapezoidal section, the depth of flow,  $d$  and area  $A$  are constant. Then  $n$  is the only variable. Best side slope will be when section is most economical or in other words,  $P$  is minimum. For  $P$  to be minimum, we must have  $\frac{dP}{dn} = 0$

Hence differentiating equation (iii) with respect to  $n$ ,

$$\frac{d}{dn} \left[ \frac{A}{d} - nd + 2d\sqrt{n^2 + 1} \right] = 0$$

$$\text{or } -d + 2d \times \frac{1}{2} \times (n^2 + 1)^{1/2-1} \times 2n = 0 \text{ or } -d + 2nd \times \frac{1}{\sqrt{n^2 + 1}} = 0$$

Cancelling  $d$  and re-arranging, we get  $2n = \sqrt{n^2 + 1}$

Squaring to both sides,

$$4n^2 = n^2 + 1 \text{ or } 3n^2 = 1 \text{ or } n = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} \quad \dots (16.14)$$

If the sloping side makes an angle  $\theta$ , with the horizontal, then we have

$$\tan \theta = \frac{1}{n} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3} = \tan 60^\circ$$

$$\therefore \theta = 60^\circ \quad \dots(16.15)$$

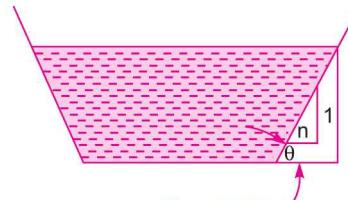


Fig. 16.18

Hence best side slope is at  $60^\circ$  to the horizontal or the value of  $n$  for the best side slope is given by equation (16.14).

For the most economical trapezoidal section, we have

Half of top width = length of one sloping side

$$\text{or } \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

Substituting the value of  $n$  from equation (16.14), we have

$$\frac{b + 2 \times \frac{1}{\sqrt{3}} \times d}{2} = d \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} = \frac{2d}{\sqrt{3}} \quad \text{or} \quad \frac{\sqrt{3}b + 2d}{2 \times \sqrt{3}} = \frac{2d}{\sqrt{3}}$$

$$\text{or } \sqrt{3}b + 2d = 2 \times \sqrt{3} \times \frac{2d}{\sqrt{3}} = 4d$$

$$\therefore b = \frac{4d - 2d}{\sqrt{3}} = \frac{2d}{\sqrt{3}} \quad \dots(iv)$$

Now, wetted perimeter,  $P = b + 2d\sqrt{n^2 + 1}$

$$= \frac{2d}{\sqrt{3}} + 2d\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} \quad \left(\because b = \frac{2d}{\sqrt{3}}, n = \frac{1}{\sqrt{3}}\right)$$

$$= \frac{2d}{\sqrt{3}} + 2d \times \frac{2}{\sqrt{3}} = \frac{2d}{\sqrt{3}} + \frac{4d}{\sqrt{3}}$$

$$\text{or } P = \frac{6d}{\sqrt{3}} = 3 \times \frac{2d}{\sqrt{3}} = 3 \times b \quad \left(\because \text{From (iv), } \frac{2d}{\sqrt{3}} = b\right)$$

For a slope of  $60^\circ$ , the length of sloping side is equal to the width of the trapezoidal section.

**Problem 16.23** A power canal of trapezoidal section has to be excavated through hard clay at the least cost. Determine the dimensions of the channel given, discharge equal to  $14 \text{ m}^3/\text{s}$ , bed slope  $1 : 2500$  and Manning's  $N = 0.020$ .

**Solution.** Given :

Discharge,  $Q = 14 \text{ m}^3/\text{s}$

Bed slope,  $i = \frac{1}{2500}$

Manning's,  $N = 0.020$

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For excavation of the canal at the least cost, the trapezoidal section should be most economical. Here side slope (*i.e.*, value of  $n$ ) is not given. Hence the best side slope for most economical trapezoidal section (*i.e.*, the value of  $n$ ) is given by equation (16.14) as  $n = \frac{1}{\sqrt{3}}$

Let  $b$  = width of channel,  $d$  = depth of flow

For most economical section,

Half of top width = length of one of sloping side

$$\text{or } \frac{b + 2nd}{2} = d\sqrt{n^2 + 1}$$

$$\text{For } n = \frac{1}{\sqrt{3}}, \quad \frac{b + 2 \times \frac{1}{\sqrt{3}}d}{2} = d\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} = \frac{2d}{\sqrt{3}}$$

$$\text{or } b = \frac{2 \times 2d}{\sqrt{3}} - \frac{2d}{\sqrt{3}} = \frac{2d}{\sqrt{3}} \quad \dots(i)$$

$$\begin{aligned} \text{Area of trapezoidal section, } A &= (b + nd) \times d = \left( \frac{2d}{\sqrt{3}} + \frac{1}{\sqrt{3}}d \right) \times d && \left( \because b = \frac{2d}{\sqrt{3}}, n = \frac{1}{\sqrt{3}} \right) \\ &= \sqrt{3}d^2 \end{aligned}$$

$$\text{Hydraulic mean depth for most economical section, } m = \frac{d}{2}$$

$$\text{Now discharge, } Q \text{ is given by } Q = AC\sqrt{mi}, \text{ where } C = \frac{1}{N} m^{1/6}$$

$$\begin{aligned} \therefore Q &= \sqrt{3}d^2 \times \frac{1}{N} m^{1/6} \sqrt{m \times \frac{1}{2500}} \\ &= \sqrt{3}d^2 \times \frac{1}{0.020} \times m^{1/6 + 1/2} \times \sqrt{\frac{1}{2500}} = 1.732 d^2 \times m^{2/3} \end{aligned}$$

$$\text{or } 14.0 = 1.732 d^2 \times \left(\frac{d}{2}\right)^{2/3} = \frac{1.732}{2^{2/3}} d^{8/3} = 1.09 d^{8/3}$$

$$\therefore d^{8/3} = \frac{14.0}{1.09} = 12.844$$

$$\therefore d = (12.844)^{3/8} = (12.844)^{0.375} = 2.605 \text{ m. Ans.}$$

$$\text{From equation (i), } b = \frac{2d}{\sqrt{3}} = \frac{2 \times 2.605}{1.732} = 3.008 \text{ m. Ans.}$$

**Problem 16.24** For a trapezoidal channel with bottom width 40 m and side slopes 2H : 1 V, Manning's  $N$  is 0.015 and bottom slope is 0.0002. If it carries 60 m<sup>3</sup>/s discharge, determine the normal depth.

**Solution.** Given :

$$\text{Bottom width, } b = 40 \text{ m}$$

Side slopes 2 horizontal to 1 vertical i.e.,  $n = 2$

$$\therefore \text{Manning's constant, } N = 0.015$$

$$\text{Bed slope, } i = 0.0002$$

$$\text{Discharge, } Q = 60 \text{ m}^3/\text{s}$$

Let  $d$  = Normal depth.

$$\text{Now } A = (b + nd) \times d = (40 + 2d) \times d$$

$$P = b + 2d\sqrt{1+n^2} = 40 + 2d\sqrt{1+2^2} = 40 + 2 \times \sqrt{5}d = 40 + 4.472d$$

$$\therefore m = \frac{A}{P} = \frac{(40 + 2d) \times d}{40 + 4.472d}$$

The discharge is given by,  $Q = \text{Area} \times \text{Velocity}$

$$= A \times \frac{1}{N} m^{2/3} t^{1/2} = \frac{A}{N} \times m^{2/3} \times i^{1/2}$$

$$60 = \frac{(40 + 2d) \times d}{0.015} \times \left[ \frac{(40 + 2d) \times d}{40 + 4.472d} \right]^{2/3} \times 0.0002^{1/2}$$

$$= \frac{[(40 + 2d) \times d]^{5/3}}{0.015 \times (40 + 4.472d)^{2/3}} \times 0.01414$$

$$\therefore \frac{60 \times 0.015 \times (40 + 4.472d)^{2/3}}{0.01414} = [(40 + 2d) \times d]^{5/3}$$

$$63.65(40 + 4.472d)^{2/3} = (40d + 2d^2)^{5/3}$$

$$(40d + 2d^2)^{5/3} - 63.65(40 + 4.472d)^{2/3} = 0 \quad \dots(i)$$

The above equation will be solved by Hit and Trial method.

(i) Assume  $d = 1 \text{ m}$ , then L.H.S of equation (i) will be as

$$\begin{aligned} \text{L.H.S.} &= (40 + 2)^{5/3} - 63.65(40 + 4.472)^{2/3} \\ &= 42^{5/3} - 63.65 \times 44.472^{2/3} = 513.838 - 808.4 = -294.56 \end{aligned}$$

(ii) Assume  $d = 2 \text{ m}$ , then L.H.S. of equation (i) will be as

$$\begin{aligned} \text{L.H.S.} &= (40 \times 2 + 2 \times 2^2)^{5/3} - 63.65(40 + 4.47 \times 2)^{2/3} \\ &= 88^{5/3} - 63.65 \times 48.944^{2/3} = 1767.2 - 862.77 = 904.43 \end{aligned}$$

where  $d = 1 \text{ m}$ , L.H.S. is -ve. But when  $d = 2 \text{ m}$ , L.H.S. is +ve. Hence value of  $d$  lies between 1 and 2.

(iii) Assume  $d = 1.3 \text{ m}$ , then L.H.S. of equation (i) will be as

$$\begin{aligned} \text{L.H.S.} &= (40 \times 1.3 + 2 \times 1.3^2)^{5/3} - 63.65(40 + 4.472 \times 1.3)^{2/3} \\ &= 55.38^{5/3} - 63.65 \times 45.8136^{2/3} = 815.45 - 825.4 = -9.95 \end{aligned}$$

(iv) Assume  $d = 1.31 \text{ m}$ , then L.H.S. of equation (i) will be

$$\begin{aligned} \text{L.H.S.} &= (40 \times 1.31 + 2 \times 1.31^2)^{5/3} - 63.65(40 + 4.472 \times 1.31)^{2/3} \\ &= 55.8322^{5/3} - 63.65 \times 45.8583^{2/3} = 826.6 - 825.9 = 0.7 \end{aligned}$$

The value of L.H.S. = 0.7 is negligible in comparison to the value of 904.43.

$\therefore$  Value of  $d = 1.31 \text{ m}$ . Ans.

**16.5.4 Flow Through Circular Channel.** The flow of a liquid through a circular pipe, when the level of liquid in the pipe is below the top of the pipe is classified as an open channel flow. The rate of flow through circular channel is determined from the depth of flow and angle subtended by the liquid surface at the centre of the circular channel.

Fig.16.19 shows a circular channel through which water is flowing.

Let  $d$  = depth of water,

$2\theta$  = angle subtended by water surface  $AB$  at the centre in radians,

$R$  = radius of the channel,

Then the wetted perimeter and wetted area is determine as :

$$\text{Wetted perimeter, } P = \frac{2\pi R}{2\pi} \times 2\theta = 2R\theta \quad \dots(16.16)$$

Wetted area,  $A$  = Area  $ADBA$

= Area of sector  $OADBO$  – Area of  $\Delta ABO$

$$= \frac{\pi R^2}{2\pi} \times 2\theta - \frac{AB \times CO}{2} = R^2\theta - \frac{2BC \times CO}{2} \quad (\because AB = 2BC)$$

$$= R^2\theta - \frac{2 \times R \sin \theta \times R \cos \theta}{2} \quad (\because BC = R \sin \theta, CO = R \cos \theta)$$

$$= R^2\theta - \frac{R^2 \times 2 \sin \theta \cos \theta}{2} = R^2\theta - \frac{R^2 \sin 2\theta}{2} \quad (\because 2 \sin \theta \cos \theta = \sin 2\theta)$$

$$= R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) \quad \dots(16.17)$$

$$\text{Then hydraulic mean depth, } m = \frac{A}{P} = \frac{R^2 \left( \theta - \frac{\sin 2\theta}{2} \right)}{2R\theta} = \frac{R}{2\theta} \left( \theta - \frac{\sin 2\theta}{2} \right)$$

And discharge,  $Q$  is given by,  $Q = AC\sqrt{mi}$ .

**Problem 16.25** Find the discharge through a circular pipe of diameter 3.0 m, if the depth of water in the pipe is 1.0 m and the pipe is laid at a slope of 1 in 1000. Take the value of Chezy's constant as 70.

**Solution.** Given :

Dia. of pipe,  $D = 3.0$

$$\therefore \text{Radius, } R = \frac{D}{2} = \frac{3.0}{2} = 1.50 \text{ m}$$

Depth of water in pipe,  $d = 1.0 \text{ m}$

$$\text{Bed slope, } i = \frac{1}{1000}$$

Chezy's constant,  $C = 70$

From Fig. 16.20, we have  $OC = OD - CD = R - 1.0$

$$= 1.5 - 1.0 = 0.5 \text{ m}$$

$$AO = R = 1.5 \text{ m}$$

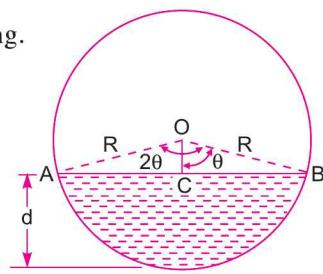


Fig. 16.19 Circular channel.

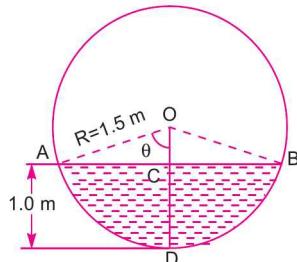


Fig. 16.20

Also  $\cos \theta = \frac{OC}{AO} = \frac{0.5}{1.5} = \frac{1}{3}$

$$\therefore \theta = 70.53^\circ = 70.53 \times \frac{\pi}{180} = 1.23 \text{ radians} \quad (\because 180^\circ = \pi \text{ radians})$$

Wetted perimeter is given by equation (16.16) as

$$P = 2R\theta = 2 \times 1.5 \times 1.23 \quad (\theta \text{ should be in radians}) \\ = 3.69 \text{ m}$$

Wetted area is given by equation (16.17) as

$$A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) = 1.5^2 \left( 1.23 - \frac{\sin (2 \times 70.53^\circ)}{2} \right) \\ = 2.25 \left[ 1.23 - \frac{\sin (141.08^\circ)}{2} \right] = 2.25 \left[ 1.23 - \frac{\sin (180^\circ - 141.08^\circ)}{2} \right] \\ = 2.25 \left[ 1.23 - \frac{\sin 38.94^\circ}{2} \right] = 2.06 \text{ m}^2$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{2.06}{3.69} = .5582$$

$$\text{The discharge is given by, } Q = AC\sqrt{mi} = 2.06 \times 70 \times \sqrt{0.5582 \times \frac{1}{1000}} = 3.407 \text{ m}^2/\text{s. Ans.}$$

**Problem 16.26** If in the problem 16.25, the depth of water in the pipe is 2.5 m, find the rate of flow through the pipe.

**Solution.** Given :

$$\text{Dia. of pipe} \quad = 3.0 \text{ m}$$

$$\therefore \text{Radius,} \quad R = 1.5 \text{ m}$$

$$\text{Depth of water,} \quad d = 2.5 \text{ m}$$

$$i = \frac{1}{1000} \text{ and } C = 70$$

$$\text{From Fig. 16.21,} \quad OC = CD - OD = 2.5 - R = 2.5 - 1.5 = 1.0 \text{ m}$$

$$OA = R = 1.5 \text{ m}$$

$$\text{From } \Delta AOC, \quad \cos \alpha = \frac{OC}{OA} = \frac{1.0}{1.5} = 0.667$$

$$\therefore \alpha = 48.16^\circ$$

$$\theta = 180^\circ - \alpha = 180^\circ - 48.16^\circ = 131.84^\circ$$

$$= 131.84 \times \frac{\pi}{180} = 2.30 \text{ radians}$$

Now wetted perimeter is given by equation (16.16) as

$$P = 2R\theta = 2 \times 1.5 \times 2.30 = 6.90 \text{ m}$$

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And wetted area is given by equation (16.17) as

$$\begin{aligned}
 A &= R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) = 1.5^2 \left( 2.30 - \frac{\sin (2 \times 131.84^\circ)}{2} \right) \\
 &= 2.25 \left( 2.30 - \frac{\sin 263.68^\circ}{2} \right) \\
 &= 2.25 \left[ 2.30 - \frac{\sin (180^\circ + 83.68^\circ)}{2} \right] \\
 &= 2.25 \left[ 2.30 - \frac{(-\sin 83.58^\circ)}{2} \right] \\
 &= 2.25 \left[ 2.30 + \frac{\sin 83.68^\circ}{2} \right] = 6.293 \text{ m}^2
 \end{aligned}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{6.293}{6.90} = 0.912 \text{ m}$$

$$\text{Discharge, } Q \text{ is given by, } Q = AC\sqrt{mi} = 6.293 \times 70 \times \sqrt{0.912 \times \frac{1}{1000}} = 13.303 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.27** Calculate the quantity of water that will be discharged at a uniform depth of 0.9 m in a 1.2 m diameter pipe which is laid at a slope 1 in 1000. Assume Chezy's C = 58.

**Solution.** Given :

$$\text{Dia. of pipe} = 1.2 \text{ m}$$

$$\therefore \text{Radius, } R = \frac{1.2}{2} = 0.6 \text{ m}$$

$$\text{Depth of water, } d = 0.9 \text{ m}$$

$$\text{Slope, } i = \frac{1}{1000}$$

$$\text{Chezy's, } C = 58$$

From Fig. 16.22, we have  $OC = CD - OD$

$$= 0.9 - R = 0.9 - 0.6 = 0.3 \text{ m}$$

$$OA = R = 0.6 \text{ m}$$

Now in triangle  $AOC$ ,

$$\cos \alpha = \frac{OC}{OA} = \frac{0.3}{0.6} = \frac{1}{2}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

$$\therefore \theta = \text{Angle } DOA = 180^\circ - \alpha$$

$$= 180^\circ - 60^\circ = 120^\circ = 120 \times \frac{\pi}{180} \text{ radians}$$

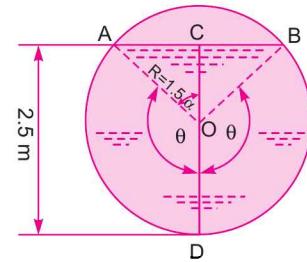


Fig. 16.21

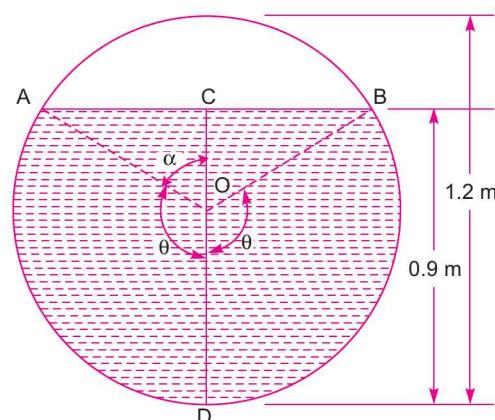


Fig. 16.22

$$= 0.667 \pi \text{ radians}$$

Now wetted perimeter is given by equation (16.16) as

$$P = 2R\theta = 2 \times 0.6 \times 0.667 \pi = 2.526 \text{ m}$$

And area of flow is given by equation (16.17) as

$$\begin{aligned} A &= R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) \\ &= 0.6^2 \left[ 0.667\pi - \frac{\sin (2 \times 120^\circ)}{2} \right] = 0.36 \left[ 0.667\pi - \frac{\sin 240^\circ}{2} \right] \\ &= 0.36 \left[ 0.667\pi - \frac{(-0.866)}{2} \right] = 0.36 [0.667\pi + 0.433] = 0.913 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Now discharge is given by, } Q &= A \times V = A \times C\sqrt{mi} = 0.913 \times 58 \sqrt{\frac{A}{P} \times \frac{1}{1000}} \quad \left( \because m = \frac{A}{P} \right) \\ &= 0.913 \times 58 \sqrt{\frac{0.913}{2.526} \times \frac{1}{1000}} = 1.007 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

**Problem 16.28** Water is flowing through a circular channel at the rate of 400 litres/s, when the channel is having a bed slope of 1 in 9000. The depth of water in the channel is 8.0 times the diameter. Find the diameter of the circular channel if the value of Manning's  $N = 0.015$ .

**Solution.** Given :

$$\text{Discharge, } Q = 400 \text{ litres/s} = 0.4 \text{ m}^3/\text{s}$$

$$\text{Bed slope, } i = \frac{1}{9000}$$

$$\text{Manning's, } N = 0.015$$

$$\text{Let the diameter of channel } = D$$

$$\text{Then depth of flow, } d = 0.8 D$$

From Fig. 16.23, we have

$$\begin{aligned} OC &= CD - OD = 0.8 D - \frac{D}{2} \\ &= (0.8 - 0.5) D = 0.3 D \end{aligned}$$

$$\text{And } AO = R = \frac{D}{2} = 0.5 D$$

$$\therefore \cos \alpha = \frac{OC}{AO} = \frac{0.3 D}{0.5 D} = 0.6$$

$$\therefore \alpha = 53.13^\circ$$

$$\text{And } \theta = 180^\circ - 53.13^\circ = 126.87^\circ = 126.87 \times \frac{\pi}{180} = 2.214 \text{ radians.}$$

From equation (16.16), wetted perimeter,

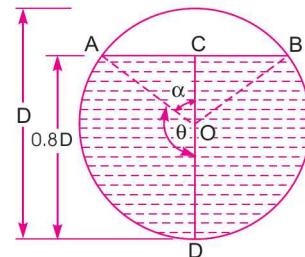


Fig. 16.23

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$$P = 2R\theta = 2 \times \frac{D}{2} \times 2.214 = 2.214 D \text{ m.}$$

From equation (16.17), wetted area,

$$\begin{aligned} A &= R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) = \left( \frac{D}{2} \right)^2 \left[ 2.214 - \frac{\sin (2 \times 126.87^\circ)}{2} \right] \\ &= \frac{D^2}{4} \left[ 2.214 - \frac{\sin 253.74^\circ}{2} \right] = \frac{D^2}{4} \left[ 2.214 - \frac{\sin (180^\circ + 73.74^\circ)}{2} \right] \\ &= \frac{D^2}{4} \left[ 2.214 - \left( \frac{-\sin 73.74^\circ}{2} \right) \right] = \frac{D^2}{4} \left[ 2.214 + \frac{\sin 73.74^\circ}{2} \right] \\ &= \frac{D^2}{4} [2.214 + .48] = 0.6735 D^2 \end{aligned}$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{0.6735 D^2}{2.214 D} = 0.3042 D$$

Discharge by Manning's formula is given by,

$$Q = \frac{1}{N} \times A \times m^{2/3} \times i^{1/2}$$

or

$$\begin{aligned} 0.4 &= \frac{1}{0.015} \times 0.6735 D^2 \times (0.3042 D)^{2/3} \times \left( \frac{1}{9000} \right)^{1/2} \\ &= \frac{0.6735}{0.015} D^2 \times 34521 \times D^{2/3} \times 0.0105 = 0.213 D^{8/3} \\ \therefore D^{8/3} &= \frac{0.40}{0.213} = 1.8779 \\ \therefore D &= (1.8779)^{3/8} = (1.8779)^{0.375} = 1.266 \text{ m. Ans.} \end{aligned}$$

**Problem 16.29** A sewer pipe is to be laid at a slope of 1 in 8100 to carry a maximum discharge of 600 litres/s, when the depth of water is 75% of the vertical diameter. Find the diameter of this pipe if the value of Manning's  $N$  is 0.025.

**Solution.** Given :

$$\text{Discharge, } Q = 600 \text{ litres/s} = 0.6 \text{ m}^3/\text{s}$$

$$\text{Bed slope, } i = \frac{1}{8100}$$

$$\text{Manning's, } N = 0.025$$

$$\text{Depth of water} = 75\% \text{ of dia. of pipe} = 0.75 \text{ dia. of pipe}$$

$$\text{Let } d = \text{depth of water, } D = \text{Dia. of pipe}$$

$$\text{Then } d = 0.75 D$$

From Fig. 16.23 (a), we have  $OC = CD - OD = 0.75 D - 0.5 D = 0.25 D$

$$AO = R = 0.5 D$$

In triangle  $AOC$ ,  $\cos \alpha = \frac{OC}{AO} = \frac{0.25 D}{0.5 D} = 0.5$

$$\therefore \alpha = \cos^{-1} 0.5 = 60^\circ$$

And  $\theta = 180^\circ - \alpha = 180^\circ - 60^\circ = 120^\circ$

$$= 120 \times \frac{\pi}{180} \text{ radians} = 2.0946 \text{ radians.}$$

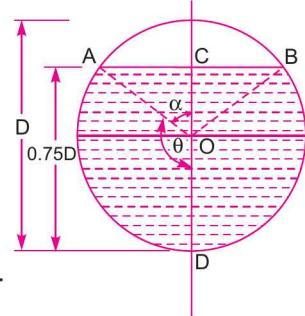


Fig. 16.23 (a)  
( $\because R = 0.5 D$ )

From equation (16.16), wetted perimeter

$$P = 2R\theta = 2 \times 0.5 D \times 2.0946 \\ = 2.0496 D$$

And from equation (16.17), the area of flow,

$$A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) \\ = (0.5 D)^2 \left[ 2.0946 - \frac{\sin (2 \times 120^\circ)}{2} \right] \\ = 0.25 D^2 \left[ 2.0946 - \left( \frac{-0.866}{2} \right) \right] = 0.25 D^2 [2.0946 + 0.433] \\ = 0.6319 D^2 \\ \therefore m = \frac{A}{P} = \frac{0.6319 D^2}{2.0496 D} = 0.308 D$$

Discharge by Manning's formula is given by

$$Q = \frac{1}{N} \times A \times m^{2/3} \times i^{1/2}$$

or  $0.6 = \frac{1}{0.025} \times 0.6319 D^2 \times (0.308 D)^{2/3} \times \left( \frac{1}{8100} \right)^{1/2} = 0.128 \times D^{8/3}$

$$\therefore D^{8/3} = \frac{0.6}{0.128} = 4.6875$$

$$\therefore D = (4.6875)^{3/8} = 1.785 \text{ m. Ans.}$$

**16.5.5 Most Economical Circular Section.** We have discussed in Art. 16.5 that for a most economical section the discharge for a constant cross-sectional area, slope of bed and resistance co-efficient, is maximum. But in case of circular channels, the area of flow cannot be maintained constant. With the change of depth of flow in a circular channel of any radius, the wetted area and wetted perimeter changes. Thus in case of circular channels, for most economical section, two separate conditions are obtained. They are :

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1. Condition for maximum velocity, and
2. Condition for maximum discharge.

**1. Condition for Maximum Velocity for Circular Section.** Fig. 16.24 shows a circular channel through which water is flowing.

Let

$d$  = depth of water,

$2\theta$  = angle subtended at the centre by water surface,

$R$  = radius of channel, and

$i$  = slope of the bed,

The velocity of flow according to Chezy's formula is given as

$$V = C \sqrt{mi} = C \sqrt{\frac{A}{P}} i \quad \left( \because m = \frac{A}{P} \right)$$

The velocity of flow through a circular channel will be maximum when the hydraulic mean depth  $m$  or  $A/P$  is maximum for a given value of  $C$  and  $i$ . In case of circular pipe, the variable is  $\theta$  only. Hence for maximum value of  $A/P$  we have the condition,

$$\frac{d \left( \frac{A}{P} \right)}{d\theta} = 0$$

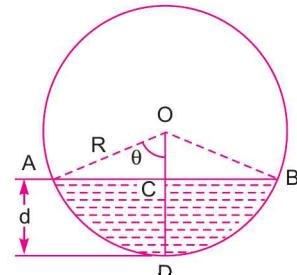


Fig. 16.24

...(i)

where  $A$  and  $P$  both are functions of  $\theta$ .

The value of wetted area,  $A$  is given by equation (16.17) as

$$A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) \quad \dots(ii)$$

The value of wetted perimeter,  $P$  is given by equation (16.16) as

$$P = 2R\theta \quad \dots(iii)$$

Differentiating equation (i) with respect to  $\theta$ , we have

$$\frac{P \frac{dA}{d\theta} - A \frac{dP}{d\theta}}{P^2} = 0 \quad \text{or} \quad P \frac{dA}{d\theta} - A \frac{dP}{d\theta} = 0 \quad \dots(iv)$$

From equation (ii),  $\frac{dA}{d\theta} = R^2 \left( 1 - \frac{\cos 2\theta}{2} \times 2 \right) = R^2 (1 - \cos 2\theta)$

From equation (iii),  $\frac{dP}{d\theta} = 2R$

Substituting the values of  $A$ ,  $P \frac{dA}{d\theta}$  and  $\frac{dP}{d\theta}$  in equation (iv),

$$2R\theta \left[ R^2 (1 - \cos 2\theta) \right] - R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) (2R) = 0$$

or  $2R^3\theta (1 - \cos 2\theta) - 2R^3 \left( \theta - \frac{\sin 2\theta}{2} \right) = 0$

or  $\theta (1 - \cos 2\theta) - \left( \theta - \frac{\sin 2\theta}{2} \right) = 0 \quad (\text{Cancelling } 2R^3)$

$$\theta - \theta \cos 2\theta - \theta + \frac{\sin 2\theta}{2} = 0$$

or  $\theta \cos 2\theta = \frac{\sin 2\theta}{2}$  or  $\frac{\sin 2\theta}{\cos 2\theta} = 2\theta$

$$\therefore \tan 2\theta = 2\theta$$

The solution of this equation by hit and trial, gives

$$2\theta = 257^\circ 30' \quad (\text{approximately})$$

or  $\theta = 128^\circ 45'$

The depth of flow for maximum velocity from Fig. 16.24, is

$$\begin{aligned} d &= OD - OC = R - R \cos \theta \\ &= R[1 - \cos \theta] = R[1 - \cos 128^\circ 45'] = R[1 - \cos (180^\circ - 51^\circ 15')] \\ &= R[1 - (-\cos 51^\circ 15')] = R[1 + \cos 51^\circ 15'] \\ &= R[1 + 0.62] = 1.62 R = 1.62 \times \frac{D}{2} = 0.81 D \end{aligned} \quad \dots(16.18)$$

where  $D$  = diameter of the circular channel.

Thus for maximum velocity of flow, the depth of water in the circular channel should be equal to 0.81 times the diameter of the channel.

Hydraulic mean depth for maximum velocity is

$$m = \frac{A}{P} = \frac{R^2 \left( \theta - \frac{\sin 2\theta}{2} \right)}{2R\theta} = \frac{R}{2\theta} \left[ \theta - \frac{\sin 2\theta}{2} \right]$$

where  $\theta = 128^\circ 45' = 128.75^\circ$

$$= 128.75 \times \frac{\pi}{180} = 2.247 \text{ radians}$$

$$\begin{aligned} \therefore m &= \frac{R}{2 \times 2.247} \left[ 2.247 - \frac{\sin 257^\circ 30'}{2} \right] = \frac{R}{4.494} \left[ 2.247 - \frac{\sin (180^\circ + 87.5^\circ)}{2} \right] \\ &= \frac{R}{4.494} \left[ 2.247 + \frac{\sin 87.5^\circ}{2} \right] = 0.611 R \\ &= 0.611 \times \frac{D}{2} = 0.3055 D = 0.3 D \end{aligned} \quad \dots(16.19)$$

Thus for maximum velocity, the hydraulic mean depth is equal to 0.3 times the diameter of circular channel.

**2. Condition for Maximum Discharge for Circular Section.** The discharge through a channel is given by

$$\begin{aligned} Q &= AC\sqrt{mi} = AC\sqrt{\frac{A}{P}i} \quad \left( \because m = \frac{A}{P} \right) \\ &= C\sqrt{\frac{A^3}{P}i} \end{aligned}$$

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The discharge will be maximum for constant values of  $C$  and  $i$ , when  $\frac{A^3}{P}$  is maximum.  $\frac{A^3}{P}$  will be maximum when  $\frac{d}{d\theta} \left( \frac{A^3}{P} \right) = 0$ .

Differentiating this equation with respect to  $\theta$  and equating the same to zero, we get

$$\frac{P \times 3A^2 \frac{dA}{d\theta} - A^3 \frac{dP}{d\theta}}{P^2} = 0 \quad \text{or} \quad 3PA^2 \frac{dA}{d\theta} - A^3 \frac{dP}{d\theta} = 0$$

$$\text{Dividing by } A^2, \quad 3P \frac{dA}{d\theta} - A \frac{dP}{d\theta} = 0 \quad \dots(i)$$

But from equation (16.16),  $P = 2R\theta$

$$\therefore \frac{dP}{d\theta} = 2R$$

$$\text{From equation (16.17),} \quad A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right)$$

$$\therefore \frac{dA}{d\theta} = R^2 (1 - \cos 2\theta)$$

Substituting the values of  $P$ ,  $A$ ,  $\frac{dP}{d\theta}$  and  $\frac{dA}{d\theta}$  in equation (i)

$$3 \times 2R\theta \times R^2 (1 - \cos 2\theta) - R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) \times 2R = 0$$

$$6R^3\theta (1 - \cos 2\theta) - 2R^3 \left( \theta - \frac{\sin 2\theta}{2} \right) = 0$$

Dividing by  $2R^3$ , we get

$$3\theta (1 - \cos 2\theta) - \left( \theta - \frac{\sin 2\theta}{2} \right) = 0 \quad \text{or} \quad 3\theta - 3\theta \cos 2\theta - \theta + \frac{\sin 2\theta}{2} = 0$$

$$\text{or} \quad 2\theta - 3\theta \cos 2\theta + \frac{\sin 2\theta}{2} = 0 \quad \text{or} \quad 4\theta - 6\theta \cos 2\theta + \sin 2\theta = 0$$

The solution of this equation by hit and trial, gives

$$2\theta = 308^\circ \quad (\text{approximately})$$

$$\therefore \theta = \frac{308^\circ}{2} = 154^\circ$$

Depth of flow for maximum discharge [See Fig. 16.24]

$$\begin{aligned} d &= OD - OC = R - R \cos \theta \\ &= R[1 - \cos \theta] = R[1 - \cos 154^\circ] \\ &= R[1 - \cos (180^\circ - 26^\circ)] = R[1 + \cos 26^\circ] = 1.898 R \\ &= 1.898 \times \frac{D}{2} = 0.948 D \approx 0.95 D \end{aligned} \quad \dots(16.20)$$

where  $D$  = Diameter of the circular channel.

Thus for maximum discharge through a circular channel the depth of flow is equal to 0.95 times its diameter.

**Problem 16.30** The rate of flow of water through a circular channel of diameter 0.6 m is 150 litres/s. Find the slope of the bed of the channel for maximum velocity. Take  $C = 60$

**Solution.** Given :

$$\text{Discharge, } Q = 150 \text{ litres/s} = 0.15 \text{ m}^3/\text{s}$$

$$\text{Dia. of channel, } D = 0.6 \text{ m}$$

$$\text{Value of } C = 60$$

Let the slope of the bed of channel for maximum velocity =  $i$

For maximum velocity through a circular channel, depth of flow is given by equation (16.18) as

$$d = 0.81 \times D = 0.81 \times 0.6 = .486 \text{ m}$$

and

$$\theta = 128^\circ 45' \text{ or } 128.75 \times \frac{\pi}{180} = 2.247 \text{ radians}$$

From equation (16.19), hydraulic mean depth for maximum velocity is given as

$$m = 0.3 \times D = 0.3 \times 0.6 = 0.18 \text{ m}$$

Wetted perimeter for circular pipe is given by equation (16.16),

$$P = 2R\theta = D \times \theta = 0.6 \times 2.247 = 1.3482 \text{ m}$$

But

$$m = \frac{A}{P} = 0.18 \text{ m}$$

∴ Area,

$$A = 0.18 \times P = 0.18 \times 1.3482 = 0.2426 \text{ m}^2$$

For discharge, using the relation

$$Q = AC\sqrt{mi} \text{ or } 0.15 = 0.2426 \times 60 \times \sqrt{0.81 \times i} = 6.175\sqrt{i}$$

$$\therefore i = \left( \frac{0.15}{6.175} \right)^2 = \frac{1}{1694.7} \text{ . Ans.}$$

∴ Bed slope is 1 in 1694.7.

**Problem 16.31** Determine the maximum discharge of water through a circular channel of diameter 1.5 m when the bed slope of the channel is 1 in 1000. Take  $C = 60$ .

**Solution.** Given :

$$\text{Dia. of channel, } D = 1.5 \text{ m}$$

$$\therefore R = \frac{1.5}{2} = 0.75 \text{ m}$$

$$\text{Bed slope, } i = \frac{1}{1000}$$

$$\text{Value of } C = 60$$

$$\text{For maximum discharge, } \theta = 154^\circ \text{ or } \frac{154 \times \pi}{180} = 2.6878 \text{ radians.}$$

Wetted perimeter for a circular channel is given by equation (16.16) as

$$P = 2R\theta = 2 \times \frac{D}{2} \times 2.6878 = 2 \times \frac{1.5}{2} \times 2.6878 = 4.0317 \text{ m}$$

Wetted area  $A$  is given by equation (16.17) as

$$A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) = 0.75^2 \left[ 2.6878 - \frac{\sin (2 \times 154)^\circ}{2} \right]$$

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$$= 0.75^2 \left[ 2.6878 - \frac{\sin 308^\circ}{2} \right] = .75^2 \left[ 2.6878 - \frac{\sin (360^\circ - 52^\circ)}{2} \right]$$

$$= 0.75^2 \left[ 2.6878 + \frac{\sin 52^\circ}{2} \right] = 1.7335$$

$$\therefore \text{Hydraulic mean depth, } m = \frac{A}{P} = \frac{1.7335}{4.0317} = 0.4299$$

$$\text{Maximum discharge is given by } Q = AC\sqrt{mi} = 1.7335 \times 60 \times \sqrt{0.4299 \times \frac{1}{1000}}$$

$$= 2.1565 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.32** A concrete lined circular channel of diameter 3 m has a bed slope of 1 in 500. Work out the velocity and flow rate for the conditions of (i) maximum velocity and (ii) maximum discharge. Assume Chezy's C = 50.

**Solution.** Given :

$$\text{Dia of channel, } D = 3 \text{ m}$$

$$\text{Bed slope, } i = \frac{1}{500}$$

$$\text{Value of } C = 50$$

(i) Velocity and discharge for maximum velocity

$$\text{For maximum velocity, } \theta = 128^\circ 45' = 128.75^\circ$$

$$= 128.75 \times \frac{\pi}{180} \text{ radians} = 2.247 \text{ radians}$$

$$\therefore \text{Wetted perimeter, } P = 2 \times R \times \theta$$

$$= 2 \times 1.5 \times 2.247 \quad \left( \because R = \frac{D}{2} = \frac{3}{2} = 1.5 \right)$$

$$= 6.741 \text{ m}$$

$$\text{Area of flow, } A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right) = 1.5^2 \left[ 2.247 - \frac{\sin (2 \times 128.75^\circ)}{2} \right]$$

$$= 2.25 [2.247 - (-0.488)] = 6.1537 \text{ m}^2$$

$$\therefore \text{Hydraulic mean depth } m^* = \frac{A}{P} = \frac{6.1537}{6.741} = 0.912$$

$$\text{Now velocity, } V = C\sqrt{m \times i} = 50 \times \sqrt{\frac{0.912 \times 1}{500}} = 2.135 \text{ m/s. Ans.}$$

and discharge,

$$Q = A \times V = 6.1537 \times 2.135 = 13.138 \text{ m}^3/\text{s. Ans.}$$

(ii) Velocity and discharge for maximum discharge

$$\text{For maximum discharge, } \theta = 154^\circ = \frac{154 \times \pi}{180} \text{ radians} = 2.6878 \text{ radians}$$

---

\* From equation (16.19),  $m$  is also equal to 0.3055  $D$ .  
Hence  $m = 0.3055 \times 3 = 0.9165$

$$\therefore A = R^2 \left[ \theta - \frac{\sin 2\theta}{2} \right] = 1.5^2 \left[ 2.6878 - \frac{\sin (2 \times 154)}{2} \right] \\ = 2.25 [2.6878 - (-0.394)] = 6.934$$

$$P = 2R \times \theta = 2 \times 1.5 \times 2.6878 = 8.0634$$

and

$$m = \frac{A}{P} = \frac{6.934}{8.0634} = 0.8599$$

Now velocity

$$V = C\sqrt{mi} = 50 \times \sqrt{0.8599 \times \frac{1}{500}} = 2.0735 \text{ m/s. Ans.}$$

and discharge,

$$Q = A \times V = 6.934 \times 2.0735 = 14.377 \text{ m}^3/\text{s. Ans.}$$

## ► 16.6 NON-UNIFORM FLOW THROUGH OPEN CHANNELS

We have defined uniform flow and non-uniform flow in Art. 16.2.2. A flow is said to be uniform if the velocity of flow, depth of flow, slope of the bed of the channel and area of cross-section remain constant for a given length of the channel. On the other hand, if velocity of flow, depth of flow, area of cross-section and slope of the bed of channel do not remain constant for a given length of pipe, the flow is said to be non-uniform.

Non-uniform is further divided into Rapidly Varied Flow (R.V.F.), and Gradually Varied Flow (G.V.F.) depending upon the change of depth of flow over the length of the channel. If the depth of flow changes abruptly over a small length of the channel, the flow is said as rapidly varied flow. And if the depth of flow in a channel changes gradually over a long length of channel, the flow is said to be gradually varied flow.

## ► 16.7 SPECIFIC ENERGY AND SPECIFIC ENERGY CURVE

The total energy of a flowing liquid per unit weight is given by,

$$\text{Total Energy} = z + h + \frac{V^2}{2g}$$

where  $z$  = Height of the bottom of channel above datum,

$h$  = Depth of liquid, and  $V$  = Mean velocity of flow.

If the channel bottom is taken as the datum as shown in Fig. 16.25, then the total energy per unit weight of liquid will be,

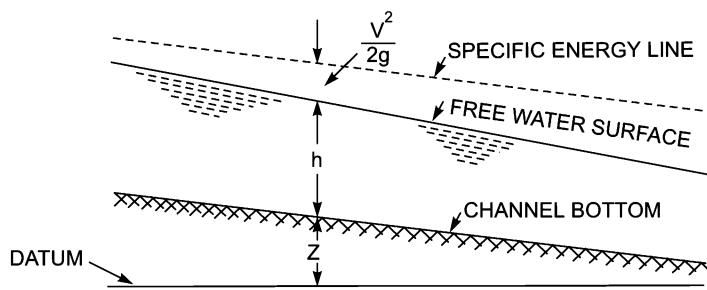


Fig. 16.25 Specific energy.

$$E = h + \frac{V^2}{2g} \quad \dots(16.21)$$

The energy given by equation (16.21) is known as **specific energy**. Hence specific energy of a flowing liquid is defined as energy per unit weight of the liquid with respect to the bottom of the channel.

**Specific Energy Curve.** It is defined as the curve which shows the variation of specific energy with depth of flow. It is obtained as :

From equation (16.21), the specific energy of a flowing liquid

$$E = h + \frac{V^2}{2g} = E_p + E_k$$

where  $E_p$  = Potential energy of flow =  $h$

$$E_k = \text{Kinetic energy of flow} = \frac{V^2}{2g}$$

Consider a rectangular channel in which a steady but non-uniform flow is taking place.

Let  $Q$  = discharge through the channel,

$b$  = width of the channel,

$h$  = depth of flow, and

$q$  = discharge per unit width.

Then

$$q = \frac{Q}{\text{width}} = \frac{Q}{b} = \text{constant} \quad (\because Q \text{ and } b \text{ are constant})$$

Velocity of flow,

$$V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{b \times h} = \frac{q}{h} \quad \left( \because \frac{Q}{b} = q \right)$$

Substituting the values of  $V$  in equation (16.21), we get

$$E = h + \frac{q^2}{2gh^2} = E_p + E_k \quad \dots(16.22)$$

Equation (16.22), gives the variation of specific energy ( $E$ ) with the depth of flow ( $h$ ). Hence for a given discharge  $Q$ , for different values of depth of flow, the corresponding values of  $E$  may be obtained. Then a graph between specific energy (along X-X axis) and depth of flow,  $h$  (along Y-Y axis) may be plotted.

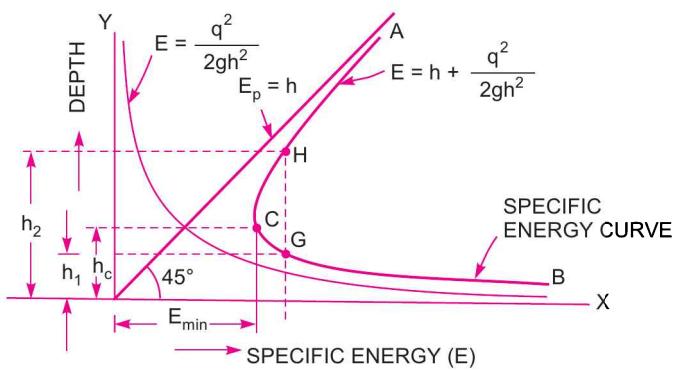


Fig. 16.26 Specific energy curve.

The specific energy curve may also be obtained by first drawing a curve for potential energy (*i.e.*,  $E_p = h$ ), which will be a straight line passing through the origin, making an angle of  $45^\circ$  with the  $X$ -axis as

shown in Fig. 16.26. Then drawing another curve for kinetic energy (*i.e.*,  $E_k = \frac{q^2}{2gh^2}$  or  $E_k = \frac{K}{h^2}$ ,

where  $K = \frac{q^2}{2g} = \text{constant}$ ) which will be a parabola as shown in Fig. 16.26. By combining these two

curves, we can obtain the specific energy curve. In Fig. 16.26, curve  $ACB$  denotes the specific energy curve.

**16.7.1 Critical Depth ( $h_c$ ).** Critical depth is defined as that depth of flow of water at which the specific energy is minimum. This is denoted by ' $h_c$ '. In Fig. 16.26, curve  $ACB$  is a specific energy curve and point  $C$  corresponds to the minimum specific energy. The depth of flow of water at  $C$  is known as critical depth. The mathematical expression for critical depth is obtained by differentiating the specific energy equation (16.22) with respect to depth of flow and equating the same to zero.

$$\text{or } \frac{dE}{dh} = 0, \quad \text{where } E = h + \frac{q^2}{2gh^2} \text{ from equation (16.22)}$$

$$\text{or } \frac{d}{dh} \left[ h + \frac{q^2}{2gh^2} \right] = 0 \quad \text{or} \quad 1 + \frac{q^2}{2g} \left( \frac{-2}{h^3} \right) = 0 \quad \left( \because \frac{q^2}{2g} \text{ is constant} \right)$$

$$\text{or } 1 - \frac{q^2}{gh^3} = 0 \quad \text{or} \quad 1 = \frac{q^2}{gh^3} \quad \text{or} \quad h^3 = \frac{q^2}{g}$$

$$\therefore h = \left( \frac{q^2}{g} \right)^{1/3}$$

But when specific energy is minimum, depth is critical and it is denoted by  $h_c$ . Hence critical depth is

$$h_c = \left( \frac{q^2}{g} \right)^{1/3} \quad \dots(16.23)$$

**16.7.2 Critical Velocity ( $V_c$ ).** The velocity of flow at the critical depth is known as critical velocity. It is denoted by  $V_c$ . The mathematical expression for critical velocity is obtained from equation (16.23) as

$$h_c = \left( \frac{q^2}{g} \right)^{1/3}$$

$$\text{Taking cube to both sides, we get } h_c^3 = \frac{q^2}{g} \quad \text{or} \quad gh_c^3 = q^2 \quad \dots(i)$$

$$\text{But } q = \text{Discharge per unit width} = \frac{Q}{b}$$

$$= \frac{\text{Area} \times V}{b} = \frac{b \times h \times V}{b} = h \times V = h_c \times V_c$$

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Substituting this value of  $q$  in (i),

∴

$$gh_c^3 = (h_c \times V_c)^2$$

or

$$gh_c^3 = h_c^2 \times V_c^2 \text{ or } gh_c = V_c^2$$

[Dividing by  $h_c^2$ ]

or

$$V_c = \sqrt{g \times h_c}$$

...(16.24)

**16.7.3 Minimum Specific Energy in Terms of Critical Depth.** Specific energy equation is given by equation (16.22)

$$E = h + \frac{q^2}{2gh^2}$$

When specific energy is minimum, depth of flow is critical and hence above equation becomes as

$$E_{\min} = h_c + \frac{q^2}{2gh_c^2} \quad \dots(ii)$$

$$\text{But from equation (16.23), } h_c = \left(\frac{q^2}{g}\right)^{1/3} \quad \text{or} \quad h_c^3 = \frac{q^2}{g}$$

Substituting the value of  $\frac{q^2}{g} = h_c^3$  in equation (ii), we get

$$E_{\min} = h_c + \frac{h_c^3}{2h_c^2} = h_c + \frac{h_c}{2} = \frac{3h_c}{2}. \quad \dots(16.25)$$

**Problem 16.33** Find the specific energy of flowing water through a rectangular channel of width 5 m when the discharge is 10 m<sup>3</sup>/s and depth of water is 3 m.

**Solution.** Given :

Width of rectangular channel,  $b = 5$  m

Discharge,  $Q = 10$  m<sup>3</sup>/s

Depth of water,  $h = 3$  m

Specific energy is given by equation (16.21), as

$$E = h + \frac{V^2}{2g}, \quad \text{where } V = \frac{Q}{\text{Area}} = \frac{10}{b \times h} = \frac{10}{5 \times 3} = \frac{2}{3}$$

$$\therefore E = 3 + \frac{\left(\frac{2}{3}\right)^2}{2 \times 9.81} = 3 + .0226 = \mathbf{3.0226 \text{ m. Ans.}}$$

**Problem 16.34** Find the critical depth and critical velocity of the water flowing through a rectangular channel of width 5 m, when discharge is 15 m<sup>3</sup>/s.

**Solution.** Given :

Width of channel,  $b = 5$  m

Discharge,  $Q = 15$  m<sup>3</sup>/s

$$\therefore \text{Discharge per unit width, } q = \frac{Q}{b} = \frac{15}{5} = 3 \text{ m}^2/\text{s}$$

Critical depth is given by equation (16.23) as

$$h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{3^2}{9.81} \right)^{1/3} = \left( \frac{9}{9.81} \right)^{1/3} = 0.972 \text{ m. Ans.}$$

Critical velocity is given by equation (16.24) as

$$V_c = \sqrt{g \times h_c} = \sqrt{9.81 \times 0.972} = 3.088 \text{ m/s. Ans.}$$

**Problem 16.35** The discharge of water through a rectangular channel of width 8 m, is 15 m<sup>3</sup>/s when depth of flow of water is 1.2 m. Calculate :

- (i) Specific energy of the flowing water,
- (ii) Critical depth and critical velocity,
- (iii) Value of minimum specific energy.

**Solution.** Given :

$$\text{Discharge, } Q = 15 \text{ m}^3/\text{s}$$

$$\text{Width, } b = 8 \text{ m}$$

$$\text{Depth, } h = 1.2 \text{ m}$$

$$\therefore \text{Discharge per unit width, } q = \frac{Q}{b} = \frac{15}{8} = 1.875 \text{ m}^2/\text{s}$$

$$\text{Velocity of flow, } V = \frac{Q}{\text{Area}} = \frac{15}{b \times h} = \frac{15}{8 \times 1.2} = 1.5625 \text{ m/s}$$

(i) Specific energy ( $E$ ) is given by equation (16.21) as

$$E = h + \frac{V^2}{2g} = 1.2 + \frac{1.5625^2}{8 \times 9.81} = 1.20 + 0.124 = 1.324 \text{ m. Ans.}$$

(ii) Critical depth ( $h_c$ ) is given by equation (16.23) as

$$h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{1.875^2}{9.81} \right)^{1/3} = 0.71 \text{ m. Ans.}$$

Critical velocity ( $V_c$ ) is given by equation (16.24) as

$$V_c = \sqrt{g \times h_c} = \sqrt{9.81 \times 0.71} = 2.639 \text{ m/s. Ans.}$$

(iii) Minimum specific energy ( $E_{\min}$ ) is given by equation (16.25)

$$E_{\min} = \frac{3h_c}{2} = \frac{3 \times 0.71}{2} = 1.065 \text{ m. Ans.}$$

**16.7.4 Critical Flow.** It is defined as that flow at which the specific energy is minimum or the flow corresponding to critical depth is defined as critical flow. Equation (16.24) gives the relation for critical velocity in terms of critical depth as

$$V_c = \sqrt{g \times h_c} \quad \text{or} \quad \frac{V_c}{\sqrt{gh_c}} = 1 \quad \left| \text{ where } \frac{V_c}{\sqrt{gh_c}} = \text{Froude number} \right.$$

$\therefore$  Froude number,  $F_e = 1.0$  for critical flow.

**16.7.5 Streaming Flow or Sub-critical Flow or Tranquil Flow.** When the depth of flow in a channel is greater than the critical depth ( $h_c$ ), the flow is said to be sub-critical flow or streaming flow or tranquil flow. For this type of flow the Froude number is less than one i.e.,  $F_e < 1.0$ .

**16.7.6 Super-critical Flow or Shooting Flow or Torrential Flow.** When the depth of flow in a channel is less than the critical depth ( $h_c$ ), the flow is said to be super-critical flow or shooting flow or torrential flow. For this type of flow the Froude number is greater than one i.e.,  $F_e > 1.0$ .

**16.7.7 Alternate Depths.** In the specific energy curve shown in Fig. 16.26, the point  $C$  corresponds to the minimum specific energy and the depth of flow at  $C$  is called critical depth. For any other value of the specific energy, there are two depths, one greater than the critical depth and other smaller than the critical depth. These two depths for a given specific energy are called the alternate depths. These depths are shown as  $h_1$  and  $h_2$  in Fig. 16.26. Or the depths corresponding to points  $G$  and  $H$  in Fig. 16.26 are called alternate depths.

**16.7.8 Condition for Maximum Discharge for a Given Value of Specific Energy.** The specific energy ( $E$ ) at any section of a channel is given by equation (16.21) as

$$E = h + \frac{V^2}{2g}, \text{ where } V = \frac{Q}{A} = \frac{Q}{b \times h}$$

$$\therefore E = h + \frac{Q^2}{b^2 \times h^2} \times \frac{1}{2g} = h + \frac{Q^2}{2gb^2h^2}$$

or  $Q^2 = (E - h) 2gb^2h^2 \quad \text{or} \quad Q = \sqrt{(E - h) 2gb^2h^2} = b\sqrt{2g(Eh^2 - h^3)}$

For maximum discharge,  $Q$ , the expression  $(Eh^2 - h^3)$  should be maximum. Or in other words,

$$\frac{d}{dh}(Eh^2 - h^3) = 0 \quad \text{or} \quad 2Eh - 3h^2 = 0 \quad (\because E \text{ is constant})$$

or  $2E - 3h = 0 \quad (\text{Dividing by } h)$

or  $h = \frac{2}{3}E \quad \dots(16.26)$

or  $E = \frac{3h}{2} \quad \dots(i)$

But from equation (16.25), specific energy is minimum when it is equal to  $\frac{3}{2}$  times the value of depth of critical flow. Here in equation (i), the specific energy ( $E$ ) is equal to  $\frac{3}{2}$  times the depth of flow. Thus equation (i) represents the minimum specific energy and  $h$  is the critical depth. Hence the condition for maximum discharge for given value of specific energy is that the depth of flow should be critical.

**Problem 16.36** The specific energy for a 3 m wide channel is to be 3 kg-m/kg. What would be the maximum possible discharge?

**Solution.** Given :

Width of channel,  $b = 3 \text{ m}$

Specific energy,  $E = 3 \text{ kg-m/kg} = 3 \text{ m}$

For the given value of specific energy, the discharge will be maximum, when depth of flow is critical. Hence from equation (16.26) for maximum discharge.

$$h_c = h = \frac{2}{3}E = \frac{2}{3} \times 3.0 = 2.0 \text{ m}$$

∴ Maximum discharge,  $Q_{\max}$  is given by

$$Q_{\max} = \text{Area} \times \text{Velocity} = (b \times \text{depth of flow}) \times \text{Velocity}$$

$$= (b \times h_c) \times V_c \quad (\because \text{At critical depth, Velocity will be critical})$$

where  $V_c$  is critical velocity and it is given by equation (16.24),

$$V_c = \sqrt{g \times h_c} = \sqrt{9.81 \times 2.0} = 4.4249 \text{ m/s}$$

$$\therefore Q_{\max} = (b \times h_c) \times V_c = (3 \times 2) \times 4.4249 = 26.576 \text{ m}^3/\text{s. Ans.}$$

**Problem 16.37** The specific energy for a 5 m wide rectangular channel is to be 4 Nm/N. If the rate of flow of water through the channel is 20 m<sup>3</sup>/s, determine the alternate depths of flow.

**Solution.** Given :

$$\text{Width of channel, } b = 5 \text{ m}$$

$$\text{Specific energy, } E = 4 \text{ Nm/N} = 4 \text{ m}$$

$$\text{Discharge, } Q = 20 \text{ m}^3/\text{s}$$

The specific energy ( $E$ ) is given by equation (16.21) as,

$$E = h + \frac{V^2}{2g}, \quad \text{where } V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{b \times h} = \frac{20}{5 \times h} = \frac{4}{h}$$

$$\therefore \text{Specific energy, } E = h + \frac{V^2}{2g} = h + \left(\frac{4}{h}\right)^2 \times \frac{1}{2g} = h + \frac{8}{g \times h^2}$$

But

$$E = 4.0$$

$$\text{Equating the two values of } E, 4 = h + \frac{8}{9.81 \times h^2} = h + \frac{0.8155}{h^2}$$

$$4h^2 = h^3 + .8155 \quad \text{or} \quad h^3 - 4h^2 + .8155 = 0$$

This is a cubic equation. Solving by trial and error, we get

$$h = 3.93 \text{ m and } 0.48 \text{ m. Ans.}$$

## ► 16.8 HYDRAULIC JUMP OR STANDING WAVE

Consider the flow of water over a dam as shown in Fig. 16.27. The height of water at the section 1-1 is small. As we move towards downstream, the height or depth of water increases rapidly over a short length of the channel. This is because at the section 1-1, the flow is a *shooting flow* as the depth of water at section 1-1 is less than critical depth. Shooting flow is an unstable type of flow and does not continue on the downstream side. Then this shooting will convert itself into a streaming or tranquil flow and hence depth of water will increase. This sudden increase of depth of water is called a hydraulic jump or a standing wave. Thus hydraulic jump is defined as :

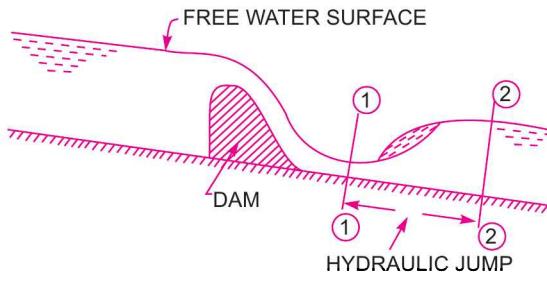


Fig. 16.27 Hydraulic jump.

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"The rise of water level, which takes place due to the transformation of the unstable shooting flow (Super-critical) to the stable streaming flow (sub-critical flow)."

When hydraulic jump takes place, a loss of energy due to eddy formation and turbulence occurs.

**16.8.1 Expression for Depth of Hydraulic Jump.** Before deriving an expression for the depth of hydraulic jump, the following assumptions are made :

1. The flow is uniform and pressure distribution is due to hydrostatic before and after the jump.
2. Losses due to friction on the surface of the bed of the channel are small and hence neglected.
3. The slope of the bed of the channel is small, so that the component of the weight of the fluid in the direction of flow is negligibly small.

Consider a hydraulic jump formed in a channel of horizontal bed as shown in Fig. 16.28. Consider two sections 1-1 and 2-2 before and after hydraulic jump.

Let  $d_1$  = Depth of flow at section 1-1,

$d_2$  = Depth of flow at section 2-2,

$V_1$  = Velocity of flow at section 1-1,

$V_2$  = Velocity of flow at section 2-2,

$\bar{Z}_1$  = Depth of centroid of area at section 1-1 below free surface,

$\bar{Z}_2$  = Depth of centroid of area at section 2-2 below free surface,

$A_1$  = Area of cross-section at section 1-1, and

$A_2$  = Area of cross-section at section 2-2.

Consider unit width of the channel.

The forces acting on the mass of water between sections 1-1 and 2-2 are :

(i) Pressure force,  $P_1$  on section 1-1,

(ii) Pressure force,  $P_2$  on section 2-2,

(iii) Frictional force on the floor of the channel, which is assumed to be negligible.

Let

$$q = \text{discharge per unit width}$$

$$= V_1 d_1 = V_2 d_2 \quad \dots(i)$$

Now pressure force  $P_1$  on section 1-1

$$= \rho g A_1 \bar{Z}_1 = \rho g \times d_1 \times 1 \times \frac{d_1}{2}$$

$$= \frac{\rho g d_1^2}{2} \quad \left( \because A_1 = d_1 \times 1, \bar{Z}_1 = \frac{d_1}{2} \right)$$

Similarly pressure force on section 2-2,

$$P_2 = \rho g A_2 \bar{Z}_2$$

$$= \rho g \times d_2 \times 1 \times \frac{d_2}{2} = \frac{\rho g d_2^2}{2}$$

Net force acting on the mass of water between sections 1-1 and 2-2

$$= P_2 - P_1 \quad | \because P_2 \text{ is greater than } P_1 \text{ and } d_2 \text{ is greater than } d_1$$

$$= \frac{\rho g d_2^2}{2} - \frac{\rho g d_1^2}{2} = \frac{\rho g}{2} [d_2^2 - d_1^2] \quad \dots(ii)$$

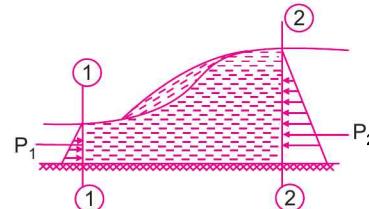


Fig. 16.28 Hydraulic jump.

But from momentum principle, the net force acting on a mass of fluid must be equal to the rate of change of momentum in the same section.

$\therefore$  Rate of change of momentum in the direction of force

= mass of water per sec  $\times$  change of velocity in direction of force

Now mass of water per second =  $\rho \times$  discharge per unit width  $\times$  width

$$= \rho \times q \times 1 = \rho q \text{ m}^3/\text{s}$$

Change of velocity in the direction of force =  $(V_1 - V_2)$

[as net force is acting from right to left, the change of velocity should be taken from right to left and hence is equal to  $(V_1 - V_2)$ ]

$\therefore$  Rate of change of momentum in the direction of force =  $\rho q(V_1 - V_2)$  ... (iii)

Hence according to momentum principle, the expression given by equation (ii) is equal to the expression given by equation (iii)

$$\text{or } \frac{\rho g}{2}(d_2^2 - d_1^2) = \rho q(V_1 - V_2)$$

$$\text{But from equation (i), } V_1 = \frac{q}{d_1} \text{ and } V_2 = \frac{q}{d_2}$$

$$\therefore \frac{\rho g}{2}(d_2^2 - d_1^2) = \rho q \left( \frac{q}{d_1} - \frac{q}{d_2} \right)$$

$$\text{or } \frac{g}{2}(d_2 + d_1)(d_2 - d_1) = q^2 \left( \frac{d_2 - d_1}{d_1 d_2} \right) \quad (\text{Dividing by } \rho)$$

$$\text{or } \frac{g}{2}(d_2 + d_1) = \frac{q^2}{d_1 d_2} \quad [\text{Dividing by } (d_2 - d_1)]$$

$$\text{or } (d_2 + d_1) = \frac{2q^2}{gd_1 d_2} \quad \dots(iv)$$

Multiplying both sides by  $d_2$ , we get

$$d_2^2 + d_1 d_2 = \frac{2q^2}{gd_1} \quad \text{or} \quad d_2^2 + d_1 d_2 - \frac{2q^2}{gd_1} = 0 \quad \dots(v)$$

Equation (v) is a quadratic equation in  $d_2$  and hence its solution is

$$d_2 = \frac{-d_1 \pm \sqrt{d_1^2 - 4 \times 1 \times \left( \frac{-2q^2}{gd_1} \right)}}{2 \times 1}$$

$$= \frac{-d_1 \pm \sqrt{d_1^2 + \frac{8q^2}{gd_1}}}{2} = -\frac{d_1}{2} \pm \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}}$$

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The two roots of the equation are  $-\frac{d_1}{2} - \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}}$  and  $-\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}}$

First root is not possible as it gives -ve depth. Hence

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}} \quad \dots(16.27)$$

$$= -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2 \times (V_1 d_1)^2}{gd_1}} \quad \{ \because q_1 = V_1 d_1 \}$$

$$= -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2V_1^2 d_1}{g}} \quad \dots(16.28)$$

$$\therefore \text{Depth of Hydraulic jump} = (d_2 - d_1). \quad \dots(16.29)$$

**16.8.2 Expression for Loss of Energy Due to Hydraulic Jump.** As mentioned in Art. 16.8 that when hydraulic jump takes place, a loss of energy due to eddies formation and turbulence occurs. This loss of energy is equal to the difference of specific energies at sections 1-1 and 2-2.

Or loss of energy due to hydraulic jump,

$$h_L = E_1 - E_2 \\ = \left( d_1 + \frac{V_1^2}{2g} \right) - \left( d_2 + \frac{V_2^2}{2g} \right) \quad \left( \because E_1 = d_1 + \frac{V_1^2}{2g} \text{ and so } E_2 \right)$$

$$= \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right) - (d_2 - d_1) \\ = \left( \frac{q^2}{2gd_1^2} - \frac{q^2}{2gd_2^2} \right) - (d_2 - d_1) \quad \left( \because V_1 = \frac{q}{d_1} \text{ and } V_2 = \frac{q}{d_2} \right)$$

$$= \frac{q^2}{2g} \left[ \frac{1}{d_1^2} - \frac{1}{d_2^2} \right] - [d_2 - d_1] = \frac{q^2}{2g} \left[ \frac{d_2^2 - d_1^2}{d_1^2 d_2^2} \right] - [d_2 - d_1] \quad \dots(vi)$$

$$\text{But from equation (iv), } q^2 = gd_1 d_2 \frac{(d_2 + d_1)}{2}$$

Substituting the value of  $q^2$  in equation (vi), we get

$$\begin{aligned} \text{Loss of energy, } h_L &= gd_1 d_2 \frac{(d_2 + d_1)}{2} \times \frac{d_2^2 - d_1^2}{2gd_1^2 d_2^2} - (d_2 - d_1) = \frac{(d_2 + d_1)(d_2^2 - d_1^2)}{4d_1 d_2} - (d_2 - d_1) \\ &= \frac{(d_2 + d_1)(d_2 + d_1)(d_2 - d_1)}{4d_1 d_2} - (d_2 - d_1) = (d_2 - d_1) \left[ \frac{(d_2 + d_1)^2}{4d_1 d_2} - 1 \right] \end{aligned}$$

$$\begin{aligned}
 &= (d_2 - d_1) \left[ \frac{d_2^2 + d_1^2 + 2d_1d_2 - 4d_1d_2}{4d_1d_2} \right] = (d_2 - d_1) \frac{[d_2 - d_1]^2}{4d_1d_2} \\
 \therefore h_L &= \frac{[d_2 - d_1]^3}{4d_1d_2}. \quad \dots(16.30)
 \end{aligned}$$

### 16.8.3 Expression for Depth of Hydraulic Jump in Terms of Upstream Froude Number.

Let  $V_1$  = Velocity of flow on the upstream side,  
and  $d_1$  = Depth of flow on upstream side,

Then Froude Number  $(F_e)_1$  on the upstream side of the jump is given by

$$(F_e)_1 = \frac{V_1}{\sqrt{gd_1}} \quad \dots(vii)$$

Now the depth of flow after the hydraulic jump is  $d_2$  and it is given by equation (16.28) as

$$\begin{aligned}
 d_2 &= -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2V_1^2d_1}{g}} = -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} \left( 1 + \frac{8V_1^2}{gd_1} \right)} \\
 &= -\frac{d_1}{2} + \frac{d_1}{2} \sqrt{1 + \frac{8V_1^2}{gd_1}} \quad \dots(viii)
 \end{aligned}$$

But from equation (vii),  $(F_e)_1 = \frac{V_1}{\sqrt{gd_1}}$  or  $(F_e)_1^2 = \frac{V_1^2}{gd_1}$

Substituting this value in equation (viii), we get

$$d_2 = -\frac{d_1}{2} + \frac{d_1}{2} \sqrt{1 + 8(F_e)_1^2} = \frac{d_1}{2} \left( \sqrt{1 + 8(F_e)_1^2} - 1 \right) \quad \dots(16.31)$$

**16.8.4 Length of Hydraulic Jump.** This is defined as the length between the two sections where one section is taken before the hydraulic jump and the second section is taken immediately after the jump. For a rectangular channel from experiments, it has been found equal to 5 to 7 times the height of the hydraulic jump.

**Problem 16.38** The depth of flow of water, at a certain section of a rectangular channel of 4 m wide, is 0.5 m. This discharge through the channel is 16 m<sup>3</sup>/s. If a hydraulic jump takes place on the downstream side, find the depth of flow after the jump.

**Solution.** Given :

Width of channel,  $b = 4 \text{ m}$

Depth of flow before jump,  $d_1 = 0.5 \text{ m}$

Discharge,  $Q = 16 \text{ m}^3/\text{s}$

$$\therefore \text{Discharge per unit width, } q = \frac{Q}{b} = \frac{16}{4} = 4 \text{ m}^2/\text{s}$$

Let the depth of flow after jump =  $d_2$

Depth of flow after the jump is given by equation (16.27), as

$$\begin{aligned}
 d_2 &= -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}} = -\frac{0.5}{2} + \sqrt{\frac{0.5^2}{4} + \frac{2 \times 4^2}{9.81 \times 0.5}} \\
 &= -0.25 + \sqrt{0.0625 + 6.5239} = -0.25 + 2.566 = 2.316 \text{ m. Ans.}
 \end{aligned}$$

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**Problem 16.39** The depth of flow of water, at a certain section of a rectangular channel of 2 m wide, is 0.3 m. The discharge through the channel is 1.5 m<sup>3</sup>/s. Determine whether a hydraulic jump will occur, and if so, find its height and loss of energy per kg of water.

**Solution.** Given :

$$\text{Depth of flow, } d_1 = 0.3 \text{ m}$$

$$\text{Width of channel, } b = 2 \text{ m}$$

$$\text{Discharge, } Q = 1.5 \text{ m}^3/\text{s}$$

$$\text{Discharge per unit width, } q = \frac{Q}{b} = \frac{1.5}{2.0} = 0.75 \text{ m}^2/\text{s.}$$

Hydraulic jump will occur if the depth of flow on the upstream side is less than the critical depth on upstream side or if the Froude number on the upstream side is more than one.

Critical depth ( $h_c$ ) is given by equation (16.23) as

$$h_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{0.75^2}{9.81} \right)^{1/3} = 0.3859$$

Now the depth on the upstream side is 0.3 m. This depth is less than critical depth and hence hydraulic jump will occur.

The depth of flow after hydraulic jump is given by equation (16.27) as

$$\begin{aligned} d_2 &= -\frac{d_1}{2} + \sqrt{\frac{d_1^2 + 2q^2}{4gd_1}} = -\frac{0.3}{2} + \sqrt{\frac{0.3^2 + 2 \times 0.75^2}{4 \times 9.81 \times 0.3}} \\ &= -1.5 + \sqrt{0.0255 + .3822} = -0.15 + .6362 = 0.4862 \text{ m} \end{aligned}$$

$$\therefore \text{Height of hydraulic jump} = d_2 - d_1 = 0.4862 - 0.30 = \mathbf{0.1862 \text{ m. Ans.}}$$

Loss of energy per kg of water due to hydraulic jump is given by equation (16.30) as

$$\begin{aligned} h_L &= \frac{(d_2 - d_1)^3}{4d_1 d_2} = \frac{[0.4862 - 0.30]^3}{4 \times 0.4862 \times 0.30} \\ &= \frac{0.1862^3}{4 \times 0.4862 \times 0.30} = \mathbf{0.01106 \text{ m-kg/kg. Ans.}} \end{aligned}$$

**Problem 16.40** A sluice gate discharges water into a horizontal rectangular channel with a velocity of 10 m/s and depth of flow of 1 m. Determine the depth of flow after the jump and consequent loss in total head.

**Solution.** Given :

$$\text{Velocity of flow before hydraulic jump, } V_1 = 10 \text{ m/s}$$

$$\text{Depth of flow before hydraulic jump, } d_1 = 1 \text{ m}$$

$$\text{Discharge per unit width, } q = V_1 \times d_1 = 10 \times 1 = 10 \text{ m}^2/\text{s}$$

The depth of flow after jump is given by equation (16.27) as

$$\begin{aligned} d_2 &= -\frac{d_1}{2} + \sqrt{\frac{d_1^2 + 2q^2}{4gd_1}} = -\frac{1.0}{2} + \sqrt{\frac{1^2 + 2 \times 10^2}{4 \times 9.81 \times 1}} \\ &= -0.50 + \sqrt{0.25 + 20.387} = \mathbf{4.043 \text{ m. Ans.}} \end{aligned}$$

Loss in total head is given by equation (16.30) as

$$h_L = \frac{(d_2 - d_1)^3}{4d_1 d_2} = \frac{(4.043 - 1.0)^3}{4 \times 1.0 \times 4.043} = \mathbf{1.742 \text{ m. Ans.}}$$

**Problem 16.41** A sluice gate discharges water into a horizontal rectangular channel with a velocity of 6 m/s and depth of flow is 0.4 m. The width of the channel is 8 m. Determine whether a hydraulic jump will occur, and if so, find its height and loss of energy per kg of water. Also determine the power lost in the hydraulic jump.

**Solution.** (i) Given :

$$\text{Velocity of flow, } V_1 = 6 \text{ m/s}$$

$$\text{Depth of flow, } d_1 = 0.4 \text{ m}$$

$$\text{Width of channel, } b = 8 \text{ m}$$

$$\therefore \text{Discharge per unit width, } q = \frac{Q}{b} = \frac{V_1 \times \text{area}}{b} = \frac{V_1 \times d_1 \times b}{b} \\ = V_1 \times d_1 = 6 \times 0.4 = 2.4 \text{ m}^2/\text{s}$$

Froude number on the upstream side,

$$(F_e)_1 = \frac{V_1}{\sqrt{gd_1}} = \frac{6.0}{\sqrt{9.81 \times 0.4}} = 3.0289 \approx 3.029.$$

As Froude number is more than one, the flow is shooting on the upstream side. Shooting flow is unstable flow and it will convert itself into streaming flow by raising its height and hence hydraulic jump will take place.

(ii) Let the depth of hydraulic jump =  $d_2$

Using equation (16.31), we have

$$d_2 = \frac{d_1}{2} \left( \sqrt{1 + 8(F_e)_1^2} - 1 \right) = \frac{0.4}{2} \left( \sqrt{1 + 8 \times 3.029^2} - 1 \right) = 1.525 \text{ m}$$

$$\therefore \text{Height of hydraulic jump} = d_2 - d_1 = 1.525 - 0.4 = 1.125 \text{ m. Ans.}$$

(iii) Loss of energy per kg of water is given by equation (16.30)

$$h_L = \frac{(d_2 - d_1)^3}{4d_1 d_2} = \frac{(1.525 - 0.4)^3}{4 \times 0.4 \times 1.525} = 0.5835 \text{ m-kg/kg of water. Ans.}$$

$$(iv) \text{Power lost in kW} = \frac{\rho g \times Q \times h_L}{1000}, \text{ where } Q = V \times \text{area}$$

$$= V_1 \times d_1 \times b = 6 \times 0.4 \times 8 = 19.2 \text{ m}^3/\text{s}$$

$$\therefore \text{Power, } P = \frac{1000 \times 9.81 \times 19.2 \times 0.5835}{1000} = 109.9 \text{ kW. Ans.}$$

**Problem 16.42** A hydraulic jump forms at the downstream end of spillway carrying 17.93 m<sup>3</sup>/s discharge. If the depth before jump is 0.80 m, determine the depth after the jump and energy loss.

**Solution.** Given :

$$\text{Discharge } Q = 17.93 \text{ m}^3/\text{s}$$

$$\text{Depth before jump, } d_1 = 0.8 \text{ m}$$

Taking width  $b = 1 \text{ m}$ , we get

$$\text{Discharge per unit width, } q = \frac{17.93}{1} = 17.93$$

Let  $d_2 = \text{Depth after jump and } h_L = \text{Loss of energy.}$

$$\text{Using equation (16.27), we get } d_2 = -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}} = -\frac{0.8}{2} + \sqrt{\frac{0.8^2}{4} + \frac{2 \times 17.93^2}{9.81 \times 0.8}}$$

$$= -0.4 + \sqrt{0.16 + 81.927} = -0.4 + 9.06 = 8.66 \text{ m. Ans.}$$

Using equation (16.30) for loss of energy,

$$h_L = \frac{(d_2 - d_1)^3}{4d_1 d_2} = \frac{(8.66 - 0.8)^3}{4 \times 0.8 \times 8.66} = 17.52 \text{ m. Ans.}$$

### ► 16.9 GRADUALLY VARIED FLOW (G.V.F.)

If the depth of flow in a channel changes gradually over a long length of the channel, the flow is said to be gradually varied flow and is denoted by G.V.F.

**16.9.1 Equation of Gradually Varied Flow.** Before deriving an equation for gradually varied flow, the following assumptions are made :

1. The bed slope of the channel is small,
2. The flow is steady and hence discharge  $Q$  is constant,
3. Accelerative effect is negligible and hence hydrostatic pressure distribution prevails over channel cross-section.
4. The energy correction factor,  $\alpha$  is unity.
5. The roughness co-efficient is constant for the length of the channel and it does not depend on the depth of flow.
6. The formulae, such as Chezy's formula, Manning's formula, which are applicable, to the uniform flow are also applicable to the gradually varied flow for determining the slope of energy line.
7. The channel is prismatic.

Consider a rectangular channel having gradually varied flow as shown in Fig. 16.29. The depth of flow is gradually decreasing in the direction of flow.

Let

$Z$  = height of bottom of channel above datum

$h$  = depth of flow,

$V$  = mean velocity of flow,

$i_b$  = slope of the channel bed,

$i_e$  = slope of the energy line,

$b$  = width of channel, and

$Q$  = discharge through the channel

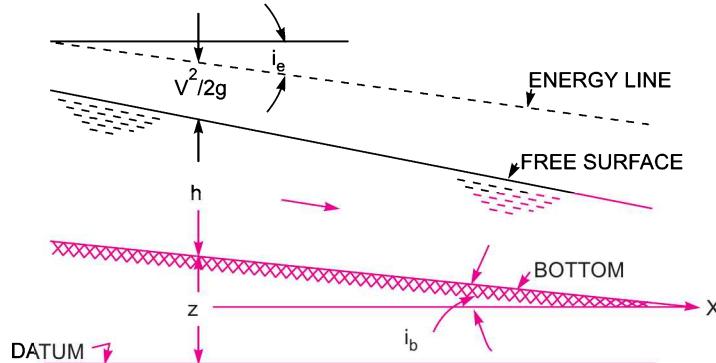


Fig. 16.29 Equation for gradually varied flow.

The energy equation at any section is given by Bernoulli's equation,

$$E = Z + h + \frac{V^2}{2g} \quad \dots(i)$$

Differentiating this equation with respect to  $x$ , where  $x$  is measured along the bottom of the channel in the direction of flow, we get

$$\frac{dE}{dx} = \frac{dZ}{dx} + \frac{dh}{dx} + \frac{d}{dx} \left( \frac{V^2}{2g} \right) \quad \dots(ii)$$

$$\begin{aligned} \text{Now } \frac{d}{dx} \left( \frac{V^2}{2g} \right) &= \frac{d}{dx} \left( \frac{Q^2}{A^2 \times 2g} \right) & \left( \because V = \frac{Q}{A} = \frac{Q}{b \times h} \right) (\text{as } A = b \times h) \\ &= \frac{d}{dx} \left( \frac{Q^2}{b^2 h^2 \times 2g} \right) = \frac{Q^2}{b^2 \times 2g} \frac{d}{dx} \left( \frac{1}{h^2} \right) (\because Q, b \text{ and } g \text{ are constant}) \\ &= \frac{Q^2}{b^2 \times 2g} \frac{d}{dh} \left[ \frac{1}{h^2} \right] \frac{dh}{dx} = \frac{Q^2}{b^2 \times 2g} \left[ \frac{-2}{h^3} \right] \frac{dh}{dx} = \frac{-2Q^2}{b^2 \times 2gh^3} \frac{dh}{dx} \\ &= -\frac{Q^2}{b^2 h^2 \times gh} \frac{dh}{dx} = -\frac{V^2}{gh} \frac{dh}{dx} & \left[ \because \frac{Q}{bh} = V \right] \end{aligned}$$

Substituting the value of  $\frac{d}{dx} \left( \frac{V^2}{2g} \right)$  in equation (ii), we get

$$\frac{dE}{dx} = \frac{dZ}{dx} + \frac{dh}{dx} - \frac{V^2}{gh} \frac{dh}{dx} = \frac{dZ}{dx} + \frac{dh}{dx} \left[ 1 - \frac{V^2}{gh} \right] \quad \dots(iii)$$

Now  $\frac{dE}{dx}$  = slope of the energy line =  $-i_e$

and  $\frac{dZ}{dx}$  = slope of the bed of the channel =  $-i_b$

-ve sign with  $i_e$  and  $i_b$  is taken as with the increase of  $x$ , the value of  $E$  and  $Z$  decreases.

Substituting the value of  $\frac{dE}{dx}$  and  $\frac{dZ}{dx}$  in equation (iii), we get

$$-i_e = -i_b + \frac{dh}{dx} \left[ 1 - \frac{V^2}{gh} \right] \quad \text{or} \quad i_b - i_e = \frac{dh}{dx} \left[ 1 - \frac{V^2}{gh} \right]$$

$$\begin{aligned} \text{or } \frac{dh}{dx} &= \frac{(i_b - i_e)}{\left[ 1 - \frac{V^2}{gh} \right]} & \dots(16.32) \\ &= \frac{(i_b - i_e)}{\left[ 1 - (F_e)^2 \right]} & \left[ \because \frac{V}{\sqrt{gh}} = F_e \right] \dots(16.33) \end{aligned}$$

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As  $h$  is the depth of flow and  $x$  is the distance measured along the bottom of the channel hence  $\frac{dh}{dx}$  represents the variation of the water depth along the bottom of the channel.

This is also called the slope of the free water surface. Thus :

(i) When  $\frac{dh}{dx} = 0$ ,  $h$  is constant or depth of the water above the bottom of channel is constant. It

means that free surface of water is parallel to the bed of the channel.

(ii) When  $\frac{dh}{dx} > 0$  or  $\frac{dh}{dx}$  is +ve, it means the depth of water increases in the direction of flow. The profile of the water so obtained is called *back water curve*.

(iii) When  $\frac{dh}{dx} < 0$  or  $\frac{dh}{dx}$  is -ve, it means that the depth of water decreases in the direction of flow. The profile of the water so obtained is called *drop down curve*.

**Problem 16.43** Find the rate of change of depth of water in a rectangular channel of 10 m wide and 1.5 m deep, when the water is flowing with a velocity of 1 m/s. The flow of water through the channel of bed slope 1 in 4000, is regulated in such a way that energy line is having a slope of .00004.

**Solution.** Given :

$$\text{Width of channel, } b = 10 \text{ m}$$

$$\text{Depth of channel, } h = 1.5 \text{ m}$$

$$\text{Velocity of flow, } V = 1 \text{ m/s}$$

$$\text{Bed slope, } i_b = \frac{1}{4000} = .00025$$

$$\text{Slope of energy line, } i_e = .00004$$

$$\text{Let rate of change of depth of water} = \frac{dh}{dx}$$

$$\text{Using equation (16.32) as } \frac{dh}{dx} = \frac{(i_b - i_e)}{\left(1 - \frac{V^2}{gh}\right)} = \frac{.00025 - .00004}{\left(1 - \frac{1 \times 1}{9.81 \times 1.5}\right)} = \frac{.00021}{.966} = .000217. \text{ Ans.}$$

**Problem 16.44** Find the slope of the free water surface in a rectangular channel of width 20 m, having depth of flow 5 m. The discharge through the channel is  $50 \text{ m}^3/\text{s}$ . The bed of the channel is having a slope of 1 in 4000. Take the value of Chezy's constant  $C = 60$ .

**Solution.** Given :

$$\text{Width of channel, } b = 20 \text{ m}$$

$$\text{Depth of flow, } h = 5 \text{ m}$$

$$\text{Discharge, } Q = 50 \text{ m}^3/\text{s}$$

$$\text{Bed slope, } i_b = \frac{1}{4000} = .00025$$

$$\text{Chezy's constant, } C = 60$$

$$\text{The discharge, } Q \text{ is given by } Q = V \times \text{Area} = C \sqrt{mi} \times A = AC \sqrt{mi}$$

$$\text{where } A = \text{Area of flow} = b \times h = 20 \times 5 = 100 \text{ m}^2,$$

$$m = \text{hydraulic mean depth} = \frac{A}{P} = \frac{100}{b + 2h} = \frac{100}{20 + 2 \times 5} = \frac{100}{30} = \frac{10}{3} \text{ m,}$$

$i = i_e = \text{slope of energy line.}$

The slope of the energy line\* is determined from Chezy's formula

$$50 = 100 \times 60 \times \sqrt{\frac{10}{3} \times i_e} = 10954.45 \sqrt{i_e}$$

or  $i_e = \left( \frac{50}{10954.45} \right)^2 = 0.0000208$

The slope of free water surface =  $\frac{dh}{dx}$

Using equation (16.32) as  $\frac{dh}{dx} = \frac{i_b - i_e}{1 - \frac{V^2}{gh}} = \frac{0.00025 - 0.0000208}{1 - \frac{V^2}{9.81 \times 5.0}}$

Now  $V = \frac{Q}{\text{Area}} = \frac{50}{b \times h} = \frac{50}{20 \times 5} = 0.5$

$$\frac{dh}{dx} = \frac{0.00025 - 0.0000208}{1 - \frac{0.5 \times 0.5}{9.81 \times 5.0}} = \frac{0.0002292}{0.9949} = \mathbf{0.00023. Ans.}$$

**16.9.2 Back Water Curve and Afflux.** Consider the flow over a dam as shown in Fig. 16.30. On the upstream side of the dam, the depth of water will be rising. If there had not been any obstruction (such as dam) in the path of flow of water in the channel, the depth of water would have been constant as shown by dotted line parallel to the bed of the channel in Fig. 16.30. Due to obstruction, the water level rises and it has maximum depth from the bed at some section.

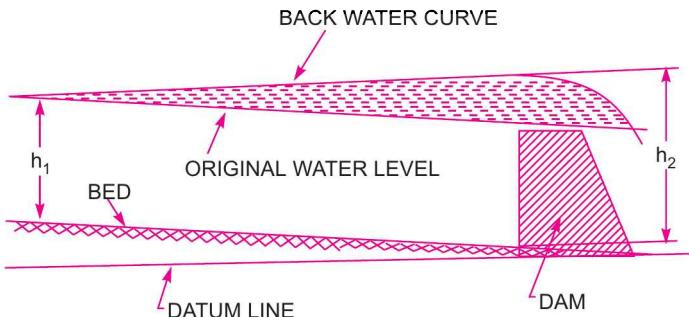


Fig. 16.30 Back water curve and afflux.

Let  $h_1$  = depth of water at the point, where the water starts rising up, and  
 $h_2$  = maximum height of rising water from bed.

Then  $(h_2 - h_1)$  = afflux. Thus afflux is defined as the maximum increase in water level due to obstruction in the path of flow of water. The profile of the rising water on the upstream side of the

\* Please refer to Art. 16.9.1 point number 6.

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dam is called *back water curve*. The distance along the bed of the channel between the section where water starts rising to the section where water is having maximum height is known as *length of back water curve*.

**16.9.3 Expression for the Length of Back Water Curve.** Consider the flow of water through a channel in which depth of water is rising as shown in Fig. 16.31. Let the two sections 1-1 and 2-2 are at such a distance that the distance between them represents the length of back water curve.

Let

$h_1$  = depth of flow at section 1-1,

$V_1$  = velocity of flow at section 1-1,

$h_2$  = depth of flow at section 2-2,

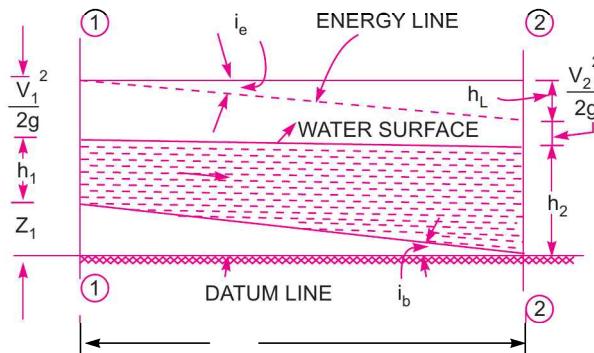


Fig. 16.31 Length of back water curve.

$V_2$  = velocity of flow at section 2-2,

$i_b$  = bed slope,

$i_e$  = energy line slope, and

$L$  = length of back water curve.

Applying Bernoulli's equation at sections 1-1 and 2-2,

$$Z_1 + h_1 + \frac{V_1^2}{2g} = Z_2 + h_2 + \frac{V_2^2}{2g} + h_L \quad \dots(i)$$

where  $h_L$  = Loss of energy due to friction =  $i_e \times L$

Also taking datum line passing through the bed of the channel at section 2-2. Then  $Z_2 = 0$

$$\therefore \text{Equation (i) becomes as } Z_1 + h_1 + \frac{V_1^2}{2g} = h_2 + \frac{V_2^2}{2g} + i_e \times L$$

From Fig. 16.31,  $Z_1 = i_b \times L$

$$\therefore i_b \times L + h_1 + \frac{V_1^2}{2g} = h_2 + \frac{V_2^2}{2g} + i_e \times L$$

or

$$i_b \times L - i_e \times L = \left( h_2 + \frac{V_2^2}{2g} \right) - \left( h_1 + \frac{V_1^2}{2g} \right)$$

or  $L(i_b - i_e) = E_2 - E_1$ , where  $E_2 = h_2 + \frac{V_2^2}{2g}$ ,  $E_1 = h_1 + \frac{V_1^2}{2g}$

$$\therefore L = \frac{E_2 - E_1}{i_b - i_e} \quad \dots(16.34)$$

Equation (16.34) is used to calculate the length of back water curve. The value of  $i_e$  (slope of energy line) is calculated either by Manning's formula or by Chezy's formula. The mean values of velocity, depth of flow, hydraulic mean depth etc., are used between sections 1-1 and 2-2 for calculating the value of  $i_e$ .

**Problem 16.45** Determine the length of the back water curve caused by an afflux of 2.0 m in a rectangular channel of width 40 m and depth 2.5 m. The slope of the bed is given as 1 in 11000. Take Manning's  $N = 0.03$ .

**Solution.** Given :

Width of channel,  $b = 40 \text{ m}$

Afflux,  $(h_2 - h_1) = 2.0 \text{ m}$

Depth of channel,  $h_1 = 2.5 \text{ m}$

$\therefore h_2 = 2.0 + 2.5 = 4.5 \text{ m}$

Bed slope,  $i_b = \frac{1}{11000} = 0.0000909$

Manning's,  $N = 0.03$

Area of flow at section 1,  $A_1 = b \times h_1 = 40 \times 2.5 = 100 \text{ m}^2$

Wetted perimeter,  $P_1 = b + 2h_1 = 40 + 2 \times 2.5 = 45 \text{ m}$

$\therefore$  Hydraulic mean depth,  $m_1 = \frac{A_1}{P_1} = \frac{100}{45} = 2.22 \text{ m}$

Using Manning's formula,  $V = \frac{1}{N} \cdot m^{2/3} i_b^{1/2}$

$\therefore$  Velocity at section 1,  $V_1 = \frac{1}{N} m_1^{2/3} i_b^{1/2} = \frac{1}{0.03} \times 2.22^{2/3} \times 0.0000909^{1/2}$

$$= \frac{1}{0.03} \times 1.7 \times 0.009534 = 0.54 \text{ m/s}$$

Specific energy at section 1,  $E_1 = \frac{V_1^2}{2g} + h_1 = \frac{0.54^2}{2 \times 9.81} + 2.5 = 2.5148 \text{ m}$

From continuity, velocity at section 2 is given as

$$V_1 A_1 = V_2 A_2$$

$\therefore V_2 = \frac{V_1 \times A_1}{A_2} = \frac{0.54 \times 100}{b \times h_2} = \frac{0.54 \times 100}{40 \times 4.5} = 0.3 \text{ m/s}$

where area  $A_2 = b \times h_2 = 40 \times 4.5 = 180 \text{ m}^2$

Wetted perimeter at section 2,  $P_2 = b + 2h_2 = 40 + 2 \times 4.5 = 49 \text{ m}$

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$$\therefore m_2 = \frac{A_2}{P_2} = \frac{180}{49} = 3.673 \text{ m}$$

$$\text{Specific energy at section 2, } E_2 = h_2 + \frac{V_2^2}{2g} = 4.5 + \frac{0.3^2}{2 \times 9.81} = 4.504 \text{ m}$$

To find average velocity ( $V_{av}$ ), first find average depth ( $h_{av}$ ) as

$$h_{av} = \frac{h_1 + h_2}{2} = \frac{2.5 + 4.5}{2} = 3.5 \text{ m}$$

$$\therefore V_{av} = \frac{V_1 A_1}{A_{av}} = \frac{V_1 \times b \times h_1}{b \times h_{av}} = \frac{V_1 \times h_1}{h_{av}} = \frac{0.54 \times 2.5}{3.5} = 0.3857 \text{ m/s}$$

$$\text{Also } m_{av} = \frac{m_1 + m_2}{2} = \frac{2.22 + 3.673}{2} = 2.9465$$

To find the value of  $i_e$ , use Manning's formula as

$$V_{av} = \frac{1}{N} m_{av}^{2/3} \times i_e^{1/2}$$

$$\text{or } 0.3857 = \frac{1}{0.03} \times 2.9465^{2/3} \times i_e^{1/2} = 68.534 i_e^{1/2}$$

$$\text{or } i_e = \left( \frac{0.3857}{68.534} \right)^2 = 0.00003167$$

The length of back water curve ( $L$ ) is obtained from equation (16.34)

$$\begin{aligned} L &= \frac{E_2 - E_1}{i_b - i_e} = \frac{4.504 - 2.5148}{0.0000909 - 0.00003167} \\ &= \frac{1.9892}{0.00005923} = 33584.3 \text{ m. Ans.} \end{aligned}$$

### HIGHLIGHTS

- If the depth of flow, velocity of flow, slope of the bed of channel and cross-section remain constant, the flow is called uniform, otherwise it is called non-uniform flow.
- Non-uniform flow is also called varied flow. If the depth of flow changes abruptly over a small length of channel, the flow is called rapidly varied flow. If the depth of flow changes gradually over a long length of the channel, the flow is said to be gradually varied flow.
- If Reynold number for open channel flow is less than 500, the flow is said to be laminar and if  $R_e$  is more than 2000, the flow is said to be turbulent.
- If Froude number is less than 1.0, the flow is sub-critical or streaming. If  $F_e = 1.0$ , the flow is critical. If  $F_e > 1.0$ , the flow is super-critical or shooting.
- Velocity of Chezy's formula is given as  $V = C \sqrt{mi}$

where  $C$  = Chezy's constant,  $m$  = Hydraulic mean depth =  $\frac{\text{Area}}{\text{Wetted perimeter}}$ ,

$i$  = Slope of the bed.

6. The value of Chezy's constant,  $C$  is given by empirical formulae as :

$$(i) \quad C = \frac{157.6}{1.81 + \frac{K}{\sqrt{m}}} \quad \dots \text{Bazin Formula}$$

where  $K$  = Bazin's constant,  $m$  = Hydraulic mean depth

$$(ii) \quad C = \frac{23 + \frac{0.00155}{i} + \frac{1}{N}}{1 + \left(23 + \frac{0.00155}{i}\right) \frac{N}{\sqrt{m}}} \quad \dots \text{Kutter's Formula}$$

where  $N$  = Kutter's constant.

$$(iii) \quad C = \frac{1}{N} m^{1/6} \quad \dots \text{Manning's Formula}$$

where  $N$  = Manning's constant = Kutter's constant.

7. Most economical section is one that gives maximum discharge for a given values of cross-section area, slope of the bed and co-efficient of resistance.

8. Conditions for maximum discharge through :

- (a) Rectangular section,

$$(i) \quad b = 2d \quad (ii) \quad m = \frac{d}{2}$$

- (b) Trapezoidal section,

$$(i) \quad \text{Half of top width} = \text{Sloping side or } \frac{b + 2nd}{2} = d \sqrt{n^2 + 1}$$

$$(ii) \quad m = \frac{d}{2}$$

(iii) A semi-circle drawn from the mid-point of the top width with radius equal to depth of flow will touch the three sides of the channel.

9. Best side slope for most economical trapezoidal section is,

$$\theta = 60^\circ \text{ or } n = \frac{1}{\sqrt{3}} = \frac{1}{\tan \theta}.$$

10. For circular sections, area cannot be maintained constant and hence there are two different conditions, one is for maximum velocity and other for maximum discharge.

11. Condition for maximum velocity through a circular channel is,

Depth of flow,  $d = 0.81$  diameter of circular channel

Hydraulic mean depth,  $m = 0.305$  diameter of circular channel

12. Condition for maximum discharge through a circular channel is,

Depth,  $d = 0.95$  diameter of circular channel.

13. For a circular channel,

Wetted perimeter,  $P = 2R\theta$

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Area of flow, 
$$A = R^2 \left( \theta - \frac{\sin 2\theta}{2} \right)$$

where  $R$  = Radius of circular channel,

$\theta$  = Half the angle subtended by the water surface at the centre.

14. Total energy of a flowing liquid per unit weight, Total energy =  $z + h + V^2 / 2g$ .
15. Specific energy of a flowing liquid per unit weight,

$$E = h + \frac{V^2}{2g}, \quad \text{where } h = \text{Depth of flow, } V = \text{Velocity of flow.}$$

16. The depth of flow at which specific energy is minimum is called critical depth, which is given by

$$h_c = \left( \frac{q^2}{g} \right)^{1/3}, \quad \text{where } q = \text{discharge per unit width} = \frac{\text{Total discharge}}{b}.$$

17. The velocity of flow at critical depth is known as critical velocity, which is given as  $V_c = \sqrt{g \times h_c}$ .
18. Minimum specific energy is related with critical depth by the relation,  $E_{\min} = \frac{3}{2} h_c$ .
19. The flow corresponding to critical depth (or when Froude number is equal to 1.0) is known as critical flow.
20. If the depth of flow in a channel is greater than the critical depth (or Froude number is less than 1.0), the flow is said sub-critical or streaming flow.
21. If the depth of flow in a channel is less than the critical depth (or Froude number is more than 1.0), the flow is known as super-critical or shooting flow.
22. The condition for maximum discharge for a given value of specific energy is that the depth of flow should be critical.
23. The rise of water-level which takes place due to the transformation of the shooting to the streaming flow is known as hydraulic jump. The depth of flow after the jump is given by

$$\begin{aligned} d_2 &= -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2q^2}{gd_1}} && \dots (\text{In terms of } q) \\ &= -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2V_1^2 d_1}{g}} && \dots (\text{In terms of } V_1) \\ &= \frac{d_1}{2} \left( \sqrt{1 + 8(F_e)_1^2} - 1 \right) && \dots (\text{In terms of } F_{e_1}) \end{aligned}$$

Depth of hydraulic jump,  $= d_2 - d_1$

where  $d_1$  = depth of flow before hydraulic jump,

$V_1$  = velocity of flow before hydraulic jump.

24. Energy lost due to hydraulic jump per kg of liquid

$$h_L = (E_1 - E_2) = \left( d_1 + \frac{V_1^2}{2g} \right) - \left( d_2 + \frac{V_2^2}{2g} \right) = \frac{(d_2 - d_1)^3}{4d_1 d_2}.$$

25. Equation of gradually varied flow,  $\frac{dh}{dx} = \frac{i_b - i_e}{1 - \frac{V^2}{gh}}$  ... (In terms of  $V$ )

$$= \frac{i_b - i_e}{(1 - F_e^2)} \quad \dots \text{ (In term of } F_e)$$

where  $\frac{dh}{dx}$  = slope of free water surface,  $i_b$  = bed slope,

$i_e$  = slope of the energy line,  $h$  = depth of flow, and  $V$  = velocity of flow.

26. Afflux is the increase in water level due to some obstruction across the flowing liquid, while back water curve is the profile of the rising water on the upstream side of the obstruction.

27. Length of back water curve is given by,  $L = \frac{E_2 - E_1}{i_b - i_e}$

where  $E_1$  = Specific energy at the section, where water starts rising =  $h_1 + \frac{V_1^2}{2g}$

and  $E_2$  = Specific energy at the end of the water curve =  $h_2 + \frac{V_2^2}{2g}$ .

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. What do you understand by 'Flow in open channel' ?
2. Differentiate between : (i) Uniform flow and non-uniform flow, (ii) Steady and unsteady flow, (iii) Laminar and turbulent flow and (iv) Critical, sub-critical and super-critical flow in a open channel.
3. Explain the terms : (i) Rapidly varied flow and (ii) Gradually varied flow.
4. Derive an expression for the discharge through a channel by Chezy's formula.
5. Explain the terms : (i) Slope of the bed, (ii) Hydraulic mean depth and (iii) Wetted perimeter.
6. (a) What are the empirical formulae for determining the value of Chezy's constant ?  
(b) What is the relation between Manning's constant and Chezy's constant.
7. State the following formulae for the values of  $C$  :  
(i) Bazin's formulae, (ii) Kutter's formula, and (iii) Manning's formula.
8. (a) Define the term most economical section of a channel. What are the conditions for the rectangular channel of the best section ?  
(b) What is meant by an economical section of a channel ?
9. Prove that for the trapezoidal channel of most economical section :  
(i) Half of top width = Length of one of the sloping sides  
(ii) Hydraulic mean depth =  $\frac{1}{2}$  depth of flow.
10. (a) Derive the condition for the best side slope of the most economical trapezoidal channel.  
(b) Find the side slope in a trapezoidal section of maximum efficiency which will carry the same flow as a half square section of the same area.
11. Prove that for a channel of circular section, the depth of flow,  
$$d = 0.81 D \text{ for maximum velocity, and}$$
  
$$= 0.95 D \text{ for maximum discharge,}$$
  
where  $D$  = Diameter of circular channel,  $d$  = depth of flow.
12. Explain the terms : Specific energy of a flowing liquid, minimum specific energy, critical depth, critical velocity and alternate depths as applied to non-uniform flow.

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13. What is specific energy curve ? Draw specific energy curve, and then derive expressions for critical depth and critical velocity.
14. (a) Derive an expression for critical depth and critical velocity.  
(b) Define critical depth in an open channel in as many ways as you can.
15. Derive the condition for maximum discharge for a given value of specific energy.
16. Explain the term hydraulic jump. Derive an expression for the depth of hydraulic jump in terms of the upstream Froude number.
17. Derive an expression for the variation of depth along the length of the bed of the channel for gradually varied flow in an open channel. State clearly all the assumptions made.
18. Find an expression for loss of energy head for a hydraulic jump.
19. Define the terms : (i) Afflux and (ii) Back water curve. Prove that the length of the back water curve is given by,

$$L = \frac{(E_2 - E_1)}{i_b - i_e}$$

where  $L$  = Length of back water curve,

$E_2$  = Specific energy at the end of back water curve,

$E_1$  = Specific energy at the section where water starts rising,

$i_b$  = Slope of bed, and  $i_e$  = Slope of the energy line.

20. Find, in terms of specific energy  $E$ , an expression for the critical depth in a trapezoidal channel with bottom width  $B$  and side slopes of 1 vertical to  $n$  horizontal.
21. Show that in a rectangular channel :
  - (i) Critical depth is two-third of specific energy, and
  - (ii) Froude number at critical depth is unity.
22. Obtain the condition for a trapezoidal channel with side slopes 2 H : 1 V to be most efficient for a given area  $A$  let  $B$  be its bed width.
23. By applying the momentum equation to open channel flow, show that the consequent depths and flow rate are related by  $2q^2/g = y_1y_2(y_1 + y_2)$ .  
State the assumptions made in the derivation.
24. Derive the differential equation for steady gradually varied flow in open channels and list all assumptions.

$$\frac{dh}{dx} = \frac{(i_b - i_e)}{\left(1 - F_e^2\right)}.$$

25. What is the essential difference between gradually varied flow and rapidly varied flow ? Illustrate with neatly drawn sketches.
26. (a) Prove that the loss of energy head in a hydraulic jump is equal to  $(d_2 - d_1)^3/4d_1d_2$ , where  $d_1$  and  $d_2$  are the conjugate depths.  
(b) Obtain the relationship between the Froude Numbers of flow before and after the hydraulic jump in a horizontal rectangular channel.

**(B) NUMERICAL PROBLEMS**

- Find the velocity of flow and rate of flow of water through a rectangular channel of 5 m wide and 2 m deep, when it is running full. The channel is having bed slope of 1 in 3000. Take Chezy's constant  $C = 50$ .  
[Ans. 0.962 m/s,  $9.62 \text{ m}^3/\text{s}$ ]
- A flow of water of 150 litres per second flows down in a rectangular flume of width 70 cm and having adjustable bottom slope. If Chezy's constant  $C$  is 60, find the bottom slope necessary for uniform flow with a depth of flow of 40 cm. Also find the conveyance  $K$  of the flume.  
[Ans. 1 in 2341.5, 7.258]
- Find the discharge through a trapezoidal channel of width 6 m and side slope of 1 horizontal to 3 vertical. The depth of flow of water is 3 m and Chezy's constant,  $C = 60$ . The slope of the bed of the channel is given 1 in 5000.  
[Ans.  $23.26 \text{ m}^3/\text{s}$ ]
- Find the rate of flow of water through a V-shaped channel having total angle between the sides as  $60^\circ$ . Take the value of  $C = 50$  and slope of the bed 1 in 1500. The depth of flow is 6 m.  
[Ans.  $32.864 \text{ m}^3/\text{s}$ ]
- Find the discharge through a rectangular channel 3 m wide, having depth of water 2 m and bed slope as 1 in 1500. Take the value of  $K = 2.36$  in Bazin's formula.  
[Ans.  $5.184 \text{ m}^3/\text{s}$ ]
- Find the discharge through the rectangular channel given in the above question, taking the value of  $N = 0.012$  in Manning's formula.  
[Ans.  $11.64 \text{ m}^3/\text{s}$ ]
- Find the bed slope of trapezoidal channel of bed width 3 m, depth of water 2.5 m and side slope of 2 horizontal to 3 vertical, when the discharge through the channel is  $10 \text{ m}^3/\text{s}$ . Taking the value of  $N = 0.03$  in Manning's formula

$$C = \frac{1}{N} m^{1/6}. \quad [\text{Ans. 1 in 1803}]$$

- Find the diameter of a circular sewer pipe which is laid at a slope of 1 in 10000 and carries a discharge of 1000 litres/s when flowing half full. Take the value of Manning's  $N = 0.02$ .  
[Ans. 2.6 m]
- A rectangular channel carries water at the rate of 500 litres/s when bed slope is 1 in 3000. Find the most economical dimensions of the channel if  $C = 60$ .  
[Ans.  $b = 1.272 \text{ m}$ ,  $d = 0.636 \text{ m}$ ]
- A trapezoidal channel has side slopes of 1 horizontal to 2 vertical and the slope of the bed is 1 in 2000. The area of the section is  $42 \text{ m}^2$ . Find the dimensions of the section if it is most economical. Determine the discharge of the most economical section if  $C = 60$ .  
[Ans.  $d = 4.918 \text{ m}$ ,  $b = 6.08 \text{ m}$ ,  $Q = 88.36 \text{ m}^3/\text{s}$ ]
- A trapezoidal channel with side slopes of 1 to 1 has to be designed to convey  $9 \text{ m}^3/\text{s}$  at a velocity of 1.5 m/s so that the amount of concrete lining for the bed and sides is the minimum. Calculate the area of lining required for one metre length of canal.  
[Ans.  $6.62 \text{ m}^2$ ]
- A power canal of trapezoidal section has to be excavated through hard clay at the least cost. Determine the dimensions of the channel given, discharge equal to  $15 \text{ m}^3/\text{s}$ , bed slope 1 : 2000 and Manning's,  $N = 0.020$ .  
[Ans.  $b = 2.956 \text{ m}$ ,  $d = 2.56 \text{ m}$ ]
- Find the discharge through a circular pipe of diameter 4.0 m, if the depth of water in the pipe is 1.33 m and pipe is laid at a slope of 1 in 1500. Take the value of Chezy's constant = 60.  
[Ans.  $4.89 \text{ m}^3/\text{s}$ ]
- Water is flowing through a circular channel at the rate of 500 litres/s. The depth of water in the channel is 0.7 times the diameter and the slope of the bed of the channel is 1 in 8000. Find the diameters of the circular channel if the value of Manning's,  $N = 0.015$ .  
[Ans. 1.425 m]
- The rate of flow of water through a circular channel of diameter 0.8 m is 200 litres/s. Find the slope of the bed of the channel for maximum velocity. Take  $C = 50$ .  
[Ans. 1/2787]
- Determine the maximum discharge of water through a circular channel of diameter 2.0 m when the bed slope of the channel is 1 in 1500. Take  $C = 50$ .  
[Ans.  $3.02 \text{ m}^3/\text{s}$ ]
- The discharge of water through a rectangular channel of width 6 m, is  $18 \text{ m}^3/\text{s}$  when depth of flow of water is 2 m. Calculate : (i) specific energy of the flowing water, (ii) critical depth and critical velocity and (iii) value of minimum specific energy.  
[Ans. (i) 2.115 m, (ii) 0.971 m, 3.087 m/s, (iii) 1.457 m]

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18. The specific energy for a 6 m wide rectangular channel is to be 5 kg-m/kg. If the rate of flow of water through the channel is  $24 \text{ m}^2/\text{s}$ , determine the alternate depths of flow. [Ans. 4.831 m, 0.169 m]
19. The depth of flow of water, at a certain section of a rectangular channel of 5 m wide is 0.6 m. The discharge through the channel is  $15 \text{ m}^3/\text{s}$ . If a hydraulic jump takes place on the downstream side, find the depth of flow after the jump. [Ans. 1.474 m]
20. For the Question 19, find the loss of energy per kg of water due to hydraulic jump. [Ans. 0.188 m]
21. A sluice gate discharges water into a horizontal rectangular channel with a velocity of 8 m/s and depth of flow is 0.5 m. The width of the channel is 6 m. Determine whether a hydraulic jump will occur, and if so, find its height and loss of energy per kg of water. Also determine the horse power lost in the hydraulic jump. [Ans. Yes, 1.816 m, 1.293 m, 413.76 h.p.]
22. Find the rate of change of depth of water in a rectangular channel of 12 m wide and 2 m deep, when the water is flowing with a velocity of 1.5 m/s. The flow of water through the channel of bed slope 1 in 300, is regulated in such a way that energy line is having a slope of 1 in 8000. [Ans. 0.000235]
23. Find the slope of the free water surface in a rectangular channel of width 15 m, having depth of flow 4 m. The discharge through the channel is  $40 \text{ m}^3/\text{s}$ . The bed of the channel is having a slope of 1 in 4000. Take the value of Chezy's constant,  $C = 50$ . [Ans. 0.000184]
24. Determine the length of the back water curve caused by an afflux of 1.5 m in a rectangular channel of width 50 m and depth 2.0 m. The slope of the bed is given as 1 in 2000. Take Manning's,  $N = 0.03$ . [Ans. 4566 m]
25. A trapezoidal channel with bottom slope 0.000169, bottom width 10 m and side slopes 1 : 1 carries  $20 \text{ m}^3/\text{s}$  when Manning's constant = 0.015. Determine the normal depth.

[Hint.  $i = 0.000169$ ,  $b = 10 \text{ m}$ ,  $n = 1$ ,  $N = 0.015$ ,  $Q = 20 \text{ m}^3/\text{s}$

$$\text{Use } Q = \frac{1}{N} m^{2/3} \times i^{1/2} \times A, \text{ where } A = (b + nd) \times d = (10 + d) d$$

$$P = b + 2d \sqrt{1 + n^2} = 10 + 2 \times \sqrt{2} \times d, m = \frac{(10 + d) d}{(10 + 2\sqrt{2} d)}$$

$$\therefore 20 = \frac{1}{0.015} \times \left[ \frac{(10 + d) d}{(10 + 2\sqrt{2} d)} \right]^{2/3} \times 0.000169^{1/2} \times (10 + d) d$$

$$\text{or } \frac{[(10 + d) d]^{5/3}}{[10 + 2\sqrt{2} d]^{2/3}} = 23. \text{ Find 'd' by hit and trial.} \quad [d = 1.65 \text{ m}]$$

# 17

## CHAPTER

# IMPACT OF JETS AND JET PROPULSION

### ► 17.1 INTRODUCTION

The liquid comes out in the form of a jet from the outlet of a nozzle, which is fitted to a pipe through which the liquid is flowing under pressure. If some plate, which may be fixed or moving, is placed in the path of the jet, a force is exerted by the jet on the plate. This force is obtained from Newton's second law of motion or from impulse-momentum equation. Thus impact of jet means the force exerted by the jet on a plate which may be stationary or moving. In this chapter, the following cases of the impact of jet *i.e.*, the force exerted by the jet on a plate, will be considered :

1. Force exerted by the jet on a stationary plate when
  - (a) Plate is vertical to the jet, (b) Plate is inclined to the jet, and (c) Plate is curved.
2. Force exerted by the jet on a moving plate, when
  - (a) Plate is vertical to the jet, (b) Plate is inclined to the jet, and (c) Plate is curved.

### ► 17.2 FORCE EXERTED BY THE JET ON A STATIONARY VERTICAL PLATE

Consider a jet of water coming out from the nozzle, strikes a flat vertical plate as shown in Fig. 17.1  
Let  $V$  = velocity of the jet,  $d$  = diameter of the jet,

$$a = \text{area of cross-section of the jet} = \frac{\pi}{4} d^2.$$

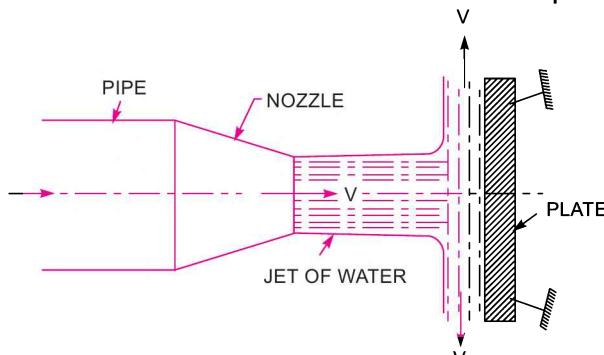


Fig. 17.1 Force exerted by jet on vertical plate.

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The jet after striking the plate, will move along the plate. But the plate is at right angles to the jet. Hence the jet after striking, will get deflected through  $90^\circ$ . Hence the component of the velocity of jet, in the direction of jet, after striking will be zero.

The force exerted by the jet on the plate in the direction of jet,

$$\begin{aligned}
 F_x &= \text{Rate of change of momentum in the direction of force} \\
 &= \frac{\text{Initial momentum} - \text{Final momentum}}{\text{Time}} \\
 &= \frac{(\text{Mass} \times \text{Initial velocity} - \text{Mass} \times \text{Final velocity})}{\text{Time}} \\
 &= \frac{\text{Mass}}{\text{Time}} [\text{Initial velocity} - \text{Final velocity}] \\
 &= (\text{Mass/sec}) \times (\text{velocity of jet before striking} - \text{velocity of jet after striking}) \\
 &= \rho a V [V - 0] \quad (\because \text{mass/sec} = \rho \times a V) \\
 &= \rho a V^2 \quad \dots(17.1)
 \end{aligned}$$

For deriving above equation, we have taken initial velocity minus final velocity and not final velocity minus initial velocity. If the force exerted on the jet is to be calculated then final minus initial velocity is taken. But if the force exerted by the jet on the plate is to be calculated, then initial velocity minus final velocity is taken.

Note. In equation (17.1), if the value of density ( $\rho$ ) is taken in S.I. units (i.e.,  $\text{kg/m}^3$ ), the force ( $F_x$ ) will be in Newton (N). The value of  $\rho$  for water in S.I. units is equal to  $1000 \text{ kg/m}^3$ .

**17.2.1 Force Exerted by a Jet on Stationary Inclined Flat Plate.** Let a jet of water, coming out from the nozzle, strikes an inclined flat plate as shown in Fig. 17.2.

Let

$$\begin{aligned}
 V &= \text{Velocity of jet in the direction of } x, \\
 \theta &= \text{Angle between the jet and plate}, \\
 a &= \text{Area of cross-section of the jet}.
 \end{aligned}$$

Then mass of water per sec striking the plate =  $\rho \times aV$ .

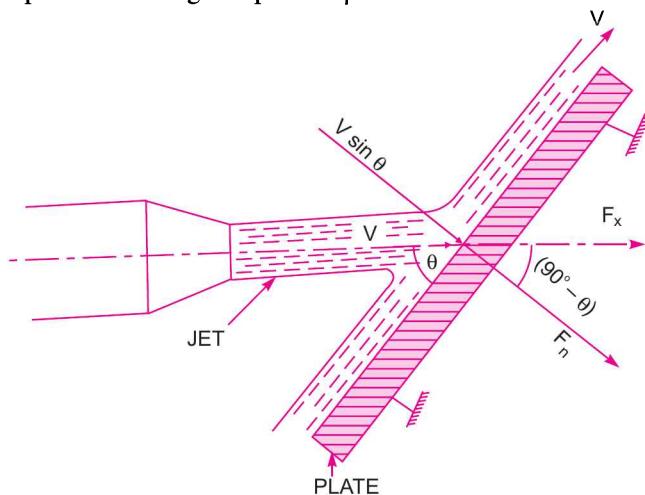


Fig. 17.2 Jet striking stationary inclined plate.

If the plate is smooth and if it is assumed that there is no loss of energy due to impact of the jet, then jet will move over the plate after striking with a velocity equal to initial velocity *i.e.*, with a velocity  $V$ . Let us find the force exerted by the jet on the plate in the direction normal to the plate. Let this force is represented by  $F_n$

Then

$$\begin{aligned} F_n &= \text{mass of jet striking per second} \\ &\times [\text{Initial velocity of jet before striking in the direction of } n] \\ &- [\text{Final velocity of jet after striking in the direction of } n] \\ &= \rho A V [V \sin \theta - 0] = \rho A V^2 \sin \theta \quad \dots(17.2) \end{aligned}$$

This force can be resolved into two components, one in the direction of the jet and other perpendicular to the direction of flow. Then we have,

$$\begin{aligned} F_x &= \text{component of } F_n \text{ in the direction of flow} \\ &= F_n \cos (90^\circ - \theta) = F_n \sin \theta = \rho A V^2 \sin \theta \times \sin \theta \quad (\therefore F_n = \rho A V^2 \sin \theta) \\ &= \rho A V^2 \sin^2 \theta \quad \dots(17.3) \end{aligned}$$

And,

$$\begin{aligned} F_y &= \text{component of } F_n, \text{ perpendicular to flow} \\ &= F_n \sin (90^\circ - \theta) = F_n \cos \theta = \rho A V^2 \sin \theta \cos \theta. \quad \dots(17.4) \end{aligned}$$

### 17.2.2 Force Exerted by a Jet on Stationary Curved Plate

(A) Jet strikes the curved plate at the centre. Let a jet of water strikes a fixed curved plate at the centre as shown in Fig. 17.3. The jet after striking the plate, comes out with the same velocity if the plate is smooth and there is no loss of energy due to impact of the jet, in the tangential direction of the curved plate. The velocity at outlet of the plate can be resolved into two components, one in the direction of jet and other perpendicular to the direction of the jet.

Component of velocity in the direction of jet =  $-V \cos \theta$ .

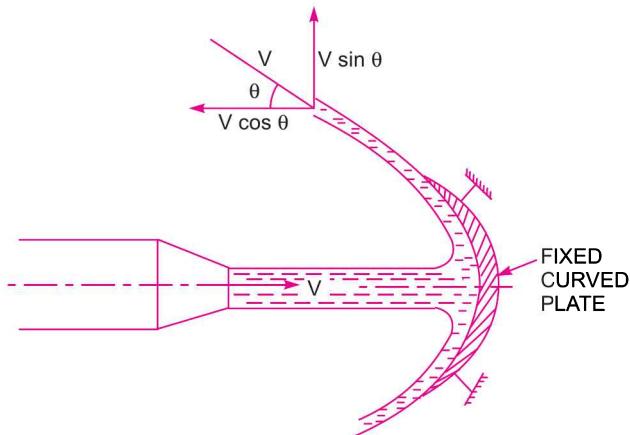


Fig. 17.3 Jet striking a fixed curved plate at centre.

(-ve sign is taken as the velocity at outlet is in the opposite direction of the jet of water coming out from nozzle).

Component of velocity perpendicular to the jet =  $V \sin \theta$

Force exerted by the jet in the direction of jet,

$$F_x = \text{Mass per sec} \times [V_{1x} - V_{2x}]$$

where  $V_{1x}$  = Initial velocity in the direction of jet =  $V$

$V_{2x}$  = Final velocity in the direction of jet =  $-V \cos \theta$

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$$\therefore F_x = \rho aV[V - (-V \cos \theta)] = \rho aV[V + V \cos \theta] \\ = \rho aV^2[1 + \cos \theta] \quad \dots(17.5)$$

Similarly,

$$F_y = \text{Mass per sec} \times [V_{1y} - V_{2y}]$$

where  $V_{1y}$  = Initial velocity in the direction of  $y = 0$

$V_{2y}$  = Final velocity in the direction of  $y = V \sin \theta$

$$\therefore F_y = \rho aV[0 - V \sin \theta] = -\rho aV^2 \sin \theta \quad \dots(17.6)$$

-ve sign means that force is acting in the downward direction. In this case the angle of deflection of the jet  $= (180^\circ - \theta)$   $\dots[17.6 \text{ (A)}]$

**(B) Jet strikes the curved plate at one end tangentially when the plate is symmetrical.** Let the jet strikes the curved fixed plate at one end tangentially as shown in Fig. 17.4. Let the curved plate is symmetrical about  $x$ -axis. Then the angle made by the tangents at the two ends of the plate will be same.

Let  $V$  = Velocity of jet of water,

$\theta$  = Angle made by jet with  $x$ -axis at inlet tip of the curved plate.

If the plate is smooth and loss of energy due to impact is zero, then the velocity of water at the outlet tip of the curved plate will be equal to  $V$ . The forces exerted by the jet of water in the directions of  $x$  and  $y$  are

$$\begin{aligned} F_x &= (\text{mass/sec}) \times [V_{1x} - V_{2x}] \\ &= \rho aV[V \cos \theta - (-V \cos \theta)] \\ &= \rho aV[V \cos \theta + V \cos \theta] \\ &= 2\rho aV^2 \cos \theta \quad \dots(17.7) \end{aligned}$$

$$\begin{aligned} F_y &= \rho aV[V_{1y} - V_{2y}] \\ &= \rho aV[V \sin \theta - V \sin \theta] = 0 \end{aligned}$$

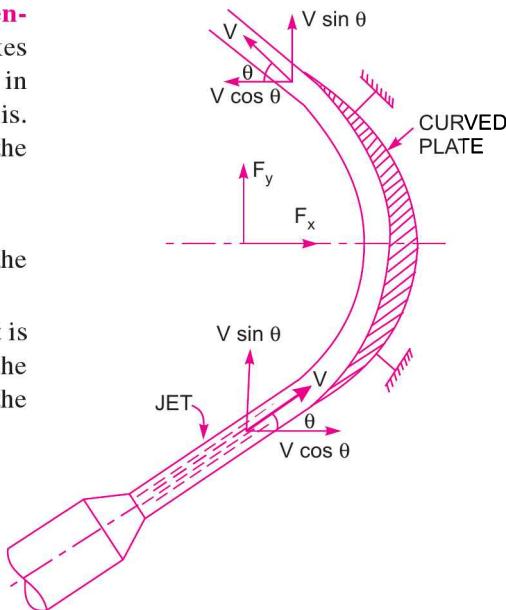


Fig. 17.4 Jet striking curved fixed plate at one end.

**(C) Jet strikes the curved plate at one end tangentially when the plate is unsymmetrical.** When the curved plate is unsymmetrical about  $x$ -axis, then angle made by the tangents drawn at the inlet and outlet tips of the plate with  $x$ -axis will be different.

Let

$\theta$  = angle made by tangent at inlet tip with  $x$ -axis,

$\phi$  = angle made by tangent at outlet tip with  $x$ -axis.

The two components of the velocity at inlet are

$$V_{1x} = V \cos \theta \text{ and } V_{1y} = V \sin \theta$$

The two components of the velocity at outlet are

$$V_{2x} = -V \cos \phi \text{ and } V_{2y} = V \sin \phi$$

$\therefore$  The forces exerted by the jet of water in the directions of  $x$  and  $y$  are

$$\begin{aligned} F_x &= \rho aV[V_{1x} - V_{2x}] = \rho aV[V \cos \theta - (-V \cos \phi)] \\ &= \rho aV[V \cos \theta + V \cos \phi] = \rho aV^2 [\cos \theta + \cos \phi] \quad \dots(17.8) \end{aligned}$$

$$\begin{aligned} F_y &= \rho aV[V_{1y} - V_{2y}] = \rho aV[V \sin \theta - V \sin \phi] \\ &= \rho aV^2 [\sin \theta - \sin \phi]. \quad \dots(17.9) \end{aligned}$$

**Problem 17.1** Find the force exerted by a jet of water of diameter 75 mm on a stationary flat plate, when the jet strikes the plate normally with velocity of 20 m/s.

**Solution.** Given :

$$\text{Diameter of jet, } d = 75 \text{ mm} = 0.075 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.075)^2 = 0.004417 \text{ m}^2$$

$$\text{Velocity of jet, } V = 20 \text{ m/s.}$$

The force exerted by the jet of water on a stationary vertical plate is given by equation (17.1) as

$$F = \rho a V^2 \quad \text{where } \rho = 1000 \text{ kg/m}^3$$

$$\therefore F = 1000 \times 0.004417 \times 20^2 \text{ N} = 1766.8 \text{ N. Ans.}$$

**Problem 17.2** Water is flowing through a pipe at the end of which a nozzle is fitted. The diameter of the nozzle is 100 mm and the head of water at the centre nozzle is 100 m. Find the force exerted by the jet of water on a fixed vertical plate. The co-efficient of velocity is given as 0.95.

**Solution.** Given :

$$\text{Diameter of nozzle, } d = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Head of water, } H = 100 \text{ m}$$

$$\text{Co-efficient of velocity, } C_v = 0.95$$

$$\text{Area of nozzle, } a = \frac{\pi}{4} (0.1)^2 = 0.007854 \text{ m}^2$$

Theoretical velocity of jet of water is given as

$$V_{th} = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 100} = 44.294 \text{ m/s}$$

$$\text{But } C_v = \frac{\text{Actual velocity}}{\text{Theoretical velocity}}$$

$$\therefore \text{Actual velocity of jet of water, } V = C_v \times V_{th} = 0.95 \times 44.294 = 42.08 \text{ m/s.}$$

Force on a fixed vertical plate is given by equation (17.1) as

$$\begin{aligned} F &= \rho a V^2 = 1000 \times 0.007854 \times 42.08^2 \quad (\because \text{In S.I. units } \rho \text{ for water} = 1000 \text{ kg/m}^3) \\ &= 13907.2 \text{ N} = 13.9 \text{ kN. Ans.} \end{aligned}$$

**Problem 17.3** A jet of water of diameter 75 mm moving with a velocity of 25 m/s strikes a fixed plate in such a way that the angle between the jet and plate is  $60^\circ$ . Find the force exerted by the jet on the plate (i) in the direction normal to the plate and (ii) in the direction of the jet.

**Solution.** Given :

$$\text{Diameter of jet, } d = 75 \text{ mm} = 0.075 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.075)^2 = 0.004417 \text{ m}^2$$

$$\text{Velocity of jet, } V = 25 \text{ m/s.}$$

$$\text{Angle between jet and plate } \theta = 60^\circ$$

(i) The force exerted by the jet of water in the direction normal to the plate is given by equation (17.2) as

$$\begin{aligned} F_n &= \rho a V^2 \sin \theta \\ &= 1000 \times 0.004417 \times 25^2 \times \sin 60^\circ = 2390.7 \text{ N. Ans.} \end{aligned}$$

(ii) The force in the direction of the jet is given by equation (17.3),

$$\begin{aligned} F_x &= \rho a V^2 \sin^2 \theta \\ &= 1000 \times .004417 \times 25^2 \times \sin^2 60^\circ = 2070.4 \text{ N. Ans.} \end{aligned}$$

**Problem 17.4** A jet of water of diameter 50 mm strikes a fixed plate in such a way that the angle between the plate and the jet is 30°. The force exerted in the direction of the jet is 1471.5 N. Determine the rate of flow of water.

**Solution.** Given :

$$\text{Diameter of jet, } d = 50 \text{ mm} = 0.05 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.05)^2 = .001963 \text{ m}^2$$

$$\text{Angle, } \theta = 30^\circ$$

$$\text{Force in the direction of jet, } F_x = 1471.5 \text{ N}$$

Force in the direction of jet is given by equation (17.3) as  $F_x = \rho a V^2 \sin^2 \theta$

As the force is given in Newton, the value of  $\rho$  should be taken equal to 1000 kg/m<sup>3</sup>

$$\therefore 1471.5 = 1000 \times .001963 \times V^2 \times \sin^2 30^\circ = .05 V^2$$

$$\therefore V^2 = \frac{150}{.05} = 3000.0$$

$$V = 54.77 \text{ m/s}$$

$$\therefore \text{Discharge, } Q = \text{Area} \times \text{Velocity}$$

$$= .001963 \times 54.77 = 0.1075 \text{ m}^3/\text{s} = 107.5 \text{ liters/s. Ans.}$$

**Problem 17.5** A jet of water of diameter 50 mm moving with a velocity of 40 m/s, strikes a curved fixed symmetrical plate at the centre. Find the force exerted by the jet of water in the direction of the jet, if the jet is deflected through an angle of 120° at the outlet of the curved plate.

**Solution.** Given :

$$\text{Diameter of the jet, } d = 50 \text{ mm} = 0.05 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.05)^2 = 0.001963 \text{ m}^2$$

$$\text{Velocity of jet, } V = 40 \text{ m/s}$$

$$\text{Angle of deflection} = 120^\circ$$

From equation [17.6 (A)], the angle of deflection =  $180^\circ - \theta$

$$\therefore 180^\circ - \theta = 120^\circ \text{ or } \theta = 180^\circ - 120^\circ = 60^\circ$$

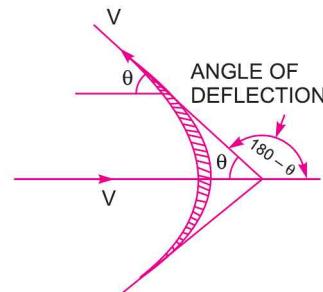


Fig. 17.5

Force exerted by the jet on the curved plate in the direction of the jet is given by equation (17.5) as

$$\begin{aligned} F_x &= \rho a V^2 [1 + \cos \theta] \\ &= 1000 \times .001963 \times 40^2 \times [1 + \cos 60^\circ] = 4711.15 \text{ N. Ans.} \end{aligned}$$

**Problem 17.6** A jet of water of diameter 75 mm moving with a velocity of 30 m/s, strikes a curved fixed plate tangentially at one end at an angle of 30° to the horizontal. The jet leaves the plate at an angle of 20° to the horizontal. Find the force exerted by the jet on the plate in the horizontal and vertical direction.

**Solution.** Given :

$$\text{Diameter of the jet, } d = 75 \text{ mm} = 0.075 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.075)^2 = .004417 \text{ m}^2$$

$$\text{Velocity of jet, } V = 30 \text{ m/s}$$

$$\text{Angle made by the jet at inlet tip with horizontal, } \theta = 30^\circ$$

$$\text{Angle made by the jet at outlet tip with horizontal, } \phi = 20^\circ$$

The force exerted by the jet of water in the direction of  $x$  is given by equation (17.8) and in the direction of  $y$  by equation (17.9),

$$\therefore F_x = \rho a V^2 [\cos \theta + \cos \phi] \\ = 1000 \times .004417 [\cos 30^\circ + \cos 20^\circ] \times 30^2 = 7178.2 \text{ N. Ans.}$$

and

$$F_y = \rho a V^2 [\sin \theta - \sin \phi] \\ = 1000 \times .004417 [\sin 30^\circ - \sin 20^\circ] \times 30^2 = 628.13 \text{ N. Ans.}$$

### ► 17.3 FORCE EXERTED BY A JET ON A HINGED PLATE

Consider a jet of water striking a vertical plate at the centre which is hinged at  $O$ . Due to the force exerted by the jet on the plate, the plate will swing through some angle about the hinge as shown in Fig. 17.6

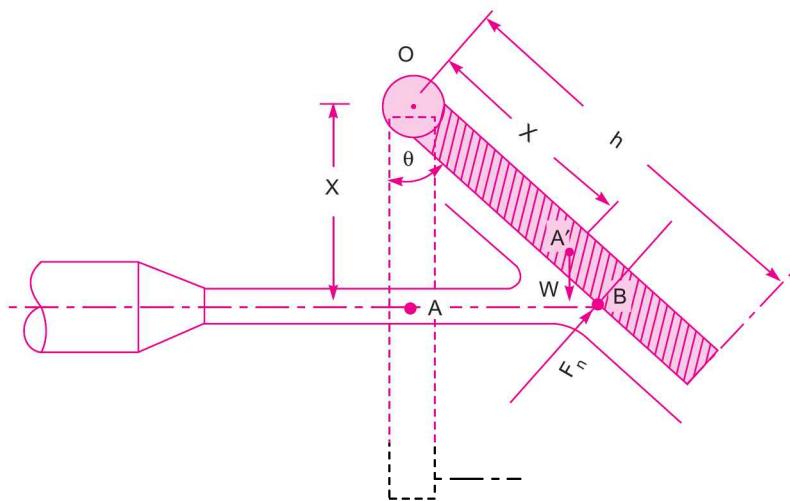


Fig. 17.6 Force on a hinged plate.

Let

$x$  = distance of the centre of jet from hinge  $O$ ,

$\theta$  = angle of swing about hinge,

$W$  = weight of plate acting at C.G. of the plate.

The dotted lines show the position of the plate, before the jet strikes the plate. The point  $A$  on the plate will be at  $A'$  after the jet strikes the plate. The distance  $OA = OA' = x$ . Let the weight of the plate is acting at  $A'$ . When the plate is in equilibrium after the jet strikes the plate, the moment of all the forces about the hinge must be zero. Two forces are acting on the plate. They are :

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1. Force due to jet of water, normal to the plate,

$$F_n = \rho a V^2 \sin \theta'$$

where  $\theta' = \text{Angle between jet and plate} = (90^\circ - \theta)$

2. Weight of the plate,  $W$

Moment of force  $F_n$  about hinge  $= F_n \times OB = \rho a V^2 \sin (90^\circ - \theta) \times OB = \rho a V^2 \cos \theta \times OB$

$$= \rho a V^2 \cos \theta \times \frac{OA}{\cos \theta} = \rho a V^2 \times OA = \rho a V^2 \times x$$

Moment of weight  $W$  about hinge  $= W \times OA' \sin \theta = W \times x \times \sin \theta$

For equilibrium of the plate,  $\rho a V^2 \times x = W \times x \times \sin \theta$

$$\therefore \sin \theta = \frac{\rho a V^2}{W} \quad \dots(17.10)$$

From equation (17.10), the angle of swing of the plate about hinge can be calculated.

**Problem 17.7** A jet of water of 2.5 cm diameter, moving with a velocity of 10 m/s, strikes a hinged square plate of weight 98.1 N at the centre of the plate. The plate is of uniform thickness. Find the angle through which the plate will swing.

**Solution.** Given :

Diameter of jet,  $d = 2.5 \text{ cm} = 0.025 \text{ m}$

Velocity of jet,  $V = 10 \text{ m/s}$

Weight of plate,  $W = 98.1 \text{ N}$

Area of jet,  $a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times (0.025)^2 = .00049 \text{ m}^2$

The angle through which the plate will swing is given by equation (17.10) as

$$\begin{aligned} \sin \theta &= \frac{\rho a V^2}{W} = 1000 \times \frac{.00049 \times 10^2}{98.1} \\ &= .499 \\ \therefore \theta &= 29.96^\circ. \text{ Ans.} \end{aligned} \quad (\because \rho = 1000)$$

**Problem 17.8** A jet of water of 30 mm diameter strikes a hinged square plate at its centre with a velocity of 20 m/s. The plate is deflected through an angle of 20°. Find the weight of the plate.

If the plate is not allowed, to swing, what will be the force required at the lower edge of the plate to keep the plate in vertical position.

**Solution.** Given :

Diameter of the jet,  $d = 30 \text{ mm} = 3 \text{ cm} = 0.03 \text{ m}$

$\therefore$  Area,  $a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.03)^2 = .0007068 \text{ m}^2$

Velocity of jet,  $V = 20 \text{ m/s}$

Angle of swing,  $\theta = 20^\circ$

Using equation (17.10) for angle of swing,

$$\sin \theta = \frac{\rho a V^2}{W}$$

or  $\sin 20^\circ = 1000 \times \frac{.0007068 \times 20^2}{W} = \frac{282.72}{W}$

$$\therefore W = \frac{282.72}{\sin 20^\circ} = 826.6 \text{ N}$$

If the plate is not allowed to swing, a force  $P$  will be applied at the lower edge of the plate as shown in Fig. 17.7. The weight of the plate is acting vertically downward through the C.G. of the plate.

Let  $F$  = Force exerted by jet of water

$h$  = Height of plate

= Distance of  $P$  from the hinge.

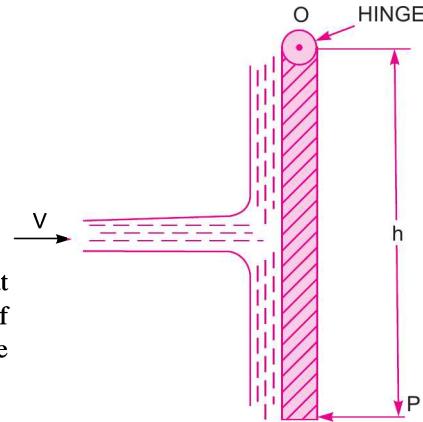


Fig. 17.7

The jet strikes at the centre of the plate and hence distance of the centre of the jet from hinge =  $\frac{h}{2}$ .

Taking moments\* about the hinge,  $O$ ,  $P \times h = F \times \frac{h}{2}$ .

$$\begin{aligned} \therefore P &= \frac{F \times h}{2 \times h} = \frac{F}{2} = \frac{\rho a V^2}{2} \quad (\because F = \rho a V^2) \\ &= 1000 \times \frac{.0007068 \times 20^2}{2} = 141.36 \text{ N. Ans.} \end{aligned}$$

**Problem 17.9** A rectangular plate, weighing 58.86 N is suspended vertically by a hinge on the top of horizontal edge. The centre of gravity of the plate is 10 cm from the hinge. A horizontal jet of water 2 cm diameter, whose axis is 15 cm below the hinge impinges normally on the plate with a velocity of 5 m/s. Find the horizontal force applied at the centre of the gravity to maintain the plate in its vertical position. Find the corresponding velocity of the jet, if the plate is deflected through  $30^\circ$  and the same force continues to act at the centre of gravity of the plate.

**Solution.** Given :

Weight of plate,  $W = 58.86 \text{ N}$

Distance of  $W$  from hinge,  $x = 10 \text{ cm} = 0.1 \text{ m}$

Diameter of jet,  $d = 2 \text{ cm} = 0.02 \text{ m}$

$$\therefore \text{Area}, \quad a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times .02^2 = .000314 \text{ m}^2$$

Distance of the axis of the jet of water from hinge = 15 cm = 0.15 m

Velocity of jet,  $V = 5 \text{ m/s}$

(i) Let the force applied at the centre of gravity of the plate to keep the plate in vertical position =  $P$  as shown in Fig. 17.8 (a).

\* The weight of the plate is passing through the hinge O. Hence moment of  $W$  about hinge is zero.

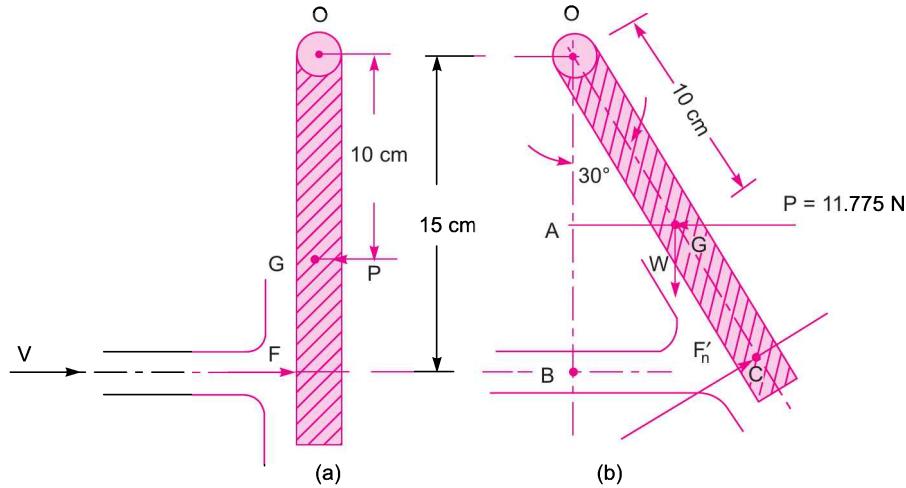


Fig. 17.8

The force exerted by a jet of water on the vertical plate,

$$F = \rho a V^2 = 1000 \times .000314 \times 5^2 = 7.85 \text{ N}$$

This force  $F$  is acting at a distance of 15 cm or 0.15 m from the hinge. Taking moments about hinge, we get

$$F \times 0.15 = P \times 0.10$$

$$\therefore P = \frac{F \times 0.15}{0.10} = \frac{7.85 \times .15}{.10} = 11.775 \text{ N. Ans.}$$

(ii) The plate is deflected through an angle of  $30^\circ$  as shown in Fig. 17.8(b).

$$\therefore \text{Angle of swing} = 30^\circ$$

$$\text{The force at the C.G.} = P = 11.775 \text{ N}$$

Let the velocity of the jet in this position =  $V$  m/s

For the deflected position of the plate as shown in Fig. 17.8 (b), the plate is in equilibrium under the action of three forces, which are :

(i) Weight of the plate,  $W$  acting at  $G$  at a distance 10 cm from  $O$ .

(ii) Horizontal force,  $P$  acting at  $G$ .

(iii) Normal force  $F'_n$  on the plate due to jet of water.

The angle between the jet and the plate,  $\theta = 90^\circ - 30^\circ = 60^\circ$

Hence,  $F'_n$  is given by equation (17.2) as

$$\begin{aligned} F'_n &= \rho a V^2 \sin \theta = \rho a V^2 \sin 60^\circ \\ &= 1000 \times .000314 \times V^2 \times \sin 60^\circ = 0.2717 V^2 \end{aligned}$$

Taking moments of all forces about hinge  $O$ , we get

$$F'_n \times OC = P \times OA + W \times AG \quad \dots(i)$$

where  $OB = OC \cos 30^\circ$

$$\therefore OC = \frac{OB}{\cos 30^\circ} = \frac{15}{\cos 30^\circ} = 17.32 \text{ cm} = 0.1732 \text{ m}$$

$$OA = OG \cos 30^\circ = 10 \times .866 = 8.66 \text{ cm} = 0.0866 \text{ m}$$

$$AG = OG \sin 30^\circ = 10 \times \frac{1}{2} = 5 \text{ cm} = .05 \text{ m}$$

Substituting these values in equation (i), we get

$$0.2717 V^2 \times .1732 = 11.775 \times .0866 + 58.86 \times 0.05 = 3.962$$

$$\therefore V = \sqrt{\frac{3.962}{0.2717 \times .1732}} = 9.175 \text{ m/s.}$$

**Problem 17.10** A jet of water of diameter 25 mm strikes a 20 cm  $\times$  20 cm square plate of uniform thickness with a velocity of 10 m/s at the centre of the plate which is suspended vertically by a hinge on its top horizontal edge. The weight of the plate is 98.1 N. The jet strikes normal to the plate. What force must be applied at the lower edge of the plate so that plate is kept vertical? If the plate is allowed to deflect freely, what will be the inclination of the plate with vertical due to the force exerted by jet of water?

**Solution.** Given :

$$\text{Diameter of the jet, } d = 25 \text{ mm} = 25 \times 10^{-3} \text{ m} = .025 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (.025)^2 = .00049 \text{ m}^2$$

$$\text{Size of the plate, } = 20 \text{ cm} \times 20 \text{ cm}$$

$$\text{Weight of the plate, } W = 98.1 \text{ N}$$

$$\text{Velocity of jet, } V = 10 \text{ m/s}$$

(i) Let the force applied at the lower edge to keep the plate in vertical position is  $P$ . See Fig. 17.9 (a).

Force exerted by the jet of water at the centre of the vertical plate,

$$\begin{aligned} F &= \rho a V^2 \\ &= 1000 \times .00049 \times 10^2 = 49 \text{ N.} \end{aligned}$$

This force is acting at a distance of  $\frac{20}{2} = 10 \text{ cm}$  from the hinge. The force  $P$  is acting at a distance of 20 cm from the hinge.

Taking moments about hinge,

$$F \times 10 = P \times 20$$

$$\therefore 49 \times 10 = P \times 20$$

$$\therefore P = \frac{49 \times 10}{20} = 24.5 \text{ N. Ans.}$$

(ii) When the plate is allowed to deflect freely about hinge.

Let the inclination of the plate with vertical =  $\theta$

In this position, the angle between the plate and jet will be

$$= (90^\circ - \theta).$$

$\therefore$  Force exerted by water normal to the plate is given by equation (17.2) as

$$F_n = \rho a V^2 \sin (90^\circ - \theta) = \rho a V^2 \cos \theta$$

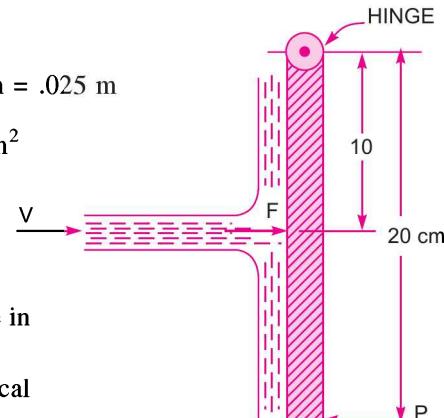


Fig. 17.9 (a)

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The distance

$$OB = \frac{OA}{\cos \theta} = \frac{10}{\cos \theta}$$

The weight  $W$  of the plate is acting at a distance 10 cm from hinge. Distance

$$DG = OG \sin \theta = 10 \times \sin \theta$$

Taking moments about hinge, we get

$$F_n \times OB = W \times GD$$

$$\text{or } \rho a V^2 \cos \theta \times \frac{10}{\cos \theta} = W \times 10 \times \sin \theta$$

$$\therefore \rho a V^2 = W \times \sin \theta$$

$$\therefore \sin \theta = \frac{\rho a V^2}{W} = 1000 \times \frac{0.00049 \times 10^2}{98.1} = 0.5$$

$$\therefore \theta = 30^\circ. \text{ Ans.}$$

**Problem 17.10 (A)** A square plate of uniform thickness and length of side 300 mm hangs vertically from hinge at its top edge. When a horizontal water jet strikes the plate at its centre, the plate is deflected and comes to rest at angle of  $30^\circ$  to the vertical. The jet is 25 mm in diameter and has a velocity of 6 m/s. Determine the weight of the plate.

**Solution.** Given :

$$\text{Length of plate, } L = 300 \text{ mm} = 0.3 \text{ m}$$

$$\text{Angle of swing, or angle made by deflected plate with the vertical, } \theta = 30^\circ$$

$$\text{Dia. of the jet, } d = 25 \text{ mm} = 0.025 \text{ m}$$

$$\therefore \text{Area of jet, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.025^2) \text{ m}^2$$

$$\text{Velocity of jet, } V = 6 \text{ m/s}$$

$$\text{Let } W = \text{Weight of plate}$$

$$\text{Using equation (17.10), we get } \sin \theta = \frac{\rho \times a \times V^2}{W}$$

$$\therefore W = \frac{\rho \times a \times V^2}{\sin \theta} = \frac{1000 \times \left( \frac{\pi}{4} \times 0.025^2 \right) \times 6^2}{\sin 30^\circ} = 35.33 \text{ N. Ans.}$$

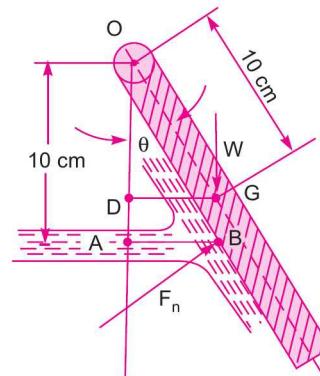


Fig. 17.9. (b)

### ► 17.4 FORCE EXERTED BY A JET ON MOVING PLATES

The following cases of the moving plates will be considered :

1. Flat vertical plate moving in the direction of the jet and away from the jet,
2. Inclined plate moving in the direction of the jet, and
3. Curved plate moving in the direction of the jet or in the horizontal direction.

\* If  $\rho = 1000 \text{ kg/m}^3$ , then weight  $W$  will be in Newton.

**17.4.1 Force on Flat Vertical Plate Moving in the Direction of Jet.** Fig. 17.10 shows a jet of water striking a flat vertical plate moving with a uniform velocity away from the jet.

Let  $V$  = Velocity of the jet (absolute),  
 $a$  = Area of cross-section of the jet,  
 $u$  = Velocity of the flat plate.

In this case, the jet does not strike the plate with a velocity  $V$ , but it strikes with a relative velocity, which is equal to the absolute velocity of jet of water minus the velocity of the plate.

Hence relative velocity of the jet with respect to plate

$$= (V - u)$$

Mass of water striking the plate per sec

$$\begin{aligned} &= \rho \times \text{Area of jet} \times \text{Velocity with} \\ &\quad \text{which jet strikes the plate} \\ &= \rho a \times [V - u] \end{aligned}$$

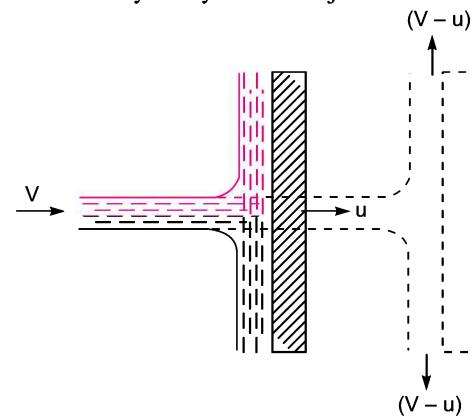


Fig. 17.10 Jet striking a flat vertical moving plate.

∴ Force exerted by the jet on the moving plate in the direction of the jet,

$$\begin{aligned} F_x &= \text{Mass of water striking per sec} \\ &\quad \times [\text{Initial velocity with which water strikes} - \text{Final velocity}] \\ &= \rho a (V - u) [(V - u) - 0] \quad (\because \text{Final velocity in the direction of jet is zero}) \\ &= \rho a (V - u)^2 \end{aligned} \quad \dots(17.11)$$

In this case, the work will be done by the jet on the plate, as plate is moving. For the stationary plates, the work done is zero.

∴ Work done per second by the jet on the plate

$$\begin{aligned} &= \text{Force} \times \frac{\text{Distance in the direction of force.}}{\text{Time}} \\ &= F_x \times u = \rho a (V - u)^2 \times u \end{aligned} \quad \dots(17.12)$$

In equation (17.12), if the value of  $\rho$  for water is taken in S.I. units (*i.e.*,  $1000 \text{ kg/m}^3$ ), the work done will be in N m/s. The term  $\frac{\text{Nm}}{\text{s}}$  is equal to W (watt).

**17.4.2 Force on the Inclined Plate Moving in the Direction of the Jet.** Let a jet of water strikes an inclined plate, which is moving with a uniform velocity in the direction of the jet as shown in Fig. 17.11.

Let  $V$  = Absolute velocity of jet of water,  
 $u$  = Velocity of the plate in the direction of jet,  
 $a$  = Cross-sectional area of jet, and  
 $\theta$  = Angle between jet and plate.

Relative velocity of jet of water =  $(V - u)$

∴ The velocity with which jet strikes =  $(V - u)$

Mass of water striking per second

$$= \rho \times a \times (V - u)$$

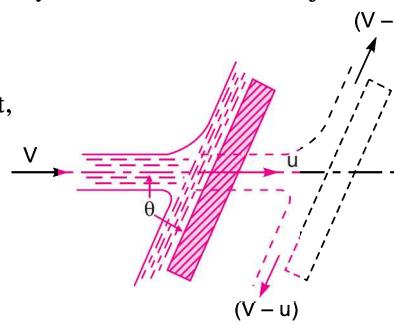


Fig. 17.11 Jet striking an inclined moving plate.

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If the plate is smooth and loss of energy due to impact of the jet is assumed zero, the jet of water will leave the inclined plate with a velocity equal to  $(V - u)$ .

The force exerted by the jet of water on the plate in the direction normal to the plate is given as

$$\begin{aligned} F_n &= \text{Mass striking per second} \times [\text{Initial velocity in the normal} \\ &\quad \text{direction with which jet strikes} - \text{Final velocity}] \\ &= \rho a (V - u) [(V - u) \sin \theta - 0] = \rho a (V - u)^2 \sin \theta \quad \dots(17.13) \end{aligned}$$

This normal force  $F_n$  is resolved into two components namely  $F_x$  and  $F_y$  in the direction of the jet and perpendicular to the direction of the jet respectively.

$$\therefore F_x = F_n \sin \theta = \rho a (V - u)^2 \sin^2 \theta \quad \dots(17.14)$$

$$F_y = F_n \cos \theta = \rho a (V - u)^2 \sin \theta \cos \theta \quad \dots(17.15)$$

$\therefore$  Work done per second by the jet on the plate

$$\begin{aligned} &= F_x \times \text{Distance per second in the direction of } x \\ &= F_x \times u = \rho a (V - u)^2 \sin^2 \theta \times u = \rho a (V - u)^2 u \sin^2 \theta \text{ N m/s.} \quad \dots(17.16) \end{aligned}$$

**Problem 17.11** A jet of water of diameter 10 cm strikes a flat plate normally with a velocity of 15 m/s. The plate is moving with a velocity of 6 m/s in the direction of the jet and away from the jet. Find:

- (i) the force exerted by the jet on the plate
- (ii) work done by the jet on the plate per second.

**Solution.** Given :

$$\text{Diameter of the jet, } d = 10 \text{ cm} = 0.1 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.1)^2 = .007854 \text{ m}^2$$

$$\text{Velocity of jet, } V = 15 \text{ m/s}$$

$$\text{Velocity of the plate, } u = 6 \text{ m/s.}$$

(i) The force exerted by the jet on a moving flat vertical plate is given by equation (17.11),

$$\begin{aligned} F_x &= \rho a (V - u)^2 \\ &= 1000 \times .007854 \times (15 - 6)^2 \text{ N} = \mathbf{636.17 \text{ N. Ans.}} \end{aligned}$$

(ii) Work done per second by the jet

$$= F_x \times u = 636.17 \times 6 = \mathbf{3817.02 \text{ Nm/s. Ans.}}$$

**Problem 17.12** For Problem 17.11, find the power and efficiency of the jet.

**Solution.** The given data from Problem 17.11 is

$$a = .007854 \text{ m}^2, V = 15 \text{ m/s, } u = 6 \text{ m/s}$$

Also work done per second by the jet = 3817.02 Nm/s

$$(i) \text{Power of the jet in kW} = \frac{\text{Work done per second}}{1000} = \frac{3817.02}{1000} = \mathbf{3.817 \text{ kW. Ans.}}$$

$$(ii) \text{Efficiency of the jet } (\eta) = \frac{\text{Output of the jet per second}}{\text{Input of the jet per second}} \quad \dots(i)$$

where output of jet/sec = Work done by jet per second = 3817.02 Nm/s

$$\begin{aligned}
 \text{And input per second} &= \text{Kinetic energy of the jet/sec} \\
 &= \frac{1}{2} \left( \frac{\text{mass}}{\text{sec}} \right) V^2 = \frac{1}{2} (\rho aV) \times V^2 = \frac{1}{2} \rho a V^3 \\
 &= \frac{1}{2} \times 1000 \times .007854 \times 15^3 \text{ Nm/s} = 13253.6 \text{ Nm/s} \\
 \therefore \eta \text{ of the jet} &= \frac{3817.02}{13253.6} = 0.288 = 28.8\%. \text{ Ans.}
 \end{aligned}$$

**Problem 17.12 (A)** A nozzle of 50 mm diameter delivers a stream of water at 20 m/s perpendicular to a plate that moves away from the jet at 5 m/s. Find :

- (i) the force on the plate,
- (ii) the work done, and
- (iii) the efficiency of jet.

(J.N.T.U., Hyderabad S 2002)

**Solution.** Given :

$$\begin{aligned}
 \text{Dia. of jet} &= 50 \text{ mm} = 0.05 \text{ m} \\
 \therefore \text{Area,} &= \frac{\pi}{4} (0.05^2) = 0.0019635 \text{ m}^2
 \end{aligned}$$

Velocity of jet,  $V = 20 \text{ m/s}$ , Velocity of plate,  $u = 5 \text{ m/s}$

(i) The force on the plate is given by equation (17.11) as,

$$\begin{aligned}
 F_x &= \rho a (V - u)^2 \\
 &= 1000 \times 0.0019635 \times (20 - 5)^2 = 441.78 \text{ N. Ans.}
 \end{aligned}$$

(ii) The work done by the jet

$$= F_x \times u = 441.78 \times 5 = 2208.9 \text{ Nm/s. Ans.}$$

(iii) The efficiency of the jet,  $\eta = \frac{\text{Output of jet}}{\text{Input of jet}}$

$$\begin{aligned}
 &= \frac{\text{Work done/s}}{\text{K.E. of jet/s}} = \frac{F_x \times u}{\frac{1}{2} m V^2} \\
 &= \frac{F_x \times u}{\frac{1}{2} (\rho A V) \times V^2} \\
 &= \frac{2208.9}{\frac{1}{2} (1000 \times 0.0019635 \times 20) \times 20^2} = \frac{2208.9}{6540} \\
 &= 0.3377 = 33.77\%. \text{ Ans.}
 \end{aligned}$$

**Problem 17.13** A 7.5 cm diameter jet having a velocity of 30 m/s strikes a flat plate, the normal of which is inclined at  $45^\circ$  to the axis of the jet. Find the normal pressure on the plate : (i) when the plate is stationary, and (ii) when the plate is moving with a velocity of 15 m/s and away from the jet. Also determine the power and efficiency of the jet when the plate is moving.

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**Solution.** Given :

$$\text{Diameter of the jet, } d = 7.5 \text{ cm} = 0.075 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.075)^2 = .004417 \text{ m}^2$$

$$\text{Angle between the jet and plate } \theta = 90^\circ - 45^\circ = 45^\circ$$

$$\text{Velocity of jet, } V = 30 \text{ m/s.}$$

(i) When the plate is stationary, the normal force on the plate is given by equation (17.2) as

$$F_n = \rho a V^2 \sin \theta = 1000 \times .004417 \times 30^2 \times \sin 45^\circ = 2810.96 \text{ N. Ans.}$$

(ii) When the plate is moving with a velocity 15 m/s and away from the jet, the normal force on the plate is given by equation (17.13) as

$$F_n = \rho a (V - u)^2 \sin \theta \quad \text{where } u = 15 \text{ m/s.}$$

$$= 1000 \times .004417 \times (30 - 15)^2 \times \sin 45^\circ = 702.74 \text{ N. Ans.}$$

(iii) The power and efficiency of the jet when plate is moving is obtained as

Work done per second by the jet

= Force in the direction of jet  $\times$  Distance moved by the plate in the direction of jet/sec

$$= F_x \times u, \text{ where } F_x = F_n \sin \theta = 702.74 \times \sin 45^\circ = 496.9 \text{ N}$$

$$\text{Work done/sec} = 496.9 \times 15 = 7453.5 \text{ Nm/s}$$

$$\therefore \text{Power in kW} = \frac{\text{Work done per second}}{1000} = \frac{7453.5}{1000} = 7.453 \text{ kW. Ans.}$$

$$\text{Efficiency of the jet} = \frac{\text{Output}}{\text{Input}} = \frac{\text{Work done per second}}{\text{Kinetic energy of the jet}}$$

$$= \frac{7453.5}{\frac{1}{2}(\rho a V) \times V^2} = \frac{7453.5}{\frac{1}{2}\rho a V^3} = \frac{7453.5}{\frac{1}{2} \times 1000 \times .004417 \times 30^3}$$

$$= 0.1249 \approx 0.125 = 12.5\%. \text{ Ans.}$$

**17.4.3 Force on the Curved Plate when the Plate is Moving in the Direction of Jet.** Let a jet of water strikes a curved plate at the centre of the plate which is moving with a uniform velocity in the direction of the jet as shown in Fig. 17.12.

Let  $V$  = Absolute velocity of jet,

$a$  = Area of jet,

$u$  = Velocity of the plate in the direction of the jet.

Relative velocity of the jet of water or the velocity with which jet strikes the curved plate =  $(V - u)$ .

If plate is smooth and the loss of energy due to impact of jet is zero, then the velocity with which the jet will be leaving the curved vane =  $(V - u)$ .

This velocity can be resolved into two components, one in the direction of the jet and other perpendicular to the direction of the jet.

Component of the velocity in the direction of jet

$$= -(V - u) \cos \theta$$

(-ve sign is taken as at the outlet, the component is in the opposite direction of the jet).

Component of the velocity in the direction perpendicular to the direction of the jet =  $(V - u) \sin \theta$ .

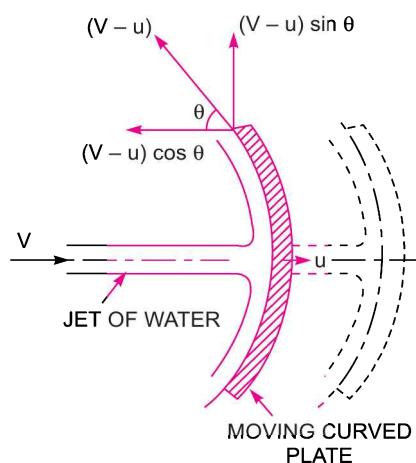


Fig. 17.12 Jet striking a curved moving plate.

Mass of the water striking the plate =  $\rho \times a \times$  Velocity with which jet strikes the plate  
 $= \rho a(V - u)$

$\therefore$  Force exerted by the jet of water on the curved plate in the direction of the jet,

$$\begin{aligned} F_x &= \text{Mass striking per sec} \times [\text{Initial velocity with which jet strikes the plate in the direction of jet} - \text{Final velocity}] \\ &= \rho a(V - u) [(V - u) - (- (V - u) \cos \theta)] \\ &= \rho a(V - u) [(V - u) + (V - u) \cos \theta] \\ &= \rho a(V - u)^2 [1 + \cos \theta] \end{aligned} \quad \dots(17.17)$$

Work done by the jet on the plate per second

$$\begin{aligned} &= F_x \times \text{Distance travelled per second in the direction of } x \\ &= F_x \times u = \rho a(V - u)^2 [1 + \cos \theta] \times u \\ &= \rho a(V - u)^2 \times u [1 + \cos \theta] \end{aligned} \quad \dots(17.18)$$

**Problem 17.14** A jet of water of diameter 7.5 cm strikes a curved plate at its centre with a velocity of 20 m/s. The curved plate is moving with a velocity of 8 m/s in the direction of the jet. The jet is deflected through an angle of 165°. Assuming the plate smooth find :

(i) Force exerted on the plate in the direction of jet, (ii) Power of the jet, and (iii) Efficiency of the jet.

**Solution.** Given :

Diameter of the jet,  $d = 7.5 \text{ cm} = 0.075 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.075)^2 = 0.004417$$

Velocity of the jet,  $V = 20 \text{ m/s}$

Velocity of the plate,  $u = 8 \text{ m/s}$

Angle of deflection of the jet,  $= 165^\circ$

$\therefore$  Angle made by the relative velocity at the outlet of the plate,

$$\theta = 180^\circ - 165^\circ = 15^\circ.$$

(i) Force exerted by the jet on the plate in the direction of the jet is given by equation (17.17) as

$$\begin{aligned} &= F_x = \rho a(V - u)^2 (1 + \cos \theta) \\ &= 1000 \times 0.004417 \times (20 - 8)^2 [1 + \cos 15^\circ] = 1250.38 \text{ N. Ans.} \end{aligned}$$

(ii) Work done by the jet on the plate per second

$$= F_x \times u = 1250.38 \times 8 = 10003.04 \text{ N m/s}$$

$$\therefore \text{Power of the jet} = \frac{10003.04}{1000} = 10 \text{ kW. Ans.}$$

(iii) Efficiency of the jet

$$= \frac{\text{Output}}{\text{Input}} = \frac{\text{Work done by jet/sec}}{\text{Kinetic energy of jet/sec}}$$

$$= \frac{1250.38 \times 8}{\frac{1}{2} (\rho a V) \times V^2} = \frac{1250.38 \times 8}{\frac{1}{2} \times 1000 \times 0.004417 \times V^3}$$

$$= \frac{1250.38 \times 8}{\frac{1}{2} \times 1000 \times 0.004417 \times 20^3} = 0.564 = 56.4\%. \text{ Ans}$$

**Problem 17.15** A jet of water from a nozzle is deflected through  $60^\circ$  from its original direction by a curved plate which it enters tangentially without shock with a velocity of 30 m/s and leaves with a mean velocity of 25 m/s. If the discharge from the nozzle is 0.8 kg/s, calculate the magnitude and direction of the resultant force on the vane, if the vane is stationary.

**Solution.** Given :

$$\text{Velocity at inlet, } V_1 = 30 \text{ m/s}$$

$$\text{Velocity at outlet, } V_2 = 25 \text{ m/s}$$

$$\text{Mass per second} = 0.8 \text{ kg/s}$$

Force in the direction of jet,

$$F_x = \text{Mass per second} \times (V_{1x} - V_{2x})$$

$$\begin{aligned} \text{where } V_{1x} &= \text{Initial velocity in the direction of } x \\ &= 30 \text{ m/s} \end{aligned}$$

$$V_{2x} = \text{Final velocity in the direction of } x$$

$$= 25 \cos 60^\circ = 25 \times \frac{1}{2} = 12.5 \text{ m/s}$$

$$\therefore F_x = 0.8[30 - 12.5] = 0.8 \times 17.5 = 14.0 \text{ N}$$

Similarly, force normal to the jet,

$$\begin{aligned} F_y &= \text{Mass per second} \times (V_{1y} - V_{2y}) \\ &= 0.8 [0 - 25 \sin 60^\circ] = -17.32 \text{ N} \end{aligned}$$

-ve sign means the force,  $F_y$ , is acting in the vertically downward direction.

$$\therefore \text{Resultant force on the vane} = \sqrt{F_x^2 + F_y^2} = \sqrt{14^2 + (-17.32)^2} = 22.27 \text{ N. Ans.}$$

The angle made by the resultant with  $x$ -axis

$$\tan \theta = \frac{F_y}{F_x} = \frac{-17.32}{14.0} = -1.237$$

-ve sign means the angle  $\theta$  is in the clockwise direction with  $x$ -axis as shown in Fig. 17.13 (a)

$$\therefore \theta = \tan^{-1} 1.237 = 51^\circ 2.86'. \text{ Ans.}$$

**Problem 17.16** (a) A stationary vane having an inlet angle of zero degree and an outlet angle of  $25^\circ$  as shown in Fig. 17.13(b), receives water at a velocity of 50 m/s. Determine the components of force acting on it in the direction of the jet velocity and normal to it. Also find the resultant force in magnitude and direction per unit weight of the flow.

(b) If the vane stated above is moving with a velocity of 20 m/s in the direction of the jet, calculate the force components in the direction of the vane velocity and across it, also the resultant force in magnitude and direction. Calculate the work done and power developed per unit weight of the flow.

**Solution.** Given :

$$(a) \text{Velocity of jet, } V = 50 \text{ m/s}$$

$$\text{Angle at outlet, } = 25^\circ$$

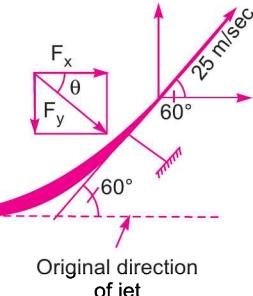


Fig. 17.13 (a)

For the stationary vane, the force in the direction of jet is given as

$$F_x = \text{Mass per sec} \times [V_{1x} - V_{2x}]$$

where  $V_{1x} = 50 \text{ m/s}$

$$V_{2x} = -50 \cos 25^\circ = -45.315$$

$\therefore$  Force in the direction of jet per unit weight of water

$$= \frac{\text{Mass/sec} [50 - (-45.315)]}{\text{Weight of water/sec}}$$

or

$$F_x = \frac{(\text{Mass/sec}) [50 + 45.315]}{(\text{Mass/sec}) \times g}$$

$$= \frac{1}{g} [50 + 45.315] \text{ N/N} = \frac{95.315}{9.81} = 9.716 \text{ N/N}$$

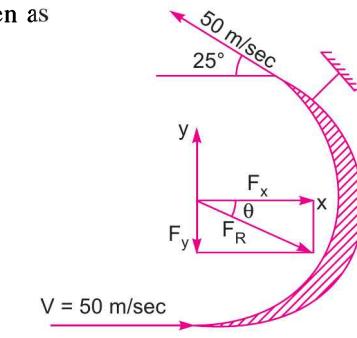


Fig. 17.13 (b)

Force exerted by jet in the direction perpendicular to the direction of the jet per unit weight of the flow,

$$\begin{aligned} F_y &= \frac{(\text{Mass per sec}) [V_{1y} - V_{2y}]}{g \times \text{Mass per sec}} \\ &= \frac{1}{g} [V_{1y} - V_{2y}] = \frac{1}{g} [0 - 50 \sin 25^\circ] \quad (\because V_{1y} = 0, V_{2y} = 50 \sin 25^\circ) \\ &= \frac{-50 \sin 25^\circ}{9.81} = -2.154. \text{ Ans.} \end{aligned}$$

-ve sign means the force  $F_y$  is acting in the downward direction.

$$\therefore \text{Resultant force per unit weight of water} = \sqrt{F_x^2 + F_y^2}$$

$$\text{or } F_R = \sqrt{(9.716)^2 + (2.154)^2} = 9.952 \text{ N. Ans.}$$

The angle made by the resultant with the  $x$ -axis,

$$\tan \theta = \frac{F_y}{F_x} = \frac{2.154}{9.716} = 0.2217$$

$$\therefore \theta = \tan^{-1} 0.2217 = 12.50^\circ. \text{ Ans.}$$

(b) Velocity of the vane = 20 m/s.

When the vane is moving in the direction of the jet, the force exerted by the jet on the plate in the direction of jet,

$$F'_x = [\text{Mass of water striking/sec}] \times [V_{1x} - V_{2x}]$$

where  $V_{1x}$  = Initial velocity of the striking water

$$= (V - u) = 50 - 20 = 30 \text{ m/s}$$

$V_{2x}$  = Final velocity in the direction of  $x$

$$= -(V - u) \cos 25^\circ = 30 \times \cos 25^\circ = -27.189 \text{ m/s.}$$

$$\therefore F'_x = \text{Mass per sec} [30 + 27.189]$$

Force in the direction of jet per unit weight,

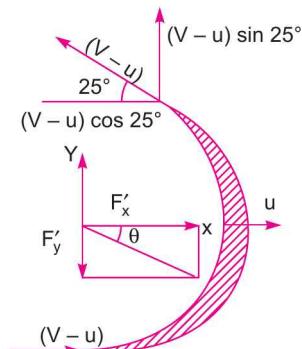


Fig. 17.14

$$F'_x = \frac{\text{Mass per sec } [30 + 27.189]}{\text{Mass per sec} \times g}$$

$$= \frac{(30 + 27.189)}{9.81} = 5.829 \text{ N.}$$

Force exerted by the jet in the direction perpendicular to direction of jet, per unit weight,

$$F'_y = \frac{1}{g} [V_{1y} - V_{2y}]$$

where  $V_{1y} = 0$ ;  $V_{2y} = (V - u) \sin 25^\circ = (50 - 20) \sin 25^\circ = 30 \sin 25^\circ$

$$F'_y = \frac{1}{9.81} [0 - 30 \sin 25^\circ] = -1.292 \text{ N}$$

$$\therefore \text{Resultant force} = \sqrt{(5.829)^2 + (1.292)^2} = 5.917 \text{ N}$$

The angle made by the resultant with  $x$ -axis,  $\tan \theta = \frac{1.292}{5.829} = 0.2217$

$$\therefore \theta = \tan^{-1} 0.2217 = 12.30^\circ$$

$\therefore$  Work done per second per unit weight of flow

$$= F'_x \times u = 5.829 \times 20 = 116.58 \text{ N m/s}$$

$$\therefore \text{Power developed} = \frac{\text{Work done per second}}{1000} = \frac{116.58}{1000} = 0.116 \text{ kW. Ans.}$$

**Problem 17.17** A jet of water of diameter 50 mm moving with a velocity of 25 m/s impinges on a fixed curved plate tangentially at one end at an angle of  $30^\circ$  to the horizontal. Calculate the resultant force of the jet on the plate if the jet is deflected through an angle of  $50^\circ$ . Take  $g = 10 \text{ m/s}^2$

**Solution.** Given :

Dia. of jet,  $d = 50 \text{ mm} = 0.05 \text{ m}^2$

$$\therefore \text{Area of jet, } a = \frac{\pi}{4} (0.05)^2 \text{ m}^2$$

Velocity of jet,  $V = 25 \text{ m/s}$

The angle made by the jet at inlet with horizontal,  $\theta = 30^\circ$

Angle of deflection =  $50^\circ$

$\therefore$  Angle made by the jet at outlet with horizontal is given by,

$$\phi = \theta + \text{Angle of deflection}$$

$$= 30^\circ + 50^\circ = 80^\circ$$

Value of  $g = 10 \text{ m/s}^2$

The force exerted by the jet of water in the direction of  $x$  is given by,

$$F_x = \rho a V [V_{1x} - V_{2x}] \quad \dots(i)$$

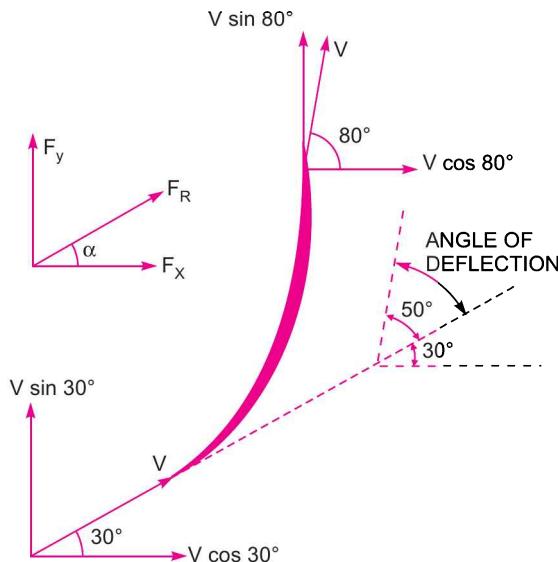


Fig. 17.14 (a)

where

$$\rho = 1000$$

( $\because g$  is given as  $10 \text{ m/s}^2$ )

$$a = \frac{\pi}{4} (0.05)^2; V = 25 \text{ m/s};$$

$$V_{1x} = V \cos 30^\circ = 25 \cos 30^\circ,$$

$$V_{2x} = V \cos 80^\circ = 25 \cos 80^\circ.$$

Substituting these values in equation (i), we get

$$F_x = 1000 \times \frac{\pi}{4} (0.05)^2 \times 25 [25 \cos 30^\circ - 25 \cos 80^\circ] = 849.7 \text{ N}$$

The force in the direction of  $y$  is given by,

$$F_y = \rho a V [V_{1y} - V_{2y}]$$

$$= 1000 \times \frac{\pi}{4} (0.05)^2 \times 25 [25 \sin 30^\circ - 25 \sin 80^\circ] = -594.9 \text{ N}$$

The  $-ve$  sign shows that force  $F_y$  is acting in the downward direction.

The resultant force is given by,

$$F_R = \sqrt{F_x^2 + F_y^2}$$

$$= \sqrt{849.7^2 + 594.9^2} = 1037 \text{ N. Ans.}$$

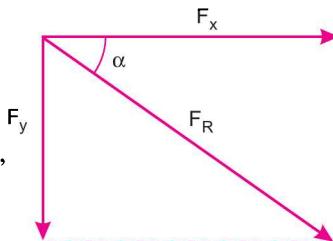


Fig. 17.14 (b)

And the angle made by the resultant with the horizontal is given by,

$$\tan \alpha = \frac{F_y}{F_x} = \frac{594.9}{849.7} = 0.7$$

$$\therefore \alpha = \tan^{-1} 0.7 = 35^\circ. \text{ Ans.}$$

#### 17.4.4 Force Exerted by a Jet of Water on an Unsymmetrical Moving Curved Plate when Jet Strikes Tangentially at one of the Tips.

Fig. 17.15 shows a jet of water striking a moving curved plate (also called vane) tangentially, at one of its tips. As the jet strikes tangentially, the loss of energy due to impact of the jet will be zero. In this case as plate is moving, the velocity with which jet of water strikes is equal to the relative velocity of the jet with respect to the plate. Also as the plate is moving in different direction of the jet, the relative velocity at inlet will be equal to the vector difference of the velocity of jet and velocity of the plate at inlet.

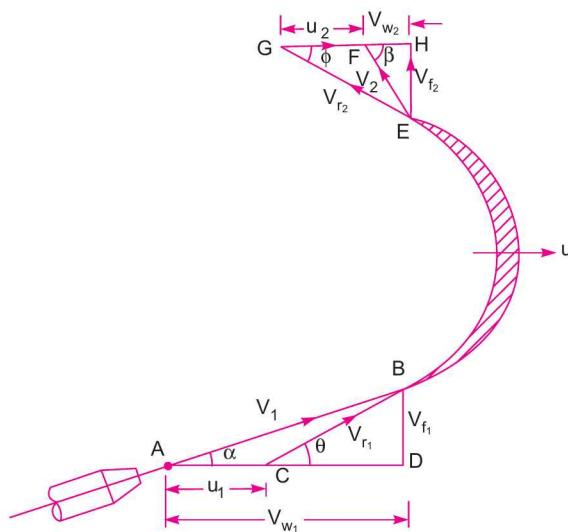


Fig. 17.15 Jet striking a moving curved vane at one of the tips.

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Let  $V_1$  = Velocity of the jet at inlet.  
 $u_1$  = Velocity of the plate (vane) at inlet.  
 $V_r_1$  = Relative velocity of jet and plate at inlet.  
 $\alpha$  = Angle between the direction of the jet and direction of motion of the plate, also called guide blade angle.  
 $\theta$  = Angle made by the relative velocity ( $V_{r_1}$ ) with the direction of motion at inlet also called vane angle at inlet.

$V_{w_1}$  and  $V_{f_1}$  = The components of the velocity of the jet  $V_1$ , in the direction of motion and perpendicular to the direction of motion of the vane respectively.

$V_{w_1}$  = It is also known as velocity of whirl at inlet.  
 $V_{f_1}$  = It is also known as velocity of flow at inlet.  
 $V_2$  = Velocity of the jet, leaving the vane or velocity of jet at outlet of the vane.  
 $u_2$  = Velocity of the vane at outlet.  
 $V_{r_2}$  = Relative velocity of the jet with respect to the vane at outlet.  
 $\beta$  = Angle made by the velocity  $V_2$  with the direction of motion of the vane at outlet.  
 $\phi$  = Angle made by the relative velocity  $V_{r_2}$  with the direction of motion of the vane at outlet and also called vane angle at outlet.

$V_{w_2}$  and  $V_{f_2}$  = Components of the velocity  $V_2$ , in the direction of motion of vane and perpendicular to the direction of motion of vane at outlet.  
 $V_{w_2}$  = It is also called the velocity of whirl at outlet.  
 $V_{f_2}$  = Velocity of flow at outlet.

The triangles  $ABD$  and  $EGH$  are called the velocity triangles at inlet and outlet. These velocity triangles are drawn as given below :

**1. Velocity Triangle at Inlet.** Take any point  $A$  and draw a line  $AB = V_1$  in magnitude and direction which means line  $AB$  makes an angle  $\alpha$  with the horizontal line  $AD$ . Next draw a line  $AC = u_1$  in magnitude. Join  $C$  to  $B$ . Then  $CB$  represents the relative velocity of the jet at inlet. If the loss of energy at inlet due to impact is zero, then  $CB$  must be in the tangential direction to the vane at inlet. From  $B$  draw a vertical line  $BD$  in the downward direction to meet the horizontal line  $AC$  produced at  $D$ .

Then  $BD$  = Represents the velocity of flow at inlet =  $V_{f_1}$

$AD$  = Represents the velocity of whirl at inlet =  $V_{w_1}$

$\angle BCD$  = Vane angle at inlet =  $\theta$ .

**2. Velocity Triangle at Outlet.** If the vane surface is assumed to be very smooth, the loss of energy due to friction will be zero. The water will be gliding over the surface of the vane with a relative velocity equal to  $V_{r_1}$  and will come out of the vane with a relative velocity  $V_{r_2}$ . This means that the relative velocity at outlet  $V_{r_2} = V_{r_1}$ . And also the relative velocity at outlet should be in tangential direction to the vane at outlet.

Draw  $EG$  in the tangential direction of the vane at outlet and cut  $EG = V_{r_2}$ . From  $G$ , draw a line  $GF$  in the direction of vane at outlet and equal to  $u_2$ , the velocity of the vane at outlet. Join  $EF$ . Then  $EF$  represents the absolute velocity of the jet at outlet in magnitude and direction. From  $E$  draw a vertical line  $EH$  to meet the line  $GF$  produced at  $H$ . Then

$EH$  = Velocity of flow at outlet =  $V_{f_2}$

$FH$  = Velocity of whirl at outlet =  $V_{w_2}$

$\phi$  = Angle of vane at outlet.

$\beta$  = Angle made by  $V_2$  with the direction of motion of vane at outlet.

If the vane is smooth and is having velocity in the direction of motion at inlet and outlet equal then we have

$u_1 = u_2 = u$  = Velocity of vane in the direction of motion and

$$V_{r_1} = V_{r_2}.$$

Now mass of water striking vane per sec =  $\rho a V_{r_1}$  ... (i)

where  $a$  = Area of jet of water,  $V_{r_1}$  = Relative velocity at inlet.

∴ Force exerted by the jet in the direction of motion

$F_x$  = Mass of water striking per sec × [Initial velocity with which jet strikes in the direction of motion – Final velocity of jet in the direction of motion] ... (ii)

But initial velocity with which jet strikes the vane =  $V_{r_1}$

The component of this velocity in the direction of motion

$$= V_{r_1} \cos \theta = (V_{w_1} - u_1) \quad (\text{See Fig. 17.15})$$

Similarly, the component of the relative velocity at outlet in the direction of motion =  $-V_{r_2} \cos \phi$

$$= -[u_2 + V_{w_2}]$$

–ve sign is taken as the component of  $V_{r_2}$  in the direction of motion is in the opposite direction.

Substituting the equation (i) and all above values of the velocities in equation (ii), we get

$$\begin{aligned} F_x &= \rho a V_{r_1} [(V_{w_1} - u_1) - \{-(u_2 + V_{w_2})\}] = \rho a V_{r_1} [V_{w_1} - u_1 + u_2 + V_{w_2}] \\ &= \rho a V_{r_1} [V_{w_1} + V_{w_2}] \quad (\because u_1 = u_2) \quad \dots (\text{iii}) \end{aligned}$$

Equation (iii) is true only when angle  $\beta$  shown in Fig. 17.15 is an acute angle. If  $\beta = 90^\circ$ , the  $V_{w_2} = 0$ , then equation (iii) becomes as,

$$F_x = \rho a V_{r_1} [V_{w_1}]$$

If  $\beta$  is an obtuse angle, the expression for  $F_x$  will become

$$F_x = \rho a V_{r_1} [V_{w_1} - V_{w_2}]$$

Thus in general,  $F_x$  is written as  $F_x = \rho a V_{r_1} [V_{w_1} \pm V_{w_2}]$  ... (17.19)

Work done per second on the vane by the jet

= Force × Distance per second in the direction of force

$$= F_x \times u = \rho a V_{r_1} [V_{w_1} \pm V_{w_2}] \times u \quad \dots (17.20)$$

∴ Work done per second per unit weight of fluid striking per second

$$\begin{aligned} &= \frac{\rho a V_{r_1} [V_{w_1} \pm V_{w_2}] \times u}{\text{Weight of fluid striking/s}} \frac{\text{Nm/s}}{\text{N/s}} = \frac{\rho a V_{r_1} [V_{w_1} \pm V_{w_2}] \times u}{g \times \rho a V_{r_1}} = \text{Nm/N} \\ &= \frac{1}{g} [V_{w_1} \pm V_{w_2}] \times u \text{ Nm/N} \quad \dots (17.21) \end{aligned}$$

Work done/sec per unit mass of fluid striking per second

$$\begin{aligned}
 &= \frac{\rho a V_r [V_{w_1} \pm V_{w_2}] \times u}{\text{Mass of fluid striking / s}} \frac{\text{Nm / s}}{\text{kg / s}} = \frac{\rho a V_r [V_{w_1} \pm V_{w_2}] \times u}{\rho a V_r} \frac{\text{Nm/kg}}{\text{kg}} \\
 &= (V_{w_1} \pm V_{w_2}) \times u \text{ Nm/kg} \quad \dots[17.21(A)]
 \end{aligned}$$

Note. Equation (17.21) gives the work done per unit weight whereas equation [17.21(A)] gives the work done per unit mass.

**3. Efficiency of Jet.** The work done by the jet on the vane given by equation (17.20), is the output of the jet whereas the initial kinetic energy of the jet is the input. Hence, the efficiency ( $\eta$ ) of the jet is expressed as

$$\eta = \frac{\text{Output}}{\text{Input}} = \frac{\text{Work done per second on the vane}}{\text{Initial K. E. per second of the jet}} = \frac{\rho a V_r (V_{w_1} \pm V_{w_2}) \times u}{\frac{1}{2} m V_1^2}$$

where  $m$  = mass of the fluid per second in the jet =  $\rho a V_1$

$V_1$  = initial velocity of jet

$$\therefore \eta = \frac{\rho a V_r [V_{w_1} \pm V_{w_2}] \times u}{\frac{1}{2} (\rho a V_1) \times V_1^2}. \quad \dots[17.21(B)]$$

**Problem 17.18** A jet of water having a velocity of 20 m/s strikes a curved vane, which is moving with a velocity of 10 m/s. The jet makes an angle of  $20^\circ$  with the direction of motion of vane at inlet and leaves at an angle of  $130^\circ$  to the direction of motion of vane at outlet. Calculate :

- (i) Vane angles, so that the water enters and leaves the vane without shock.
- (ii) Work done per second per unit weight of water striking (or work done per unit weight of water striking) the vane per second.

**Solution.** Given :

Velocity of jet,  $V_1 = 20 \text{ m/s}$

Velocity of vane,  $u_1 = 10 \text{ m/s}$

Angle made by jet at inlet, with direction of motion of vane,

$$\alpha = 20^\circ$$

Angle made by the leaving jet, with the direction of motion

$$= 130^\circ$$

$$\therefore \beta = 180^\circ - 130^\circ = 50^\circ$$

In this problem,  $u_1 = u_2 = 10 \text{ m/s}$

$$V_{r_1} = V_{r_2}$$

(i) **Vane Angle** means angle made by the relative velocities at inlet and outlet, i.e.,  $\theta$  and  $\phi$ .

From Fig. 17.16, in  $\Delta ABD$ , we have  $\tan \theta = \frac{BD}{CD}$

$$= \frac{V_{f_1}}{AD - AC} = \frac{V_{f_1}}{V_{w_1} - u_1} \quad \dots(i)$$

where  $V_{f_1} = V_1 \sin \alpha = 20 \times \sin 20^\circ = 6.84 \text{ m/s}$

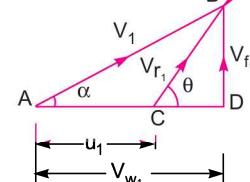
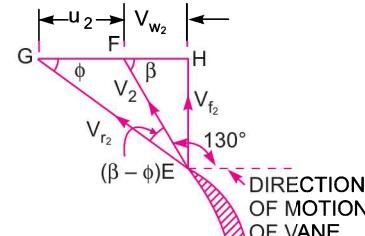


Fig. 17.16

$$V_{w_1} = V_1 \cos \alpha = 20 \times \cos 20^\circ = 18.794 \text{ m/s.}$$

$$u_1 = 10 \text{ m/s}$$

$$\therefore \tan \theta = \frac{6.84}{18.794 - 10} = .7778 \text{ or } \theta = 37.875^\circ$$

$$\therefore \theta = 37^\circ 52.5'. \text{ Ans.}$$

$$\text{From, } \Delta ABC, \quad \sin \theta = \frac{V_{f_1}}{V_{r_1}} \quad \text{or} \quad V_{r_1} = \frac{V_{f_1}}{\sin \theta} = \frac{6.84}{\sin 37.875^\circ} = 11.14$$

$$\therefore V_{r_2} = V_{r_1} = 11.14 \text{ m/s.}$$

From,  $\Delta EFG$ , applying sine rule, we have

$$\frac{V_{r_2}}{\sin (180^\circ - \beta)} = \frac{u_2}{\sin (\beta - \phi)}$$

$$\text{or} \quad \frac{11.14}{\sin \beta} = \frac{10}{\sin [\beta - \phi]} \quad \text{or} \quad \frac{11.14}{\sin 50^\circ} = \frac{10}{\sin [50^\circ - \phi]} \quad (\because \beta = 50^\circ)$$

$$\therefore \sin (50^\circ - \phi) = \frac{10 \times \sin 50^\circ}{11.14} = 0.6876 = \sin 43.44^\circ$$

$$\therefore 50^\circ - \phi = 43.44^\circ \quad \text{or} \quad \phi = 50^\circ - 43.44^\circ = 6.56^\circ$$

$$\therefore \phi = 6^\circ 33.6'. \text{ Ans.}$$

(ii) Work done per second per unit weight of the water striking the vane per second is given by equation (17.21) as

$$= \frac{1}{g} [V_{w_1} + V_{w_2}] \times u \text{ Nm/N} \quad (+ \text{ve sign is taken as } \beta \text{ is an acute angle})$$

$$\text{where } V_{w_1} = 18.794 \text{ m/s, } V_{w_2} = GH - GF = V_{r_2} \cos \phi - u_2 = 11.14 \times \cos 6.56^\circ - 10 = 1.067 \text{ m/s}$$

$$u = u_1 = u_2 = 10 \text{ m/s}$$

$\therefore$  Work done per unit weight of water

$$= \frac{1}{9.81} [18.794 + 1.067] \times 10 \text{ Nm/N} = 20.24 \text{ Nm/N. Ans.}$$

**Problem 17.19** A jet of water having a velocity of 40 m/s strikes a curved vane, which is moving with a velocity of 20 m/s. The jet makes an angle of  $30^\circ$  with the direction of motion of vane at inlet and leaves at an angle of  $90^\circ$  to the direction of motion of vane at outlet. Draw the velocity triangles at inlet and outlet and determine the vane angles at inlet and outlet so that the water enters and leaves the vane without shock.

**Solution.** Given :

Velocity of jet,

$$V_1 = 40 \text{ m/s}$$

Velocity of vane,

$$u_1 = 20 \text{ m/s}$$

Angle made by jet at inlet,

$$\alpha = 30^\circ$$

Angle made by leaving jet

$$= 90^\circ$$

$\therefore$

$$\beta = 180^\circ - 90^\circ = 90^\circ$$

For this problem, we have

$$u_1 = u_2 = u = 20 \text{ m/s}$$

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Vane angles at inlet and outlet are  $\theta$  and  $\phi$  respectively.

From  $\Delta ABCD$ , we have

$$\tan \theta = \frac{BD}{CD} = \frac{BD}{AD - AC} = \frac{V_{f_1}}{V_{w_1} - u_1}$$

where  $V_{f_1} = V_1 \sin \alpha = 40 \times \sin 30^\circ = 20 \text{ m/s}$

$$V_{w_1} = V_1 \cos \alpha = 40 \times \cos 30^\circ = 34.64 \text{ m/s}$$

$$u_1 = 20 \text{ m/s}$$

$$\therefore \tan \theta = \frac{20}{34.64 - 20} = \frac{20}{14.64} = 1.366 = \tan 53.79^\circ$$

$$\therefore \theta = 53.79^\circ \text{ or } 53^\circ 47.4'. \text{ Ans.}$$

Also from  $\Delta ABCD$ ,

$$\sin \theta = \frac{V_{f_1}}{V_{r_1}} \text{ or } V_{r_1} = \frac{V_{f_1}}{\sin \theta} = \frac{20}{\sin 53.79^\circ}$$

$$\therefore V_{r_1} = 24.78$$

$$\text{But } V_{r_2} = V_{r_1} = 24.78$$

$$\text{Hence, from } \Delta EFG, \cos \phi = \frac{u_2}{V_{r_2}} = \frac{20}{24.78} = 0.8071 = \cos 36.18^\circ$$

$$\therefore \phi = 36.18^\circ \text{ or } 36^\circ 10.8'. \text{ Ans.}$$

**Problem 17.20** A jet of water of diameter 50 mm, having a velocity of 20 m/s strikes a curved vane which is moving with a velocity of 10 m/s in the direction of the jet. The jet leaves the vane at an angle of  $60^\circ$  to the direction of motion of vane at outlet. Determine :

- The force exerted by the jet on the vane in the direction of motion.
- Work done per second by the jet.

**Solution.** Given :

Diameter of the jet,  $d = 50 \text{ mm} = 0.05 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.05)^2 = .001963 \text{ m}^2$$

$$\text{Velocity of jet, } V_1 = 20 \text{ m/s}$$

$$\text{Velocity of vane, } u_1 = 10 \text{ m/s}$$

As jet and vane are moving in the same direction,

$$\therefore \alpha = 0$$

Angle made by the leaving jet, with the direction of motion =  $60^\circ$

$$\therefore \beta = 180^\circ - 60^\circ = 120^\circ$$

For this problem, we have

$$u_1 = u_2 = u = 10 \text{ m/s}$$

$$V_{r_1} = V_{r_2}$$

From Fig. 17.18, we have

$$\begin{aligned} V_{r_1} &= AB - AC = V_1 - u_1 \\ &= 20 - 10 = 10 \text{ m/s} \end{aligned}$$

$$V_{w_1} = V_1 = 20 \text{ m/s}$$

$$\therefore V_{r_2} = V_{r_1} = 10 \text{ m/s}$$

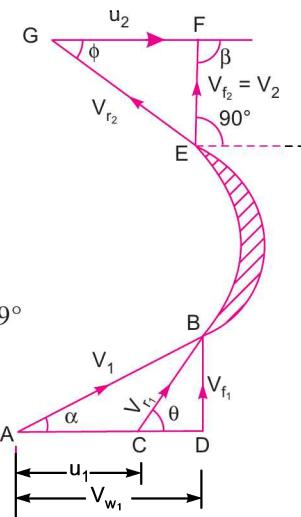


Fig. 17.17

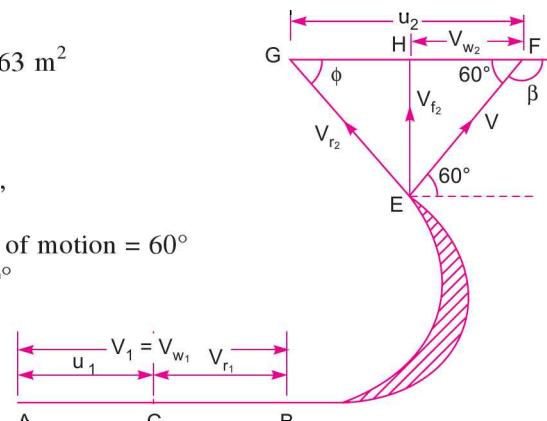


Fig. 17.18

Now in  $\Delta EFG$ ,  $EG = V_{r_2} = 10 \text{ m/s}$ ,

$$GF = u_2 = 10 \text{ m/s}$$

$$\angle GEF = 180^\circ - (60^\circ + \phi) = (120^\circ - \phi)$$

From sine rule, we have

$$\frac{EG}{\sin 60^\circ} = \frac{GF}{\sin (120^\circ - \phi)} \quad \text{or} \quad \frac{10}{\sin 60^\circ} = \frac{10}{\sin (120^\circ - \phi)}$$

or

$$\sin 60^\circ = \sin (120^\circ - \phi)$$

$$\therefore 60^\circ = 120^\circ - \phi \quad \text{or} \quad \phi = 120^\circ - 60^\circ = 60^\circ$$

Now

$$V_{w_2} = HF = GF - GH$$

$$= u_2 - V_{r_2} \cos \phi = 10 - 10 \times \cos 60^\circ = 10 - 5 = 5 \text{ m/s.}$$

(i) The force exerted by the jet on the vane in the direction of motion is given by equation (17.19) as

$$F_x = \rho a V_h [V_{w_1} - V_{w_2}] \quad (\text{-ve sign is taken as } \beta \text{ is an obtuse angle}) \\ = 1000 \times .001963 \times 10 [20 - 5] \text{ N} = 294.45 \text{ N. Ans.}$$

(ii) Work done per second by the jet

$$= F_x \times u = 294.45 \times 10 = 2944.5 \text{ N m/s}$$

$$= 2944.5 \text{ W. Ans.} \quad [\because \text{Nm / s} = \text{W (watt)}]$$

**Problem 17.21** A jet of water having a velocity of 15 m/s strikes a curved vane which is moving with a velocity of 5 m/s. The vane is symmetrical and is so shaped that the jet is deflected through  $120^\circ$ . Find the angle of the jet at inlet of the vane so that there is no shock. What is the absolute velocity of the jet at outlet in magnitude and direction and the work done per unit weight of water. Assume the vane to be smooth.

**Solution.** Given :

$$\text{Velocity of jet, } V_1 = 15 \text{ m/s}$$

$$\text{Velocity of vane, } u_1 = 5 \text{ m/s}$$

As vane is symmetrical. Hence angle  $\theta = \phi$

$$\text{Angle of deflection of the jet} = 120^\circ = 180^\circ - (\theta + \phi)$$

$$\therefore \theta + \phi = 60^\circ \text{ or each angle, i.e., } \theta = \phi = 30^\circ$$

Let the angle of jet at inlet =  $\alpha$

Absolute velocity of jet at outlet =  $V_2$

Angle made by  $V_2$  at outlet with direction of motion of vane =  $\beta^*$ .

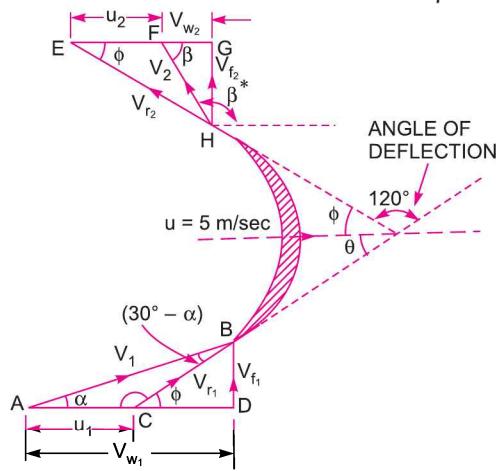


Fig. 17.19

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For this problem,  $u_1 = u_2 = u = 5 \text{ m/s}$

$$V_{r_1} = V_{r_2} \text{ (as vane is smooth)}$$

Applying the sine rule to  $\Delta ACB$ ,

$$\frac{AB}{\sin(180^\circ - \theta)} = \frac{AC}{\sin(30^\circ - \alpha)} \quad \text{or} \quad \frac{V_1}{\sin(180^\circ - 30^\circ)} = \frac{u_1}{\sin(30^\circ - \alpha)}$$

or

$$\frac{15}{\sin 30^\circ} = \frac{5}{\sin(30^\circ - \alpha)} \quad \text{or} \quad \sin(30^\circ - \alpha) = \frac{5 \sin 30^\circ}{15}$$

$$= \frac{1}{3} \times 0.5 = .1667 = \sin 9.596^\circ$$

$$\therefore 30^\circ - \alpha = 9.596^\circ \quad \text{or} \quad \alpha = 30^\circ - 9.596^\circ = 20.404^\circ \quad \text{or} \quad 20^\circ 24'. \text{ Ans.}$$

Also from sine rule to  $\Delta ACB$ , we have

$$\frac{AB}{\sin(180^\circ - 30^\circ)} = \frac{CB}{\sin \alpha} \quad \text{or} \quad \frac{V_1}{\sin 30^\circ} = \frac{V_{r_1}}{\sin 20.404^\circ}$$

$$\therefore V_{r_1} = \frac{V_1 \sin 20.404^\circ}{\sin 30^\circ} = 10.46 \text{ m/s}$$

$$\therefore V_{r_2} = V_{r_1} = 10.46 \text{ m/s}$$

From velocity  $\Delta HEG$  at outlet,

$$V_{r_2} \cos \phi = u_2 + V_{w_2} \quad \text{or} \quad 10.46 \cos 30^\circ = 5.0 + V_{w_2}$$

$$\therefore V_{w_2} = 10.46 \cos 30^\circ - 5.0 = 4.06 \text{ m/s}$$

Also, we have

$$V_{r_2} \sin \phi = V_{f_2} \quad \text{or} \quad V_{f_2} = 10.46 \sin 30^\circ = 5.23 \text{ m/s}$$

In  $\Delta HFG$ ,

$$V_2 = \sqrt{V_{f_2}^2 + V_{w_2}^2} = \sqrt{5.23^2 + 4.06^2}$$

$$= \sqrt{27.353 + 16.483} = 6.62 \text{ m/s. Ans.}$$

$$\tan \beta = \frac{V_{f_2}}{V_{w_2}} = \frac{5.23}{4.06} = 1.288 = \tan 52.17^\circ$$

$$\therefore \beta = 52.17^\circ \quad \text{or} \quad 52^\circ 10.2'$$

$$\therefore \text{Angle made by absolute velocity at outlet with the direction of motion } \beta^* \\ = 180^\circ - \beta = 180^\circ - (52^\circ 10.2') = 127^\circ 49.8'$$

$$\therefore \beta^* = 127^\circ 49.8. \text{ Ans.}$$

Work done\* per unit weight of the water striking

$$\begin{aligned} &= \frac{1}{g} [V_{w_1} + V_{w_2}] \times u \text{ Nm} \quad (\because +ve sign taken as } \beta \text{ is an acute angle) \\ &= \frac{1}{9.81} [V_1 \cos \alpha + 4.06] \times 5 \quad (\because V_{w_1} = V_1 \cos \alpha) \\ &= \frac{5}{9.81} [15 \cos 20.404^\circ + 4.06] = 9.225 \text{ Nm/N. Ans.} \end{aligned}$$

---

\* Work done per unit weight of water striking is the same as work done per second per unit weight of water striking per second refer to equation (17.21).

**Problem 17.22** A jet of water moving at 12 m/s impinges on vane shaped to deflect the jet through  $120^\circ$  when stationary. If the vane is moving at 5 m/s, find the angle of the jet so that there is no shock at inlet. What is the absolute velocity of the jet at exit in magnitude and direction and the work done per second per unit weight of water striking per second? Assume that the vane is smooth.

**Solution.** Given :

$$\text{Velocity of jet, } V_1 = 12 \text{ m/s}$$

$$\text{Velocity of vane, } u = u_1 = u_2 = 5 \text{ m/s}$$

$$\text{Angle of deflection of jet} = 120^\circ$$

$$\therefore \theta + \phi = 180^\circ - 120^\circ = 60^\circ.$$

It is not given that the vane is symmetrical and without this condition problem cannot be solved.

Assuming vane to be symmetrical, we have  $\theta = \phi$

$$\text{Then } \theta = \phi = 30^\circ$$

(i) Angle of jet at inlet with the direction of motion of vane =  $\alpha$

In  $\Delta ABC$ , applying sine rule, we have

$$\frac{AB}{\sin(180^\circ - \theta)} = \frac{AC}{\sin(30^\circ - \alpha)} \text{ or } \frac{V_1}{\sin \theta} = \frac{u_1}{\sin(30^\circ - \alpha)} \text{ or } \frac{12}{\sin 30^\circ} = \frac{5}{\sin(30^\circ - \alpha)}$$

$$\therefore \sin(30^\circ - \alpha) = \frac{5 \sin 30^\circ}{12} = 0.2083 = \sin 12.02^\circ$$

$$\therefore 30^\circ - \alpha = 12.02^\circ \text{ or } \alpha = 30^\circ - 12.02^\circ = 17.98^\circ \text{ or } 17^\circ 59'. \text{ Ans.}$$

Again applying sine rule to  $\Delta ABC$ , we have

$$\frac{V_1}{\sin(180^\circ - \theta)} = \frac{V_{r_1}}{\sin \alpha} \text{ or } \frac{12}{\sin \theta} = \frac{V_{r_1}}{\sin 17.98^\circ}$$

$$\therefore V_{r_1} = \frac{12 \sin 17.98^\circ}{\sin \theta} = \frac{12 \times \sin 17.98^\circ}{\sin 30^\circ} = 7.41 \text{ m/s}$$

$$\text{In } \Delta ABD, V_{w_1} = V_1 \times \cos \alpha = 12 \cos 17.98^\circ = 11.41 \text{ m/s}$$

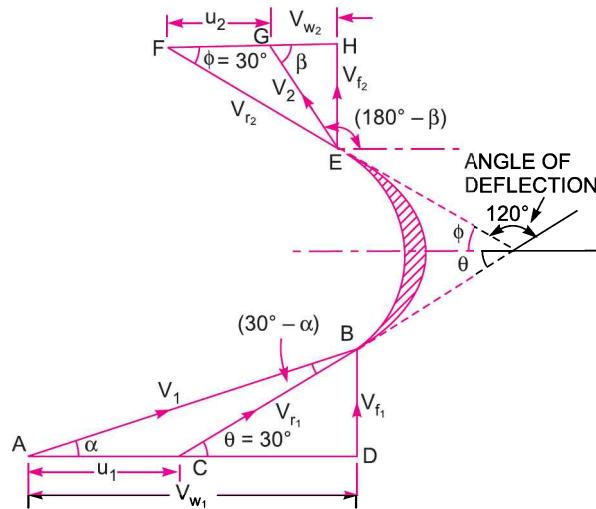


Fig. 17.20

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(ii) The absolute velocity of jet at outlet =  $V_2$

The angle made by  $V_2$  at outlet with the direction of the motion of vane =  $180^\circ - \beta$

Now as vane is given smooth,  $V_{r_2} = V_{f_1} = 7.41 \text{ m/s}$

At outlet, from  $\Delta E F H$ , we have  $V_{r_2} \cos \phi = u_2 + V_{w_2}$  or  $7.41 \cos 30^\circ = 5 + V_{w_2}$

$$\therefore V_{w_2} = 7.41 \cos 30^\circ - 5.0 = 1.417 \text{ m/s}$$

$$\text{Also } V_{f_2} = V_{r_2} \sin 30^\circ = 7.41 \sin 30^\circ = 3.705 \text{ m/s}$$

$$\text{And } \tan \beta = \frac{V_{f_2}}{V_{w_2}} = \frac{3.705}{1.417} = 2.614 = \tan 69.07^\circ$$

$$\therefore \beta = 69.07^\circ \text{ or } 69^\circ 4.2'$$

$$\therefore \text{Angle made by } V_2 \text{ at outlet with the direction of motion of vane} \\ = 180^\circ - \beta = 180^\circ - (69^\circ 4.2') = 110^\circ 55.8'. \text{ Ans.}$$

$$\text{Also } V_2 = \sqrt{V_{f_2}^2 + V_{w_2}^2} = \sqrt{(3.705)^2 + (1.417)^2} = \sqrt{13.727 + 2.007} \\ = 3.96 \text{ m/s. Ans.}$$

(iii) Work done per second per unit weight of water striking per second

$$= \frac{1}{g} [V_{w_1} + V_{w_2}] u = \frac{1}{9.81} [11.41 + 1.417] \times 5 = 6.537 \text{ Nm/N. Ans.}$$

**Problem 17.23** A jet of water having a velocity of 15 m/s, strikes a curved vane which is moving with a velocity of 5 m/s in the same direction as that of the jet at inlet. The vane is so shaped that the jet is deflected through  $135^\circ$ . The diameter of jet is 100 mm. Assuming the vane to be smooth, find :

(i) Force exerted by the jet on the vane in the direction of motion,

(ii) Power exerted on the vane, and

(iii) Efficiency of the vane.

**Solution.** Given :

Velocity of jet,  $V_1 = 15 \text{ m/s}$

Velocity of vane,  $u = u_1 = u_2 = 5 \text{ m/s}$

At inlet jet and vane are in the same direction, hence  $\alpha = 0$

Diameter of jet,  $d = 100 \text{ mm} = 0.1 \text{ m}$

$$\therefore \text{Area, } a = \frac{\pi}{4} (0.1)^2 = .007854 \text{ m}^2$$

$$\text{Angle of deflection of the jet} = 135^\circ = 180^\circ - \phi \quad (\because \theta = 0^\circ) \\ \therefore \phi = 180^\circ - 135^\circ = 45^\circ$$

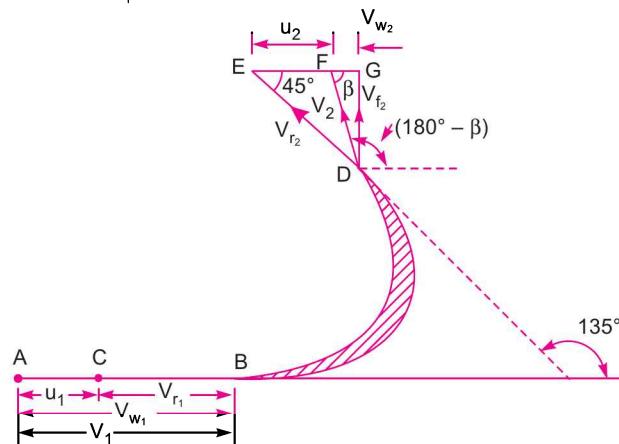


Fig. 17.21

As vane is given smooth hence  $V_{r_1} = V_{r_2}$

From the inlet velocity triangle, which is a straight line in this case, we have

$$V_{r_1} = V_1 - u_1 = 15 - 5 = 10 \text{ m/s}$$

$$V_{w_1} = V_1 = 15 \text{ m/s}$$

From the outlet velocity triangle  $DEG$ , we have

$$V_{r_2} = V_{r_1} = 10 \text{ m/s}$$

$$u_2 = u_1 = u = 5 \text{ m/s}$$

$$V_{r_2} \cos \phi = u_2 + V_{w_2} \quad \text{or} \quad 10 \cos 45^\circ = 5 + V_{w_2}$$

$$\therefore V_{w_2} = 10 \cos 45^\circ - 5 = 7.07 - 5 = 2.07 \text{ m/s.}$$

(i) Force exerted by the jet on the vane in the direction of motion is given by equation (17.19) as

$$F_x = \rho a V_{r_1} [V_{w_1} + V_{w_2}] \quad (\text{+ve sign is taken as } \beta \text{ is an acute angle}) \\ = 1000 \times .007854 \times 10[15 + 2.07] = 1340.6 \text{ N. Ans.}$$

(ii) Power of the vane is given as

$$= F_x \times u \text{ N m/s} = 1340.6 \times 5 = 6703 \text{ W} = 6.703 \text{ kW. Ans.}$$

(iii) Efficiency of the vane

$$= \frac{\text{Work done per second on vane}}{\text{Kinetic energy supplied by jet per second}}$$

$$= \frac{F_x \times u}{\frac{1}{2} \times (\text{mass per second}) \times V^2} = \frac{F_x \times u}{\frac{1}{2} (\rho a V_1) \times V_1^2}$$

$$= \frac{1340.6 \times 5.0}{\frac{1}{2} \times (1000 \times .007854 \times 15) \times 15^2} = 0.505 = 50.5\%. \text{ Ans.}$$

**17.4.5 Force Exerted by a Jet of Water on a Series of Vanes.** The force exerted by a jet of water on a *single* moving plate (which may be flat or curved) is not practically feasible. This case is only a theoretical one. In actual practice, a large number of plates are mounted on the circumference of a wheel at a fixed distance apart as shown in Fig. 17.22. The jet strikes a plate and due to the force exerted by the jet on the plate, the wheel starts moving and the 2nd plate mounted on the wheel appears before the jet, which again exerts the force on the 2nd plate. Thus each plate appears successively before the jet and the jet exerts force on each plate. The wheel starts moving at a constant speed.

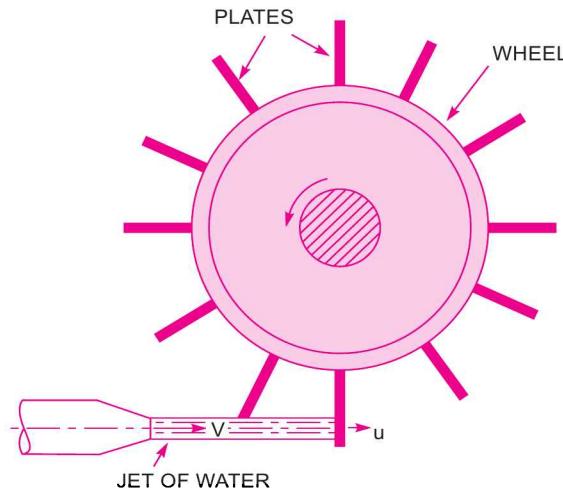


Fig. 17.22 Jet striking a series of vanes.

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Let

$V$  = Velocity of jet,  
 $d$  = Diameter of jet,  
 $a$  = Cross-sectional area of jet,

$$= \frac{\pi}{4} d^2$$

$u$  = Velocity of vane.

In this case the mass of water coming out from the nozzle per second is always in contact with the plates, when all the plates are considered. Hence mass of water per second striking the series of plates =  $\rho aV$ .

Also the jet strikes the plate with a velocity =  $(V - u)$ .

After striking, the jet moves tangential to the plate and hence the velocity component in the direction of motion of plate is equal to zero.

∴ The force exerted by the jet in the direction of motion of plate,

$$\begin{aligned} F_x &= \text{Mass per second [Initial velocity - Final velocity]} \\ &= \rho aV[(V - u) - 0] = \rho aV[V - u] \end{aligned} \quad \dots(17.22)$$

Work done by the jet on the series of plates per second

$$\begin{aligned} &= \text{Force} \times \text{Distance per second in the direction of force} \\ &= F_x \times u = \rho aV[V - u] \times u \end{aligned}$$

Kinetic energy of the jet per second

$$= \frac{1}{2} mV^2 = \frac{1}{2} (\rho aV) \times V^2 = \frac{1}{2} \rho aV^3$$

$$\therefore \text{Efficiency, } \eta = \frac{\text{Work done per second}}{\text{Kinetic energy per second}} = \frac{\rho aV[V - u] \times u}{\frac{1}{2} \rho aV^3} = \frac{2u[V - u]}{V^2} \quad \dots(17.23)$$

**Condition for Maximum Efficiency.** Equation (17.23) gives the value of the efficiency of the wheel. For a given jet velocity  $V$ , the efficiency will be maximum when

$$\frac{d\eta}{du} = 0 \quad \text{or} \quad \frac{d}{du} \left[ \frac{2u(V - u)}{V^2} \right] = 0 \quad \text{or} \quad \frac{d}{du} \left[ \frac{2uV - 2u^2}{V^2} \right] = 0$$

$$\text{or} \quad \frac{2V - 2 \times 2u}{V^2} = 0 \quad \text{or} \quad 2V - 4u = 0 \quad \text{or} \quad V = \frac{4u}{2} = 2u \quad \text{or} \quad u = \frac{V}{2}. \quad \dots(17.24)$$

**Maximum Efficiency.** Substituting the value of  $V = 2u$  in equation (17.23), we get the maximum efficiency as

$$\eta_{\max} = \frac{2u[2u - u]}{(2u)^2} = \frac{2u \times u}{2u \times 2u} = \frac{1}{2} = 0.5 \text{ or } 50%. \quad \dots(17.25)$$

**17.4.6 Force Exerted on a Series of Radial Curved Vanes.** For a radial curved vane, the radius of the vane at inlet and outlet is different and hence the tangential velocities of the radial vane at inlet and outlet will not be equal. Consider a series of radial curved vanes mounted on a wheel as shown in Fig. 17.23. The jet of water strikes the vanes and the wheel starts rotating at a constant angular speed.

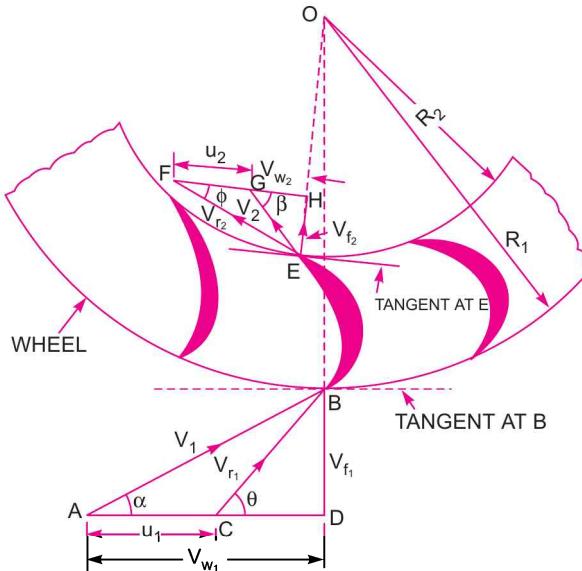


Fig. 17.23 Series of radial curved vanes mounted on a wheel.

Let  $R_1$  = Radius of wheel at inlet of the vane,

$R_2$  = Radius of the wheel at the outlet of the vane,

$\omega$  = Angular speed of the wheel.

Then  $u_1 = \omega R_1$  and  $u_2 = \omega R_2$

The velocity triangles at inlet and outlet are drawn as shown in Fig. 17.23.

The mass of water striking per second for a series of vanes

= Mass of water coming out from nozzle per second

=  $\rho a V_1$ , where  $a$  = Area of jet and  $V_1$  = Velocity of jet.

Momentum of water striking the vanes in the tangential direction per sec at inlet

= Mass of water per second  $\times$  Component of  $V_1$  in the tangential direction

=  $\rho a V_1 \times V_{w_1}$  ( $\because$  Component of  $V_1$  in tangential direction =  $V_1 \cos \alpha = V_{w_1}$ )

Similarly, momentum of water at outlet per sec

=  $\rho a V_1 \times$  Component of  $V_2$  in the tangential direction

=  $\rho a V_1 \times (-V_2 \cos \beta) = -\rho a V_1 \times V_{w_2}$  ( $\because V_2 \cos \beta = V_{w_2}$ )

-ve sign is taken as the velocity  $V_2$  at outlet is in opposite direction.

Now, angular momentum per second at inlet

= Momentum at inlet  $\times$  Radius at inlet

=  $\rho a V_1 \times V_{w_1} \times R_1$

Angular momentum per second at outlet

= Momentum of outlet  $\times$  Radius at outlet

=  $-\rho a V_1 \times V_{w_2} \times R_2$

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Torque exerted by the water on the wheel,

$$\begin{aligned} T &= \text{Rate of change of angular momentum} \\ &= [\text{Initial angular momentum per second} - \text{Final angular momentum per second}] \\ &= \rho a V_1 \times V_{w_1} \times R_1 - (-\rho a V_1 \times V_{w_2} \times R_2) = \rho a V_1 [V_{w_1} \times R_1 + V_{w_2} \times R_2] \end{aligned}$$

Work done per second on the wheel

$$\begin{aligned} &= \text{Torque} \times \text{Angular velocity} = T \times \omega \\ &= \rho a V_1 [V_{w_1} \times R_1 + V_{w_2} \times R_2] \times \omega = \rho a V_1 [V_{w_1} \times R_1 \times \omega + V_{w_2} \times R_2 \times \omega] \\ &= \rho a V_1 [V_{w_1} u_1 + V_{w_2} u_2] \quad (\because u_1 = \omega R_1 \text{ and } u_2 = \omega R_2) \end{aligned}$$

If the angle  $\beta$  in Fig. 17.23 is an obtuse angle then work done per second will be given as

$$\begin{aligned} &= \rho a V_1 [V_{w_1} u_1 - V_{w_2} u_2] \\ \therefore \text{The general expression for the work done per second on the wheel} \\ &= \rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2] \quad \dots(17.26) \end{aligned}$$

If the discharge is radial at outlet, then  $\beta = 90^\circ$  and work done becomes as

$$= \rho a V_1 [V_{w_1} u_1] \quad (\because V_{w_2} = 0) \quad \dots(17.27)$$

### Efficiency of the Radial Curved Vane

The work done per second on the wheel is the output of the system whereas the initial kinetic energy per second of the jet is the input. Hence, efficiency of the system is expressed as

$$\begin{aligned} \text{Efficiency, } \eta &= \frac{\text{Work done per second}}{\text{Kinetic energy per second}} = \frac{\rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2]}{\frac{1}{2} (\text{mass/sec}) \times V_1^2} \\ &= \frac{\rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2]}{\frac{1}{2} (\rho a V_1) \times V_1^2} = \frac{2 [V_{w_1} u_1 \pm V_{w_2} u_2]}{V_1^2}. \quad \dots(17.28) \end{aligned}$$

If there is no loss of energy when water is flowing over the vanes, the work done on the wheel per second is also equal to the change in kinetic energy of the jet per second. Hence, the work done per second on the wheel is also given as

Work done per second on the wheel

$$\begin{aligned} &= \text{Change of K.E. per second of the jet} \\ &= (\text{Initial K.E. per second} - \text{Final K.E. per second}) \text{ of the jet} \\ &= \left( \frac{1}{2} m V_1^2 - \frac{1}{2} m V_2^2 \right) \\ &= \frac{1}{2} m (V_1^2 - V_2^2) = \frac{1}{2} (\rho a V_1^2) (V_1^2 - V_2^2) \quad (\because \text{mass/second} = \rho a V_1) \end{aligned}$$

Hence efficiency,  $\eta = \frac{\text{Work done per second on the wheel}}{\text{Initial K.E. per second of the jet}}$

$$\begin{aligned} &= \frac{\frac{1}{2} \rho a V_1^2 (V_1^2 - V_2^2)}{\frac{1}{2} (\rho a V_1^2) \cdot V_1^2} \end{aligned}$$

$$= \frac{V_1^2 - V_2^2}{V_1^2} = \left(1 - \frac{V_2^2}{V_1^2}\right) \quad \dots(17.28A)$$

From the above equation, it is clear that for a given initial velocity of the jet (*i.e.*,  $V_1$ ), the efficiency will be maximum, when  $V_2$  is minimum. But  $V_2$  cannot be zero as in that case the incoming jet will not move out of the vane. Equation (17.28) also gives the efficiency of the system. From this equation, it is clear that efficiency will be maximum when  $V_{w_2}$  is added to  $V_{w_1}$ . This is only possible if  $\beta$  is an acute\* angle. Also for maximum efficiency  $V_{w_2}$  should also be maximum. This is only possible if  $\beta = 0$ . In that case  $V_{w_2} = V_2$  and angle  $\phi$  will be zero. But in actual practice  $\phi$  cannot be zero. Hence for maximum efficiency, the angle  $\phi$  should be minimum.

**Problem 17.24** If in Problem 17.23, the jet of water instead of striking a single plate, strikes a series of curved vanes, find for the data given Problem 17.23,

- (i) Force exerted by the jet on the vane in the direction of motion,
- (ii) Power exerted on the vane, and
- (iii) Efficiency of the vane.

**Solution.** Given :

From Problem 17.23,  $V_1 = 15 \text{ m/s}$ ,  $u = u_1 = u_2 = 5 \text{ m/s}$ ,  
 $\alpha = 0$ ,  $a = .007854 \text{ m}^2$   
 $\phi = 45^\circ$ ,  $V_{w_1} = 15 \text{ m/s}$  and  $V_{w_2} = 2.07 \text{ m/s}$ .

For the series of vanes, mass of water striking per second

$$\begin{aligned} &= \text{Mass of water coming out from nozzle} \\ &= \rho a V_1 = 1000 \times .007854 \times 15 = 117.72 \end{aligned}$$

- (i) Force exerted by the jet on the vane in the direction of motion

$$F_x = \rho a V_1 [V_{w_1} + V_{w_2}] = 117.72 [15 + 2.07] = 2009.5 \text{ N. Ans.}$$

- (ii) Power of the vane in kW

$$\begin{aligned} &= \frac{\text{Work done per second}}{1000} = \frac{F_x \times u}{1000} \text{ kW} = \frac{2009.5 \times 5}{1000} \\ &= 10.05 \text{ kW. Ans.} \end{aligned}$$

- (iii) Efficiency,

$$\begin{aligned} \eta &= \frac{\text{Work done per second}}{\frac{1}{2} (\text{mass of water per sec}) \times V_1^2} \\ &= \frac{2009.5 \times 5.0}{\frac{1}{2} \times 117.72 \times 15^2} = 0.7586 \text{ or } 75.86\%. \text{ Ans.} \end{aligned}$$

**Problem 17.25** A jet of water having a velocity of 35 m/s impinges on a series of vanes moving with a velocity of 20 m/s. The jet makes an angle of  $30^\circ$  to the direction of motion of vanes when entering and leaves at an angle of  $120^\circ$ . Draw the triangles of velocities at inlet and outlet and find :

- (a) the angles of vanes tips so that water enters and leaves without shock,
- (b) the work done per unit weight of water entering the vanes, and
- (c) the efficiency.

**Solution.** Given :

Velocity of jet,	$V_1 = 35 \text{ m/s}$
Velocity of vane,	$u_1 = u_2 = 20 \text{ m/s}$
Angle of jet at inlet,	$\alpha = 30^\circ$

---

\* The work done is equal to torque multiplied by  $\omega$  (angular velocity). Torque is the rate of change of angular momentum. Due to change of angular momentum (*i.e.*, initial angular momentum – final angular momentum),  $V_{w_2}$  should be in opposite direction so that it can be added to  $V_{w_1}$ . This is possible if  $\beta < 90^\circ$ .

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Angle made by the jet at outlet with the direction of motion of vanes =  $120^\circ$

$$\therefore \text{Angle} \quad \beta = 180^\circ - 120^\circ = 60^\circ$$

(a) Angle of vane tips.

From inlet velocity triangle

$$V_{w_1} = V_1 \cos \alpha = 35 \cos 30^\circ = 30.31 \text{ m/s}$$

$$V_{f_1} = V_1 \sin \alpha = 35 \sin 30^\circ = 17.50 \text{ m/s}$$

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{17.50}{30.31 - 20} = 1.697$$

$$\therefore \theta = \tan^{-1} 1.697 = 60^\circ. \text{ Ans.}$$

$$\text{By sine rule, } \frac{V_{r_1}}{\sin 90^\circ} = \frac{V_{f_1}}{\sin \theta} \quad \text{or} \quad \frac{V_{r_1}}{1} = \frac{17.50}{\sin 60^\circ}$$

$$\therefore V_{r_1} = \frac{17.50}{0.866} = 20.25 \text{ m/s.}$$

$$\text{Now, } V_{r_2} = V_{r_1} = 20.25 \text{ m/s}$$

From outlet velocity triangle, by sine rule

$$\frac{V_{r_2}}{\sin 120^\circ} = \frac{u_2}{\sin (60^\circ - \phi)} \quad \text{or} \quad \frac{20.25}{0.866} = \frac{20}{\sin (60^\circ - \phi)}$$

$$\therefore \sin (60^\circ - \phi) = \frac{20 \times 0.866}{20.25} = 0.855 = \sin (58.75^\circ)$$

$$60^\circ - \phi = 58.75^\circ$$

$$\therefore \phi = 60^\circ - 58.75^\circ = 1.25^\circ. \text{ Ans.}$$

$$(b) \text{Work done per unit weight of water entering} = \frac{1}{g} (V_{w_1} + V_{w_2}) \times u_1 \quad \dots(i)$$

$$V_{w_1} = 30.31 \text{ m/s and } u_1 = 30 \text{ m/s}$$

The value of  $V_{w_2}$  is obtained from outlet velocity triangle

$$V_{w_2} = V_{r_2} \cos \phi - u_2 = 20.25 \cos 1.25^\circ - 20.0 = 0.24 \text{ m/s}$$

$$\therefore \text{Work done/unit weight} = \frac{1}{9.81} [30.31 + 0.24] \times 20 = 62.28 \text{ Nm/N. Ans.}$$

$$(c) \text{Efficiency} = \frac{\text{Work done per kg}}{\text{Energy supplied per kg}}$$

$$= \frac{62.28}{\frac{V_1^2}{2g}} = \frac{62.28 \times 2 \times 9.81}{35 \times 35} = 99.74\% \text{ Ans.}$$

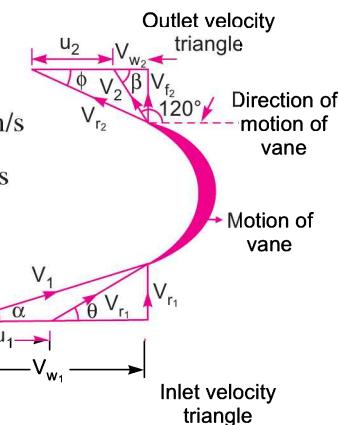


Fig. 17.23(a)

**Problem 17.26** A jet of water having a velocity of 30 m/s strikes a series of radial curved vanes mounted on a wheel which is rotating at 200 r.p.m. The jet makes an angle of  $20^\circ$  with the tangent to the wheel at inlet and leaves the wheel with a velocity of 5 m/s at an angle of  $130^\circ$  to the tangent to the wheel at outlet. Water is flowing from outward in a radial direction. The outer and inner radii of the wheel are 0.5 m and 0.25 m respectively. Determine :

- (i) Vane angles at inlet and outlet, (ii) Work done per unit weight of water, and
- (iii) Efficiency of the wheel.

**Solution.** Given :

$$\text{Velocity of jet, } V_1 = 30 \text{ m/s}$$

$$\text{Speed of wheel, } N = 200 \text{ r.p.m.}$$

$$\therefore \text{Angular speed, } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.94 \text{ rad/s}$$

$$\text{Angle of jet at inlet, } \alpha = 20^\circ$$

$$\text{Velocity of jet at outlet, } V_2 = 5 \text{ m/s}$$

$$\text{Angle made by the jet at outlet with the tangent to wheel} = 130^\circ$$

$$\therefore \text{Angle, } \beta = 180^\circ - 130^\circ = 50^\circ$$

$$\text{Outer radius, } R_1 = 0.5 \text{ m}$$

$$\text{Inner radius, } R_2 = 0.25 \text{ m}$$

$$\therefore \text{Velocity } u_1 = \omega \times R_1 = 20.94 \times 0.5 = 10.47 \text{ m/s}$$

$$\text{And } u_2 = \omega \times R_2 = 20.94 \times 0.25 = 5.235 \text{ m/s.}$$

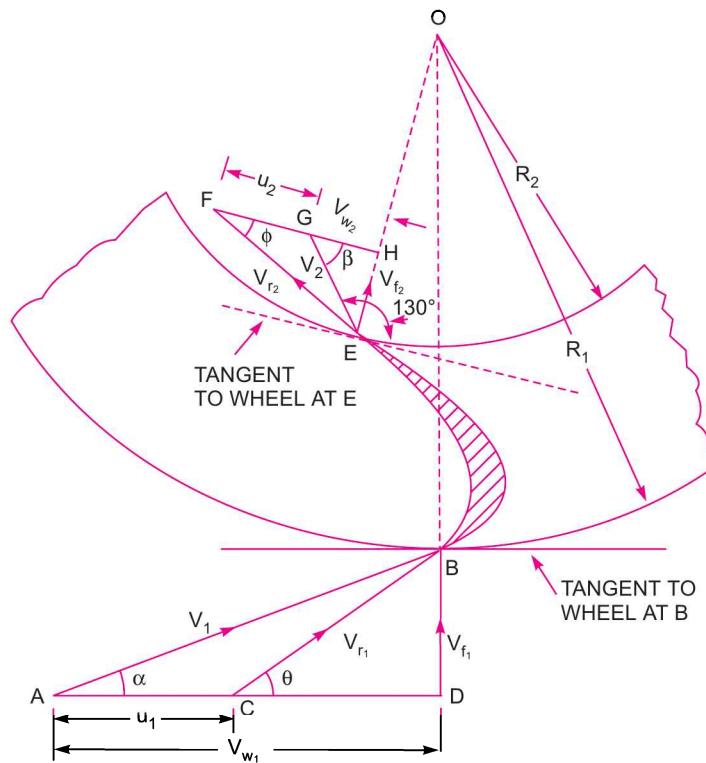


Fig. 17.24

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(i) **Vane angles** at inlet and outlet means the angle made by the relative velocities  $V_{r_1}$  and  $V_{r_2}$ , i.e., angle  $\theta$  and  $\phi$ .

From  $\Delta ABD$ ,

$$V_{w_1} = V_1 \cos \alpha = 30 \times \cos 20^\circ = 28.19 \text{ m/s}$$

$$V_{f_1} = V_1 \sin \alpha = 30 \times \sin 20^\circ = 10.26 \text{ m/s}$$

$$\text{In } \Delta CBD, \quad \tan \theta = \frac{BD}{CD} = \frac{V_{f_1}}{AD - AC} = \frac{10.26}{V_{w_1} - u_1} = \frac{10.26}{28.19 - 10.47} = 0.579 = \tan 30.07$$

$$\therefore \theta = 30.07^\circ \text{ or } 30^\circ 4.2'. \text{ Ans.}$$

From outlet velocity  $\Delta$ ,  $V_{w_2} = V_2 \cos \beta = 5 \times \cos 50^\circ = 3.214 \text{ m/s}$

$$V_{f_2} = V_2 \times \sin \beta = 5 \sin 50^\circ = 3.83 \text{ m/s}$$

$$\text{In } \Delta EFH, \quad \tan \phi = \frac{V_{f_2}}{u_2 + V_{w_2}} = \frac{3.83}{5.235 + 3.214} = 0.453 = \tan 24.385^\circ$$

$$\therefore \phi = 24.385^\circ \text{ or } 24^\circ 23.1'. \text{ Ans.}$$

(ii) Work done per second by water is given by equation (17.26)

$$= \rho a V_1 [V_{w_1} u_1 + V_{w_2} u_2]$$

(+ ve sign is taken as  $\beta$  is acute angle in Fig.17.24)

$\therefore$  Work done\* per second per unit weight of water striking per second

$$= \frac{\rho a V_1 [V_{w_1} u_1 + V_{w_2} u_2]}{\text{Weight of water/s}} = \frac{\rho a V_1 [V_{w_1} u_1 + V_{w_2} u_2]}{\rho a V_1 \times g}$$

$$= \frac{1}{g} [V_{w_1} u_1 + V_{w_2} u_2] \text{ Nm/N} = \frac{1}{9.81} [28.19 \times 10.47 + 3.214 \times 5.235]$$

$$= \frac{1}{9.81} [295.15 + 16.82] = 31.8 \text{ Nm/N. Ans.}$$

(iii) Efficiency,  $\eta$  is given by equation (17.28) as

$$\eta = \frac{2 [V_{w_1} u_1 + V_{w_2} u_2]}{V_1^2} = \frac{2 [28.19 \times 10.47 + 3.214 \times 5.235]}{30^2}$$

$$= \frac{2 [295.15 + 16.82]}{30 \times 30} = 0.6932 \text{ or } 69.32\%. \text{ Ans.}$$

## ► 17.5 JET PROPULSION

Jet propulsion means the propulsion or movement of the bodies such as ships, aircrafts, rocket etc., with the help of jet. The reaction of the jet coming out from the orifice provided in the bodies is used to move the bodies. This is explained as given below.

\* Work done per second per unit weight striking per second is same as work done per unit weight of water.

A jet of fluid coming out from an orifice or nozzle, when strikes a plate, exerts a force on the plate. The magnitude of the force exerted on the plate can be determined depending upon whether plate is flat, inclined, curved, stationary or moving. This force exerted by the jet on the plate is called as '*action of the jet*'. But according to Newton's third law of motion, every action is accompanied by an equal and opposite reaction. Hence the jet while coming out of the orifice or nozzle, exerts a force on the orifice or nozzle in the opposite direction in which jet is coming out. The magnitude of the force exerted is equal to the '*action of the jet*'. This force which is acting on the orifice or nozzle in the opposite direction is called the '*reaction of the jet*'. If the body in which orifice or nozzle is fitted, is free to move, the body will start moving in the direction opposite to the jet. The following cases are important where this principle is used :

- (a) Jet propulsion of a tank to which orifice is fitted, and
- (b) Jet propulsion of ships.

**17.5.1 Jet Propulsion of a Tank with an Orifice.** Consider a large tank fitted with an orifice in one of its sides as shown in Fig. 17.25.

Let

$$H = \text{Constant head of water in tank from the centre of orifice},$$

$$a = \text{Area of orifice},$$

$$V = \text{Velocity of the jet of water},$$

$$C_v = \text{Co-efficient of the velocity of orifice}.$$

Then

$$V = C_v \sqrt{2gH}$$

And mass of water coming out from the orifice per second

$$= \rho \times \text{Volume per second} = \rho \times (\text{Area} \times \text{Velocity})$$

$$= \rho \times a \times V$$

Force acting on the water is equal to the rate of change of momentum.

or

$$F = \text{Mass per second} \times [\text{Change of velocity}]$$

$$= \text{Mass per second} \times [\text{Final velocity} - \text{Initial velocity}].$$

**Note.** Here change of velocity is to be taken as final minus initial as we are finding force on water and not force exerted by water.

Initial velocity of water in the tank is zero and final velocity of water when it comes out in the form of jet is equal to  $V$ .

$$\therefore F = \rho a V [V - 0] = \rho a V^2 \quad \dots(17.29)$$

Thus,  $F$  is the force exerted on the jet of water. This jet of water will exert a force on the tank which is equal to  $F$  but opposite in direction as shown in Fig. 17.25. The force will be acting at  $A$ , the point on the tank in the horizontal line of the centre of the orifice. If the tank is free to move or the tank is fitted with frictionless wheel, it will start moving with some velocity say, ' $u$ ' in the direction opposite to the direction of the jet. When the tank starts moving, the velocity of the jet with which it comes out of the orifice will not be equal to  $V$  but it will be equal to the relative velocity of the jet with respect to tank.

Hence if

$$V = \text{Absolute velocity of jet},$$

$$u = \text{Velocity of tank},$$

$$V_r = \text{Velocity of jet with respect to tank}$$

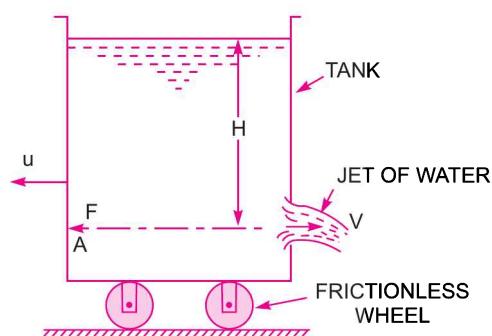


Fig. 17.25 Jet propulsion of a tank with an orifice.

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Then  $V_r$  = Vectorial difference of absolute velocity ( $V$ ) and velocity of tank ( $u$ )  
 $= V - (-u)$  (as  $u$  is in opposite direction to  $V$  hence velocity of tank is taken as  $-u$ )  
 $= V + u$

Hence when the tank is moving, the velocity with which jet comes out from the orifice is  $(V + u)$ .

Mass of water coming out from the orifice per sec

$$\begin{aligned} &= \rho \times a \times \text{Velocity with which water comes out} \\ &= \rho \times a \times (V_r) = \rho a (V + u) \end{aligned}$$

$\therefore$  Force exerted on the tank is given as

$$\begin{aligned} F_x &= \text{Mass of water coming out from orifice per second} \times [\text{Change of velocity}]^* \\ &= \rho a (V + u) \times [(V + u) - u] = \rho a [V + u] [V] \\ &= \rho a [V + u] \times V \end{aligned} \quad \dots(17.30)$$

Thus, the force given by equation (17.30) is used for propelling the tank.

$\therefore$  Work done on the moving tank by jet per second

$$= F_x \times u = \rho a (V + u) \times V \times u$$

$\therefore$  Efficiency of propulsion is given as,

$$\begin{aligned} \eta &= \frac{\text{Work done per second}}{\text{Kinetic energy of the issuing jet per second}} \\ &= \frac{\rho a (V + u) \times V \times u}{\frac{1}{2} (\text{Mass of water issuing per second}) \times (\text{Velocity of issuing jet})} \\ &= \frac{\rho a (V + u) \times V \times u}{\frac{1}{2} [\rho a (V + u)] \times (V + u)^2} = \frac{2Vu}{(V + u)^2} \end{aligned} \quad \dots(17.31)$$

**Condition for Maximum Efficiency and Expression for Maximum  $\eta$ .** For a given value of  $V$ , the efficiency will be maximum when  $\frac{d\eta}{du} = 0$

or  $\frac{d}{du} \left[ \frac{2Vu}{(V + u)^2} \right] = 0 \quad \text{or} \quad \frac{d}{du} [2Vu \times (V + u)^{-2}] = 0$

or  $2Vu \times (-2)(V + u)^{-3} + (V + u)^{-2} \times 2V = 0$

or  $\frac{-4Vu}{(V + u)^3} + \frac{2V}{(V + u)^2} = 0 \quad \text{or} \quad -4Vu + 2V(V + u) = 0$

Dividing by  $2V$ ,  $-2u + (V + u) = 0 \quad \text{or} \quad -u + V = 0 \quad \text{or} \quad u = V$   $\dots(17.32)$

Equation (17.32) is the condition for maximum efficiency. Substituting equation (17.32) in equation (17.31), the value of maximum efficiency is obtained as

$$\eta_{\max} = \frac{2 \times u \times u}{(u + u)^2} = \frac{2u^2}{4u^2} = \frac{1}{2} = 0.5 \text{ or } 50\%. \quad \dots(17.32A)$$

---

\* Change of velocity is the final velocity minus initial velocity of jet of water coming out from the orifice. The final velocity of the jet with respect to tank is  $(V + u)$ . This velocity is obtained by applying a velocity  $u$  to the whole system (i.e., tank, water in the tank and jet of water) in a direction opposite to the motion of tank. Then final velocity of jet becomes as  $(V + u)$  and initial velocity of water as  $u$ . Hence change of velocity is  $(V + u) - u$ .

**Problem 17.27** The head of water from the centre of the orifice which is fitted to one side of the tank is maintained at 2 m of water. The tank is not allowed to move and the diameter of orifice is 100 mm. Find the force exerted by the jet of water on the tank. Take  $C_v = 0.97$  cm.

**Solution.** Given :

$$\text{Head of water, } H = 2 \text{ m}$$

$$\text{Diameter of orifice, } d = 100 \text{ mm} = 0.1 \text{ m}$$

$$\therefore \text{Area, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.1)^2 = .007854 \text{ m}^2$$

$$\text{Value of } C_v = 0.97$$

$$\therefore \text{Velocity of jet, } V = C_v \times \sqrt{2gH} = 0.97 \times \sqrt{2 \times 9.81 \times 2.0} = 6.07 \text{ m/s}$$

Force exerted on the tank is given by equation (17.29) as

$$F = \rho a V^2 = 1000 \times .007854 \times 6.072 = 289.3 \text{ N. Ans.}$$

**Problem 17.28** If in the above problem, the tank is fitted with frictionless wheels and allowed to move, determine

- (i) Propelling force on tank,
- (ii) Work done by the propelling force per second, and
- (iii) Efficiency of propulsion.

The tank is moving with a velocity of 2 m/s.

**Solution.** Given :

$$H = 2 \text{ m}, d = 100 \text{ mm}, a = .007854 \text{ m}^2, C_v = 0.97$$

$$\text{and velocity of jet, } V = 6.07 \text{ m/s.}$$

$$\text{Velocity of tank, } u = 2 \text{ m/s.}$$

(i) Propelling force is given by equation (17.30) as

$$\begin{aligned} F_x &= \rho a (V + u) \times V \\ &= 1000 \times .007854 \times (6.07 + 2.0) \times 6.07 = 384.65 \text{ N. Ans.} \end{aligned}$$

(ii) Work done by the propelling force per second

$$= F_x \times u = 384.65 \times 2.0 = 769.3 \text{ N m/s. Ans.}$$

(iii) Efficiency of propulsion is given by equation (17.31) as

$$\eta = \frac{2Vu}{(V+u)^2} = \frac{2 \times 6.07 \times 2.0}{(6.07+2.0)^2} = 0.3728 \text{ or } 37.28\%. \text{ Ans.}$$

**17.5.2 Jet Propulsion of Ships.** By the application of the jet propulsion principle, a ship is driven through water. A jet of water which is discharged at the back (also called stern) of the ship, exerts a propulsive force on the ship. The ship carries centrifugal pumps which draw water from the surrounding sea. This water is discharged through the orifice provided at the back of the ship in the form of a jet. The reaction of the jet coming out at the back of the ship propels the ship in the opposite direction of the jet. The water from the surrounding sea by the centrifugal pump is taken by the following two ways :

1. Through inlet orifices which are at right angles to the direction of the motion of the ship, and
2. Through the inlet orifices, which are facing the direction of motion of the ship.

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**1st Case. Jet propulsion of the ship when the inlet orifices are at right angles to the direction of the motion of the ship.**

Fig. 17.26 shows a ship which is having the inlet orifices at right angles to its direction.

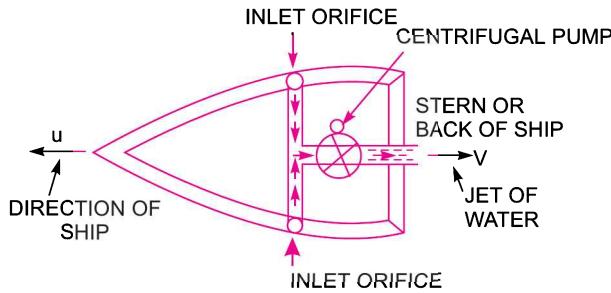


Fig. 17.26 Inlet orifices are at right angles.

Let

$$\begin{aligned}V &= \text{Absolute velocity of jet of water coming at the back of the ship}, \\u &= \text{Velocity of the ship}, \\V_r &= \text{Relative velocity of jet with respect to ship} \\&= (V + u).\end{aligned}$$

As the velocity  $V$  and  $u$  are in opposite direction and hence relative velocity will be equal to the sum of these two velocities.

Mass of water issuing from the orifice at the back of the ship  $= \rho a V_r = \rho a(V + u)$ , where  $a$  = Area of the jet of water

$\therefore$  Propulsive force exerted on the ship

$$\begin{aligned}F &= \text{Mass of water issuing per sec} \times \text{Change of velocity}^* \\&= \rho a(V + u) [V_r - u] = \rho a(V + u) [(V + u) - u] = \rho a(V + u) \times V \quad \dots(17.33)\end{aligned}$$

$$\text{Work done per second} = F \times u = \rho a (V + u) \times V \times u \quad \dots(17.34)$$

The efficiency of propulsion, the condition of maximum efficiency and expression for maximum efficiency are given by equations (17.31), (17.32) and (17.32 A) respectively.

**Note.** (i) When the inlet orifices are at right angles to the direction of motion of the ship, then this case is also known as water is drawn **AMID SHIP** which means the water is drawn at the middle of the ship.

(ii) The centrifugal pump draws the water from the surrounding sea and discharges through orifice. The kinetic energy of the issuing jet is  $\frac{1}{2} \times \text{mass} \times \text{velocity}^2$  i.e.,  $\frac{1}{2} [\rho a(V + u)] \times [V + u]^2 = \frac{1}{2} \rho a(V + u)^3$ . This energy is provided by centrifugal pump i.e., work is done by pump to provide this energy.

**Problem 17.29** Find the propelling force acting on a ship which takes water through inlet orifices which are at right angles to the direction of motion of ship, and discharges at the back through orifices having effective areas of  $0.04 \text{ m}^2$ . The water is flowing at the rate of  $1000 \text{ litres/s}$  and ship is moving with a velocity of  $8 \text{ m/s}$ .

**Solution.** Given :

$$\text{Effective areas of orifices, } a = 0.04 \text{ m}^2$$

$$\text{Discharge of water, } Q = 1000 \text{ litres/s} = 1 \text{ m}^3/\text{s}$$

$$\therefore \text{Velocity of jet relative to water} = \frac{Q}{a} = \frac{1}{0.04} = \frac{100}{4} = 25 \text{ m/s or } V_r = 25 \text{ m/s}$$

\* To find the change of velocity, apply a velocity  $u$  to the whole system (i.e., ship, jet of water and surrounding water in the sea) in a direction opposite to the motion of ship. Then final velocity of jet of water becomes as  $(V + u)$ . And velocity of water in sea becomes as  $u$ . Hence change of velocity becomes  $(V + u) - u$ .

$$\text{Velocity of ship, } u = 8 \text{ m/s}$$

$$\text{Now, } V_r = u + V, \text{ where } V = \text{Absolute velocity of jet}$$

$$\therefore 25 = 8 + V \text{ or } V = 25 - 8 = 17 \text{ m/s}$$

Propelling force is given by equation (17.33) as,

$$\begin{aligned} F &= \rho a (V + u) \times V \\ &= 1000 \times .04 \times (17 + 8) \times 17 = 16999.94 \text{ N. Ans.} \end{aligned}$$

**Problem 17.30** The water in a jet propelled boat is drawn amid-ship and discharged at the back with an absolute velocity of 20 m/s. The cross-sectional area of the jet at the back is  $0.02 \text{ m}^2$  and the boat is moving in sea water with a speed of 30 km/hour. Determine :

(i) Propelling force on the boat, (ii) Power required to drive the pump, and

(iii) Efficiency of the jet propulsion.

**Solution.** Given :

'Water is drawn amid-ship' means water is drawn at the middle of the ship and inlet orifices are at right angles to the motion of ship.

$$\text{Absolute velocity of jet, } V = 20 \text{ m/s}$$

$$\text{Area of the jet, } a = 0.02 \text{ m}^2$$

$$\text{Speed of boat, } u = 30 \text{ km/hr} = \frac{30 \times 1000}{60 \times 60} = 8.33 \text{ m/s.}$$

(i) Propelling force is given by equation (17.33) as

$$\begin{aligned} F &= \rho a (V + u) \times V \\ &= 1000 \times .02(20 + 8.33) \times 20 = 11332 \text{ N. Ans.} \end{aligned}$$

(ii) Power required to drive the pump in kW

$$= \frac{\text{Work done per sec}}{1000} = \frac{F \times u}{1000} = \frac{11332 \times 8.33}{1000} = 94.395 \text{ kW. Ans.}$$

(iii) Efficiency of the jet propulsion is given by (17.31) as

$$\eta = \frac{2Vu}{(V+u)^2} = \frac{2 \times 20 \times 8.33}{(20+8.33)^2} = 0.415 \text{ or } 41.5\%. \text{ Ans.}$$

**Problem 17.31** A small ship is fitted with jets of total area  $0.65 \text{ m}^2$ . The velocity through the jet is 9 m/s and speed of the ship is 18 km p.h. in sea-water. The efficiencies of the engine and pump are 85% and 65% respectively. If the water is taken amid-ships, determine the propelling force and the overall efficiency, assuming the pipe losses to be 10% of the kinetic energy of the jets.

**Solution.** Given :

$$\text{Total area of jets, } a = 0.65 \text{ m}^2$$

$$\text{Velocity through the jet relative to ship, } V_r = 9 \text{ m/s}$$

$$\text{Speed of ship, } u = 18 \text{ km/hour} = \frac{18 \times 1000}{60 \times 60} \text{ m/s} = 5 \text{ m/s}$$

$$\text{Efficiency of the engine, } \eta_E = 85\% = 0.85$$

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Efficiency of the pump,  $\eta_p = 65\% = 0.65$

Pipe losses,  $h_f = 10\%$  of kinetic energy of the jet

$$= \frac{10}{100} \times \frac{V_r^2}{2g} = \frac{V_r^2}{20g}$$

Now,

$$V_r = u + V, \quad \text{where } V = \text{Absolute velocity of jet}$$

$\therefore$

$$9 = 5 + V \text{ or } V = 9 - 5 = 4 \text{ m/s.}$$

(i) Propelling force is given by equation (17.33) as

$$\begin{aligned} F &= \rho a (V + u) \times V \\ &= 1000 \times 0.65 \times (4 + 5) \times 4 = 23400 \text{ N. Ans.} \end{aligned}$$

(ii) Work done by the jets per second

$$= F \times u = 23400 \times 5 = 117000 \text{ Nm/s}$$

Weight of water issuing from the jets per second

$$= g \times \text{Mass of water per second}$$

$$= g \times \rho a V_r = 9.81 \times 1000 \times 0.65 \times 9 = 57388.5 \text{ N/s.}$$

The pump should have the output which will give the jet a relative velocity ( $V_r$ ) and also overcome the pipe losses.

$\therefore$  Output of the pump per unit weight of water

$$= \text{Kinetic energy of jet} + \text{Pipe losses}$$

$$= \frac{V_r^2}{2g} + \frac{V_r^2}{20g} = \frac{V_r^2}{2g} (1 + 0.1) = \frac{V_r^2}{2g} \times 1.1$$

Input to the pump per unit weight of water

$$= \frac{\text{Output of pump}}{\text{Efficiency of pump}} = \frac{1.1 V_r^2}{2g \times 0.65}$$

The input to the pump is equal to the output of the engine. Hence input to the engine per unit weight of water

$$= \frac{1.1 V_r^2}{2g \times 0.65 \times \text{Efficiency of engine}}$$

$$= \frac{1.1 V_r^2}{2g \times 0.65 \times 0.85} = \frac{1.1 \times 9^2}{2 \times 9.81 \times 0.65 \times 0.85} = 8.22 \text{ Nm/N}$$

$\therefore$  Total input to the engine = Weight of water  $\times$  Input per unit weight of water

$$= 57388.5 \times 8.22 = 471733.5 \text{ Nm}$$

$\therefore$  Overall efficiency,  $\eta_o = \frac{\text{Work done by jets}}{\text{Total input to engine}} = \frac{117000}{471733.5} = 0.248 = 24.80\%. \text{ Ans.}$

**Problem 17.32** A jet propelled boat, moving with a velocity of 5 m/s, draws water amid-ship. The water is discharged through two jets provided at the back of the ship. The diameter of each jet is 150 mm. The total resistance offered to the motion of the boat is 4905 N ( $500 \times 9.81$  N). Determine :

(i) Volume of water drawn by the pump per second, and

(ii) Efficiency of the jet propulsion.

**Solution.** Given :

$$\text{Velocity of boat, } u = 5 \text{ m/s}$$

$$\text{Diameter of each jet, } d = 150 \text{ mm} = 0.15 \text{ m}$$

$$\text{Area of each jet} = \frac{\pi}{4}(0.15)^2 = 0.01767 \text{ m}^2$$

$$\therefore \text{Total area of the jets, } a = 2 \times 0.01767 = 0.03534 \text{ m}^2$$

$$\text{Total resistance to motion} = 4905 \text{ N} (500 \times 9.81 \text{ N})$$

The propelling force must be equal to the resistance to the motion.

$$\therefore \text{Propelling force, } F = 4905 \text{ N or } (500 \times 9.81 \text{ N})$$

Propelling force is given by equation (17.33) as

$$F = \rho a (V + u) V$$

$$500 \times 9.81 = 1000 \times 0.03534 \times (V + 5) \times V$$

$$\text{or } 500 = \frac{1000}{9.81} \times 0.03534 \times (V + 5) \times V$$

$$= 3.6 (V + 5) V = 3.6V^2 + 3.6 \times 5V = 3.6 V^2 + 18V$$

$$\text{or } 3.6 V^2 + 18V - 500 = 0$$

The above equation is quadratic and its solution is

$$V = \frac{-18 \pm \sqrt{18^2 + 4 \times 3.6 \times 500}}{2 \times 3.6} = \frac{-18 \pm 86.74}{7.2}$$

$$= \frac{86.74 - 18}{2} = 34.37 \text{ m/s} \quad [-\text{ve value is not possible}]$$

(i) Volume of water drawn by the pump per second is equal to the volume of water discharged through the orifices at the back in the form of jets and this volume

$$= aV_r = a(V + u)$$

$$= 0.03534 \times (34.37 + 5.0) = 1.39 \text{ m}^3/\text{s. Ans.}$$

(ii) Efficiency of the jet propulsion is given by equation (17.31) as

$$\eta = \frac{2Vu}{(V+u)^2} = \frac{2 \times 34.37 \times 5.0}{(34.37 + 5.0)^2} = .2217 \text{ or } 22.17\%. \text{ Ans.}$$

**2nd Case. Jet propulsion of ship when the inlet orifices face the direction of motion of the ship.**

Fig. 17.27 shows a ship which is having the inlet orifices facing the direction of the motion of the ship. In this case the expression for propelling force and work done per second will be same as in the 1st case in which inlet orifices are at right angles to the ship. But the energy supplied by the jet will be different,

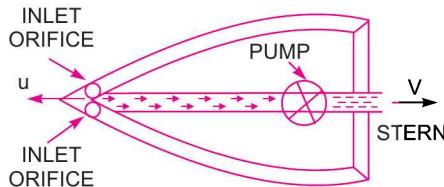


Fig. 17.27 Inlet orifices facing the direction of ship.

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as in this case the water enters with a velocity equal to the velocity of the ship, i.e., with a velocity  $u$ . Hence the expression for the energy supplied by the jet.

$$\begin{aligned} &= \frac{1}{2} (\text{Mass of water supplied per sec}) \times [V_r^2 - u^2] \\ &= \frac{1}{2} (\rho a V_r) \times [V_r^2 - u^2] \end{aligned}$$

where  $V_r = (V + u)$  as in the previous case

$$\therefore \text{K.E. supplied by jet} = \frac{1}{2} \rho a (V + u) [(V + u)^2 - u^2] \quad \dots(17.35)$$

$$\therefore \text{Efficiency of propulsion, } \eta = \frac{\text{Work done per sec by jet}}{\text{Energy supplied by jet}}$$

$$\begin{aligned} &= \frac{\rho a (V + u) \times V \times u}{\frac{1}{2} \rho a (V + u) [(V + u)^2 - u^2]} \\ &\quad \left\{ \because \text{From equation (17.34) work done} = \rho a (V + u) V \times u \right\} \end{aligned}$$

$$= \frac{2V \times u}{(V + u)^2 - u^2} = \frac{2Vu}{V^2 + u^2 + 2Vu - u^2} = \frac{2Vu}{V^2 + 2Vu} = \frac{2u}{V + 2u} \quad \dots(17.36)$$

**Problem 17.33** The water in a jet propelled boat is drawn through inlet openings facing the direction of motion of the ship. The boat is moving in sea-water with a speed of 30 km/hour. The absolute velocity of the jet of the water discharged at the back is 20 m/s and the area of the jet of water is  $0.03 \text{ m}^2$ . Find the propelling force and efficiency of propulsion.

**Solution.** Given :

$$\text{Speed of boat, } u = 30 \text{ km/hr} = \frac{30 \times 1000}{60 \times 60} = 8.33 \text{ m/s}$$

$$\text{Absolute velocity of jet, } V = 20 \text{ m/s}$$

$$\text{Area of the jet, } a = 0.03 \text{ m}^2.$$

(i) Propelling force is given by equation (17.33) as

$$F = \rho a (V + u) \times V = 1000 \times 0.03 \times (20 + 8.33) \times 20 = 16997.98 \text{ N.}$$

(ii) Efficiency of propulsion is given by equation (17.36) as

$$\eta = \frac{2u}{V + 2u} = \frac{2 \times 8.33}{20 + 2 \times 8.33} = 0.4544 \text{ or } 45.44\%. \text{ Ans.}$$

## HIGHLIGHTS

1. The force exerted by a jet of water on a stationary plate in the direction of the jet is given by

$$\begin{aligned} F_x &= \rho a V^2 && \dots \text{for a vertical plate} \\ &= \rho a V^2 \sin^2 \theta && \dots \text{for an inclined plate} \\ &= \rho a V^2 (1 + \cos \theta) && \dots \text{for a curved plate and jet strikes at the centre} \\ &= 2\rho a V^2 \cos \theta && \dots \text{for a curved plate and jet strikes at one of the tips of the jet.} \end{aligned}$$

where  $V$  = Velocity of the jet,

$\theta$  = Angle between the jet and the plate for inclined plate,

= Angle made by the jet with the direction of motion for curved plates.

2. When a jet of water strikes a vertical hinged plate, the angle of swing about the hinge is given by

$$\sin \theta = \frac{\rho a V^2}{W}$$

where  $V$  = Velocity of the jet of water,  $W$  = Weight of the hinged plate.

3. The force exerted by a jet of water on a moving plate, in the direction of the motion of the plate, is given by

$$\begin{aligned} F_x &= \rho a (V - u)^2 && \dots \text{for a moving vertical plate,} \\ &= \rho a (V - u)^2 \sin^2 \theta && \dots \text{for an inclined moving plate,} \\ &= \rho a (V - u)^2 (1 + \cos \theta) && \dots \text{when jet strikes the curved plate at the centre} \end{aligned}$$

4. When a jet of water strikes a curved moving plate at one of its tips and comes out at the other tip, the force exerted and work done are obtained from velocity triangles at inlet and outlet. The expression for force and work done are

$$F_x = \rho a V_{r_1} [V_{w_1} \pm V_{w_2}]$$

$$\text{Work done per second} = \rho a V_{r_1} [V_{w_1} \pm V_{w_2}] \times u$$

+ve sign is taken when  $\beta$  is an acute angle. If  $\beta$  is an obtuse angle then -ve sign is taken. If  $\beta$  is  $90^\circ$ ,

$$V_{w_2} = 0.$$

$$\text{Work done per second per kg of fluid} = \frac{1}{g} [V_{w_1} \pm V_{w_2}] .$$

5. For a series of vanes, the force and work done are given as  $F_x = \rho a V_{r_1} [V_{w_1} \pm V_{w_2}]$

$$\text{Work done/sec} = \rho a V_{r_1} [V_{w_1} \pm V_{w_2}] \times u$$

$$\text{Work done/sec per kg} = \frac{1}{g} [V_{w_1} \pm V_{w_2}] \times u .$$

6. Efficiency of a series of vanes is given as  $\eta = \frac{2u(V - u)}{V^2}$

and condition of max.  $\eta$  is  $u = \frac{V}{2}$

Max.  $\eta = 50\%$ .

7. For a curved radial vane, the work done per second =  $\rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2]$

where  $V_1$  = Absolute velocity of jet at inlet,  $V_{w_1}$  = Velocity of whirl at inlet

$u_1$  = Tangential velocity of vane at inlet,  $V_{w_2}$  = Velocity of whirl at outlet

$u_2$  = Tangential velocity of vane at outlet.

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8. For a curved radial vane the efficiency is given by

$$\eta = \frac{\rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2]}{\frac{1}{2} (\rho a V_1) \times V_1^2} = \frac{2 [V_{w_1} u_1 \pm V_{w_2} u_2]}{V_1^2}.$$

9. Jet propulsion means the propulsion of a vessel with the help of the jet. The reaction of the jet is used for propelling the vessel. The propelling force exerted on a tank with a orifice is given by

$$F_x = \rho a (V + u) \times V$$

where  $V$  = Absolute velocity of the jet of water,  $u$  = Velocity of the tank.

10. The efficiency of propulsion is given by  $\eta = \frac{2Vu}{(V+u)^2}$  and  $u = V$  for maximum efficiency

$$\therefore \text{Maximum } \eta = 50\%.$$

11. Ships are also propelled by jets. The intake water by the centrifugal pump is taken by two ways. In one case, the water is taken from orifices which are at right angles to the direction of the motion of the ship and in the other case the water is taken through orifices which are facing the direction of motion of the ship.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define the terms : (a) Impact of jets, and (b) Jet propulsion.

2. Obtain an expression for the force exerted by a jet of water on a fixed vertical plate in the direction of the jet.

3. Show that the force exerted by a jet of water on an inclined fixed plate in the direction of the jet is given by,

$$F_x = \rho a V^2 \sin^2 \theta$$

where  $a$  = Area of the jet,  $V$  = Velocity of the jet  
and  $\theta$  = Inclination of the plate with the jet.

4. Prove that the force exerted by a jet of water on a fixed semi-circular plate in the direction of the jet when the jet strikes at the centre of the semi-circular plate is two times the force exerted by the jet on an fixed vertical plate.

5. Show that the angle of swing of a vertical hinged plate is given by  $\sin \theta = \frac{\rho a V^2}{W}$

where  $V$  = Velocity of the jet striking the plate,  $a$  = Area of the jet, and  $W$  = Weight of the plate.

6. Differentiate between : (i) the force exerted by a jet of water on a fixed vertical plate and moving vertical plate, and (ii) the force exerted by a jet on a single curved moving plate and a series of curved moving plate.

7. Prove that the work done per second on a series of moving curved vanes by a jet of water striking at one of the tips of the vane is given by,

$$\text{Work done/sec} = \rho a V_1 [V_{w_1} \pm V_{w_2}] \times u.$$

8. Find an expression for the efficiency of a series of moving curved vanes when a jet of water strikes the vanes at one of its tips. Prove that maximum efficiency is when  $u = V$  and the value of maximum efficiency is 50%.

9. Show that for a curved radial vane, the work done per second is given by,  $\rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2]$ .
10. Find an expression for the propelling force and the work done per second on a tank which is provided with an orifice through which jet of water is coming out and tank is free to move.
11. Show that the efficiency of a free jet striking normally on a series of flat plates mounted on the periphery of a wheel can never exceed 50%.
12. Show that the force exerted by a jet of water on moving inclined plate in the direction of jet is given by

$$F_x = \rho a (V - u)^2 \sin^2 \theta$$

where  $a$  = area of jet,

$\theta$  = inclination of the plate with the jet, and

$V$  = velocity of jet.

*(J.N.T.U., Hyderabad, S 2002)*

### (B) NUMERICAL PROBLEMS

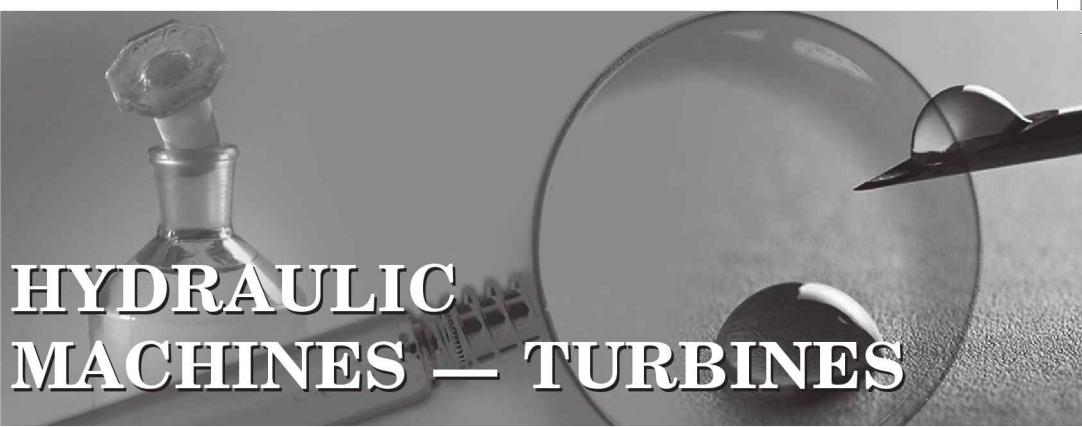
1. Find the force exerted by a jet of water of diameter 100 mm on a stationary flat plate, when the jet strikes the plate normally with a velocity of 30 m/s. [Ans. 7068.6 N]
2. A jet of water of diameter 50 mm moving with a velocity of 20 m/s strikes a fixed plate in such a way that the angle between the jet and the plate is 60°. Find the force exerted by the jet on the plate (i) in the direction normal to the plate, and (ii) in the direction of the jet. [Ans. (i) 680.13 N, (ii) 589 N]
3. A jet of water of diameter 100 mm moving with a velocity of 30 m/s strikes a curved fixed symmetrical plate at the centre. Find the force exerted by the jet of water in the direction of the jet, if the jet is deflected through an angle of 120° at the outlet of the curved plate. [Ans. 10602.7 N]
4. A jet of water of the diameter 100 mm moving with a velocity of 20 m/s strikes a curved fixed plate tangentially at one end at an angle of 30° to the horizontal. The jet leaves the plate at an angle of 20° to the horizontal. Find the force exerted by the jet on the plate in the horizontal and vertical directions. [Ans. 5672.34 N, 496.3 N]
5. A jet of water of 30 mm diameter, moving with a velocity of 15 m/s, strikes a hinged square plate of weight 245.25 N at the centre of the plate. The plate is of uniform thickness. Find the angle through which the plate will swing. [Ans.  $\theta = 40^\circ 25.6'$ ]
6. A plate is acted upon at its centre by a jet of water of diameter 20 mm with a velocity of 20 m/s. The plate is hinged and is deflected through an angle of 15°. Find the weight of the plate. If the plate is not allowed to swing, what will be the force required at the lower edge of the plate to keep the plate in vertical position. [Ans. 485.5 N, 62.8 N]
7. A jet of water of diameter 150 mm strikes a flat plate normally with a velocity of 12 m/s. The plate is moving with a velocity of 6 m/s in the direction of the jet and away from the jet. Find : (i) the force exerted by the jet on the plate, (ii) work done by the jet on the plate per second, (iii) power of the jet, and (iv) efficiency of the jet. [Ans. (i) 636.3 N, (ii) 3817.6 Nm/s, (iii) 3.82 kW, (iv) 25%]
8. If in the problem 7, the jet strikes the plate in such a way that the normal on the plate makes an angle of 30° to the axis of the jet, find : (i) The normal force exerted on the plate, (ii) power, and (iii) efficiency of the jet. [Ans. (i) 551 N, (ii) 2.86 kW, (iii) 18.74%]
9. A jet of water of diameter 100 mm strikes a curved plate at its centre with a velocity of 15 m/s. The curved plate is moving with a velocity of 7 m/s in the direction of the jet. The jet is deflected through an angle of 150°. Assuming the plate smooth find : (i) force exerted on the plate in the direction of the jet, (ii) power of the jet, and (iii) efficiency. [Ans. (i) 938 N (ii) 6.56 kW, (iii) 49.53%]
10. A jet of water having a velocity of 30 m/s strikes a curved vane, which is moving with a velocity of 15 m/s. The jet makes an angle of 30° with the direction of motion of vane at inlet and leaves at an angle of 120° to the direction of motion of vane at outlet. Calculate : (i) Vane angles, if the water enters and leaves the vane without shock, (ii) Work done per second per unit weight of water striking the vanes per second. [Ans. (i) 53° 47.7', 15° 41', (ii) 44.15 Nm/N]

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11. A jet of water of diameter 50 mm, having a velocity of 30 m/s strikes a curved vane which is moving with a velocity of 15 m/s in the direction of the jet. The jet leaves the vane at an angle of  $60^\circ$  to the direction of motion of vanes at outlet. Determine : (i) the force exerted by the jet on the vane in the direction of motion, (ii) work done per second by the jet. [Ans. (i) 662.5 N, (ii) 9937.5 Nm/s]
12. A jet of water having a velocity of 20 m/s strikes a curved vane which is moving with a velocity of 9 m/s. The vane is symmetrical and is so shaped that the jet is deflected through  $120^\circ$ . Find the angle of the jet at inlet of the vane so that there is no shock. What is the absolute velocity of the jet at outlet in magnitude and direction and the work done per second per unit weight of water striking? Assume the vane to be smooth. [Ans.  $17^\circ$ , 5.95 m/s,  $\beta = 79^\circ 6'$ , 18.57 Nm/N]
13. A jet of water, having a velocity of 15 m/s, strikes a curved vane which is moving with a velocity of 6 m/s in the same direction as that of the jet at inlet. The vane is so shaped that the jet is deflected through  $135^\circ$ . The diameter of the jet is 150 mm. Assuming the vane to be smooth, find : (i) the force exerted by the jet on the vane in the direction of motion, (ii) power of the vane, and (iii) efficiency of the vane. [Ans. (i) 2443.5 N, (ii) 14.65 kW, (iii) 49.16%]
14. If in the above problem, the jet of water instead of striking a single plate, strikes a series of curved vanes, find : (i) force exerted by the jet on the vanes in the direction of motion, (ii) power of the vane, and (iii) efficiency of the vane. [Ans. (i) 4072.5 N, (ii) 24.43 kW, (iii) 81.9%]
15. A jet of water having a velocity of 30 m/s, strikes a series of radial curved vanes mounted on a wheel which is rotating at 300 r.p.m. The jet makes an angle of  $30^\circ$  with the tangent to wheel at inlet and leaves the wheel with a velocity of 4 m/s at an angle of  $120^\circ$  to the tangent to the wheel at outlet. Water is flowing from outward in a radial direction. The outer and inner radii of the wheel are 0.6 m and 0.3 m respectively. Determine : (i) vane angles at inlet and outlet, (ii) work done per second per kg of water, and (iii) efficiency of the wheel. [Ans. (i)  $42^\circ 10.7'$ ,  $27^\circ 17.8'$ , (ii) 52.92, (iii) 56.5%]
16. The head of water from the centre of the orifice fitted to a tank is maintained at 6 m of water. The diameter of the orifice is 150 mm. The tank is fitted with frictionless wheels at the bottom and the tank is moving with a velocity of 4 m/s due to the reaction of the jet coming out from the orifice. Determine : (i) propelling force on the tank, (ii) work done per second, and (iii) efficiency of propulsion [Ans. (i) 2847.3 N, (ii) 11389 Nm/s, (iii)  $\eta = 39.36\%$ ]
17. The water in a jet propelled boat is drawn mid-ship and is discharged at the back with an absolute velocity of 30 m/s. The cross-sectional area of the jet at the back is  $0.04 \text{ m}^2$  and the boat is moving in sea-water with a speed of 30 km/hour. Determine : (i) propelling force of the boat, (ii) power, and (iii) efficiency of the jet propulsion. [Ans. (i) 45995.6 N, (ii) 383.14 kW, (iii) 34.02%]
18. The water in a jet propelled boat is drawn through inlet openings facing the direction of motion of the ship. The boat is moving in sea-water with a speed of 40 km/hr. The absolute velocity of the jet of the water discharged at the back is 40 m/s and the area of the jet of water is  $0.04 \text{ m}^2$ . Find the propelling force and efficiency of propulsion. [Ans. 81775.3 N,  $\eta = 35.71\%$ ]

# 18

## CHAPTER



# HYDRAULIC MACHINES — TURBINES

### ► 18.1 INTRODUCTION

Hydraulic machines are defined as those machines which convert either hydraulic energy (energy possessed by water) into mechanical energy (which is further converted into electrical energy ) or mechanical energy into hydraulic energy. The hydraulic machines, which convert the hydraulic energy into mechanical energy, are called *turbines* while the hydraulic machines which convert the mechanical energy into hydraulic energy are called *pumps*. Thus the study of hydraulic machines consists of study of turbines and pumps. Turbines consists of mainly study of Pelton turbine, Francis Turbine and Kaplan Turbine while pumps consist of study of centrifugal pump and reciprocating pumps.

### ► 18.2 TURBINES

Turbines are defined as the hydraulic machines which convert hydraulic energy into mechanical energy. This mechanical energy is used in running an electric generator which is directly coupled to the shaft of the turbine. Thus the mechanical energy is converted into electrical energy. The electric power which is obtained from the hydraulic energy (energy of water) is known as *Hydroelectric power*. At present the generation of hydroelectric power is the cheapest as compared by the power generated by other sources such as oil, coal etc.

### ► 18.3 GENERAL LAYOUT OF A HYDROELECTRIC POWER PLANT

Fig. 18.1 shows a general layout of a hydroelectric power plant which consists of :

- (i) A dam constructed across a river to store water.
- (ii) Pipes of large diameters called penstocks, which carry water under pressure from the storage reservoir to the turbines. These pipes are made of steel or reinforced concrete.
- (iii) Turbines having different types of vanes fitted to the wheels.
- (iv) Tail race, which is a channel which carries water away from the turbines after the water has worked on the turbines. The surface of water in the tail race channel is also known as tail race.

### ► 18.4 DEFINITIONS OF HEADS AND EFFICIENCIES OF A TURBINE

**1. Gross Head.** The difference between the head race level and tail race level when no water is flowing is known as Gross Head. It is denoted by ' $H_g$ ' in Fig. 18.1.

**2. Net Head.** It is also called effective head and is defined as the head available at the inlet of the turbine. When water is flowing from head race to the turbine, a loss of head due to friction between the water and penstocks occurs. Though there are other losses also such as loss due to bend, pipe fittings, loss at the entrance of penstock etc., yet they are having small magnitude as compared to head loss due to friction. If ' $h_f$ ' is the head loss due to friction between penstocks and water then net head on turbine is given by

$$H = H_g - h_f \quad \dots(18.1)$$

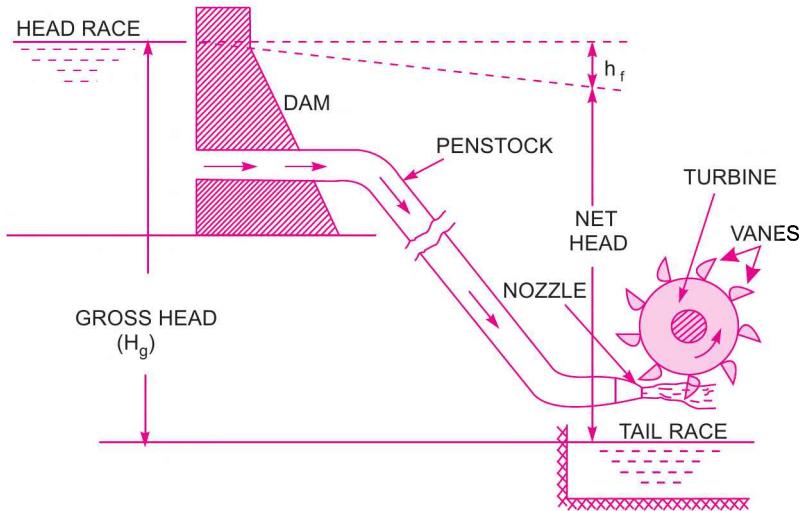


Fig. 18.1 Layout of a hydroelectric power plant.

where  $H_g$  = Gross head,  $h_f = \frac{4 \times f \times L \times V^2}{D \times 2g}$ ,

in which

$V$  = Velocity of flow in penstock,

$L$  = Length of penstock,

$D$  = Diameter of penstock.

**3. Efficiencies of a Turbine.** The following are the important efficiencies of a turbine.

- (a) Hydraulic Efficiency,  $\eta_h$
- (b) Mechanical Efficiency,  $\eta_m$
- (c) Volumetric Efficiency,  $\eta_v$  and (d) Overall Efficiency,  $\eta_o$

(a) **Hydraulic Efficiency ( $\eta_h$ )**. It is defined as the ratio of power given by water to the runner of a turbine (runner is a rotating part of a turbine and on the runner vanes are fixed) to the power supplied by the water at the inlet of the turbine. The power at the inlet of the turbine is more and this power goes on decreasing as the water flows over the vanes of the turbine due to hydraulic losses as the vanes are not smooth. Hence, the power delivered to the runner of the turbine will be less than the power available at the inlet of the turbine. Thus, mathematically, the hydraulic efficiency of a turbine is written as

$$\eta_h = \frac{\text{Power delivered to runner}}{\text{Power supplied at inlet}} = \frac{\text{R.P.}}{\text{W.P.}} \quad \dots(18.2)$$

where R.P. = Power delivered to runner *i.e.*, runner power

$$= \frac{W [V_{w_1} \pm V_{w_2}] \times u}{g \times 1000} \text{ kW} \quad \dots \text{for Pelton Turbine}$$

$$= \frac{W [V_{w_1} u_1 \pm V_{w_2} u_2]}{g \times 1000} \text{ kW} \quad \dots \text{for a radial flow turbine}$$

W.P. = Power supplied at inlet of turbine and also called water power

$$= \frac{W \times H}{1000} \text{ kW} \quad \dots(18.3)$$

where  $W$  = Weight of water striking the vanes of the turbine per second

$= \rho g \times Q$  in which  $Q$  = Volume of water/s,

$V_{w_1}$  = Velocity of whirl at inlet,

$V_{w_2}$  = Velocity of whirl at outlet,

$u$  = Tangential velocity of vane,

$u_1$  = Tangential velocity of vane at inlet for radial vane,

$u_2$  = Tangential velocity of vane at outlet for radial vane,

$H$  = Net head on the turbine.

Power supplied at the inlet of turbine in S.I.units is known as water power. It is given by

$$W.P. = \frac{\rho \times g \times Q \times H}{1000} \text{ kW} \quad \dots(18.3A)$$

For water

$$\rho = 1000 \text{ kg/m}^3$$

$$\therefore W.P. = \frac{1000 \times g \times Q \times H}{1000} = g \times Q \times H \text{ kW} \quad \dots(18.3B)$$

The relation (18.3B) is only used when the flowing fluid is water. If the flowing fluid is other than the water, then relation (18.3A) is used.

(b) **Mechanical Efficiency ( $\eta_m$ )**. The power delivered by water to the runner of a turbine is transmitted to the shaft of the turbine. Due to mechanical losses, the power available at the shaft of the turbine is less than the power delivered to the runner of a turbine. The ratio of the power available at the shaft of the turbine (known as S.P. or B.P.) to the power delivered to the runner is defined as mechanical efficiency. Hence, mathematically, it is written as

$$\eta_m = \frac{\text{Power at the shaft of the turbine}}{\text{Power delivered by water to the runner}} = \frac{\text{S.P.}}{\text{R.P.}} \quad \dots(18.4)$$

(c) **Volumetric Efficiency ( $\eta_v$ )**. The volume of the water striking the runner of a turbine is slightly less than the volume of the water supplied to the turbine. Some of the volume of the water is discharged to the tail race without striking the runner of the turbine. Thus the ratio of the volume of the water actually striking the runner to the volume of water supplied to the turbine is defined as volumetric efficiency. It is written as

$$\eta_v = \frac{\text{Volume of water actually striking the runner}}{\text{Volume of water supplied to the turbine}} \quad \dots(18.5)$$

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(d) **Overall Efficiency ( $\eta_o$ )**. It is defined as the ratio of power available at the shaft of the turbine to the power supplied by the water at the inlet of the turbine. It is written as :

$$\begin{aligned}
 \eta_o &= \frac{\text{Volume available at the shaft of the turbine}}{\text{Power supplied at the inlet of the turbine}} = \frac{\text{Shaft power}}{\text{Water power}} \\
 &= \frac{\text{S.P.}}{\text{W.P.}} \\
 &= \frac{\text{S.P.}}{\text{W.P.}} \times \frac{\text{R.P.}}{\text{R.P.}} \quad (\text{where R.P.} = \text{Power delivered to runner}) \\
 &= \frac{\text{S.P.}}{\text{R.P.}} \times \frac{\text{R.P.}}{\text{W.P.}} \\
 &= \eta_m \times \eta_h \quad \left. \begin{array}{l} \left( \because \text{From equation (18.4), } \frac{\text{S.P.}}{\text{R.P.}} = \eta_m \right) \\ \text{and from equation (18.2), } \frac{\text{R.P.}}{\text{W.P.}} = \eta_h \end{array} \right\} \dots(18.6)
 \end{aligned}$$

If shaft power (S.P.) is taken in kW then water power should also be taken in kW. Shaft power is commonly represented by P. But from equation (18.3A),

$$\begin{aligned}
 \text{Water power in kW} &= \frac{\rho \times g \times Q \times H}{1000}, \text{ where } \rho = 1000 \text{ kg/m}^3 \\
 \therefore \eta_o &= \frac{\text{Shaft power in kW}}{\text{Water power in kW}} = \frac{P}{\left( \frac{\rho \times g \times Q \times H}{1000} \right)} \quad \dots(18.6A)
 \end{aligned}$$

where  $P$  = Shaft power.

### ► 18.5 CLASSIFICATION OF HYDRAULIC TURBINES

The hydraulic turbines are classified according to the type of energy available at the inlet of the turbine, direction of flow through the vanes, head at the inlet of the turbine and specific speed of the turbines. Thus the following are the important classifications of the turbines :

1. According to the type of energy at inlet :
  - (a) Impulse turbine, and (b) Reaction turbine.
2. According to the direction of flow through runner :
 

(a) Tangential flow turbine,	(b) Radial flow turbine,
(c) Axial flow turbine, and	(d) Mixed flow turbine.
3. According to the head at the inlet of turbine :
 

(a) High head turbine,	(b) Medium head turbine, and
(c) Low head turbine.	
4. According to the specific speed of the turbine :
 

(a) Low specific speed turbine,	(b) Medium specific speed turbine, and
(c) High specific speed turbine.	

If at the inlet of the turbine, the energy available is only kinetic energy, the turbine is known as **impulse turbine**. As the water flows over the vanes, the pressure is atmospheric from inlet to outlet of

the turbine. If at the inlet of the turbine, the water possesses kinetic energy as well as pressure energy, the turbine is known as **reaction turbine**. As the water flows through the runner, the water is under pressure and the pressure energy goes on changing into kinetic energy. The runner is completely enclosed in an air-tight casing and the runner and casing is completely full of water.

If the water flows along the tangent of the runner, the turbine is known as **tangential flow turbine**. If the water flows in the radial direction through the runner, the turbine is called **radial flow turbine**. If the water flows from outwards to inwards, radially, the turbine is known as **inward radial flow turbine**, on the other hand, if water flows radially from inwards to outwards, the turbine is known as **outward radial flow turbine**. If the water flows through the runner along the direction parallel to the axis of rotation of the runner, the turbine is called **axial flow turbine**. If the water flows through the runner in the radial direction but leaves in the direction parallel to axis of rotation of the runner, the turbine is called **mixed flow turbine**.

#### ► 18.6 PELTON WHEEL (OR TURBINE)

The Pelton wheel or Pelton turbine is a tangential flow impulse turbine. The water strikes the bucket along the tangent of the runner. The energy available at the inlet of the turbine is only kinetic energy. The pressure at the inlet and outlet of the turbine is atmospheric. This turbine is used for high heads and is named after L.A. Pelton, an American Engineer.

Fig. 18.1 shows the layout of a hydroelectric power plant in which the turbine is Pelton wheel. The water from the reservoir flows through the penstocks at the outlet of which a nozzle is fitted. The nozzle increases the kinetic energy of the water flowing through the penstock. At the outlet of the nozzle, the water comes out in the form of a jet and strikes the buckets (vanes) of the runner. The main parts of the Pelton turbine are :

1. Nozzle and flow regulating arrangement (spear),
2. Runner and buckets,
3. Casing, and
4. Breaking jet.

**1. Nozzle and Flow Regulating Arrangement.** The amount of water striking the buckets (vanes) of the runner is controlled by providing a spear in the nozzle as shown in Fig. 18.2. The spear is a conical needle which is operated either by a hand wheel or automatically in an axial direction depending upon the size of the unit. When the spear is pushed forward into the nozzle the amount of water striking the runner is reduced. On the other hand, if the spear is pushed back, the amount of water striking the runner increases.

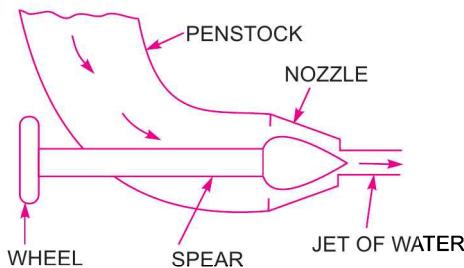


Fig. 18.2 Nozzle with a spear to regulate flow.

**2. Runner with Buckets.** Fig. 18.3 shows the runner of a Pelton wheel. It consists of a circular disc on the periphery of which a number of buckets evenly spaced are fixed. The shape of the buckets is of a double hemispherical cup or bowl. Each bucket is divided into two symmetrical parts by a dividing wall which is known as splitter.

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The jet of water strikes on the splitter. The splitter divides the jet into two equal parts and the jet comes out at the outer edge of the bucket. The buckets are shaped in such a way that the jet gets deflected through  $160^\circ$  or  $170^\circ$ . The buckets are made of cast iron, cast steel bronze or stainless steel depending upon the head at the inlet of the turbine.

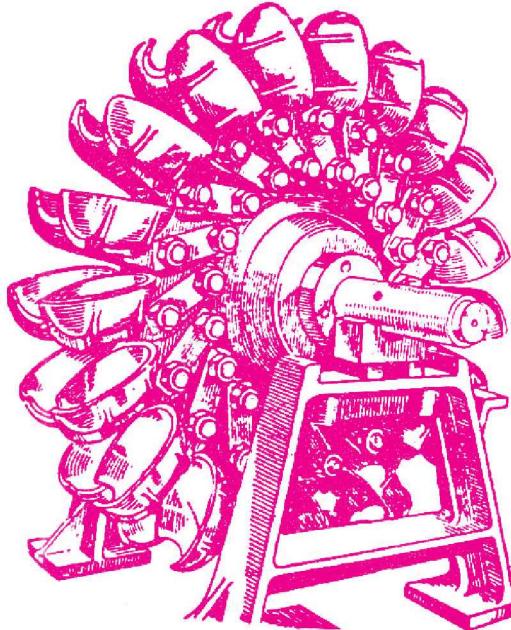


Fig. 18.3 Runner of a pelton wheel.

**3. Casing.** Fig. 18.4 shows a Pelton turbine with a casing. The function of the casing is to prevent the splashing of the water and to discharge water to tail race. It also acts as safeguard against accidents. It is made of cast iron or fabricated steel plates. The casing of the Pelton wheel does not perform any hydraulic function.

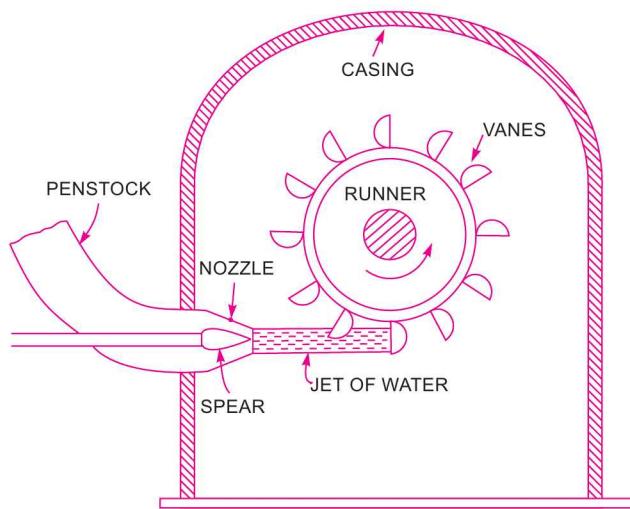


Fig. 18.4 Pelton turbine.

**4. Breaking Jet.** When the nozzle is completely closed by moving the spear in the forward direction, the amount of water striking the runner reduces to zero. But the runner due to inertia goes on revolving for a long time. To stop the runner in a short time, a small nozzle is provided which directs the jet of water on the back of the vanes. This jet of water is called breaking jet.

**18.6.1 Velocity Triangles and Work done for Pelton Wheel.** Fig. 18.5 shows the shape of the vanes or buckets of the Pelton wheel. The jet of water from the nozzle strikes the bucket at the splitter, which splits up the jet into two parts. These parts of the jet, glides over the inner surfaces and comes out at the outer edge. Fig. 18.5 (b) shows the section of the bucket at Z-Z. The splitter is the inlet tip and outer edge of the bucket is the outlet tip of the bucket. The inlet velocity triangle is drawn at the splitter and outlet velocity triangle is drawn at the outer edge of the bucket, by the same method as explained in Chapter 17.

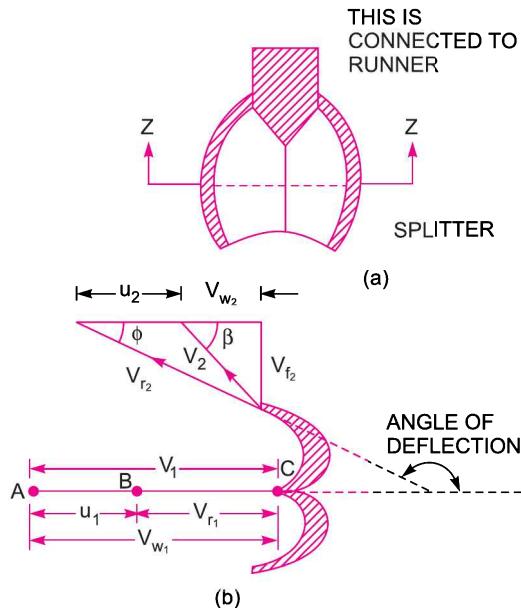


Fig. 18.5 Shape of bucket.

Let

$$\begin{aligned} H &= \text{Net head acting on the Pelton wheel} \\ &= H_g - h_f \end{aligned}$$

where  $H_g$  = Gross head and  $h_f = \frac{4fLV^2}{D^* \times 2g}$

where  $D^*$  = Dia. of Penstock,  $N$  = Speed of the wheel in r.p.m.,  
 $D$  = Diameter of the wheel,  $d$  = Diameter of the jet.

Then

$$V_1 = \text{Velocity of jet at inlet} = \sqrt{2gH} \quad \dots(18.7)$$

$$u = u_1 = u_2 = \frac{\pi DN}{60}.$$

The velocity triangle at inlet will be a straight line where

$$V_{r_1} = V_1 - u_1 = V_1 - u$$

$$V_{w_1} = V_1$$

$$\alpha = 0^\circ \text{ and } \theta = 0^\circ$$

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From the velocity triangle at outlet, we have

$$V_{r_2} = V_{r_1} \text{ and } V_{w_2} = V_{r_2} \cos \phi - u_2.$$

The force exerted by the jet of water in the direction of motion is given by equation (17.19) as

$$F_x = \rho a V_1 [V_{w_1} + V_{w_2}] \quad \dots(18.8)$$

As the angle  $\beta$  is an acute angle, +ve sign should be taken. Also this is the case of series of vanes, the mass of water striking is  $\rho a V_1$  and not  $\rho a V_{r_1}$ . In equation (18.8), 'a' is the area of the jet which is given as

$$a = \text{Area of jet} = \frac{\pi}{4} d^2.$$

Now work done by the jet on the runner per second

$$= F_x \times u = \rho a V_1 [V_{w_1} + V_{w_2}] \times u \text{ Nm/s} \quad \dots(18.9)$$

Power given to the runner by the jet

$$= \frac{\rho a V_1 [V_{w_1} + V_{w_2}] \times u}{1000} \text{ kW} \quad \dots(18.10)$$

Work done/s per unit weight of water striking/s

$$\begin{aligned} &= \frac{\rho a V_1 [V_{w_1} + V_{w_2}] \times u}{\text{Weight of water striking/s}} \\ &= \frac{\rho a V_1 [V_{w_1} + V_{w_2}] \times u}{\rho a V_1 \times g} = \frac{1}{g} [V_{w_1} + V_{w_2}] \times u \end{aligned} \quad \dots(18.11)$$

The energy supplied to the jet at inlet is in the form of kinetic energy and is equal to  $\frac{1}{2} m V^2$

$$\therefore \text{K.E. of jet per second} = \frac{1}{2} (\rho a V_1) \times V_1^2$$

$$\therefore \text{Hydraulic efficiency, } \eta_h = \frac{\text{Work done per second}}{\text{K.E. of jet per second}}$$

$$= \frac{\rho a V_1 [V_{w_1} + V_{w_2}] \times u}{\frac{1}{2} (\rho a V_1) \times V_1^2} = \frac{2 [V_{w_1} + V_{w_2}] \times u}{V_1^2} \quad \dots(18.12)$$

Now

$$V_{w_1} = V_1, V_{r_1} = V_1 - u_1 = (V_1 - u)$$

$$\therefore V_{r_2} = (V_1 - u)$$

and

$$V_{w_2} = V_{r_2} \cos \phi - u_2 = V_{r_2} \cos \phi - u = (V_1 - u) \cos \phi - u$$

Substituting the values of  $V_{w_1}$  and  $V_{w_2}$  in equation (18.12),

$$\begin{aligned} \eta_h &= \frac{2[V_1 + (V_1 - u) \cos \phi - u] \times u}{V_1^2} \\ &= \frac{2[V_1 - u + (V_1 - u) \cos \phi] \times u}{V_1^2} = \frac{2(V_1 - u)[1 + \cos \phi]u}{V_1^2}. \end{aligned} \quad \dots(18.13)$$

The efficiency will be maximum for a given value of  $V_1$  when

$$\frac{d}{du}(\eta_h) = 0 \quad \text{or} \quad \frac{d}{du}\left[\frac{2u(V_1 - u)(1 + \cos \phi)}{V_1^2}\right] = 0$$

$$\text{or} \quad \frac{(1 + \cos \phi)}{V_1^2} \frac{d}{du}(2uV_1 - 2u^2) = 0 \quad \text{or} \quad \frac{d}{du}[2uV_1 - 2u^2] = 0 \quad \left(\because \frac{1 + \cos \phi}{V_1^2} \neq 0\right)$$

$$\text{or} \quad 2V_1 - 4u = 0 \quad \text{or} \quad u = \frac{V_1}{2} \quad \dots(18.14)$$

Equation (18.14) states that hydraulic efficiency of a Pelton wheel will be maximum when the velocity of the wheel is half the velocity of the jet of water at inlet. The expression for maximum efficiency will be obtained by substituting the value of  $u = \frac{V_1}{2}$  in equation (18.13).

$$\therefore \text{Max. } \eta_h = \frac{2\left(V_1 - \frac{V_1}{2}\right)(1 + \cos \phi) \times \frac{V_1}{2}}{V_1^2}$$

$$= \frac{2 \times \frac{V_1}{2}(1 + \cos \phi) \frac{V_1}{2}}{V_1^2} = \frac{(1 + \cos \phi)}{2}. \quad \dots(18.15)$$

### 18.6.2 Points to be Remembered for Pelton Wheel

(i) The velocity of the jet at inlet is given by  $V_1 = C_v \sqrt{2gH}$   
where  $C_v$  = Co-efficient of velocity = 0.98 or 0.99

$H$  = Net head on turbine

(ii) The velocity of wheel ( $u$ ) is given by  $u = \phi \sqrt{2gH}$   
where  $\phi$  = Speed ratio. The value of speed ratio varies from 0.43 to 0.48.

(iii) The angle of deflection of the jet through buckets is taken at  $165^\circ$  if no angle of deflection is given.

(iv) The mean diameter or the pitch diameter  $D$  of the Pelton wheel is given by

$$u = \frac{\pi DN}{60} \text{ or } D = \frac{60u}{\pi N}$$

(v) **Jet Ratio.** It is defined as the ratio of the pitch diameter ( $D$ ) of the Pelton wheel to the diameter of the jet ( $d$ ). It is denoted by ‘ $m$ ’ and is given as

$$m = \frac{D}{d} \quad (= 12 \text{ for most cases}) \quad \dots(18.16)$$

(vi) Number of buckets on a runner is given by

$$Z = 15 + \frac{D}{2d} = 15 + 0.5 \text{ m} \quad \dots(18.17)$$

where  $m$  = Jet ratio

(vii) **Number of Jets.** It is obtained by dividing the total rate of flow through the turbine by the rate of flow of water through a single jet.

**Problem 18.1** A Pelton wheel has a mean bucket speed of 10 metres per second with a jet of water flowing at the rate of 700 litres/s under a head of 30 metres. The buckets deflect the jet through an angle of  $160^\circ$ . Calculate the power given by water to the runner and the hydraulic efficiency of the turbine. Assume co-efficient of velocity as 0.98.

**Solution.** Given :

Speed of bucket,  $u = u_1 = u_2 = 10 \text{ m/s}$

Discharge,  $Q = 700 \text{ litres/s} = 0.7 \text{ m}^3/\text{s}$ , Head of water,  $H = 30 \text{ m}$

Angle of deflection  $= 160^\circ$

$\therefore$  Angle,  $\phi = 180^\circ - 160^\circ = 20^\circ$

Co-efficient of velocity,  $C_v = 0.98$ .

The velocity of jet,  $V_1 = C_v \sqrt{2gH} = 0.98 \sqrt{2 \times 9.81 \times 30} = 23.77 \text{ m/s}$

$$\therefore V_{r_1} = V_1 - u_1 = 23.77 - 10 \\ = 13.77 \text{ m/s}$$

$$V_{w_1} = V_1 = 23.77 \text{ m/s}$$

From outlet velocity triangle,

$$V_{r_2} = V_{r_1} = 13.77 \text{ m/s}$$

$$V_{w_2} = V_{r_2} \cos \phi - u_2 \\ = 13.77 \cos 20^\circ - 10.0 = 2.94 \text{ m/s}$$

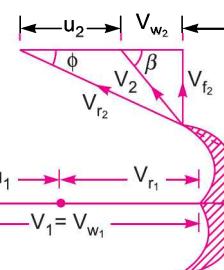


Fig. 18.6

Work done by the jet per second on the runner is given by equation (18.9) as

$$= \rho a V_1 [V_{w_1} + V_{w_2}] \times u \\ = 1000 \times 0.7 \times [23.77 + 2.94] \times 10 \quad (\because aV_1 = Q = 0.7 \text{ m}^3/\text{s}) \\ = 186970 \text{ Nm/s}$$

$$\therefore \text{Power given to turbine} = \frac{186970}{1000} = 186.97 \text{ kW. Ans.}$$

The hydraulic efficiency of the turbine is given by equation (18.12) as

$$\eta_h = \frac{2[V_{w_1} + V_{w_2}] \times u}{V_1^2} = \frac{2[23.77 + 2.94] \times 10}{23.77 \times 23.77} \\ = 0.9454 \text{ or } 94.54\%. \text{ Ans.}$$

**Problem 18.2** A Pelton wheel is to be designed for the following specifications :

Shaft power = 11,772 kW ; Head = 380 metres ; Speed = 750 r.p.m. ; Overall efficiency = 86% ; Jet diameter is not to exceed one-sixth of the wheel diameter. Determine :

- (i) The wheel diameter,      (ii) The number of jets required, and
- (iii) Diameter of the jet.

Take  $K_{v_1} = 0.985$  and  $K_{u_1} = 0.45$

**Solution.** Given :

Shaft power, S.P. = 11,772 kW

Head,  $H = 380 \text{ m}$

Speed,  $N = 750 \text{ r.p.m.}$

Overall efficiency,  $\eta_o = 86\% \text{ or } 0.86$

$$\text{Ratio of jet dia. to wheel dia.} = \frac{d}{D} = \frac{1}{6}$$

Co-efficient of velocity,  $K_{v_1} = C_v = 0.985$

Speed ratio,  $K_{u_1} = 0.45$

$$\text{Velocity of jet, } V_1 = C_v \sqrt{2gH} = 0.985 \sqrt{2 \times 9.81 \times 380} = 85.05 \text{ m/s}$$

The velocity of wheel,  $u = u_1 = u_2$

$$= \text{Speed ratio} \times \sqrt{2gH} = 0.45 \times \sqrt{2 \times 9.81 \times 380} = 38.85 \text{ m/s}$$

But

$$u = \frac{\pi DN}{60} \quad \therefore \quad 38.85 = \frac{\pi DN}{60}$$

or

$$D = \frac{60 \times 38.85}{\pi \times N} = \frac{60 \times 38.85}{\pi \times 750} = \mathbf{0.989 \text{ m. Ans.}}$$

But

$$\frac{d}{D} = \frac{1}{6}$$

$\therefore$  Dia. of jet,

$$d = \frac{1}{6} \times D = \frac{0.989}{6} = \mathbf{0.165 \text{ m. Ans.}}$$

Discharge of one jet,

$q = \text{Area of jet} \times \text{Velocity of jet}$

$$= \frac{\pi}{4} d^2 \times V_1 = \frac{\pi}{4} (.165) \times 85.05 \text{ m}^3/\text{s} = 1.818 \text{ m}^3/\text{s} \quad \dots(i)$$

Now

$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{11772}{\frac{\rho g \times Q \times H}{1000}}$$

$$0.86 = \frac{11772 \times 1000}{1000 \times 9.81 \times Q \times 380}, \text{ where } Q = \text{Total discharge}$$

$\therefore$  Total discharge,

$$Q = \frac{11772 \times 1000}{1000 \times 9.81 \times 380 \times 0.86} = 3.672 \text{ m}^3/\text{s}$$

$\therefore$  Number of jets

$$= \frac{\text{Total discharge}}{\text{Discharge of one jet}} = \frac{Q}{q} = \frac{3.672}{1.818} = \mathbf{2 \text{ jets. Ans.}}$$

**Problem 18.3** The penstock supplies water from a reservoir to the Pelton wheel with a gross head of 500 m. One third of the gross head is lost in friction in the penstock. The rate of flow of water through the nozzle fitted at the end of the penstock is 2.0 m<sup>3</sup>/s. The angle of deflection of the jet is 165°. Determine the power given by the water to the runner and also hydraulic efficiency of the Pelton wheel. Take speed ratio = 0.45 and  $C_v = 1.0$ .

**Solution.** Given :

Gross head,

$$H_g = 500 \text{ m}$$

Head lost in friction,

$$h_f = \frac{H_g}{3} = \frac{500}{3} = 166.7 \text{ m}$$

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$$\therefore \text{Net head, } H = H_g - h_f = 500 - 166.7 = 333.30 \text{ m}$$

$$\text{Discharge, } Q = 2.0 \text{ m}^3/\text{s}$$

$$\text{Angle of deflection} = 165^\circ$$

$$\therefore \text{Angle, } \phi = 180^\circ - 165^\circ = 15^\circ$$

$$\text{Speed ratio} = 0.45$$

$$\text{Co-efficient of velocity, } C_v = 1.0$$

$$\text{Velocity of jet, } V_1 = C_v \sqrt{2gH} = 1.0 \times \sqrt{2 \times 9.81 \times 333.3} = 80.86 \text{ m/s}$$

$$\text{Velocity of wheel, } u = \text{Speed ratio} \times \sqrt{2gH}$$

$$\text{or } u = u_1 = u_2 = 0.45 \times \sqrt{2 \times 9.81 \times 333.3} = 36.387 \text{ m/s}$$

$$\therefore V_{r_1} = V_1 - u_1 = 80.86 - 36.387 \\ = 44.473 \text{ m/s}$$

$$\text{Also } V_{w_1} = V_1 = 80.86 \text{ m/s}$$

From outlet velocity triangle, we have

$$V_{r_2} = V_{r_1} = 44.473 \\ V_{r_2} \cos \phi = u_2 + V_{w_2}$$

$$\text{or } 44.473 \cos 15^\circ = 36.387 + V_{w_2}$$

$$\text{or } V_{w_2} = 44.473 \cos 15^\circ - 36.387 = 6.57 \text{ m/s.}$$

Work done by the jet on the runner per second is given by equation (18.9) as

$$\rho a V_1 [V_{w_1} + V_{w_2}] \times u = \rho Q [V_{w_1} + V_{w_2}] \times u \\ = 1000 \times 2.0 \times [80.86 + 6.57] \times 36.387 = 6362630 \text{ Nm/s}$$

$\therefore$  Power given by the water to the runner in kW

$$= \frac{\text{Work done per second}}{1000} = \frac{6362630}{1000} = 6362.63 \text{ kW. Ans.}$$

Hydraulic efficiency of the turbine is given by equation (18.12) as

$$\eta_h = \frac{2[V_{w_1} + V_{w_2}] \times u}{V_1^2} = \frac{2[80.86 + 6.57] \times 36.387}{80.86 \times 80.86} \\ = 0.9731 \text{ or } 97.31\%. \text{ Ans.}$$

**Problem 18.4** A Pelton wheel is having a mean bucket diameter of 1 m and is running at 1000 r.p.m. The net head on the Pelton wheel is 700 m. If the side clearance angle is  $15^\circ$  and discharge through nozzle is  $0.1 \text{ m}^3/\text{s}$ , find :

- (i) Power available at the nozzle, and (ii) Hydraulic efficiency of the turbine.

**Solution.** Given :

$$\text{Diameter of wheel, } D = 1.0 \text{ m}$$

$$\text{Speed of wheel, } N = 1000 \text{ r.p.m.}$$

$$\therefore \text{Tangential velocity of the wheel, } u = \frac{\pi D N}{60} = \frac{\pi \times 1.0 \times 1000}{60} = 52.36 \text{ m/s}$$

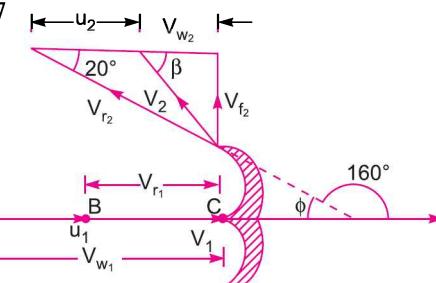


Fig. 18.7

Net head on turbine  $H = 700 \text{ m}$

Side clearance angle,  $\phi = 15^\circ$

Discharge,  $Q = 0.1 \text{ m}^3/\text{s}$

$$\text{Velocity of jet at inlet, } V_1 = C_v \sqrt{2gH} = 1 \times \sqrt{2 \times 9.81 \times 700}$$

( $\because$  Value of  $C_v$  is not given. Take it = 1.0)

$$\text{or } V_1 = 117.19 \text{ m/s}$$

(i) Power available at the nozzle is given by equation (18.3) as

$$\begin{aligned} \text{W.P.} &= \frac{W \times H}{1000} = \frac{\rho \times g \times Q \times H}{1000} \\ &= \frac{1000 \times 9.81 \times 0.1 \times 700}{1000} = 686.7 \text{ kW. Ans.} \end{aligned}$$

(ii) Hydraulic efficiency is given by equation (18.13) as

$$\begin{aligned} \eta_h &= \frac{2(V_1 - u)(1 + \cos \phi) u}{V_1^2} \\ &= \frac{2(117.19 - 52.36)(1 + \cos 15^\circ) \times 52.36}{117.19 \times 117.19} \\ &= \frac{2 \times 64.83 \times 1.966 \times 52.36}{117.19 \times 117.19} = 0.9718 = 97.18 \% \text{, Ans.} \end{aligned}$$

**Problem 18.5** A Pelton wheel is working under a gross head of 400 m. The water is supplied through penstock of diameter 1 m and length 4 km from reservoir to the Pelton wheel. The co-efficient of friction for the penstock is given as .008. The jet of water of diameter 150 mm strikes the buckets of the wheel and gets deflected through an angle of  $165^\circ$ . The relative velocity of water at outlet is reduced by 15% due to friction between inside surface of the bucket and water. If the velocity of the buckets is 0.45 times the jet velocity at inlet and mechanical efficiency as 85% determine :

- (i) Power given to the runner,
- (ii) Shaft power,
- (iii) Hydraulic efficiency and overall efficiency.

**Solution.** Given :

Gross head,  $H_g = 400 \text{ m}$

Diameter of penstock,  $D = 1.0 \text{ m}$

Length of penstock,  $L = 4 \text{ km} = 4 \times 1000 = 4000 \text{ m}$

Co-efficient of friction,  $f = .008$

Diameter of jet,  $d = 150 \text{ mm} = 0.15 \text{ m}$

Angle of deflection  $= 165^\circ$

$\therefore$  Angle,  $\phi = 180^\circ - 165^\circ = 15^\circ$

Relative velocity at outlet,  $V_{r_2} = 0.85 V_{r_1}$

Velocity of bucket,  $u = 0.45 \times \text{Jet velocity}$

Mechanical efficiency,  $\eta_m = 85\% = 0.85$

Let  $V^*$  = Velocity of water in penstock, and

$V_1 = \text{Velocity of jet of water.}$

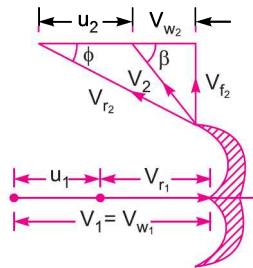


Fig. 18.8

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Using continuity equation, we have

$$\text{Area of penstock} \times V^* = \text{Area of jet} \times V_1$$

or

$$\frac{\pi}{4} D^2 \times V^* = \frac{\pi}{4} d^2 \times V_1$$

$$\therefore V^* = \frac{d^2}{D^2} \times V_1 = \frac{0.15^2}{1.0^2} \times V_1 = .0225 V_1 \quad \dots(i)$$

Applying Bernoulli's equation to the free surface of water in the reservoir and outlet of the nozzle, we get

$$H_g = \text{Head lost due to friction} + \frac{V_1^2}{2g}$$

or

$$400 = \frac{4 f L V^*^2}{D \times 2g} + \frac{V_1^2}{2g} = \frac{4 \times .008 \times 4000 \times V^*^2}{1.0 \times 2 \times 9.81} + \frac{V_1^2}{2g}$$

Substituting the value of  $V^*$  from equation (i), we get

$$\begin{aligned} 400 &= \frac{4 \times .008 \times 4000}{2 \times 9.81} \times (0.0225 V_1)^2 + \frac{V_1^2}{2g} \\ &= .0033 V_1^2 + .051 V_1^2 \text{ or } 400 = .0543 V_1^2 \end{aligned}$$

$$\therefore V_1 = \sqrt{\frac{400}{.0543}} = 85.83 \text{ m/s.}$$

Now velocity of bucket,  $u_1 = 0.45 V_1 = 0.45 \times 85.83 = 38.62 \text{ m/s}$

From inlet velocity triangle,  $V_{r_1} = V_1 - u_1 = 85.83 - 38.62 = 47.21 \text{ m/s}$

$$V_{w_1} = V_1 = 85.83 \text{ m/s}$$

From outlet velocity triangle,  $V_{r_2} = 0.85 \times V_{r_1} = 0.85 \times 47.21 = 40.13 \text{ m/s}$

$$\begin{aligned} V_{w_2} &= V_{r_2} \cos \phi - u_2 = 40.13 \cos 15^\circ - 38.62 \\ &= 0.143 \text{ m/s} \quad (\because u = u_1 = u_2 = 38.62) \end{aligned}$$

Discharge through nozzle is given as

$$Q = \text{Area of jet} \times \text{Velocity of jet} = a \times V_1$$

$$= \frac{\pi}{4} d^2 \times V_1 = \frac{\pi}{4} (.15)^2 \times 85.83 = 1.516 \text{ m}^3/\text{s}$$

Work done on the wheel per second is given by equation (18.9) as

$$\begin{aligned} &= \rho a V_1 [V_{w_1} + V_{w_2}] \times u = \rho Q [V_{w_1} + V_{w_2}] \times u \\ &= 1000 \times 1.516 [85.83 + .143] \times 38.62 = 5033540 \text{ Nm/s} \end{aligned}$$

(i) Power given to the runner in kW

$$= \frac{\text{Work done per second}}{1000} = \frac{5033540}{1000} = 5033.54 \text{ kW. Ans.}$$

(ii) Using equation (18.4) for mechanical efficiency,

$$\eta_m = \frac{\text{Power at the shaft}}{\text{Power given to the runner}} = \frac{\text{S.P.}}{5033.54}$$

$$\therefore \text{S.P.} = \eta_m \times 5033.54 = 0.85 \times 5033.54 = 4278.5 \text{ kW. Ans.}$$

(iii) Hydraulic efficiency is given by equation (18.12) as

$$\begin{aligned}\eta_h &= \frac{2[V_{w_1} + V_{w_2}] \times u}{V_1^2} \\ &= \frac{2[85.83 + .143] \times 38.62}{85.83 \times 85.83} = 0.9014 = 90.14\%. \text{ Ans.}\end{aligned}$$

Overall efficiency is given by equation (18.6) as

$$\eta_0 = \eta_m \times \eta_h = 0.85 \times 0.9014 = 0.7662 \text{ or } 76.62\%. \text{ Ans.}$$

**Problem 18.6** A Pelton wheel nozzle, for which  $C_v = 0.97$ , is 400 m below the water surface of a lake. The jet diameter is 80 mm, the pipe diameter is 0.6 m, its length is 4 km and  $f = 0.032$  in the formula  $h_f = \frac{fLV^2}{2g \times D}$ . The buckets, deflect the jet through  $165^\circ$  and they run at 0.48 times the jet speed, bucket friction reducing the relative velocity at outlet by 15% of the relative velocity at inlet. Mechanical efficiency = 90%. Find the flow rate and the shaft power developed by the turbine.

**Solution.** Given :

Gross head,

$$C_v = 0.97$$

$$H_g = 400 \text{ m}$$

Dia. of jet,

$$d = 800 \text{ mm} = \frac{80}{1000} \text{ m}$$

$$= .08 \text{ m}$$

Dia. of pipe,

$$D = 0.6 \text{ m}$$

Length of pipe

$$L = 4 \text{ km} = 4000 \text{ m}$$

$$f = .032$$

Angle,

$$\phi = 180^\circ - 165^\circ = 15^\circ$$

Bucket speed,

$$u = 0.48 \text{ times jet speed}$$

Relative velocity at outlet

$$= 0.85 \text{ times relative velocity at inlet}$$

or

$$V_{r_2} = 0.85 V_{r_1}$$

Mechanical efficiency,  $\eta_m = 0.90$ .

**Find.** (i) Flow rate, and (ii) Shaft power, S.P.

Let

$V$  = Velocity of water in pipe, and

$V_1$  = Velocity of jet of water.

From continuity equation, we have

$$\text{Area of pipe} \times V = \text{Area of jet} \times V_1$$

or

$$\frac{\pi}{4} D^2 \times V = \frac{\pi}{4} d^2 \times V_1$$

$$= \frac{d^2}{D^2} \times V_1 = \left( \frac{.08}{0.60} \right)^2 \times V_1 = 0.0177 V_1 \quad \dots(i)$$

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Applying Bernoulli's equation to the free surface of water in the reservoir and the outlet of the nozzle, we get

Head at reservoir = Kinetic head of jet of water + Head lost due to friction in pipe + Head lost in nozzle

$$= \frac{V_1^2}{2g} + \frac{fLV^2}{D \times 2g} + \text{Head lost in nozzle} \quad \dots(ii)$$

Let  $V^*$  = Theoretical velocity at the outlet of nozzle,  $V_1$  = Actual velocity of jet of water

Then  $\frac{V_1}{V^*} = C_v$  or  $V^* = \frac{V_1}{C_v}$ .

Now head lost in nozzle = Head corresponding to  $V^*$  – Head corresponding to  $V_1$

$$= \frac{V^{*2}}{2g} - \frac{V_1^2}{2g} = \left( \frac{V_1}{C_v} \right)^2 \times \frac{1}{2g} - \frac{V_1^2}{2g} = \frac{V_1^2}{2g} \left( \frac{1}{C_v^2} - 1 \right)$$

Substituting this value in equation (ii), we get

Head at reservoir  $= \frac{V_1^2}{2g} + \frac{fLV^2}{2g \times D} + \frac{V_1^2}{2g} \left( \frac{1}{C_v^2} - 1 \right)$

or  $400 = \frac{V_1^2}{2g} + \frac{0.032 \times 4000 \times V^2}{0.6 \times 2 \times 9.81} + \frac{V_1^2}{2g} \times \frac{1}{C_v^2} - \frac{V_1^2}{2g}$

$$\begin{aligned} &= \frac{0.032 \times 4000 \times (0.0177 V_1)^2}{0.6 \times 2 \times 9.81} + \frac{V_1^2}{2 \times 9.81} \times \frac{1}{.97^2} \\ &= .0034 V_1^2 + .054 V_1^2 \quad (\because V = .0177 V_1) \\ &= 0.0574 V_1^2 \end{aligned}$$

$\therefore V_1 = \sqrt{\frac{400}{0.0574}} = 83.47 \text{ m/s}$

Now velocity of bucket,  $u_1 = 0.48 \times V_1 = 0.48 \times 83.47 = 40.06 \text{ m/s}$

Refer to Fig. 18.8 (a), we have from inlet velocity triangle

$$V_{r_1} = V_1 - u_1 = 83.47 - 40.06 = 43.41 \text{ m/s}$$

$$V_{w_1} = V_1 = 83.47$$

From outlet velocity triangle,

$$V_{r_2} = 0.85 V_{r_1} = 0.85 \times 43.41 = 36.898 \text{ m/s}$$

$$\begin{aligned} V_{w_2} &= u_2 - V_{r_2} \cos \phi \\ &= 40.06 - 36.898 \times \cos 15^\circ \\ &\quad (\because u_1 = u_2 = 40.06) \\ &= 4.42 \end{aligned}$$

Flow rate,

$$Q = \text{Area of jet} \times \text{Velocity of jet}$$

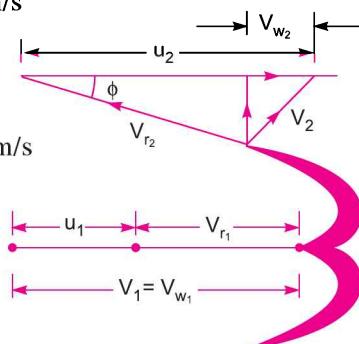


Fig. 18.8 (a)

$$= \frac{\pi}{4} d^2 \times V_1 = \frac{\pi}{4} (.08)^2 \times 83.47 = 0.419 \text{ m}^3/\text{s. Ans.}$$

Now using equation (18.4),  $\eta_m = \frac{\text{S.P.}}{\text{Power given to the runner}}$

$$\therefore \text{S.P.} = \eta_m \times \text{Power given to the runner}$$

where power given to the runner in kW

$$= \frac{\text{Work done per second}}{1000}$$

$$\text{Work done per second} = \frac{W}{g} [V_{w_1} - V_{w_2}] \times u$$

(Here –ve sign is taken as  $V_{w_1}$  and  $V_{w_2}$  are in the same direction)

$$\begin{aligned} &= \frac{\rho \times g \times Q}{g} [V_{w_1} - V_{w_2}] \times u_1 && \{ \because u = u_1 \} \\ &= \frac{1000 \times 9.81 \times .419}{9.81} [83.47 - 4.42] \times 40.06 = 1326865 \text{ Nm/s} \end{aligned}$$

$$\therefore \text{Power given to the runner} = \frac{1326865}{1000} = 1326.865 \text{ kW}$$

$$\begin{aligned} \therefore \text{S.P.} &= \eta_m \times \text{Power given to runner} \\ &= 0.90 \times 1326.865 = 1194.18 \text{ kW. Ans.} \end{aligned}$$

**Problem 18.7** A 137 mm diameter jet of water issuing from a nozzle impinges on the buckets of a Pelton wheel and the jet is deflected through an angle of 165° by the buckets. The head available at the nozzle is 400 m. Assuming co-efficient of velocity as 0.97, speed ratio as 0.46, and reduction in relative velocity while passing through buckets as 15%, find :

- (i) The force exerted by the jet on buckets in tangential direction,
- (ii) The power developed.

**Solution.** Given :

$$\text{Dia. of jet, } d = 137 \text{ mm} = 0.137 \text{ m}$$

$$\therefore \text{Area of jet, } a = \frac{\pi}{4} \times 0.137 = 0.01474 \text{ m}^2$$

$$\text{Angle of deflection} = 165^\circ$$

$$\therefore \text{Angle, } \phi = 180^\circ - 165^\circ = 15^\circ$$

$$\text{Head of water, } H = 400 \text{ m}$$

$$\text{Co-efficient of velocity, } C_v = 0.97$$

$$\text{Speed ratio} = 0.46$$

$$\text{Relative velocity at outlet} = 0.85 \times \text{relative velocity at inlet}$$

$$\text{or } V_{r_2} = 0.85 V_{r_1}$$

$$\text{Now velocity of jet, } V_1 = C_v \sqrt{2gH} = 0.97 \sqrt{2 \times 9.81 \times 400} = 85.93 \text{ m/s}$$

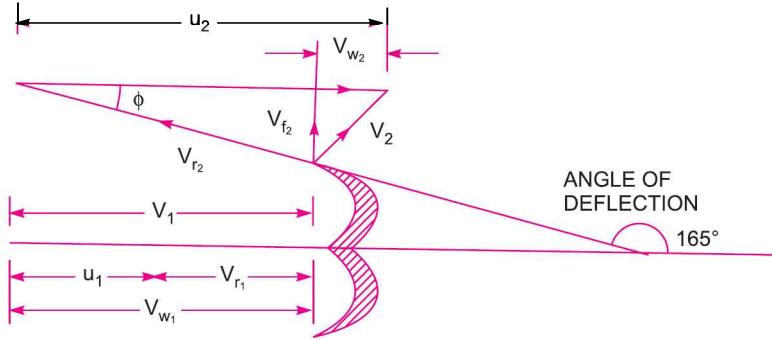


Fig. 18.8 (b)

$$\text{Speed ratio} = \frac{u_1}{\sqrt{2gH}} \text{ or } 0.46 = \frac{u_1}{\sqrt{2 \times 9.81 \times 400}}$$

$$\therefore u_1 = 0.46 \times \sqrt{2 \times 9.81 \times 400} = 40.75 \text{ m/s}$$

$$\text{Hence } V_{r_1} = V_1 - u_1 = 85.93 - 40.75 = 45.18 \text{ m/s}$$

$$\text{and } V_{r_2} = 0.85 V_{r_1} = 0.85 \times 45.18 = 38.40 \text{ m/s}$$

$$\text{For Pelton turbine, } u_1 = u_2 = u = 40.75 \text{ m/s}$$

$$V_{r_2} \cos \phi = 38.40 \times \cos 15^\circ = 37.092$$

Here  $V_{r_2} \cos \phi$  is less than  $u_2$ . Hence velocity triangle at outlet will be as shown in Fig. 18.8 (b)

$$\therefore V_{w_2} = u_2 - V_{r_2} \cos \phi = 40.75 - 37.092 = 3.658 \text{ m/s.}$$

(i) Force exerted by jet on buckets in tangential direction is given by,

$$F_x = \rho a V_1 [V_{w_1} - V_{w_2}]$$

(Here -ve sign is taken as  $V_{w_1}$  and  $V_{w_2}$  are in the same direction)

$$\therefore F_x = 1000 \times 0.01474 \times 85.93 (85.93 - 3.658) \text{ N} = 104206 \text{ N. Ans.}$$

(ii) Power developed is given by,

$$\text{Power} = \frac{F_x \times u}{1000} \text{ kW} = \frac{104206 \times 40.75}{1000} = 4246.4 \text{ kW. Ans.}$$

**Problem 18.8** Two jets strike the buckets of a Pelton wheel, which is having shaft power as 15450 kW. The diameter of each jet is given as 200 mm. If the net head on the turbine is 400 m, find the overall efficiency of the turbine. Take  $C_v = 1.0$ .

**Solution.** Given :

$$\text{Number of jets} = 2$$

$$\text{Shaft power, S.P.} = 15450 \text{ kW}$$

$$\text{Diameter of each jet, } d = 200 \text{ mm} = 0.20 \text{ m}$$

$$\therefore \text{Area of each jet, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (.2)^2 = 0.031416 \text{ m}^2$$

$$\text{Net head, } H = 400 \text{ m}$$

$$\text{Co-efficient of velocity, } C_v = 1.0$$

$$\text{Velocity of each jet, } V_1 = C_v \sqrt{2gH} = 1.0 \times \sqrt{2 \times 9.81 \times 400} = 88.58 \text{ m/s}$$

$$\text{Discharge of each jet} = a \times V_1 = .031416 \times 88.58 = 2.78 \text{ m}^3/\text{s}$$

$$\therefore \text{Total discharge, } Q = 2 \times 2.78 = 5.56 \text{ m}^3/\text{s}$$

Power at the inlet of turbine,

$$\begin{aligned} \text{W.P.} &= \frac{\rho \times g \times Q \times H}{1000} \text{ kW} \\ &= \frac{1000 \times 9.81 \times 5.56 \times 400}{1000} = 21817.44 \text{ kW} \end{aligned}$$

$\therefore$  Overall efficiency is given as

$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{15450}{21817.44} = 0.708 = 70.8\% \text{ Ans.}$$

**Problem 18.9** The water available for a Pelton wheel is 4 cumec and the total head from the reservoir to the nozzle is 250 metres. The turbine has two runners with two jets per runner. All the four jets have the same diameters. The pipe line is 3000 metres long. The efficiency of power transmission through the pipe line and the nozzle is 91% and efficiency of each runner is 90%. The velocity co-efficient of each nozzle is 0.975 and co-efficient of friction '4f' for the pipe is 0.0045. Determine :

- (i) The power developed by the turbine, (ii) The diameter of the jet, and
- (iii) The diameter of the pipe line.

**Solution.** Given :

$$\text{Total discharge, } Q = 4 \text{ cumec} = 4.0 \text{ m}^3/\text{s}$$

$$\text{Total or gross head, } H_g = 250 \text{ m}$$

$$\text{Total number of jets} = 2 \times 2 = 4$$

$$\text{Length of pipe, } L = 3000 \text{ m}$$

$$\text{Efficiency of the pipe line and nozzle} = 91\% \text{ or } 0.91$$

$$\text{Efficiency of runner* or } \eta_h = 90\% \text{ or } 0.90$$

$$\text{Co-efficient of velocity, } C_v = 0.975$$

$$\text{Co-efficient of friction, } 4f = 0.0045$$

Efficiency of power transmission through pipe lines and nozzle is given by

$$\eta = \frac{H_g - h_f}{H_g} \text{ or } 0.91 = \frac{250 - h_f}{250}$$

where  $h_f$  = Head lost due to friction.

$$\therefore h_f = 250 - 0.91 \times 250 = 22.5 \text{ m}$$

$$\therefore \text{Net head on the turbine, } H = H_g - h_f = 250 - 22.50 = 227.5 \text{ m}$$

$$\text{Velocity of jet, } V_1 = C_v \sqrt{2gH} = 0.975 \sqrt{2 \times 9.81 \times 227.5} = 65.14 \text{ m/s.}$$

---

\* Efficiency of runner means the ratio of power delivered to the runner to the power at the inlet of turbine i.e., hydraulic efficiency.

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(i) Power at the inlet of the turbine is given as,

$$\text{W.P.} = \text{Kinetic energy of the jet/s}$$

$$= \frac{\frac{1}{2} m V_1^2}{1000} = \frac{\frac{1}{2} (\rho \times Q) V_1^2}{1000} = \frac{1}{2} \times 1000 \times \frac{4.0 \times 65.14^2}{1000} = 8486.44 \text{ kW}$$

But

$$\eta_h = \frac{\text{Power developed by turbine}}{\text{W.P.}}$$

$$\therefore 0.90 = \frac{\text{Power developed by turbine}}{8486.44}$$

$$\therefore \text{Power developed by turbine} = 0.90 \times 8486.44 = 7637.8 \text{ kW. Ans.}$$

$$(ii) \text{ Discharge per jet, } q = \frac{\text{Total discharge}}{\text{No. of jets}} = \frac{4.0}{4.0} = 1.0 \text{ m}^3/\text{s}$$

But

$$q = \text{Area of one jet} \times \text{Velocity of jet}$$

$$= \frac{\pi}{4} d^2 \times V_1, \quad \text{where } d = \text{Diameter of each jet}$$

$$\therefore 1.0 = \frac{\pi}{4} d^2 \times 65.14$$

$$\therefore d = \sqrt{\frac{4 \times 1.0}{\pi \times 65.14}} = 0.14 \text{ m. Ans.}$$

(iii) Let

$$D = \text{Diameter of pipe line}$$

$$\text{Then } h_f = \frac{4 \times f \times L \times V^{*2}}{D \times 2g}, \quad \text{where } V^* = \text{Velocity through pipe}$$

$$\therefore V^* = \frac{Q}{\text{Area}} = \frac{Q}{\frac{\pi}{4} D^2} = \frac{4Q}{\pi D^2}$$

$$\text{And } h_f = \frac{.0045 \times 3000 \times \left(\frac{4Q}{\pi D^2}\right)^2}{D \times 2g}$$

$$\text{or } 22.50 = \frac{.0045 \times 3000 \times 16 \times Q^2}{D \times 2 \times 9.81 \times \pi^2 \times D^4} = \frac{.0045 \times 3000 \times 16 \times (4)^2}{D^5 \times 2 \times 9.81 \times \pi^2} = \frac{17.84}{D^5}$$

$$\therefore D^5 = \frac{17.84}{22.50} = 0.7933$$

$$\therefore D = (0.7933)^{1/5} = 0.955 \text{ m. Ans.}$$

**Problem 18.10** The following data is related to a Pelton wheel :

$$\text{Head at the base of the nozzle} = 80 \text{ m}$$

$$\text{Diameter of the jet} = 100 \text{ mm}$$

*Discharge of the nozzle* =  $0.30 \text{ m}^3/\text{s}$

*Power at the shaft* =  $206 \text{ kW}$

*Power absorbed in mechanical resistance* =  $4.5 \text{ kW}$

Determine (i) Power lost in nozzle and (ii) Power lost due to hydraulic resistance in the runner.

**Solution.** Given :

Head at the base of the nozzle,  $H_1 = 80 \text{ m}$

Diameter of the jet,  $d = 100 \text{ mm} = 0.1 \text{ m}$

$$\therefore \text{Area of the jet, } a = \frac{\pi}{4} = (0.1)^2 = .007854$$

Discharge of the nozzle,  $Q = 0.30 \text{ m}^3/\text{s}$

Shaft power, S.P. =  $206 \text{ kW}$

Power absorbed in mechanical resistance =  $4.5 \text{ kW}$

Now discharge  $Q = \text{area of jet} \times \text{velocity of jet} = a \times V_1$

$$0.30 = .007854 \times V_1$$

$$\therefore V_1 = \frac{0.30}{.007854} = 38.197 \text{ m/s}$$

Power at the base of the nozzle in kW

$$= \frac{\rho \times g \times Q \times H_1}{1000} = \frac{1000 \times 9.81 \times 0.30 \times 80}{1000} = 235.44$$

Power corresponding to kinetic energy of the jet in kW

$$\begin{aligned} &= \frac{1}{2} \frac{(\rho \times V_1^2)}{1000} = \frac{1}{2} \frac{(\rho \times aV_1 \times V_1^2)}{1000} = \frac{1}{2} \frac{\rho \times Q \times V_1^2}{1000} \\ &= \frac{1}{2} \times 1000 \times \frac{0.3 \times 38.197^2}{1000} = 218.85 \text{ kW.} \end{aligned}$$

(i) Power at the base of the nozzle

$$= \text{Power of the jet} + \text{Power lost in nozzle}$$

or

$$235.44 = 218.85 + \text{Power lost in nozzle}$$

$$\therefore \text{Power lost in nozzle} = 235.44 - 218.85 = 16.59 \text{ kW. Ans.}$$

(ii) Also power at the base of nozzle = power at the shaft + power lost in nozzle + power lost in runner + power lost due to mechanical resistance

$$\therefore 235.44 = 206 + 16.59 + \text{Power lost in runner} + 4.5$$

$$\therefore \text{Power lost in runner} = 235.44 - (206 + 16.59 + 4.5) = 235.44 - 227.09 = 8.35 \text{ kW. Ans.}$$

**18.6.3 Design of Pelton Wheel.** Design of Pelton wheel means the following data is to be determined :

1. Diameter of the jet ( $d$ ),
2. Diameter of wheel ( $D$ ),
3. Width of the buckets which is  $= 5 \times d$ ,
4. Depth of the buckets which is  $= 1.2 \times d$ , and
5. Number of buckets on the wheel.

Size of buckets means the width and depth of the buckets.

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**Problem 18.11** A Pelton wheel is to be designed for a head of 60 m when running at 200 r.p.m. The Pelton wheel develops 95.6475 kW shaft power. The velocity of the buckets = 0.45 times the velocity of the jet, overall efficiency = 0.85 and co-efficient of the velocity is equal to 0.98.

**Solution.** Given :

$$\text{Head, } H = 60 \text{ m}$$

$$\text{Speed } N = 200 \text{ r.p.m}$$

$$\text{Shaft power, S.P.} = 95.6475 \text{ kW}$$

$$\text{Velocity of bucket, } u = 0.45 \times \text{Velocity of jet}$$

$$\text{Overall efficiency, } \eta_o = 0.85$$

$$\text{Co-efficient of velocity, } C_v = 0.98$$

Design of Pelton wheel means to find diameter of jet ( $d$ ), diameter of wheel ( $D$ ), Width and depth of buckets and number of buckets on the wheel.

$$(i) \text{ Velocity of jet, } V_1 = C_v \times \sqrt{2gH} = 0.98 \times \sqrt{2 \times 9.81 \times 60} = 33.62 \text{ m/s}$$

$$\therefore \text{ Bucket velocity, } u = u_1 = u_2 = 0.45 \times V_1 = 0.45 \times 33.62 = 15.13 \text{ m/s}$$

$$\text{But } u = \frac{\pi D N}{60}, \quad \text{where } D = \text{Diameter of wheel}$$

$$\therefore 15.13 = \frac{\pi \times D \times 200}{60} \quad \text{or} \quad D = \frac{60 \times 15.13}{\pi \times 200} = 1.44 \text{ m. Ans.}$$

(ii) Diameter of the jet ( $d$ )

$$\text{Overall efficiency } \eta_o = 0.85$$

$$\text{But } \eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{95.6475}{\left( \frac{\text{W.P.}}{1000} \right)} = \frac{95.6475 \times 1000}{\rho \times g \times Q \times H} \quad (\because \text{ W.P.} = \rho g Q H)$$

$$= \frac{95.6475 \times 1000}{1000 \times 9.81 \times Q \times 60}$$

$$\therefore Q = \frac{95.6475 \times 1000}{\eta_o \times 1000 \times 9.81 \times 60} = \frac{95.6475 \times 1000}{0.85 \times 1000 \times 9.81 \times 60} = 0.1912 \text{ m}^3/\text{s.}$$

But the discharge,  $Q = \text{Area of jet} \times \text{Velocity of jet}$

$$\therefore 0.1912 = \frac{\pi}{4} d^2 \times V_1 = \frac{\pi}{4} d^2 \times 33.62$$

$$\therefore d = \sqrt{\frac{4 \times 0.1912}{\pi \times 33.62}} = 0.085 \text{ m} = 85 \text{ mm. Ans.}$$

(iii) Size of buckets

$$\text{Width of buckets} = 5 \times d = 5 \times 85 = 425 \text{ mm}$$

$$\text{Depth of buckets} = 1.2 \times d = 1.2 \times 85 = 102 \text{ mm. Ans.}$$

(iv) Number of buckets on the wheel is given by equation (18.17) as

$$Z = 15 + \frac{D}{2d} = 15 + \frac{1.44}{2 \times 0.085} = 15 + 8.5 = 23.5 \text{ say 24. Ans.}$$

**Problem 18.12** Determine the power given by the jet of water to the runner of a Pelton wheel which is having tangential velocity as 20 m/s. The net head on the turbine is 50 m and discharge through the jet water is 0.03 m<sup>3</sup>/s. The side clearance angle is 15° and take  $C_v = 0.975$ .

**Solution.** Given :

Tangential velocity of wheel,  $u = u_1 = u_2 = 20 \text{ m/s}$

Net head,  $H = 50 \text{ m}$

Discharge,  $Q = 0.03 \text{ m}^3/\text{s}$

Side clearance angle,  $\phi = 15^\circ$

Co-efficient of velocity,  $C_v = 0.975$

$$\begin{aligned}\text{Velocity of the jet, } V_1 &= C_v \times \sqrt{2gH} \\ &= 0.975 \times \sqrt{2 \times 9.81 \times 50} \\ &= 30.54 \text{ m/s}\end{aligned}$$

$$\text{From inlet triangle, } V_{w_1} = V_1 = 30.54 \text{ m/s}$$

$$V_{r_1} = V_{w_1} - u_1 = 30.54 - 20.0 = 10.54 \text{ m/s}$$

From outlet velocity triangle, we have

$$V_{r_2} = V_{r_1} = 10.54 \text{ m/s}$$

$$V_{r_2} \cos \phi = 10.54 \cos 15^\circ = 10.18 \text{ m/s}$$

As  $V_{r_2} \cos \phi$  is less than  $u_2$ , the velocity triangle at outlet will be as shown in Fig. 18.9.

$$\therefore V_{w_2} = u_2 - V_{r_2} \cos \phi = 20 - 10.18 = 9.82 \text{ m/s.}$$

Also as  $\beta$  is an obtuse angle, the work done per second on the runner,

$$\begin{aligned}&= \rho a V_1 [V_{w_1} - V_{w_2}] \times u = \rho Q [V_{w_1} - V_{w_2}] \times u \\ &= 1000 \times 0.03 \times [30.54 - 9.82] \times 20 = 12432 \text{ Nm/s}\end{aligned}$$

$$\therefore \text{Power given to the runner in kW} = \frac{\text{Work done per second}}{1000} = \frac{12432}{1000} = 12.432 \text{ kW. Ans.}$$

**Problem 18.13** The three-jet Pelton turbine is required to generate 10,000 kW under a net head of 400 m. The blade angle at outlet is 15° and the reduction in the relative velocity while passing over the blade is 5%. If the overall efficiency of the wheel is 80%,  $C_v = 0.98$  and speed ratio = 0.46, then find: (i) the diameter of the jet, (ii) total flow in m<sup>3</sup>/s and (iii) the force exerted by a jet on the buckets.

If the jet ratio is not to be less than 10, find the speed of the wheel for a frequency of 50 hertz/sec and the corresponding wheel diameter.

**Solution.** Given :

No. of jets = 3

Total power,  $P = 10000 \text{ kW}$

Net head,  $H = 400 \text{ m}$

Blade angle at outlet,  $\phi = 15^\circ$

Relative velocity at outlet = 0.95 of relative velocity at inlet

or  $V_{r_2} = 0.95 V_{r_1}$

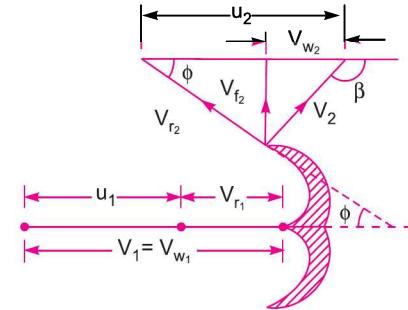


Fig. 18.9

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Overall efficiency,	$\eta_o = 0.80$
Value of	$C_v = 0.98$
Speed ratio	$= 0.46$
Frequency,	$f = 50 \text{ hertz/sec}$

Now using equation (18.6 A),  $\eta_o = \frac{P}{(\rho \times g \times Q \times H)}$

where  $Q$  = Total discharge through three nozzles and  $\rho = 1000 \text{ kg/m}^3$

$$\therefore 0.80 = \frac{10000}{\left( \frac{1000 \times 9.81 \times Q \times 400}{1000} \right)}$$

$$\therefore Q = \frac{10000}{0.8 \times 9.81 \times 400} = 3.18 \text{ m}^3/\text{s. Ans.}$$

Discharge through one nozzle  $= \frac{3.18}{3} = 1.06 \text{ m}^3/\text{s.}$

(i) **Diameter of the jet (d).**

Discharge through one nozzle = Area of one jet  $\times$  Velocity

But velocity of jet,  $V_1 = C_v \times \sqrt{2gH} = 0.98 \times \sqrt{2 \times 9.81 \times 400} = 87 \text{ m/s}$

$$\therefore 1.06 = \frac{\pi}{4} d^2 \times 87$$

$$\therefore d = \sqrt{\frac{4 \times 1.06}{\pi \times 87}} = 0.125 \text{ m} = 125 \text{ mm. Ans.}$$

(ii) **Total flow in m<sup>3</sup>/s**  $= 3.18 \text{ m}^3/\text{s.}$

(iii) **Force exerted by a jet on the wheel.**

$$\text{Speed ratio} = \frac{u_1}{\sqrt{2gH}}$$

$$\therefore u_1 = \text{Speed ratio} \times \sqrt{2gH} = 0.46 \times \sqrt{2 \times 9.81 \times 400} = 40.75 \text{ m/s}$$

Now  $V_{r_1} = V_1 - u_1 = 87 - 40.75 = 46.25 \text{ m/s}$

and

$$V_{r_2} = 0.95 V_{r_1} = 0.95 \times 46.25 = 44.0 \text{ m/s}$$

$$V_{w_1} = V_1 = 87 \text{ m/s}$$

$$V_{w_2} = V_{r_2} \cos \phi - u_2 = 44 \times \cos 15^\circ - 40.75 \quad (\because u_1 = u_2 = 40.75 \text{ m/s}) \\ = 1.75 \text{ m/s}$$

Force exerted by a single jet on the buckets

$$= \rho \times \text{discharge through one jet} \times (V_{w_1} + V_{w_2}) \\ = 1000 \times 1.06 (87 + 1.75) = 94075 \text{ N} = 94.075 \text{ kN. Ans.}$$

$$(iv) \text{ Jet ratio} = 10 \text{ or } \frac{D}{d} = 10$$

$$\therefore \text{ Dia. of wheel, } D = 10 \times d = 10 \times 0.125 = 1.25 \text{ m}$$

$$\text{But, } u_1 = \frac{\pi DN}{60}$$

$$\therefore N = \frac{60 \times u_1}{\pi \times D} = \frac{60 \times 40.75}{\pi \times 1.25} = 620 \text{ r.p.m.}$$

$$\text{Now using the relation, } N = \frac{60 \times f}{p}$$

where  $f$  = frequency in hertz per second,

$p$  = pairs of poles, and  $N$  = speed.

$$\therefore p = \frac{60 \times f}{N} = \frac{60 \times 50}{620} = 4.85$$

Take the next whole number *i.e.*, 5. Hence, pairs of poles are 5.

Now corresponding to five pairs of poles, the speed of the turbine will become as given below :

$$N = \frac{60 \times f}{p} = \frac{60 \times 50}{5} = 600 \text{ r.p.m.}$$

$$\text{But } u = \frac{\pi DN}{60}$$

As the peripheral velocity is constant. Hence with the change of speed, diameter of wheel will change.

$$\therefore D = \frac{60 \times u}{\pi \times N} = \frac{60 \times 40.75}{\pi \times 600} = 1.3 \text{ m}$$

$$\therefore \text{ Jet ratio becomes } = \frac{D}{d} = \frac{1.30}{0.125} > 10$$

Hence the given condition is satisfied.

## ► 18.7 RADIAL FLOW REACTION TURBINES

Radial flow turbines are those turbines in which the water flows in the radial direction. The water may flow radially from outwards to inwards (*i.e.*, towards the axis of rotation) or from inwards to outwards. If the water flows from outwards to inwards through the runner, the turbine is known as inward radial flow turbine. And if the water flows from inwards to outwards, the turbine is known as outward radial flow turbine.

Reaction turbine means that the water at the inlet of the turbine possesses kinetic energy as well as pressure energy. As the water flows through the runner, a part of pressure energy goes on changing into kinetic energy. Thus the water through the runner is under pressure. The runner is completely enclosed in an air-tight casing and casing and the runner is always full of water.

**18.7.1 Main Parts of a Radial Flow Reaction Turbine.** The main parts of a radial flow reaction turbine are :

- |                |                     |
|----------------|---------------------|
| 1. Casing,     | 2. Guide mechanism, |
| 3. Runner, and | 4. Draft-tube.      |

**1. Casing.** As mentioned above that in case of reaction turbine, casing and runner are always full of water. The water from the penstocks enters the casing which is of spiral shape in which area of cross-section of the casing goes on decreasing gradually. The casing completely surrounds the runner of the turbine. The casing as shown in Fig. 18.10 is made of spiral shape, so that the water may enter the runner at constant velocity throughout the circumference of the runner. The casing is made of concrete, cast steel or plate steel.

**2. Guide Mechanism.** It consists of a stationary circular wheel all round the runner of the turbine. The stationary guide vanes are fixed on the guide mechanism. The guide vanes allow the water to strike the vanes fixed on the runner without shock at inlet. Also by a suitable arrangement, the width between two adjacent vanes of guide mechanism can be altered so that the amount of water striking the runner can be varied.

**3. Runner.** It is a circular wheel on which a series of radial curved vanes are fixed. The surface of the vanes are made very smooth. The radial curved vanes are so shaped that the water enters and leaves the runner without shock. The runners are made of cast steel, cast iron or stainless steel. They are keyed to the shaft.

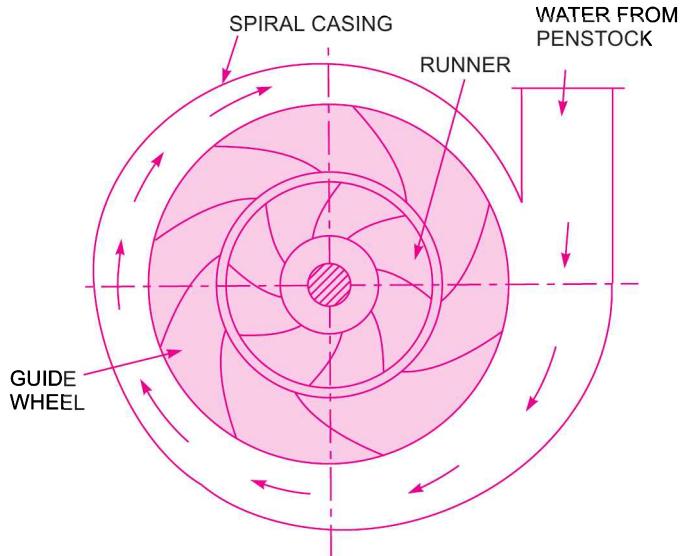


Fig. 18.10 Main parts of a radial reaction turbines.

**4. Draft-tube.** The pressure at the exit of the runner of a reaction turbine is generally less than atmospheric pressure. The water at exit cannot be directly discharged to the tail race. A tube or pipe of gradually increasing area is used for discharging water from the exit of the turbine to the tail race. This tube of increasing area is called draft tube.

**18.7.2 Inward Radial Flow Turbine.** Fig. 18.11 shows inward radial flow turbine, in which case the water from the casing enters the stationary guiding wheel. The guiding wheel consists of guide vanes which direct the water to enter the runner which consists of moving vanes. The water flows over the moving vanes in the inward radial direction and is discharged at the inner diameter of the runner. The outer diameter of the runner is the inlet and the inner diameter is the outlet.

**Velocity Triangles and Work done by Water on Runner.** In Chapter 17 (Art. 17.4.6), we have discussed in detail the force exerted by the water on the radial curved vanes fixed on a wheel. From the force exerted on the vanes, the work done by water, the horse power given by the water to the vanes and

efficiency of the vanes can be obtained. Also we have drawn velocity triangles at inlet and outlet of the moving radial vanes in Fig. 17.23. From the velocity triangles, the work done by the water on the runners, horse power and efficiency of the turbine can be obtained.

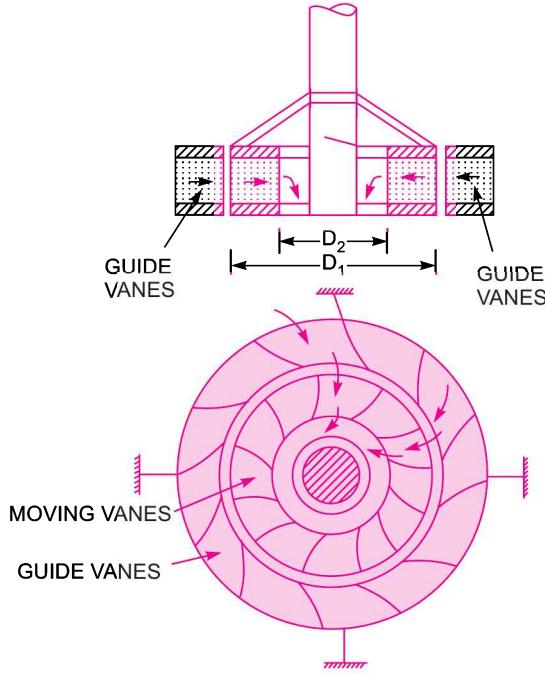


Fig. 18.11 Inward radial flow turbine.

The work done per second on the runner by water is given by equation (17.26) as

$$\begin{aligned} &= \rho a V_1 [V_{w_1} u_1 \pm V_{w_2} u_2] \\ &= \rho Q [V_{w_1} u_1 \pm V_{w_2} u_2] \quad (\because aV_1 = Q) \quad \dots(18.18) \end{aligned}$$

The equation (18.18) also represents the energy transfer per second to the runner.

where  $V_{w_1}$  = Velocity of whirl at inlet,

$V_{w_2}$  = Velocity of whirl at outlet,

$u_1$  = Tangential velocity of wheel at inlet

$$= \frac{\pi D_1 \times N}{60}, \text{ where } D_1 = \text{Outer dia. of runner,}$$

$u_2$  = Tangential velocity of wheel at outlet

$$= \frac{\pi D_2 \times N}{60}, \text{ where } D_2 = \text{Inner dia. of runner, } N = \text{Speed of the turbine in r.p.m.}$$

The work done per second per unit weight of water per second.

$$\begin{aligned} &= \frac{\text{Work done per second}}{\text{Weight of water striking per second}} \\ &= \frac{\rho Q [V_{w_1} u_1 \pm V_{w_2} u_2]}{\rho Q \times g} = \frac{1}{g} [V_{w_1} u_1 \pm V_{w_2} u_2] \quad \dots(18.19) \end{aligned}$$

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The equation (18.19) represents the energy transfer per unit weight/s to the runner. This equation is known by **Euler's equation** of hydrodynamics machines. This is also known as fundamental equation of hydrodynamic machines. This equation was given by Swiss scientist *L. Euler*.

In equation (18.19), +ve sign is taken if angle  $\beta$  is an acute angle. If  $\beta$  is an obtuse angle then -ve sign is taken. If  $\beta = 90^\circ$ , then  $V_{w_2} = 0$  and work done per second per unit weight of water striking/s become as

$$= \frac{1}{g} V_{w_1} u_1 \quad \dots(18.20)$$

**Hydraulic efficiency** is obtained from equation (18.2) as

$$\eta_h = \frac{\text{R.P.}}{\text{W.P.}} = \frac{\frac{W}{1000g} [V_{w_1} u_1 \pm V_{w_2} u_2]}{\frac{W \times H}{1000}} = \frac{(V_{w_1} u_1 \pm V_{w_2} u_2)}{gH} \quad \dots(18.20A)$$

where R.P. = Runner power i.e., power delivered by water to the runner

W.P. = Water power

If the discharge is radial at outlet, then  $V_{w_2} = 0$

$$\eta_h = \frac{V_{w_1} u_1}{gH} \quad \dots(18.20B)$$

**18.7.3 Degree of Reaction.** Degree of reaction is defined as the ratio of pressure energy change inside a runner to the total energy change inside the runner. It is represented by 'R'. Hence mathematically it can be written as

$$R = \frac{\text{Change of pressure energy inside the runner}}{\text{Change of total energy inside the runner}} \quad \dots(18.20C)$$

The equation (18.19) which is the fundamental equation of hydrodynamic machines, represents the energy transfer per unit weight to the runner. This is also known as the total energy change inside the runner per unit weight.

∴ Change of total energy per unit weight inside the runner

$$= \frac{1}{g} [V_{w_1} u_1 \pm V_{w_2} u_2]$$

Let  $H_e$  = Change of total energy per unit weight inside the runner.

$$\text{Then } H_e = \frac{1}{g} [V_{w_1} u_1 \pm V_{w_2} u_2] \quad \dots(18.20D)$$

Let us find the values of  $V_{w_1} u_1$  and  $V_{w_2} u_2$  from inlet and outlet velocity triangles.

Now from inlet velocity triangle, we know that [Refer to Fig. 18.11(a)]

$$\begin{aligned} V_{w_1} &= u_1 + V_{r_1x}, \quad \text{where } V_{r_1x} = V_{r_1} \cos \theta = \sqrt{V_{r_1}^2 - V_{f_1}^2} \\ &= u_1 + \sqrt{V_{r_1}^2 - V_{f_1}^2} \\ &= u_1 + \sqrt{V_{r_1}^2 - (V_1^2 - V_{w_1}^2)} \quad [\because \text{From triangle } ABC, V_{f_1}^2 = V_1^2 - V_{w_1}^2] \\ \therefore (V_{w_1} - u_1) &= \sqrt{V_{r_1}^2 - (V_1^2 - V_{w_1}^2)} \end{aligned}$$

Squaring both sides, we get

$$(V_{w_1} - u_1)^2 = V_r^2 - (V_1^2 - V_{w_1}^2)$$

or

$$V_{w_1}^2 + u_1^2 - 2V_{w_1}u_1 = V_r^2 - V_1^2 + V_{w_1}^2$$

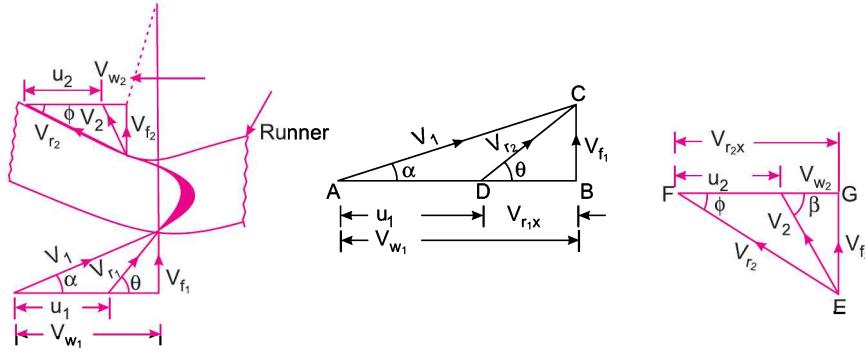


Fig. 18.11 (a)

$$\text{or } V_{w_1}^2 + u_1^2 - V_r^2 + V_1^2 - V_{w_1}^2 = 2V_{w_1}u_1$$

$$\text{or } u_1^2 - V_r^2 + V_1^2 = 2V_{w_1}u_1$$

$$\text{or } 2V_{w_1}u_1 = u_1^2 - V_r^2 + V_1^2$$

$$\text{or } V_{w_1}u_1 = \frac{1}{2}[u_1^2 - V_r^2 + V_1^2] \quad \dots(i)$$

Similarly from outlet triangle, we know that [Refer to Fig. 18.11(a)]

$$\begin{aligned} V_{w_2} &= V_{r_2x} - u_2 \\ &= \sqrt{V_{r_2}^2 - V_{f_2}^2} - u_2, \text{ where } V_{r_2x} = V_{r_2} \cos \theta = \sqrt{V_{r_2}^2 - V_{f_2}^2} \\ &= \sqrt{V_{r_2}^2 - (V_2^2 - V_{w_2}^2)} - u_2 \quad \therefore \quad V_{f_2}^2 = V_2^2 - V_{w_2}^2 \\ \therefore \quad V_{w_2} + u_2 &= \sqrt{V_{r_2}^2 - V_2^2 + V_{w_2}^2} \end{aligned}$$

Squaring both sides, we get

$$(V_{w_2} + u_2)^2 = V_{r_2}^2 - V_2^2 + V_{w_2}^2$$

$$\text{or } V_{w_2}^2 + u_2^2 + 2V_{w_2}u_2 = V_{r_2}^2 - V_2^2 + V_{w_2}^2$$

$$\text{or } 2V_{w_2}u_2 = V_{r_2}^2 - V_2^2 + V_{w_2}^2 - V_{w_2}^2 - u_2^2$$

$$\text{or } 2V_{w_2}u_2 = V_{r_2}^2 - V_2^2 - u_2^2$$

$$\text{or } V_{w_2}u_2 = \frac{1}{2}[V_{r_2}^2 - V_2^2 - u_2^2] \quad \dots(ii)$$

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In the above case of velocity triangles under consideration, the change of total energy per unit weight inside the runner is equal to  $\frac{1}{g} [V_{w_1} u_1 + V_{w_2} u_2]$

Substituting the values of  $V_{w_1} u_1$  and  $V_{w_2} u_2$  from equations (i) and (ii) into equation (18.20 D), we get Change of total energy per unit weight inside the runner as

$$\begin{aligned} H_e &= \frac{1}{g} \left[ \frac{1}{2} (u_1^2 - V_{r_1}^2 + V_1^2) + \frac{1}{2} (V_{r_2}^2 - V_2^2 - u_2^2) \right] \\ &= \frac{1}{2g} [(u_1^2 - u_2^2) + (V_1^2 - V_2^2) + (V_{r_2}^2 - V_{r_1}^2)] \\ &= \frac{V_1^2 - V_2^2}{2g} + \frac{u_1^2 - u_2^2}{2g} + \frac{V_{r_2}^2 - V_{r_1}^2}{2g} \end{aligned} \quad \dots(18.20E)$$

The above equation consists of three terms. The first term represents the change in kinetic energy of the fluid per unit weight and the second term represents the change of energy per unit weight due to centrifugal action. The third term represents the change in static pressure energy per unit weight, as per Bernoulli's equation applied to relative flow through runner passage by reducing the rotating system into stationary system. We know that the energy change due to centrifugal action takes place in the form of pressure energy. [When a container containing a liquid is rotated, then due to centrifugal

action there is change of pressure energy i.e.,  $h = \frac{\Delta p}{\rho g} = \frac{u_2^2 - u_1^2}{2g}$ . Hence, the last two terms in equation (18.20E) represents the change in pressure energy inside the runner passage per unit weight.

$$\therefore \text{Change in pressure energy inside the runner per unit weight} = \frac{u_1^2 - u_2^2}{2g} + \frac{V_{r_2}^2 - V_{r_1}^2}{2g} \quad \dots(iii)$$

Now the equation (18.20C) becomes as

$$\begin{aligned} R &= \frac{\text{Change of pressure energy inside the runner per unit weight}}{\text{Change of total energy inside the runner per unit weight}} \\ &= \left( \frac{(u_1^2 - u_2^2)}{2g} + \frac{V_{r_2}^2 - V_{r_1}^2}{2g} \right) \Big/ \left[ \left( \frac{V_1^2 - V_2^2}{2g} \right) + \left( \frac{u_1^2 - u_2^2}{2g} \right) + \left( \frac{V_{r_2}^2 - V_{r_1}^2}{2g} \right) \right] \end{aligned}$$

$$\text{or } R = \frac{(u_1^2 - u_2^2) + (V_{r_2}^2 - V_{r_1}^2)}{(V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r_2}^2 - V_{r_1}^2)} \quad \dots(18.20F)$$

$$\begin{aligned} \text{or } R &= \frac{(V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r_2}^2 - V_{r_1}^2) - (V_1^2 - V_2^2)}{(V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r_2}^2 - V_{r_1}^2)} \\ &= 1 - \frac{(V_1^2 - V_2^2)}{(V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r_2}^2 - V_{r_1}^2)} \end{aligned} \quad \dots(18.20G)$$

From equation (18.20E), we know that

$$H_e = \frac{V_1^2 - V_2^2}{2g} + \frac{u_1^2 - u_2^2}{2g} + \frac{V_{r_2}^2 - V_{r_1}^2}{2g}$$

or  $2gH_e = (V_1^2 - V_2^2) + (u_1^2 - u_2^2) + (V_{r_2}^2 - V_{r_1}^2)$

Now the equation (18.20G) can be written as

$$R = 1 - \frac{(V_1^2 - V_2^2)}{2g H_e} \quad \dots(18.20H)$$

### Values of R for Pelton turbine and other actual reaction turbines

(i) For a Pelton turbine,

$$u_1 = u_2 \text{ and } V_{r_2} = V_{r_1}$$

$\therefore$  From equation (18.20G)

$$R = 1 - \frac{(V_1^2 - V_2^2)}{(V_1^2 - V_2^2)} = 1 - 1 = 0$$

(ii) For an actual reaction turbine, generally, the angle  $\beta$  is  $90^\circ$  so that the loss of kinetic energy at outlet is minimum (i.e.,  $V_2$  is minimum).

Hence in outlet velocity triangle,  $V_{w_2}$  becomes zero

(i.e.,  $V_{w_2} = 0$ ). Also  $V_2 = V_{f_2}$  [Refer to Fig. 18.11(b)]

Also there is not much change in velocity of flow. This means  $V_{f_1} = V_{f_2}$

From equation (18.20D), we know that

$$\begin{aligned} H_e &= \frac{1}{g} [V_{w_1} u_1 + V_{w_2} u_2] \\ &= \frac{1}{g} V_{w_1} u_1 \quad (\because V_{w_2} = 0) \\ &= \frac{1}{g} [V_{f_1} \cot \alpha] [V_{f_1} \cot \alpha - V_{f_1} \cot \theta] \quad [\text{Refer to Fig. 18.11(a)}] \\ &[\because V_{w_1} = V_{f_1} \cot \alpha \text{ and } u_1 = V_{w_1} - V_{f_1} \cot \theta = V_{f_1} \cot \alpha - V_{f_1} \cot \theta] \\ &= \frac{1}{g} V_{f_1}^2 \cot \alpha [\cot \alpha - \cot \theta] \end{aligned}$$

Now  $V_1^2 - V_2^2 = (V_{f_1} \cosec \alpha)^2 - V_{f_2}^2 \quad (\because V_2 = V_{f_2})$

$$= V_{f_1}^2 \cosec^2 \alpha - V_{f_1}^2 \quad (\because V_{f_2} = V_{f_1})$$

or  $V_1^2 - V_2^2 = V_{f_1}^2 (\cosec^2 \alpha - 1)$   
 $= V_{f_1}^2 \cot^2 \alpha \quad (\because \cosec^2 \alpha - 1 = \cot^2 \alpha)$

Substituting the value of  $H_e$  and  $(V_1^2 - V_2^2)$  in equation (18.20H), we get

$$\begin{aligned} R &= 1 - \frac{V_{f_1}^2 \cot^2 \alpha}{2g \times [\frac{1}{g} V_{f_1}^2 \cot \alpha (\cot \alpha - \cot \theta)]} \\ &= 1 - \frac{\cot \alpha}{2(\cot \alpha - \cot \theta)} \quad \dots(18.20I) \end{aligned}$$

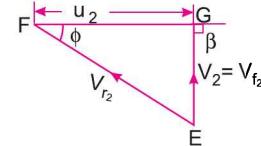


Fig. 18.11 (b)

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**18.7.4 Definitions.** The following terms are generally used in case of reaction radial flow turbines which are defined as :

(i) **Speed Ratio.** The speed ratio is defined as  $\frac{u_1}{\sqrt{2gH}}$   
where  $u_1$  = Tangential velocity of wheel at inlet.

(ii) **Flow Ratio.** The ratio of the velocity of flow at inlet ( $V_{f_1}$ ) to the velocity given  $\sqrt{2gH}$  is known as flow ratio or it is given as

$$= \frac{V_{f_1}}{\sqrt{2gH}}, \text{ where } H = \text{Head on turbine}$$

(iii) **Discharge of the Turbine.** The discharge through a reaction radial flow turbine is given by

$$Q = \pi D_1 B_1 \times V_{f_1} = \pi D_2 \times B_2 \times V_{f_2} \quad \dots(18.21)$$

where  $D_1$  = Diameter of runner at inlet,

$B_1$  = Width of runner at inlet,

$V_{f_1}$  = Velocity of flow at inlet, and

$D_2, B_2, V_{f_2}$  = Corresponding values at outlet.

If the thickness of vanes are taken into consideration, then the area through which flow takes place is given by  $(\pi D_1 - n \times t)$

where  $n$  = Number of vanes on runner and  $t$  = Thickness of each vane

The discharge  $Q$ , then is given by  $Q = (\pi D_1 - n \times t) B_1 \times V_{f_1}$   $\dots(18.22)$

(iv) The head ( $H$ ) on the turbine is given by  $H = \frac{p_1}{\rho \times g} + \frac{V_1^2}{2g}$   $\dots(18.23)$

where  $p_1$  = Pressure at inlet.

(v) **Radial Discharge.** This means the angle made by absolute velocity with the tangent on the wheel is  $90^\circ$  and the component of the whirl velocity is zero. Radial discharge at outlet means  $\beta = 90^\circ$  and  $V_{w_2} = 0$ , while radial discharge at outlet means  $\alpha = 90^\circ$  and  $V_{w_1} = 0$ .

(vi) If there is no loss of energy when water flows through the vanes then we have

$$H - \frac{V_2^2}{2g} = \frac{1}{g} [V_{w_1} u_1 \pm V_{w_2} u_2]. \quad \dots(18.24)$$

**Problem 18.14** An inward flow reaction turbine has external and internal diameters as 1 m and 0.5 m respectively. The velocity of flow through the runner is constant and is equal to 1.5 m/s. Determine :

(i) Discharge through the runner, and

(ii) Width of the turbine at outlet if the width of the turbine at inlet = 200 mm.

**Solution.** Given :

External diameter of turbine,  $D_1 = 1$  m

Internal diameter of turbine,  $D_2 = 0.5$  m

Velocity of flow at inlet and outlet,  $V_{f_1} = V_{f_2} = 1.5$  m/s

Width of turbine at inlet,  $B_1 = 200$  mm = 0.20 m

Let the width at outlet  $= B_2$

Using equation (18.21) for discharge,

$$Q = \pi D_1 B_1 \times V_{f_1} = \pi \times 1 \times 0.20 \times 1.5 = \mathbf{0.9425 \text{ m}^3/\text{s. Ans.}}$$

Also  $\pi D_1 B_1 V_{f_1} = \pi D_2 B_2 V_{f_2}$  or  $D_1 B_1 = D_2 B_2$   $(\because \pi V_{f_1} = \pi V_{f_2})$

$$\therefore B_2 = \frac{D_1 \times B_1}{D_2} = \frac{1 \times 0.20}{0.5} = 0.40 \text{ m} = \mathbf{400 \text{ mm. Ans.}}$$

**Problem 18.15** An inward flow reaction turbine has external and internal diameters as 0.9 m and 0.45 m respectively. The turbine is running at 200 r.p.m. and width of turbine at inlet is 200 mm. The velocity of flow through the runner is constant and is equal to 1.8 m/s. The guide blades make an angle of 10° to the tangent of the wheel and the discharge at the outlet of the turbine is radial. Draw the inlet and outlet velocity triangles and determine:

- The absolute velocity of water at inlet of runner,
- The velocity of whirl at inlet, (iii) The relative velocity at inlet,
- The runner blade angles, (v) Width of the runner at outlet,
- (vi) Mass of water flowing through the runner per second,
- (vii) Head at the inlet of the turbine,
- (viii) Power developed and hydraulic efficiency of the turbine.

**Solution.** Given :

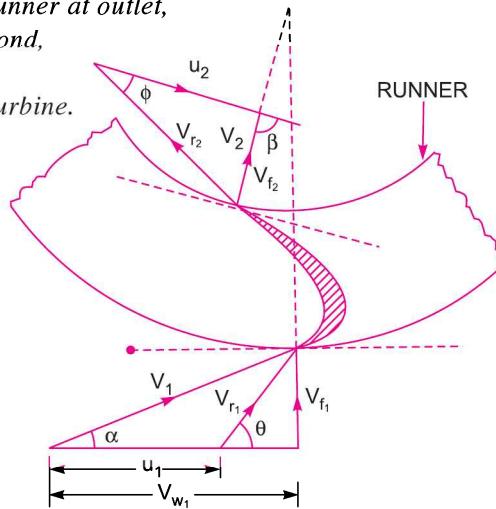
$$\begin{aligned}
 \text{External Dia.,} & D_1 = 0.9 \text{ m} \\
 \text{Internal Dia.,} & D_2 = 0.45 \text{ m} \\
 \text{Speed,} & N = 200 \text{ r.p.m.} \\
 \text{Width at inlet,} & B_1 = 200 \text{ mm} = 0.2 \text{ m} \\
 \text{Velocity of flow,} & V_{f_1} = V_{f_2} = 1.8 \text{ m/s} \\
 \text{Guide blade angle,} & \alpha = 10^\circ \\
 \text{Discharge at outlet} & = \text{Radial} \\
 \therefore & \beta = 90^\circ \text{ and } V_{w_2} = 0
 \end{aligned}$$

Tangential velocity of wheel at inlet and outlet  
are :

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.9 \times 200}{60} = 9.424 \text{ m/s}$$

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.45 \times 200}{60} = 4.712 \text{ m/s.}$$

Fig 18.12



(i) Absolute velocity of water at inlet of the runner i.e.,  $V_1$

From inlet velocity triangle,

$$V_1 \sin \alpha = V_{f_1}$$

$$\therefore V_1 = \frac{V_{f_1}}{\sin \alpha} = \frac{18}{\sin 10^\circ} = 10.365 \text{ m/s. Ans.}$$

(ii) Velocity of whirl at inlet, i.e.,  $V_{w_1}$

$$V_{w_1} = V_1 \cos \alpha = 10.365 \times \cos 10^\circ = 10.207 \text{ m/s. Ans.}$$

(iii) Relative velocity at inlet, i.e.,  $V_{r_1}$

$$\begin{aligned}
 V_{r_1} &= \sqrt{V_{r_1}^2 + (V_{w_1} - u_1)^2} = \sqrt{1.8^2 + (10.207 - 9.424)^2} \\
 &= \sqrt{3.24 + .613} = 1.963 \text{ m/s. Ans.}
 \end{aligned}$$

(iv) The runner blade angles means the angle  $\theta$  and  $\phi$

$$\text{Now } \tan \theta = \frac{V_{f_1}}{(V_{w_1} - u_1)} = \frac{1.8}{(10.207 - 9.424)} = 2.298$$

$$\therefore \theta = \tan^{-1} 2.298 = 66.48^\circ \text{ or } 66^\circ 29'. \text{ Ans.}$$

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From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2} = \frac{1.8}{4.712} = \tan 20.9^\circ$$

$$\therefore \phi = 20.9^\circ \text{ or } 20^\circ 54.4'. \text{ Ans.}$$

(v) Width of runner at outlet, i.e.,  $B_2$

From equation (18.21), we have

$$\pi D_1 B_1 V_{f_1} = \pi D_2 B_2 V_{f_2} \text{ or } D_1 B_1 = D_2 B_2 \quad (\because \pi V_{f_1} = \pi V_{f_2} \text{ as } V_{f_1} = V_{f_2})$$

$$\therefore B_2 = \frac{D_1 B_1}{D_2} = \frac{0.90 \times 0.20}{0.45} = 0.40 \text{ m} = 400 \text{ mm. Ans.}$$

(vi) Mass of water flowing through the runner per second.

$$\text{The discharge, } Q = \pi D_1 B_1 V_{f_1} = \pi \times 0.9 \times 0.20 \times 1.8 = 1.0178 \text{ m}^3/\text{s.}$$

$$\therefore \text{Mass} = \rho \times Q = 1000 \times 1.0178 \text{ kg/s} = 1017.8 \text{ kg/s. Ans.}$$

(vii) Head at the inlet of turbine, i.e.,  $H$ .

Using equation (18.24), we have

$$H - \frac{V_2^2}{2g} = \frac{1}{g} (V_{w_1} u_1 \pm V_{w_2} u_2) = \frac{1}{g} (V_{w_1} u_1) \quad (\because \text{Here } V_{w_2} = 0)$$

$$\begin{aligned} H &= \frac{1}{g} V_{w_1} u_1 + \frac{V_2^2}{2g} = \frac{1}{9.81} \times 10.207 \times 9.424 + \frac{1.8^2}{2 \times 9.81} \quad (\because V_2 = V_{f_2}) \\ &= 9.805 + 0.165 = 9.97 \text{ m. Ans.} \end{aligned}$$

(viii) Power developed, i.e.,  $P = \frac{\text{Work done per second on runner}}{1000}$

$$\begin{aligned} &= \frac{\rho Q [V_{w_1} u_1]}{1000} \quad [\text{Using equation (18.18)}] \\ &= 1000 \times \frac{1.0178 \times 10.207 \times 9.424}{1000} = 97.9 \text{ kW. Ans.} \end{aligned}$$

Hydraulic efficiency is given by equation (18.20B) as

$$\eta_h = \frac{V_{w_1} u_1}{gH} = \frac{10.207 \times 9.424}{9.81 \times 9.97} = 0.9834 = 98.34\%. \text{ Ans.}$$

**Problem 18.16** A reaction turbine works at 450 r.p.m. under a head of 120 metres. Its diameter at inlet is 120 cm and the flow area is 0.4 m<sup>2</sup>. The angles made by absolute and relative velocities at inlet are 20° and 60° respectively with the tangential velocity. Determine :

- (a) The volume flow rate, (b) The power developed, and
- (c) Hydraulic efficiency.

Assume whirl at outlet to be zero.

**Solution.** Given :

Speed of turbine,  $N = 450 \text{ r.p.m.}$

Head,  $H = 120 \text{ m}$

Diameter at inlet,  $D_1 = 120 \text{ cm} = 1.2 \text{ m}$

Flow area,  $\pi D_1 \times B_1 = 0.4 \text{ m}^2$

Angle made by absolute velocity at inlet,  $\alpha = 20^\circ$

Angle made by the relative velocity at inlet,  $\theta = 60^\circ$

Whirl at outlet,  $V_{w_2} = 0$

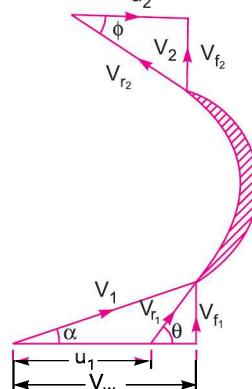


Fig. 18.13

Tangential velocity of the turbine at inlet,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 1.2 \times 450}{60} = 28.27 \text{ m/s}$$

From inlet velocity triangle,

$$\tan \alpha = \frac{V_{f_1}}{V_{w_1}} \text{ or } \tan 20^\circ = \frac{V_{f_1}}{V_{w_1}} \text{ or } \frac{V_{f_1}}{V_{w_1}} = \tan 20^\circ = 0.364$$

∴

$$V_{f_1} = 0.364 V_{w_1}$$

$$\text{Also } \tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{0.364 V_{w_1}}{V_{w_1} - 28.27} \quad (\because V_{f_1} = 0.364 V_{w_1})$$

or

$$\frac{0.364 V_{w_1}}{V_{w_1} - 28.27} = \tan \theta = \tan 60^\circ = 1.732$$

∴

$$0.364 V_{w_1} = 1.732(V_{w_1} - 28.27) = 1.732 V_{w_1} - 48.96$$

or

$$(1.732 - 0.364) V_{w_1} = 48.96$$

∴

$$V_{w_1} = \frac{48.96}{(1.732 - 0.364)} = 35.789 = 35.79 \text{ m/s.}$$

From equation (i),

$$V_{f_1} = 0.364 \times V_{w_1} = 0.364 \times 35.79 = 13.027 \text{ m/s.}$$

(a) Volume flow rate is given by equation (18.21) as  $Q = \pi D_1 B_1 \times V_{f_1}$

But

$$\pi D_1 \times B_1 = 0.4 \text{ m}^2$$

(given)

$$Q = 0.4 \times 13.027 = 5.211 \text{ m}^3/\text{s. Ans.}$$

(b) Work done per sec on the turbine is given by equation (18.18),

$$\begin{aligned} &= \rho Q [V_{w_1} u_1] \quad (\because V_{w_2} = 2) \\ &= 1000 \times 5.211 [35.79 \times 28.27] = 5272402 \text{ Nm/s} \end{aligned}$$

$$\therefore \text{Power developed in kW} = \frac{\text{Work done per second}}{1000} = \frac{5272402}{1000} = 5272.402 \text{ kW. Ans.}$$

(c) The hydraulic efficiency is given by equation (18.20B) as

$$\eta_h = \frac{V_{w_1} u_1}{gH} = \frac{35.79 \times 28.27}{9.81 \times 120} = 0.8595 = 85.95\%. \text{ Ans.}$$

**Problem 18.17** As inward flow reaction turbine has external and internal diameters as 1.0 m and 0.6 m respectively. The hydraulic efficiency of the turbine is 90% when the head on the turbine is 36 m. The velocity of flow at outlet is 2.5 m/s and discharge at outlet is radial. If the vane angle at outlet is 15° and width of the wheel is 100 mm at inlet and outlet, determine : (i) the guide blade angle, (ii) speed of the turbine, (iii) vane angle of the runner at inlet, (iv) volume flow rate of turbine and (v) power developed.

**Solution.** Given :

External diameter,  $D_1 = 1.0 \text{ m}$

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Internal diameter,  $D_2 = 0.6 \text{ m}$   
 Hydraulic efficiency,  $\eta_h = 90\% = 0.90$

Head,  $H = 36 \text{ m}$

Velocity of flow at outlet,  $V_{f_2} = 2.5 \text{ m/s}$

Discharge is radial,  $V_{w_2} = 0$

Vane angle at outlet,  $\phi = 15^\circ$

Width of wheel,  $B_1 = B_2 = 100 \text{ mm} = 0.1 \text{ m}$

Using equation (18.20 B) for hydraulic efficiency as

$$\eta_h = \frac{V_{w_1} u_1}{gH} \text{ or } 0.90 = \frac{V_{w_1} \cdot u_1}{9.81 \times 36}$$

$$\therefore V_{w_1} u_1 = 0.90 \times 9.81 \times 36 = 317.85$$

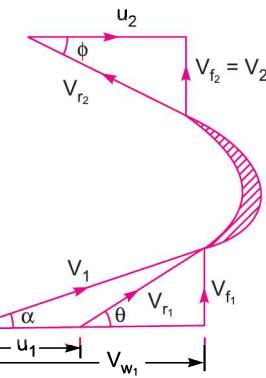


Fig. 18.14

... (i)

$$\text{From outlet velocity triangle, } \tan \phi = \frac{V_{f_2}}{u_2} = \frac{2.5}{u_2}$$

$$\therefore u_2 = \frac{2.5}{\tan \phi} = \frac{2.5}{\tan 15^\circ} = 9.33$$

$$\text{But } u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.6 \times N}{60}$$

$$\therefore 9.33 = \frac{\pi \times 0.6 \times N}{60} \text{ or } N = \frac{60 \times 9.33}{\pi \times 0.6} = 296.98. \text{ Ans.}$$

$$\therefore u_1 = \frac{\pi \times D_1 \times N}{60} = \frac{\pi \times 1.0 \times 296.98}{60} = 15.55 \text{ m/s.}$$

Substituting this value of ' $u_1$ ' in equation (i),

$$V_{w_1} \times 15.55 = 317.85$$

$$\therefore V_{w_1} = \frac{317.85}{15.55} = 20.44 \text{ m/s}$$

$$\text{Using equation (18.21), } \pi D_1 B_1 V_{f_1} = \pi D_2 B_2 V_{f_2} \quad \text{or} \quad D_1 V_{f_1} = D_2 V_{f_2} \quad (\because B_1 = B_2)$$

$$\therefore V_{f_1} = \frac{D_2 \times V_{f_2}}{D_1} = \frac{0.6 \times 2.5}{1.0} = 1.5 \text{ m/s.}$$

(i) Guide blade angle ( $\alpha$ ).

$$\text{From inlet velocity triangle, } \tan \alpha = \frac{V_{f_1}}{V_{w_1}} = \frac{1.5}{20.44} = 0.07338$$

$$\therefore \alpha = \tan^{-1} 0.07338 = 4.19^\circ \text{ or } 4^\circ 11.8'. \text{ Ans.}$$

(ii) Speed of the turbine,  $N = 296.98 \text{ r.p.m. Ans.}$

(iii) Same angle of runner at inlet ( $\theta$ )

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{1.5}{(20.44 - 15.55)} = 0.3067$$

$$\therefore \theta = \tan^{-1} 0.3067 = 17.05^\circ \text{ or } 17^\circ 3'. \text{ Ans.}$$

(iv) Volume flow rate of turbine is given by equation (18.21) as

$$= \pi D_1 B_1 V_{f_1} = \pi \times 1.0 \times 0.1 \times 1.5 = 0.4712 \text{ m}^3/\text{s. Ans.}$$

(v) Power developed (in kW)

$$= \frac{\text{Work done per second}}{1000} = \frac{\rho Q [V_{w_1} u_1]}{1000}$$

[Using equation (18.18) and  $V_{w_2} = 0$ ]

$$= 1000 \times \frac{0.4712 \times 20.44 \times 15.55}{1000} = 149.76 \text{ kW. Ans.}$$

**Problem 18.18** An inward flow reaction turbine has an exit diameter of 1 metre and its breadth at inlet is 250 mm. If the velocity of flow at inlet is 2 metres/s, find the mass of water passing through the turbine per second. Assume 10% of the area of flow is blocked by blade thickness. If the speed of the runner is 210 r.p.m. and guide blades make an angle of  $10^\circ$  to the wheel tangent, draw the inlet velocity triangle, and find :

- (i) the runner vane angle at inlet, (ii) velocity of wheel at inlet,
- (iii) the absolute velocity of water leaving the guide vanes, and
- (iv) the relative velocity of water entering the runner blade.

**Solution.** Given :

Exit or External diameter,  $D_1 = 1.0 \text{ m}$

Breadth at inlet,  $B_1 = 250 \text{ mm} = 0.25 \text{ m}$

Velocity of flow at inlet,  $V_{f_1} = 2.0 \text{ m/s}$

Area blocked by vanes = 10%

Speed,  $N = 210 \text{ r.p.m.}$

Guide blade angle,  $\alpha = 10^\circ$

Tangential velocity of wheel at inlet,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 1.0 \times 210}{60} = 10.99 \text{ m/s}$$

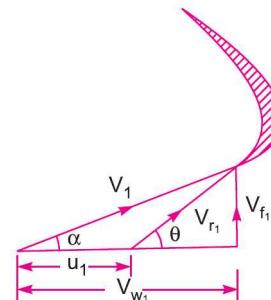


Fig. 18.15

$$\text{Area blocked by vane thickness} = \frac{10}{100} \times \pi D_1 B_1 = 0.1 \pi D_1 B_1$$

$\therefore$  Actual area through which flow takes place,

$$a = \pi D_1 B_1 - 0.1 \pi D_1 B_1 = 0.9 \pi D_1 B_1$$

$$= 0.9 \times \pi \times 1.0 \times 0.25 = 0.7068 \text{ m}^2$$

$\therefore$  Mass of water passing per second

$$= \rho \times a \times V_{f_1} = 1000 \times 0.7068 \times 2.0 = 1413.6 \text{ kg. Ans.}$$

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(i) The runner vane angle at inlet ( $\theta$ ).

$$\text{From inlet velocity triangle } \tan \alpha = \frac{V_{f_1}}{V_{w_1}} = \frac{2.0}{V_{w_1}}$$

$$\therefore V_{w_1} = \frac{2.0}{\tan \alpha} = \frac{2.0}{\sin 10^\circ} = 11.34 \text{ m/s}$$

$$\therefore \tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{2.0}{11.34 - 10.99} = 5.714$$

$$\therefore \theta = \tan^{-1} 5.714 = 80.07^\circ \text{ or } 80^\circ 4.2'. \text{ Ans.}$$

(ii) Velocity of wheel at inlet,  $u_1 = 10.99 \text{ m/s. Ans.}$

(iii) The absolute velocity of water leaving the guide vanes ( $V_1$ ):

$$\text{From inlet triangle, } \sin \alpha = \frac{V_{f_1}}{V_1}$$

$$\therefore V_1 = \frac{V_{f_1}}{\sin \alpha} = \frac{2.0}{\sin 10^\circ} = 11.517 \text{ m/s. Ans.}$$

(iv) The relative velocity of water entering the runner blade ( $V_{r_1}$ )

$$\sin \theta = \frac{V_{f_1}}{V_{r_1}}$$

$$\therefore V_{r_1} = \frac{V_{f_1}}{\sin \theta} = \frac{2.0}{\sin 80.07^\circ} = 2.03 \text{ m/s. Ans.}$$

**Problem 18.19** The external and internal diameters of an inward flow reaction turbines are 1.20 m and 0.6 m respectively. The head on the turbine is 22 m and velocity of flow through the runner is constant and equal to 2.5 m/s. The guide blade angle is given as  $10^\circ$  and the runner vanes are radial at inlet. If the discharge at outlet is radial, determine :

- (i) The speed of the turbine,
- (ii) The vane angle at outlet of the runner, and
- (iii) Hydraulic efficiency.

**Solution.** Given :

$$\text{External diameter, } D_1 = 1.20 \text{ m}$$

$$\text{Internal diameter, } D_2 = 0.60 \text{ m}$$

$$\text{Head, } H = 22.0 \text{ m}$$

$$\text{Velocity of flow, } V_{f_1} = V_{f_2} = 2.5 \text{ m/s}$$

$$\text{Guide blade angle, } \alpha = 10^\circ$$

Runner vanes radial at inlet means,  $\theta = 90^\circ$

$$\therefore V_{w_1} = u_1, V_{r_1} = V_{f_1} = 2.5 \text{ m/s}$$

Discharge is radial

$$\therefore V_{w_2} = 0, V_2 = V_{f_2} = 2.5 \text{ m/s}$$

From inlet velocity triangle,

$$\tan \alpha = \frac{V_{f_1}}{u_1}$$

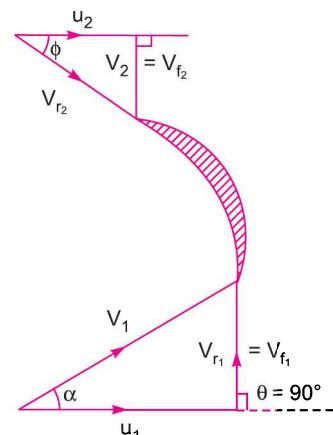


Fig. 18.16

$$\therefore u_1 = \frac{V_{f_1}}{\tan \alpha} = \frac{1.5}{\tan 10^\circ} = 14.178 \text{ m/s}$$

$$\therefore V_{w_1} = u_1 = 14.178 \text{ m/s}$$

(i) The speed of the turbine (N)

Using

$$u_1 = \frac{\pi D_1 \times N}{60}$$

$$\therefore N = \frac{60 \times u_1}{\pi D_1} = \frac{60 \times 14.178}{\pi \times 1.20} = 225.65 \text{ r.p.m. Ans.}$$

(ii) The vane angle at outlet of the runner ( $\phi$ ).

$$u_2 = \frac{\pi D_2 \times N}{60} = \frac{\pi \times 0.6 \times 225.65}{60} = 7.09 \text{ m/s.}$$

$$\text{From outlet velocity triangle, } \tan \phi = \frac{V_{f_2}}{u_2} = \frac{2.5}{7.09} = 0.3526$$

$$\therefore \phi = \tan^{-1} 0.3526 = 19.42^\circ \text{ or } 19^\circ 25.2'. \text{ Ans.}$$

$$(iii) \text{ Hydraulic efficiency is given by } \eta_h = \frac{V_{w_1} u_1}{g H} = \frac{14.178 \times 14.178}{9.81 \times 22.0} = 0.9314 = 93.14\%. \text{ Ans.}$$

**Problem 18.20** 233 litres of water per second are supplied to an inward flow reaction turbine. The head available is 11 m. The wheel vanes are radial at inlet and the inlet diameter is twice the outlet diameter. The velocity of flow is constant and equal to 1.83 m/s. The wheel makes 370 r.p.m. Find :

(a) Guide vane angle, (b) Inlet and outlet diameter of the wheel,

(c) The width of the wheel at inlet and exit. Neglect the thickness of the vanes.

Assume that the discharge is radial and there are no losses in the wheel. Take speed ratio = 0.7.

**Solution.** Given :

Discharge,  $Q = 233 \text{ lit/s} = 0.233 \text{ m}^3/\text{s}$

Head,  $H = 11 \text{ m}$

Wheel vanes are radial at inlet. This means angle

$$\theta = 90^\circ \text{ and } V_{r_1} = V_{f_1}$$

Inlet diameter  $= 2 \times \text{Outlet diameter}$

$$\therefore D_1 = 2D_2$$

Velocity of flow at inlet and outlet = 1.83 m/s

$$\therefore V_{f_1} = V_{f_2} = 1.83 \text{ m/s}$$

Speed,  $N = 370 \text{ r.p.m.}$

$$\text{Speed ratio} = 0.7 \text{ or } \frac{u_1}{\sqrt{2gH}} = 0.7$$

$$\therefore u_1 = 0.7 \times \sqrt{2gH} \\ = 0.7 \times \sqrt{2 \times 9.81 \times 11} = 10.28 \text{ m/s}$$

Discharge is radial at outlet. This means angle  $\beta = 90^\circ$  and  $V_{w_2} = 0$

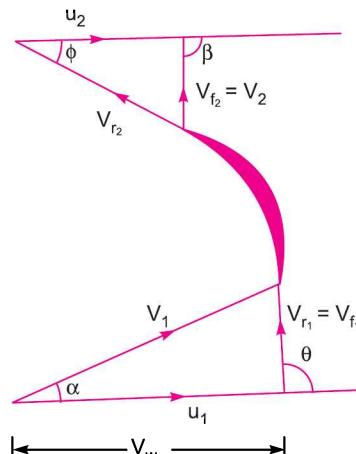


Fig. 18.17

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(a) Guide vane angle ( $\alpha$ )

$$\tan \alpha = \frac{V_{f_1}}{u_1} = \frac{1.83}{10.28} = 0.178$$

$$\therefore \alpha = \tan^{-1} 0.178 = 10^\circ 6'. \text{ Ans.}$$

(b) Inlet and outlet diameter of the wheel

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times D_1 \times 370}{60}$$

$$\therefore D_1 = \frac{60 \times u_1}{\pi \times 370} = \frac{60 \times 10.28}{\pi \times 370} = .532 \text{ m} = 53.2 \text{ cm. Ans.}$$

$$D_2 = \frac{D_1}{2} = \frac{53.2}{2} = 26.6 \text{ cm. Ans.}$$

(c) Width of wheel at inlet and outlet

$$Q = \pi D_1 \times B_1 \times V_{f_1} = \pi D_2 \times B_2 \times V_{f_2}$$

But

$$V_{f_1} = V_{f_2} \quad \therefore D_1 \times B_1 = D_2 \times B_2$$

As

$$D_1 = 2D_2, B_2 = 2B_1$$

Now

$$Q = \pi D_1 \times B_1 \times V_{f_1} = \pi \times .532 \times B_1 \times 1.83$$

$$\therefore B_1 = \frac{Q}{\pi \times .532 \times 1.83} = \frac{0.233}{\pi \times .532 \times 1.83} = 0.0762 \text{ m} = 7.62 \text{ cm. Ans.}$$

$$B_2 = 2 \times B_1 = 2 \times 7.62 = 15.24 \text{ cm. Ans.}$$

**18.7.5 Outward Radial Flow Reaction Turbine.** Fig. 18.18 shows outward radial flow reaction turbine in which the water from casing enters the stationary guide wheel. The guide wheel consists of guide vanes which direct the water to the runner.

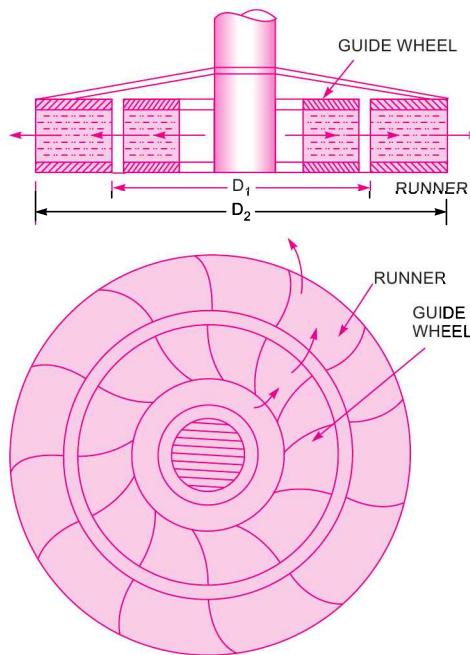


Fig. 18.18 Outward radial flow turbine.

vanes which direct water to enter the runner which is around the stationary guide wheel. The water flows through the vanes of the runner in the outward radial direction and is discharged at the outer diameter of the runner. The inner diameter of the runner is inlet and outer diameter is the outlet.

The velocity triangles at inlet and outlet will be drawn by the same procedure as adopted for inward flow turbine. The work done by the water on the runner per second, the horse power developed and hydraulic efficiency will be obtained from the velocity triangles. In this case as inlet of the runner is at the inner diameter of the runner, the tangential velocity at inlet will be less than that of at outlet, i.e.,

$$u_1 < u_2 \text{ as } D_1 < D_2.$$

**Problem 18.21** An outward flow reaction turbine has internal and external diameters of the runner as 0.6 m and 1.2 m respectively. The guide blade angle is  $15^\circ$  and velocity of flow through the runner is constant and equal to 4 m/s. If the speed of the turbine is 200 r.p.m., head on the turbine is 10 m and discharge at outlet is radial, determine :

- (i) The runner vane angles at inlet and outlet,
- (ii) Work done by the water on the runner per second per unit weight of water striking per second ,
- (iii) Hydraulic efficiency, and
- (iv) The degree of reaction.

**Solution.** Given :

Internal diameter,	$D_1 = 0.6 \text{ m}$
External diameter,	$D_2 = 1.2 \text{ m}$
Guide blade angle,	$\alpha = 15^\circ$
Velocity of flow,	$V_{f_1} = V_{f_2} = 4 \text{ m/s}$
Speed,	$N = 200 \text{ r.p.m.}$
Head,	$H = 10 \text{ m}$
Discharge at outlet	= Radial
$\therefore$	$V_{w_2} = 0, V_{f_2} = V_2$

Tangential velocity of runner at inlet and outlet are :

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.6 \times 200}{60} = 6.283 \text{ m/s}$$

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 1.2 \times 200}{60} = 12.566 \text{ m/s.}$$

From the inlet velocity triangle,  $\tan \alpha = \frac{V_{f_1}}{V_{w_1}}$

$$\therefore V_{w_1} = \frac{V_{f_1}}{\tan \alpha} = \frac{4.0}{\tan 15^\circ} = 14.928 \text{ m/s.}$$

(i) Runner Vane Angles at inlet and outlet are  $\theta$  and  $\phi$

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{4.0}{(14.928 - 6.283)} = 0.4627$$

$$\theta = \tan^{-1} 0.4627 = 24.83 \text{ or } 24^\circ 49.8'. \text{ Ans.}$$

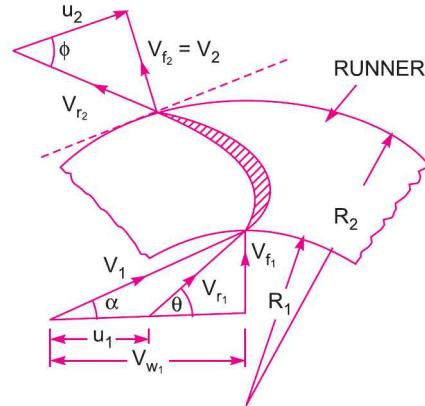


Fig. 18.19

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From outlet velocity triangle,  $\tan \phi = \frac{V_{f_2}}{u_2} = \frac{4.0}{12.566} = 0.3183$

$$\therefore \phi = \tan^{-1} 0.3183 = 17.65^\circ \text{ or } 17^\circ 39.4'. \text{ Ans.}$$

(ii) Work done by water per second per unit weight of water striking per second

$$= \frac{1}{g} V_{w_1} u_1 \quad (\because V_{w_2} = 0)$$

$$= \frac{1}{9.81} \times 14.928 \times 6.283 = 9.561 \text{ Nm/N. Ans.}$$

(iii) Hydraulic efficiency is given by equation (18.20B) as

$$\eta_h = \frac{V_{w_1} u_1}{gH} = \frac{14.928 \times 6.283}{9.81 \times 10} = 0.9561 \text{ or } 95.61\%. \text{ Ans.}$$

(iv) Given : In this question, the velocity of flow is constant through the runner (i.e.,  $V_{f_1} = V_{f_2}$ ) and the discharge is radial at outlet (i.e.,  $\beta = 90^\circ$  or  $V_{w_2} = 0$ ), the degree of reaction ( $R$ ) is given by equation (18.20I) as

$$R = 1 - \frac{\cot \alpha}{2(\cot \alpha - \cot \theta)}$$

Here  $\alpha = 13.928^\circ$  and  $\theta = 41.09^\circ$  (calculated)

Substituting the value of  $\alpha$  and  $\theta$ , we get

$$R = 1 - \frac{\cot 13.928^\circ}{2(\cot 13.928^\circ - \cot 41.09^\circ)} = 1 - \frac{4.032}{2(4.032 - 1.146)} \\ = 1 - 0.698 = 0.302 \approx 0.3. \text{ Ans.}$$

For Francis turbine, the degree of reaction varies from 0 to 1 i.e.,  $0 \leq R \leq 1$ .

**Problem 18.22** The internal and external diameters of an outward flow reaction turbine are 2 m and 2.75 m respectively. The turbine is running at 250 r.p.m. and rate of flow of water through the turbine is 5 m<sup>3</sup>/s. The width of the runner is constant at inlet and outlet and is equal to 250 mm. The head on the turbine is 150 m. Neglecting thickness of the vanes and taking discharge radial at outlet determine :

- (i) Vane angles at inlet and outlet, and
- (ii) Velocity of flow at inlet and outlet.

**Solution.** Given :

Internal diameter,  $D_1 = 2.0 \text{ m}$

External diameter,  $D_2 = 2.75 \text{ m}$

Speed of turbine,  $N = 250 \text{ r.p.m.}$

Discharge,  $Q = 5 \text{ m}^3/\text{s}$

Width at inlet and outlet,  $B_1 = B_2 = 250 \text{ mm} = 0.25 \text{ m}$

Head,  $H = 150 \text{ m}$

Discharge at outlet = radial

$\therefore V_{w_2} = 0 \text{ and } V_{f_2} = V_2$ .

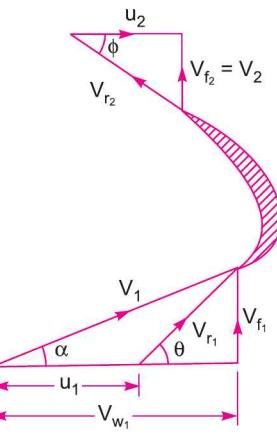


Fig. 18.20

The tangential velocity of the turbine at inlet and outlet,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 2.0 \times 250}{60} = 26.18 \text{ m/s}$$

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 2.75 \times 250}{60} = 36.0 \text{ m/s.}$$

The discharge through turbine is given by equation (18.21) as

$$Q = \pi D_1 B_1 V_{f_1} = \pi D_2 B_2 V_{f_2}$$

$$\therefore V_{f_1} = \frac{Q}{\pi D_1 B_1} = \frac{5.0}{\pi \times 2.0 \times .25} = 3.183 \text{ m/s.}$$

$$\text{And } V_{f_2} = \frac{Q}{\pi D_2 B_2} = \frac{5.0}{\pi \times 2.75 \times .25} = 2.315 \text{ m/s}$$

$$\text{Using equation (18.24), } H - \frac{V_2^2}{2g} = \frac{V_{w_1} u_1}{g} \quad (\because V_{w_2} = 0)$$

But for radial discharge,  $V_2 = V_{f_2} = 2.315 \text{ m/s}$

$$\therefore 150 - \frac{2.315^2}{2 \times 9.81} = \frac{V_{w_2} \times 26.18}{9.81} \text{ or } 149.73 = \frac{V_{w_1} \times 26.18}{9.81}$$

$$\therefore V_{w_1} = \frac{149.73 \times 9.81}{26.18} = 56.1 \text{ m/s.}$$

(i) *Vane angles at inlet and outlet*

$$\text{From inlet velocity triangle, } \tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{3.183}{56.1 - 26.18} = 0.1064$$

$$\therefore \theta = \tan^{-1} 0.1064 = 6.072^\circ \text{ or } 6^\circ 4.32'. \text{ Ans.}$$

$$\text{From outlet velocity triangle, } \tan \phi = \frac{V_{f_2}}{u_2} = \frac{2.315}{36.0} = 0.0643$$

$$\therefore \phi = \tan^{-1} 0.0643 = 3.68^\circ \text{ or } 3^\circ 40.8'. \text{ Ans.}$$

(ii) *Velocity of flow at inlet and outlet*

$$V_{f_1} = 3.183 \text{ m/s and } V_{f_2} = 2.315 \text{ m/s. Ans.}$$

## ► 18.8 FRANCIS TURBINE

The inward flow reaction turbine having radial discharge at outlet is known as Francis Turbine, after the name of J.B. Francis, an American engineer who in the beginning designed inward radial flow reaction type of turbine. In the modern Francis turbine, the water enters the runner of the turbine in the radial direction at outlet and leaves in the axial direction at the inlet of the runner. Thus the modern Francis Turbine is a mixed flow type turbine.

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The velocity triangle at inlet and outlet of the Francis turbine are drawn in the same way as in case of inward flow reaction turbine. As in case of Francis turbine, the discharge is radial at outlet, the velocity of whirl at outlet (*i.e.*,  $V_{w_2}$ ) will be zero. Hence the work done by water on the runner per second will be

$$= \rho Q [V_{w_1} u_1]$$

And work done per second per unit weight of water striking/s =  $\frac{1}{g} [V_{w_1} u_1]$

Hydraulic efficiency will be given by,  $\eta_h = \frac{V_{w_1} u_1}{gH}$ .

**18.8.1 Important Relations for Francis Turbines.** The following are the important relations for Francis Turbines :

1. The ratio of width of the wheel to its diameter is given as  $n = \frac{B_1}{D_1}$ . The value of  $n$  varies from 0.10

to .40.

2. The flow ratio is given as,

$$\text{Flow ratio} = \frac{V_{f_1}}{\sqrt{2gH}} \text{ and varies from 0.15 to 0.30.}$$

3. The speed ratio =  $\frac{u_1}{\sqrt{2gH}}$  varies from 0.6 to 0.9.

**Problem 18.23** A Francis turbine with an overall efficiency of 75% is required to produce 148.25 kW power. It is working under a head of 7.62 m. The peripheral velocity =  $0.26 \sqrt{2gH}$  and the radial velocity of flow at inlet is  $0.96 \sqrt{2gH}$ . The wheel runs at 150 r.p.m. and the hydraulic losses in the turbine are 22% of the available energy. Assuming radial discharge, determine :

- |   |                                     |
|---|-------------------------------------|
| (i) The guide blade angle,                | (ii) The wheel vane angle at inlet, |
| (iii) Diameter of the wheel at inlet, and | (iv) Width of the wheel at inlet.   |

**Solution.** Given :

$$\text{Overall efficiency} \quad \eta_o = 75\% = 0.75$$

$$\text{Power produced,} \quad \text{S.P.} = 148.25 \text{ kW}$$

$$\text{Head,} \quad H = 7.62 \text{ m}$$

$$\text{Peripheral velocity,} \quad u_1 = 0.26 \sqrt{2gH} = 0.26 \times \sqrt{2 \times 9.81 \times 7.62} = 3.179 \text{ m/s}$$

$$\text{Velocity of flow at inlet,} \quad V_{f_1} = 0.96 \sqrt{2gH} = 0.96 \times \sqrt{2 \times 9.81 \times 7.62} = 11.738 \text{ m/s.}$$

$$\text{Speed,} \quad N = 150 \text{ r.p.m.}$$

$$\text{Hydraulic losses} \quad = 22\% \text{ of available energy}$$

$$\text{Discharge at outlet} \quad = \text{Radial}$$

$$V_{w_2} = 0 \text{ and } V_{f_2} = V_2$$

Hydraulic efficiency is given as

$$\eta_h = \frac{\text{Total head at inlet} - \text{Hydraulic loss}}{\text{Head at inlet}}$$

$$= \frac{H - .22 H}{H} = \frac{0.78 H}{H} = 0.78$$

But

$$\eta_h = \frac{V_{w_1} u_1}{g H}$$

$$\therefore \frac{V_{w_1} u_1}{g H} = 0.78$$

$$\therefore V_{w_1} = \frac{0.78 \times g \times H}{u_1}$$

$$= \frac{0.78 \times 9.81 \times 7.62}{3.179} = 18.34 \text{ m/s.}$$

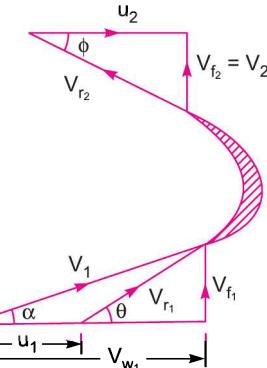


Fig. 18.21

(i) The guide blade angle, i.e.,  $\alpha$ . From inlet velocity triangle,

$$\tan \alpha = \frac{V_{f_1}}{V_{w_1}} = \frac{11.738}{18.34} = 0.64$$

$$\therefore \alpha = \tan^{-1} 0.64 = 32.619^\circ \text{ or } 32^\circ 37'. \text{ Ans.}$$

(ii) The wheel vane angle at inlet, i.e.,  $\theta$

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{11.738}{18.34 - 3.179} = 0.774$$

$$\therefore \theta = \tan^{-1} 0.774 = 37.74^\circ \text{ or } 37^\circ 44.4'. \text{ Ans.}$$

(iii) Diameter of wheel at inlet ( $D_1$ ).

$$\text{Using the relation, } u_1 = \frac{\pi D_1 N}{60}$$

$$D_1 = \frac{60 \times u_1}{\pi \times N} = \frac{60 \times 3.179}{\pi \times 50} = 0.4047 \text{ m. Ans.}$$

(iv) Width of the wheel at inlet ( $B_1$ )

$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \frac{148.25}{\text{W.P.}}$$

But

$$\text{W.P.} = \frac{\text{WH}}{1000} = \frac{\rho \times g \times Q \times H}{1000} = \frac{1000 \times 9.81 \times Q \times 7.62}{1000}$$

$$\therefore \eta_o = \frac{148.25}{\frac{1000 \times 9.81 \times Q \times 7.62}{1000}} = \frac{148.25 \times 1000}{1000 \times 9.81 \times Q \times 7.62}$$

$$\text{or } Q = \frac{148.25 \times 1000}{1000 \times 9.81 \times 7.62 \times \eta_o} = \frac{148.25 \times 1000}{1000 \times 9.81 \times 7.62 \times 0.75} = 2.644 \text{ m}^3/\text{s}$$

Using equation (18.21),  $Q = \pi D_1 \times B_1 \times V_{f_1}$

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$$\therefore 2.644 = \pi \times .4047 \times B_1 \times 11.738$$

$$\therefore B_1 = \frac{2.644}{\pi \times .4047 \times 11.738} = 0.177 \text{ m. Ans.}$$

**Problem 18.24** The following data is given for a Francis Turbine. Net head  $H = 60 \text{ m}$ ; Speed  $N = 700 \text{ r.p.m.}$ ; shaft power = 294.3 kW;  $\eta_o = 84\%$ ;  $\eta_h = 93\%$ ; flow ratio = 0.20; breadth ratio  $n = 0.1$ ; Outer diameter of the runner = 2  $\times$  inner diameter of runner. The thickness of vanes occupy 5% of circumferential area of the runner, velocity of flow is constant at inlet and outlet and discharge is radial at outlet. Determine :

- (i) Guide blade angle,
- (ii) Runner vane angles at inlet and outlet,
- (iii) Diameters of runner at inlet and outlet, and
- (iv) Width of wheel at inlet.

**Solution.** Given :

Net head,

$$H = 60 \text{ m}$$

Speed,

$$N = 700 \text{ r.p.m.}$$

Shaft power

$$= 294.3 \text{ kW}$$

Overall efficiency,

$$\eta_o = 84\% = 0.84$$

Hydraulic efficiency,

$$\eta_h = 93\% = 0.93$$

Flow ratio,

$$\frac{V_{f_1}}{\sqrt{2gH}} = 0.20$$

$$\therefore$$

$$V_{f_1} = 0.20 \times \sqrt{2gH}$$

$$= 0.20 \times \sqrt{2 \times 9.81 \times 60} = 6.862 \text{ m/s}$$

Breadth ratio,

$$\frac{B_1}{D_1} = 0.1$$

Outer diameter,

$$D_1 = 2 \times \text{Inner diameter} = 2 \times D_2$$

Velocity of flow,

$$V_{f_1} = V_{f_2} = 6.862 \text{ m/s.}$$

Thickness of vanes

= 5% of circumferential area of runner

$\therefore$  Actual area of flow

$$= 0.95 \pi D_1 \times B_1$$

Discharge at outlet

= Radial

$$\therefore$$

$$V_{w_2} = 0 \text{ and } V_{f_2} = V_2$$

Using relation,

$$\eta_o = \frac{\text{S.P.}}{\text{W.P.}}$$

$$0.84 = \frac{294.3}{\text{W.P.}}$$

$$\therefore$$

$$\text{W.P.} = \frac{294.3}{0.84} = 350.357 \text{ kW.}$$

But

$$\text{W.P.} = \frac{WH}{1000} = \frac{\rho \times g \times Q \times H}{1000} = \frac{1000 \times 9.81 \times Q \times 60}{1000}$$

$$\therefore$$

$$\frac{1000 \times 9.81 \times Q \times 60}{1000} = 350.357$$

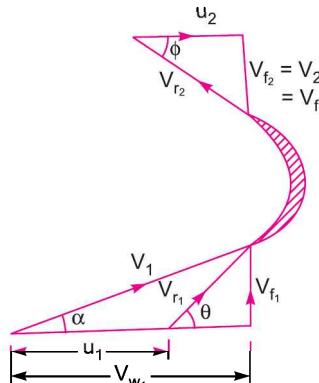


Fig. 18.22

$$\therefore Q = \frac{350.357 \times 1000}{60 \times 1000 \times 9.81} = 0.5952 \text{ m}^3/\text{s.}$$

Using equation (18.21),

$$\begin{aligned} Q &= \text{Actual area of flow} \times \text{Velocity of flow} \\ &= 0.95 \pi D_1 \times B_1 \times V_{f_1} \\ &= 0.95 \times \pi \times D_1 \times 0.1 D_1 \times V_{f_1} \quad (\because B_1 = 0.1 D_1) \end{aligned}$$

or

$$0.5952 = 0.95 \times \pi \times D_1 \times 0.1 \times D_1 \times 6.862 = 2.048 D_1^2$$

$$\therefore D_1 = \sqrt{\frac{0.5952}{2.048}} = 0.54 \text{ m}$$

But  $\frac{B_1}{D_1} = 0.1$

$$\therefore B_1 = 0.1 \times D_1 = 0.1 \times 0.54 = 0.054 \text{ m} = 54 \text{ mm}$$

Tangential speed of the runner at inlet,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.54 \times 700}{60} = 19.79 \text{ m/s.}$$

Using relation for hydraulic efficiency,

$$\eta_h = \frac{V_{w_1} u_1}{g H} \text{ or } 0.93 = \frac{V_{w_1} \times 19.79}{9.81 \times 60}$$

$$\therefore V_{w_1} = \frac{0.93 \times 9.81 \times 60}{19.79} = 27.66 \text{ m/s.}$$

(i) Guide blade angle ( $\alpha$ )

$$\text{From inlet velocity triangle, } \tan \alpha = \frac{V_{f_1}}{V_{w_1}} = \frac{6.862}{27.66} = 0.248$$

$$\therefore \alpha = \tan^{-1} 0.248 = 13.928^\circ \text{ or } 13^\circ 55.7'. \text{ Ans.}$$

(ii) Runner vane angles at inlet and outlet ( $\theta$  and  $\phi$ )

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{6.862}{27.66 - 19.79} = 0.872$$

$$\therefore \theta = \tan^{-1} 0.872 = 41.09^\circ \text{ or } 41^\circ 5.4'. \text{ Ans.}$$

$$\text{From outlet velocity triangle, } \tan \phi = \frac{V_{f_2}}{u_2} = \frac{V_{f_1}}{u_2} = \frac{6.862}{u_2} \quad \dots(i)$$

But

$$\begin{aligned} u_2 &= \frac{\pi D_2 N}{60} = \frac{\pi \times D_1}{2} \times \frac{N}{60} \quad \left( \because D_2 = \frac{D_1}{2} \text{ given} \right) \\ &= \pi \times \frac{0.54}{2} \times \frac{700}{60} = 9.896 \text{ m/s.} \end{aligned}$$

Substituting the value of  $u_2$  in equation (i),

$$\tan \phi = \frac{6.862}{9.896} = 0.6934$$

$$\therefore \phi = \tan^{-1} 0.6934^\circ = 34.74 \text{ or } 34^\circ 44.4'. \text{ Ans.}$$

(iii) Diameters of runner at inlet and outlet

$$D_1 = 0.54 \text{ m}, D_2 = 0.27 \text{ m. Ans.}$$

(iv) Width of wheel at inlet

$$B_1 = 54 \text{ mm. Ans.}$$

**Problem 18.24 (A)** For the above problem, find the degree of reaction for the given Francis Turbine.

**Solution.** Given :

In this question, the velocity of flow is constant through the runner (i.e.,  $V_{f_1} = V_{f_2}$ ) and the discharge is radial at outlet (i.e.,  $\beta = 90^\circ$  or  $V_{w_2} = 0$ ), the degree of reaction ( $R$ ) is given by equation (18.20 I) as

$$R = 1 - \frac{\cot \alpha}{2(\cot \alpha - \cot \theta)}$$

Here  $\alpha = 13.928^\circ$  and  $\theta = 41.09^\circ$  (calculated)

Substituting the value of  $\alpha$  and  $\theta$ , we get

$$\begin{aligned} R &= 1 - \frac{\cot 13.928^\circ}{2(\cot 13.928^\circ - \cot 41.09^\circ)} = 1 - \frac{4.032}{2(4.032 - 1.146)} \\ &= 1 - 0.698 = 0.302 \approx 0.3. \text{ Ans.} \end{aligned}$$

For Francis Turbine, the degree of reaction varies from 0 to 1 i.e.,  $0 \leq R \leq 1$ .

**Problem 18.25** (a) Show that the hydraulic efficiency for a Francis turbine having velocity of flow through runner as constant, is given by the relation.

$$\eta_h = \frac{I}{I + \frac{2}{\left(1 - \frac{\tan \alpha}{\tan \theta}\right) \tan^2 \alpha}}$$

where  $\alpha$  = Guide blade angle and  $\theta$  = Runner vane angle at inlet.

The turbine is having radial discharge at outlet.

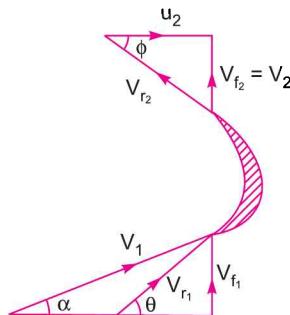


Fig. 18.23

**Solution.** Given :

Velocity of flow = Constant.

$$\therefore V_{f_1} = V_{f_2}$$

Discharge is radial at outlet.

$$\therefore V_{w_2} = 0 \text{ and } V_{f_2} = V_2$$

(a) From the inlet velocity triangle,

$$\tan \alpha = \frac{V_{f_1}}{V_{w_1}} \therefore V_{f_1} = V_{w_1} \tan \alpha \quad \dots(i)$$

Also

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1}$$

$$\text{or } (V_{w_1} - u_1) = \frac{V_{f_1}}{\tan \theta} = \frac{V_{w_1} \tan \alpha}{\tan \theta} \quad (\because V_{f_1} = V_{w_1} \tan \alpha)$$

$$\therefore u_1 = V_{w_1} - \frac{V_{w_1} \tan \alpha}{\tan \theta} = V_{w_1} \left(1 - \frac{\tan \alpha}{\tan \theta}\right) \quad \dots(ii)$$

Using equation (18.24), we have

$$H - \frac{V_2^2}{2g} = \frac{1}{g}(V_{w_1} u_1) \quad (\because V_{w_2} = 0)$$

$$\therefore H = \frac{1}{g} V_{w_1} u_1 + \frac{V_2^2}{2g} = \frac{1}{g} V_{w_1} u_1 + \frac{V_{f_1}^2}{2g} \quad (\because V_2 = V_{f_2} = V_{f_1})$$

Substituting the values of  $V_{f_1}$  and  $u_1$  from equations (i) and (ii),

$$\begin{aligned} H &= \frac{1}{g} V_{w_1} \times V_{w_1} \left(1 - \frac{\tan \alpha}{\tan \theta}\right) + \frac{[V_{w_1} \tan \alpha]^2}{2g} \\ &= \frac{V_{w_1}^2}{g} \left(1 - \frac{\tan \alpha}{\tan \theta}\right) + \frac{V_{w_1}^2 \tan^2 \alpha}{2g} = \frac{V_{w_1}^2}{g} \left[1 - \frac{\tan \alpha}{\tan \theta} + \frac{\tan^2 \alpha}{2}\right]. \end{aligned}$$

Now, hydraulic efficiency is given by

$$\begin{aligned} \eta_h &= \frac{V_{w_1} u_1}{gH} = \frac{V_{w_1} u_1}{g \times \frac{V_{w_1}^2}{g} \left[1 - \frac{\tan \alpha}{\tan \theta} + \frac{\tan^2 \alpha}{2}\right]} \\ &= \frac{V_{w_1} \times V_{w_1} \left(1 - \frac{\tan \alpha}{\tan \theta}\right)}{V_{w_1}^2 \left[1 - \frac{\tan \alpha}{\tan \theta} + \frac{\tan^2 \alpha}{2}\right]} \quad \left[\because u_1 = V_{w_1} \left(1 - \frac{\tan \alpha}{\tan \theta}\right)\right] \\ &= \frac{\left(1 - \frac{\tan \alpha}{\tan \theta}\right)}{\left[1 - \frac{\tan \alpha}{\tan \theta} + \frac{\tan^2 \alpha}{2}\right]} = \frac{1}{1 + \frac{\frac{1}{2} \tan^2 \alpha}{\left(1 - \frac{\tan \alpha}{\tan \theta}\right)}}. \text{ Ans.} \end{aligned}$$

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(b) If vanes are radial at inlet, then  $\theta = 90^\circ$

$$\therefore \eta_h = \frac{1}{1 + \frac{\frac{1}{2} \tan^2 \alpha}{\left(1 - \frac{\tan \alpha}{\tan 90^\circ}\right)}} = \frac{1}{1 + \frac{\frac{1}{2} \tan^2 \alpha}{(1-0)}} = \frac{2}{2 + \tan^2 \alpha} \cdot \text{Ans.}$$

**Problem 18.26** A Francis turbine working under a head of 30 m has a wheel diameter of 1.2 m at the entrance and 0.6 m at the exit. The vane angle at the entrance is  $90^\circ$  and guide blade angle is  $15^\circ$ . The water at the exit leaves the vanes without any tangential velocity and the velocity of flow in the runner is constant. Neglecting the effect of draft tube and losses in the guide and runner passages, determine the speed of wheel in r.p.m. and vane angle at the exit. State whether the speed calculated is synchronous or not. If not, what speed would you recommend to couple the turbine with an alternator of 50 cycles?

**Solution.** Given :

Head on turbine,	$H = 30 \text{ m}$
Inlet dia.,	$D_1 = 1.2 \text{ m}$
Outlet dia.,	$D_2 = 0.6 \text{ m}$
Vane angle at inlet,	$\theta = 90^\circ$
Guide blade angle,	$\alpha = 15^\circ$

The water at exit leaves the vanes without any tangential velocity.

$$\therefore V_{w_2} = 0 \text{ and } V_2 = V_{f_2}$$

Velocity of flow is constant in runner.

$$\therefore V_{f_1} = V_{f_2}$$

(i) Speed of turbine in r.p.m.

Using equation (18.24), we have

$$\begin{aligned} H - \frac{V_2^2}{2g} &= \frac{1}{g} (V_{w_1} u_1 \pm V_{w_2} u_2) \\ &= \frac{1}{g} (V_{w_1} \times u_1) \quad (\because V_{w_2} = 0) \\ &= \frac{1}{g} u_1 \times u_1 \quad (\because V_{w_1} = u_1) \\ \text{or} \quad 30 - \frac{V_2^2}{2g} &= \frac{1}{g} u_1^2 \quad (\because V_2 = V_{f_2} = V_{f_1}) \dots(i) \end{aligned}$$

But from inlet velocity triangle, we have

$$\tan \alpha = \frac{V_{f_1}}{u_1} \text{ or } u_1 = \frac{V_{f_1}}{\tan \alpha} = \frac{V_{f_1}}{\tan 15^\circ} = 3.732 V_{f_1} \quad \dots(ii)$$

Substituting the values of  $u_1$  in equation (i), we get

$$30 - \frac{V_{f_2}^2}{2g} = \frac{1}{g} \times (3.732 V_{f_1})^2 \text{ or } 30 - \frac{V_{f_2}^2}{2g} = \frac{13.928 V_{f_2}^2}{g} \quad (\because V_{f_2} = V_{f_1})$$

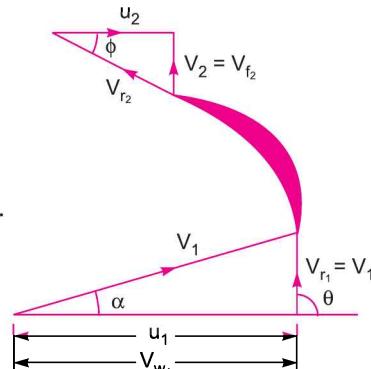


Fig. 18.24

or 
$$30 = \frac{14.928 V_{f_1}^2}{g}$$

$$\therefore V_{f_1} = \sqrt{\frac{30 \times 9.81}{14.928}} = 4.44 \text{ m/s.}$$

Substituting the value of  $V_{f_1}$  in equation (ii), we get

$$u_1 = 3.732 \times 4.44 = 16.57 \text{ m/s}$$

But  $u_1 = \frac{\pi D_1 N}{60}$  or  $16.57 = \frac{\pi \times 1.2 \times N}{60}$

$$\therefore N = \frac{16.57 \times 60}{\pi \times 1.2} = 263.72 \text{ r.p.m. Ans.}$$

(ii) Vane angle at exit (i.e.,  $\phi$ )

$$u_2 = \frac{\pi D_2 \times N}{60} = \frac{\pi \times 0.6 \times 263.72}{60} = 8.285 \text{ m/s}$$

$$V_{f_2} = V_{f_1} = 4.44$$

Now, from velocity triangle at outlet,

$$\tan \phi = \frac{V_{f_2}}{u_2} = \frac{4.44}{8.285} = 0.5359$$

$$\therefore \phi = 28.187^\circ. \text{ Ans.}$$

(iii) For a turbine, which is directly coupled to the alternator of 50 cycles, the synchronous speed ( $N^*$ ) is given by  $f = \frac{p \cdot N^*}{60}$

where  $f$  = Frequency of alternator in cycles/s,  $p$  = Number of pair of poles for the alternator.

Assuming the number of pair of poles = 12, we get

$$50 = \frac{12 \times N^*}{60}$$

$$\therefore N^* = \frac{60 \times 50}{12} = 250 \text{ r.p.m.}$$

But the speed of turbine is 263.72. And synchronous speed ( $N^*$ ) is equal to 250. Hence, the speed of turbine is not synchronous. The speed of turbine should be 250 r.p.m.

## ► 18.9 AXIAL FLOW REACTION TURBINE

If the water flows parallel to the axis of the rotation of the shaft, the turbine is known as axial flow turbine. And if the head at the inlet of the turbine is the sum of pressure energy and kinetic energy and during the flow of water through runner a part of pressure energy is converted into kinetic energy, the turbine is known as reaction turbine.

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For the axial flow reaction turbine, the shaft of the turbine is vertical. The lower end of the shaft is made larger which is known as 'hub' or 'boss'. The vanes are fixed on the hub and hence hub acts as a runner for axial flow reaction turbine. The following are the important type of axial flow reaction turbines :

### 1. Propeller Turbine, and

When the vanes are fixed to the hub and they are not adjustable, the turbine is known as propeller turbine. But if the vanes on the hub are adjustable, the turbine is known as a *Kaplan Turbine*, after the name of V Kaplan, an Austrian Engineer. This turbine is suitable where a large quantity of water at low head is available. Fig. 18.25 shows the runner of a Kaplan turbine, which consists of a hub fixed to the shaft. On the hub, the adjustable vanes are fixed as shown in Fig. 18.25.

The main parts of a Kaplan turbine are :

1. Scroll casing,
2. Guide vanes mechanism,
3. Hub with vanes or runner of the turbine, and
4. Draft tube.

Fig. 18.26 shows all main parts of a Kaplan turbine. The water from penstock enters the scroll casing and then moves to the guide vanes. From the guide vanes, the water turns through  $90^\circ$  and flows axially through the runner as shown in Fig. 18.26. The discharge through the runner is obtained as

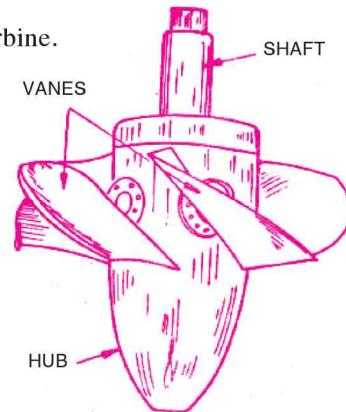


Fig. 18.25 *Kaplan turbine runner.*

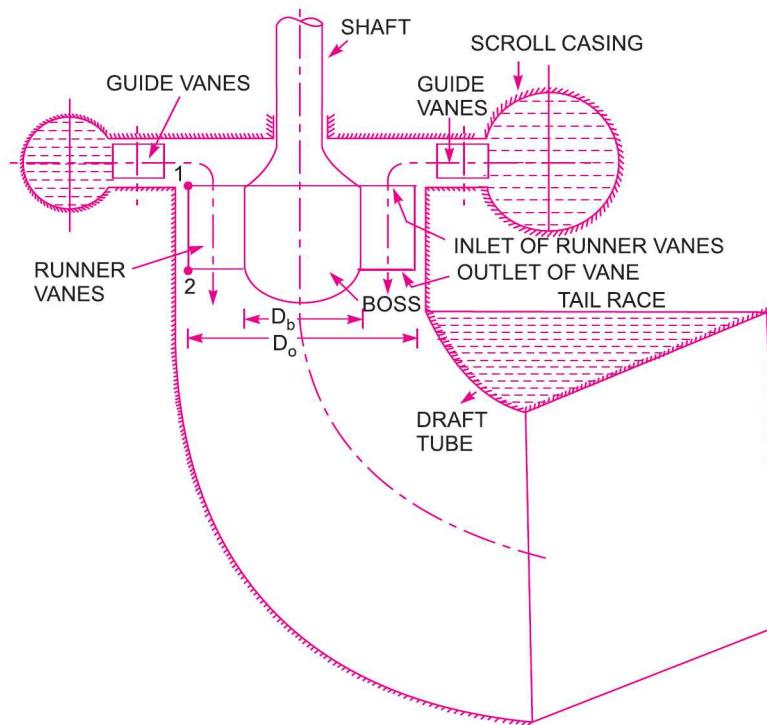


Fig. 18.26 *Main components of Kaplan turbine.*

$$Q = \frac{\pi}{4} (D_o^2 - D_b^2) \times V_{f_1} \quad \dots(18.25)$$

where  $D_o$  = Outer diameter of the runner,

$D_b$  = Diameter of hub, and

$V_{f_1}$  = Velocity of flow at inlet.

The inlet and outlet velocity triangles are drawn at the extreme edge of the runner vane corresponding to the points 1 and 2 as shown in Fig. 18.26.

**18.9.1 Some Important Point for Propeller (Kaplan Turbine).** The following are the important points for propeller or Kaplan turbine :

1. The peripheral velocity at inlet and outlet are equal

$$\therefore u_1 = u_2 = \frac{\pi D_o N}{60}, \text{ where } D_o = \text{Outer dia. of runner}$$

2. Velocity of flow at inlet and outlet are equal

$$\therefore V_{f_1} = V_{f_2}.$$

3. Area of flow at inlet = Area of flow at outlet

$$= \frac{\pi}{4} (D_o^2 - D_b^2).$$

**Problem 18.27** A Kaplan turbine working under a head of 20 m develops 11772 kW shaft power. The outer diameter of the runner is 3.5 m and hub diameter is 1.75 m. The guide blade angle at the extreme edge of the runner is 35°. The hydraulic and overall efficiencies of the turbines are 88% and 84% respectively. If the velocity of whirl is zero at outlet, determine :

- (i) Runner vane angles at inlet and outlet at the extreme edge of the runner, and
- (ii) Speed of the turbine.

**Solution.** Given :

Head,	$H = 20 \text{ m}$
Shaft power,	S.P. = 11772 kW
Outer dia. of runner,	$D_o = 3.5 \text{ m}$
Hub diameter,	$D_b = 1.75 \text{ m}$
Guide blade angle,	$\alpha = 35^\circ$
Hydraulic efficiency,	$\eta_h = 88\%$
Overall efficiency,	$\eta_o = 84\%$
Velocity of whirl at outlet	= 0.

$$\text{Using the relation, } \eta_o = \frac{\text{S.P.}}{\text{W.P.}}$$

where  $\text{W.P.} = \frac{\text{W.P.}}{1000} = \frac{\rho \times g \times Q \times H}{1000}$ , we get

$$0.84 = \frac{11772}{\frac{\rho \times g \times Q \times H}{1000}}$$

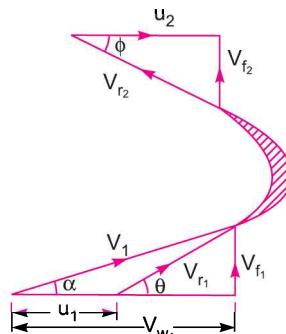


Fig. 18.27

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$$= \frac{11772 \times 1000}{1000 \times 9.81 \times Q \times 20} \quad (\because \rho = 1000)$$

$$\therefore Q = \frac{11772 \times 1000}{0.84 \times 1000 \times 9.81 \times 20} = 71.428 \text{ m}^3/\text{s.}$$

Using equation (18.25),  $Q = \frac{\pi}{4} (D_o^2 - D_b^2) \times V_{f_1}$

$$\text{or } 71.428 = \frac{\pi}{4} (3.5^2 - 1.75^2) \times V_{f_1} = \frac{\pi}{4} (12.25 - 3.0625) V_{f_1}$$

$$= 7.216 V_{f_1}$$

$$\therefore V_{f_1} = \frac{71.428}{7.216} = 9.9 \text{ m/s.}$$

From inlet velocity triangle,  $\tan \alpha = \frac{V_{f_2}}{V_{w_1}}$

$$\therefore V_{w_1} = \frac{V_{f_1}}{\tan \alpha} = \frac{9.9}{\tan 35^\circ} = \frac{9.9}{.7} = 14.14 \text{ m/s}$$

Using the relation for hydraulic efficiency,

$$\eta_h = \frac{V_{w_1} u_1}{g H} \quad (\because V_{w_2} = 0)$$

$$0.88 = \frac{14.14 \times u_1}{9.81 \times 20}$$

$$\therefore u_1 = \frac{0.88 \times 9.81 \times 20}{14.14} = 12.21 \text{ m/s.}$$

(i) Runner vane angles at inlet and outlet at the extreme edge of the runner are given as :

$$\tan \theta = \frac{V_{f_1}}{V_{w_1} - u_1} = \frac{9.9}{(14.14 - 12.21)} = 5.13$$

$$\therefore \theta = \tan^{-1} 5.13 = 78.97^\circ \text{ or } 78^\circ 58'. \text{ Ans.}$$

For Kaplan turbine,  $u_1 = u_2 = 12.21 \text{ m/s and } V_{f_1} = V_{f_2} = 9.9 \text{ m/s}$

$$\therefore \text{From outlet velocity triangle, } \tan \phi = \frac{V_{f_2}}{u_2} = \frac{9.9}{12.21} = 0.811$$

$$\therefore \phi = \tan^{-1} 0.811 = 39.035^\circ \text{ or } 39^\circ 2'. \text{ Ans.}$$

(ii) Speed of turbine is given by  $u_1 = u_2 = \frac{\pi D_o N}{60}$

$$12.21 = \frac{\pi \times 3.5 \times N}{60}$$

$$\therefore N = \frac{60 \times 12.21}{\pi \times 3.5} = 66.63 \text{ r.p.m. Ans.}$$

**Problem 18.28** A Kaplan turbine develops 24647.6 kW power at an average head of 39 metres. Assuming a speed ratio of 2, flow ratio of 0.6, diameter of the boss equal to 0.35 times the diameter of the runner and an overall efficiency of 90%, calculate the diameter, speed and specific speed of the turbine.

**Solution.** Given :

Shaft power,

$$S.P. = 24647.6 \text{ kW}$$

Head,

$$H = 39 \text{ m}$$

Speed ratio,

$$u_1 \sqrt{2gH} = 2.0$$

∴

$$u_1 = 2.0 \times \sqrt{2gH} = 2.0 \times \sqrt{2 \times 9.81 \times 39} = 55.32 \text{ m/s}$$

Flow ratio,

$$\frac{V_{f_i}}{\sqrt{2gH}} = 0.6$$

∴

$$V_{f_i} = 0.6 \times \sqrt{2gH} = 0.6 \times \sqrt{2 \times 9.81 \times 39} = 16.59 \text{ m/s}$$

Diameter of boss

$$= 0.35 \times \text{Diameter of runner}$$

∴

$$D_b = 0.35 \times D_o$$

Overall efficiency,

$$\eta_o = 90\% = 0.90$$

Using the relation,

$$\eta_o = \frac{S.P.}{W.P.}, \text{ where W.P.} = \frac{\rho \times g \times Q \times H}{1000}$$

∴

$$0.90 = \frac{24647.6}{\frac{\rho \times g \times Q \times H}{1000}} = \frac{24647.6 \times 1000}{1000 \times 9.81 \times Q \times 39}$$

∴

$$Q = \frac{24647.6 \times 1000}{0.9 \times 1000 \times 9.81 \times 39} = 71.58 \text{ m}^3/\text{s.}$$

But from equation (18.25), we have

$$Q = \frac{\pi}{4} (D_o^2 - D_b^2) \times V_{f_i}$$

∴

$$71.58 = \frac{\pi}{4} [D_o^2 - (0.35 D_o)^2] \times 16.59 \quad (\because D_b = 0.35 D_o, V_{f_i} = 16.59)$$

$$= \frac{\pi}{4} [D_o^2 - 0.1225 D_o^2] \times 16.59$$

$$= \frac{\pi}{4} \times 0.8775 D_o^2 \times 16.59 = 11.433 D_o^2$$

(i) ∴

$$D_o = \sqrt{\frac{71.58}{11.433}} = 2.5 \text{ m. Ans.}$$

∴

$$D_b = 0.35 \times D_o = 0.35 \times 2.5 = 0.875 \text{ m. Ans.}$$

(ii) Speed of the turbine is given by  $u_1 = \frac{\pi D_o N}{60}$

∴

$$55.32 = \frac{\pi \times 2.5 \times N}{60}$$

## 908 Fluid Mechanics

$$\therefore N = \frac{60 \times 55.32}{\pi \times 2.5} = 422.61 \text{ r.p.m. Ans.}$$

(iii) Specific speed \* is given by  $N_s = \frac{N\sqrt{P}}{H^{5/4}}$ , where  $P$  = Shaft power in kW

$$\therefore N_s = \frac{422.61 \times \sqrt{24647.6}}{(39)^{5/4}} = \frac{422.61 \times 156.99}{97.461} = 680.76 \text{ r.p.m. Ans.}$$

**Problem 18.29** A Kaplan turbine runner is to be designed to develop 9100 kW. The net available head is 5.6 m. If the speed ratio = 2.09, flow ratio = 0.68, overall efficiency = 86% and the diameter of the boss is 1/3 the diameter of the runner. Find the diameter of the runner, its speed and the specific speed of the turbine.

**Solution.** Given :

$$\text{Power, } P = 9100 \text{ kW}$$

$$\text{Net head, } H = 5.6 \text{ m}$$

$$\text{Speed ratio} = 2.09$$

$$\text{Flow ratio} = 0.68$$

$$\text{Overall efficiency, } \eta_o = 86\% = 0.86$$

$$\text{Diameter of boss} = \frac{1}{3} \text{ of diameter of runner}$$

$$\text{or } D_b = \frac{1}{3} D_o$$

$$\text{Now, speed ratio} = \frac{u_1}{\sqrt{2gH}}$$

$$\therefore u_1 = 2.09 \times \sqrt{2 \times 9.81 \times 5.6} = 21.95 \text{ m/s}$$

$$\text{Flow ratio} = \frac{V_{f_1}}{\sqrt{2gH}}$$

$$\therefore V_{f_1} = 0.68 \times \sqrt{2 \times 9.81 \times 5.6} = 7.12 \text{ m/s}$$

$$\text{The overall efficiency is given by, } \eta_o = \frac{P}{\left( \frac{\rho \times g \cdot Q \cdot H}{1000} \right)}$$

$$\text{or } Q = \frac{P \times 1000}{\rho \times g \times H \times \eta_o} = \frac{9100 \times 1000}{1000 \times 9.81 \times 5.6 \times 0.86} \\ (= 192.5 \text{ m}^3/\text{s.})$$

The discharge through a Kaplan turbine is given by

$$Q = \frac{\pi}{4} [D_o^2 - D_b^2] \times V_{f_1}$$

---

\* For the definition and derivation, please refer to page 920 Arts. 18.11 and 18.11.1.

or

$$192.5 = \frac{\pi}{4} \left[ D_o^2 - \left( \frac{D_o}{3} \right)^2 \right] \times 7.12 \quad \left( \because D_b = \frac{D_o}{3} \right)$$

$$= \frac{\pi}{4} \left[ 1 - \frac{1}{9} \right] D_o^2 \times 7.12$$

$$\therefore D_o = \sqrt{\frac{4 \times 192.5 \times 9}{\pi \times 8 \times 7.12}} = 6.21 \text{ m. Ans.}$$

The speed of turbine is given by,  $u_1 = \frac{\pi D N}{60}$

$$\therefore N = \frac{60 \times u_1}{\pi \times D} = \frac{60 \times 21.95}{\pi \times 6.21} = 67.5 \text{ r.p.m. Ans.}$$

The specific speed is given by,  $N_s = \frac{N \sqrt{P}}{H^{5/4}} = \frac{67.5 \times \sqrt{9100}}{5.6^{5/4}} = 746$ . Ans.

**Problem 18.30** The hub diameter of a Kaplan turbine, working under a head of 12 m, is 0.35 times the diameter of the runner. The turbine is running at 100 r.p.m. If the vane angle of the extreme edge of the runner at outlet is  $15^\circ$  and flow ratio is 0.6, find :

- (i) Diameter of the runner, (ii) Diameter of the boss, and
- (iii) Discharge through the runner.

The velocity of whirl at outlet is given as zero.

**Solution.** Given :

Head,

$$H = 12 \text{ m}$$

Hub diameter,

$$D_b = 0.35 \times D_o, \text{ where } D_o = \text{Dia. of runner}$$

Speed,

$$N = 100 \text{ r.p.m.}$$

Vane angle at outlet,

$$\phi = 15^\circ$$

Flow ratio

$$= \frac{V_{f_2}}{\sqrt{2gH}} = 0.6$$

∴

$$V_{f_1} = 0.6 \times \sqrt{2gH} = 0.6 \times \sqrt{2 \times 9.81 \times 12} \\ = 9.2 \text{ m/s.}$$

From the outlet velocity triangle,  $V_{w_2} = 0$

$$\tan \phi = \frac{V_{f_2}}{u_2} = \frac{V_{f_1}}{u_2} \quad (\because V_{f_2} = V_{f_1} = 9.2)$$

∴

$$\tan 15^\circ = \frac{9.2}{u_2}$$

∴

$$u_2 = \frac{9.2}{\tan 15^\circ} = 34.33 \text{ m/s.}$$

But for Kaplan turbine,

$$u_1 = u_2 = 34.33$$

Now, using the relation,

$$u_1 = \frac{\pi D_o \times N}{60} \text{ or } 34.33 = \frac{\pi \times D_o \times 100}{60}$$

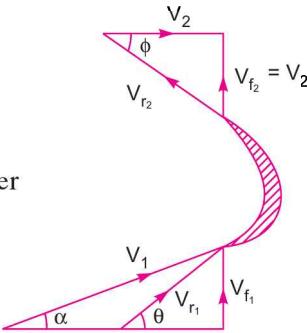


Fig. 18.28

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$$D_o = \frac{60 \times 34.33}{\pi \times 100} = 6.55 \text{ m. Ans.}$$

$$\therefore D_b = 0.35 \times D_o = 0.35 \times 6.35 = 2.3 \text{ m. Ans.}$$

Discharge through turbine is given by equation (18.25) as

$$\begin{aligned} Q &= \frac{\pi}{4} [D_o^2 - D_b^2] \times V_{f_1} = \frac{\pi}{4} [6.55^2 - 2.3^2] \times 9.2 \\ &= \frac{\pi}{4} (42.9026 - 5.29) \times 9.2 = 271.77 \text{ m}^3/\text{s. Ans.} \end{aligned}$$

**Problem 18.31** A Kaplan turbine runner is to be designed to develop 7357.5 kW shaft power. The net available head is 5.50 m. Assume that the speed ratio is 2.09 and flow ratio is 0.68, and the overall efficiency is 60%. The diameter of the boss is  $\frac{1}{3}$ rd of the diameter of the runner. Find the diameter of the runner, its speed and its specific speed.

**Solution.** Given :

$$\text{Shaft power, } P = 7357.5 \text{ kW}$$

$$\text{Head, } H = 5.50 \text{ m}$$

$$\text{Speed ratio } = \frac{u_1}{\sqrt{2gH}} = 2.09$$

$$\therefore u_1 = 2.09 \times \sqrt{2 \times 9.81 \times 5.50} = 21.71 \text{ m/s}$$

$$\text{Flow ratio } = \frac{V_{f_1}}{\sqrt{2gH}} = 0.68$$

$$\therefore V_{f_1} = 0.68 \times \sqrt{2 \times 9.81 \times 5.50} = 7.064 \text{ m/s}$$

$$\text{Overall efficiency, } \eta_o = 60\% = 0.60$$

$$\text{Diameter of boss, } D_b = \frac{1}{3} \times D_o$$

$$\text{Using relation, } \eta_o = \frac{\text{Shaft power}}{\text{Water power}} = \frac{7357.5}{\rho \times g \times Q \times H} \times 1000$$

$$\text{or } 0.60 = \frac{7357.5 \times 1000}{\rho \times g \times Q \times H} = \frac{7357.5 \times 1000}{1000 \times 9.81 \times Q \times 5.5}$$

$$\therefore Q = \frac{7357.5 \times 1000}{1000 \times 9.81 \times 5.5 \times 0.60} = 227.27 \text{ m}^3/\text{s.}$$

Using equation (18.25) for discharge,

$$Q = \frac{\pi}{4} (D_o^2 - D_b^2) \times V_{f_1}$$

$$\begin{aligned} \text{or } 227.27 &= \frac{\pi}{4} \left[ D_o^2 - \left( \frac{D_o}{3} \right)^2 \right] \times 7.064 & \left( \because D_b = \frac{D_o}{3} \right) \\ &= \frac{\pi}{4} \times \frac{8}{9} D_o^2 \times 7.064 = 4.9316 D_o^2 \end{aligned}$$

$$\therefore D_o = \sqrt{\frac{227.27}{4.9316}} = 6.788 \text{ m. Ans.}$$

$$\text{And } D_b = \frac{1}{3} \times 6.788 = 2.262 \text{ m. Ans.}$$

$$\text{Using the relation, } u_1 = \frac{\pi D_o \times N}{60}$$

$$\therefore N = \frac{60 \times u_1}{\pi D_o} = \frac{60 \times 21.71}{\pi \times 6.788} = 61.08 \text{ r.p.m. Ans.}$$

The specific speed ( $N_s$ ) is given by,

$$N_s = \frac{N \sqrt{P}}{H^{5/4}} = \frac{61.08 \times \sqrt{7357.5}}{5.50^{5/4}} = 622 \text{ r.p.m. Ans.}$$

**Problem 18.32** In a tidal power plant, a bulb turbine (which is basically an axial flow turbine) operates a 5 MW generator at 150 r.p.m. under a head of 5.5 m. The generator efficiency is 93% and the overall efficiency of the turbine is 88%. The tip diameter of the runner is 4.5 m and hub diameter is 2 m. Assuming hydraulic efficiency of 94% and no exit whirl, determine the runner vane angles at inlet and exit at the mean diameter of the vanes.

**Solution.** Given :

$$\text{Output of generator} = 5 \text{ M* W} = 5 \times 10^6 \text{ W}$$

$$\text{Speed of turbine, } N = 150 \text{ r.p.m.}$$

$$\text{Head on turbine, } H = 5.5 \text{ m}$$

$$\text{Generator efficiency, } \eta_g = 93\% \text{ or } 0.93$$

$$\text{Overall efficiency of the turbine, } \eta_o = 88\% \text{ or } 0.88$$

$$\text{Tip dia. of runner, } D_o = 4.5 \text{ m}$$

$$\text{Hub dia. of runner, } D_b = 2 \text{ m}$$

$$\text{Hydraulic efficiency, } \eta_h = 94\% \text{ or } 0.94$$

No exit whirl means the velocity of whirl at outlet is zero i.e.,  $V_{w2} = 0$ . And hence angle  $\beta = 90^\circ$  as shown in Fig. 18.29.

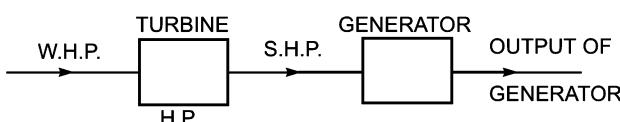


Fig. 18.29

Now, generator efficiency is given by,

$$\begin{aligned} \eta_g &= \frac{\text{Output of generator}}{\text{Input of generator}} \\ &= \frac{\text{Output of generator}}{\text{Output of turbine}} \quad (\because \text{Output of turbine} = \text{Input of generator}) \end{aligned}$$

or

$$0.93 = \frac{5 \times 10^6}{\text{S.P.}}$$

$$\therefore \text{S.P.} = \frac{5 \times 10^6}{0.93} \text{ W} \quad \dots(i)$$

\* MW stands for Mega Watt which is equal to  $10^6$  Watt or  $10^6$  W.

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Now, overall efficiency of turbine is given by,  $\eta_o = \frac{\text{S.P.}}{\text{W.P.}}$

But W.P. in Watt =  $\rho \times Q \times H \times 9.81 = 1000 \times Q \times 5.5 \times 9.81 \text{ W}$

$$\therefore \eta_o = \frac{\text{S.P.}}{1000 \times Q \times 5.5 \times 9.81}$$

$$\therefore \text{S.P.} = \eta_o \times 1000 \times Q \times 5.5 \times 9.81 \quad \dots(ii)$$

Equating the two values of S.P. given by equations (i) and (ii), we get

$$\frac{5 \times 10^6}{0.93} = 0.88 \times 1000 \times Q \times 5.5 \times 9.81 \quad (\because \eta_o = 0.88)$$

$$\therefore Q = \frac{5 \times 10^6}{0.93 \times 0.88 \times 1000 \times 5.5 \times 9.81} = 113.23 \text{ m}^3/\text{s}$$

The vane angles are to be calculated at the mean diameter of the runner.

$$\therefore \text{Mean diameter, } D_m = \frac{D_o + D_b}{2} = \frac{4.5 + 2.0}{2} = 3.25 \text{ m}$$

Inlet vane velocity corresponding to mean dia. is given by,

$$u_1 = \frac{\pi D_m \times N}{60} = \frac{\pi \times 3.25 \times 150}{60} = 25.52 \text{ m/s}$$

For axial flow turbine,  $u_1 = u_2 = 25.52 \text{ m/s}$  and  $V_{f_1} = V_{f_2}$

For no whirl velocity at outlet, the hydraulic efficiency is given by,

$$\eta_h = \frac{V_{w_2} \times u_1}{g \times H} \text{ or } 0.94 = \frac{V_{w_1} \times 25.52}{9.81 \times 5.5}$$

$$\therefore V_{w_1} = \frac{0.94 \times 9.81 \times 5.5}{25.52} = 1.987 \text{ m/s}$$

This value of  $V_{w_1}$  is less than  $u_1$ . Hence, the velocity triangle at inlet will be as shown in Fig. 18.30.

Now, using equation (18.25), we get

$$Q = \frac{\pi}{4} (D_o^2 - D_b^2) \times V_{f_1}$$

$$\text{or } 113.23 = \frac{\pi}{4} (4.5^2 - 2^2) \times V_{f_1} = \frac{\pi}{4} \times 16.25 \times V_{f_1}$$

$$\therefore V_{f_1} = \frac{113.23 \times 4}{\pi \times 16.25} = 8.87 \text{ m/s}$$

$$\therefore V_{f_2} = V_{f_1} = 8.87 \text{ m/s.}$$

Let  $\theta$  = Runner vane angle at inlet and

$\phi$  = Runner vane angle at outlet.

From inlet velocity triangle,

$$\begin{aligned} \tan \theta &= \frac{V_{f_1}}{(u_1 - V_{w_1})} = \frac{8.87}{25.52 - 1.987} \\ &= 0.3769 \end{aligned}$$

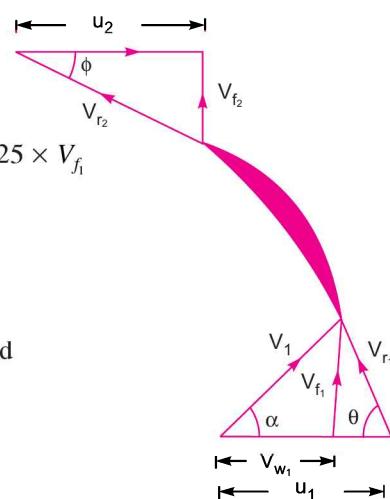


Fig. 18.30

$$\therefore \theta = \tan^{-1} 0.3769 = 20.65^\circ. \text{ Ans.}$$

Now, from outlet velocity triangle,

$$\begin{aligned} \tan \phi &= \frac{V_{f_2}}{u_2} = \frac{8.87}{25.52} & (\because V_{f_2} = V_{f_1} \text{ and } u_1 = u_2) \\ &= 0.347 \\ \therefore \phi &= \tan^{-1} 0.347 = 19.16^\circ. \text{ Ans.} \end{aligned}$$

**Problem 18.33** A propeller reaction turbine of runner diameter 4.5 m is running at 40 r.p.m. The guide blade angle at inlet is  $145^\circ$  and runner blade angle at outlet is  $25^\circ$  to the direction of vane. The axial flow area of water through runner is  $25 \text{ m}^2$ . If the runner blade angle at inlet is radial determine :

- (i) Hydraulic efficiency of the turbine, (ii) Discharge through turbine,
- (iii) Power developed by the turbine, and (iv) Specific speed of the turbine.

**Solution.** Given :

$$\text{Runner diameter, } D_o = 4.5 \text{ m}$$

$$\text{Speed, } N = 40 \text{ r.p.m.}$$

$$\text{Guide blade angle, } \alpha = 145^\circ$$

$$\text{Runner blade angle at outlet, } \phi = 25^\circ$$

$$\text{Flow area, } a = 25 \text{ m}^2$$

Runner blade angle at inlet is radial

$$\therefore \theta = 90^\circ, V_{r_1} = V_{f_1} \text{ and } u_1 = V_{w_1}$$

For Kaplan turbine, the discharge is given by the product of area of flow and velocity of flow.

$$\text{As area of flow is constant and hence } V_{f_1} = V_{f_2} (\because Q = \text{Area of flow} \times V_{f_1} = \text{Area of flow} \times V_{f_2})$$

The tangential speed of turbine at inlet,

$$\begin{aligned} u_1 &= \frac{\pi D_o N}{60} = \frac{\pi \times 4.5 \times 40}{60} \\ &= 9.42 \text{ m/s} \end{aligned}$$

Also

$$u_2 = u_1 = 9.42 \text{ m/s.}$$

From inlet velocity triangle,

$$\tan(180^\circ - \alpha) = \frac{V_{f_1}}{u_1}$$

$$\text{or } \tan(180^\circ - 145^\circ) = \tan 35^\circ = \frac{V_{f_1}}{u_1}$$

$$\therefore V_{f_1} = u_1 \tan 35^\circ = 9.42 \tan 35^\circ = 6.59$$

$$\text{Also } V_{w_1} = u_1 = 9.42 \text{ m/s.}$$

From outlet velocity triangle,

$$\tan \phi = \frac{V_{f_2}}{u_2 + V_{w_2}} \quad (\text{where } V_{f_2} = V_{f_1} = 6.59 \text{ and } u_2 = u_1 = 9.42)$$

$$\therefore \tan 25^\circ = \frac{6.59}{9.42 + V_{w_2}}$$

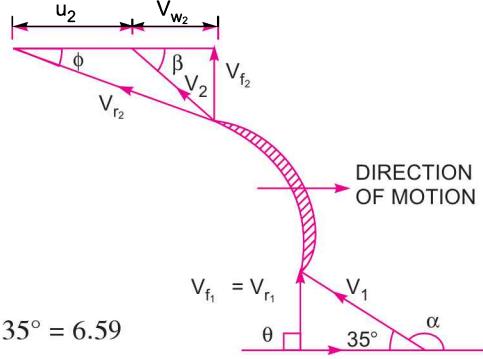


Fig. 18.31

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$$\therefore V_{w_2} + 9.42 = \frac{6.59}{\tan 25^\circ} = 14.13$$

$$\therefore V_{w_2} = 14.13 - 9.42 = 4.71 \text{ m/s}$$

$$\therefore V_2 = \sqrt{V_{f_2}^2 + V_{w_2}^2} = \sqrt{6.59^2 + 4.71^2} = \sqrt{43.43 + 22.18} = 8.1 \text{ m/s.}$$

Using equation (18.24),

$$H - \frac{V_2^2}{2g} = \frac{1}{g} [V_{w_1} u_1 - V_{w_2} u_2].$$

Here -ve sign is taken as the absolute velocity at inlet and outlet (i.e.,  $V_1$  and  $V_2$ ) are in the same direction and hence change of velocity will be with a -ve sign

$$\therefore H - \frac{8.1^2}{2 \times 9.81} = \frac{1}{9.81} [9.42 \times 9.42 - 4.71 \times 9.42]$$

$$H - 3.344 = \frac{1}{9.81} [88.736 - 44.368] = 4.522 \text{ m}$$

$$\therefore H = 4.522 + 3.344 = 7.866 \text{ m.}$$

(i) Hydraulic efficiency is given by equation (18.20A) as

$$\begin{aligned} \eta_h &= \frac{V_{w_1} u_1 - V_{w_2} u_2}{g \times H} \\ &= \frac{(9.42 \times 9.42 - 4.71 \times 9.42)}{9.81 \times 7.866} = 0.575 = 57.5\%. \text{ Ans.} \end{aligned}$$

(ii) Discharge through turbine is given by,

$$Q = \text{Area of flow} \times \text{Velocity of flow}$$

$$= 25 \times V_{f_1} = 25 \times 6.59 = 164.75 \text{ m}^3/\text{s. Ans.}$$

(iii) Power developed by turbine =  $\frac{\text{Work done per second}}{1000}$

$$= \frac{1}{g} \frac{[V_{w_1} u_1 - V_{w_2} u_2]}{1000} \times \text{Weight of water}$$

$$= \frac{1}{9.81} \left[ \frac{9.42 \times 9.42 - 4.71 \times 9.42}{1000} \right] \times \rho \times g \times Q$$

$$= \frac{1}{9.81} \left[ \frac{9.42 \times 9.42 - 4.71 \times 9.42}{1000} \right] \times 1000 \times 9.81 \times 164.75$$

$$= 6867 \text{ kW. Ans.}$$

(iv) Specific speed is given by the relation,

$$\begin{aligned} N_s &= \frac{N \sqrt{P}}{H^{5/4}} = \frac{N \sqrt{6867}}{7.866^{5/4}} = \frac{40 \times \sqrt{6867}}{7.866^{5/4}} \\ &= \frac{40 \times 82.867}{13.173} = 251.62 \text{ r.p.m. Ans.} \end{aligned}$$

### ► 18.10 DRAFT-TUBE

The draft-tube is a pipe of gradually increasing area which connects the outlet of the runner to the tail race. It is used for discharging water from the exit of the turbine to the tail race. This pipe of gradually increasing area is called a draft-tube. One end of the draft-tube is connected to the outlet of the runner while the other end is sub-merged below the level of water in the tail race. The draft-tube, in addition to serve a passage for water discharge, has the following two purposes also :

1. It permits a negative head to be established at the outlet of the runner and thereby increase the net head on the turbine. The turbine may be placed above the tail race without any loss of net head and hence turbine may be inspected properly.
2. It converts a large proportion of the kinetic energy ( $V_2^2/2g$ ) rejected at the outlet of the turbine into useful pressure energy. Without the draft tube, the kinetic energy rejected at the outlet of the turbine will go waste to the tail race.

Hence by using draft-tube, the net head on the turbine increases. The turbine develops more power and also the efficiency of the turbine increases.

If a reaction turbine is not fitted with a draft-tube, the pressure at the outlet of the runner will be equal to atmospheric pressure. The water from the outlet of the runner will discharge freely into the tail race. The net head on the turbine will be less than that of a reaction turbine fitted with a draft-tube.

Also without a draft-tube, the kinetic energy  $\left(\frac{V_2^2}{2g}\right)$  rejected at the outlet of the runner will go waste to the tail race.

**18.10.1 Types of Draft-Tubes.** The following are the important types of draft-tubes which are commonly used :

1. Conical draft-tubes,
2. Simple elbow tubes,
3. Moody spreading tubes, and
4. Elbow draft-tubes with circular inlet and rectangular outlet.

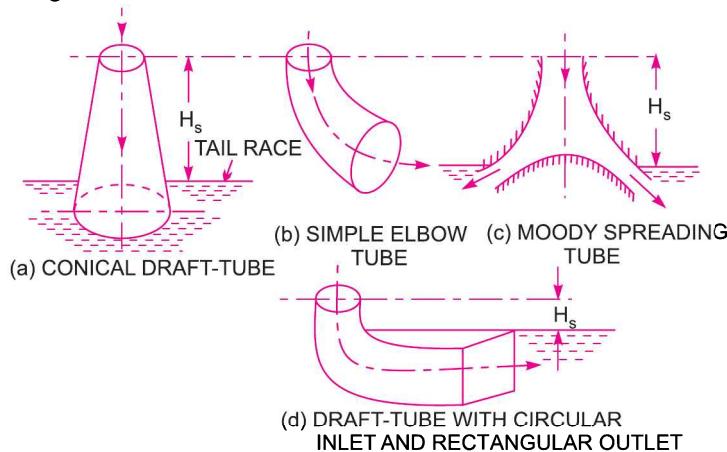


Fig. 18.32 Types of draft-tubes.

These different types of draft-tubes are shown in Fig. 18.32. The conical draft-tubes and Moody spreading draft-tubes are most efficient while simple elbow tubes and elbow draft-tubes with circular inlet and rectangular outlet require less space as compared to other draft-tubes.

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**18.10.2 Draft-Tube Theory.** Consider a capital draft-tube as shown in Fig. 18.33.

Let

$H_s$  = Vertical height of draft-tube above the tail race,

$y$  = Distance of bottom of draft-tube from tail race.

Applying Bernoulli's equation to inlet (section 1-1) and outlet (section 2-2) of the draft-tube and taking section 2-2 as the datum line, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + (H_s + y) = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + 0 + h_f \quad \dots(i)$$

where  $h_f$  = loss of energy between sections 1-1 and 2-2.

But

$$\begin{aligned} \frac{p_2}{\rho g} &= \text{Atmospheric pressure head} + y \\ &= \frac{p_a}{\rho g} + y. \end{aligned}$$

Substituting this value of  $\frac{p_2}{\rho g}$  in equation (i), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + (H_s + y) = \frac{p_a}{\rho g} + y + \frac{V_2^2}{2g} + h_f$$

$$\text{or } \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + H_s = \frac{p_a}{\rho g} + \frac{V_2^2}{2g} + h_f$$

$$\begin{aligned} \therefore \frac{p_1}{\rho g} &= \frac{p_a}{\rho g} + \frac{V_2^2}{2g} + h_f - \frac{V_1^2}{2g} - H_s \\ &= \frac{p_a}{\rho g} - H_s - \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - h_f \right) \end{aligned} \quad \dots(18.26)$$

In equation (18.26),  $\frac{p_1}{\rho g}$  is less than atmospheric pressure.

**18.10.3 Efficiency of Draft-Tube.** The efficiency of a draft-tube is defined as the ratio of actual conversion of kinetic head into pressure head in the draft-tube to the kinetic head at the inlet of the draft-tube. Mathematically, it is written as

$$\eta_d = \frac{\text{Actual conversion of kinetic head into pressure head}}{\text{Kinetic head at the inlet of draft-tube}}$$

Let

$V_1$  = Velocity of water at inlet of draft-tube,

$V_2$  = Velocity of water at outlet of draft-tube, and

$h_f$  = Loss of head in the draft-tube.

$$\text{Theoretical conversion of kinetic head into pressure head in draft-tube} = \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right).$$

$$\text{Actual conversion of kinetic head into pressure head} = \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right) - h_f$$

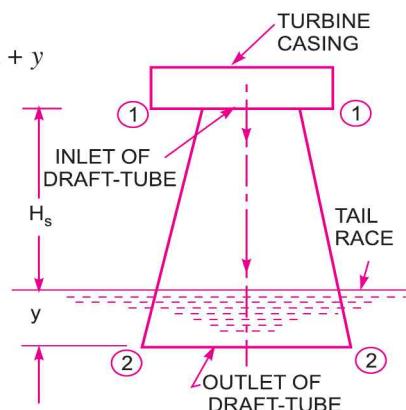


Fig. 18.33 Draft-tube theory.

$$\therefore \eta_d = \frac{\left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right) - h_f}{\left( \frac{V_1^2}{2g} \right)} \quad \dots(18.27)$$

**Problem 18.33 (A)** A water turbine has a velocity of 6 m/s at the entrance to the draft-tube and a velocity of 1.2 m/s at the exit. For friction losses of 0.1 m and a tail water 5 m below the entrance to the draft-tube, find the pressure head at the entrance.

**Solution.** Given :

$$\text{Velocity at inlet, } V_1 = 6 \text{ m/s}$$

$$\text{Velocity at outlet, } V_2 = 1.2 \text{ m/s}$$

$$\text{Friction loss, } h_f = 0.1 \text{ m}$$

Vertical height between tail race and inlet of draft-tube = 5 m

Let  $y$  = Vertical height between tail race and outlet of draft-tube.

Applying Bernoulli's equation at the inlet and outlet of draft-tube and taking reference line passing through section (2-2), we get

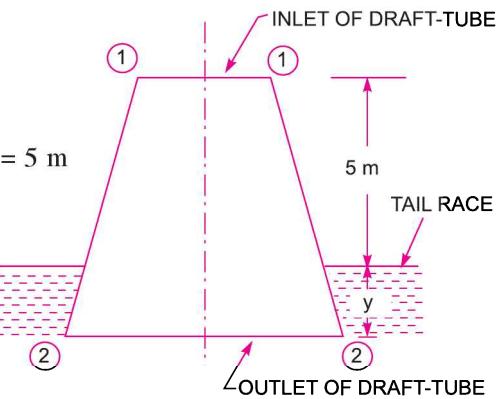


Fig. 18.33 (a)

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 + h_f$$

where  $Z_1 = (5 + y)$ ;  $V_1 = 6 \text{ m/s}$ ;  $V_2 = 1.2 \text{ m/s}$ ,

$$h_f = 0.1$$

$$\frac{p_2}{\rho g} = \text{Atmospheric pressure head} + y = \frac{p_a}{\rho g} + y$$

$$Z_2 = 0$$

Substituting the values, we get

$$\frac{p_1}{\rho g} + \frac{6^2}{2 \times 9.81} + (5 + y) = \left( \frac{p_a}{\rho g} + y \right) + \frac{1.2^2}{2 \times 9.81} + 0 + 0.1$$

$$\text{or } \frac{p_1}{\rho g} + 1.835 + 5 + y = \frac{p_a}{\rho g} + y + 0.0734 + 0.1$$

$$\text{or } \frac{p_1}{\rho g} + 6.835 = \frac{p_a}{\rho g} + 0.1734 \quad \dots(i)$$

If  $\frac{p_a}{\rho g}$  (i.e., atmospheric pressure head) is taken zero, then we will get  $\frac{p_1}{\rho g}$  as vacuum pressure head at inlet of draft-tube.

But if  $\frac{p_a}{\rho g} = 10.3 \text{ m}$  of water, then we will get  $\frac{p_1}{\rho g}$  as absolute pressure head at inlet of draft-tube.

Taking  $\frac{p_a}{\rho g} = 0$  and substituting this value in equation (i), we get

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$$\frac{p_1}{\rho g} + 6.835 = 0 + 0.1734$$

$$\therefore \frac{p_1}{\rho g} = -6.835 + 0.1734 = -6.6616 \text{ m. Ans.}$$

Negative sign means vacuum pressure head.

**Problem 18.34** A conical draft-tube having diameter at the top as 2.0 m and pressure head at 7 m of water (vacuum), discharges water at the outlet with a velocity of 1.2 m/s at the rate of 25 m<sup>3</sup>/s. If atmospheric pressure head is 10.3 m of water and losses between the inlet and outlet of the draft-tubes are negligible, find the length of draft-tube immersed in water. Total length of tube is 5 m.

**Solution.** Given :

Diameter at top,  $D_1 = 2.0 \text{ m}$

Pressure head,  $\frac{p_1}{\rho g} = 7 \text{ m (Vacuum)}$   
 $= 10.3 - 7.0 = 3.3 \text{ m (abs.)}$

Velocity at outlet,  $V_2 = 1.2 \text{ m/s}$

Discharge,  $Q = 25 \text{ m}^3/\text{s}$

Loss of energy,  $h_f = \text{Negligible}$

Let the length of the tube immersed in water =  $y \text{ m}$ .

Total length of the tube = 5 m

The velocity at inlet,  $V_1 = \frac{\text{Discharge}}{\text{Area at inlet}}$   
 $= \frac{Q}{\frac{\pi D_1^2}{4}} = \frac{25}{\frac{\pi (2.0)^2}{4}} = 7.957 \text{ m/s.}$

Using equation (18.26), we have

$$\begin{aligned} \frac{p_1}{\rho g} &= \frac{p_a}{\rho g} - H_s - \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - h_f \right) \\ 3.30 &= 10.3 - H_s - \left( \frac{7.957^2}{2 \times 9.81} - \frac{1.2^2}{2 \times 9.81} - 0 \right) \\ &\quad \left( \because h_f = 0 \text{ and } \frac{p_a}{\rho g} = 10.3 \right) \\ &= 10.3 - H_s - (3.227 - 0.0734) \end{aligned}$$

or

$$3.3 = 10.3 - H_s - 3.1536$$

$\therefore$

$$H_s = 10.3 - 3.1536 - 3.3 = 3.8464 \text{ m}$$

$\therefore$

$$y = \text{Total length} - H_s = 5 - 3.8464 = 1.1536 \text{ m. Ans.}$$

**Problem 18.35** A conical draft-tube having inlet and outlet diameters 1 m and 1.5 m discharges water at outlet with a velocity of 2.5 m/s. The total length of the draft-tube is 6 m and 1.20 m of the length of draft-tube is immersed in water. If the atmospheric pressure head is 10.3 m of water and loss of head due to friction in the draft-tube is equal to  $0.2 \times \text{velocity head at outlet of the tube}$ , find :

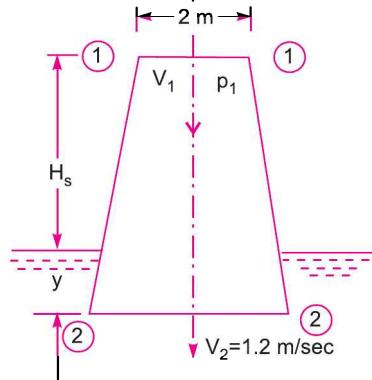


Fig. 18.34

(i) Pressure head at inlet, and (ii) Efficiency of the draft-tube.

**Solution.** Given :

$$\text{Diameter at inlet, } D_1 = 1.0 \text{ m}$$

$$\text{Diameter at outlet } D_2 = 1.5 \text{ m}$$

$$\text{Velocity at outlet, } V_2 = 2.5 \text{ m/s}$$

$$\text{Total length of tube, } H_s + y = 6.0 \text{ m}$$

$$\text{Length of tube in water, } y = 1.20 \text{ m}$$

$$\therefore H_s = 6.0 - 1.20 = 4.80 \text{ m}$$

$$\text{Atmospheric pressure head, } \frac{p_a}{\rho g} = 10.3 \text{ m}$$

Loss of head due to friction,  $h_f = 0.2 \times \text{Velocity head at outlet}$

$$= 0.2 \times \frac{V_2^2}{2g}$$

$$\text{Discharge through tube, } Q = A_2 V_2 = \frac{\pi}{4} D_2^2 \times 2.5 = \frac{\pi}{4} (1.5)^2 \times 2.5 = 4.4178 \text{ m}^3/\text{s}$$

$$\text{Velocity at inlet, } V_1 = \frac{Q}{A_1} = \frac{4.4178}{\frac{\pi}{4} \times 1^2} = 5.625 \text{ m/s}$$

(i) Pressure head at inlet  $\left( \frac{p_1}{\rho g} \right)$ .

$$\begin{aligned} \text{Using equation (18.26), } \frac{p_1}{\rho g} &= \frac{p_a}{\rho g} - H_s - \left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - h_f \right) \\ &= 10.3 - 4.8 - \left( \frac{5.625^2}{2 \times 9.81} - \frac{2.5^2}{2 \times 9.81} - 0.2 \times \frac{V_2^2}{2g} \right) \\ &= 10.3 - 4.8 - \left( 1.6126 - .3185 - \frac{0.2 \times 2.5^2}{2 \times 9.81} \right) \\ &= 10.3 - 4.8 - (1.6126 - .3185 - .0637) = 5.5 - (1.2304) = 4.269 \\ &\approx \mathbf{4.27 \text{ m (abs.) Ans.}} \end{aligned}$$

(ii) Efficiency of Draft-tube ( $\eta_d$ )

$$\begin{aligned} \text{Using equation (18.27), } \eta_d &= \frac{\left( \frac{V_1^2}{2g} - \frac{V_2^2}{2g} \right) - h_f}{\frac{V_1^2}{2g}} = \frac{\frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{0.2 V_2^2}{2g}}{\frac{V_1^2}{2g}} \\ &= \frac{\frac{V_1^2}{2g} - 1.2 \frac{V_2^2}{2g}}{\frac{V_1^2}{2g}} = 1 - 1.2 \left( \frac{V_2}{V_1} \right)^2 = 1 - 1.2 \left( \frac{2.5}{5.625} \right)^2 = 1 - 0.237 \\ &= \mathbf{0.763 \text{ or } 76.3\%. Ans.} \end{aligned}$$

### ► 18.11 SPECIFIC SPEED

It is defined as the speed of a turbine which is identical in shape, geometrical dimensions, blade angles, gate opening etc., with the actual turbine but of such a size that it will develop unit power when working under unit head. It is denoted by the symbol  $N_s$ . The specific speed is used in comparing the different types of turbines as every type of turbine has different specific speed.

In M.K.S. units, unit power is taken as one horse power and unit head as one metre. But in S.I. units, unit power is taken as one kilowatt and unit head as one metre.

**18.11.1 Derivation of the Specific Speed.** The overall efficiency ( $\eta_o$ ) of any turbine is given by,

$$\eta_o = \frac{\text{Shaft power}}{\text{Water power}} = \frac{\text{Power developed}}{\rho \times g \times Q \times H} = \frac{P}{\rho \times g \times Q \times H} \quad \dots(i)$$

where  $H$  = Head under which the turbine is working,

$Q$  = Discharge through turbine,

$P$  = Power developed or shaft power.

From equation (i),  $P = \eta_o \times \frac{\rho \times g \times Q \times H}{1000}$

$$\propto Q \times H \quad (\text{as } \eta_o \text{ and } \rho \text{ are constant}) \quad \dots(ii)$$

Now let

$D$  = Diameter of actual turbine,

$N$  = Speed of actual turbine,

$u$  = Tangential velocity of the turbine,

$N_s$  = Specific speed of the turbine,

$V$  = Absolute velocity of water.

The absolute velocity, tangential velocity and head on the turbine are related as,

$$\begin{aligned} u &\propto V, \text{ where } V \propto \sqrt{H} \\ &\propto \sqrt{H} \end{aligned} \quad \dots(iii)$$

But the tangential velocity  $u$  is given by

$$\begin{aligned} u &= \frac{\pi D N}{60} \\ &\propto D N \end{aligned} \quad \dots(iv)$$

∴ From equations (iii) and (iv), we have

$$\sqrt{H} \propto D N \text{ or } D \propto \frac{\sqrt{H}}{N} \quad \dots(v)$$

The discharge through turbine is given by

$$Q = \text{Area} \times \text{Velocity}$$

But  $\text{Area} \propto B \times D$  (where  $B$  = Width)  
 $\propto D^2$  ( $\because B \propto D$ )

And  $\text{Velocity} \propto \sqrt{H}$

$$\therefore Q \propto D^2 \times \sqrt{H}$$

$$\begin{aligned} &\propto \left( \frac{\sqrt{H}}{N} \right)^2 \times \sqrt{H} \quad \left( \because \text{From equation (v), } D \propto \frac{\sqrt{H}}{N} \right) \\ &\propto \frac{H}{N^2} \times \sqrt{H} \propto \frac{H^{3/2}}{N^2} \end{aligned} \quad \dots(vi)$$

Substituting the value of  $Q$  in equation (ii), we get

$$\begin{aligned} P &\propto \frac{H^{3/2}}{N^2} \times H \propto \frac{H^{5/2}}{N^2} \\ \therefore P &= K \frac{H^{5/2}}{N^2}, \text{ where } K = \text{Constant of proportionality.} \end{aligned}$$

If  $P = 1$ ,  $H = 1$ , the speed  $N = \text{Specific speed } N_s$ . Substituting these values in the above equation, we get

$$\begin{aligned} 1 &= \frac{K \times 1^{5/2}}{N_s^2} \quad \text{or} \quad N_s^2 = K \\ \therefore P &= N_s^2 \frac{H^{5/2}}{N^2} \quad \text{or} \quad N_s^2 = \frac{N^2 P}{H^{5/2}} \\ \therefore N_s &= \sqrt{\frac{N^2 P}{H^{5/2}}} = \frac{N \sqrt{P}}{H^{5/4}} \end{aligned} \quad \dots(18.28)$$

In equation (18.28), if  $P$  is taken in metric horse power the specific speed is obtained in M.K.S. units. But if  $P$  is taken in kilowatts, the specific speed is obtained in S.I. units.

**18.11.2 Significance of Specific Speed.** Specific speed plays an important role for selecting the type of the turbine. Also the performance of a turbine can be predicted by knowing the specific speed of the turbine. The type of turbine for different specific speed is given in Table 18.1 as :

Table 18.1

S. No.	Specific speed		Types of turbine
	(M.K.S.)	(S.I.)	
1.	10 to 35	8.5 to 30	Pelton wheel with single jet
2.	35 to 60	30 to 51	Pelton wheel with two or more jets
3.	60 to 300	51 to 225	Francis turbine
4.	300 to 1000	255 to 860	Kaplan or Propeller turbine

**Problem 18.36** A turbine develops 7225 kW power under a head of 25 metres at 135 r.p.m. Calculate the specific speed of the turbine and state the type of the turbine.

**Solution.** Given :

Power developed,  $P = 7225 \text{ kW}$

Head,  $H = 25 \text{ m}$

Speed,  $N = 135 \text{ r.p.m.}$

Specific speed of the turbine ( $N_s$ )

Using equation (18.28),

$$N_s = \frac{N \sqrt{P}}{H^{5/4}} = \frac{135 \times \sqrt{7225}}{25^{5/4}} = 205.28. \text{ Ans.}$$

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From Table 18.1, for specific speeds (S.I.) between 51 and 255 the type of turbine is Francis. As the specific speed 205.28 lies in this range and hence type of turbine is Francis. **Ans.**

**Problem 18.37** A turbine is to operate under a head of 25 m at 200 r.p.m. The discharge is 9 cumec. If the efficiency is 90%, determine :

- (i) Specific speed of the machine,
- (ii) Power generated, and
- (iii) Type of turbine.

**Solution.** Given :

Head,	$H = 25 \text{ m}$
Speed,	$N = 200 \text{ r.p.m.}$
Discharge,	$Q = 9 \text{ cumec} = 9 \text{ m}^3/\text{s}$
Efficiency,	$\eta_o = 90\% = 0.90$ (Take the efficiency as overall $\eta$ )

Now using relation, 
$$\eta_o = \frac{\text{Power developed}}{\text{Water power}} = \frac{P}{\frac{\rho \times g \times Q \times H}{1000}}$$

$$\therefore P = \eta_o \times \frac{\rho \times g \times Q \times H}{1000} = \frac{0.90 \times 9.81 \times 1000 \times 9 \times 25}{1000} = 1986.5 \text{ kW}$$

(i) Specific speed of the machine ( $N_s$ )

Using equation (18.28), 
$$N_s = \frac{N \sqrt{P}}{H^{5/4}} = \frac{200 \times \sqrt{1986.5}}{25^{5/4}} = 159.46 \text{ r.p.m. Ans.}$$

(ii) Power generated

$$P = 1986.5 \text{ kW. Ans.}$$

(iii) As the specific speed lies between 51 and 255, the turbine is a Francis turbine. **Ans.**

**Problem 18.38** A turbine develops 9000 kW when running at a speed of 140 r.p.m. and under a head of 30 m. Determine the specific speed of the turbine.

**Solution.** Given :

Power developed,	$P = 9000 \text{ kW}$
Head,	$H = 30 \text{ m}$
Speed ,	$N = 140 \text{ r.p.m.}$

The specific speed is given by equation (18.28) as

$$N_s = \frac{N \sqrt{P}}{H^{5/4}} = \frac{140 \times \sqrt{9000}}{30^{5/4}} = \frac{13281.56}{70.21} = 189.167 \text{ (S.I. units). Ans.}$$

**Problem 18.39** A Pelton wheel develops 8000 kW under a net head of 130 m at a speed of 200 r.p.m. Assuming the co-efficient of velocity for the nozzle 0.98, hydraulic efficiency 87%, speed ratio 0.46 and jet diameter to wheel diameter ratio  $\frac{I}{9}$ , determine :

- (i) the discharge required,
- (ii) the diameter of the wheel,
- (iii) the diameter and number of jets required, and
- (iv) the specific speed.

Mechanical efficiency is 75%.

**Solution.** Given :

Power developed,	$P = 8000 \text{ kW}$
------------------	-----------------------

Net head,  $H = 130 \text{ m}$   
 Speed,  $N = 200 \text{ r.p.m.}$   
 Co-efficient of velocity,  $C_v = 0.98$   
 Hydraulic efficiency,  $\eta_h = 87\% = 0.87$

Speed ratio,  $\frac{u_1}{\sqrt{2gH}} = 0.46$

$$\therefore u_1 = 0.46 \times \sqrt{2gH} = 0.46 \times \sqrt{2 \times 9.81 \times 130} = 23.23 \text{ m/s.}$$

Jet diameter to wheel diameter  $= \frac{d}{D} = \frac{1}{9}$

Mechanical efficiency,  $\eta_m = 75\% = 0.75$

Overall efficiency is given by equation (18.6) as

$$\eta_o = \eta_h \times \eta_m = 0.87 \times 0.75 = 0.6525$$

Also  $\eta_o = \frac{\text{Power developed}}{\text{Water power}} = \frac{8000}{\text{W.P. in kW}}$  or  $0.6525 = \frac{8000}{\text{W.P. in kW}}$

$$\therefore \text{W.P. in kW} = \frac{8000}{0.6525} = 12260.536 \text{ kW}$$

$$\begin{aligned}
 \text{But W.P. in kW} &= \frac{\rho \times g \times Q \times H}{1000} \\
 &= \frac{1000 \times 9.81 \times Q \times H}{1000} \quad (\because \rho g \text{ in S.I.} = 1000 \times 9.81) \\
 &= Q \times H \times 9.81 = Q \times 130 \times 9.81
 \end{aligned}$$

$$\therefore 12260.536 = Q \times 130 \times 9.81$$

$$\therefore Q = \frac{12260.536}{130 \times 9.81} = 9.614 \text{ m}^3/\text{s. Ans.}$$

(i) Discharge required

$$Q = 9.614 \text{ m}^3/\text{s. Ans.}$$

(ii) Diameter of wheel ( $D$ )

Using the relation,  $u_1 = \frac{\pi DN}{60}$

$$\therefore D = \frac{60 \times u_1}{\pi \times N} = \frac{60 \times 23.23}{\pi \times 200} = 2.218 \text{ m. Ans.}$$

(iii) Diameter of jet ( $d$ ) and number of jets required

$$\frac{d}{D} = \frac{1}{9}$$

$$\therefore d = \frac{D}{9} = \frac{2.218}{9} = 0.2464 \text{ m} = 246.4 \text{ mm. Ans.}$$

$$\therefore \text{Area of jet, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.2464)^2 = .04768 \text{ m}^2.$$

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Velocity of jet is given by,  $V_1 = C_v \sqrt{2gH} = 0.98 \sqrt{2 \times 9.81 \times 130} = 49.49 \text{ m/s}$

$$\therefore \text{Discharge through one jet} = \text{Area of jet} \times \text{Velocity of jet} = a \times V_1 \\ = 0.04768 \times 49.49 = 2.359 \text{ m}^3/\text{s}$$

$$\therefore \text{Number of jets} = \frac{\text{Total discharge}}{\text{Discharge through one jet}} \\ = \frac{Q}{2.359} = \frac{9.614}{2.359} = 4.07 \text{ say } \mathbf{4.0. \text{ Ans.}}$$

(iv) Specific speed is given by equation (18.28) as

$$N_s (\text{S.I. units}) = \frac{N \sqrt{P}}{H^{5/4}} = \frac{200 \times \sqrt{8000}}{130^{5/4}} = \frac{17888.54}{438.96} = \mathbf{40.75. \text{ Ans.}}$$

**Problem 18.40** A Pelton turbine develops 3000 kW under a head of 300 m. The overall efficiency of the turbine is 83%. If speed ratio = 0.46,  $C_v = 0.98$  and specific speed is 16.5, then find :

(i) Diameter of the turbine, and (ii) Diameter of the jet.

**Solution.** Given :

Power,	$P = 3000 \text{ kW}$
Net head,	$H = 300 \text{ m}$
Overall efficiency,	$\eta_o = 83\% \text{ or } 0.83$
Speed ratio	= 0.46
Value of $C_v$ ,	= 0.98
Specific speed*,	$N_s = 16.5$

$$\text{Using equation, } N_s = \frac{N \sqrt{P}}{H^{5/4}} \text{ or } N = \frac{N_s H^{5/4}}{\sqrt{P}} = \frac{16.5 \times 300^{5/4}}{\sqrt{3000}} = 375 \text{ r.p.m.}$$

The velocity ( $V$ ) at the outlet of nozzle is given by,

$$V = C_v \sqrt{2 \times g \times H} = 0.98 \sqrt{2 \times 9.81 \times 300} = 75.1 \text{ m/s}$$

$$\begin{aligned} \text{Now speed ratio} &= \frac{u}{\sqrt{2gH}} \text{ or } u = \text{Speed ratio} \times \sqrt{2gH} \\ &= 0.46 \times \sqrt{2 \times 9.81 \times 300} = 34.95 \text{ m/s.} \end{aligned}$$

(i) Diameter of the turbine ( $D$ )

$$\text{Using, } u = \frac{\pi D N}{60} \text{ or } D = \frac{60 \times u}{\pi \times N} = \frac{60 \times 34.95}{\pi \times 375} = \mathbf{1.78 \text{ m. Ans.}}$$

(ii) Diameter of the jet ( $d$ )

Let  $Q = \text{Discharge through turbine in m}^3/\text{s}$

$$\text{Using the relation, } \eta_o = \frac{P}{(\rho \times g \times Q \times H)}, \quad \text{where } \rho \times g = 1000 \times 9.81 \text{ N/m}^3 \text{ for water}$$

$$\therefore 0.83 = \frac{3000}{\left( \frac{1000 \times 9.81 \times Q \times 300}{1000} \right)}$$

\* Specific speed is the speed of the turbine working under a unit head and develops one kilowatt power.

$$\therefore Q = \frac{3000}{9.81 \times 300 \times 0.83} = 1.23 \text{ m}^3/\text{s}$$

But discharge through a Pelton turbine is given by,

$$Q = \text{Area of jet} \times \text{Velocity}$$

or

$$1.23 = \frac{\pi}{4} d^2 \times 75.1$$

$$\therefore d = \sqrt{\frac{4 \times 1.23}{\pi \times 75.1}} = 0.142 \text{ m} = 142 \text{ mm. Ans.}$$

**Problem 18.41** Water under a head of 300 m is available for a hydel-plant situated at a distance of 2.35 km from the source. The frictional losses of energy for transporting water is equivalent to 26 (J/N). A number of Pelton wheels are to be installed generating a total output of 18 MW. Determine the number of units to be installed, diameter of Pelton wheel and the jet diameter when the following are available : Wheel speed 650 r.p.m.; ratio of bucket to jet speed 0.46 ; specific speed not to exceed 30 (m, kW, r.p.m.) ;  $C_v$  and  $C_d$  for the nozzle 0.97 and 0.94 respectively and the overall efficiency of the wheel 87%.

**Solution.** Given :

Total head

$$= 300 \text{ m}$$

Length

$$= 2.35 \text{ km} = 2350 \text{ m}$$

Frictional losses

$$= 26 \text{ (J/N)} = 26 \text{ (Nm/N)} \text{ (as J = Nm)} = 26 \text{ m}$$

∴ Net head,

$$H = 300 - 26 = 274 \text{ m}$$

Total output

$$= 18 \text{ MW} = 18 \times 10^3 \text{ kW}$$

$$N = 650 \text{ r.p.m.}$$

Ratio of bucket to jet speed

$$= 0.46$$

$$C_v = 0.97, C_d = 0.94$$

$$\eta_o = 87\% = 0.87$$

and

$$N_s = 30,$$

where  $H$  is in m,  $P$  in kW and  $N$  in r.p.m.

Find : (i) Number of units to be installed

(ii) Dia. of Pelton wheel ( $D$ )

(iii) Dia. of jet of water ( $d$ )

(i) Number of units to be installed

Let

$$P = \text{Power output of each unit in kW}$$

$$\text{Using equation (18.28) as } N_s = \frac{N\sqrt{P}}{H^{5/4}} \text{ or } 30 = \frac{650 \times \sqrt{P}}{274^{5/4}} \text{ or } \sqrt{P} = \frac{30 \times 274^{5/4}}{650}$$

$$\text{Squaring both sides, we get } P = \frac{30^2 \times 274^{5/2}}{650^2} = 2647.2 \text{ kW}$$

$$\therefore \text{No. of units} = \frac{\text{Total output in kW}}{\text{Output of one unit in kW}}$$

$$= \frac{18 \times 10^3}{2647.2} = 6.799 \approx 7 \text{ units. Ans.}$$

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(ii) Dia. of Pelton wheel ( $D$ )

$$\text{Velocity of jet is given by, } V_1 = C_v \times \sqrt{2gH} \\ = 0.97 \times \sqrt{2 \times 9.81 \times 274} = 71.12 \text{ (m/s)}$$

But ratio of bucket to jet speed = 0.46\*

$$\text{or } \frac{\text{Speed of bucket}}{\text{Speed of jet}} = 0.46 \text{ or } \frac{u_1}{V_1} = 0.46$$

$$\therefore u_1 = 0.46 \times V_1 = 0.46 \times 71.12 = 32.715 \text{ m/s.}$$

$$\text{But } u_1 = \frac{\pi DN}{60}$$

$$\therefore 32.715 = \frac{\pi \times D \times 650}{60} \text{ or } \frac{32.715 \times 60}{\pi \times 650} = D \text{ or } 0.945 \text{ m} = D$$

$\therefore$  Dia. of Pelton wheel = **0.945. Ans.**

(iii) Dia. of jet ( $d$ )

$$\text{We know } \eta_o = \frac{\text{Total power output}}{\text{Total water power in kW}}$$

$$\text{or } 0.87 = \frac{18 \times 10^3}{\text{Total water power in kW}}$$

$$\therefore \text{Total water power in kW} = \frac{18 \times 10^3}{0.87} = 20.689 \times 10^3 \text{ kW}$$

$$\therefore \text{Water power in kW per unit} = \frac{\text{Total water power}}{\text{No. of units}} = \frac{20.689 \times 10^3}{7} = 2.955 \times 10^3 \text{ kW}$$

But water power in kW per unit is given by equation (18.3 A) as,

$$\text{Water power} = \frac{\rho \times g \times Q \times H}{1000} \text{ kW}$$

$$\therefore 2.955 \times 10^3 = \frac{\rho \times g \times Q \times H}{1000} = \frac{(1000 \times 9.81) \times Q \times H}{1000} \\ (\because \rho \times g = 1000 \times 9.81) \\ = 9.81 \times Q \times 274$$

$$\therefore Q = \frac{2.955 \times 10^3}{9.81 \times 274} = 1.099 \text{ m}^3/\text{s}$$

But discharge ( $Q$ ) through one unit is also given by

$$Q = C_d \times \frac{\pi}{4} d^2 \times \sqrt{2gH}$$

---

\* It is not speed ratio. It is the ratio of bucket speed to jet speed i.e., ratio of  $u_1$  and  $V_1$ . Speed ratio is  $u_1/\sqrt{2gH}$ .

or  $1.099 = 0.94 \times \frac{\pi}{4} d^2 \times \sqrt{2 \times 9.81 \times 274}$

or  $d^2 = \frac{1.099 \times 4}{0.94 \times \pi \times \sqrt{2 \times 9.81 \times 274}} = 0.0203 \text{ m}$   
 $\therefore d = \sqrt{0.0203} = 0.1424 \text{ m} = 142.4 \text{ mm. Ans.}$

## ► 18.12 UNIT QUANTITIES

In order to predict the behaviour of a turbine working under varying conditions of head, speed, output and gate opening, the results are expressed in terms of quantities which may be obtained when the head on the turbine is reduced to unity. The conditions of the turbine under unit head are such that the efficiency of the turbine remains unaffected. The following are the three important unit quantities which must be studied under unit head :

1. Unit speed,
2. Unit discharge, and
3. Unit power.

**18.12.1 Unit Speed.** It is defined as the speed of a turbine working under a unit head (*i.e.*, under a head of 1 m). It is denoted by ' $N_u$ '. The expression for unit speed ( $N_u$ ) is obtained as :

Let  $N$  = Speed of a turbine under a head  $H$ ,

$H$  = Head under which a turbine is working,

$u$  = Tangential velocity.

The tangential velocity, absolute velocity of water and head on the turbine are related as

$$\begin{aligned} u &\propto V, & \text{where } V &\propto \sqrt{H} \\ &\propto \sqrt{H} & & \dots(i) \end{aligned}$$

Also tangential velocity ( $u$ ) is given by

$$u = \frac{\pi D N}{60}, \quad \text{where } D = \text{Diameter of turbine.}$$

For a given turbine, the diameter ( $D$ ) is constant.

$$\therefore u \propto N \text{ or } N \propto u \text{ or } N \propto \sqrt{H} \quad (\because \text{From (i), } u \propto \sqrt{H})$$

$$\therefore N = K_1 \sqrt{H} \quad \dots(ii)$$

where  $K_1$  is a constant of proportionality.

If head on the turbine becomes unity, the speed becomes unit speed or  
when  $H = 1, N = N_u$

Substituting these values in equation (ii), we get

$$N_u = K_1 \sqrt{1.0} = K_1$$

Substituting the value of  $K_1$  in equation (ii),

$$N = N_u \sqrt{H} \text{ or } N_u = \frac{N}{\sqrt{H}}. \quad \dots(18.29)$$

**18.12.2 Unit Discharge.** It is defined as the discharge passing through a turbine, which is working under a unit head (*i.e.*, 1 m). It is denoted by the symbol ' $Q_u$ '. The expression for unit discharge is given as :

Let  $H$  = Head of water on the turbine,

$Q$  = Discharge passing through turbine when head is  $H$  on the turbine,

$a$  = Area of flow of water.

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The discharge passing through a given turbine under a head ' $H$ ' is given by,

$$Q = \text{Area of flow} \times \text{Velocity}$$

But for a turbine, area of flow is constant and velocity is proportional to  $\sqrt{H}$ .

$$\therefore Q \propto \text{Velocity} \propto \sqrt{H}$$

or 
$$Q = K_2 \sqrt{H} \quad \dots(iii)$$

where  $K_2$  is constant of proportionality.

If  $H = 1$ ,  $Q = Q_u$ .

(By definition)

Substituting these values in equation (iii), we get

$$Q_u = K_2 \sqrt{1.0} = K_2.$$

Substituting the value of  $K_2$  in equation (iii), we get

$$Q = Q_u \sqrt{H}$$

$$\therefore Q_u = \frac{Q}{\sqrt{H}} \quad \dots(18.30)$$

**18.12.3 Unit Power.** It is defined as the power developed by a turbine, working under a unit head (*i.e.*, under a head of 1 m). It is denoted by the symbol ' $P_u$ '. The expression for unit power is obtained as :

Let

$H$  = Head of water on the turbine,

$P$  = Power developed by the turbine under a head of  $H$ ,

$Q$  = Discharge through turbine under a head  $H$ .

The overall efficiency ( $\eta_o$ ) is given as

$$\eta_o = \frac{\text{Power developed}}{\text{Water power}} = \frac{P}{\frac{\rho \times g \times Q \times H}{1000}}$$

$$\begin{aligned} \therefore P &= \eta_o \times \frac{\rho \times g \times Q \times H}{1000} \\ &\propto Q \times H \\ &\propto \sqrt{H} \times H \\ &\propto H^{3/2} \\ \therefore P &= K_3 H^{3/2} \end{aligned} \quad (\because Q \propto \sqrt{H}) \quad \dots(iv)$$

where  $K_3$  is a constant of proportionality.

When

$$H = 1 \text{ m} \quad P = P_u$$

$$\therefore P_u = K_3 (1)^{3/2} = K_3.$$

Substituting the value of  $K_3$  in equation (iv), we get

$$P = P_u H^{3/2}$$

$$\therefore P_u = \frac{P}{H^{3/2}}. \quad \dots(18.31)$$

**18.12.4 Use of Unit Quantities ( $N_u$ ,  $Q_u$ ,  $P_u$ ).** If a turbine is working under different heads, the behaviour of the turbine can be easily known from the values of the unit quantities, *i.e.*, from the values of unit speed, unit discharge and unit power.

Let

$H_1, H_2, \dots$  are the heads under which a turbine works,  
 $N_1, N_2, \dots$  are the corresponding speeds,  
 $Q_1, Q_2, \dots$  are the discharge, and  
 $P_1, P_2, \dots$  are the power developed by the turbine.

Using equations (18.29), (18.30) and (18.31) respectively,

$$\left. \begin{aligned} N_u &= \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}} \\ Q_u &= \frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}} \\ P_u &= \frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}} \end{aligned} \right\} \quad \dots(18.32)$$

Hence, if the speed, discharge and power developed by a turbine under a head are known, then by using equation (18.32) the speed, discharge and power developed by the same turbine under a different head can be obtained easily.

**Problem 18.41 (A)** A turbine develops 9000 kW when running at 10 r.p.m. The head on the turbine is 30 m. If the head on the turbine is reduced to 18 m, determine the speed and power developed by the turbine.

**Solution.** Given :

Power developed,	$P_1 = 9000 \text{ kW}$
Speed,	$N_1 = 100 \text{ r.p.m.}$
Head,	$H_1 = 30 \text{ m}$
Let for a head,	$H_2 = 18 \text{ m}$
Speed	$= N_2$
Power	$= P_2$

Using equation (18.32),  $\frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$

$$N_2 = \frac{N_1 \sqrt{H_2}}{\sqrt{H_1}} = \frac{100 \sqrt{18}}{\sqrt{30}} = \frac{100 \times 4.2426}{5.4772} = 77.46 \text{ r.p.m. Ans.}$$

Also we have

$$\frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$$

$$\therefore P_2 = \frac{P_1 H_2^{3/2}}{H_1^{3/2}} = \frac{9000 \times 18^{3/2}}{30^{3/2}} = \frac{687307.78}{164.316} = 4182.84 \text{ kW. Ans.}$$

**Problem 18.42** A turbine develops 500 kW power under a head of 100 metres at 200 r.p.m. What would be its normal speed and output under a head of 81 metres ?

**Solution.** Given :

Power,	$P_1 = 500 \text{ kW}$
Head,	$H_1 = 100 \text{ m}$

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Speed,  $N_1 = 200 \text{ r.p.m.}$

For a head,  $H_2 = 81 \text{ m}$

Let,  $N_2 = \text{Speed}$

$P_2 = \text{Power}$

Using equation (18.32) for speed, we have

$$\frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$$

$$\therefore N_2 = \sqrt{H_2} \times \frac{N_1}{\sqrt{H_1}} = \sqrt{\frac{H_2}{H_1}} \times N_1 = \sqrt{\frac{81}{100}} \times 200 \\ = \frac{9}{10} \times 20 = 180 \text{ r.p.m. Ans.}$$

Using equation (18.32) for power, we have

$$\frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$$

$$\therefore P_2 = H_2^{3/2} \times \frac{P_1}{H_1^{3/2}} = \frac{81^{3/2}}{100^{3/2}} \times 500 \\ = \frac{729}{1000} \times 500 = 364.5 \text{ kW. Ans.}$$

**Problem 18.43** A turbine is to operate under a head of 25 m at 200 r.p.m. The discharge is 9 cumec. If the efficiency is 90%, determine the performance of the turbine under a head of 20 metres.

**Solution.** Given :

Head on turbine,  $H_1 = 25 \text{ m}$

Speed,  $N_1 = 200 \text{ r.p.m.}$

Discharge,  $Q_1 = 9 \text{ m}^3/\text{s}$

Overall efficiency,  $\eta_o = 90\% \text{ or } 0.90$ .

Performance of the turbine under a head,  $H_2 = 20 \text{ m}$ , means to find the speed, discharge and power developed by the turbine when working under the head of 20 m.

Let for the head,  $H_2 = 20 \text{ m}$ , Speed =  $N_2$ , discharge =  $Q_2$  and power =  $P_2$

Using the relation,

$$\eta_o = \frac{P}{W.P.} = \frac{P_1}{\rho \times g \times Q_1 \times H_1} \times 1000$$

$$\therefore P_1 = \frac{\eta_o \times \rho \times g \times Q_1 \times H_1}{1000} = \frac{0.90 \times 1000 \times 9.81 \times 9 \times 25}{1000} = 1986.5 \text{ kW}$$

Using equation (18.32),  $\frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$

$$\therefore N_2 = \frac{N_1 \sqrt{H_2}}{\sqrt{H_1}} = 200 \times \frac{\sqrt{20}}{\sqrt{25}} = 178.88 \text{ r.p.m. Ans.}$$

Also

$$\frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$$

∴

$$Q_2 = Q_1 \times \frac{\sqrt{H_2}}{\sqrt{H_1}} = 9.0 \times \sqrt{\frac{20}{25}} = 8.05 \text{ m}^3/\text{s. Ans.}$$

And

$$\frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$$

∴

$$P_2 = \frac{P_1 H_2^{3/2}}{H_1^{3/2}} = P_1 \left( \frac{H_2}{H_1} \right)^{3/2} = 1986.5 \left( \frac{20}{25} \right)^{3/2} = 1421.42 \text{ kW. Ans.}$$

**Problem 18.44** A Pelton wheel is revolving at a speed of 190 r.p.m. and develops 5150.25 kW when working under a head of 220 m with an overall efficiency of 80%. Determine unit speed, unit discharge and unit power. The speed ratio for the turbine is given as 0.47. Find the speed, discharge and power when this turbine is working under a head of 140 m.

**Solution.** Given :

Speed,

$$N_1 = 190 \text{ r.p.m.}$$

Power,

$$P_1 = 5150.25 \text{ kW}$$

Head,

$$H_1 = 220 \text{ m}$$

Overall efficiency,

$$\eta_o = 80\% = 0.80$$

Speed ratio

$$= 0.47$$

New head of water,

$$H_2 = 140 \text{ m}$$

$$\text{Overall efficiency is given by } \eta_o = \frac{P_1}{\rho \times g \times Q_1 \times H_1} = \frac{1000 \times P_1}{\rho \times g \times Q_1 \times H_1}$$

$$1000$$

∴

$$Q_1 = \frac{1000 \times P_1}{\eta_o \times \rho \times g \times H_1} = \frac{1000 \times 5150.25}{0.80 \times 1000 \times 9.81 \times 220} = 2.983 \text{ m}^3/\text{s}$$

Unit speed is given by equation (18.29),

$$N_u = \frac{N_1}{\sqrt{H_1}} = \frac{190}{\sqrt{220}} = 12.81 \text{ r.p.m. Ans.}$$

Unit discharge is given by equation (18.30),

$$Q_u = \frac{Q_1}{\sqrt{H_1}} = \frac{2.983}{\sqrt{220}} = 0.201 \text{ m}^3/\text{s. Ans.}$$

Unit power is given by equation (18.31),

$$P_u = \frac{P_1}{H_1^{3/2}} = \frac{5150.25}{220^{3/2}} = 1.578 \text{ kW. Ans.}$$

When the turbine is working under a new head of 140 m, the speed, discharge and power are given by equation (18.32) as

For speed,

$$\frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$$

∴

$$N_2 = \frac{N_1 \sqrt{H_2}}{\sqrt{H_1}} = N_1 \sqrt{\frac{H_2}{H_1}} = 190 \sqrt{\frac{140}{220}} = 151.56 \text{ r.p.m. Ans.}$$

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For discharge,

$$\frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$$

$$\therefore Q_2 = Q_1 \frac{\sqrt{H_2}}{\sqrt{H_1}} = Q_1 \sqrt{\frac{H_2}{H_1}} = 2.983 \sqrt{\frac{140}{220}} = 2.379 \text{ m}^3/\text{s. Ans.}$$

For power,

$$\frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$$

$$\therefore P_2 = P_1 \frac{H_2^{3/2}}{H_1^{3/2}} = P_1 \left( \frac{H_2}{H_1} \right)^{3/2} = 5150.25 \left( \frac{140}{220} \right)^{3/2} = 2614.48 \text{ kW. Ans.}$$

**Problem 18.45** A Pelton wheel is supplied with water under a head of 35 m at the rate of 40.5 kilo litre/min. The bucket deflects the jet through an angle of 160° and the mean bucket speed is 13 m/s. Calculate the power and hydraulic efficiency of the turbine.

**Solution.** Given :

Net head,  $H = 35 \text{ m}$

Discharge,  $Q = 40.5 \text{ kilo litre/min}$   
 $= 40.5 \times 1000 \text{ litre/min}$

$$= \frac{40.5 \times 1000}{1000} \text{ m}^3/\text{min} \quad \left( 1 \text{ litre} = \frac{1}{1000} \text{ m}^3 \right)$$

$$= 40.5 \text{ m}^3/\text{min} = \frac{40.5}{60} \text{ m}^3/\text{s} = 0.675 \text{ m}^3/\text{s}$$

Angle of deflection  $= 160^\circ$

$\therefore$  Angle,  $\phi = 180^\circ - 160^\circ = 20^\circ$

Mean bucket speed,  $u = u_1 = u_2 = 13 \text{ m/s}$

Calculate : (i) Power at runner and (ii) Hydraulic efficiency.

Taking the value of  $C_v = 1.0$

The velocity of jet,  $V_1 = C_v \sqrt{2gH} = 1 \times \sqrt{2 \times 9.81 \times 35} = 26.2 \text{ m/s}$

$$\therefore V_{r_1} = V_1 - u_1 = 26.2 - 13 = 13.2 \text{ m/s}$$

Also  $V_{w_1} = V_1 = 26.2 \text{ m/s}$

$$V_{r_2} = V_{r_1} = 13.2 \text{ m/s}$$

and

$$V_{w_2} = V_{r_2} \cos \phi - u_2 \\ = 13.2 \times \cos 20^\circ - 13 = 12.554 - 13 = -0.446 \text{ m/s}$$

(i) *Power at runner*

Using equation (18.9), we get the work done by the jet on the runner per second.

$$\therefore \text{Work done/s} = \rho \times a \times V_1 [V_{w_1} + V_{w_2}] \times u \\ = \rho \times Q \times [V_{w_1} + V_{w_2}] \times u \quad (\because a \times V_1 = Q)$$

$$= 1000 \times 0.675 [26.2 + (-0.446)] \times 13 \frac{\text{Nm}}{\text{s}} = 225991 \text{ W}$$

(∴ Nm/s = W)

$$= 225.991 \text{ kW}$$

$\therefore$  Power at runner **= 225.991 kW. Ans.**

(ii) *Hydraulic efficiency*

Input power in kW is given by equation (18.3A).

$$\begin{aligned} \therefore \text{Input power} &= \frac{\rho \times g \times Q \times H}{1000}, \quad \text{where } \rho = 1000 \text{ kg/m}^3 \\ &= \frac{1000 \times 9.81 \times 0.675 \times 35}{1000} = 231.761 \text{ kW} \\ \therefore \text{Hydraulic efficiency} &= \frac{\text{Power at runner}}{\text{Input power}} \\ &= \frac{225.991}{231.761} = 0.975 = 0.975 \times 100 = \mathbf{97.5\% \text{. Ans.}} \end{aligned}$$

### ► 18.13 CHARACTERISTIC CURVES OF HYDRAULIC TURBINES

Characteristic curves of a hydraulic turbine are the curves, with the help of which the exact behaviour and performance of the turbine under different working conditions, can be known. These curves are plotted from the results of the tests performed on the turbine under different working conditions.

The important parameters which are varied during a test on a turbine are :

- |                  |  |                      |
|------------------|--|----------------------|
| 1. Speed ( $N$ ) | 2. Head ( $H$ )                        | 3. Discharge ( $Q$ ) |
| 4. Power ( $P$ ) | 5. Overall efficiency ( $\eta_o$ ) and | 6. Gate opening.     |

Out of the above six parameters, three parameters namely speed ( $N$ ), head ( $H$ ) and discharge ( $Q$ ) are independent parameters.

Out of the three independent parameters, ( $N, H, Q$ ) one of the parameter is kept constant (say  $H$ ) and the variation of the other four parameters with respect to any one of the remaining two independent variables (say  $N$  and  $Q$ ) are plotted and various curves are obtained. These curves are called characteristic curves. The following are the important characteristic curves of a turbine.

1. Main Characteristic Curves or Constant Head Curves.
2. Operating Characteristic Curves or Constant Speed Curves.
3. Muschel Curves or Constant Efficiency Curves.

**18.13.1 Main Characteristic Curves or Constant Head Curves.** Main characteristic curves are obtained by maintaining a constant head and a constant gate opening (G.O.) on the turbine. The speed of the turbine is varied by changing load on the turbine. For each value of the speed, the corresponding values of the power ( $P$ ) and discharge ( $Q$ ) are obtained. Then the overall efficiency ( $\eta_o$ ) for each value of the speed is calculated. From these readings the values of unit speed ( $N_u$ ), unit power ( $P_u$ ) and unit discharge ( $Q_u$ ) are determined. Taking  $N_u$  as abscissa, the values of  $Q_u$ ,  $P_u$ ,  $P$  and  $\eta_o$  are plotted as shown in Figs. 18.35 and 18.36. By changing the gate opening, the values of  $Q_u$ ,  $P_u$  and  $\eta_o$  and  $N_u$  are determined and taking  $N_u$  as abscissa, the values of  $Q_u$ ,  $P_u$  and  $\eta_o$  are plotted. Fig. 18.35

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shows the main characteristic curves for Pelton wheel and Fig. 18.36 shows the main characteristic curves for reaction (Francis and Kaplan) turbines.

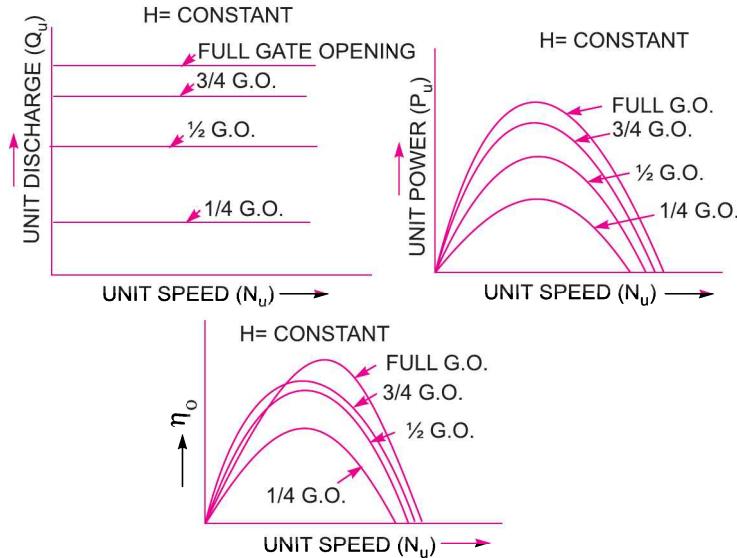


Fig. 18.35 Main characteristic curves for a Pelton wheel.

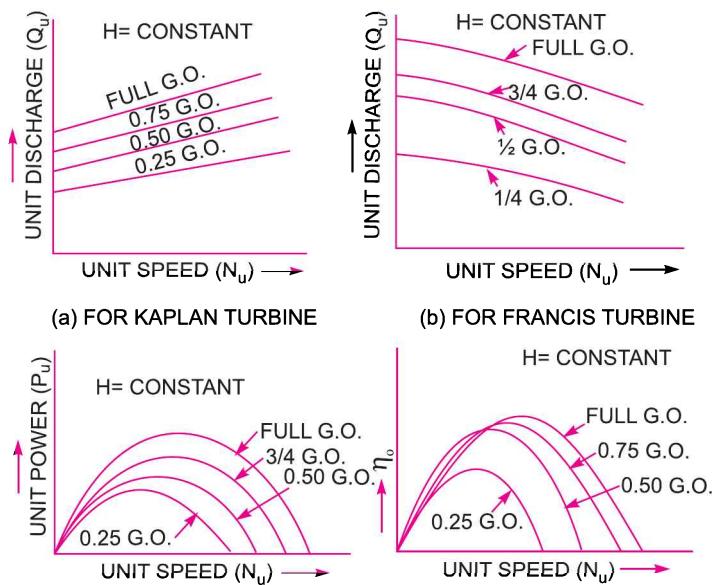


Fig. 18.36 Main characteristic curves for reaction turbine.

**18.13.2 Operating Characteristic Curves or Constant Speed Curves.** Operating characteristic curves are plotted when the speed on the turbine is constant. In case of turbines, the head is generally constant. As mentioned in Art. 18.13, there are three independent parameters namely  $N$ ,  $H$  and  $Q$ . For operating characteristics  $N$  and  $H$  are constant and hence the variation of power and efficiency with respect to discharge  $Q$  are plotted. The power curve for turbines shall not pass through

the origin because certain amount of discharge is needed to produce power to overcome initial friction. Hence the power and efficiency curves will be slightly away from the origin on the  $x$ -axis, as to overcome initial friction certain amount of discharge will be required. Fig. 18.37 shows the variation of power and efficiency with respect to discharge.

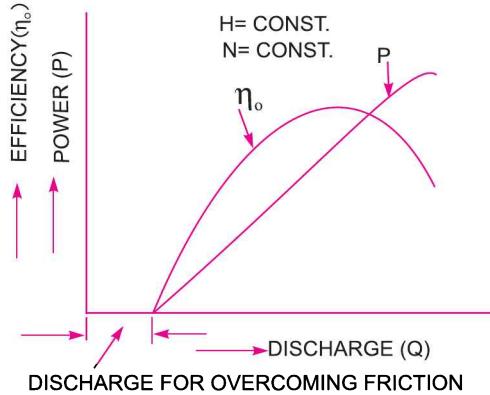


Fig. 18.37 Operating characteristic curves.

### 18.13.3 Constant Efficiency Curves or Muschel Curves or Iso-Efficiency Curves.

These curves are obtained from the speed vs. efficiency and speed vs. discharge curves for different gate openings. For a given efficiency from the  $N_u$  vs.  $\eta_o$  curves, there are two speeds. From the  $N_u$  vs.  $Q_u$  curves, corresponding to two values of speeds there are two values of discharge. Hence for a given efficiency there are two values of discharge for a particular gate opening. This means for a given efficiency there are two values of speeds and two values of the discharge for a given gate opening. If the efficiency is maximum there is only one value. These two values of speed and two values of discharge corresponding to a particular gate opening are plotted as shown in Fig. 18.38 (b). The procedure is repeated for different gate openings and the curves  $Q$  vs.  $N$  are plotted. The points having the same efficiencies are joined. The curves having same efficiency are called iso-efficiency curves. These curves are helpful for determining the zone of constant efficiency and for predicting the performance of the turbine at various efficiencies.

For plotting the iso-efficiency curves, horizontal lines representing the same efficiency are drawn on the  $\eta_o$  ~ speed curves. The points at which these lines cut the efficiency curves at various gate openings are transferred to the corresponding  $Q$  ~ speed curves. The points having the same efficiency are then joined by a smooth curves. These smooth curves represent the iso-efficiency curve.

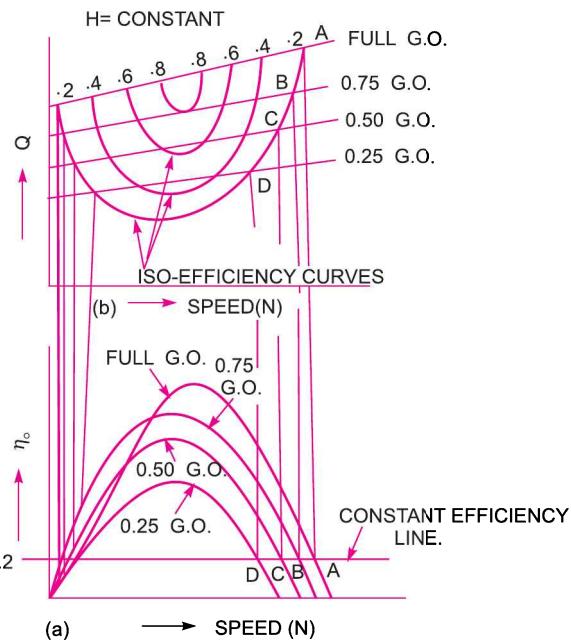


Fig. 18.38 Constant efficiency curve.

### ► 18.14 GOVERNING OF TURBINES

The governing of a turbine is defined as the operation by which the speed of the turbine is kept constant under all conditions of working. It is done automatically by means of a governor, which regulates the rate of flow through the turbines according to the changing load conditions on the turbine.

Governing of a turbine is necessary as a turbine is directly coupled to an electric generator, which is required to run at constant speed under all fluctuating load conditions. The frequency of power generation by a generator of constant number of pair of poles under all varying conditions should be constant. This is only possible when the speed of the generator, under all changing load condition, is constant. The speed of the generator will be constant, when the speed of the turbine (which is coupled to the generator) is constant.

When the load on the generator decreases, the speed of the generator increases beyond the normal speed (constant speed). Then the speed of the turbine also increases beyond the normal speed. If the turbine or the generator is to run at constant (normal) speed, the rate of flow of water to the turbine should be decreased till the speed becomes normal. This process by which the speed of the turbine (and hence of generator) is kept constant under varying condition of load is called governing.

#### **Governing of Pelton Turbine (Impulse Turbine)**

Governing of Pelton turbine is done by means of oil pressure governor, which consists of the following parts :

1. Oil sump.
2. Gear pump also called oil pump, which is driven by the power obtained from turbine shaft.
3. The Servomotor also called the relay cylinder.
4. The control valve or the distribution valve or relay valve.
5. The centrifugal governor or pendulum which is driven by belt or gear from the turbine shaft.
6. Pipes connecting the oil sump with the control valve and control valve with servomotor and
7. The spear rod or needle.

Fig. 18.39 shows the position of the piston in the relay cylinder, position of control or relay valve and fly-balls of the centrifugal governor, when the turbine is running at the normal speed.

When the load on the generator decreases, the speed of the generator increases. This increases the speed of the turbine beyond the normal speed. The centrifugal governor, which is connected to the turbine main shaft, will be rotating at an increased speed. Due to increase in the speed of the centrifugal governor, the fly-balls move upward due to the increased centrifugal force on them. Due to the upward movement of the fly-balls, the sleeve will also move upward. A horizontal lever, supported over a fulcrum, connects the sleeve and the piston rod of the control valve. As the sleeve moves up, the lever turns about the fulcrum and the piston rod of the control valve moves downward. This closes the valve  $V_1$  and opens the valve  $V_2$  as shown in Fig. 18.39.

The oil, pumped from the oil pump to the control valve or relay valve, under pressure will flow through the valve  $V_2$  to the servomotor (or relay cylinder) and will exert force on the face A of the piston of the relay cylinder. The piston along with piston rod and spear will move towards right. This will decrease the area of flow of water at the outlet of the nozzle. This decrease of area of flow will reduce the rate of flow of water to the turbine which consequently reduces the speed of the turbine. When the speed of the turbine becomes normal, the fly-balls, sleeve, lever and piston rod of control valve come to its normal position as shown in Fig. 18.39.

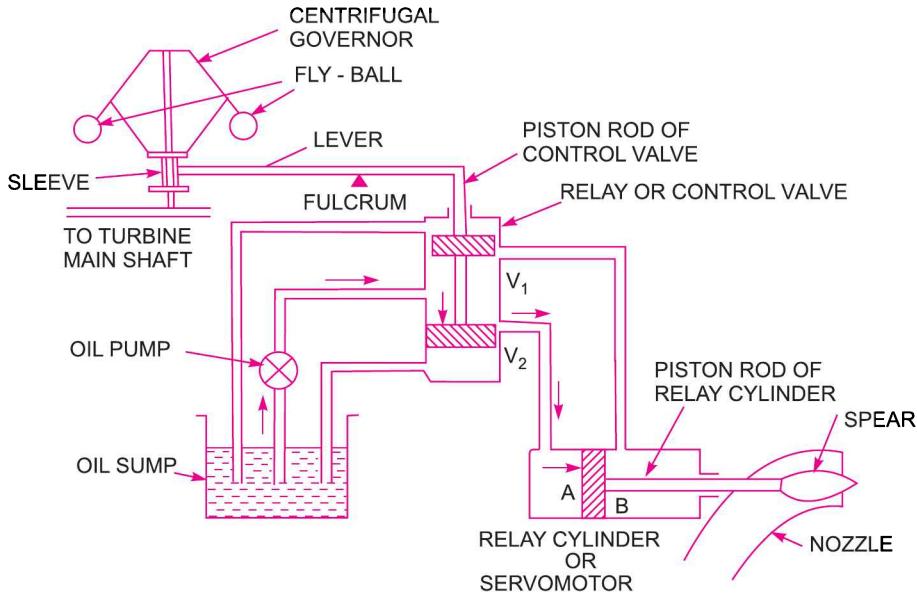


Fig. 18.39. Governing of Pelton turbine.

When the load on the generator increases, the speed of the generator and hence of the turbine decreases. The speed of the centrifugal governor also decreases and hence centrifugal force acting on the fly-balls also reduces. This brings the fly-balls in the downward direction. Due to this, the sleeve moves downward and the lever turns about the fulcrum, moving the piston rod of the control valve in the upward direction. This closes the valve  $V_2$  and opens the valve  $V_1$ . The oil under pressure from the control valve, will move through valve  $V_1$  to the servomotor and will exert a force on the face  $B$  of the piston. This will move the piston along with the piston rod and spear towards left, increasing the area of flow of water at the outlet of the nozzle. This will increase the rate of flow of water to the turbine and consequently, the speed of the turbine will also increase, till the speed of the turbine becomes normal.

### HIGHLIGHTS

1. The hydraulic machines, which convert the hydraulic energy into mechanical energy, are called turbines.
2. Gross head is the vertical difference between the head race and tail race levels. Net head or effective head is the head, available at the inlet of the turbine. It is given by

$$H = H_g - h_f$$

where  $H_g$  = Gross head, and

$h_f$  = Loss of head due to friction in penstocks

$$= \frac{4f \times L \times V^2}{D \times 2g}$$

where  $D$  = Dia. of penstock.

3. The efficiencies of a turbine are : (i) Hydraulic efficiency,  $\eta_h$ , (ii) Mechanical efficiency,  $\eta_m$ , and (iii) Overall efficiency,  $\eta_o$ .
4. Hydraulic efficiency,  $\eta_h$  is given by

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$$\begin{aligned}\eta_h &= \frac{\text{Power delivered to runner}}{\text{Power supplied at inlet}} = \frac{\text{R.P.}}{\text{W.P.}} \\ &= \frac{W}{g} \frac{(V_{w_1} u_1 \pm V_{w_2} u_2)}{1000} \Big/ \frac{(W \times H)}{1000} = \frac{(V_{w_1} u_1 \pm V_{w_2} u_2)}{gH}\end{aligned}$$

where W.P. = Water power,

R.P. = Runner power *i.e.*, power available at the runner of the turbine,

S.P. = Shaft power *i.e.*, power at the shaft of the turbine.

5. Mechanical efficiency,  $\eta_m$  is given by  $\eta_m = \frac{\text{S.P.}}{\text{R.P.}}$
6. Overall efficiency,  $\eta_o$  is given by  $\eta_o = \frac{\text{S.P.}}{\text{W.P.}} = \eta_m \times \eta_h$ .
7. If at the inlet of a turbine, the energy available is only kinetic energy, the turbine is known as impulse turbine. But if at the inlet of the turbine, the energy available is kinetic energy as well as pressure energy, the turbine is called reaction turbine.
8. Pelton wheel (or turbine) is a tangential flow impulse turbine and is used for high head. In this turbine,

$$V_1 = C_v \sqrt{2gH}, u_1 = u_2 = u.$$

9. For the maximum efficiency of Pelton wheel the condition is  $u = \frac{V_1}{2}$   
Max. efficiency is given by  $\eta_{\max} = \frac{(1 + \cos\phi)}{2}$ , where  $\phi$  = Vane angle at outlet.
10. The jet ratio ( $m$ ) is defined as ratio of the pitch diameter ( $D$ ) of the Pelton wheel to the diameter of the jet ( $d$ ) or  $m = \frac{D}{d}$ .
11. Francis turbine is an inward radial flow reaction turbine having discharge radial at outlet, which means the angle made by absolute velocity at outlet is  $90^\circ$ , *i.e.*,  $\beta = 90^\circ$ . Then  $V_{w_2} = 0$  and work done by water on the runner per second per unit weight of water becomes as  $= \frac{1}{g} V_{w_1} \times u_1$ .
12. Speed ratio is the ratio of the velocity of wheel at inlet to the velocity given by  $\sqrt{2gH}$  whereas the flow ratio is the ratio of velocity of flow at inlet to the velocity given by  $\sqrt{2gH}$ .
13. Kaplan turbine is an axial flow reaction turbine in which the vanes on the hub are adjustable. The peripheral velocity at inlet and outlet are equal, *i.e.*,  $u_1 = u_2$ .
14. The discharge through a turbine is given by

$$\begin{aligned}Q &= \frac{\pi}{4} d^2 \times \sqrt{2gH} && \dots \text{For a Pelton wheel} \\ &= \pi D_1 B_1 \times V_f && \dots \text{For a Francis turbine} \\ &= \frac{\pi}{4} (D_o^2 - D_b^2) \times V_f && \dots \text{For a Kaplan turbine.}\end{aligned}$$

15. Draft-tube is a pipe of gradually increasing area used for discharging water from the exit of a reaction turbine. They may be conical or simple elbow type. The efficiency of the draft-tube is given by

$$\eta_d = \frac{\left( \frac{V_1}{2g} - \frac{V_2^2}{2g} \right) - h_f}{\left( \frac{V_1^2}{2g} \right)}$$

where  $V_1$  = Velocity of water at the inlet of the draft-tube,  
 $V_2$  = Velocity of water at the outlet of the draft-tube,  
 $h_f$  = Loss of head in draft-tube.

16. Specific speed of a turbine is defined as the speed at which a turbine runs when it is working under a unit head and develops unit power. The expression for specific speed ( $N_s$ ) is given as

$$N_s = \frac{N\sqrt{P}}{H^{5/4}}$$

where  $P$  = shaft power in kW,  $H$  = Net head on the turbine.

17. Unit quantities are the quantities (like speed, discharge, power, etc.) which are obtained when the head on the turbine is unity. They are unit speed ( $N_u$ ), unit power ( $P_u$ ) and unit discharge ( $Q_u$ ). They are given as

$$N_u = \frac{N}{\sqrt{H}}, Q_u = \frac{Q}{\sqrt{H}}, P_u = \frac{P}{H^{3/2}} .$$

18. The important characteristic curves of a turbine are :

- (a) Main characteristic curves or Constant head curves.
- (b) Operating characteristic curves or Constant speed curves, and
- (c) Muschel curves or Constant efficiency curves.

19. Governing of a turbine is defined as the operation by which the speed of the turbine is kept constant under all conditions of working. It is done by oil pressure governor.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define the terms : Hydraulic machines, Turbines and Pumps.
  2. Differentiate between the turbines and pumps.
  3. (a) What do you mean by gross head, net head and efficiency of turbine ? Explain the different types of the efficiency of a turbine.  
 (b) Explain clearly the following terms as they are applied to a Pelton wheel :  
 (i) Gross head ; (ii) Net head.
  4. How will you classify the turbines ?
  5. Differentiate between : (a) The impulse and reaction turbines, (b) Radial and axial flow turbines, (c) Inward and outward radial flow turbine, and (d) Kaplan and propeller turbines.
  6. Obtain an expression for the work done per second by water on the runner of a Pelton wheel. Hence derive an expression for maximum efficiency of the Pelton wheel giving the relationship between the jet speed and bucket speed.
- Draw inlet and outlet velocity triangles for a Pelton turbine and indicate the direction of various velocities.
7. Prove that the work done per second per unit weight of water in a reaction turbine is given as

$$= \frac{1}{g} (V_{w_1} u_1 \pm V_{w_2} u_2)$$

where  $V_{w_1}$  and  $V_{w_2}$  = Velocities of whirl at inlet and outlet,

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$u_1$  and  $u_2$  = Peripheral velocities at inlet and outlet.

8. Define the terms : speed ratio, flow ratio and jet ratio.
9. (a) What is a draft-tube ? Why is it used in a reaction turbine ? Describe with sketch two different types of draft-tubes.  
(b) What are the uses of a draft-tube ? Describe with neat sketches different types of draft-tubes.  
*(J.N.T.U., Hyderabad S 2002).*
10. What is the basis of selection of a turbine at a particular place ?
11. Define the specific speed of a turbine ? Derive an expression for the specific speed. What is the significance of the specific speed?
12. What are unit quantities ? Define the unit quantities for a turbine. Why are they important ?
13. Obtain an expression for unit speed, unit discharge and unit power for a turbine.
14. What do you understand by the characteristic curves of a turbine ? Name the important types of characteristic curves.
15. Define the term 'Governing of a turbine'. Describe with a neat sketch the working of an oil pressure governor.
16. Give the range of specific speed values of the Kaplan, Francis turbines and Pelton wheels.  
What factors decide whether Kaplan, Francis, or a Pelton type turbine would be used in a hydroelectric project ?
17. (a) Draw neat sketches of the Pelton turbine and Francis Turbine.  
(b) Describe briefly the function of various main components of Pelton turbine with neat sketches.
18. What is cavitation ? How can it be avoided in reaction turbine ?
19. Define the terms 'unit power', 'unit speed' and 'unit discharge' with reference to a hydraulic turbine. Also derive expressions for these terms.
20. (a) Define specific speed of a turbine and derive an expression for the same. Show that Pelton turbine is a low specific speed turbine.  
(b) What is specific speed ? State its significance in the study of hydraulic machines.
21. (a) By means of a neat sketch explain the governing mechanism of Francis Turbine.  
(b) Explain the difference between Kaplan turbine and propeller turbine.
22. Define and explain hydraulic efficiency, mechanical efficiency and overall efficiency of a turbine.
23. Define the terms : specific speed of a turbine, unit speed, unit power and unit rate of flow of a turbine. Derive the expressions for specific speed and unit speed.
24. (a) What is meant by the speed ratio of a Pelton wheel ?  
(b) What is a draft-tube ? What are its functions ?  
(c) Differentiate between an inward and an outward flow reaction turbine.

### (B) NUMERICAL PROBLEMS

1. A Pelton wheel has a mean bucket speed of 35 m/s with a jet of water flowing at the rate of 1 m<sup>3</sup>/s under a head of 270 m. The buckets deflect the jet through an angle of 170° . Calculate the power delivered to the runner and the hydraulic efficiency of the turbine. Assume co-efficient of velocity as 0.98.  
**[Ans. 2523.8 kW, 95.3%]**
2. A Pelton wheel is to be designed for the following specifications. Power = 735.75 kW, S.P. Head = 200 m, Speed = 800 r.p.m. ,  $\eta_o$  = 0.86 and jet diameter is not to exceed one-tenth the wheel diameter.  
Determine : (i) Wheel diameter, (ii) The number of jets required, and (iii) Diameter of the jet. Take  $C_v$  = 0.98 and speed ratio = 0.45.  
**[Ans. (i) 0.673 m, (ii) 2, (iii) 67.3 mm]**

3. A Pelton wheel is having a mean bucket diameter of 0.8 m and is running at 1000 r.p.m. The net head on the Pelton wheel is 400 m. If the side clearance angle is  $15^\circ$  and discharge through nozzle is 150 litres/s, find : (i) Power available at the nozzle, and (ii) Hydraulic efficiency of the turbine.  
**[Ans. (i) 588.6 kW, (ii) 98%]**
4. Two jets strike at buckets of a Pelton wheel, which is having shaft power as 14,715 kW. The diameter of each jet is given as 150 mm. If the net head on the turbine is 500 m, find the overall efficiency of the turbine. Take  $C_v = 1.0$ .  
**[Ans. 85.7%]**
5. The following data is related to the Pelton wheel :  
 Head at the base of the nozzle = 110 m,  
 Diameter of the jet = 7.5 cm,  
 Discharge of the nozzle = 200 litres/s,  
 Shaft power = 191.295 kW  
 Power absorbed in mechanical resistance = 3.675 kW.  
 Determine : (i) Power lost in nozzle and, (ii) Power lost due to hydraulic resistance in the runner.  
**[Ans. (i) 10.874 kW, (ii) 9.97 kW]**
6. Design a Pelton wheel for a head of 80 m and speed 300 r.p.m. The Pelton wheel develops 103 kW S.P. Take  $C_v = 0.98$ , speed ratio = 0.45 and overall efficiency = 0.80.  
**[Ans.  $D = 1.135$  m,  $d = 72.6$  mm, size =  $36.3 \times 8.7$ ,  $Z = 23$ ]**
7. An inward flow reaction turbine has external and internal diameters as 1.2 m and 0.6 m respectively. The velocity of flow through the runner is constant and is equal to 1.8 m/s. Determine : (i) Discharge through the runner, and (ii) Width at outlet if the width at inlet = 200 mm.  
**[Ans. (i)  $1.357 \text{ m}^3/\text{s}$ , (ii) 400 mm]**
8. A reaction turbine works at 500 r.p.m. under a head of 100 m. The diameter of turbine at inlet is 100 cm and flow area is  $0.35 \text{ m}^2$ . The angles made by absolute and relative velocities at inlet are  $15^\circ$  and  $60^\circ$  respectively with the tangential velocity. Determine :  
 (i) The volume flow rate, (ii) The power developed, and (iii) Efficiency. Assume whirl at outlet to be zero.  
**[Ans. (i)  $2.905 \text{ m}^3/\text{s}$ , (ii) 2355.35 kW, (iii) 82.6% ]**
9. An inward flow reaction turbine has an external diameter of 1 m and its breadth at inlet is 200 mm. If the velocity of flow at inlet is 1.5 m/s, find the mass of water passing through the turbine per second. Assume 15% of the area of flow is blocked by blade thickness. If the speed of the runner is 200 r.p.m. and guide blades make an angle of  $15^\circ$  to the wheel tangent, draw the inlet velocity triangle and find : (i) The runner vane angle at inlet (ii) Velocity of wheel at inlet, (iii) The absolute velocity of water leaving the guide vanes, and (iv) The relative velocity of water entering the runner blade.  
**[Ans. 1602.2 kg/s, (i) 76.19, (ii) 10.47 m/s, (iii) 11.59 m/s, (iv) 3.087 m/s]**
10. An outward flow reaction turbine has internal and external diameters of the runner as 0.5 m and 1.0 m respectively. The guide blade angle is  $15^\circ$  and velocity of flow through the runner is constant and equal to 3 m/s. If the speed of the turbine is 250 r.p.m., head on turbine is 10 m and discharge at outlet is radial, determine : (i) The runner vane angles at inlet and outlet, (ii) Work done by the water on the runner per second per unit weight of water striking per second and (iii) Hydraulic efficiency.  
**[Ans. (i)  $32^\circ 48'$ ,  $12^\circ 55'$ , (ii) 7.47 m, (iii) 7.47%]**
11. A Francis turbine with an overall efficiency of 70% is required to produce 147.15 kW. It is working under a head of 8 m. The peripheral velocity =  $0.30 \sqrt{2gH}$  and the radial velocity of flow at inlet is  $0.96 \sqrt{2gH}$ . The wheel runs at 200 r.p.m. and the hydraulic losses in the turbine are 20% of the available energy. Assume radial discharge, determine : (i) The guide blade angle, (ii) The wheel vane angle at inlet, (iii) Diameter of the wheel at inlet, and (iv) Width of wheel at inlet.  
**[Ans. (i)  $35^\circ 45'$ , (ii)  $42^\circ 54'$ , (iii) 35.9 cm, (iv) 19.75 cm]**
12. The following data is given for a Francis turbine : Net head = 70 m, speed = 600 r.p.m., shaft power = 367.875 kW,  $\eta_o = 85\%$ ,  $\eta_h = 95\%$ , flow ratio = 0.25, breadth ratio = 0.1, outer diameter of the runner =  $2 \times$  inner diameter of runner. The thickness of vanes occupy 10% of the circumferential area of the

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runner. Velocity of flow is constant at inlet and outlet and discharge is radial at outlet. Determine : (i) Guide blade angle, (ii) Runner vane angles at inlet and outlet, (iii) Diameters of runner at inlet and outlet, and (iv) Width of wheel at inlet.

[Ans. (i)  $12^\circ 20'$ , (ii)  $18^\circ 57'$ ,  $50^\circ 17'$ , (iii)  $49.245$  m, (iv)  $49$  mm]

13. A Kaplan turbine working under a head of  $15$  m develops  $7357.5$  kW shaft power. The outer diameter of the runner is  $4$  m and hub diameter is  $2$  m. The guide blade angle at the extreme edge of the runner is  $30^\circ$ . The hydraulic and overall efficiencies of the turbine are  $90\%$  and  $85\%$  respectively. If the velocity of whirl is zero at outlet, determine : (i) runner vane angles at inlet and outlet at the extreme edge of the runner and (ii) speed of the turbine.

[Ans. (i)  $103^\circ 10'$ ,  $26^\circ 58.5'$ , (ii)  $58.5$ ]

14. A Kaplan turbine runner is to be designed to develop  $7357.5$  kW S.P. The net available head is  $10$  m. Assume that the speed ratio is  $1.8$  and flow ratio is  $0.6$ . If the overall efficiency is  $70\%$  and diameter of the boss is  $0.4$  times the diameter of the runner, find the diameter of the runner, its speed and specific speed.

[Ans.  $4.39$  m,  $109.63$  r.p.m.,  $528.82$ ]

15. A conical draft-tube having inlet and outlet diameters  $0.8$  m and  $1.2$  m discharges water at outlet with a velocity of  $3$  m/s. The total length of the draft-tube is  $8$  m and  $2$  m of the length of draft-tube is immersed in water. If the atmospheric pressure head is  $10.3$  m of water and loss of head due to friction in the draft-tube is equal to  $0.25$  times the velocity head at outlet of the tube, find : (i) Pressure head at inlet, and (ii) Efficiency of the draft-tube.

[Ans. (i)  $2.551$  m (abs.), (ii)  $75.3\%$ ]

16. A turbine is to operate under a head of  $30$  m at  $300$  r.p.m. The discharge is  $10 \text{ m}^3/\text{s}$ . If the efficiency is  $90\%$ , determine : (i) specific speed of the machine, (ii) power generated, and (iii) types of the turbine.

[Ans. (i)  $219.9$ , (ii)  $2648.7$  kW (iii) Francis]

17. A turbine develops  $7357.5$  kW S.P. when running at  $200$  r.p.m. The head on the turbine is  $40$  m. If the head on the turbine is reduced to  $25$  m, determine the speed and power developed by the turbine.

[Ans.  $158.11$ ,  $3635.34$  kW]

18. A Pelton wheel is revolving at a speed of  $200$  r.p.m. and develops  $5886$  kW S.P. when working under a head of  $200$  m with an overall efficiency of  $80\%$ . Determine unit speed, unit discharge and unit power. The speed ratio for the turbine is given as  $0.48$ . Find the speed, discharge and power when this turbine is working under a head of  $150$  m.

[Ans.  $14.14$ ,  $0.265 \text{ m}^3/\text{s}$ ,  $2.08$  kW and  $173.2$  r.p.m.,  $3.247 \text{ m}^3/\text{s}$ ,  $3823$  kW]

19. A Kaplan turbine working under a head of  $29$  m develops  $1287.5$  kW S.P. If the speed ratio is equal to  $2.1$ , flow ratio =  $0.62$ , diameter of boss =  $0.34$  times the diameter of the runner and overall efficiency of the turbine =  $89\%$ , find the diameter of the runner and the speed of turbine.

[Ans.  $0.705$  m,  $1162.3$ ]

20. A Kaplan turbine working under a head of  $25$  m develops  $16000$  kW shaft power. The outer diameter of the runner is  $4$  m and hub diameter is  $2$  m. The guide blade angle is  $35^\circ$ . The hydraulic and overall efficiency are  $90\%$  and  $85\%$  respectively. If the velocity of whirl is zero at outlet, determine runner vane angles at inlet and outlet, and speed of turbine.

[Ans.  $35^\circ 27'$ ,  $88^\circ 36'$ ,  $9.236$ ]

21. A Kaplan turbine develops  $9000$  kW under a net head of  $7.5$  m. Mechanical efficiency of the wheel is  $86\%$ . The speed ratio based on the outer diameter is  $2.2$  and the flow ratio is  $0.66$ . Diameter of the boss is  $0.35$  times the external diameter of the wheel. Determine the diameter of the runner and the specific speed of the runner.

(J.N.T.U., Hyderabad, S 2002)

**Hint.** Given : S.P. =  $9000$  kW;  $H = 7.5$  m ;  $\eta_m = 86\% = 0.86$ ; Speed ratio =  $2.2$ ;

flow ratio =  $0.66$  ; Dia. of boss =  $0.35 \times$  External dia. of wheel i.e.,  $D_b = 0.35 D_o$ .

$$\frac{u_1}{\sqrt{2gH}} = 2.2 \quad \therefore \quad u_1 = 2.2 \times \sqrt{2gH} = 2.2 \times \sqrt{2 \times 9.81 \times 7.5} = 26.68 \text{ m/s}$$

$$\frac{V_{f_i}}{\sqrt{2gH}} = 0.66 \quad \therefore \quad V_{f_i} = 0.66 \times \sqrt{2gH} = 0.66 \times \sqrt{2 \times 9.81 \times 7.5} = 8 \text{ m/s}$$

**Note.** In this question either data is incomplete or instead of mechanical efficiency it should be overall efficiency. The question is solved taking the given efficiency as overall efficiency.

$$\text{Now } \eta_o = 0.86 \text{ But } \eta_o = \frac{\text{S.P.}}{\text{W.P.}} \therefore \text{W.P.} = \frac{\text{S.P.}}{\eta_o} = \frac{9000}{0.86} \text{ kW}$$

$$\text{But } \text{W.P.} = \frac{\rho \times Q \times g \times H}{1000} \text{ kW} = \frac{1000 \times Q \times 9.81 \times 7.5}{1000} = Q \times 9.81 \times 7.5 \text{ kW}$$

$$\therefore Q \times 9.81 \times 7.5 = \frac{9000}{0.86} \text{ or } Q = \frac{9000}{0.86} \times \frac{1}{9.81 \times 7.5} = 142.237 \text{ m}^3/\text{s}$$

$$\text{But } Q = \frac{\pi}{4} [D_o^2 - D_b^2] \times V_{f_1} \therefore 142.237 = \frac{\pi}{4} [D_o^2 - D_b^2] \times V_{f_1} = \frac{\pi}{4} [D_o^2 - (0.35D_o)^2] \times 8$$

$$\text{or } 142.237 = \frac{\pi}{4} \times 0.8775 D_o^2 \times 8 \text{ or } D_o = \sqrt{\frac{142.237 \times 4}{\pi \times 0.8775 \times 8}} = 5.079 \text{ m} \approx 5 \text{ m.}$$

Specific speed of turbine,  $N_s = \frac{N \sqrt{\text{S.P.}}}{H^{5/4}}$ , where  $N$  is obtained from  $u_1$

$$\therefore u_1 = \frac{\pi D_o N}{60} \text{ or } 26.68 = \frac{\pi \times 5 \times N}{60} \text{ or } N = \frac{26.68 \times 60}{\pi \times 5} = 101.91 \text{ r.p.m.}$$

$$\therefore N_s = \frac{101.91 \times \sqrt{9000}}{7.5^{1.25}} = 778.95 ]$$



# 19

## CHAPTER



## CENTRIFUGAL PUMPS

### ► 19.1 INTRODUCTION

The hydraulic machines which convert the mechanical energy into hydraulic energy are called pumps. The hydraulic energy is in the form of pressure energy. If the mechanical energy is converted into pressure energy by means of centrifugal force acting on the fluid, the hydraulic machine is called centrifugal pump.

The centrifugal pump acts as a reverse of an inward radial flow reaction turbine. This means that the flow in centrifugal pumps is in the radial outward directions. The centrifugal pump works on the principle of forced vortex flow which means that when a certain mass of liquid is rotated by an external torque, the rise in pressure head of the rotating liquid takes place. The rise in pressure head at any point of the rotating liquid is proportional to the square of tangential velocity of the liquid at that point (*i.e.*, rise in pressure head  $= \frac{V^2}{2g}$  or  $\frac{\omega^2 r^2}{2g}$ ). Thus at the outlet of the impeller, where radius is more, the rise in pressure head will be more and the liquid will be discharged at the outlet with a high pressure head. Due to this high pressure head, the liquid can be lifted to a high level.

### ► 19.2 MAIN PARTS OF A CENTRIFUGAL PUMP

The following are the main parts of a centrifugal pump :

1. Impeller.
2. Casing.
3. Suction pipe with a foot valve and a strainer.
4. Delivery pipe.

All the main parts of the centrifugal pump are shown in Fig. 19.1.

**1. Impeller.** The rotating part of a centrifugal pump is called ‘impeller’. It consists of a series of backward curved vanes. The impeller is mounted on a shaft which is connected to the shaft of an electric motor.

**2. Casing.** The casing of a centrifugal pump is similar to the casing of a reaction turbine. It is an air-tight passage surrounding the impeller and is designed in such a way that the kinetic energy of the water discharged at the outlet of the impeller is converted into pressure energy before the water leaves the casing and enters the delivery pipe. The following three types of the casings are commonly adopted :

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- (a) Volute casing as shown in Fig. 19.1.
- (b) Vortex casing as shown in Fig. 19.2 (a).
- (c) Casing with guide blades as shown in Fig. 19.2 (b).

**(a) Volute Casing.** Fig 19.1 shows the volute casing, which surrounds the impeller. It is of spiral type in which area of flow increases gradually. The increase in area of flow decreases the velocity of flow. The decrease in velocity increases the pressure of the water flowing through the casing. It has been observed that in case of volute casing, the efficiency of the pump increases slightly as a large amount of energy is lost due to the formation of eddies in this type of casing.

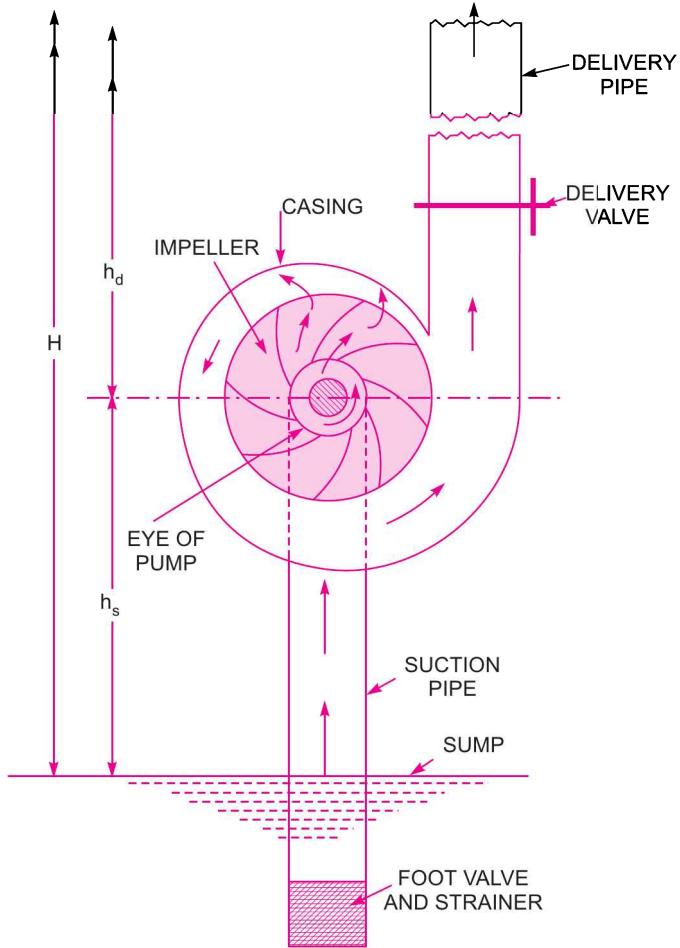


Fig. 19.1 Main parts of a centrifugal pump.

**(b) Vortex Casing.** If a circular chamber is introduced between the casing and the impeller as shown in Fig. 19.2 (a), the casing is known as Vortex Casing. By introducing the circular chamber, the loss of energy due to the formation of eddies is reduced to a considerable extent. Thus the efficiency of the pump is more than the efficiency when only volute casing is provided.

**(c) Casing with Guide Blades.** This casing is shown in Fig. 19.2 (b) in which the impeller is surrounded by a series of guide blades mounted on a ring which is known as diffuser. The guide vanes are designed in such a way that the water from the impeller enters the guide vanes without stock.

Also the area of the guide vanes increases, thus reducing the velocity of flow through guide vanes and consequently increasing the pressure of water. The water from the guide vanes then passes through the surrounding casing which is in most of the cases concentric with the impeller as shown in Fig. 19.2 (b).

**3. Suction Pipe with a Foot valve and a Strainer.** A pipe whose one end is connected to the inlet of the pump and other end dips into water in a sump is known as suction pipe. A foot valve which is a non-return valve or one-way type of valve is fitted at the lower end of the suction pipe. The foot valve opens only in the upward direction. A strainer is also fitted at the lower end of the suction pipe.

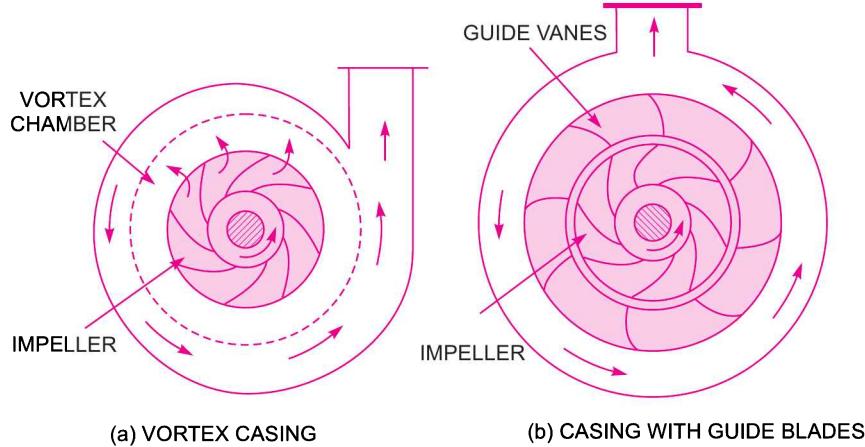


Fig. 19.2 *Different types of casing.*

**4. Delivery Pipe.** A pipe whose one end is connected to the outlet of the pump and other end delivers the water at a required height is known as delivery pipe.

### ► 19.3 WORK DONE BY THE CENTRIFUGAL PUMP (OR BY IMPELLER) ON WATER

In case of the centrifugal pump, work is done by the impeller on the water. The expression for the work done by the impeller on the water is obtained by drawing velocity triangles at inlet and outlet of the impeller in the same way as for a turbine. The water enters the impeller radially at inlet for best efficiency of the pump, which means the absolute velocity of water at inlet makes an angle of  $90^\circ$  with the direction of motion of the impeller at inlet. Hence angle  $\alpha = 90^\circ$  and  $V_{w1} = 0$ . For drawing the velocity triangles, the same notations are used as that for turbines. Fig. 19.3 shows the velocity triangles at the inlet and outlet tips of the vanes fixed to an impeller.

Let  $N$  = Speed of the impeller in r.p.m.,

$D_1$  = Diameter of impeller at inlet,

$u_1$  = Tangential velocity of impeller at inlet,

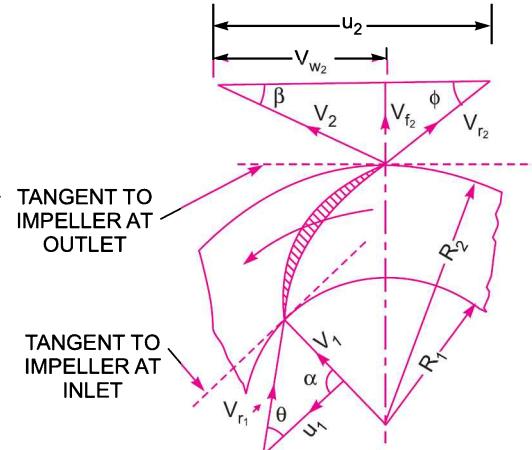


Fig. 19.3 *Velocity triangles at inlet and outlet.*

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$$= \frac{\pi D_1 N}{60}$$

$D_2$  = Diameter of impeller at outlet,

$u_2$  = Tangential velocity of impeller at outlet

$$= \frac{\pi D_2 N}{60}$$

$V_1$  = Absolute velocity of water at inlet,

$V_{r_1}$  = Relative velocity of water at inlet,

$\alpha$  = Angle made by absolute velocity ( $V_1$ ) at inlet with the direction of motion of vane,

$\theta$  = Angle made by relative velocity ( $V_{r_1}$ ) at inlet with the direction of motion of vane, and  $V_2$ ,

$V_{r_2}$ ,  $\beta$  and  $\phi$  are the corresponding values at outlet.

As the water enters the impeller radially which means the absolute velocity of water at inlet is in the radial direction and hence angle  $\alpha = 90^\circ$  and  $V_{w_1} = 0$ .

A centrifugal pump is the reverse of a radially inward flow reaction turbine. But in case of a radially inward flow reaction turbine, the work done by the water on the runner per second per unit weight of the water striking per second is given by equation (18.19) as

$$= \frac{1}{g} [V_{w_1} u_1 - V_{w_2} u_2]$$

∴ Work done by the impeller on the water per second per unit weight of water striking per second

$$= - [\text{Work done in case of turbine}]$$

$$= - \left[ \frac{1}{g} (V_{w_1} u_1 - V_{w_2} u_2) \right] = \frac{1}{g} [V_{w_2} u_2 - V_{w_1} u_1]$$

$$= \frac{1}{g} V_{w_2} u_2 \quad (\because V_{w_1} = 0 \text{ here}) \dots(19.1)$$

Work done by impeller on water per second

$$= \frac{W}{g} \cdot V_{w_2} u_2 \dots(19.2)$$

where  $W$  = Weight of water =  $\rho \times g \times Q$

where  $Q$  = Volume of water

and

$$\begin{aligned} Q &= \text{Area} \times \text{Velocity of flow} = \pi D_1 B_1 \times V_{f_1} \\ &= \pi D_2 B_2 \times V_{f_2} \end{aligned} \dots(19.2A)$$

where  $B_1$  and  $B_2$  are width of impeller at inlet and outlet and  $V_{f_1}$  and  $V_{f_2}$  are velocities of flow at inlet and outlet.

Equation (19.1) gives the head imparted to the water by the impeller or energy given by impeller to water per unit weight per second.

### ► 19.4 DEFINITIONS OF HEADS AND EFFICIENCIES OF A CENTRIFUGAL PUMP

**1. Suction Head ( $h_s$ ).** It is the vertical height of the centre line of the centrifugal pump above the water surface in the tank or pump from which water is to be lifted as shown in Fig. 19.1. This height is also called suction lift and is denoted by ' $h_s$ '.

**2. Delivery Head ( $h_d$ ).** The vertical distance between the centre line of the pump and the water surface in the tank to which water is delivered is known as delivery head. This is denoted by ' $h_d$ '.

**3. Static Head ( $H_s$ ).** The sum of suction head and delivery head is known as static head. This is represented by ' $H_s$ ' and is written as

$$H_s = h_s + h_d \quad \dots(19.3)$$

**4. Manometric Head ( $H_m$ ).** The manometric head is defined as the head against which a centrifugal pump has to work. It is denoted by ' $H_m$ '. It is given by the following expressions :

$$(a) \quad H_m = \text{Head imparted by the impeller to the water} - \text{Loss of head in the pump}$$

$$= \frac{V_{w_2} u_2}{g} - \text{Loss of head in impeller and casing} \quad \dots(19.4)$$

$$= \frac{V_{w_2} u_2}{g} \quad \dots\text{if loss of pump is zero} \quad \dots(19.5)$$

$$(b) \quad H_m = \text{Total head at outlet of the pump} - \text{Total head at the inlet of the pump}$$

$$= \left( \frac{P_o}{\rho g} + \frac{V_o^2}{2g} + Z_o \right) - \left( \frac{P_i}{\rho g} + \frac{V_i^2}{2g} + Z_i \right) \quad \dots(19.6)$$

where  $\frac{P_o}{\rho g}$  = Pressure head at outlet of the pump =  $h_d$

$\frac{V_o^2}{2g}$  = Velocity head at outlet of the pump

= Velocity head in delivery pipe =  $\frac{V_d^2}{2g}$

$Z_o$  = Vertical height of the outlet of the pump from datum line, and

$\frac{P_i}{\rho g}, \frac{V_i^2}{2g}, Z_i$  = Corresponding values of pressure head, velocity head and datum head at the inlet of the pump,

i.e.,  $h_s, \frac{V_s^2}{2g}$  and  $Z_s$  respectively.

$$(c) \quad H_m = h_s + h_d + h_{f_s} + h_{f_d} + \frac{V_d^2}{2g} \quad \dots(19.7)$$

where  $h_s$  = Suction head,  $h_d$  = Delivery head,

$h_{f_s}$  = Frictional head loss in suction pipe,  $h_{f_d}$  = Frictional head loss in delivery pipe, and

$V_d$  = Velocity of water in delivery pipe.

**5. Efficiencies of a Centrifugal Pump.** In case of a centrifugal pump, the power is transmitted from the shaft of the electric motor to the shaft of the pump and then to the impeller. From the impeller, the power is given to the water. Thus power is decreasing from the shaft of the pump to the impeller and then to the water. The following are the important efficiencies of a centrifugal pump :

(a) Manometric efficiency,  $\eta_{man}$  (b) Mechanical efficiency,  $\eta_m$  and

(c) Overall efficiency,  $\eta_o$ .

(a) **Manometric Efficiency ( $\eta_{man}$ ).** The ratio of the manometric head to the head imparted by the impeller to the water is known as manometric efficiency. Mathematically, it is written as

$$\eta_{man} = \frac{\text{Manometric head}}{\text{Head imparted by impeller to water}}$$

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$$= \frac{H_m}{\left( \frac{V_{w_2} u_2}{g} \right)} = \frac{g H_m}{V_{w_2} u_2} \quad \dots(19.8)$$

The power at the impeller of the pump is more than the power given to the water at outlet of the pump. The ratio of the power given to water at outlet of the pump to the power available at the impeller, is known as manometric efficiency.

The power given to water at outlet of the pump =  $\frac{W H_m}{1000}$  kW

The power at the impeller =  $\frac{\text{Work done by impeller per second}}{1000}$  kW

$$= \frac{W}{g} \times \frac{V_{w_2} \times u_2}{1000} \text{ kW}$$

$$\eta_{man} = \frac{\frac{W \times H_m}{1000}}{\frac{W}{g} \times \frac{V_{w_2} \times u_2}{1000}} = \frac{g \times H_m}{V_{w_2} \times u_2}.$$

(b) **Mechanical Efficiency ( $\eta_m$ )**. The power at the shaft of the centrifugal pump is more than the power available at the impeller of the pump. The ratio of the power available at the impeller to the power at the shaft of the centrifugal pump is known as mechanical efficiency. It is written as

$$\eta_m = \frac{\text{Power at the impeller}}{\text{Power at the shaft}}$$

The power at the impeller in kW =  $\frac{\text{Work done by impeller per second}}{1000}$

$$= \frac{W}{g} \times \frac{V_{w_2} u_2}{1000} \quad [\text{Using equation (19.2)}]$$

$$\eta_m = \frac{\frac{W}{g} \left( \frac{V_{w_2} u_2}{1000} \right)}{\text{S.P.}} \quad \dots(19.9)$$

where S.P. = Shaft power.

(c) **Overall Efficiency ( $\eta_o$ )**. It is defined as ratio of power output of the pump to the power input to the pump. The power output of the pump in kW

$$= \frac{\text{Weight of water lifted} \times H_m}{1000} = \frac{W H_m}{1000}$$

Power input to the pump = Power supplied by the electric motor  
= S.P. of the pump.

$$\therefore \eta_o = \frac{\left( \frac{W H_m}{1000} \right)}{\text{S.P.}} \quad \dots(19.10)$$

Also  $\eta_o = \eta_{man} \times \eta_m$ . \dots(19.11)

**Problem 19.1** The internal and external diameters of the impeller of a centrifugal pump are 200 mm and 400 mm respectively. The pump is running at 1200 r.p.m. The vane angles of the impeller at inlet and outlet are  $20^\circ$  and  $30^\circ$  respectively. The water enters the impeller radially and velocity of flow is constant. Determine the work done by the impeller per unit weight of water.

**Solution.** Given :

Internal diameter of impeller,  $D_1 = 200 \text{ mm} = 0.20 \text{ m}$

External diameter of impeller,  $D_2 = 400 \text{ mm} = 0.40 \text{ m}$

Speed,  $N = 1200 \text{ r.p.m.}$

Vane angle at inlet,  $\theta = 20^\circ$

Vane angle at outlet,  $\phi = 30^\circ$

Water enters radially\* means,  $\alpha = 90^\circ$  and  $V_{w1} = 0$

Velocity of flow,  $V_{f1} = V_{f2}$

Tangential velocity of impeller at inlet and outlet are,

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.20 \times 1200}{60} = 12.56 \text{ m/s}$$

$$\text{and } u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.40 \times 1200}{60} = 25.13 \text{ m/s.}$$

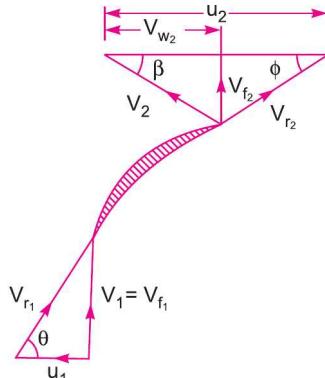


Fig. 19.4

$$\text{From inlet velocity triangle, } \tan \theta = \frac{V_{f1}}{u_1} = \frac{V_{f1}}{12.56}$$

$$\therefore V_{f1} = 12.56 \tan 20^\circ = 12.56 \times \tan 20^\circ = 4.57 \text{ m/s}$$

$$\therefore V_{f2} = V_{f1} = 4.57 \text{ m/s.}$$

$$\text{From outlet velocity triangle, } \tan \phi = \frac{V_{f2}}{u_2 - V_{w2}} = \frac{4.57}{25.13 - V_{w2}}$$

$$\text{or } 25.13 - V_{w2} = \frac{4.57}{\tan \phi} = \frac{4.57}{\tan 30^\circ} = 7.915$$

$$\therefore V_{w2} = 25.13 - 7.915 = 17.215 \text{ m/s.}$$

The work done by impeller per kg of water per second is given by equation (19.1) as

$$= \frac{1}{g} V_{w2} u_2 = \frac{17.215 \times 25.13}{9.81} = 44.1 \text{ Nm/N. Ans.}$$

**Problem 19.2** A centrifugal pump is to discharge  $0.118 \text{ m}^3/\text{s}$  at a speed of 1450 r.p.m. against a head of 25 m. The impeller diameter is 250 mm, its width at outlet is 50 mm and manometric efficiency is 75%. Determine the vane angle at the outer periphery of the impeller.

**Solution.** Given :

Discharge,  $Q = 0.118 \text{ m}^3/\text{s}$

Speed,  $N = 1450 \text{ r.p.m.}$

Head,  $H_m = 25 \text{ m}$

Diameter at outlet,  $D_2 = 250 \text{ mm} = 0.25 \text{ m}$

Width at outlet,  $B_2 = 50 \text{ mm} = 0.05 \text{ m}$

Manometric efficiency,  $\eta_{man} = 75\% = 0.75$ .

Let vane angle at outlet  $= \phi$

Tangential velocity of impeller at outlet,

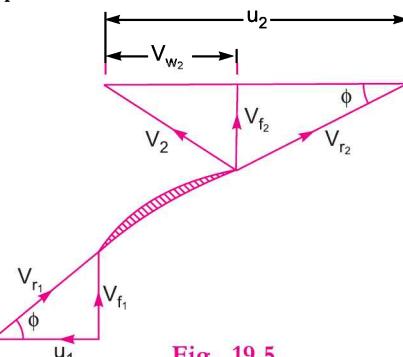


Fig. 19.5

\* If in the problem, this condition is not given even then the water is assumed to be entering radially unless stated otherwise

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$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.25 \times 1450}{60} = 18.98 \text{ m/s}$$

Discharge is given by

$$Q = \pi D_2 B_2 \times V_{f_2}$$

∴

$$V_{f_2} = \frac{Q}{\pi D_2 B_2} = \frac{0.118}{\pi \times 0.25 \times 0.05} = 3.0 \text{ m/s.}$$

Using equation (19.8),

$$\eta_{man} = \frac{gH_m}{V_{w_2} u_2} = \frac{9.81 \times 25}{V_{w_2} \times 18.98}$$

∴

$$V_{w_2} = \frac{9.81 \times 25}{\eta_{man} \times 18.98} = \frac{9.81 \times 25}{0.75 \times 18.98} = 17.23.$$

From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{(u_2 - V_{w_2})} = \frac{3.0}{(18.98 - 17.23)} = 1.7143$$

$$\therefore \phi = \tan^{-1} 1.7143 = 59.74^\circ \text{ or } 59^\circ 44'. \text{ Ans.}$$

**Problem 19.3** A centrifugal pump delivers water against a net head of 14.5 metres and a design speed of 1000 r.p.m. The vanes are curved back to an angle of  $30^\circ$  with the periphery. The impeller diameter is 300 mm and outlet width is 50 mm. Determine the discharge of the pump if manometric efficiency is 95%.

**Solution.** Given :

Net head,

$$H_m = 14.5 \text{ m}$$

Speed,

$$N = 1000 \text{ r.p.m.}$$

Vane angle at outlet,

$$\phi = 30^\circ$$

Impeller diameter means the diameter of the impeller at outlet

$$\therefore \text{Diameter, } D_2 = 300 \text{ mm} = 0.30 \text{ m}$$

$$\text{Outlet width, } B_2 = 50 \text{ mm} = 0.05 \text{ m}$$

$$\text{Manometric efficiency, } \eta_{man} = 95\% = 0.95$$

Tangential velocity of impeller at outlet,

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.30 \times 1000}{60} = 15.70 \text{ m/s.}$$

$$\text{Now using equation (19.8), } \eta_{man} = \frac{gH_m}{V_{w_2} \times u_2}$$

$$\therefore 0.95 = \frac{9.81 \times 14.5}{V_{w_2} \times 15.70}$$

$$\therefore V_{w_2} = \frac{0.95 \times 14.5}{0.95 \times 15.70} = 9.54 \text{ m/s.}$$

Refer to Fig. 19.5. From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{(u_2 - V_{w_2})} \text{ or } \tan 30^\circ = \frac{V_{f_2}}{(15.70 - 9.54)} = \frac{V_{f_2}}{6.16}$$

$$\therefore V_{f_2} = 6.16 \times \tan 30^\circ = 3.556 \text{ m/s.}$$

∴ Discharge,

$$Q = \pi D_2 B_2 \times V_{f_2} \\ = \pi \times 0.30 \times 0.05 \times 3.556 \text{ m}^3/\text{s} = \mathbf{0.1675 \text{ m}^3/\text{s. Ans.}}$$

**Problem 19.4** A centrifugal pump having outer diameter equal to two times the inner diameter and running at 1000 r.p.m. works against a total head of 40 m. The velocity of flow through the impeller is constant and equal to 2.5 m/s. The vanes are set back at an angle of  $40^\circ$  at outlet. If the outer diameter of the impeller is 500 mm and width at outlet is 50 mm, determine :

- (i) Vane angle at inlet,      (ii) Work done by impeller on water per second, and
- (iii) Manometric efficiency.

**Solution.** Given :

Speed,

$$N = 1000 \text{ r.p.m.}$$

Head,

$$H_m = 40 \text{ m}$$

Velocity of flow,

$$V_{f_1} = V_{f_2} = 2.5 \text{ m/s}$$

Vane angle at outlet,

$$\phi = 40^\circ$$

Outer dia. of impeller,

$$D_2 = 500 \text{ mm} = 0.50 \text{ m}$$

Inner dia. of impeller,

$$D_1 = \frac{D_2}{2} = \frac{0.50}{2} = 0.25 \text{ m}$$

Width at outlet,

$$B_2 = 50 \text{ mm} = 0.05 \text{ m}$$

Tangential velocity of impeller at inlet and outlet are

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.25 \times 1000}{60} = 13.09 \text{ m/s}$$

$$\text{and } u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.50 \times 1000}{60} = 26.18 \text{ m/s.}$$

Discharge is given by,  $Q = \pi D_2 B_2 \times V_{f_2} = \pi \times 0.50 \times 0.05 \times 2.5 = 0.1963 \text{ m}^3/\text{s.}$

(i) Vane angle at inlet ( $\theta$ ).

$$\text{From inlet velocity triangle } \tan \theta = \frac{V_{f_1}}{u_1} = \frac{2.5}{13.09} = 0.191$$

$$\therefore \theta = \tan^{-1} 0.191 = 10.81^\circ \text{ or } 10^\circ 48'. \text{ Ans.}$$

(ii) Work done by impeller on water per second is given by equation (19.2) as

$$\begin{aligned} &= \frac{W}{g} \times V_{w_2} u_2 = \frac{\rho \times g \times Q}{g} \times V_{w_2} \times u_2 \\ &= \frac{1000 \times 9.81 \times 0.1963}{9.81} \times V_{w_2} \times 26.18 \end{aligned} \quad \dots(i)$$

But from outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{2.5}{(26.18 - V_{w_2})}$$

$$\therefore 26.18 - V_{w_2} = \frac{2.5}{\tan \phi} = \frac{2.5}{\tan 40^\circ} = 2.979$$

$$\therefore V_{w_2} = 26.18 - 2.979 = 23.2 \text{ m/s.}$$

Substituting this value of  $V_{w_2}$  in equation (i), we get the work done by impeller as

$$\begin{aligned} &= \frac{1000 \times 9.81 \times 0.1963}{9.81} \times 23.2 \times 26.18 \\ &= 119227.9 \text{ Nm/s. Ans.} \end{aligned}$$

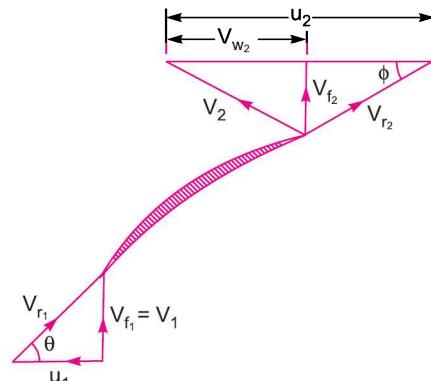


Fig. 19.6

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(iii) **Manometric efficiency ( $\eta_{man}$ )**. Using equation (19.8), we have

$$\eta_{man} = \frac{gH_m}{V_{w_2} u_2} = \frac{9.81 \times 40}{23.2 \times 26.18} = 0.646 = 64.4\% \text{ Ans.}$$

**Problem 19.5** A centrifugal pump discharges  $0.15 \text{ m}^3/\text{s}$  of water against a head of  $12.5 \text{ m}$ , the speed of the impeller being  $600 \text{ r.p.m}$ . The outer and inner diameters of impeller are  $500 \text{ mm}$  and  $250 \text{ mm}$  respectively and the vanes are bent back at  $35^\circ$  to the tangent at exit. If the area of flow remains  $0.07 \text{ m}^2$  from inlet to outlet, calculate :

- (i) Manometric efficiency of pump,
- (ii) Vane angle at inlet, and
- (iii) Loss of head at inlet to impeller when the discharge is reduced by 40% without changing the speed.

**Solution.** Given :

Discharge,

$$Q = 0.15 \text{ m}^3/\text{s}$$

Head,

$$H_m = 12.5 \text{ m}$$

Speed,

$$N = 600 \text{ r.p.m.}$$

Outer dia.,

$$D_2 = 500 \text{ mm} = 0.50 \text{ m}$$

Inner dia.,

$$D_1 = 250 \text{ mm} = 0.25 \text{ m}$$

Vane angle at outlet,

$$\phi = 35^\circ$$

Area of flow,

$$= 0.07 \text{ m}^2$$

As area of flow is constant from inlet to outlet, then velocity of flow will be constant from inlet to outlet.

$$\begin{aligned} \text{Discharge} &= \text{Area of flow} \times \text{Velocity of flow} \\ \text{or} \quad 0.15 &= 0.07 \times \text{Velocity of flow} \end{aligned}$$

$$\therefore \text{Velocity of flow} = \frac{0.15}{0.07} = 2.14 \text{ m/s.}$$

$$\therefore V_{f_1} = V_{f_2} = 2.14 \text{ m/s.}$$

Tangential velocity of impeller at inlet and outlet are

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.25 \times 600}{60} = 7.85 \text{ m/s}$$

$$\text{and} \quad u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.50 \times 600}{60} = 15.70 \text{ m/s}$$

$$\text{From outlet velocity triangle, } V_{w_2} = u_2 - \frac{V_{f_2}}{\tan \phi} = 15.70 - \frac{2.14}{\tan 35^\circ} = 12.64 \text{ m/s}$$

(i) **Manometric efficiency of the pump**

$$\text{Using equation (19.8), we have } \eta_{man} = \frac{g \times H_m}{V_{w_2} \times u_2} = \frac{9.81 \times 12.5}{12.64 \times 15.7} = 0.618 \text{ or } 61.8\% \text{ Ans.}$$

(ii) **Vane angle at inlet ( $\theta$ )**

$$\text{From inlet velocity triangle, } \tan \theta = \frac{V_{f_1}}{u_1} = \frac{2.14}{7.85} = 0.272$$

$$\therefore \theta = \tan^{-1} 0.272 = 15^\circ 12' \text{ Ans.}$$

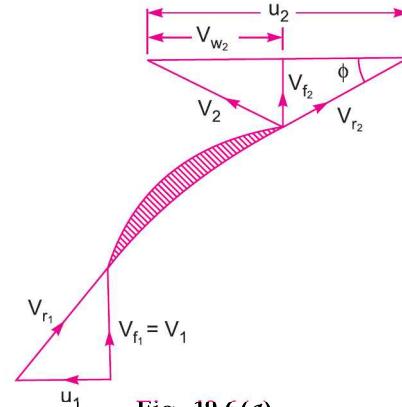


Fig. 19.6(a)

(iii) Loss of head at inlet to impeller when discharge is reduced by 40% without changing the speed.

When there is an increase or decrease in the discharge from the normal discharge, a loss of head occurs at entry due to shock. In this case, discharge is reduced by 40%. Hence the new discharge is given by,

$$Q^* = 0.6 \times Q$$

where  $Q = 0.15 \text{ m}^3/\text{s}$

As area of flow is constant, hence new velocity of flow ( $V_{f_1}^*$ ) will be given by,

$$\begin{aligned} V_{f_1}^* &= \frac{Q^*}{\text{Area of flow}} \\ &= \frac{0.6 \times Q}{0.07} = \frac{0.6 \times 0.15}{0.07} = 1.284 \text{ m/s.} \end{aligned}$$

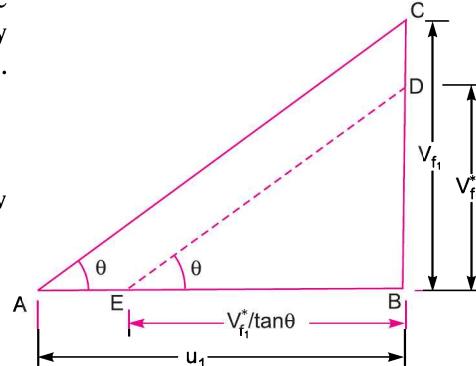


Fig. 19.6 (b)

Fig. 19.6 (b) shows the velocity triangle at inlet corresponding to normal discharge and reduced discharge.  $ABC$  is the velocity triangle due to normal discharge. Triangle  $BDE$  is corresponding to reduced discharge  $BD = 1.284 \text{ m/s}$  and  $DE$  is parallel to  $AC$ .

The blade angle  $\theta$  at inlet cannot change and hence  $DE$  will be parallel to  $AC$ .

There will be a sudden change in the tangential velocity from  $AB$  to  $BE$ . Hence due to this shock, there will be a loss of head at inlet.

$$\begin{aligned} \therefore \text{Head lost at inlet} &= \frac{(\text{change in tangential velocity at inlet})^2}{2g} \\ &= \frac{(AB - BE)^2}{2g} = \frac{\left(u_1 - \frac{V_{f_1}^*}{\tan \theta}\right)^2}{2g} = \frac{\left(7.85 - \frac{1.284}{\tan 15.2^\circ}\right)^2}{2 \times 9.81} = 0.5 \text{ m. Ans.} \end{aligned}$$

**Problem 19.6** The outer diameter of an impeller of a centrifugal pump is 400 mm and outlet width is 50 mm. The pump is running at 800 r.p.m. and is working against a total head of 15 m. The vanes angle at outlet is  $40^\circ$  and manometric efficiency is 75%. Determine :

- (i) velocity of flow at outlet, (ii) velocity of water leaving the vane,
- (iii) angle made by the absolute velocity at outlet with the direction of motion at outlet, and
- (iv) discharge.

**Solution.** Given :

Outer diameter,	$D_2 = 400 \text{ mm} = 0.4 \text{ m}$
Width at outlet,	$B_2 = 50 \text{ mm} = 0.05 \text{ m}$
Speed,	$N = 800 \text{ r.p.m.}$
Head,	$H_m = 15 \text{ m}$
Vane angle at outlet,	$\phi = 40^\circ$
Manometric efficiency,	$\eta_{man} = 75\% = 0.75$
Tangential velocity of impeller at outlet,	

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.4 \times 800}{60} = 16.75 \text{ m/s.}$$

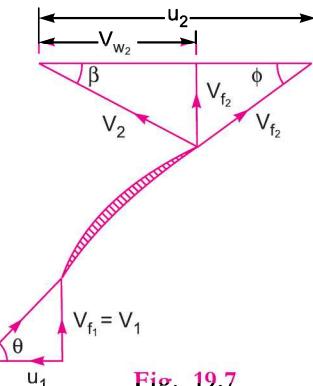


Fig. 19.7

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Using equation (19.8),  $\eta_{man} = \frac{gH_m}{V_{w_2} u_2}$

$$0.75 = \frac{9.81 \times 15}{V_{w_2} \times 16.75}$$

$$\therefore V_{w_2} = \frac{9.81 \times 15}{0.75 \times 16.75} = 11.71 \text{ m/s.}$$

From the outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{V_{f_2}}{(16.75 - 11.71)} = \frac{V_{f_2}}{5.04}$$

(i)  $V_{f_2} = 5.04 \tan \phi = 5.04 \times \tan 40^\circ = 4.23 \text{ m/s. Ans.}$

(ii) Velocity of water leaving the vane ( $V_2$ ).

$$V_2 = \sqrt{V_{f_2}^2 + V_{w_2}^2} = \sqrt{4.23^2 + 11.71^2}$$

$$= \sqrt{17.89 + 137.12} = 12.45 \text{ m/s. Ans.}$$

(iii) Angle made by absolute velocity at outlet ( $\beta$ ),

$$\tan \beta = \frac{V_{f_2}}{V_{w_2}} = \frac{4.23}{11.71} = 0.36$$

$\therefore \beta = \tan^{-1} 0.36 = 19.80^\circ \text{ or } 19^\circ 48'. \text{ Ans.}$

(iv) Discharge through pump is given by,

$$Q = \pi D_2 B_2 \times V_{f_2} = \pi \times 0.4 \times 0.05 \times 4.23 = 0.265 \text{ m}^3/\text{s. Ans.}$$

**Problem 19.7** A centrifugal pump is running at 1000 r.p.m. The outlet vane angle of the impeller is  $45^\circ$  and velocity of flow at outlet is 2.5 m/s. The discharge through the pump is 200 litres/s when the pump is working against a total head of 20 m. If the manometric efficiency of the pump is 80%, determine :

(i) the diameter\* of the impeller, and (ii) the width of the impeller at outlet.

**Solution.** Given :

Speed,  $N = 1000 \text{ r.p.m.}$

Outlet vane angle,  $\phi = 45^\circ$

Velocity of flow at outlet,  $V_{f_2} = 2.5 \text{ m/s}$

Discharge,  $Q = 200 \text{ litres/s} = 0.2 \text{ m}^3/\text{s}$

Head,  $H_m = 20 \text{ m}$

Manometric efficiency,  $\eta_{man} = 80\% = 0.80$

From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}}$$

or  $u_2 - V_{w_2} = \frac{V_{f_2}}{\tan \phi} = \frac{2.5}{\tan 45^\circ} = 2.5$

$\therefore V_{w_2} = (u_2 - 2.5)$

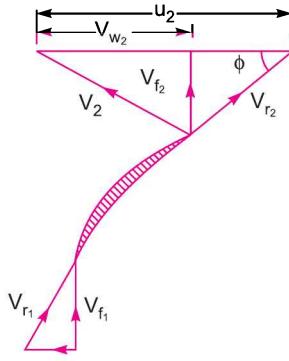


Fig. 19.8

... (i)

\* Diameter of impeller means the outside diameter.

Using equation (19.8),  $\eta_{man} = \frac{gH_m}{V_{w_2} u_2}$

$$0.80 = \frac{9.81 \times 20}{V_{w_2} u_2}$$

$$\therefore V_{w_2} u_2 = \frac{9.81 \times 20}{0.80} = 245.25 \quad \dots(ii)$$

Substituting the value of  $V_{w_2}$  from equation (i) in (ii), we get

$$(u_2 - 2.5)u_2 = 245.25$$

$$u_2^2 - 2.5u_2 - 245.25 = 0$$

which is a quadratic equation in  $u_2$  and its solution is

$$u_2 = \frac{2.5 \pm \sqrt{(2.5)^2 + 4 \times 245.25}}{2} = \frac{2.5 + \sqrt{6.25 + 981}}{2}$$

$$= \frac{2.5 \pm 31.42}{2} = 16.96 \text{ or } -14.46$$

$$\therefore u_2 = 16.96 \quad (\because -ve \text{ value is not possible})$$

(i) Diameter of impeller ( $D_2$ ).

Using,  $u_2 = \frac{\pi D_2 N}{60}$

$$\therefore 16.96 = \frac{\pi D_2 N}{60} = \frac{\pi \times D_2 \times 1000}{60}$$

$$\therefore D_2 = \frac{16.96 \times 60}{\pi \times 1000} = 0.324 \text{ m} = 324 \text{ mm. Ans.}$$

(ii) Width of impeller at outlet ( $B_2$ ).

Discharge,  $Q = \pi D_2 B_2 V_{f_2}$

$$0.2 = \pi \times .324 \times B_2 \times 2.5$$

$$\therefore B_2 = \frac{0.2}{\pi \times .324 \times 2.5} = 0.0786 \text{ m} = 78.6 \text{ mm. Ans.}$$

**Problem 19.7 (A)** A centrifugal pump has the following dimensions : inlet radius = 80 mm ; outlet radius = 160 mm ; width of impeller at the inlet = 50 mm ;  $\beta_1 = 0.45$  radians ;  $\beta_2 = 0.25$  radians ; width of impeller at outlet = 50 mm.

Assuming shockless entry determine the discharge and the head developed by the pump when the impeller rotates at 90 radians/second.

**Solution.** Given :

Inlet radius,	$R_1 = 80 \text{ mm} = 0.08 \text{ m}$
Outlet radius	$R_2 = 160 \text{ mm} = 0.16 \text{ m}$
Width at inlet,	$B_1 = 50 \text{ mm} = 0.05 \text{ m}$
Width at outlet,	$B_2 = 50 \text{ mm} = 0.05 \text{ m}$
Angles,	$\beta_1 = 0.45 \text{ radians}$ and $\beta_2 = 0.25 \text{ radians}$ .

Here  $\beta_1$  is the vane angle at inlet and  $\beta_2$  is the vane angle at outlet.

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∴ Vane angle at inlet,  $\theta = \beta_1 = 0.45$  radians

Vane angle at outlet,  $\phi = \beta_2 = 0.25$  radians.

Angular velocity,  $\omega = 90$  rad/s

Find :

(i) discharge, and (ii) head developed.

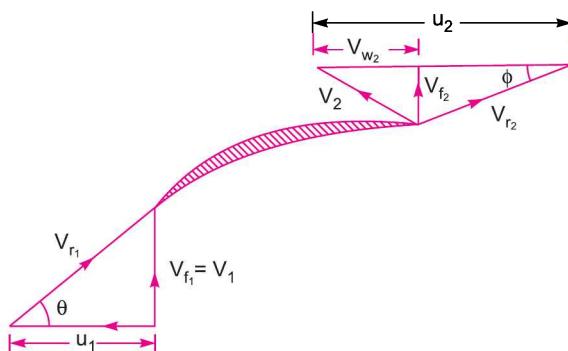


Fig. 19.8 (a)

Now tangential velocity of impeller at inlet and outlet are,

$$u_1 = \frac{\pi D_1 \times N}{60} = \frac{2\pi N}{60} \times \frac{D_1}{2} = \omega \times R_1 = 90 \times 0.08 = 7.2 \text{ m/s}$$

and

$$u_2 = \omega \times R_2 = 90 \times 0.16 = 14.4 \text{ m/s}$$

From inlet velocity triangle,  $\tan \theta = \frac{V_{f_1}}{u_1}$

$$\therefore V_{f_1} = u_1 \times \tan \theta = 7.2 \times \tan (0.45 \text{ rad}) = 7.2 \times 0.483 = 3.478 \text{ m/s}$$

(i) Discharge ( $Q$ )

$$\begin{aligned} \text{Discharge is given by, } Q &= \pi D_1 \times B_1 \times V_{f_1} = \pi \times (2R_1) \times B_1 \times V_{f_1} \\ &= \pi \times 2 \times 0.08 \times 0.05 \times 3.478 \text{ m}^3/\text{s} = \mathbf{0.0874 \text{ m}^3/\text{s. Ans.}} \end{aligned}$$

(ii) Head developed ( $H_m$ )

For the shockless entry, the losses of the pump will be zero. Hence, the head developed ( $H_m$ ) will be given by equation (19.5).

$$\therefore H_m = \frac{V_{w_2} \times u_2}{g} \quad \dots(i)$$

where from outlet velocity triangle,  $V_{w_2} = u_2 - V_{f_2} \times \cot \phi$

The value of  $V_{f_2}$  is obtained from  $Q = \pi D_2 \times B_2 \times V_{f_2}$

$$\begin{aligned} \text{or } 0.0874 &= \pi \times (2R_2) \times B_2 \times V_{f_2} \\ &= \pi \times (2 \times 0.16) \times 0.05 \times V_{f_2} \end{aligned}$$

$$\therefore V_{f_2} = \frac{0.0874}{\pi \times 2 \times 0.16 \times 0.05} = 1.7387 \text{ m/s}$$

$$\therefore V_{w_2} = u_2 - V_{f_2} \times \cot \phi$$

$$= 14.4 - 1.7387 \times \cot(0.25 \text{ radians})$$

$$= 14.4 - 1.7387 \times 3.9163 = 14.4 - 6.809 = 7.591 \text{ m/s}$$

Substituting this value in equation (i) above, we get

$$H_m = \frac{V_{w_2} \times u_2}{g} = \frac{7.591 \times 14.4}{9.81} = 11.142 \text{ m. Ans.}$$

**Problem 19.8** The internal and external diameter of an impeller of a centrifugal pump which is running at 1000 r.p.m., are 200 mm and 400 mm respectively. The discharge through pump is 0.04 m<sup>3</sup>/s and velocity of flow is constant and equal to 2.0 m/s. The diameters of the suction and delivery pipes are 150 mm and 100 mm respectively and suction and delivery heads are 6 m (abs.) and 30 m (abs.) of water respectively. If the outlet vane angle is 45° and power required to drive the pump is 16.186 kW, determine :

- (i) Vane angle of the impeller at inlet, (ii) The overall efficiency of the pump, and
- (iii) Manometric efficiency of the pump.

**Solution.** Given :

Speed,	$N = 1000 \text{ r.p.m.}$
Internal dia.,	$D_1 = 200 \text{ mm} = 0.2 \text{ m}$
External dia.,	$D_2 = 400 \text{ mm} = 0.4 \text{ m}$
Discharge,	$Q = 0.04 \text{ m}^3/\text{s}$
Velocity of flow,	$V_{f_1} = V_{f_2} = 2.0 \text{ m/s}$
Dia. of suction pipe,	$D_s = 150 \text{ mm} = 0.15 \text{ m}$
Dia. of delivery pipe,	$D_d = 100 \text{ mm} = 0.10 \text{ m}$
Suction head,	$h_s = 6 \text{ m} (\text{abs.})$
Delivery head,	$h_d = 30 \text{ m} (\text{abs.})$
Outlet vane angle,	$\phi = 45^\circ$
Power required to drive the pump, $P = 16.186 \text{ kW}$	

(i) Vane angle of the impeller at inlet ( $\theta$ ).

From inlet velocity, we have  $\tan \theta = \frac{V_{f_1}}{u_1} = \frac{2.0}{u_1}$ , where  $u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.2 \times 1000}{60} = 10.47 \text{ m/s}$

$$\therefore \tan \theta = \frac{2.0}{10.47} = 0.191 \text{ or } \theta = \tan^{-1} 0.191 = 10^\circ 48'. \text{ Ans.}$$

(ii) Overall efficiency of the pump ( $\eta_o$ ).

$$\text{Using equation (19.10), we have } \eta_o = \frac{\left( \frac{WH_m}{1000} \right)}{\text{S.P.}}$$

where S.P. = Power required to drive the pump and equal to  $P$  here.

$$\begin{aligned} \eta_o &= \frac{\left( \frac{\rho \times g \times Q \times H_m}{1000} \right)}{P} = \frac{\rho g \times Q \times H_m}{1000 \times P} \\ &= \frac{1000 \times 9.81 \times 0.04 \times H_m}{1000 \times 16.186} = 0.02424 H_m \end{aligned} \quad \dots(i)$$

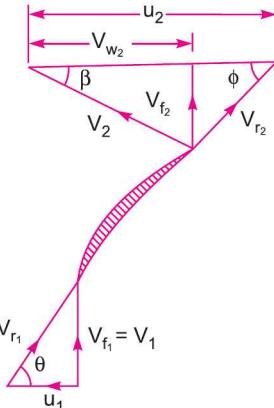


Fig. 19.9

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Now  $H_m$  is given by equation (19.6) as

$$H_m = \left( \frac{p_o}{\rho g} + \frac{V_o^2}{2g} + Z_o \right) - \left( \frac{p_i}{\rho g} + \frac{V_i^2}{2g} + Z_i \right) \quad \dots(ii)$$

where  $\frac{p_o}{\rho g}$  = Pressure head at outlet of pump =  $h_d = 30$  m

$\frac{V_o^2}{2g}$  = Velocity head at outlet of pump =  $\frac{V_d^2}{2g}$

$\frac{p_i}{\rho g}$  = Pressure head at inlet of pump =  $h_s = 6$  m

$\frac{V_i^2}{2g}$  = Velocity head at inlet of pump =  $\frac{V_s^2}{2g}$

$Z_o$  and  $Z_i$  = Vertical height at outlet and inlet of the pump from datum line.

If  $Z_o = Z_i$  then equation(ii) becomes as

$$H_m = \left( 30 + \frac{V_d^2}{2g} \right) - \left( 6 + \frac{V_s^2}{2g} \right) \quad \dots(iii)$$

Now  $V_d = \frac{\text{Discharge}}{\text{Area of delivery pipe}} = \frac{0.04}{\frac{\pi}{4}(D_d)^2} = \frac{0.04}{\frac{\pi}{4} \times 1^2} = 5.09 \text{ m/s}$

And  $V_s = \frac{.04}{\text{Area of suction pipe}} = \frac{.04}{\frac{\pi}{4} D_s^2} = \frac{.04}{\frac{\pi}{4} \times 1.5^2} = 2.26 \text{ m/s.}$

Substituting these values in equation (iii), we get

$$\begin{aligned} H_m &= \left( 30 + \frac{5.09^2}{2 \times 9.81} \right) - \left( 6 + \frac{2.26^2}{2 \times 9.81} \right) \\ &= (30 + 1.32) - (6 + .26) = 31.32 - 6.26 = 25.06 \text{ m.} \end{aligned}$$

Substituting the value of ' $H_m$ ' in equation (i), we get

$$\eta_o = .02424 \times 25.06 = 0.6074 = \mathbf{60.74\% \text{ Ans.}}$$

(iii) **Manometric efficiency of the pump ( $\eta_{man}$ )**.

Tangential velocity at outlet is given by

$$u_2 = \frac{\pi D_2 \times N}{60} = \frac{\pi \times 0.4 \times 1000}{60} = 20.94 \text{ m/s.}$$

From outlet velocity triangle, we have

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}} = \frac{2.0}{20.94 - V_{w_2}}$$

$$\therefore 20.94 - V_{w_2} = \frac{2.0}{\tan \phi} = \frac{2.0}{\tan 45} = 2.0$$

$$\therefore V_{w_2} = 20.94 - 2.0 = 18.94.$$

Using equation (19.8),  $\eta_{man} = \frac{g H_m}{V_{w_2} u_2} = \frac{9.81 \times 25.06}{18.94 \times 20.94} = 0.6198 = \mathbf{61.98\% \text{ Ans.}}$

**Problem 19.9** Find the power required to drive a centrifugal pump which delivers  $0.04 \text{ m}^3/\text{s}$  of water to a height of 20 m through a 15 cm diameter pipe and 100 m long. The overall efficiency of the pump is 70% and co-efficient of friction ' $f$ ' = 0.15 in the formula  $h_f = \frac{4fLV^2}{d \times 2g}$ .

**Solution.** Given :

Discharge,

$$Q = 0.04 \text{ m}^3/\text{s}$$

Height,

$$H_s = h_s + h_d = 20 \text{ m}$$

Dia. of pipe,

$$D_s = D_d = 15 \text{ cm} = 0.15 \text{ m}$$

Length,

$$L_s + L_d = L = 100 \text{ m}$$

Overall efficiency,

$$\eta_o = 70\% = 0.70$$

Co-efficient of friction,

$$f = .015$$

$$\text{Velocity of water in pipe, } V_s = V_d = V = \frac{\text{Discharge}}{\text{Area of pipe}} = \frac{0.04}{\frac{\pi}{4}(0.15)^2} = 2.26 \text{ m/s.}$$

Frictional head loss in pipe,

$$(h_{fs} + h_{fd}) = \frac{4fLV^2}{d \times 2g} = \frac{4 \times 0.15 \times 100 \times 2.26^2}{0.15 \times 2 \times 9.81} = 10.41 \text{ m.}$$

Using equation (19.7), we get manometric head as

$$\begin{aligned} H_m &= (h_s + h_d) + (h_{fs} + h_{fd}) + \frac{V_d^2}{2g} \\ &= 20 + 10.41 + \frac{2.26^2}{2 \times 9.81} \quad (\because h_s + h_d = H_s = 20 \text{ m}) \\ &= 30.41 + 0.26 = 30.67 \text{ m.} \end{aligned}$$

Overall efficiency is given by equation (19.10) as

$$\eta_o = \frac{\left( \frac{WH_m}{1000} \right)}{\text{S.P.}} = \frac{\rho g \times Q \times H_m}{1000 \times \text{S.P.}}$$

$$\therefore \text{S.P.} = \frac{\rho g \times Q \times H_m}{1000 \times \eta_o} = \frac{1000 \times 9.81 \times 0.04 \times 30.67}{1000 \times 0.70} = 17.19 \text{ kW. Ans.}$$

S.P. is the power required to drive the centrifugal pump.

**Problem 19.10** Show that the pressure rise in the impeller of a centrifugal pump when frictional and other losses in the impeller are neglected is given by

$$\frac{I}{2g} [V_{f_1}^2 + u_2^2 - V_{f_2}^2 \cosec^2 \phi]$$

where  $V_{f_1}$  and  $V_{f_2}$  are velocity of flow at inlet and outlet,

$u_2$  = tangential velocity of impeller at outlet, and  $\phi$  = vane angle at outlet.

**Solution.** Let suffix 1 represents the values at the inlet and suffix 2 represents the values at the outlet of the impeller.

Applying Bernoulli's equation at the inlet and outlet of the impeller and neglecting losses from inlet to outlet,

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$$\text{Total energy at inlet} = \text{Total energy at outlet} - \text{Work done by impeller on water}$$

$$\begin{aligned} \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 &= \left( \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 \right) - \text{Work done by impeller on water per kg of water} \\ &= \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2 - \frac{V_{w2} u_2}{g} \quad (\text{taking flow radial at inlet}) \end{aligned}$$

If inlet and outlet of the impeller are at the same height,

$$\begin{aligned} \frac{p_1}{\rho g} + \frac{V_1^2}{2g} &= \frac{p_2}{\rho g} + \frac{V_2^2}{2g} - \frac{V_{w2} u_2}{g} \quad (\because Z_1 = Z_2) \\ \therefore \left( \frac{p_2}{\rho g} - \frac{p_1}{\rho g} \right) &= \frac{V_1^2}{2g} - \frac{V_2^2}{2g} + \frac{V_{w2} u_2}{g} \end{aligned}$$

$$\text{But } \frac{p_2}{\rho g} - \frac{p_1}{\rho g} = \text{Pressure rise in impeller}$$

$$\therefore \text{Pressure rise in impeller} = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} + \frac{V_{w2} u_2}{g} \quad \dots(i)$$

From Fig. 19.9, we have

$$\text{From inlet velocity triangle, } V_1 = V_{f1} \quad \dots(ii)$$

$$\text{From outlet velocity triangle, } \tan \phi = \frac{V_{f2}}{(u_2 - V_{w2})} \text{ or } u_2 - V_{w2} = \frac{V_{f2}}{\tan \phi}$$

$$\therefore V_{w2} = u_2 - \frac{V_{f2}}{\tan \phi} = u_2 - V_{f2} \cot \phi \quad \dots(iii)$$

$$\begin{aligned} \text{Also } V_2^2 &= V_{f2}^2 + V_{w2}^2 = V_{f2}^2 + (u_2 - V_{f2} \cot \phi)^2 \\ &= V_{f2}^2 + (u_2^2 + V_{f2}^2 \cot^2 \phi - 2u_2 V_{f2} \cot \phi) \\ &= V_{f2}^2 + V_{f2}^2 \cot^2 \phi + u_2^2 - 2u_2 V_{f2} \cot \phi \\ &= V_{f2}^2 (1 + \cot^2 \phi) + u_2^2 - 2u_2 V_{f2} \cot \phi \\ &= V_{f2}^2 \cosec^2 \phi + u_2^2 - 2u_2 V_{f2} \cot \phi \quad (\because 1 + \cot^2 \phi = \cosec^2 \phi) \dots(iv) \end{aligned}$$

Substituting the values of  $V_1$ ,  $V_{w2}$  and  $V_2^2$  given by equations (ii), (iii) and (iv) in equation (i), we get

$$\begin{aligned} \text{Pressure rise} &= \frac{V_{f1}^2}{2g} - \frac{1}{2g} (V_{f2}^2 \cosec^2 \phi + u_2^2 - 2u_2 V_{f2} \cot \phi) + \frac{(u_2 - V_{f2} \cot \phi) \times u_2}{g} \\ &= \frac{1}{2g} [V_{f1}^2 - V_{f2}^2 \cosec^2 \phi - u_2^2 + 2u_2 V_{f2} \cot \phi + 2u_2^2 - 2u_2 V_{f2} \times \cot \phi] \\ &= \frac{1}{2g} [V_{f1}^2 - V_{f2}^2 \cosec^2 \phi + u_2^2] \\ &= \frac{1}{2g} [V_{f1}^2 + u_2^2 - V_{f2}^2 \cosec^2 \phi]. \quad \dots(19.12) \end{aligned}$$

**Problem 19.11** Find the rise in pressure in the impeller of a centrifugal pump through which water is flowing at the rate of  $0.01 \text{ m}^3/\text{s}$ . The internal and external diameters of the impeller are 15 cm and 30 cm respectively. The widths of the impeller at inlet and outlet are 1.2 cm and 0.6 cm. The pump is running at 1500 r.p.m. The water enters the impeller radially at inlet and impeller vane angle at outlet is  $45^\circ$ . Neglect losses through the impeller.

**Solution.** Given :

$$\text{Discharge, } Q = 0.10 \text{ m}^3/\text{s}$$

$$\text{Internal dia., } D_1 = 15 \text{ cm} = 0.15 \text{ m}$$

$$\text{External dia., } D_2 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\text{Width at inlet, } B_1 = 1.2 \text{ cm} = 0.012 \text{ m}$$

$$\text{Width at outlet, } B_2 = 0.6 \text{ cm} = 0.006 \text{ m}$$

$$\text{Speed, } N = 1500 \text{ r.p.m.}$$

$$\text{Vane angle at inlet, } \phi = 45^\circ$$

$$\text{Velocity of flow at outlet, } V_{f_1} = \frac{Q}{\text{Area of flow at inlet}} = \frac{0.01}{\pi D_1 B_1} = \frac{0.01}{\pi \times 0.15 \times 0.012} = 1.768 \text{ m/s}$$

$$\text{Velocity of flow at outlet, } V_{f_2} = \frac{Q}{\pi D_2 B_2} = \frac{0.01}{\pi \times 0.30 \times 0.006} = 1.768 \text{ m/s.}$$

Tangential velocity of impeller at outlet,

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.30 \times 1500}{60} = 23.56 \text{ m/s.}$$

Using equation (19.12),

$$\begin{aligned} \text{Pressure rise} &= \frac{1}{2g} [V_{f_1}^2 + u_2^2 - V_{f_2}^2 \operatorname{cosec}^2 \phi] \\ &= \frac{1}{2g} [1.768^2 + 23.56^2 - 1.768^2 \operatorname{cosec}^2 45^\circ] \end{aligned}$$

$$\text{But } \operatorname{cosec}^2 45^\circ = 1 + \cot^2 45^\circ = 1 + \frac{1}{\tan^2 45^\circ} = 1 + 1 = 2$$

$$\begin{aligned} \therefore \text{ Pressure rise} &= \frac{1}{2 \times 9.81} [1.768^2 + 23.56^2 - 1.768^2 \times 2.0] \\ &= \frac{1}{2 \times 9.81} [3.1258 + 555.07 - 6.25] = 28.13 \text{ m. Ans.} \end{aligned}$$

**Problem 19.12** Prove that the manometric head of a centrifugal pump running at speed  $N$  and giving a discharge  $Q$  may be written as :

$$H_{mano} = AN^2 + BNQ + CQ^2$$

where  $A$ ,  $B$  and  $C$  are constants.

**Solution.** From equation (19.4), we know that the manometric head is equal to the head imparted by the impeller to the water minus the losses of head in the impeller and casing.

$$\therefore \text{ Manometric head} = \frac{V_{w_2} u_2}{g} - \text{Losses of head in impeller and casing}$$

$$\text{or } H_{mano} = \frac{V_{w_2} u_2}{g} - \frac{KV_2^2}{2g} \quad \dots(i)$$

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where  $V_2$  = Absolute velocity of water at outlet of impeller and

$K \frac{V_2^2}{2g}$  is the part of head not converted into pressure head and is actually lost in eddies.

Now

$$\begin{aligned} u_2 &= \text{Velocity of impeller at outlet} \\ &= \frac{\pi D_2 N}{60} \\ &= K_1 N \end{aligned}$$

where  $K_1 = \frac{\pi D_2}{60}$  and is a constant.

From equation (19.2 A), we know that

$$\begin{aligned} Q &= \pi D_2 B_2 \times V_{f_2} \\ \therefore V_{f_2} &= \frac{Q}{\pi D_2 B_2} = K_2 Q \end{aligned}$$

where  $K_2 = \frac{1}{\pi D_2 B_2}$  and is a constant for a given pump.

From Fig. 19.10, it is clear that

$$\begin{aligned} \tan \phi &= \frac{V_{f_2}}{u_2 - V_{w_2}} \\ \therefore u_2 - V_{w_2} &= \frac{V_{f_2}}{\tan \phi} = V_{f_2} \cot \phi \\ \text{or } V_{w_2} &= u_2 - V_{f_2} \cot \phi \\ &= u_2 - K_2 Q \cot \phi \quad (\because V_{f_2} = K_2 Q) \\ &= u_2 - K_3 Q \quad \text{where } K_3 = K_2 \cot \phi \\ &= K_1 N - K_3 Q \quad (\because u_2 = K_1 N) \end{aligned}$$

Now from outlet velocity triangle, we know that

$$V_2^2 = V_{f_2}^2 + V_{w_2}^2 = (K_2 Q)^2 + (K_1 N - K_3 Q)^2 \quad (\because V_{f_2} = K_2 Q \text{ and } V_{w_2} = K_1 N - K_3 Q)$$

Substituting the values of  $V_{w_2}$ ,  $u_2$  and  $V_2$  in equation (i), we get

$$\begin{aligned} H_{mano} &= \frac{(K_1 N - K_3 Q)(K_1 N)}{g} - \frac{K[K_2^2 Q^2 + K_1^2 N^2 + K_3^2 Q^2 - 2K_1 N \times K_3 Q]}{2g} \\ &= \frac{1}{2g} [2(K_1^2 N^2 - K_1 K_3 N Q) - K K_2^2 Q^2 - K K_1^2 N^2 - K K_3^2 Q^2 + 2K K_1 K_3 N Q] \\ &= \frac{1}{2g} [N^2(2K_1^2 - K K_1^2) + N Q(2K K_1 K_3 - 2K_1 K_3) + Q^2(-K K_2^2 - K K_3^2)] \\ &= AN^2 + BNQ + CQ^2 \end{aligned}$$

where  $A = \frac{2K_1^2 - K K_1^2}{2g}$ ,  $B = \frac{2K K_1 K_3 - 2K_1 K_3}{2g}$  and  $C = \frac{-K K_2^2 - K K_3^2}{2g}$  and they are constant.

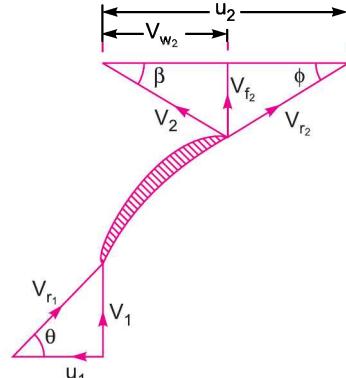


Fig. 19.10

## ► 19.5 MINIMUM SPEED FOR STARTING A CENTRIFUGAL PUMP

If the pressure rise in the impeller is more than or equal to manometric head ( $H_m$ ), the centrifugal pump will start delivering water. Otherwise, the pump will not discharge any water, though the impeller is rotating. When impeller is rotating, the water in contact with the impeller is also rotating. This is the case of forced vortex. In case of forced vortex, the centrifugal head or head due to pressure rise in the impeller

$$= \frac{\omega^2 r_2^2}{2g} - \frac{\omega^2 r_1^2}{2g} \quad \dots(i)$$

where  $\omega r_2$  = Tangential velocity of impeller at outlet =  $u_2$ , and  
 $\omega r_1$  = Tangential velocity of impeller at inlet =  $u_1$ .

$$\therefore \text{Head due to pressure rise in impeller} = \frac{u_2^2}{2g} - \frac{u_1^2}{2g}$$

The flow of water will commence only if

$$\text{Head due to pressure rise in impeller} \geq H_m \quad \text{or} \quad \frac{u_2^2}{2g} - \frac{u_1^2}{2g} \geq H_m.$$

$$\text{For minimum speed, we must have } \frac{u_2^2}{2g} - \frac{u_1^2}{2g} = H_m \quad \dots(19.13)$$

$$\text{But from equation (19.8), we have} \quad \eta_{man} = \frac{gH_m}{V_{w_2} u_2}$$

$$\therefore H_m = \eta_{man} \times \frac{V_{w_2} u_2}{g}.$$

Substituting this value of  $H_m$  in equation (19.13),

$$\frac{u_2^2}{2g} - \frac{u_1^2}{2g} = \eta_{man} \times \frac{V_{w_2} u_2}{g} \quad \dots(19.14)$$

$$\text{Now} \quad u_2 = \frac{\pi D_2 N}{60} \text{ and } u_1 = \frac{\pi D_1 N}{60}.$$

Substituting the values of  $u_2$  and  $u_1$  in equation (19.14),

$$\frac{1}{2g} \left( \frac{\pi D_2 N}{60} \right)^2 - \frac{1}{2g} \left( \frac{\pi D_1 N}{60} \right)^2 = \eta_{man} \times \frac{V_{w_2} \times \pi D_2 N}{g \times 60}$$

$$\text{Dividing by } \frac{\pi N}{g \times 60}, \text{ we get} \quad \frac{\pi N D_2^2}{120} - \frac{\pi N D_1^2}{120} = \eta_{man} \times V_{w_2} \times D_2$$

$$\text{or} \quad \frac{\pi N}{120} [D_2^2 - D_1^2] = \eta_{man} \times V_{w_2} \times D_2$$

$$\therefore N = \frac{120 \times \eta_{man} \times V_{w_2} \times D_2}{\pi [D_2^2 - D_1^2]} \quad \dots(19.15)$$

Equation (19.15) gives the minimum starting speed of the centrifugal pump.

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**Problem 19.13** The diameters of an impeller of a centrifugal pump at inlet and outlet are 30 cm and 60 cm respectively. Determine the minimum starting speed of the pump if it works against a head of 30 m.

**Solution.** Given :

$$\text{Dia. of impeller at inlet, } D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\text{Dia. of impeller at outlet, } D_2 = 60 \text{ cm} = 0.60 \text{ m}$$

$$\text{Head, } H_m = 30 \text{ cm}$$

Let the minimum starting speed =  $N$

Using equation (19.13) for minimum speed,

$$\frac{u_2^2}{2g} - \frac{u_1^2}{2g} = H_m$$

$$\text{where } u_2 = \frac{\pi \times D_2 \times N}{60} = \frac{\pi \times 0.6 \times N}{60} = 0.03141 N$$

$$u_1 = \frac{\pi \times D_1 \times N}{60} = \frac{\pi \times 0.3 \times N}{60} = 0.0157 N$$

$$\therefore \frac{1}{2g} (0.3141 N)^2 - \frac{1}{2g} (0.0157 N)^2 = 30$$

$$\text{or } (0.3141 N)^2 - (0.0157 N)^2 = 30 \times 2 \times g = 30 \times 2 \times 9.81$$

$$\text{or } N^2 = \frac{30 \times 2 \times 9.81}{(0.3141^2 - 0.0157^2)} = \frac{588.6}{.0009866 - .0002465} = 795297.9$$

$$\therefore N = \sqrt{795297.9} = 891.8 \text{ r.p.m. Ans.}$$

**Problem 19.14** The diameters of an impeller of a centrifugal pump at inlet and outlet are 30 cm and 60 cm respectively. The velocity of flow at outlet is 2.0 m/s and the vanes are set back at an angle of  $45^\circ$  at the outlet. Determine the minimum starting speed of the pump if the manometric efficiency is 70%.

**Solution.** Given :

$$\text{Diameter at inlet, } D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\text{Diameter at outlet, } D_2 = 60 \text{ cm} = 0.60 \text{ m}$$

$$\text{Velocity of flow at outlet, } V_{f_2} = 2.0 \text{ m/s}$$

$$\text{Vane angle at outlet, } \phi = 45^\circ$$

$$\text{Manometric efficiency, } \eta_{man} = 70\% = 0.70.$$

Let the minimum starting speed =  $N$ .

From Fig. 19.9, for velocity triangle at outlet, we have

$$\tan \phi = \frac{V_{f_2}}{(u_2 - V_{w_2})} \quad \text{or} \quad u_2 - V_{w_2} = \frac{V_{f_2}}{\tan \phi} = \frac{2.0}{\tan 45^\circ} = 2.0$$

$$\therefore V_{w_2} = u_2 - 2.0$$

$$\text{But } u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.60 \times N}{60} = 0.03141 N$$

$$\therefore V_{w_2} = (0.03141N - 2.0).$$

Using equation (19.15) for minimum starting speed,

$$\begin{aligned} N &= \frac{120 \times \eta_{man} \times V_{w_2} \times D_2}{\pi [D_2^2 - D_1^2]} = \frac{120 \times 0.70 \times (0.03141 N - 2.0) \times 0.6}{\pi [0.6^2 - 0.3^2]} \\ &= \frac{50.4(0.03141 N - 2.0)}{\pi [0.36 - 0.09]} = 59.417 [0.03141 N - 2.0] \\ &= 1.866 N - 118.834 \end{aligned}$$

or

$$1.866 N - N = 118.834 \text{ or } .866 N = 118.834$$

$$\therefore N = \frac{118.834}{0.866} = 137.22 \text{ r.p.m. Ans.}$$

**Problem 19.15** A centrifugal pump with 1.2 m diameter runs at 200 r.p.m. and pumps 1880 litres/s, the average lift being 6 m. The angle which the vanes make at exit with the tangent to the impeller is 26° and the radial velocity of flow is 2.5 m/s. Determine the manometric efficiency and the least speed to start pumping against a head of 6 m, the inner diameter of the impeller being 0.6 m.

**Solution.** Given :

Dia. at outlet,

$$D_2 = 1.2 \text{ m}$$

Speed,

$$N = 200 \text{ r.p.m.}$$

Discharge,

$$Q = 1880 \text{ litres/s} = 1.88 \text{ m}^3/\text{s}$$

Manometric head,

$$H_m = 6 \text{ m}$$

Angle of vane at outlet,

$$\phi = 26^\circ$$

Velocity of flow at outlet,

$$V_{f_2} = 2.5 \text{ m/s}$$

Dia. at inlet,

$$D_1 = 0.6 \text{ m}$$

(i) Manometric efficiency ( $\eta_{man}$ )

$$\text{Using equation (19.8), } \eta_{man} = \frac{g H_m}{V_{w_2} \times u_2} \quad \dots(i)$$

$$\text{But } u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 1.2 \times 200}{60} = 12.56 \text{ m/s}$$

and

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}} \quad \text{or} \quad u_2 - V_{w_2} = \frac{V_{f_2}}{\tan \phi} = \frac{2.5}{\tan 26^\circ} = 5.13$$

$$\therefore V_{w_2} = u_2 - 5.13 = 12.56 - 5.13 = 7.43 \text{ m/s.}$$

Substituting these values in equation (i), we get

$$\eta_{man} = \frac{9.81 \times 6.0}{7.43 \times 12.56} = 0.63 = 63\% \text{ Ans.}$$

(ii) Least speed to start the pump :

Least speed to start the pump is given by equation (19.13),

$$\frac{u_2^2}{2g} - \frac{u_1^2}{2g} = H_m \quad \dots(ii)$$

where  $u_2$  and  $u_1$  are the tangential velocities of the vane at outlet and inlet respectively, corresponding to least speed of the pump.

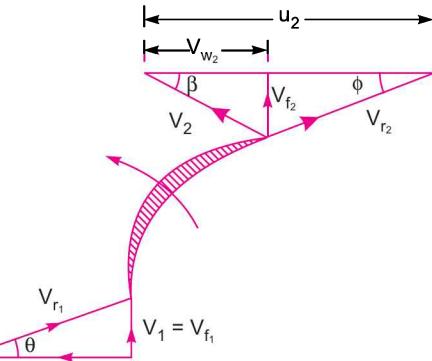


Fig. 19.11

But  $u_2 = \omega \times r_2$  and  $u_1 = \omega \times r_1$ .

Substituting these values in equation (ii), we get

$$\frac{(\omega \times r_2)^2}{2g} - \frac{(\omega \times r_1)^2}{2g} = H_m = 6.0 \text{ or } \frac{\omega^2}{2g} [r_2^2 - r_1^2] = 6.0$$

or  $\frac{\omega^2}{2 \times 9.81} [0.6^2 - 0.3^2] = 6.0 \quad (\because r_2 = \frac{D_2}{2} = \frac{1.2}{2} = 0.6 \text{ m and } r_1 = \frac{D_1}{2} = \frac{0.6}{2} = 0.3 \text{ m})$

$$\therefore \omega^2 = \frac{6.0 \times 2.0 \times 9.81}{0.36 - 0.09} = 436 \quad \therefore \omega = \sqrt{436} = 20.88 = \frac{2\pi N}{60}$$

$$\therefore N = \frac{60 \times 20.88}{2 \times \pi} = 200 \text{ r.p.m. Ans.}$$

## ► 19.6 MULTISTAGE CENTRIFUGAL PUMPS

If a centrifugal pump consists of two or more impellers, the pump is called a multistage centrifugal pump. The impellers may be mounted on the same shaft or on different shafts. A multistage pump is having the following two important functions :

1. To produce a high head, and
2. To discharge a large quantity of liquid.

If a high head is to be developed, the impellers are connected in series (or on the same shaft) while for discharging large quantity of liquid, the impellers (or pumps) are connected in parallel.

**19.6.1 Multistage Centrifugal Pumps for High Heads.** For developing a high head, a number of impellers are mounted in series or on the same shaft as shown in Fig. 19.12.

The water from suction pipe enters the 1st impeller at inlet and is discharged at outlet with increased pressure. The water with increased pressure from the outlet of the 1st impeller is taken to the inlet of the 2nd impeller with the help of a connecting pipe as shown in Fig. 19.12. At the outlet of the 2nd impeller, the pressure of water will be more than the pressure of water at the outlet of the 1st impeller. Thus if more impellers are mounted on the same shaft, the pressure at the outlet will be increased further.

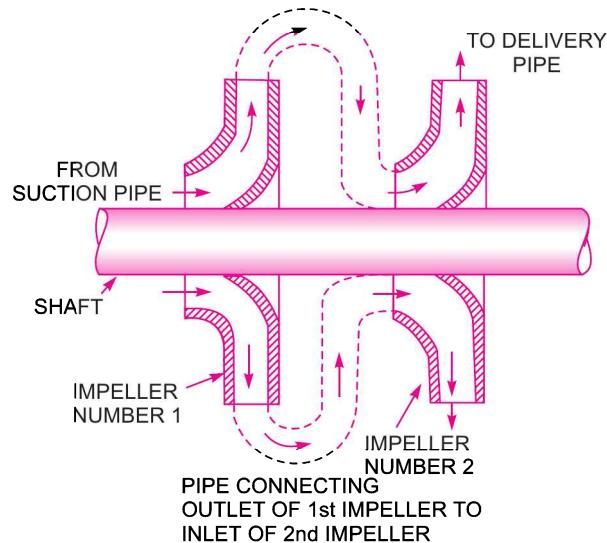


Fig. 19.12 Two-stage pumps with impellers in series.

Let

$n$  = Number of identical impellers mounted on the same shaft,

$H_m$  = Head developed by each impeller.

Then total head developed

$$= n \times H_m \quad \dots(19.16)$$

The discharge passing through each impeller is same

**19.6.2 Multistage Centrifugal Pumps for High Discharge.** For obtaining high discharge, the pumps should be connected in parallel as shown in Fig. 19.13. Each of the pumps lifts the water from a common pump and discharges water to a common pipe to which the delivery pipes of each pump is connected. Each of the pump is working against the same head.

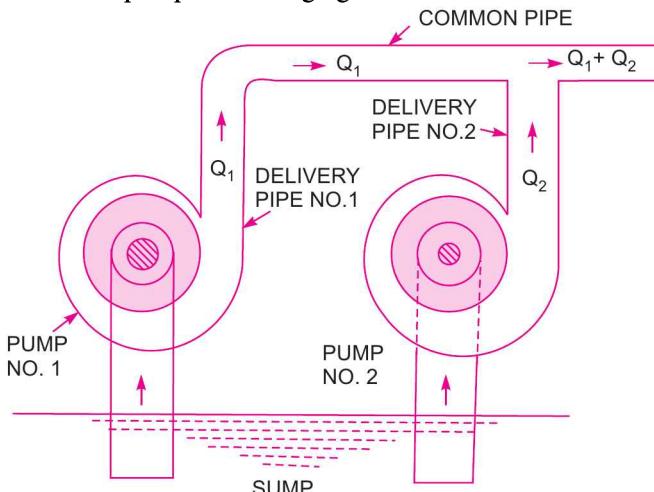


Fig. 19.13 Pumps in parallel.

Let

$n$  = Number of identical pumps arranged in parallel.

$Q$  = Discharge from one pump.

$$\therefore \text{Total discharge} = n \times Q \quad \dots(19.17)$$

**Problem 19.16** A three stage centrifugal pump has impellers 40 cm in diameter and 2 cm wide at outlet. The vanes are curved back at the outlet at  $45^\circ$  and reduce the circumferential area by 10%. The manometric efficiency is 90% and the overall efficiency is 80%. Determine the head generated by the pump when running at 1000 r.p.m. delivering 50 litres per second. What should be the shaft horse power?

**Solution.** Given :

Number of stages,  $n = 3$

Dia. of impeller at outlet,  $D_2 = 40 \text{ cm} = 0.40 \text{ m}$

Width at outlet,  $B_2 = 2 \text{ cm} = 0.02 \text{ m}$

Vane angle at outlet,  $\phi = 45^\circ$

Reduction in area at outlet  $= 10\% = 0.1$

$\therefore$  Area of flow at outlet  $= 0.9 \times \pi D_2 \times B_2 = 0.9 \times \pi \times .4 \times .02 = 0.02262 \text{ m}^2$

Manometric efficiency,  $\eta_{man} = 90\% = 0.90$

Overall efficiency,  $\eta_o = 80\% = 0.80$

Speed,  $N = 1000 \text{ r.p.m.}$

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Discharge,  $Q = 50 \text{ litres/s} = 0.05 \text{ m}^3/\text{s}$

Determine : (i) Head generated by the pump and

(ii) Shaft power.

$$\text{Velocity of flow at outlet, } V_{f_2} = \frac{\text{Discharge}}{\text{Area of flow}} = \frac{0.05}{0.02262} = 2.21 \text{ m/s}$$

Tangential velocity of impeller at outlet,

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.4 \times 1000}{60} = 20.94 \text{ m/s}$$

Refer to Fig. 19.9. From velocity triangle at outlet,

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}}$$

$$\therefore u_2 - V_{w_2} = \frac{V_{f_2}}{\tan \phi} = \frac{2.21}{\tan 45^\circ} = 2.21 \text{ m/s}$$

$$\therefore V_{w_2} = u_2 - 2.21 = 20.94 - 2.21 = 18.73 \text{ m/s}$$

$$\text{Using equation (19.8), } \eta_{man} = \frac{gH_m}{V_{w_2} u_2}, 0.90 = \frac{9.81 \times H_m}{18.73 \times 20.94}$$

$$\therefore H_m = \frac{0.90 \times 18.73 \times 20.94}{9.81} = 35.98 \text{ m.}$$

Using equation (19.16) for total head generated by pump,

$$= n \times H_m = 3 \times 35.98 = 107.94 \text{ m. Ans.}$$

$$\therefore \text{Power output of the pump} = \frac{\text{Weight of water lifted} \times \text{Total head}}{1000}$$

$$= \frac{\rho g \times Q \times 107.94}{1000} = \frac{1000 \times 9.81 \times 0.05 \times 107.94}{1000} = 52.94 \text{ kW.}$$

$$\text{Using equation (19.10), we have } \eta_o = \frac{\text{Power output of pump}}{\text{Power input to the pump}} = \frac{52.94}{\text{S.P.}}$$

$$\therefore \text{Shaft power} = \frac{52.94}{\eta_o} = \frac{52.94}{0.80} = 66.175 \text{ kW. Ans.}$$

**Problem 19.17** A four-stage centrifugal pump has four identical impellers, keyed to the same shaft. The shaft is running at 400 r.p.m. and the total manometric head developed by the multistage pump is 40 m. The discharge through the pump is 0.2 m<sup>3</sup>/s. The vanes of each impeller are having outlet angle as 45°. If the width and diameter of each impeller at outlet is 5 cm and 60 cm respectively, find the manometric efficiency.

**Solution.** Given :

Number of stage,  $n = 4$

Speed,  $N = 400 \text{ r.p.m.}$

Total manometric head = 40 m

$$\therefore \text{Manometric head for each stage, } H_m = \frac{40}{4} = 10.0 \text{ m}$$

Discharge,  $Q = 0.2 \text{ m}^3/\text{s}$

Outlet vane angle,  $\phi = 45^\circ$

Width at outlet,  $B_2 = 5 \text{ cm} = 0.05 \text{ m}$

Dia. at outlet,  $D_2 = 60 \text{ cm} = 0.6 \text{ m}$

$$\text{Tangential velocity of impeller at outlet, } u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.6 \times 400}{60} = 12.56 \text{ m/s}$$

$$\text{Velocity of flow at outlet, } V_{f_2} = \frac{\text{Discharge}}{\text{Area of flow}} = \frac{0.20}{\pi D_2 B_2} = \frac{0.20}{\pi \times 0.6 \times 0.5} = 2.122 \text{ m/s}$$

Refer to Fig. 19.9. From velocity triangle at outlet,

$$\tan \phi = \frac{V_{f_2}}{u_2 - V_{w_2}}$$

$$u_2 - V_{w_2} = \frac{V_{f_2}}{\tan \phi} = \frac{2.122}{\tan 45^\circ} = 2.122 \text{ m/s}$$

$$\therefore V_{w_2} = u_2 - 2.122 = 12.56 - 2.122 = 10.438$$

$$\text{Using equation (19.8), } \eta_{man} = \frac{g H_m}{V_{w_2} u_2} = \frac{9.81 \times 10.0}{10.438 \times 12.56} = 0.7482 \text{ or } 74.82\%. \text{ Ans.}$$

## ► 19.7 SPECIFIC SPEED OF A CENTRIFUGAL PUMP ( $N_s$ )

The specific speed of a centrifugal pump is defined as the speed of a geometrically similar pump which would deliver *one cubic metre* of liquid per second against a head of *one metre*. It is denoted by ' $N_s$ '.

**19.7.1 Expression for Specific Speed for a Pump.** The discharge,  $Q$ , for a centrifugal pump is given by the relation

$$\begin{aligned} Q &= \text{Area} \times \text{Velocity of flow} \\ &= \pi D \times B \times V_f \text{ or } Q \propto D \times B \times V_f \end{aligned} \quad \dots(i)$$

where  $D$  = Diameter of the impeller of the pump and

$B$  = Width of the impeller.

We know that  $B \propto D$

$$\therefore \text{From equation (i), we have } Q \propto D^2 \times V_f \quad \dots(ii)$$

We also know that tangential velocity is given by

$$u = \frac{\pi D N}{60} \propto D N \quad \dots(iii)$$

Now the tangential velocity ( $u$ ) and velocity of flow ( $V_f$ ) are related to the manometric head ( $H_m$ ) as

$$u \propto V_f \propto \sqrt{H_m} \quad \dots(iv)$$

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Substituting the value of  $u$  in equation (iii), we get

$$\sqrt{H_m} \propto DN \text{ or } D \propto \frac{\sqrt{H_m}}{N}$$

Substituting the values of  $D$  in equation (ii),

$$\begin{aligned} Q &\propto \frac{H_m}{N^2} \times V_f \\ &\propto \frac{H_m}{N^2} \times \sqrt{H_m} \quad [\because \text{From equation (iv), } V_f \propto \sqrt{H_m}] \\ &\propto \frac{H_m^{3/2}}{N^2} \\ \therefore Q &= K \frac{H_m^{3/2}}{N^2} \end{aligned} \quad \dots(v)$$

where  $K$  is a constant of proportionality.

If  $H_m = 1 \text{ m}$  and  $Q = 1 \text{ m}^3/\text{s}$ ,  $N$  becomes  $= N_s$ .

Substituting these values in equation (v), we get

$$\begin{aligned} 1 &= K \frac{1^{3/2}}{N_s^2} = \frac{K}{N_s^2} \\ \therefore K &= N_s^2 \end{aligned}$$

Substituting the value of  $K$  in equation (v), we get

$$\begin{aligned} Q &= N_s^2 \frac{H_m^{3/2}}{N^2} \quad \text{or} \quad N_s^2 = \frac{N^2 Q}{H_m^{3/2}} \\ \therefore N_s &= \frac{N \sqrt{Q}}{H_m^{3/4}} \end{aligned} \quad \dots(19.18)$$

### ► 19.8 MODEL TESTING OF CENTRIFUGAL PUMPS

Before manufacturing the large sized pumps, their models which are in complete similarity with the actual pumps (also called prototypes) are made. Tests are conducted on the models and performance of the prototypes are predicted. The complete similarity between the model and actual pump (prototype) will exist if the following conditions are satisfied :

1. Specific speed of model = Specific speed of prototype

$$(N_s)_m = (N_s)_p \quad \text{or} \quad \left( \frac{N \sqrt{Q}}{H_m^{3/4}} \right)_m = \left( \frac{N \sqrt{Q}}{H_m^{3/4}} \right)_p \quad \dots(19.19)$$

2. Tangential velocity ( $u$ ) is given by  $u = \frac{\pi D N}{60}$  also  $u \propto \sqrt{H_m}$

$$\therefore \sqrt{H_m} \propto DN$$

$$\therefore \frac{\sqrt{H_m}}{DN} = \text{Constant} \quad \dots(19.19 A)$$

$$\text{or } \left( \frac{\sqrt{H_m}}{DN} \right)_m = \left( \frac{\sqrt{H_m}}{DN} \right)_p \quad \dots(19.20)$$

3. From equation (ii) of Art. 19.7.1, we have

$$\begin{aligned} Q &\propto D^2 \times V_f \\ &\propto D^2 \times D \times N \\ &\propto D^3 \times N \end{aligned} \quad \text{where } V_f \propto u \propto DN$$

$$\therefore \frac{Q}{D^3N} = \text{Constant} \quad \text{or} \quad \left( \frac{Q}{D^3N} \right)_m = \left( \frac{Q}{D^3N} \right)_p \quad \dots(19.21)$$

$$4. \text{ Power of the pump, } P = \frac{\rho \times g \times Q \times H_m}{75}$$

$$\begin{aligned} \therefore P &\propto Q \times H_m \\ &\propto D^3 \times N \times H_m \\ &\propto D^3N \times D^2N^2 \\ &\propto D^5N^3 \end{aligned} \quad (\because Q \propto D^3N) \quad (\because \sqrt{H_m} \propto DN)$$

$$\therefore \frac{P}{D^5N^3} = \text{Constant} \quad \text{or} \quad \left( \frac{P}{D^5N^3} \right)_m = \left( \frac{P}{D^5N^3} \right)_p \quad \dots(19.22)$$

**Problem 19.18** A single-stage centrifugal pump with impeller diameter of 30 cm rotates at 2000 r.p.m. and lifts 3 m<sup>3</sup> of water per second to a height of 30 m with an efficiency of 75%. Find the number of stages and diameter of each impeller of a similar multistage pump to lift 5 m<sup>3</sup> of water per second to a height of 200 metres when rotating at 1500 r.p.m.

**Solution.** Given :

Single-stage pump :

$$\begin{aligned} \text{Diameter of impeller, } D_1 &= 30 \text{ cm} = 0.30 \text{ m} \\ \text{Speed, } N_1 &= 2000 \text{ r.p.m.} \\ \text{Discharge, } Q_1 &= 3 \text{ m}^3/\text{s} \\ \text{Height, } H_{m_1} &= 30 \text{ m} \\ \text{Efficiency, } \eta_{man} &= 75\% = 0.75. \end{aligned}$$

Multistage similar pump :

$$\begin{aligned} \text{Discharge, } Q_2 &= 5 \text{ m}^3/\text{s} \\ \text{Total height} &= 200 \text{ m} \\ \text{Let the height per stage} &= H_{m_2} \\ \text{Speed, } N_2 &= 1500 \text{ r.p.m.} \\ \text{Diameter of each impeller} &= D_2 \end{aligned}$$

Specific speed should be same. Hence, applying equation (19.19) as

$$\left( \frac{N\sqrt{Q}}{H_m^{3/4}} \right)_1 = \left( \frac{N\sqrt{Q}}{H_m^{3/4}} \right)_2$$

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$$\therefore \frac{N_1 \sqrt{Q_1}}{H_{m_1}^{3/4}} = \frac{N_2 \sqrt{Q_2}}{H_{m_2}^{3/4}} \quad \text{or} \quad \frac{2000 \times \sqrt{3}}{30^{3/4}} = \frac{1500 \times \sqrt{5}}{H_{m_2}^{3/4}}$$

$$\therefore H_{m_2}^{3/4} = \frac{1500 \times \sqrt{5} \times 30^{3/4}}{2000 \times \sqrt{3}} = \frac{1500}{2000} \times \sqrt{\frac{5}{3}} \times 12.818 = 12.411$$

$$\therefore H_{m_2} = (12.411)^{4/3} = 28.71 \text{ m}$$

$$\therefore \text{Number of stages} = \frac{\text{Total head}}{\text{Head per stage}} = \frac{200}{28.71} = 6.96 \approx 7. \text{ Ans.}$$

Using equation (19.20), we have

$$\frac{\sqrt{H_{m_1}}}{D_1 N_1} = \frac{\sqrt{H_{m_2}}}{D_2 N_2} \quad \text{or} \quad \frac{\sqrt{30}}{0.30 \times 2000} = \frac{\sqrt{28.71}}{D_2 \times 1500}$$

$$\therefore D_2 = \frac{0.30 \times 2000 \times \sqrt{28.71}}{1500 \times \sqrt{30}} = 0.3913 \text{ m} = 391.3 \text{ mm. Ans.}$$

**Problem 19.19** Find the number of pumps required to take water from a deep well under a total head of 89 m. All the pumps are identical and are running at 800 r.p.m. The specific speed of each pump is given as 25 while the rated capacity of each pump is  $0.16 \text{ m}^3/\text{s}$ .

**Solution.** Given :

Total head	= 89 m
Speed,	$N = 800 \text{ r.p.m.}$
Specific speed,	$N_s = 25$
Rate capacity,	$Q = 0.16 \text{ m}^3/\text{s}$
Let	$H_m = \text{Head developed by each pump.}$

Using equation (19.18),  $N_s = \frac{N \sqrt{Q}}{H_m^{3/4}}$

$$25 = \frac{800 \times \sqrt{0.16}}{H_m^{3/4}}$$

$$\therefore H_m^{3/4} = \frac{800 \times \sqrt{0.16}}{25} = 12.8$$

$$\therefore H_m = (12.8)^{4/3} = 29.94 \text{ m}$$

$$\therefore \text{Number of pumps required} = \frac{\text{Total head}}{\text{Head developed by one pump}} = \frac{89}{29.94} \approx 3. \text{ Ans.}$$

As the total head is more than the head developed by one pump, the pumps should be connected in series.

**Problem 19.20** Two geometrically similar pumps are running at the same speed of 1000 r.p.m. One pump has an impeller diameter of 0.30 metre and lifts water at the rate of 20 litres per second against a head of 15 metres. Determine the head and impeller diameter of the other pump to deliver half the discharge.

**Solution.** Given :

For pump No. 1,

$$\text{Speed, } N_1 = 1000 \text{ r.p.m.}$$

$$\text{Diameter, } D_1 = 0.30 \text{ m}$$

$$\text{Discharge, } Q_2 = 20 \text{ litres/s} = 0.02 \text{ m}^3/\text{s}$$

$$\text{Head, } H_{m_1} = 15 \text{ m}$$

For pump No. 2,

$$\text{Speed, } N_2 = 1000 \text{ r.p.m.}$$

$$\text{Discharge, } Q_2 = \frac{Q_1}{2} = \frac{20}{2} = 10 \text{ litres/s} = 0.01 \text{ m}^3/\text{s.}$$

Let  $D_2$  = Diameter of impeller

$H_{m_2}$  = Head developed.

$$\text{Using equation (19.19), } \frac{N_1 \sqrt{Q_1}}{H_{m_1}^{3/4}} = \frac{N_2 \sqrt{Q_2}}{H_{m_2}^{3/4}}$$

$$\therefore \frac{1000 \times \sqrt{0.02}}{15^{3/4}} = \frac{1000 \times \sqrt{0.01}}{H_{m_2}^{3/4}}$$

$$\text{or } H_{m_2}^{3/4} = \frac{1000 \times \sqrt{0.01} \times 15^{3/4}}{1000 \times \sqrt{0.02}} = \sqrt{\frac{0.01}{0.02}} \times 7.622 = 5.389$$

$$\therefore H_{m_2} = (5.389)^{4/3} = 9.44 \text{ m. Ans.}$$

$$\text{Using equation (19.20), } \left( \frac{\sqrt{H_m}}{DN} \right)_1 = \left( \frac{\sqrt{H_m}}{DN} \right)_2 \text{ or } \frac{\sqrt{H_{m_1}}}{D_1 N_1} = \frac{\sqrt{H_{m_2}}}{D_2 N_2}$$

$$\frac{\sqrt{15}}{0.3 \times 1000} = \frac{\sqrt{9.44}}{D_2 \times 1000}$$

$$\therefore D_2 = \frac{\sqrt{9.44} \times 0.3}{\sqrt{15}} = 0.238 \text{ m} = 238.0 \text{ mm. Ans.}$$

**Problem 19.21** The diameter of a centrifugal pump, which is discharging 0.03 m<sup>3</sup>/s of water against a total head of 20 m is 0.40 m. The pump is running at 1500 r.p.m. Find the head, discharge and ratio of powers of a geometrically similar pump of diameter 0.25 m when it is running at 3000 r.p.m.

**Solution.** Given :

Centrifugal pump,

$$\text{Discharge, } Q_1 = 0.03 \text{ m}^3/\text{s}$$

$$\text{Head, } H_{m_1} = 20 \text{ m}$$

$$\text{Diameter, } D_1 = 0.40 \text{ m}$$

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Speed,  $N_1 = 1500 \text{ r.p.m.}$

Geometrically similar pump,

Diameter,  $D_2 = 0.25 \text{ m}$

Speed,  $N_2 = 3000 \text{ r.p.m.}$

Let Head on similar group  $= H_{m_2}$

Discharge on similar pump  $= Q_2$

$$\text{Using equation (19.21), } \left( \frac{Q}{D^3 N} \right)_1 = \left( \frac{Q}{D^3 N} \right)_2$$

$$\therefore \frac{Q_1}{D_1^3 N_1} = \frac{Q_2}{D_2^3 N_2}$$

$$\frac{.03}{.40^3 \times 1500} = \frac{Q_2}{0.25^3 \times 3000}$$

$$\therefore Q_2 = \frac{.03 \times .25^3 \times 3000}{.40^3 \times 1500} = .03 \times \left( \frac{.25}{.40} \right)^3 \times 2.0 = \mathbf{0.01465 \text{ m}^3/\text{s. Ans.}}$$

Using equation (19.20), we have

$$\left( \frac{\sqrt{H_m}}{DN} \right)_1 = \left( \frac{\sqrt{H_m}}{DN} \right)_2$$

or

$$\frac{\sqrt{H_{m_1}}}{D_1 N_1} = \frac{\sqrt{H_{m_2}}}{D_2 N_2} \quad \therefore \quad \frac{\sqrt{20}}{0.40 \times 1500} = \frac{\sqrt{H_{m_2}}}{0.25 \times 3000}$$

or

$$\sqrt{H_{m_2}} = \frac{\sqrt{20} \times 0.25 \times 3000}{0.40 \times 1500} = 5.59$$

$$\therefore H_{m_2} = (5.59)^2 = \mathbf{31.25 \text{ m. Ans.}}$$

Using equation (19.22), we have

$$\left( \frac{P}{D^5 N^3} \right)_1 = \left( \frac{P}{D^5 N^3} \right)_2$$

$$\therefore \frac{P_1}{D_1^5 N_1^3} = \frac{P_2}{D_2^5 N_2^3} \quad \text{or} \quad \frac{P_1}{P_2} = \frac{D_1^5 N_1^3}{D_2^5 N_2^3} = \left( \frac{D_1}{D_2} \right)^5 \times \left( \frac{N_1}{N_2} \right)^3$$

$$= \left( \frac{0.40}{0.25} \right)^5 \times \left( \frac{1500}{3000} \right)^3 = 10.485 \times .125 = \mathbf{1.31. Ans.}$$

**Problem 19.22** A one-fifth scale model of a pump was tested in a laboratory at 1000 r.p.m. The head developed and the power input at the best efficiency point were found to be 8 m and 30 kW respectively. If the prototype pump has to work against a head of 25 m, determine its working speed, the power required to drive it and the ratio of the flow rates handled by the two pumps.

**Solution.** Given :

One-fifth scale model means that the ratio of linear dimensions of a model and its prototype is equal to 1/5.

Speed of model,

$$N_m = 1000 \text{ r.p.m.}$$

Head of model,

$$H_m = 8 \text{ m}$$

Power of model,

$$P_m = 30 \text{ kW}$$

Head of prototype,

$$H_p = 25 \text{ m}$$

Let

$$N_p = \text{Speed of prototype}$$

$$P_p = \text{Power of prototype}$$

$$Q_p = \text{Flow rate of prototype}$$

$$Q_m = \text{Flow rate of model}$$

(i) Speed of prototype

Using equation (19.20), we get

$$\left( \frac{\sqrt{H}}{DN} \right)_m = \left( \frac{\sqrt{H}}{DN} \right)_p \quad \text{or} \quad \frac{\sqrt{H_m}}{D_m N_m} = \frac{\sqrt{H_p}}{D_p N_p}$$

or

$$N_p = \frac{\sqrt{H_p}}{\sqrt{H_m}} \times \frac{D_m}{D_p} \times N_m$$

$$= \frac{\sqrt{25}}{\sqrt{8}} \times \frac{1}{5} \times 1000 \quad \left( \because \frac{D_m}{D_p} = \frac{1}{5} \right)$$

$$= 353.5 \text{ r.p.m. Ans.}$$

(ii) Power developed by prototype

Using equation (19.22), we get

$$\left( \frac{P}{D^5 N^3} \right)_m = \left( \frac{P}{D^5 N^3} \right)_p \quad \text{or} \quad \frac{P_m}{D_m^5 N_m^3} = \frac{P_p}{D_p^5 N_p^3}$$

or

$$P_p = P_m \times \left( \frac{D_p}{D_m} \right)^5 \times \left( \frac{N_p}{N_m} \right)^3 = 30 \times 5^5 \times \left( \frac{353.5}{1000} \right)^3 \quad \left( \because \frac{D_p}{D_m} = \frac{5}{1} \right)$$

$$= 30 \times 3125 \times 0.04419 = 4143 \text{ kW. Ans.}$$

(iii) Ratio of the flow rates of two pumps (i.e., model and prototype)

$$\left( \frac{Q}{D^3 N} \right)_m = \left( \frac{Q}{D^3 N} \right)_p \quad \text{or} \quad \frac{Q_m}{D_m^3 N_m} = \frac{Q_p}{D_p^3 N_p}$$

or

$$\frac{Q_p}{Q_m} = \frac{D_p^3 N_p}{D_m^3 N_m} = \left( \frac{D_p}{D_m} \right)^3 \times \frac{N_p}{N_m} = 5^3 \times \frac{353.5}{1000} \quad \left( \because \frac{D_p}{D_m} = \frac{5}{1} \right)$$

$$= 44.1875. \text{ Ans.}$$

### ► 19.9 PRIMING OF A CENTRIFUGAL PUMP

Priming of a centrifugal pump is defined as the operation in which the suction pipe, casing of the pump and a portion of the delivery pipe upto the delivery valve is completely filled up from outside source with the liquid to be raised by the pump before starting the pump. Thus the air from these parts of the pump is removed and these parts are filled with the liquid to be pumped.

The work done by the impeller per unit weight of liquid per sec is known as the head generated by

the pump. Equation (19.1) gives the head generated by the pump as  $\frac{1}{g} V_{w_2} u_2$  metre. This equation is independent of the density of the liquid. This means that when pump is running in air, the head generated is in terms of metre of air. If the pump is primed with water, the head generated is same metre of water. But as the density of air is very low, the generated head of air in terms of equivalent metre of water head is negligible and hence the water may not be sucked from the pump. To avoid this difficulty, priming is necessary.

### ► 19.10 CHARACTERISTIC CURVES OF CENTRIFUGAL PUMPS

Characteristic curves of centrifugal pumps are defined as those curves which are plotted from the results of a number of tests on the centrifugal pump. These curves are necessary to predict the behaviour and performance of the pump when the pump is working under different flow rate, head and speed. The following are the important characteristic curves for pumps :

1. Main characteristic curves,
2. Operating characteristic curves, and
3. Constant efficiency or Muschel curves.

**19.10.1 Main Characteristic Curves.** The main characteristic curves of a centrifugal pump consists of variation of head (manometric head,  $H_m$ ), power and discharge with respect to speed. For plotting curves of manometric head *versus* speed, discharge is kept constant. For plotting curves of discharge *versus* speed, manometric head ( $H_m$ ) is kept constant. And for plotting curves of power *versus* speed, the manometric head and discharge are kept constant. Fig. 19.14 shows main characteristic curves of a pump.

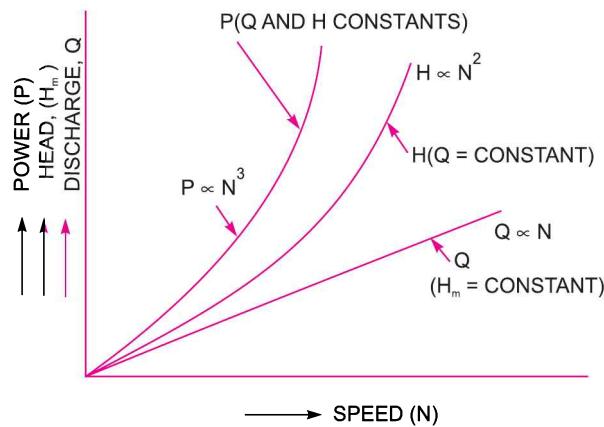


Fig. 19.14 Main characteristic curves of a pump.

For plotting the graph of  $H_m$  versus speed ( $N$ ), the discharge is kept constant. From equation (19.19 A), it is clear that  $\sqrt{H_m}/DN$  is a constant or  $H_m \propto N^2$ . This means that head developed by a pump is proportional to  $N^2$ . Hence the curve of  $H_m$  v/s  $N$  is a parabolic curves as shown in Fig. 19.14.

From equation (19.22), it is clear that  $P/D^5N^3$  is a constant. Hence  $P \propto N^3$ . This means that the curve  $P$  v/s  $N$  is a cubic curve as shown in Fig. 19.14.

Equation (19.21), shows that  $\frac{Q}{D^3N} = \text{constant}$ . This means  $Q \propto N$  for a given pump. Hence the curve  $Q$  v/s  $N$  is a straight line as shown in Fig. 19.14.

**19.10.2 Operating Characteristic Curves.** If the speed is kept constant, the variation of manometric head, power and efficiency with respect to discharge gives the operating characteristics of the pump. Fig. 19.15 shows the operating characteristic curves of a pump.

The input power curve for pumps shall not pass through the origin. It will be slightly away from the origin on the  $y$ -axis, as even at zero discharge some power is needed to overcome mechanical losses.

The head curve will have maximum value of head when discharge is zero.

The output power curve will start from origin as at  $Q = 0$ , output power ( $\rho QgH$ ) will be zero.

The efficiency curve will start from origin as at  $Q = 0$ ,  $\eta = 0$

$$\left( \because \eta = \frac{\text{Output}}{\text{Input}} \right)$$

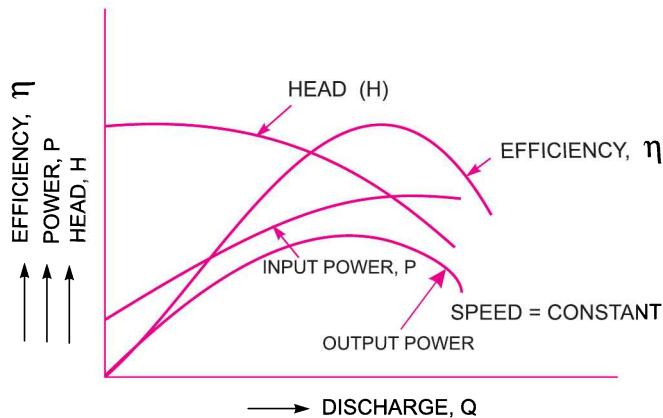


Fig. 19.15 Operating characteristic curves of a pump.

**19.10.3 Constant Efficiency Curves.** For obtaining constant efficiency curves for a pump, the head *versus* discharge curves and efficiency *versus* discharge curves for different speed are used. Fig. 19.16 (a) shows the head *versus* discharge curves for different speeds. The efficiency *versus* discharge curves for the different speeds are as shown in Fig. 19.16 (b). By combining these curves ( $H \sim Q$  curves and  $\eta \sim Q$  curves), constant efficiency curves are obtained as shown in Fig. 19.16 (a).

For plotting the constant efficiency curves (also known as iso-efficiency curves), horizontal lines representing constant efficiencies are drawn on the  $\eta \sim Q$  curves. The points, at which these lines cut the efficiency curves at various speeds, are transferred to the corresponding  $H \sim Q$  curves. The points having the same efficiency are then joined by smooth curves. These smooth curves represents the iso-efficiency curves.

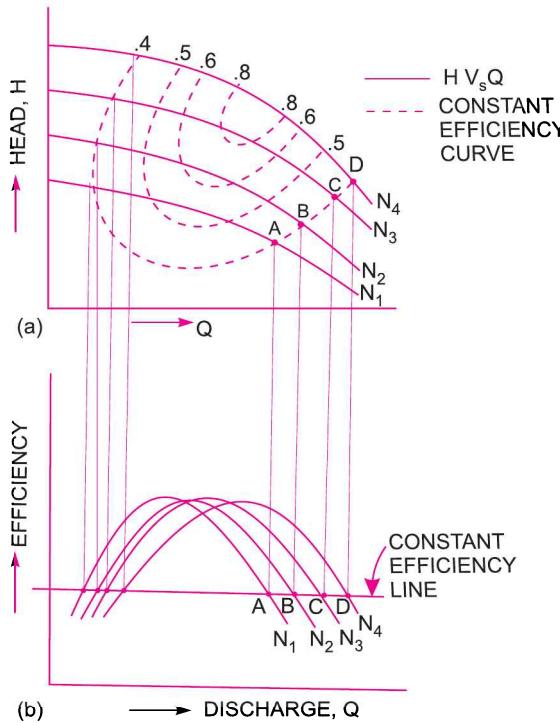


Fig. 19.16 Constant efficiency curves of a pump.

## ► 19.11 CAVITATION

Cavitation is defined as the phenomenon of formation of vapour bubbles of a flowing liquid in a region where the pressure of the liquid falls below its vapour pressure and the sudden collapsing of these vapour bubbles in a region of higher pressure. When the vapour bubbles collapse, a very high pressure is created. The metallic surfaces, above which these vapour bubbles collapse, is subjected to these high pressures, which cause pitting action on the surface. Thus cavities are formed on the metallic surface and also considerable noise and vibrations are produced.

Cavitation includes formation of vapour bubbles of the flowing liquid and collapsing of the vapour bubbles. Formation of vapour bubbles of the flowing liquid take place only whenever the pressure in any region falls below vapour pressure. When the pressure of the flowing liquid is less than its vapour pressure, the liquid starts boiling and vapour bubbles are formed. These vapour bubbles are carried along with the flowing liquid to higher pressure zones where these vapours condense and bubbles collapse. Due to sudden collapsing of the bubbles on the metallic surface, high pressure is produced and metallic surfaces are subjected to high local stresses. Thus the surfaces are damaged.

**19.11.1 Precaution Against Cavitation.** The following precautions should be taken against cavitation :

- The pressure of the flowing liquid in any part of the hydraulic system should not be allowed to fall below its vapour pressure. If the flowing liquid is water, then the absolute pressure head should not be below 2.5 m of water.

(ii) The special materials or coatings such as aluminium-bronze and stainless steel, which are cavitation resistant materials, should be used.

#### 19.11.2 Effects of Cavitation.

The following are the effects of cavitation :

- (i) The metallic surfaces are damaged and cavities are formed on the surfaces.
- (ii) Due to sudden collapse of vapour bubble, considerable noise and vibrations are produced.
- (iii) The efficiency of a turbine decreases due to cavitation. Due to pitting action, the surface of the turbine blades becomes rough and the force exerted by water on the turbine blades decreases. Hence, the work done by water or output horse power becomes less and thus efficiency decreases.

#### 19.11.3 Hydraulic Machines Subjected to Cavitation.

The hydraulic machines subjected to cavitation are reaction turbines and centrifugal pumps.

**19.11.4 Cavitation in Turbines.** In turbines, only reaction turbines are subjected to cavitation. In reaction turbines the cavitation may occur at the outlet of the runner or at the inlet of the draft-tube where the pressure is considerably reduced (*i.e.*, which may be below the vapour pressure of the liquid flowing through the turbine). Due to cavitation, the metal of the runner vanes and draft-tube is gradually eaten away, which results in lowering the efficiency of the turbine. Hence, the cavitation in a reaction turbine can be noted by a sudden drop in efficiency. In order to determine whether cavitation will occur in any portion of a reaction turbine, the critical value of Thoma's cavitation factor ( $\sigma$ , sigma) is calculated.

**Thoma's Cavitation Factor for Reaction Turbines.** Prof. D. Thoma suggested a dimensionless number, called after his name Thoma's cavitation factor  $\sigma$  (sigma), which can be used for determining the region where cavitation takes place in reaction turbines. The mathematical expression for the Thoma's cavitation factor is given by

$$\sigma = \frac{H_b - H_s}{H} = \frac{(H_{atm} - H_v) - H_s}{H} \quad \dots(19.23)$$

where  $H_b$  = Barometric pressure head in m of water,

$H_{atm}$  = Atmospheric pressure head in m of water,

$H_v$  = Vapour pressure head in m of water,

$H_s$  = Suction pressure at the outlet of reaction turbine in m of water or height of turbine runner above the tail water surface,

$H$  = Net head on the turbine in m.

**19.11.5 Cavitation in Centrifugal Pumps.** In centrifugal pumps the cavitation may occur at the inlet of the impeller of the pump, or at the suction side of the pumps, where the pressure is considerably reduced. Hence if the pressure at the suction side of the pump drops below the vapour pressure of the liquid then the cavitation may occur. The cavitation in a pump can be noted by a sudden drop in efficiency and head. In order to determine whether cavitation will occur in any portion of the suction side of the pump, the critical value of Thoma's cavitation factor ( $\sigma$ ) is calculated.

**Thoma's Cavitation Factor for Centrifugal Pumps.** The mathematical expression for Thoma's cavitation factor for centrifugal pump is given by

$$\sigma = \frac{(H_b) - H_s - h_{LS}}{H} = \frac{(H_{atm} - H_v) - H_s - h_{LS}}{H} \quad \dots(19.24)$$

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where  $H_{atm}$  = Atmospheric pressure head in m of water or absolute pressure head at the liquid surface in pump,

$H_V$  = Vapour pressure head in m of water,

$H_S$  = Suction pressure head in m of water,

$h_{LS}$  = Head lost due to friction in suction pipe, and

$H$  = Head developed by the pump.

The value of Thoma's cavitation factor ( $\sigma$ ) for a particular type of turbine or pump is calculated from equations (19.23) or (19.24). This value of Thoma's cavitation factor ( $\sigma$ ) is compared with critical cavitation factor ( $\sigma_c$ ) for that type of turbine pump. If the value of  $\sigma$  is greater than  $\sigma_c$ , the cavitation will not occur in that turbine or pump. The critical cavitation factor ( $\sigma_c$ ) may be obtained from tables or empirical relationships.

The following empirical relationships are used for obtaining the value of  $\sigma_c$  for different turbines :

$$\text{For Francis turbines, } \sigma_c = 0.625 \left( \frac{N_s}{380.78} \right)^2 \quad \dots(19.25)$$

$$\approx 431 \times 10^{-8} \text{ Ns}^2 \quad \dots(19.26)$$

$$\text{For Propeller turbines, } \sigma_c = 0.28 + \left[ \frac{1}{7.5} \left( \frac{N_s}{380.78} \right)^3 \right] \quad \dots(19.27)$$

In the above expressions  $N_s$  is in (r.p.m., kW, m) units. If  $N_s$  is in (r.p.m., h.p., m) units, the empirical relationships would be as follows :

$$\text{For Francis turbines, } \sigma_c = 0.625 \left( \frac{N_s}{444} \right)^2 \quad \dots(19.28)$$

$$\approx 317 \times 10^{-8} \times N_s^2 \quad \dots(19.29)$$

$$\text{For Propeller turbines, } \sigma_c = 0.28 + \left[ \frac{1}{7.5} \left( \frac{N_s}{444} \right)^3 \right] \quad \dots(19.30)$$

**Problem 19.23** A Francis turbine has been manufactured to develop 15000 horse power at the head of 81 m and speed 375 r.p.m. The mean atmospheric pressure at the site is 1.03 kgf/cm<sup>2</sup> and vapour pressure 0.03 kgf/cm<sup>2</sup>. Calculate the maximum permissible height of the runner above the tail water level to ensure cavitation free operation. The critical cavitation factor for Francis turbine is given by

$$\sigma_c = 317 \times 10^{-8} \times N_s^2$$

where  $N_s$  is the specific speed of the turbine in M.K.S. units.

**Solution.** Given :

Horse power developed,  $P = 15000$

Head,  $H = 81 \text{ m}$

Speed,  $N = 375 \text{ r.p.m.}$

Atmospheric pressure,  $p_a = 1.03 \text{ kgf/cm}^2 = 1.03 \times 9.81 \text{ N/cm}^2$   
 $= 1.03 \times 9.81 \times 10^4 \text{ N/m}^2$

$\therefore$  Atmospheric pressure head in meter of water,

$$H_{atm} = \frac{p_a}{\rho g} = \frac{1.03 \times 9.81 \times 10^4}{1000 \times 9.81} = 10.3 \text{ m}$$

Vapour pressure,  $p_v = 0.03 \text{ kgf/cm}^2 = 0.03 \times 9.81 \text{ N/cm}^2 = 0.03 \times 9.81 \times 10^4 \text{ N/m}^2$

$\therefore$  Vapour pressure head in meter of water,

$$H_v = \frac{p_v}{\rho g} = \frac{0.03 \times 9.81 \times 10^4}{1000 \times 9.81} = 0.3 \text{ m}$$

Critical cavitation factor,  $\sigma_c = 317 \times 10^{-8} N_s^2$  ... (i)

where  $N_s$  is the specific speed of the turbine in M.K.S. units i.e., (r.p.m., h.p., m) units.

Now specific speed of turbine is given by

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{375 \times \sqrt{15000}}{81^{5/4}} * = 189 \text{ r.p.m.}$$

Substituting this value in equation (i), we get

$$\sigma_c = 317 \times 10^{-8} \times 189^2 = 0.1132$$

Now let  $H_s$  = Suction pressure head in meter of water at the outlet of Francis turbine or height of the turbine runner above the tail water surface.

Now using equation (19.23), we get

$$\sigma_c = \frac{H_{atm} - H_v - H_s}{H} \quad \text{or} \quad 0.1132 = \frac{10.3 - 0.3 - H_s}{81}$$

or  $0.1132 \times 81 = 10 - H_s \quad \text{or} \quad H_s = 10 - 0.1132 \times 81 = 0.8308 \text{ m. Ans.}$

Hence, maximum permissible height is 0.83 m above the tail water level.

### ► 19.12 MAXIMUM SUCTION LIFT (or SUCTION HEIGHT)

Fig. 19.17 shows a centrifugal pump that lifts a liquid from a sump. The free surface of liquid is at a depth of  $h_s$  below the pump axis. The liquid is flowing with a velocity  $v_s$  in the suction pipe.

Let  $h_s$  = Suction height (or lift)

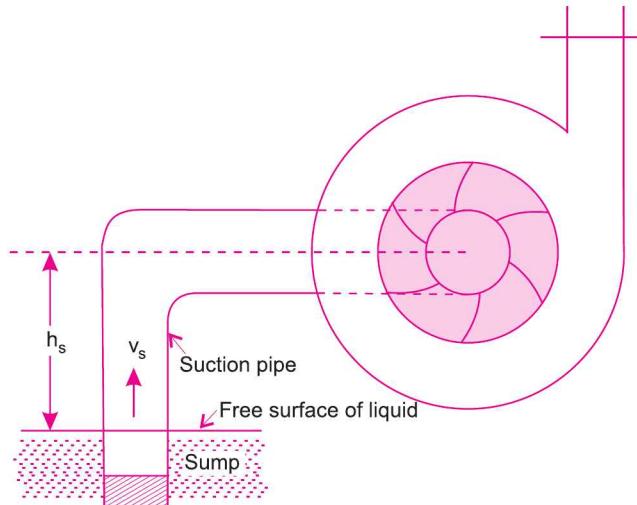


Fig. 19.17

\* Here power  $P$  should be taken in horse power and not in kW.

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Applying Bernoulli's equation at the free surface of liquid in the sump and section 1 in the suction pipe just at the inlet of the pump and taking the free surface of liquid as datum line, we get

$$\frac{p_a}{\rho g} + \frac{V_a^2}{2g} + Z_a = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 + h_L \quad \dots(i)$$

where  
 $p_a$  = Atmospheric pressure on the free surface of liquid,  
 $V_a$  = Velocity of liquid at the free surface of liquid  $\approx 0$ ,  
 $Z_a$  = Height of free surface from datum line = 0,  
 $p_1$  = Absolute pressure at the inlet of pump,  
 $V_1$  = Velocity of liquid through suction pipe =  $v_s$ ,  
 $Z_1$  = Height of inlet of pump from datum line =  $h_s$ ,  
 $h_L$  = Loss of head in the foot valve, strainer and suction pipe =  $h_{fs}$ .

Hence equation (i), after substituting the above values becomes as

$$\frac{p_a}{\rho g} + 0 + 0 = \frac{p_1}{\rho g} + \frac{v_s^2}{2g} + h_s + h_{fs}$$

or 
$$\frac{p_a}{\rho g} = \frac{p_1}{\rho g} + \frac{v_s^2}{2g} + h_s + h_{fs}$$

or 
$$\frac{p_1}{\rho g} = \frac{p_a}{\rho g} - \left( \frac{v_s^2}{2g} + h_s + h_{fs} \right) \quad \dots(ii)$$

For finding the maximum suction lift, the pressure at the inlet of the pump should not be less than the vapour pressure of the liquid. Hence for the limiting case, taking the pressure at the inlet of pump equal to vapour pressure of the liquid, we get

$p_1 = p_v$ , where  $p_v$  = vapour pressure of the liquid in absolute units.

Now the equation (ii) becomes as

$$\frac{p_v}{\rho g} = \frac{p_a}{\rho g} - \left( \frac{v_s^2}{2g} + h_s + h_{fs} \right)$$

or 
$$\frac{p_a}{\rho g} = \frac{p_v}{\rho g} + \frac{v_s^2}{2g} + h_s + h_{fs} \quad (\because p_1 = p_v) \dots(iii)$$

Now  $\frac{p_a}{\rho g}$  = Atmospheric pressure head =  $H_a$  (meter of liquid)

$\frac{p_v}{\rho g}$  = Vapour pressure head =  $H_v$  (meter of liquid)

Now, equation (iii) becomes as

$$H_a = H_v + \frac{v_s^2}{2g} + h_s + h_{fs}$$

or 
$$h_s = H_a - H_v - \frac{v_s^2}{2g} - h_{fs} \quad \dots(19.31)$$

Equation (19.31) gives the value of maximum suction lift (or maximum suction height) for a centrifugal pump. Hence, the suction height of any pump should not be more than that given by equation (19.31). If the suction height of the pump is more, then vaporization of liquid at inlet of pump will take place and there will be a possibility of cavitation.

### ► 19.13 NET POSITIVE SUCTION HEAD (NPSH)

The term NPSH ( Net Positive Suction Head) is very commonly used in the pump industry. Actually the minimum suction conditions are more frequently specified in terms of NPSH.

The net positive suction head (NPSH) is defined as the *absolute* pressure head at the inlet to the pump, minus the vapour pressure head ( in absolute units) plus the velocity head.

∴ NPSH = Absolute pressure head at inlet of the pump – vapour pressure head (absolute units) + velocity head

$$= \frac{p_1}{\rho g} - \frac{p_v}{\rho g} + \frac{v_s^2}{2g} \quad (\because \text{Absolute pressure at inlet of pump} = p_1) \dots (19.32)$$

But from equation (ii) of Art. 19.12, the absolute pressure head at inlet of the pump is given by as

$$\frac{p_1}{\rho g} = \frac{p_a}{\rho g} - \left( \frac{v_s^2}{2g} + h_s + h_{f_s} \right)$$

Substituting this value in equation (19.32) , we get

$$\begin{aligned} \text{NPSH} &= \left[ \frac{p_a}{\rho g} - \left( \frac{v_s^2}{2g} + h_s + h_{f_s} \right) \right] - \frac{p_v}{\rho g} + \frac{v_s^2}{2g} \\ &= \frac{p_a}{\rho g} - \frac{p_v}{\rho g} - h_s - h_{f_s} \\ &= H_a - H_v - h_s - h_{f_s} \\ &\quad \left( \because \frac{p_a}{\rho g} = H_a = \text{Atmospheric pressure head}, \frac{p_v}{\rho g} = H_v = \text{Vapour pressure head} \right) \\ &= \left[ (H_a - h_s - h_{f_s}) - H_v \right] \end{aligned} \dots (19.33)$$

The right hand side of equation (19.33) is the total suction head. Hence NPSH is equal to total suction head. Thus NPSH may also be defined as the total head required to make the liquid flow through the suction pipe to the pump impeller.

For any pump installation, a distinction is made between the required NPSH and the available NPSH. The value of required NPSH is given by the pump manufacturer. This value can also be determined experimentally. For determining its value, the pump is tested and minimum value of  $h_s$  is obtained at which the pump gives maximum efficiency without any objectional noise (*i.e.*, cavitation free). The required NPSH varies with the pump design, speed of the pump and capacity of the pump.

When the pump is installed, the available NPSH is calculated from equation (19.33). In order to have cavitation free operation of centrifugal pump, the available NPSH should be greater than the required NPSH.

### ► 19.14 CAVITATION IN CENTRIFUGAL PUMP

Thoma's cavitation factor is used to indicate whether cavitation will occur in pumps. Equation (19.24) gives the value of Thoma's cavitation factor for pumps as

$$\sigma = \frac{(H_{atm} - H_v) - H_s - h_{f_s}}{H}$$

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$$= \frac{H_a - H_v - h_s - h_{f_s}}{H_m} \quad (\because H_s = h_s \text{ and } h_{L_s} = h_{f_s}) \quad (H = H_m \text{ for pumps})$$

But from equation (19.33), we have

$$\begin{aligned} H_a - H_v - h_s - h_{f_s} &= \text{NPSH} && \text{(Net position suction head)} \\ \therefore \sigma &= \frac{\text{NPSH}}{H_m} && \dots(19.34) \end{aligned}$$

If the value of  $\sigma$  (calculated from equation 19.34) is less than the critical value,  $\sigma_c$  then cavitation will occur in the pumps. The value of  $\sigma_c$  depends upon the specific speed of the pump  $\left(N_s = \frac{N\sqrt{Q}}{H_m^{3/4}}\right)$ .

The following empirical relation is used to determine the value of  $\sigma_c$ .

$$\begin{aligned} \sigma_c &= 0.103 \left( \frac{N_s}{1000} \right)^{4/3} \\ &= 0.103 \frac{N_s^{4/3}}{(10^3)^{4/3}} = \frac{0.103 N_s^{4/3}}{10^4} \\ &= 1.03 \times 10^{-3} N_s^{4/3} && \dots(19.35) \end{aligned}$$

**Problem 19.24** A centrifugal pump rotating at 1000 r.p.m. delivers 160 litres/s of water against a head of 30 m. The pump is installed at a place where atmospheric pressure is  $1 \times 10^5 \text{ Pa}$  (abs.) and vapour pressure of water is 3 kPa (abs.). The head loss in suction pipe is equivalent to 0.2 m of water. Calculate :

- (i) Minimum NPSH, and
- (ii) Maximum allowable height of the pump from free surface of water in the sump.

**Solution.** Given :

$$\begin{aligned} N &= 1000 \text{ r.p.m.}; Q = 160 \text{ litres/s} = 0.16 \text{ m}^3/\text{s}; H_m = 30 \text{ m} \\ p_a &= 1 \times 10^5 \text{ Pa (abs.)} = 1 \times 10^5 \text{ N/m}^2 \text{ (abs.)}; p_v = 3 \text{ kPa (abs.)} = 3 \times 10^3 \text{ N/m}^2 \text{ (abs.)} \\ h_{f_s} &= 0.2 \text{ m.} \end{aligned}$$

(i) Minimum NPSH

Using equation (19.34), we get

$$\sigma = \frac{\text{NPSH}}{H_m}$$

From the above equation, it is clear that NPSH is directly proportional to Thoma's cavitation factor ( $\sigma$ ). NPSH will be minimum when  $\sigma$  is minimum. But the minimum value of  $\sigma$  for no cavitation is  $\sigma_c$ . Hence when  $\sigma = \sigma_c$  then NPSH will be minimum.

$$\therefore \sigma_c = \frac{(\text{NPSH})_{\min}}{H_m}$$

$$\text{or} \quad (\text{NPSH})_{\min} = H_m \times \sigma_c \quad \dots(i)$$

Now the critical value of  $\sigma$  i.e.,  $\sigma_c$  is given by equation (19.35) as

$$\sigma_c = 1.03 \times 10^{-3} \times N_s^{4/3} \quad \dots(ii)$$

where  $N_s$  = Specific speed of pump =  $\frac{N\sqrt{Q}}{H_m^{3/4}}$

$$= 1000 \times \frac{\sqrt{0.16}}{30^{3/4}} \quad (\because N = 1000 \text{ r.p.m.}, Q = 0.16 \text{ m}^3/\text{s} \text{ and } H_m = 30 \text{ m})$$

Substituting the value of  $N_s$  in equation (ii), we get

$$\begin{aligned}\sigma_c &= 1.03 \times 10^{-3} \times \left[ \frac{1000 \times \sqrt{0.16}}{30^{3/4}} \right]^{4/3} \\ &= 1.03 \times 10^{-3} \times \frac{1000^{4/3} \times 0.16^{2/3}}{30} = \frac{1.03 \times 10^{-3} \times 10^4 \times 0.2947}{30} \\ &= 0.1012\end{aligned}$$

Substituting the value of  $\sigma_c$  in equation (i), we get

$$\begin{aligned}(\text{NPSH})_{\min.} &= H_m \times 0.1012 \\ &= 30 \times 0.1012 = 3.036 \text{ m. Ans.} \quad (\because H_m = 30 \text{ m})\end{aligned}$$

(ii) Maximum allowable height of the pump from free surface of water in the sump (i.e.,  $h_s$ )

Let  $(h_s)_{\max.}$  = Max. allowable height of pump from free surface of water.

Using equation (19.33)

$$\text{NPSH} = H_a - H_v - h_s - h_{f_s} \quad \dots(i)$$

From the above equation, it is clear that for a given value of atmospheric pressure head  $\left( H_a = \frac{p_a}{\rho g} \right)$ , given vapour pressure head  $\left( H_v = \frac{p_v}{\rho g} \right)$  and given loss of head due to friction ( $h_{f_s}$ ), the value of suction head ( $h_s$ ) will be maximum if NPSH is minimum.

$$\begin{aligned}\therefore \quad (\text{NPSH})_{\min.} &= H_a - H_v - (h_s)_{\max.} - h_{f_s} \quad \dots(ii) \\ \therefore \quad (h_s)_{\max.} &= H_a - H_v - h_{f_s} - (\text{NPSH})_{\min.} \quad \dots(iii)\end{aligned}$$

$$\text{Now } H_a = \frac{p_a}{\rho g} = \frac{1 \times 10^5}{1000 \times 9.81} = 10.193 \text{ m of water}$$

$$H_v = \frac{p_v}{\rho g} = \frac{3 \times 10^3}{1000 \times 9.81} = 0.305 \text{ m of water}$$

$$h_{f_s} = 0.2 \text{ m and } (\text{NPSH})_{\min.} = 3.036 \text{ m}$$

Substituting the values of  $H_a$ ,  $H_v$ ,  $h_{f_s}$  and  $(\text{NPSH})_{\min.}$  in equation (iii), we get

$$\begin{aligned}(h_s)_{\max.} &= 10.193 - 0.305 - 0.2 - 3.036 \\ &= 6.652 \text{ m. Ans.}\end{aligned}$$

## HIGHLIGHTS

1. The hydraulic machine which converts the mechanical energy into pressure energy by means of centrifugal force is called centrifugal pump.
2. The centrifugal pump acts as a reverse of an inward radial flow reaction turbine. The work done by the impeller (rotating part of the pump) on the water per second per unit weight of water per second flowing through the pump is given as

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$$= \frac{1}{g} V_{w_2} \times u_2$$

where  $V_{w_2}$  = Velocity of whirl at outlet, and

$u_2$  = Tangential velocity of wheel at outlet.

3. The vertical height of the centre-line of the centrifugal pump from the water surface in the pump is called the suction head ( $h_s$ ).
4. Delivery head ( $h_d$ ) is the vertical distance between the centre-line of the pump and the water surface in the tank to which water is lifted.
5. Manometric head ( $H_m$ ) is the head against which a centrifugal pump has to work. It is given as

$$(a) \quad H_m = \frac{V_{w_2} \times u_2}{g} - \text{Loss of head in impeller and casing}$$

$$= \frac{V_{w_2} \times u_2}{g} \dots \text{if losses in pump is zero}$$

$$(b) \quad H_m = \text{Total head at outlet} - \text{Total head at inlet of pump}$$

$$= \left( \frac{p_o}{\rho g} + \frac{V_o^2}{2g} + Z_o \right) - \left( \frac{p_i}{\rho g} + \frac{V_i^2}{2g} + Z_i \right)$$

$$(c) \quad H_m = h_s + h_d + h_{f_s} + h_{f_d} + \frac{V_d^2}{2g}.$$

6. The efficiencies of a pump are : (i) Manometric efficiency ( $\eta_{man}$ ), (ii) Mechanical efficiency ( $\eta_m$ ), and (iii) Overall efficiency ( $\eta_o$ ). Mathematically they are given as

$$\eta_{man} = \frac{gH_m}{V_{w_2} \times u_2}$$

$$\eta_m = \frac{W \left( \frac{V_{w_2} \times u}{75} \right)}{\text{S.P.}}, \text{ where } W = w \times Q$$

$$\eta_o = \frac{W \times H_m}{1000 \times \text{S.P.}}$$

7. The minimum speed for starting a centrifugal pump is given by  $N = \frac{120 \times \eta_{man} \times V_{w_2} \times D_2}{\pi [D_2^2 - D_1^2]}.$

8. If a centrifugal pump consists of two or more impellers, the pump is called a multistage pump. To produce a high head, the impellers are connected in series while to discharge a large quantity of liquid, the impellers are connected in parallel.
9. The specific speed of a centrifugal pump is defined as the speed at which a pump runs when the head developed is one metre and discharge is one cubic metre. Mathematically, it is given as

$$N_s = \frac{N \sqrt{Q}}{H_m^{3/4}}, \text{ where } H_m = \text{Manometric hand.}$$

10. For complete similarity between the model and actual centrifugal pump (prototype) the following conditions should be satisfied :

$$(a) \left( \frac{N\sqrt{Q}}{H_m^{3/4}} \right)_{\text{model}} = \left( \frac{N\sqrt{Q}}{H_m^{3/4}} \right)_{\text{prototype}} \quad (b) \left( \frac{\sqrt{H_m}}{DN} \right)_{\text{model}} = \left( \frac{\sqrt{H_m}}{DN} \right)_{\text{prototype}}$$

$$(c) \left( \frac{Q}{D^3 N} \right)_{\text{model}} = \left( \frac{Q}{D^3 N} \right)_{\text{prototype}} \quad (d) \left( \frac{P}{D^5 N^3} \right)_{\text{model}} = \left( \frac{P}{D^5 N^3} \right)_{\text{prototype}}$$

11. Characteristic curves are used for predicting the behaviour and performance of a pump when it is working under different flow rate, head and speed.
12. Cavitation is defined as the phenomenon of formation of vapour bubbles and sudden collapsing of the vapour bubbles.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. Define a centrifugal pump. Explain the working of a single-stage centrifugal pump with sketches.
2. Differentiate between the volute casing and vortex casing for the centrifugal pump.
3. Obtain an expression for the work done by impeller of a centrifugal pump on water per second per unit weight of water.
4. Define the terms : suction head, delivery head, static head and manometric head.
5. What do you mean by manometric efficiency, mechanical efficiency and overall efficiency of a centrifugal pump?
6. How will you obtain an expression for the minimum speed for starting a centrifugal pump?
7. What is the difference between single-stage and multistage pumps? Describe multistage pump with (a) impellers in parallel, and (b) impellers in series.
8. Define specific speed of a centrifugal pump. Derive an expression for the same.
9. How does the specific speed of a centrifugal pump differ from that of a turbine ?
10. What is priming ? Why is it necessary ?
11. How the model testing of the centrifugal pumps are made?
12. What do you understand by characteristic curves of a pump? What is the significance of the characteristic curves?
13. Define cavitation. What are the effects of cavitation ? Give the necessary precautions against cavitation.
14. How will you determine the possibility of the cavitation to occur in the installation of a turbine or a pump?
15. Why are centrifugal pumps used sometimes in series and sometimes in parallel ? Draw the following characteristic curves for a centrifugal pump :
  - Head, power and efficiency *versus* discharge with constant speed.
16. Draw and discuss the operating characteristics of a centrifugal pump.
17. (a) What is cavitation and what are its causes ? How will you prevent the cavitation in hydraulic machines ?
  - (b) What is cavitation? State its effects on the performance of water turbines and also state how to prevent cavitation in water turbines.
18. Briefly state the significance of similarity parameters in hydraulic pumps.
19. The frictional torque  $T$  of a disc of diameter  $D$  rotating at a speed of  $N$  in a fluid of viscosity  $\mu$  and density  $\rho$  in a turbulent flow is given by :

$$T = D^5 N^2 \rho \phi \left[ \frac{\mu}{D^2 N \rho} \right]$$

Prove this by method of dimensions.

20. With a neat sketch, explain the principle and working of a centrifugal pump.
21. Explain the following terms as they are applied to a centrifugal pump :
  - (i) Static suction lift ; (ii) static suction head ; (iii) static discharge head ; and (iv) total static head.
22. (a) How does a volute casing differ from a vortex casing for the centrifugal pump ?  
 (b) What is priming ? Why is it necessary ?  
 (c) What do you mean by pump characteristics ? Briefly explain the uses of such characteristics.

(Jawaharlal Nehru Technical University, S 2002)

### (B) NUMERICAL PROBLEMS

1. The internal and external diameters of the impeller of a centrifugal pump are 300 mm and 600 mm respectively. The pump is running at 1000 r.p.m. The vane angles at inlet and outlet are  $20^\circ$  and  $30^\circ$  respectively. The water enters the impeller radially and velocity of flow is constant. Determine the work done by the impeller per unit weight of water. **[Ans. 68.89 Nm/N]**
2. A centrifugal pump having outer diameter equal to two times the inner diameter and running at 1200 r.p.m. works against a total head of 75 m. The velocity of flow through the impeller is constant and equal to 3 m/s. The vanes are set back at an angle of  $30^\circ$  at outlet. If the outer diameter of the impeller is 600 mm and width at outlet is 50 mm, determine : (a) vane angle at inlet, (b) work done per second by impeller, (c) manometric efficiency. **[Ans. (a)  $9^\circ 2'$ , (b) 346.37 kW, (c) 60%]**
3. A centrifugal pump is running at 1000 r.p.m. The outlet vane angle of the impeller is  $30^\circ$  and velocity of flow at outlet is 3 m/s. The pump is working against a total head of 30 m and the discharge through the pump is  $0.3 \text{ m}^3/\text{s}$ . If the manometric efficiency of the pump is 75%, determine: (i) the diameter of the impeller, and (ii) the width of the impeller at outlet. **[Ans. (i) 43.1 cm, (ii) 7.4 cm]**
4. Find the power required to drive a centrifugal pump which delivers  $0.02 \text{ m}^3/\text{s}$  of water to a height of 30 m through a 10 cm diameter pipe and 90 m long. The overall efficiency of the pump is 70% and  $f = .009$  in the formula

$$h_f = \frac{4fV^2}{d \times 2g}. \quad \text{[Ans. 11.5 kW]}$$

5. Find the rise in pressure in the impeller of a centrifugal pump through which water is flowing at the rate of 15 litre/s. The internal and external diameters of the impeller are 20 cm and 40 cm respectively. The widths of impeller at inlet and outlet are 1.6 cm and 0.8 cm. The pump is running at 1200 r.p.m. The water enters the impeller radially at inlet and impeller vane angle at outlet is  $30^\circ$ . Neglect losses through the impeller. **[Ans. 31.85 m]**
6. The diameters of an impeller of a centrifugal pump at inlet and outlet are 20 cm and 40 cm respectively. Determine the minimum speed for starting the pump if it works against a head of 25 m. **[Ans. 1221.2 r.p.m.]**
7. The diameter of an impeller of a centrifugal pump at inlet and outlet are 300 mm and 600 mm respectively. The velocity of flow at outlet is 2.5 m/s and vanes are set back at an angle of  $45^\circ$  at outlet. Determine the minimum starting speed of the pump if the manometric efficiency is 75%. **[Ans. 159.31 r.p.m.]**
8. A three-stage centrifugal pump has impeller 40 cm in diameter and 2.5 cm wide at outlet. The vanes are curved back at the outlet at  $30^\circ$  and reduce the circumferential area by 15%. The manometric efficiency is 85% and overall efficiency is 75%. Determine the head generated by the pump when running at 12000 r.p.m. and discharge is  $0.06 \text{ m}^3/\text{s}$ . Find the shaft power also. **[Ans. 138.75 m, 108.89 kW]**

9. Find the number of pumps required to take water from a deep well under a total head of 156 m. Also the pumps are identical and are running at 1000 r.p.m. The specific speed of each pump is given as 20 while the rated capacity of each pump is 150 litre/s. [Ans. 3]
10. The diameter of a centrifugal pump, which is discharging  $0.035 \text{ m}^3/\text{s}$  of water against a total head of 25 m is 0.05 m. The pump is running at 1200 r.p.m. Find the head, discharge and ratio of powers of a geometrically similar pump of diameter 0.3 m when it is running at 2000 r.p.m. [Ans. 25 m,  $0.0126 \text{ m}^3/\text{s}$ , 2.777]
11. A centrifugal pump is to discharge  $0.12 \text{ m}^3$  at a speed of 1400 r.p.m. against a head of 30 m. The diameter and width of the impeller at outlet are 25 cm and 5 cm respectively. If the manometric efficiency is 75%, determine the vane angle at outlet.



# 20

## CHAPTER

# RECIPROCATING PUMPS



### ► 20.1 INTRODUCTION

In the last chapter, we have defined the pumps as the hydraulic machines which convert the mechanical energy into hydraulic energy which is mainly in the form of pressure energy. If the mechanical energy is converted into hydraulic energy, by means of centrifugal force acting on the liquid, the pump is known as centrifugal pump. But if the mechanical energy is converted into hydraulic energy (or pressure energy) by sucking the liquid into a cylinder in which a piston is reciprocating (moving backwards and forwards), which exerts the thrust on the liquid and increases its hydraulic energy (pressure energy), the pump is known as reciprocating pump.

### ► 20.2 MAIN PARTS OF A RECIPROCATING PUMP

The following are the main parts of a reciprocating pump as shown in Fig. 20.1 :

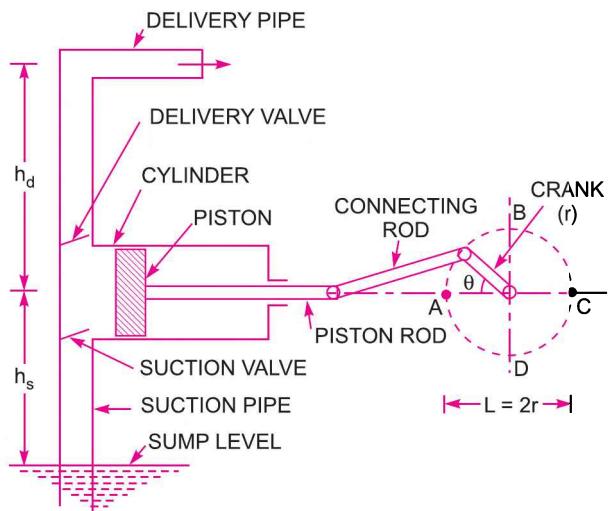


Fig. 20.1 Main parts of a reciprocating pump.

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1. A cylinder with a piston, piston rod, connecting rod and a crank,
2. Suction pipe,                   3. Delivery pipe,
4. Suction valve, and           5. Delivery valve.

### ► 20.3 WORKING OF A RECIPROCATING PUMP

Fig. 20.1 shows a single acting reciprocating pump, which consists of a piston which moves forwards and backwards in a close fitting cylinder. The movement of the piston is obtained by connecting the piston rod to crank by means of a connecting rod. The crank is rotated by means of an electric motor. Suction and delivery pipes with suction valve and delivery valve are connected to the cylinder. The suction and delivery valves are one way valves or non-return valves, which allow the water to flow in one direction only. Suction valve allows water from suction pipe to the cylinder which delivery valve allows water from cylinder to delivery pipe only.

When crank starts rotating, the piston moves to and fro in the cylinder. When crank is at  $A$ , the piston is at the extreme left position in the cylinder. As the crank is rotating from  $A$  to  $C$ , (i.e., from  $\theta = 0^\circ$  to  $\theta = 180^\circ$ ), the piston is moving towards right in the cylinder. The movement of the piston towards right creates a partial vacuum in the cylinder. But on the surface of the liquid in the sump atmospheric pressure is acting, which is more than the pressure inside the cylinder. Thus, the liquid is forced in the suction pipe from the sump. This liquid opens the suction valve and enters the cylinder.

When crank is rotating from  $C$  to  $A$  (i.e., from  $\theta = 180^\circ$  to  $\theta = 360^\circ$ ), the piston from its extreme right position starts moving towards left in the cylinder. The movement of the piston towards left increases the pressure of the liquid inside the cylinder more than atmospheric pressure. Hence suction valve closes and delivery valve opens. The liquid is forced into the delivery pipe and is raised to a required height.

**20.3.1 Discharge Through a Reciprocating Pump.** Consider a single\* acting reciprocating pump as shown in Fig. 20.1.

Let  $D$  = Diameter of the cylinder

$A$  = Cross-sectional area of the piston or cylinder

$$= \frac{\pi}{4} D^2$$

$r$  = Radius of crank

$N$  = r.p.m. of the crank

$L$  = Length of the stroke =  $2 \times r$

$h_s$  = Height of the axis of the cylinder from water surface in sump.

$h_d$  = Height of delivery outlet above the cylinder axis (also called delivery head)

Volume of water delivered in one revolution or discharge of water in one revolution

$$= \text{Area} \times \text{Length of stroke} = A \times L$$

$$\text{Number of revolution per second, } = \frac{N}{60}$$

∴ Discharge of the pump per second,

$$\begin{aligned} Q &= \text{Discharge in one revolution} \times \text{No. of revolution per second} \\ &= A \times L \times \frac{N}{60} = \frac{ALN}{60} \end{aligned} \quad \dots(20.1)$$

\* Single acting means the water is acting on one side of the piston only.

Weight of water delivered per second,

$$W = \rho \times g \times Q = \frac{\rho g A L N}{60}. \quad \dots(20.2)$$

**20.3.2 Work done by Reciprocating Pump.** Work done by the reciprocating pump per second is given by the reaction as

$$\begin{aligned} \text{Work done per second} &= \text{Weight of water lifted per second} \times \text{Total height through which water is lifted} \\ &= W \times (h_s + h_d) \end{aligned} \quad \dots(i)$$

where  $(h_s + h_d)$  = Total height through which water is lifted.

From equation (20.2), Weight,  $W$ , is given by

$$W = \frac{\rho g \times A L N}{60}.$$

Substituting the value of  $W$  in equation (i), we get

$$\text{Work done per second} = \frac{\rho g \times A L N}{60} \times (h_s + h_d) \quad \dots(20.3)$$

∴ Power required to drive the pump, in kW

$$\begin{aligned} P &= \frac{\text{Work done per second}}{1000} = \frac{\rho g \times A L N \times (h_s + h_d)}{60 \times 1000} \\ &= \frac{\rho g \times A L N \times (h_s + h_d)}{60,000} \text{ kW} \end{aligned} \quad \dots(20.4)$$

**20.3.3 Discharge, Work done and Power Required to Drive a Double-acting Pump.** In case of double-acting pump, the water is acting on both sides of the piston as shown in Fig. 20.2. Thus, we require two suction pipes and two delivery pipes for double-acting pump. When there is a suction stroke on one side of the piston, there is at the same time a delivery stroke on the other side of the piston. Thus for one complete revolution of the crank there are two delivery strokes and water is delivered to the pipes by the pump during these two delivery strokes.

Let  $D$  = Diameter of the piston,

$d$  = Diameter of the piston rod

∴ Area on one side of the piston,

$$A = \frac{\pi}{4} D^2$$

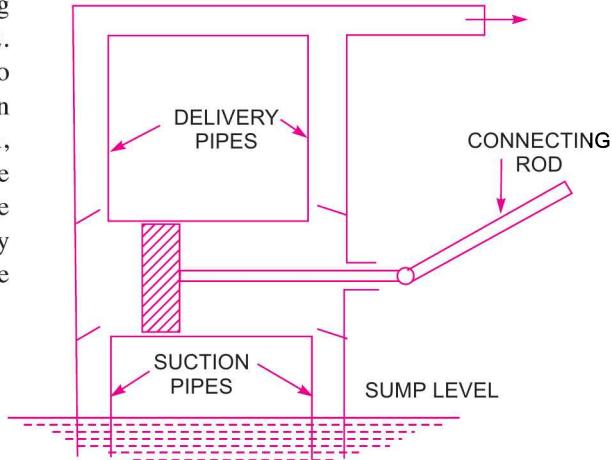


Fig. 20.2

Area on the other side of the piston, where piston rod is connected to the piston,

$$A_1 = \frac{\pi}{4} D^2 - \frac{\pi}{4} d^2 = \frac{\pi}{4} (D^2 - d^2).$$

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$$\begin{aligned}
 \therefore \text{Volume of water delivered in one revolution of crank} \\
 &= A \times \text{Length of stroke} + A_1 \times \text{Length of stroke} \\
 &= AL + A_1 L = (A + A_1)L = \left[ \frac{\pi}{4}D^2 + \frac{\pi}{4}(D^2 - d^2) \right] \times L \\
 \therefore \text{Discharge of pump per second} \\
 &= \text{Volume of water delivered in one revolution} \times \text{No. of revolution} \\
 &\quad \text{per second} \\
 &= \left[ \frac{\pi}{4}D^2 + \frac{\pi}{4}(D^2 - d^2) \right] \times L \times \frac{N}{60}
 \end{aligned}$$

If 'd' the diameter of the piston rod is very small as compared to the diameter of the piston, then it can be neglected and discharge of pump per second,

$$Q = \left( \frac{\pi}{4}D^2 + \frac{\pi}{4}D^2 \right) \times \frac{L \times N}{60} = 2 \times \frac{\pi}{4}D^2 \times \frac{L \times N}{60} = \frac{2ALN}{60} \dots(20.5)$$

Equation (20.5) gives the discharge of a double-acting reciprocating pump. This discharge is two times the discharge of a single-acting pump.

### Work done by double-acting reciprocating pump

Work done per second = Weight of water delivered × Total height

$$\begin{aligned}
 &= \rho g \times \text{Discharge per second} \times \text{Total height} \\
 &= \rho g \times \frac{2ALN}{60} \times (h_s + h_d) = 2\rho g \times \frac{ALN}{60} \times (h_s + h_d) \quad \dots(20.6)
 \end{aligned}$$

$\therefore$  Power required to drive the double-acting pump in kW,

$$\begin{aligned}
 P &= \frac{\text{Work done per second}}{1000} = 2\rho g \times \frac{ALN}{60} \times \frac{(h_s + h_d)}{1000} \\
 &= \frac{2\rho g \times ALN \times (h_s + h_d)}{60,000} \quad \dots(20.7)
 \end{aligned}$$

## ► 20.4 SLIP OF RECIPROCATING PUMP

Slip of a pump is defined as the difference between the theoretical discharge and actual discharge of the pump. The discharge of a single-acting pump given by equation (20.1) and of a double-acting pump given by equation (20.5) are theoretical discharge. The actual discharge of a pump is less than the theoretical discharge due to leakage. The difference of the theoretical discharge and actual discharge is known as slip of the pump. Hence, mathematically,

$$\text{Slip} = Q_{th} - Q_{act} \quad \dots(20.8)$$

But slip is mostly expressed as percentage slip which is given by,

$$\begin{aligned}
 \text{Percentage slip} &= \frac{Q_{th} - Q_{act}}{Q_{th}} \times 100 = \left( 1 - \frac{Q_{act}}{Q_{th}} \right) \times 100 \\
 &= (1 - C_d) \times 100 \quad \left( \because \frac{Q_{act}}{Q_{th}} = C_d \right) \quad \dots(20.9)
 \end{aligned}$$

where  $C_d$  = Co-efficient of discharge.

**20.4.1 Negative Slip of the Reciprocating Pump.** Slip is equal to the difference of theoretical discharge and actual discharge. If actual discharge is more than the theoretical discharge, the slip of the pump will become -ve. In that case, the slip of the pump is known as negative slip.

Negative slip occurs when delivery pipe is short, suction pipe is long and pump is running at high speed.

## ► 20.5 CLASSIFICATION OF RECIPROCATING PUMPS

The reciprocating pumps may be classified as :

1. According to the water being in contact with one side or both sides of the piston, and
  2. According to the number of cylinders provided.

If the water is in contact with one side of the piston, the pump is known as single-acting. On the other hand, if the water is in contact with both sides of the piston, the pump is called double-acting. Hence, classification according to the contact of water is :

- (i) Single-acting pump, and (ii) Double-acting pump.

According to the number of cylinder provided, the pumps are classified as :

- (i) Single cylinder pump,  
(ii) Double cylinder pump, and  
(iii) Triple cylinder pump.

**Problem 20.1** A single-acting reciprocating pump, running at 50 r.p.m., delivers  $0.01 \text{ m}^3/\text{s}$  of water. The diameter of the piston is 200 mm and stroke length 400 mm. Determine :

(i) The theoretical discharge of the pump, (ii) Co-efficient of discharge, and (iii) Slip and the percentage slip of the pump.

**Solution.** Given :

Speed of the pump,  $N = 50$  r.p.m.

Actual discharge,  $O_{act} = .01 \text{ m}^3/\text{s}$

$$\text{Dia. of piston, } D = 200 \text{ mm} = .20 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} (.2)^2 = .031416 \text{ m}^2$$

Stroke,  $L = 400 \text{ mm} = 0.40 \text{ m.}$

(i) Theoretical discharge for single-acting reciprocating pump is given by equation (20.1) as

$$Q_{th} = \frac{A \times L \times N}{60} = \frac{.031416 \times .40 \times 50}{60} = \mathbf{0.01047 \text{ m}^3/\text{s. Ans.}}$$

(ii) Co-efficient of discharge is given by

$$C_d = \frac{Q_{act}}{Q_{st}} = \frac{0.01}{.01047} = \mathbf{0.955. \text{ Ans.}}$$

(iii) Using equation (20.8), we get

$$\text{Slip} = Q_{st} - Q_{act} \equiv .01047 - .01 \equiv \mathbf{0.00047 \text{ m}^3/\text{s.}} \text{ Ans.}$$

And percentage slip

$$\begin{aligned}
 &= \frac{(Q_{th} - Q_{act})}{Q_{th}} \times 100 = \frac{(.01047 - .01)}{.01047} \times 100 \\
 &= \frac{.00047}{.01047} \times 100 = \mathbf{4.489\%}. \text{ Ans.}
 \end{aligned}$$

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**Problem 20.2** A double-acting reciprocating pump, running at 40 r.p.m., is discharging 1.0 m<sup>3</sup> of water per minute. The pump has a stroke of 400 mm. The diameter of the piston is 200 mm. The delivery and suction head are 20 m and 5 m respectively. Find the slip of the pump and power required to drive the pump.

**Solution.** Given:

Speed of pump,

$$N = 40 \text{ r.p.m.}$$

Actual discharge,

$$Q_{act} = 1.0 \text{ m}^3/\text{min} = \frac{1.0}{60} \text{ m}^3/\text{s} = 0.01666 \text{ m}^3/\text{s}$$

Stroke,

$$L = 400 \text{ mm} = 0.40 \text{ m}$$

Diameter of piston,

$$D = 200 \text{ mm} = 0.20 \text{ m}$$

∴ Area,

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.2)^2 = 0.031416 \text{ m}^2$$

Suction head,

$$h_s = 5 \text{ m}$$

Delivery head,

$$h_d = 20 \text{ m.}$$

Theoretical discharge for double-acting pump is given by equation (20.5) as,

$$Q_{th} = \frac{2ALN}{60} = \frac{2 \times 0.031416 \times 0.4 \times 40}{60} = .01675 \text{ m}^3/\text{s.}$$

Using equation (20.8), Slip =  $Q_{th} - Q_{act} = .01675 - .01666 = .00009 \text{ m}^3/\text{s. Ans.}$

Power required to drive the double-acting pump is given by equation (20.7) as,

$$\begin{aligned} P &= \frac{2 \times \rho g \times ALN \times (h_s + h_d)}{60,000} = \frac{2 \times 1000 \times 9.81 \times 0.031416 \times 0.4 \times 40 \times (5 + 20)}{60,000} \\ &= 4.109 \text{ kW. Ans.} \end{aligned}$$

### ► 20.6 VARIATION OF VELOCITY AND ACCELERATION IN THE SUCTION AND DELIVERY PIPES DUE TO ACCELERATION OF THE PISTON

It is mentioned in Art. 20.3 that when crank starts rotating, the piston moves forwards and backwards in the cylinder. At the extreme left position and right position of the piston in the cylinder, the velocity of the piston is zero. The velocity of the piston is maximum at the centre of the cylinder. This means that at the start of a stroke (may be suction or delivery stroke), the velocity of the piston is zero and this velocity becomes maximum at the centre of each stroke and again becomes zero at the end of each stroke. Thus at the beginning of each stroke, the piston will be having an acceleration and at the end of each stroke, the piston will be having a retardation. The water in the cylinder is in contact with the piston and hence the water, flowing from the suction pipe or to the delivery pipe will have an acceleration at the beginning of each stroke and a retardation at the end of each stroke. This means the velocity of flow of water in the suction and delivery pipe will not be uniform. Hence, an accelerative or retarding head will be acting on the water flowing through the suction or delivery pipe. This accelerative or retarding head will change the pressure inside the cylinder.

If the ratio of length of connecting rod to the radius of crank (*i.e.*,  $L/r$ ) is very large, then the motion of the piston can be assumed as simple harmonic in nature. Fig. 20.3 shows the cylinder of a reciprocating single-acting pump, fitted with a piston which is connected to the crank. Let the crank is rotating at a constant angular speed.

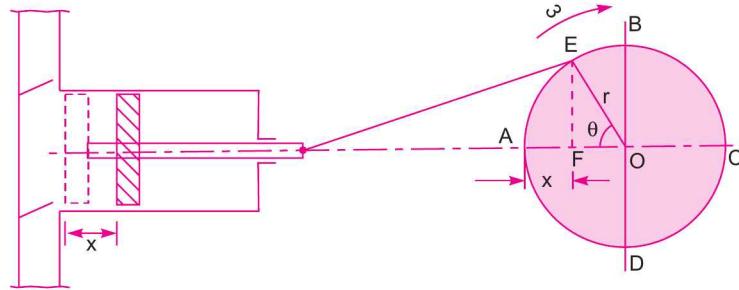


Fig. 20.3 Velocity and acceleration of piston.

Let  $\omega$  = Angular speed of the crank in rad./s,

$A$  = Area of the cylinder,

$a$  = Area of the pipe (suction or delivery),

$l$  = Length of the pipe (suction or delivery), and

$r$  = Radius of the crank.

In the beginning, the crank is at  $A$  (which is called inner dead centre) and the piston in the cylinder is at a position shown by dotted lines. The crank is rotating with an angular velocity  $\omega$  and let in time ' $t$ ' seconds, the crank turns through an angle  $\theta$  (in radians) from  $A$  (*i.e.*, inner dead centre). The displacement of the piston in time ' $t$ ' is ' $x$ ' as shown in Fig. 20.3.

Now

$$\begin{aligned}\theta &= \text{Angle turned by crank in radians in time } 't' \\ &= \omega t\end{aligned} \quad \dots(i)$$

The distance  $x$  travelled by the piston is given as

$$\begin{aligned}x &= \text{Distance } AF = AO - FO \\ &= r - r \cos \theta \quad (\because AO = r, FO = r \cos \theta) \\ &= r - r \cos (\omega t) \quad (\because \text{From (i), } \theta = \omega t) \dots(ii)\end{aligned}$$

The velocity of the piston is obtained by differentiating equation (ii) with respect to ' $t$ ' .

$$\begin{aligned}\therefore \text{Velocity of piston, } V &= \frac{dx}{dt} = \frac{d}{dt} [r - r \cos (\omega t)] \\ &= 0 - r [-\sin \omega t] \times \omega \quad (\because r \text{ is constant}) \\ &= \omega r \sin \omega t.\end{aligned} \quad \dots(20.10)$$

Now from continuity equation, the volume of water flowing into cylinder per second is equal to the volume of water flowing from the pipe per second.

$\therefore$  Velocity of water in cylinder  $\times$  Area of cylinder

$$= \text{Velocity of water in pipe} \times \text{Area of pipe}$$

or  $V \times A = v \times a$  ( $\because$  Velocity of water in cylinder = Velocity of piston =  $V$ )

where  $v$  = Velocity of water in pipe

$$\begin{aligned}\therefore v &= \frac{V \times A}{a} = \frac{A}{a} \times V \\ &= \frac{A}{a} \omega r \sin \omega t \quad [\because \text{From (20.10), } V = \omega r \sin \omega t]\end{aligned} \quad \dots(20.11)$$

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The acceleration of water in pipe is obtained by differentiating equation (20.11) with respect to 't'.

∴ Acceleration of water in pipe

$$= \frac{dv}{dt} = \frac{d}{dt} \left( \frac{A}{a} \omega r \sin \omega t \right) = \frac{A}{a} \omega^2 r \cos \omega t \quad \dots(20.12)$$

Mass of water in pipe =  $\rho \times$  Volume of water in pipe

$$= \rho \times [\text{Area of pipe} \times \text{Length of pipe}] = \rho \times [a \times l] = \rho al$$

∴ Force required to accelerate the water in the pipe

$$= \text{Mass of water in pipe} \times \text{Acceleration of water in pipe}$$

$$= \rho al \times \frac{A}{a} \omega^2 r \cos \omega t$$

∴ Intensity of pressure due to acceleration

$$= \frac{\text{Force required to accelerate the water}}{\text{Area of pipe}}$$

$$= \frac{\rho al \times \frac{A}{a} \omega^2 r \cos \omega t}{a} = \rho l \times \frac{A}{a} \omega^2 r \cos \omega t$$

$$= \rho l \times \frac{A}{a} \omega^2 r \cos \theta \quad (\because \omega t = \theta)$$

∴ Pressure head ( $h_a$ ) due to acceleration

$$h_a = \frac{\text{Intensity of pressure due to acceleration}}{\text{Weight density of liquid}}$$

$$= \frac{\rho l \times \frac{A}{a} \omega^2 r \cos \theta}{\rho g} = \frac{l}{g} \times \frac{A}{a} \omega^2 r \cos \theta. \quad \dots(20.13)$$

The pressure head due to acceleration in the suction and delivery pipes is obtained from equation (20.13) by using subscripts 's' and 'd' as

$$h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r \cos \theta \quad \dots(20.14)$$

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 r \cos \theta. \quad \dots(20.15)$$

The pressure head ( $h_a$ ) due to acceleration, given by equation (20.13) varies with  $\theta$ . The values of ' $h_a$ ' for different values of  $\theta$  are :

- |                                |  |                          |
|--------------------------------|--|--------------------------|
| 1. When $\theta = 0^\circ$ ,   | $h_a = \frac{l}{g} \times \frac{A}{a} \omega^2 r$  | as $\cos 0^\circ = 1$    |
| 2. When $\theta = 90^\circ$ ,  | $h_a = 0$  | as $\cos 90^\circ = 0$   |
| 3. When $\theta = 180^\circ$ , | $h_a = -\frac{l}{g} \times \frac{A}{a} \omega^2 r$ | as $\cos 180^\circ = -1$ |

∴ Maximum pressure head due to acceleration

$$(h_a)_{max} = \frac{l}{g} \times \frac{A}{a} \omega^2 r \quad \dots(20.16)$$

#### ► 20.7 EFFECT OF VARIATION OF VELOCITY ON FRICTION IN THE SUCTION AND DELIVERY PIPES

The velocity of water in suction or delivery pipe is given by equation (20.11) as

$$v = \frac{A}{a} \omega r \sin \theta = \frac{A}{a} \omega r \sin \theta \quad \dots(i)$$

Loss of head due to friction in pipes is given by

$$h_f = \frac{4flv^2}{d \times 2g} \quad \dots(ii)$$

where  $f$  = Co-efficient of friction,  $l$  = Length of pipe,

$d$  = Diameter of pipe, and  $v$  = Velocity of water in pipe.

Substituting equation (i) into equation (ii), we get

$$h_f = \frac{4fl}{d \times 2g} \times \left[ \frac{A}{a} \omega r \sin \theta \right]^2 \quad \dots(20.17)$$

The variation of  $h_f$  with  $\theta$  is parabolic. The loss of head due to friction in suction and delivery pipes is obtained from equation (20.17) by using subscripts 's' for suction pipe and 'd' for delivery pipe as

$$h_{fs} = \frac{4fl_s}{d_s \times 2g} \times \left[ \frac{A}{a_s} \omega r \sin \theta \right]^2 \quad \dots(20.18)$$

$$h_{fd} = \frac{4fl_d}{d_d \times 2g} \times \left[ \frac{A}{a_d} \omega r \sin \theta \right]^2 \quad \dots(20.19)$$

The loss of head due to friction in pipes given by equation (20.17) varies with  $\theta$  as :

$$1. \text{ When } \theta = 0^\circ, \quad \sin \theta = 0 \quad \therefore \quad h_f = \frac{4fl}{d \times 2g} \times 0 = 0$$

$$2. \text{ When } \theta = 90^\circ, \quad \sin 90^\circ = 1 \quad \therefore \quad h_f = \frac{4fl}{d \times 2g} \times \left[ \frac{A}{a} \omega r \right]^2$$

$$3. \text{ When } \theta = 180^\circ, \quad \sin 180^\circ = 0 \quad \therefore \quad h_f = 0$$

∴ Maximum value of loss of head due to friction ;

$$(h_f)_{max} = \frac{4fl}{d \times 2g} \times \left[ \frac{A}{a} \omega r \right]^2 \quad \dots(20.20)$$

**Problem 20.3** The cylinder bore diameter of a single-acting reciprocating pump is 150 mm and its stroke is 300 mm. The pump runs at 50 r.p.m. and lifts water through a height of 25 m. The delivery pipe is 22 m long and 100 mm in diameter. Find the theoretical discharge and the theoretical power required to run the pump. If the actual discharge is 4.2 litres/s, find the percentage slip. Also determine the acceleration head at the beginning and middle of the delivery stroke.

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**Solution.** Given :

$$\text{Dia. of cylinder, } D = 150 \text{ mm} = 0.15 \text{ m}$$

$$\therefore \text{Area, } A = \left( \frac{\pi}{4} \right) \times 0.15^2 = 0.01767 \text{ m}^2$$

$$\text{Stroke, } L = 300 \text{ mm} = 0.3 \text{ m}$$

$$\text{Speed of pump, } N = 50 \text{ r.p.m.}$$

Total height through which water is lifted,

$$H = 25 \text{ m}$$

$$\text{Length of delivery pipe, } l_d = 22 \text{ m}$$

$$\text{Diameter of delivery pipe, } d_d = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Actual discharge, } Q_{act} = 4.2 \text{ litres/s} = \frac{4.2}{1000} \text{ m}^3/\text{s} = 0.0042 \text{ m}^3/\text{s.}$$

(i) *Theoretical discharge ( $Q_{th}$ )*

Theoretical discharge for a single-acting reciprocating pump is given by equation (20.1), as

$$Q_{th} = \frac{A \times L \times N}{60} = \frac{0.01767 \times 0.3 \times 50}{60} = 0.0044175 \text{ m}^3/\text{s}$$

$$= 0.0044175 \times 1000 \text{ litres/s} = \mathbf{4.4175 \text{ litres/s. Ans.}}$$

(ii) *Theoretical power ( $P_t$ )*

$$\text{Theoretical power is given by, } P_t = \frac{(\text{Theoretical weight of water lifted/s}) \times \text{Total height}}{1000}$$

$$= \frac{\rho \times g \times Q_{th} \times H}{1000}$$

$$= \frac{1000 \times 9.81 \times 0.0044175 \times 25}{1000} \quad (\because Q_{th} = 0.0044175 \text{ m}^3/\text{s})$$

$$= \mathbf{1.0833 \text{ kW. Ans.}}$$

(iii) *The percentage slip*

The percentage slip is given by,

$$\% \text{ slip} = \left( \frac{Q_{th} - Q_{act}}{Q_{th}} \right) \times 100 = \left( \frac{4.4175 - 4.2}{4.4175} \right) \times 100 = \mathbf{4.92\%. Ans.}$$

(iv) *Acceleration head at the beginning of delivery stroke.*

The acceleration head in the delivery pipe is given by equation (20.15) as :

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 r \times \cos \theta$$

$$\text{where } a_d = \text{Area of delivery pipe} = \frac{\pi}{4} \times (0.1)^2 = 0.007854$$

$$\omega = \text{Angular speed} = \frac{2\pi N}{60} = \frac{2\pi \times 50}{60} = 5.236$$

$$r = \text{Crank radius} = \frac{L}{2} = \frac{0.3}{2} = 0.15 \text{ m}$$

$$\therefore h_{ad} = \frac{22}{9.81} \times \frac{0.01767}{0.007854} \times 5.236^2 \times 0.15 \times \cos \theta = 20.75 \times \cos \theta$$

At the beginning of delivery stroke,  $\theta = 0^\circ$  and hence  $\cos \theta = 1$

$$\therefore h_{ad} = 20.75 \text{ m. Ans.}$$

(v) *Acceleration head at the middle of delivery stroke.*

At the middle of delivery stroke,  $\theta = 90^\circ$  and hence  $\cos \theta = 0$ .

$$\therefore h_{ad} = 20.75 \times 0 = 0. \text{ Ans.}$$

## ► 20.8 INDICATOR DIAGRAM

The indicator diagram for a reciprocating pump is defined as the graph between the pressure head in the cylinder and the distance travelled by piston from inner dead centre for one complete revolution of the crank. As the maximum distance travelled by the piston is equal to the stroke length and hence the indicator diagram is a graph between pressure head and stroke length of the piston for one complete revolution. The pressure head is taken as ordinate and stroke length as abscissa.

**20.8.1 Ideal Indicator Diagram.** The graph between pressure head in the cylinder and stroke length of the piston for one complete revolution of the crank under ideal conditions is known as ideal indicator diagram. Fig. 20.4 shows the ideal indicator diagram, in which line  $EF$  represents the atmospheric pressure head equal to 10.3 m of water.

Let  $H_{atm}$  = Atmospheric pressure head  
= 10.3 m of water,

$L$  = Length of the stroke,

$h_s$  = Suction head, and

$h_d$  = Delivery head.

During suction stroke, the pressure head in the cylinder is constant and equal to suction head ( $h_s$ ), which is below the atmospheric pressure head ( $H_{atm}$ ) by a height of  $h_s$ . The pressure head during suction stroke is represented by a horizontal line  $AB$  which is below the line  $EF$  by a height of ' $h_s$ '.

During delivery stroke, the pressure head in the cylinder is constant and equal to delivery head ( $h_d$ ), which is above the atmospheric head by a height of ( $h_d$ ). Thus, the pressure head during delivery stroke is represented by a horizontal line  $CD$  which is above the line  $EF$  by a height of  $h_d$ . Thus, for one complete revolution of the crank, the pressure head in the cylinder is represented by the diagram  $A-B-C-D-A$ . This diagram is known as ideal indicator diagram.

Now from equation (20.3), we know that the work done by the pump per second

$$\begin{aligned}
 &= \frac{\rho \times g \times ALN}{60} \times (h_s + h_d) \\
 &= K \times L(h_s + h_d) \quad \left( \text{where } K = \frac{\rho g A N}{60} = \text{Constant} \right) \\
 &\propto L \times (h_s + h_d) \quad \dots(i)
 \end{aligned}$$

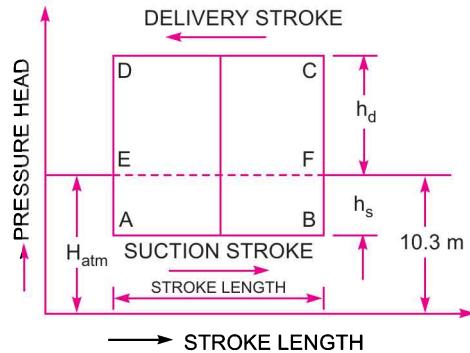


Fig. 20.4 *Ideal indicator diagram.*

But from Fig. 20.4, area of indicator diagram

$$= AB \times BC = AB \times (BF + FC) = L \times (h_s + h_d).$$

Substituting this value in equation (i), we get

$$\text{Work done by pump} \propto \text{Area of indicator diagram.} \quad \dots(20.21)$$

### 20.8.2 Effect of Acceleration in Suction and Delivery Pipes on Indicator Diagram.

The pressure head due to acceleration in the suction pipe is given by equation (20.14) as

$$h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r \cos \theta$$

$$\text{When } \theta = 0^\circ, \quad \cos \theta = 1, \quad \text{and} \quad h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r$$

$$\text{When } \theta = 90^\circ, \quad \cos \theta = 0, \quad \text{and} \quad h_{as} = 0$$

$$\text{When } \theta = 180^\circ, \quad \cos \theta = -1, \quad \text{and} \quad h_{as} = -\frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r.$$

Thus, the pressure head inside the cylinder during suction stroke will not be equal to ' $h_s$ ', as was the case for ideal indicator diagram, but it will be equal to the sum of ' $h_s$ ' and ' $h_{as}$ '. At the beginning of suction stroke  $\theta = 0^\circ$ , ' $h_{as}$ ' is +ve and hence the pressure head in the cylinder will be ( $h_s + h_{as}$ ) below the atmospheric pressure head. At the middle of suction stroke  $\theta = 90^\circ$  and  $h_{as} = 0$  and hence pressure head in the cylinder will be  $h_s$  below the atmospheric pressure head. At the end of suction stroke,  $\theta = 180^\circ$  and  $h_{as}$  is -ve and hence the pressure head in the cylinder will be ( $h_s - h_{as}$ ) below the atmospheric pressure head. For suction stroke, the indicator diagram will be shown by  $A'GB'$ . Also the area of  $A'AG$  = Area of  $BGB'$ .

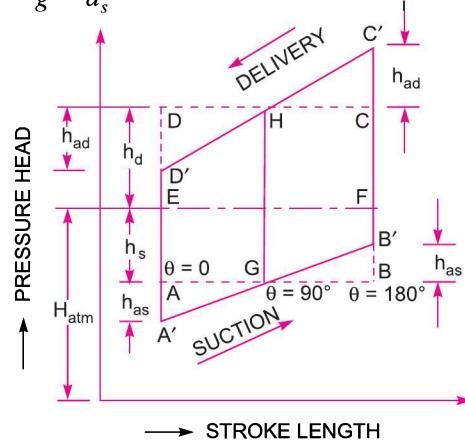


Fig. 20.5 Effect of acceleration on indicator diagram.

Similarly, the indicator diagram for the delivery stroke can be drawn. At the beginning of delivery stroke,  $h_{ad}$  is +ve and hence the pressure head in the cylinder will be ( $h_d + h_{ad}$ ) above the atmospheric pressure head. At the middle of the delivery stroke,  $h_{ad} = 0$  and hence pressure head in the cylinder is equal to  $h_d$  above the atmospheric pressure head. At the end of the delivery stroke,  $h_{ad}$  is -ve and hence pressure in the cylinder will be ( $h_d - h_{ad}$ ) above the atmospheric pressure head. And thus the indicator diagram for delivery stroke is represented by the line  $C'HD'$ . Also, the area of  $CC'H$  = Area of  $DD'H$ .

From Fig. 20.5, it is now clear that due to acceleration in suction and delivery pipe, the indicator diagram has changed from  $ABCD$  to  $A'B'C'D'$ . But the area of indicator diagram  $ABCD$  = Area  $A'B'C'D'$ . Now from equation (20.21), work done by pump is proportional to the area of indicator diagram. Hence the work done by the pump on the water remains same.

**Problem 20.4** The length and diameter of a suction pipe of a single-acting reciprocating pump are 5 m and 10 cm respectively. The pump has a plunger of diameter 15 cm and a stroke length of 35 cm. The centre of the pump is 3 m above the water surface in the pump. The atmospheric pressure head is 10.3 m of water and pump is running at 35 r.p.m. Determine :

- (i) Pressure head due to acceleration at the beginning of the suction stroke,
- (ii) Maximum pressure head due to acceleration, and
- (iii) Pressure head in the cylinder at the beginning and at the end of the stroke.

**Solution.** Given :

$$\text{Length of suction pipe, } l_s = 5 \text{ m}$$

$$\text{Diameter of suction pipe, } d_s = 10 \text{ cm} = 0.1 \text{ m}$$

$$\therefore \text{Area, } a_s = \frac{\pi}{4} d_s^2 = \frac{\pi}{4} \times 0.1^2 = .007854 \text{ m}^2$$

$$\text{Diameter of plunger, } D = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \text{Area of plunger, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times .15^2 = .01767 \text{ m}^2$$

$$\text{Stroke length, } L = 35 \text{ cm} = 0.35 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{0.35}{2} = 0.175 \text{ m}$$

$$\text{Suction head, } h_s = 3 \text{ m}$$

$$\text{Atmospheric pressure head, } H_{atm} = 10.3 \text{ m of water}$$

$$\text{Speed of pump, } N = 35 \text{ r.p.m.}$$

Angular speed of the crank is given by,

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 35}{60} = 3.665 \text{ rad/s.}$$

(i) The pressure head due to acceleration in the suction pipe is given by equation (20.14) as

$$h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r \cos \theta$$

At the beginning of stroke,  $\theta = 0^\circ$  and hence  $\cos \theta = 1$

$$\therefore h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r = \frac{5}{9.81} \times \frac{.01767}{.007854} \times 3.665^2 \times .175 = 2.695 \text{ m. Ans.}$$

(ii) Maximum pressure head due to acceleration in suction pipe is given by equation (20.16), as

$$(h_{as})_{\max} = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r = 2.695 \text{ m. Ans.}$$

(iii) Pressure head in the cylinder at the beginning of the suction stroke (Refer to Fig. 20.5)

$$= h_s + h_{as} = 3.0 + 2.695 = 5.695.$$

This pressure head in the cylinder is below the atmospheric pressure head.

$\therefore$  Absolute pressure head in the cylinder at the beginning of suction stroke

$$= \text{Atmospheric pressure head} - 5.695$$

$$= 10.3 - 5.695 = 4.605 \text{ m of water (abs.) Ans.}$$

Similarly, pressure head in the cylinder at the end of suction stroke

$$= h_s - h_{as} = 3.0 - 2.695 = 0.305 \text{ m below atmospheric pressure head}$$

$$= 10.3 - 0.305 = 9.995 \text{ m of water (abs.) Ans.}$$

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**Problem 20.5** If in Problem 20.4, the length and diameter of the delivery pipe are 30 m and 10 cm respectively and water is delivered by the pump to a tank which is 20 m above the centre of the pump, determine :

- Pressure head due to acceleration at the beginning of delivery stroke,
- Pressure head in the cylinder at the beginning of the delivery stroke, and
- Pressure head in the cylinder at the end of the delivery stroke.

**Solution.** Given :

$$\text{Length of delivery pipe, } l_d = 30 \text{ m}$$

$$\text{Diameter of delivery pipe, } d_d = 10 \text{ cm} = 0.1 \text{ m}$$

$$\therefore \text{Area of delivery pipe, } a_d = \frac{\pi}{4} d_d^2 = \frac{\pi}{4} (0.1)^2 = .007854 \text{ m}^2$$

$$\text{Diameter of plunger, } D = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \text{Area of plunger, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.15)^2 = .01767 \text{ m}^2$$

$$\text{Stroke length, } L = 35 \text{ cm} = 0.35 \text{ m}$$

$$\text{Crank radius, } r = \frac{L}{2} = \frac{0.35}{2} = 0.175 \text{ m}$$

$$\text{Delivery head, } h_d = 20 \text{ m}$$

$$\text{Speed of pump, } N = 35 \text{ r.p.m.}$$

$$\text{Angular speed, } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 35}{60} = 3.665 \text{ rad/s.}$$

(i) Using equation (20.15), we get the pressure head due to acceleration in delivery pipe as

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 r \cos \theta$$

At the beginning of delivery stroke,  $\theta = 0^\circ$  and hence  $\cos \theta = 1$ .

$\therefore$  Pressure head due to acceleration at the beginning of delivery stroke becomes as

$$\begin{aligned} h_{ad} &= \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 r \\ &= \frac{30}{9.81} \times \frac{.01767}{.007854} \times (3.665)^2 \times 0.175 = 16.17 \text{ m. Ans.} \end{aligned}$$

(ii) From Fig. 20.5, the pressure head in the cylinder at the beginning of the delivery stroke

$$\begin{aligned} &= FC' = FC + CC' = (h_d + h_{ad}) \text{ m of water above atmospheric head} \\ &= 20 + 16.17 = 36.17 \text{ m of water above atms.} \\ &= 36.17 + \text{Atmospheric pressure head} \\ &= 36.17 + 10.3 = 46.47 \text{ m (abs.) Ans.} \end{aligned}$$

(iii) The pressure head in the cylinder at the end of delivery stroke

$$\begin{aligned} &= ED' \text{ above atmospheric pressure head} \\ &= (ED - DD') = (h_d - h_{ad}) \\ &= 20 - 16.17 = 3.83 \text{ m of water above atms.} \\ &= 3.83 + 10.3 = 14.13 \text{ m (abs.) Ans.} \end{aligned}$$

**Problem 20.6** A single-acting reciprocating pump has piston diameter 12.5 cm and stroke length 30 cm. The centre of the pump is 4 m above the water level in the sump. The diameter and length of suction pipe are 7.5 cm and 7 m respectively. The separation occurs if the absolute pressure head in the cylinder during suction stroke falls below 2.5 m of water. Calculate the maximum speed at which the pump can run without separation. Take atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

$$\text{Diameter of piston, } D = 12.5 \text{ cm} = 0.125 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4}(12.5)^2 = .01227 \text{ m}^2$$

$$\text{Stroke length, } L = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{0.30}{2} = 0.15 \text{ m}$$

$$\text{Suction head, } h_s = 4.0 \text{ m}$$

$$\text{Diameter of suction pipe, } d_s = 7.5 \text{ cm} = 0.075 \text{ m}$$

$$\therefore \text{Area of suction pipe, } a_s = \frac{\pi}{4}(0.075)^2 = .004418 \text{ m}^2$$

$$\text{Length of suction pipe, } l_s = 7.0 \text{ m}$$

$$\text{Separation pressure head, } h_{sep} = 2.5 \text{ m (absolute)}$$

$$\text{Atmospheric pressure head, } H_{atm} = 10.3 \text{ m}$$

From the indicator diagram, drawn in Fig. 20.5, it is clear that the absolute pressure head during suction stroke is minimum at the beginning of the stroke. Thus, the separation can take place at the beginning of the stroke only. In that case the pressure head in the cylinder at the beginning of stroke becomes =  $h_{sep}$ .

But pressure head in the cylinder at the beginning of suction stroke

$$= (h_s + h_{as}) \text{ m below atmospheric pressure head}$$

$$= \text{Atmospheric pressure head} - (h_s + h_{as}) \text{ m absolute}$$

$$= H_{atm} - (h_s + h_{as}) \text{ m (abs.)}$$

$$= 10.3 - (4.0 + h_{as})$$

$$\therefore h_{sep} = 10.3 - (4.0 + h_{as})$$

$$2.5 = 10.3 - 4.0 - h_{as}$$

$$\text{or } h_{as} = 10.3 - 4.0 - 2.5 = 3.80 \text{ m.} \quad \dots(i)$$

But from equation (20.14),  $h_{as}$  at the beginning of suction stroke is given by the relation

$$h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r \quad (\because \theta = 0^\circ, \therefore \cos \theta = 1) \dots(ii)$$

Equating equations (i) and (ii), we get

$$3.80 = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r = \frac{7.0}{9.81} \times \frac{.01227}{.004418} \times \omega^2 \times .15$$

$$\therefore \omega^2 = \frac{3.80 \times 9.81 \times .004418}{7.0 \times .01227 \times .15} = 12.783$$

$$\text{or } \omega = \sqrt{12.783} = 3.575 \text{ radian/s.}$$

But

$$\omega = \frac{2\pi N}{60}$$

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$$\therefore N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 3.575}{2\pi} = 34.14 \text{ r.p.m. Ans.}$$

Thus, the maximum speed at which the pump can run without separation is 34.14 r.p.m.

**Problem 20.7** The diameter and stroke length of a single-acting reciprocating pump are 100 mm and 300 mm respectively. The water is lifted to a height of 20 m above the centre of the pump. Find the maximum speed at which the pump may be run so that no separation occurs during the delivery stroke if the diameter and length of delivery pipe are 50 mm and 25 m respectively. Separation occurs if the absolute pressure head in the cylinder during delivery stroke falls below 2.50 m of water.

Take atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

$$\text{Diameter of pump, } D = 100 \text{ mm} = 0.1 \text{ m}$$

$$\text{Stroke length, } L = 300 \text{ mm} = 0.30 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{0.30}{2} = 0.15 \text{ m}$$

$$\text{Delivery head, } h_d = 20 \text{ m}$$

$$\text{Diameter of delivery pipe, } d_d = 50 \text{ mm} = 0.05 \text{ m}$$

$$\text{Length of delivery pipe, } l_d = 25 \text{ m}$$

$$\text{Separation pressure head, } h_{sep} = 2.5 \text{ m (abs.)}$$

$$\text{Atmospheric pressure head, } H_{atm} = 10.3 \text{ m of water.}$$

Fig. 20.6 show the indicator diagram for delivery stroke only. The absolute pressure head during delivery stroke is minimum at the end of the stroke only. It means, if separation is to take place, it will occur only at the end of the delivery stroke where pressure head will be equal to separation pressure head ( $h_{sep}$ ). The absolute pressure head at the end of delivery stroke from Fig. 20.6 is equal to  $D'M$ . where  $D'M = DM - DD'$

$$= (DE + EM) - DD' \quad (\because DM = DE + EM)$$

$$= (h_d + H_{atm}) - h_{ad}$$

$$\therefore h_{sep} = (h_d + H_{atm}) - h_{ad}$$

$$\text{or} \quad 2.5 = (20 + 10.3) - h_{ad}$$

$$\therefore h_{ad} = (20 + 10.3) - 2.5 = 27.8 \text{ m}$$

But the acceleration head ( $h_{ad}$ ) at the end of delivery stroke is given by

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 r$$

$$27.8 = \frac{25}{9.81} \times \frac{\pi}{4} \frac{D^2}{d_d^2} \times \omega^2 \times 0.15 = \frac{25}{9.81} \times \frac{D^2}{d_d^2} \times \omega^2 \times 15$$

$$= \frac{25}{9.81} \times \left( \frac{0.1}{0.05} \right)^2 \times \omega^2 \times 15 = 1.529 \omega^2$$

$$\therefore \omega = \sqrt{\frac{27.8}{1.529}} = 4.264 \text{ radians/s.}$$

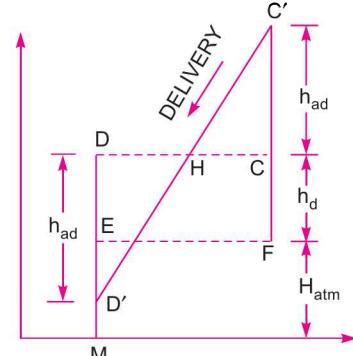


Fig. 20.6

But we know,

$$\omega = \frac{2\pi N}{60}$$

$$\therefore H = \frac{60 \times \omega}{2\pi} = \frac{60 \times 4.264}{2\pi} = 40.72 \text{ r.p.m. Ans.}$$

**Problem 20.8** A single-acting reciprocating pump raises water to a height of 20 m through a delivery pipe 35 m long and 140 mm in diameter. The bore and stroke of piston are 250 mm and 400 mm respectively. Cavitation occurs at 2.5 m of water absolute. Find the speed at which the pump can run without separation on delivery side if the pipe rises first vertically and then runs horizontally. Will there be any change in the maximum speed if the pipe first runs horizontally and then rises vertically.

**Solution.** Given :

$$\text{Delivery head, } h_d = 20 \text{ m}$$

$$\text{Length of delivery pipe, } l_d = 35 \text{ m}$$

$$\text{Dia. of delivery pipe, } d_d = 140 \text{ mm} = 0.14 \text{ m}$$

$$\text{Dia. of piston, } D = 250 \text{ mm} = 0.25 \text{ m}$$

$$\text{Stroke length, } L = 400 \text{ mm} = 0.40 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{0.40}{2} = 0.20 \text{ m}$$

$$\text{Separation pressure head, } h_{sep} = 2.5 \text{ m (abs.)}$$

$$\text{Atmospheric pressure head, } H_{atm} = 10.3 \text{ m}$$

The separation on delivery side can occur only at the end of delivery stroke as the pressure head during delivery stroke is minimum at the end of delivery stroke only. The acceleration head ( $h_{ad}$ ) at the end of delivery stroke is given by,

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 r = \frac{35}{9.81} \times \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} d_d^2} \times \omega^2 \times 0.20$$

$$= \frac{35}{9.81} \times \frac{0.25^2}{0.14^2} \times \omega^2 \times 0.20.$$

**1st Case.** The pipe rises first vertically and then horizontally as shown in Fig. 20.6 (a). In this case, the possibility of separation is at the point C at the end of the delivery stroke.

The pressure head at the end of delivery stroke at B will be equal to atmospheric pressure head plus delivery head minus acceleration head.

$$\therefore \text{Pressure head at } B = H_{atm} + h_d - h_{ad}$$

$$\text{The pressure head at } C = \text{Pressure head at } B - h_d$$

$$= (H_{atm} + h_d - h_{ad}) - h_d = H_{atm} - h_{ad}$$

Now if separation is to take place at C, then the pressure head at C is 2.5 m.

Equating the two pressure heads at C, we get

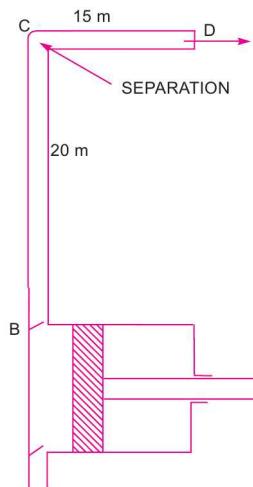


Fig. 20.6(a)

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$$2.5 = H_{atm} - h_{ad}$$

or  $h_{ad} = H_{atm} - 2.5 = 10.3 - 2.5 = 7.8 \text{ m}$

or  $\frac{35}{9.81} \times \frac{0.25^2}{0.14^2} \times \omega^2 \times 0.20 = 7.8$

or  $\omega = \sqrt{\frac{9.81 \times 0.14^2 \times 7.8}{35 \times 0.25^2 \times 0.20}} = 1.85 \text{ rad/s}$

$\therefore N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 1.85}{2\pi} = 17.68 \text{ r.p.m. Ans.}$

**2nd Case.** The pipe first runs horizontally and then rises vertically as shown in Fig. 20.6 (b).

In this case, the possibility of separation is at the point C at the end of delivery stroke. But the pressure head at C is same as pressure head at B and C are in the horizontal plane. Hence, at the end of delivery stroke, the pressure head at B is equal to  $H_{atm} + h_d - h_{ad}$ .

For this condition

or  $h_{sep} = H_{atm} + h_d - h_{ad}$

or  $2.5 = 10.3 + 20 - h_{ad}$

or  $h_{ad} = 10.3 + 20 - 2.5 = 27.8$

or  $\frac{35}{9.81} \times \frac{0.25^2}{0.14^2} \times \omega^2 \times 0.20 = 27.8$

or  $\omega = \sqrt{\frac{9.81 \times 0.14^2 \times 27.8}{35 \times 0.25^2 \times 0.20}} = 3.495$

$\therefore N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 3.495}{2\pi}$   
 $= 33.37 \text{ r.p.m. Ans.}$

$\therefore$  Change in maximum speed

$$= 33.36 - 17.68 = 15.69 \text{ r.p.m. Ans.}$$

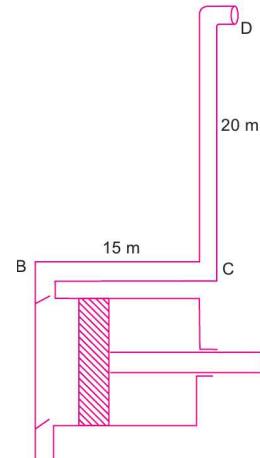


Fig. 20.6(b)

**Problem 20.9** A single-acting reciprocating pump has a plunger of 10 cm diameter and a stroke of length 200 mm. The centre of the pump is 4 m above the water level in the sump and 14 m below the level of water in a tank to which water is delivered by the pump. The diameter and length of suction pipe are 40 mm and 6 m while of the delivery pipe are 30 mm and 18 m respectively. Determine the maximum speed at which the pump may be run without separation, if separation occurs at  $7.848 \text{ N/cm}^2$  below the atmospheric pressure. Take atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

Diameter of plunger,  $D = 100 \text{ mm} = 0.10 \text{ m}$

Stroke length,  $L = 200 \text{ mm} = 0.20 \text{ m}$

$\therefore$  Crank radius,  $r = \frac{L}{2} = \frac{0.20}{2} = 0.10 \text{ m}$

Suction head,  $h_s = 4 \text{ m}$

Delivery head,  $h_d = 14 \text{ m}$

Dia. of suction pipe,  $d_s = 40 \text{ mm} = 0.04 \text{ m}$

Length of suction pipe,  $l_s = 6 \text{ m}$

Dia. of delivery pipe,  $d_d = 30 \text{ mm} = .03 \text{ m}$

Length of delivery pipe,  $l_d = 18 \text{ m}$

$$\text{Separation pressure, } p_{sep} = \frac{7.848 N}{\text{cm}^2} = \frac{7.848 \times 10^4}{\text{m}^2}$$

$$\therefore \text{Separation pressure head, } h_{sep} = \frac{p_{sep}}{\rho g} = \frac{7.848 \times 10^4}{1000 \times 9.81} = 8.0 \text{ m below atmosphere}$$

$$= (H_{atm} - 8.0) \text{ absolute} = (10.3 - 8.0) = 2.3 \text{ m (abs.)}$$

where  $H_{atm}$  = Atmospheric pressure head = 10.3 m.

(i) **Speed of pump without separation during suction stroke.** During suction stroke, possibility of separation is only at the beginning of the stroke. The pressure head in the cylinder at the beginning of suction stroke

$$= (h_s + h_{as}) \text{ m below atmospheric pressure head}$$

$$= 10.3 - (h_s + h_{as}) \text{ m (abs.)}$$

$$\therefore h_{sep} = 10.3 - (h_s + h_{as})$$

$$\text{or } 2.3 = 10.3 - (h_s + h_{as}) = 10.3 - 4 - h_{as}$$

$$\therefore h_{as} = 10.3 - 4 - 2.3 = 4 \text{ m.} \quad \dots(i)$$

But ' $h_{as}$ ' at the beginning of suction stroke is given by,

$$\begin{aligned} h_{as} &= \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r = \frac{6}{9.81} \times \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} d_s^2} \times \omega^2 \times .10 \\ &= \frac{6}{9.81} \times \left( \frac{0.1}{.04} \right)^2 \times \omega^2 \times .1 = 0.3822 \omega^2 \end{aligned} \quad \dots(ii)$$

Equating the values of  $h_{as}$  given by equations (i) and (ii),

$$4 = 0.3822 \omega^2$$

$$\therefore \omega = \sqrt{\frac{4}{.382}} = 3.235 \text{ rad/s}$$

$$\text{But } \omega \text{ is also} \quad = \frac{2\pi N}{60}$$

$$\therefore N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 3.235}{2\pi} = 30.89 \text{ r.p.m.}$$

$\therefore$  Maximum speed of the pump without separation during suction stroke only is 30.89 r.p.m.

(ii) **Speed of pump without separation during delivery stroke.** During delivery stroke, the possibility of separation is only at the end of the delivery stroke. The pressure head in the cylinder at the end of the delivery stroke from Fig. 20.6

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$$= (H_{atm} + h_d) - h_{ad} \text{ m (abs.)} = (10.3 + 14) - h_{ad}$$

If separation is to be avoided this pressure should be equal to separation pressure head.

$$\therefore h_{sep} = (10.3 + 14) - h_{ad} \text{ or } 2.30 = (10.3 + 14) - h_{ad}$$

$$\therefore h_{ad} = 10.3 + 14.0 - 2.30 = 22.0 \text{ m}$$

But ' $h_{ad}$ ' is given by the relation

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 \times r$$

$$\therefore 22.0 = \frac{18.0}{9.81} \times \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} d_d^2} \times \omega^2 \times r = \frac{18.0}{9.81} \times \frac{D^2}{d_d^2} \times \omega^2 \times r$$

$$= \frac{18.0}{9.81} \times \left( \frac{.1}{.03} \right)^2 \times \omega^2 \times 0.10 = 2.04 \omega^2$$

$$\therefore \omega = \sqrt{\frac{22.0}{2.04}} = 3.284 \text{ rad/s.}$$

$$\text{But } \omega = \frac{2\pi N}{60}$$

$$\therefore N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 3.284}{2 \times \pi} = 31.36 \text{ r.p.m.}$$

$\therefore$  Maximum speed of the pump without separation during delivery stroke is 31.36 r.p.m.

Thus the maximum speed of the pump without separation during suction and delivery stroke is the minimum of these two speeds, i.e., minimum of 30.89 and 31.36 r.p.m.

$\therefore$  Maximum speed = 30.89 r.p.m. Ans.

**20.8.3 Effect of Friction in Suction and Delivery Pipes on Indicator Diagram.** The loss of head due to friction in suction and delivery pipes is given by equations (20.18) and (20.19) as

$$h_{fs} = \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \sin \theta \right)^2 \text{ and } h_{fd} = \frac{4fl_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \sin \theta \right)^2$$

It is clear from the above equations that the variation of  $h_{fs}$  or  $h_{fd}$  is parabolic with  $\theta$ .

During the suction or delivery stroke, the pressure head inside the cylinder will change as given below :

- (i) At the beginning of the suction or delivery stroke,  $\theta = 0^\circ$  and hence  $\sin \theta = 0$ . This means  $h_{fs}$  and  $h_{fd} = 0$ .
- (ii) At the middle of the suction or delivery stroke,  $\theta = 90^\circ$  and hence  $\sin \theta = 1$ . This means

$$h_{fs} = \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \right)^2 \text{ and } h_{fd} = \frac{4fl_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \right)^2.$$

(iii) At the end of the suction or delivery stroke,  $\theta = 180^\circ$  and hence  $\sin \theta = 0^\circ$ . This means  $h_{fs}$  and  $h_{fd} = 0$ .

As the variation of  $h_{fs}$  or  $h_{fd}$  with  $\theta$  is parabolic in nature, the indicator diagram during suction and delivery stroke with friction in suction pipe and delivery pipe will be as shown in Fig. 20.7.

The area of the parabolas  $AGB$  and  $CHD$  represents the work done against friction in suction and delivery pipes.

Now

$$\begin{aligned}\text{area } AGB &= AB \times \frac{2}{3} GG' \\ &= AB \times \frac{2}{3} h_{fs} = L \times \frac{2}{3} h_{fs}\end{aligned}$$

$$\text{where } h_{fs} = \frac{4 \times f \times l_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \right)^2$$

Similarly,

$$\begin{aligned}\text{area } CHD &= CD \times \frac{2}{3} \times HH' = CD \times \frac{2}{3} h_{fd} \\ &= L \times \frac{2}{3} h_{fd} \quad (\because CD = L = \text{Length of stroke})\end{aligned}$$

$$\text{where } h_{fd} = \frac{4 f l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \right)^2.$$

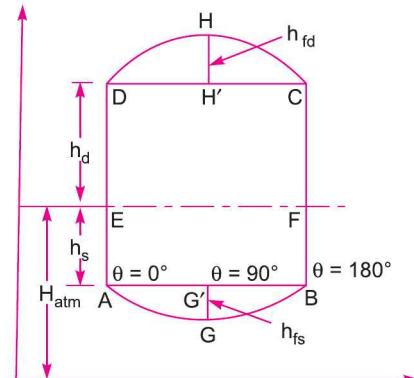


Fig. 20.7 Effect of friction on indicator diagram.

**20.8.4 Effect of Acceleration and Friction in Suction and Delivery Pipes on Indicator Diagram.** Fig. 20.8 shows the combined effect of acceleration and friction in suction and delivery pipes. The pressure head in the cylinder during suction and delivery strokes will change as given below :

(i) At the beginning of the suction stroke,  $\theta = 0^\circ$  and hence  $h_{as}$  from equation (20.14) is equal to  $\frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r$ . But  $h_{fs} = 0$ . Thus, the pressure head in the cylinder will be  $(h_s + h_{as})$  below the atmospheric pressure head.

(ii) At the middle of the suction stroke,  $h_{as} = 0$  but  $h_{fs} = \frac{4 \times f \times l_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \right)^2$ . Thus, the pressure head in the cylinder will be  $(h_s + h_{fs})$  below the atmospheric pressure head.

(iii) At the end of the suction stroke,  $h_{as} = -\frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r$  but  $h_{fs} = 0$ . Thus, the pressure head in the cylinder will be  $(h_s - h_{as})$  below the atmospheric pressure head.

(iv) At the beginning of the delivery stroke,  $h_{ad} = -\frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 r$  but  $h_{fd} = 0$ . Thus, the pressure head in the cylinder will be  $(h_d + h_{ad})$  above the atmospheric pressure head.

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(v) In the middle of the delivery stroke,  $h_{ad} = 0$  and  $h_{fd} = \frac{4fl_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \right)^2$ . Thus the pressure head in the cylinder will be  $(h_d + h_{fd})$  above the atmospheric pressure head.

(vi) At the end of the delivery stroke,  $h_{ad} = -\frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 r$  and  $h_{fd} = 0$ . Thus, the pressure head in the cylinder will be  $(h_d - h_{ad})$  above the atmospheric pressure head.

Thus, the indicator diagram with acceleration and friction in suction and delivery pipes will become as shown in Fig. 20.8.

Area of the indicator diagram  $A'GB'C'HD'$

$$= \text{Area of } A'G'B'C'H'D' + \text{Area of parabola } A'GB' + \text{Area of parabola } C'HD'$$

But area of  $A'GB'C'H'D'$

$$= \text{Area of } ABCD$$

$$= (h_s + h_d) \times L$$

Area of parabola  $A'GB'$

$$= A'B' \times \frac{2}{3} \times G'I = \frac{2}{3} \times (A'B' \times GI)$$

$$= \frac{2}{3} \times (AB \times GG') = \frac{2}{3} \times L'h_{fs}$$

Similarly, area of parabola  $C'HD'$

$$= C'D' \times \frac{2}{3} H'J = \frac{2}{3} (C'D' \times H'J)$$

$$= \frac{2}{3} \times (CD \times HH') = \frac{2}{3} (L \times h_{fd}) = \frac{2}{3} L \times h_{fd}$$

$\therefore$  Area of indicator diagram

$$= (h_s + h_d) \times L + \frac{2}{3} \times L \times h_{fs} + \frac{2}{3} \times L \times h_{fd}$$

$$= \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) \times L$$

But from equation (20.21), we know that work done by pump is proportional to the area of the indicator diagram.

$$\therefore \text{Work done by pump per second} \propto \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) \times L$$

$$= KL \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right)$$

where  $K$  = a constant of proportionality

$$= \frac{\rho g A N}{60} \quad \dots \text{for a single-acting}$$

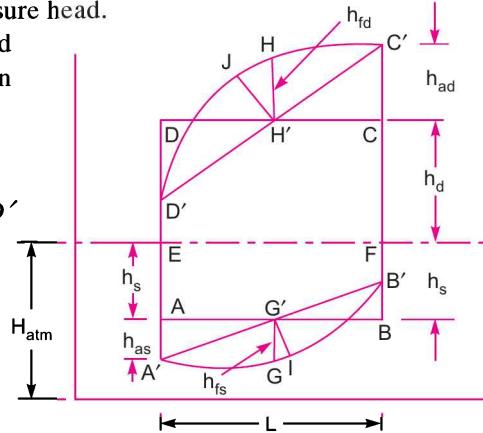


Fig. 20.8 Effect of acceleration and friction on indicator diagram.

$$= \frac{2\rho g A N}{60} \quad \text{...for a double-acting}$$

$\therefore$  Work done by pump per second for a single-acting

$$= \frac{\rho g A L N}{60} \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) \quad \dots(20.22)$$

$$\text{and for a double-acting} \quad = \frac{2\rho g A L N}{60} \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) \quad \dots(20.23)$$

**Problem 20.10** A single-acting reciprocating pump has a stroke length of 15 cm. The suction pipe is 7 metre long and the ratio of the suction diameter to the plunger diameter is 3/4. The water level in the sump is 2.5 metres below the axis of the pump cylinder, and the pipe connecting the sump and pump cylinder is 7.5 cm diameter. If the crank is running at 75 r.p.m., determine the pressure head on the piston :

- (i) in the beginning of the suction stroke,
- (ii) in the end of the suction stroke, and
- (iii) in the middle of the suction stroke.

Take co-efficient of friction as 0.01.

**Solution.** Given :

Stroke length,  $L = 15 \text{ cm} = 0.15 \text{ m}$

$\therefore$  Crank radius,  $r = \frac{L}{2} = \frac{0.15}{2} = 0.075 \text{ m}$

Length of suction pipe,  $l_s = 7.0 \text{ m}$

$$\frac{\text{Suction pipe diameter}}{\text{Plunger diameter}} = \frac{d_s}{D} = \frac{3}{4}$$

$$\therefore \frac{\text{Area of suction pipe}}{\text{Area of plunger}} = \frac{a_s}{A} = \left( \frac{3}{4} \right)^2 = \frac{9}{16}$$

Suction head,  $h_s = 2.5$

Diameter of suction pipe,  $d_s = 7.5 \text{ cm} = 0.075 \text{ m}$

Crank speed,  $N = 75 \text{ r.p.m.}$

$\therefore$  Angular speed,  $\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 75}{60} = 2.5\pi \text{ rad/s.}$

Friction co-efficient,  $f = 0.01$

The pressure head due to acceleration in suction pipe is given by equation (20.14), as

$$\begin{aligned} h_{as} &= \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 \times r \cos \theta \\ &= \frac{7.0}{9.81} \times \frac{16}{9} \times (2.5\pi)^2 \times 0.075 \cos \theta \\ &= 5.87 \cos \theta \end{aligned} \quad \left( \because \frac{A}{a_s} = \frac{16}{9} \right)$$

The loss of head due to friction in suction pipe is given by equation (20.18) as

$$h_{fs} = \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \sin \theta \right)^2$$

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$$= \frac{4 \times 0.01 \times 7.0}{0.075 \times 2 \times 9.81} \times \left( \frac{16}{9} \times 2.5\pi \times 0.075 \times \sin \theta \right)^2 = 0.208 \sin^2 \theta.$$

(i) Pressure head on the piston in the beginning of suction stroke :

(Refer to Fig. 20.8). At the beginning of suction stroke,  $\theta = 0^\circ$  and pressure head

$$\begin{aligned} &= (h_s + h_{as}) \text{ m below atmospheric pressure head} \\ &= 2.5 + 5.87 = 8.37 \text{ m vacuum. Ans.} \end{aligned}$$

(ii) Pressure head on the piston at the end of suction stroke :

At the end of the suction stroke,  $\theta = 180^\circ$  and hence  $\cos \theta$

$$= -1 \text{ and } \sin \theta = 0$$

The pressure head

$$\begin{aligned} &= (h_s - h_{as}) \text{ m below atmospheric pressure head} \\ &= H_{atm} - (h_s - h_{as}) \text{ m abs.} = H_{atm} - (2.5 - 5.87) \text{ m abs.} \\ &= H_{atm} - (-3.37) \text{ m abs.} = H_{atm} + 3.37 \text{ m abs.} = 3.37 \text{ m (gauge). Ans.} \end{aligned}$$

(iii) Pressure head on the piston in the middle of suction stroke :

In the middle of suction stroke,  $\theta = 90^\circ$  and hence  $\cos \theta = 0$  and  $\sin \theta = 1$ . The pressure head

$$\begin{aligned} &= (h_s + h_{fs}) \text{ m below atmospheric pressure head} \\ &= (2.5 + 0.208) \text{ m vacuum} = 2.708 \text{ m vacuum. Ans.} \end{aligned}$$

**Problem 20.11** The diameter and stroke length of a single-acting reciprocating pump are 12 cm and 20 cm respectively. The lengths of suction and delivery pipes are 8 m and 25 m respectively and their diameters are 7.5 cm. If the pump is running at 40 r.p.m. and suction and delivery heads are 4 m and 14 m respectively, find the pressure head in the cylinder :

- (i) at the beginning of the suction and delivery stroke,
- (ii) in the middle of suction and delivery stroke, and
- (iii) at the end of the suction and delivery stroke.

Take atmospheric pressure head = 10.30 metres of water and  $f = .009$  for both pipes.

**Solution.** Given :

Diameter of cylinder,  $D = 12 \text{ cm} = 0.12 \text{ m}$

Stroke length,  $L = 20 \text{ cm} = 0.20 \text{ m}$

∴ Crank radius,  $r = \frac{L}{2} = \frac{0.20}{2} = 0.10 \text{ m}$

Length of suction pipe,  $l_s = 8 \text{ m}$

Length of delivery pipe,  $l_d = 25 \text{ m}$

Dia. of suction pipe,  $d_s = 7.5 \text{ cm} = 0.075 \text{ m}$

Dia. of delivery pipe,  $d_d = 7.5 \text{ cm} = 0.075 \text{ m}$

Speed of pump,  $N = 40 \text{ r.p.m.}$

Suction head,  $h_s = 4 \text{ m}$

Delivery head,  $h_d = 14 \text{ m}$

Atmospheric pressure head,  $H_{atm} = 10.3 \text{ m of water}$

Co-efficient of friction,  $f = .009$ .

We know that angular velocity,  $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 40}{60} = 4.188 \text{ rad/s.}$

Using equation (20.14), the pressure head due to acceleration in suction pipe is obtained as

$$\begin{aligned}
 h_{as} &= \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r \cos \theta = \frac{8}{9.81} \times \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} d_s^2} \times 4.188^2 \times 0.1 \times \cos \theta \\
 &= \frac{8}{9.81} \times \left( \frac{D}{d_s} \right)^2 \times 4.188^2 \times 0.1 \times \cos \theta \\
 &= \frac{8}{9.81} \times \left( \frac{.120}{.075} \right)^2 \times 4.188^2 \times 0.1 \times \cos \theta = 3.66 \times \cos \theta \text{ m.}
 \end{aligned}$$

Similarly, the pressure head due to acceleration in delivery pipe is obtained from equation (20.15) as

$$\begin{aligned}
 h_{ad} &= \frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 r \cos \theta \\
 &= \frac{25}{9.81} \times \frac{D^2}{d_d^2} \times 4.188^2 \times 0.1 \times \cos \theta \\
 &= \frac{25}{9.81} \times \left( \frac{.120}{.075} \right)^2 \times 4.188^2 \times 0.1 \times \cos \theta = 11.44 \times \cos \theta \text{ m.}
 \end{aligned}$$

Using equation (20.18) for the loss of head due to friction in suction pipe,

$$\begin{aligned}
 h_{fs} &= \frac{4 f l_s}{d_s \times 2 g} \times \left( \frac{A}{a_s} \omega r \sin \theta \right)^2 \\
 &= \frac{4 \times .009 \times 8}{.075 \times 2 \times 9.81} \times \left( \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} d_s^2} \times 4.188 \times .1 \times \sin \theta \right)^2 \\
 &= \frac{4 \times .009 \times 8}{.075 \times 2 \times 9.81} \times \left( \frac{.12^2}{.075^2} \times 4.188 \times .1 \right)^2 \times \sin^2 \theta = 0.225 \sin^2 \theta.
 \end{aligned}$$

Similarly, loss of head due to friction in delivery pipe is obtained from equation (20.19) as

$$\begin{aligned}
 h_{fd} &= \frac{4 \times f \times l_d}{d_d \times 2 g} \times \left[ \frac{A}{a_d} \omega r \sin \theta \right]^2 \\
 &= \frac{4 \times .009 \times 25}{0.075 \times 2 \times 9.81} \left[ \frac{D^2}{d_d^2} \times 4.188 \times 0.1 \times \sin \theta \right]^2 \\
 &= 0.6116 \left[ \frac{.12^2}{.075^2} \times 4.188 \times 0.1 \right]^2 \times \sin^2 \theta = 0.703 \sin^2 \theta.
 \end{aligned}$$

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(i) **Pressure head in the cylinder at the beginning of suction and delivery strokes (Fig. 20.8).** At the beginning of suction and delivery strokes,  $\theta = 0^\circ$ .

$$\therefore h_{as} = 3.66 \cos 0^\circ = 3.66 \text{ m}$$

and  $h_{ad} = 11.44 \cos 0^\circ = 11.44 \text{ m}$

Pressure head in the cylinder at the beginning of suction stroke

$$\begin{aligned} &= (h_s + h_{as}) \text{ below atmospheric pressure head} \\ &= H_{atm} - (h_s + h_{as}) \text{ m (abs.)} \\ &= 10.3 - (1 + 3.66) = 10.3 - 7.66 = 2.64 \text{ m (abs.) Ans.} \end{aligned}$$

Pressure head in the cylinder at the beginning of delivery stroke

$$\begin{aligned} &= (h_d + h_{ad}) \text{ above atmospheric pressure head} \\ &= H_{atm} + (h_d + h_{ad}) \text{ m (abs.)} \\ &= 10.3 + (14 + 11.44) = 35.74 \text{ m (abs.) Ans.} \end{aligned}$$

(ii) **Pressure head in the cylinder in the middle of suction and delivery strokes.** In the middle of suction and delivery strokes,  $\theta = 90^\circ$ .

$$\therefore h_{as} = 0, h_{ad} = 0, h_{fs} = 0.225 \sin^2 90 = 0.225 \text{ m and}$$

$$h_{fd} = 0.703 \sin^2 90 = 0.703 \text{ m.}$$

Pressure head in the cylinder at the beginning of suction stroke

$$\begin{aligned} &= (h_s + h_{fs}) \text{ below the atmospheric pressure head} \\ &= H_{atm} - (h_s + h_{fs}) \text{ m (abs.)} \\ &= 10.3 - (4 + 0.225) = 10.3 - 4.225 = 6.075 \text{ m (abs.) Ans.} \end{aligned}$$

Pressure head in the cylinder at the beginning of delivery stroke

$$\begin{aligned} &= (h_d + h_{fd}) \text{ above atmospheric pressure head} \\ &= H_{atm} + (h_d + h_{fd}) \text{ m (abs.)} \\ &= 10.3 + (14 + 0.703) = 125.003 \text{ m (abs.) Ans.} \end{aligned}$$

(iii) **Pressure head in the cylinder at the end of suction and delivery strokes.** At the end of suction and delivery strokes,  $\theta = 180^\circ$ .

$$\therefore \cos \theta = -1 \text{ and } \sin \theta = 0$$

$$\therefore h_{as} = 3.66 \times (-1) = -3.66 \text{ m}$$

and  $h_{ad} = 11.44 \times (-1) = -11.44 \text{ m.}$

Pressure head in the cylinder at the end of suction stroke

$$\begin{aligned} &= (h_s - 3.66) \text{ below the atmospheric pressure head} \\ &= H_{atm} - (h_s - 3.66) \text{ m (abs.)} = 10.3 - (4 - 3.66) = 9.96 \text{ m (abs.)} \\ &\quad \text{Ans.} \end{aligned}$$

Pressure head in the cylinder at the end of delivery stroke

$$\begin{aligned} &= (h_d - 11.44) \text{ above the atmospheric pressure head} \\ &= H_{atm} + (h_d - 11.44) \text{ m (abs.)} \\ &= 10.3 + (14 - 11.44) = 12.86 \text{ m (abs.) Ans.} \end{aligned}$$

**Problem 20.12** For the single-acting reciprocating pump, given in Problem 20.11, find the power required to drive the pump, if water is flowing through the pump.

**Solution.** Using equation (20.22) for the work done by the pump per second for a single-acting, we get

$$\text{Work done per sec} = \frac{\rho g A L N}{60} \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right)$$

$$\text{where } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times .12^2 = 0.01131 \text{ m}^2$$

$L$  = Stroke length = 0.20 m,  $N$  = Speed = 40 r.p.m.

$\rho$  = Density of water = 1000 kg/m<sup>3</sup>,  $h_s$  = 4 m,  $h_d$  = 14 m

$h_{fs}$  = Maximum loss of head due to friction in suction pipe = 0.225 m

$h_{fd}$  = Maximum loss of head due to friction in delivery pipe = 0.703 m

$$\therefore \text{Work done per second} = \frac{1000 \times 9.81 \times 0.01131 \times 0.20 \times 40}{60} \left( 4 + 14 + \frac{2}{3} \times .225 + \frac{2}{3} \times .703 \right)$$

$$= 14.793 \times (4 + 14 + 0.15 + 0.468) = 275.42 \text{ Nm/s}$$

$\therefore$  Power required to drive the pump in kW

$$= \frac{\text{Work done per second}}{1000} = \frac{275.42}{1000} = 0.2754 \text{ kW.}$$

**20.8.5 Maximum Speed of a Reciprocating Pump.** Maximum speed of a reciprocating pump is determined from the fact that the pressure in the cylinder during suction and delivery stroke, should not fall below the vapour pressure of the liquid, flowing through suction and delivery pipe. If the pressure in the cylinder is below the vapour pressure, the dissolved gases will be liberated from the liquid and cavitation \* will take place. Also the continuous flow of liquid will not exist which means separation of liquid will take place. The pressure at which separation takes place is known as separation pressure and the head corresponding to separation pressure is called separation pressure head. It is denoted by  $h_{sep}$ . For water, the limiting value of separation pressure head ( $h_{sep}$ ) is 7.8 below atmospheric pressure head or  $10.3 - 7.8 = 2.5$  m abs. The separation may take place during the suction stroke or during delivery stroke. The maximum speed of the reciprocating pump during suction and delivery strokes is calculated as :

(a) **Maximum Speed during Suction Stroke.** From the indicator diagram, drawn in Fig. 20.8, it is clear that the absolute pressure head during suction stroke is minimum at the beginning of the stroke. Thus, the separation can take place at the beginning of the stroke only. In that case, the abs. pressure head in the cylinder at the beginning of the stroke will be equal to separation pressure head ( $h_{sep}$ ).

$$\therefore h_{sep} = H_{atm} - (h_s + h_{as}) \text{ (abs.)}$$

or  $h_{as} = H_{atm} - h_s - h_{sep}$  ... (i)

Generally, the values of  $h_{sep}$  and  $h_s$  (suction head) are given and hence ' $h_{as}$ ', the pressure head due to acceleration at the beginning of suction stroke can be obtained. The value of ' $h_{as}$ ' is also given by equation (20.14) as

$$h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r \quad \dots (ii)$$

Equating the two values of  $h_{as}$  given by equations (i) and (ii)

$$H_{atm} - h_s - h_{sep} = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r \quad \dots (iii)$$

\* Please refer to Art. 19.11 on page 980 for definition.

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From equation (iii), the value of the  $\omega$  (and hence speed  $N$ ) can be obtained. This speed is the maximum speed of the reciprocating pump without separation during suction stroke.

(b) **Maximum Speed during Delivery Stroke.** During delivery stroke, the probability of separation is only at the end of the delivery stroke. The pressure head in the cylinder at the end of the delivery stroke from Fig. 20.8.

$$= (H_{atm} + h_d) - h_{ad} \text{ m (abs.)}$$

If separation is to be avoided, the above pressure head should be more than the separation pressure head ( $h_{sep}$ ). In the limiting case

$$h_{sep} = (H_{atm} + h_d) - h_{ad} \text{ or } h_{ad} = (H_{atm} + h_d) - h_{sep}$$

But ' $h_{ad}$ ', the pressure head due to acceleration at the end of the delivery stroke is also given by

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 \times r$$

Equating the two values of  $h_{ad}$ , we get

$$(H_{atm} + h_d) - h_{sep} = \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 \times r \quad \dots(iv)$$

From the above equation (iv), the value of  $\omega$  and hence speed  $N$ , can be calculated. This is the maximum speed of the reciprocating pump without separation during delivery stroke only.

The minimum of the two speeds given by above two cases (a) and (b) is the maximum speed of the reciprocating pump without separation during suction and delivery strokes.

**Problem 20.13** Find the maximum speed of a single-acting reciprocating pump to avoid separation, which occurs at 3.0 m of water (abs.). The pump has a cylinder of diameter 10 cm and a stroke length of 20 cm. The pump draws water from a sump and delivers to a tank. The water level in the sump is 3.5 m below the pump axis and in the tank the water level is 13 m above the pump axis. The diameter and length of the suction pipe are 4 cm and 5 m while of delivery pipe the diameter and length are 3 cm, 20 m respectively. Take atmospheric pressure head = 10.3 m of water.

**Solution.** Given :

Separation pressure head,  $h_{sep} = 3.0 \text{ m of water (abs.)}$

Dia. of cylinder,  $D = 10 \text{ cm} = 0.10 \text{ m}$

Stroke length,  $L = 20 \text{ cm} = 0.20 \text{ m}$

$\therefore$  Crank radius,  $r = \frac{L}{2} = \frac{0.20}{2} = 0.10 \text{ m}$

Suction head,  $h_s = 3.5 \text{ m}$

Delivery head,  $h_d = 13 \text{ m}$

Dia. of suction pipe,  $d_s = 4 \text{ cm} = .04 \text{ m}$

Length of suction pipe,  $l_s = 5 \text{ m}$

Dia. of delivery pipe,  $d_d = 3 \text{ cm} = .03 \text{ m}$

Length of delivery pipe,  $l_d = 20 \text{ m}$

Atmos. pressure head,  $H_{atm} = 10.3 \text{ m.}$

(a) Maximum speed during suction stroke without separation is obtained from the relation, given by equation (iii),

$$H_{atm} - h_s - h_{sep} = \frac{l_s}{g} \times \frac{A}{a_s} \times \omega^2 r$$

or  $10.3 - 3.5 - 3.0 = \frac{5}{9.81} \times \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} d_s^2} \times \omega^2 \times 0.10 \quad \left( \because A = \frac{\pi}{4} D^2, a_s = \frac{\pi}{4} d_s^2 \right)$

or  $3.8 = \frac{5}{9.81} \times \frac{.1 \times .1}{.04 \times .04} \times \omega^2 \times 0.1 = .3185 \omega^2$

$\therefore \omega = \sqrt{\frac{3.8}{.3185}} = 3.454 \text{ rad/s.}$

But  $\omega = \frac{2\pi N}{60} \quad \therefore \frac{2\pi N}{60} = 3.454$

$\therefore N = \frac{60 \times 3.454}{2\pi} = 32.98 \text{ r.p.m.} \quad \dots(i)$

(b) Maximum speed during delivery stroke without separation is obtained from equation (iv),

$$(H_{atm} + h_d) - h_{sep} = \frac{l_d}{g} \times \frac{A}{a_d} \times \omega^2 \times r$$

$$(10.3 + 13.0) - 3.0 = \frac{20}{9.81} \times \frac{D^2}{d_d^2} \times \omega^2 \times .1 = \frac{20}{9.81} \times \frac{0.1 \times .1}{.03 \times .03} \times \omega^2 \times .1$$

or  $20.3 = 2.265 \omega^2$

$\therefore \omega = \sqrt{\frac{20.3}{2.265}} = 2.994 \text{ rad/s.}$

But  $\omega = \frac{2\pi N}{60} = 2.994$

$\therefore N = \frac{60 \times 2.994}{2\pi} = 28.59 \text{ r.p.m.} \quad \dots(ii)$

The minimum of the two speeds given by equations (i) and (ii) is the maximum speed of the pump, without separation.

$\therefore$  Maximum speed without separation = **28.59 r.p.m. Ans.**

## ► 20.9 AIR VESSELS

An air vessel is a closed chamber containing compressed air in the top portion and liquid (or water) at the bottom of the chamber. At the base of the chamber there is an opening through which the liquid (or water) may flow into the vessel or out from the vessel. When the liquid enters the air vessel, the air gets compressed further and when the liquid flows out the vessel, the air will expand in the chamber.

An air vessel is fitted to the suction pipe and to the delivery pipe at a point close to the cylinder of a single-acting reciprocating pump :

- (i) to obtain a continuous supply of liquid at a uniform rate,
- (ii) to save a considerable amount of work in overcoming the frictional resistance in the suction and delivery pipes, and

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(iii) to run the pump at a high speed without separation.

Fig. 20.9 shows the single-acting reciprocating pump to which air vessels are fitted to the suction and delivery pipes. The air vessels act like an intermediate reservoir. During the first half of the suction stroke, the piston moves with acceleration, which means the velocity of water in the suction pipe is more than the mean velocity and hence the discharge of water entering the cylinder will be more than the mean discharge. This excess quantity of water will be supplied from the air vessel to the cylinder in such a way that the velocity in the suction pipe below the air vessel is equal to mean velocity of flow. During the second half of the suction stroke, the piston moves with retardation and hence velocity of flow in the suction pipe is less than the mean velocity of flow. Thus, the discharge entering the cylinder will be less than the mean discharge. The velocity of water in the suction pipe due to air vessel is equal to mean velocity of flow and discharge required in cylinder is less than the mean discharge. Thus, the excess water flowing in suction pipe will be stored into air vessel, which will be supplied during the first half of the next suction stroke.

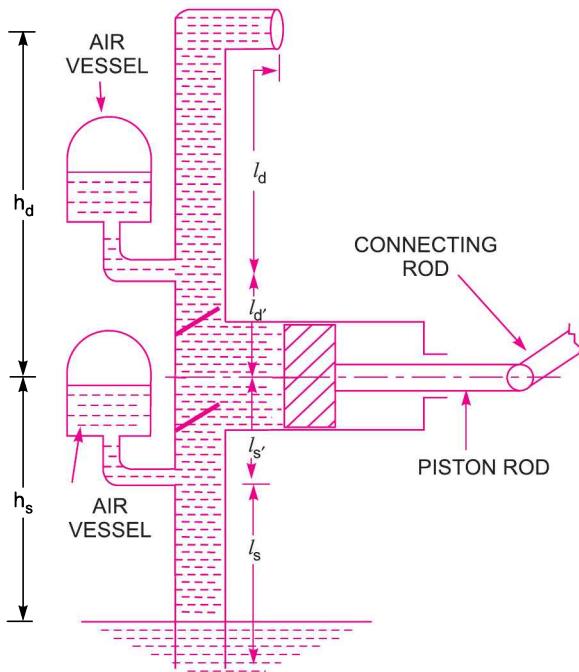


Fig. 20.9 Air vessels fitted to reciprocating pump.

When the air vessel is fitted to the delivery pipe, during the first half of delivery stroke, the piston moves with acceleration and forces the water into the delivery pipe with a velocity more than the mean velocity. The quantity of water in excess of the mean discharge will flow into the air vessel. This will compress the air inside the vessel. During the second half of the delivery stroke, the piston moves with retardation and the velocity of water in the delivery pipe will be less than the mean velocity. The water already stored into the air vessel will start flowing into the delivery pipe and the velocity of flow in the delivery pipe beyond the point to which air vessel is fitted will become equal to the mean velocity. Hence, the rate of flow of water in the delivery pipe will be uniform.

Let  $A$  = Cross-sectional area of the cylinder,

$a$  = Cross-sectional area of suction or delivery pipe,

$l_d$  = Length of delivery pipe beyond the air vessel,

$l'_d$  = Length of delivery pipe between cylinder and air vessel,

$l'_s$  = Length of suction pipe between cylinder and air vessel,

$l_s$  = Length of suction pipe below air vessel,

$h_{ad}$  = Pressure head due to acceleration in delivery pipe,

$h_{as}$  = Pressure head due to acceleration in suction pipe,

$h_{fd}$  = Loss of head due to friction in delivery pipe beyond the air vessel,

$h'_{fd}$  = Loss of head due to friction in delivery pipe between cylinder and air vessel,

$h_{fs}$  = Loss of head due to friction in suction pipe below the air vessel, and

$h'_{fs}$  = Loss of head due to friction in suction pipe between cylinder and air vessel.

The effect of acceleration will be observed only in the lengths  $l'_d$  and  $l'_s$  which may be made very small by fitting air vessels very close to the cylinder. The velocity of flow of water in the length  $l_d$  and  $l_s$  will be equal to mean velocity of flow.

For a single-acting, discharge per second is given by equation (20.1), as

$$Q = \frac{ALN}{60}, \quad \text{where } L = \text{Length of stroke}$$

$$\begin{aligned} \therefore \text{Mean velocity, } \bar{V} &= \frac{\text{Discharge}}{\text{Area of pipe}} = \frac{Q}{a} = \frac{ALN}{60a} \\ &= \frac{AL}{60a} \times \frac{60\omega}{2\pi} \quad \left( \because \omega = \frac{2\pi N}{60} \text{ or } N = \frac{60\omega}{2\pi} \right) \\ &= \frac{A}{a} \times L \times \frac{\omega}{2\pi} = \frac{A}{a} \times 2r \times \frac{\omega}{2\pi} \quad (\because L = 2r) \\ &= \frac{A}{a} \times \frac{\omega r}{\pi} \end{aligned} \quad \dots(20.24)$$

The velocity of water in the suction or delivery pipes for the lengths  $l'_s$  and  $l'_d$  due to acceleration and retardation of the piston is given by equation (20.11) as

$$v = \frac{A}{a} \omega r \sin \omega t = \frac{A}{a} \omega r \sin \theta \quad (\because \theta = \omega t)$$

(a) **Pressure head in the cylinder during delivery stroke.** The pressure head due to acceleration in the delivery pipe of length  $l'_d$  (between air vessel and cylinder) is given by equation (20.15) as

$$h_{ad} = \frac{l'_d}{g} \times \frac{A}{a_d} \omega^2 r \cos \theta \quad \dots(i)$$

Loss of head due to friction in the delivery pipe for lengths  $l'_d$  is given as

$$h'_{fd} = \frac{4f \times l'_d \times v^2}{d \times 2g}$$

where for delivery pipe,

$$v = \frac{A}{a_d} \omega r \sin \theta$$

$$d = \text{dia. of delivery pipe} = d_d$$

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$$\therefore h_{fd}' = \frac{4f \times l_d'}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \sin \theta \right)^2 \quad \dots(ii)$$

Loss of head due to friction in the delivery pipe for the length beyond the air vessel (*i.e.*, length  $l_d$ ),

$$h_{fd} = \frac{4f \times l_d \times (\bar{V}_d)^2}{d_d \times 2g}$$

where  $\bar{V}_d$  = Mean velocity in delivery pipe =  $\frac{A}{a_d} \times \frac{\omega r}{\pi}$  [from equation (20.24)]

$$\therefore h_{fd} = \frac{4f \times l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(iii)$$

The pressure head in the cylinder during delivery stroke is given as :

(i) At the beginning of the delivery stroke,  $\theta = 0^\circ$ ,  $\sin \theta = 0$  and  $\cos \theta = 1$  and hence total pressure head

$$\begin{aligned} &= (h_d + h_{ad} + h_{fd}' + h_{fd}) + \text{velocity head at the outlet of delivery} \\ &= h_d + h_{ad} + h_{fd}' + h_{fd} + \frac{\bar{V}_d^2}{2g} \quad (\because \text{Velocity at outlet is equal to mean velocity}) \\ &= h_d + \frac{l_d'}{g} \times \frac{A}{a_d} \omega^2 r + 0 + \frac{4f \times l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{\left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2}{2g} \quad \left( \because \bar{V}_d = \frac{A}{a_d} \times \frac{\omega r}{\pi} \right) \\ &= h_d + \frac{l_d'}{g} \times \frac{A}{a_d} \omega^2 r + \frac{4f l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(20.25) \end{aligned}$$

(ii) In the middle of the stroke,  $\theta = 90^\circ$ ,  $\sin \theta = 1$  and  $\cos \theta = 0$  and hence total pressure head

$$\begin{aligned} &= h_d + h_{ad} + h_{fd}' + h_{fd} + \frac{\bar{V}_d^2}{2g} \text{ above atmospheric pressure head} \\ &= h_d + 0 + \frac{4f \times l_d'}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \right)^2 + \frac{4f \times l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 \\ &= h_d + \frac{4f \times l_d'}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \right)^2 + \frac{4f \times l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(20.26) \end{aligned}$$

(iii) At the end of the delivery stroke,  $\theta = 180^\circ$ ,  $\sin \theta = 0$  and  $\cos \theta = -1$  and hence total pressure head

$$= h_d - \frac{l_d'}{g} \times \frac{A}{a_d} \omega^2 r + \frac{4f \times l_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(20.27)$$

In equations (20.25), (20.26) and (20.27), the quantities

$$\left[ \frac{l_d'}{g} \times \frac{A}{a_d} \omega^2 r \right] \text{ and } \left[ \frac{4f \times l_d'}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \right)^2 \right]$$

are very small and can be neglected.

(b) **Pressure head in the cylinder during suction stroke.** The pressure head due to acceleration in the suction pipe of length  $l_s'$  (between air vessel and cylinder) is given as

$$h_{as}' = \frac{l_s'}{g} \times \frac{A}{a_s} \omega^2 r \cos \theta$$

Loss of friction head in the suction pipe of length  $l_s'$  given as

$$h_{fs}' = \frac{4f \times l_s' \times v_s^2}{d_s \times 2g}$$

where for suction pipe, the velocity ( $v_s$ ) is given by equation (20.11) as  $v_s = \frac{A}{a_s} \omega r \sin \theta$

Substituting the value of  $v_s$  in  $h_{fs}'$ , we get

$$h_{fs}' = \frac{4 \times f \times l_s'}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \sin \theta \right)^2$$

Loss of head due to friction in the suction pipe for the length below the air vessel (i.e., lengths  $l_s$ ),

$$h_{fs} = \frac{4f \times l_s}{d_s \times 2g} \times (\bar{V}_s)^2$$

where  $\bar{V}_s$  is the mean velocity of flow and is given by equation (20.24) as  $\bar{V}_s = \frac{A}{a_s} \times \frac{\omega r}{\pi}$

$$\therefore h_{fs} = \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2$$

Hence the pressure head in the cylinder during suction stroke is given as

(i) At the beginning of suction stroke  $\theta = 0^\circ$ ,  $\sin \theta = 0$  and  $\cos \theta = 1$  and hence pressure head

$$\begin{aligned} &= (h_s + h_{as}' + h_{fs}' + h_{fs})_{\theta=0^\circ} + \frac{\bar{V}_s^2}{2g} \text{ below atmospheric pressure head} \\ &= h_s + \frac{l_s'}{g} \times \frac{A}{a_s} \omega^2 r + 0 + \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 \\ &= h_s + \frac{l_s'}{g} \times \frac{A}{a_s} \omega^2 r + \frac{4fl_s}{d_s \times 2g} \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(20.28) \end{aligned}$$

below atmospheric pressure head.

(ii) In the middle of suction stroke,  $\theta = 90^\circ$ ,  $\sin \theta = 1$  and  $\cos \theta = 0$  and hence pressure head

$$\begin{aligned} &= (h_s + h_{as}' + h_{fs}' + h_{fs})_{\theta=90^\circ} + \frac{\bar{V}_s^2}{2g} \text{ below atmospheric pressure head} \\ &= h_s + 0 + \frac{4f \times l_s'}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \right)^2 + \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 \end{aligned}$$

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$$= h_s + \frac{4fl'_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \right)^2 + \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(20.29)$$

(iii) At the end of the suction stroke,  $\theta = 180^\circ$ ,  $\sin \theta = 0$  and  $\cos \theta = -1$ .

Hence pressure head =  $(h_s + h'_{as} + h'_{fs} + h_{fd})_{\theta=180^\circ} + \frac{\bar{V}_s^2}{2g}$  below atmospheric pressure head

$$= h_s - \frac{l'_s}{g} \times \frac{A}{a_s} \omega^2 r + 0 + \frac{4fl_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2} \left( \frac{A}{a_s} \times \frac{\omega r}{\pi} \right)^2 \quad \dots(20.30)$$

In equations (20.28), (20.29) and (20.30), the quantities  $\left( \frac{l'_s}{g} \times \frac{A}{a_s} \omega^2 r \right)$  and  $\left[ \frac{4f \times l'_s}{d_s \times 2g} \times \left( \frac{A}{a_s} \omega r \right)^2 \right]$  are very small and may be neglected.

(c) **Work done by reciprocating pump with air vessels.** Work done by reciprocating pump fitted with air vessels to the suction and delivery pipe

= Weight of water discharged per second

$$\times \left[ h_s + h_d + h_{fs} + h_{fd} + \frac{\bar{V}_s^2}{2g} + \frac{\bar{V}_d^2}{2g} + \frac{2}{3} h'_{fs} + \frac{2}{3} h'_{fd} \right] \text{ N-m/s.}$$

where weight of water discharged per second for a single-acting pump,  $W = \frac{wALN}{60} = \frac{\rho g ALN}{60}$ .

Also the values of  $h'_{fs}$ ,  $h'_{fd}$ ,  $\frac{\bar{V}_s^2}{2g}$  and  $\frac{\bar{V}_d^2}{2g}$  are very small and hence they can be neglected.

$$\therefore \text{Work done per sec} = \frac{\rho g ALN}{60} [h_s + h_d + h_{fs} + h_{fd}] \quad \dots(20.31)$$

(d) **Work saved by fitting air vessel.** By fitting air vessel the head loss due to friction in suction and delivery pipe is reduced. This reduction in the head loss saves a certain amount of energy, which can be calculated by finding the work done against friction without air vessel and with air vessel. The difference of the two gives the saving in work done.

(i) **Work done against friction without air vessels.** Consider a single-acting reciprocating pump without any air vessels on the pipes. The velocity of flow through the pipes is given by equation (20.11) as

$$v = \frac{A}{a} \omega \times r \sin \theta$$

and loss of head due to friction is given by equation (20.17) as  $h_f = \frac{4fl}{d \times 2g} \times \left[ \frac{A}{a} \omega r \sin \theta \right]^2$ .

The variation of  $h_f$  with  $\theta$  is parabolic in nature and hence indicator diagram for the loss of head due to friction in pipes will be a parabola. The work done by pump against friction per stroke is equal to the area of the indicator diagram due to friction.

∴ Work done by pump per stroke against friction,

$$\begin{aligned}
 W_1 &= \text{Area of the parabola} = \frac{2}{3} \times \text{Base} \times \text{Height} \\
 &= \frac{2}{3} \times L \times \left[ \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 \right] \quad (\because \text{Height} = h_f \text{ at } \theta = 90^\circ) \\
 &= \frac{2}{3} \times L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2. \quad \dots(20.32)
 \end{aligned}$$

(ii) **Work done against friction with air vessels.** By fitting an air vessel to the pump, the velocity of flow through the pipes (except for lengths  $l_s'$  and  $l_d'$  which may be considered negligible) is uniform and equal to mean velocity of flow, which is given by equation (20.24) as

$$\bar{V} = \frac{A}{a} \times \frac{\omega r}{\pi}$$

∴ Loss of head due to friction with air vessel is given as

$$= \frac{4fl \times \bar{V}^2}{d \times 2g} = \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \times \frac{\omega r}{\pi} \right)^2$$

The head loss due to friction with air vessel is independent of  $\theta$  and hence indicator diagram will be a rectangle.

∴ Work done by pump per stroke against friction,

$$\begin{aligned}
 W_2 &= \text{Area of the rectangle} \\
 &= \text{Base} \times \text{Height} = L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \times \frac{\omega r}{\pi} \right)^2 \\
 &= \frac{1}{\pi^2} \times L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 \quad \dots(20.33)
 \end{aligned}$$

The work given by equation (20.33) is less than the work given by equation (20.32). Hence, by fitting an air vessel work is saved.

(iii) **Work saved in a single-acting reciprocating pump.** Hence, saving in work done per stroke is obtained by subtracting equation (20.33) from equation (20.32).

∴ Work saved per stroke =  $W_1 - W_2$

$$\begin{aligned}
 &= \frac{2}{3} L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 - \frac{1}{\pi^2} L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 \\
 &= L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 \left[ \frac{2}{3} - \frac{1}{\pi^2} \right] \quad \dots(20.34)
 \end{aligned}$$

The percentage of the work saved per stroke

$$\begin{aligned}
 &= \left( \frac{W_1 - W_2}{W_1} \right) \times 100 = \frac{L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 \left[ \frac{2}{3} - \frac{1}{\pi^2} \right]}{\frac{2}{3} L \times \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2} \times 100
 \end{aligned}$$

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$$= \frac{\left(\frac{2}{3}\right) - \left(\frac{1}{\pi^2}\right)}{\left(\frac{2}{3}\right)} \times 100 = 84.8\%.$$

(iv) **Work saved in a double-acting reciprocating pump.** The work lost in friction per stroke in case of double-acting reciprocating pump without air vessel is the same as given in case of single-acting reciprocating pump. Hence, it is given by equation (20.32) as

$$W_1 = \frac{2}{3} \times L \times \frac{4fl}{d \times 2g} \times \left(\frac{A}{a} \omega r\right)^2$$

When the air vessel is fitted to the pipe near the cylinder, the mean velocity of flow,  $\bar{V}$  for double-acting is given by,

$$\begin{aligned} \bar{V} &= \frac{\text{Discharge}}{\text{Area of pipe}} = \frac{Q}{a} = \frac{2ALN}{60a} \\ &= \frac{2A \times 2r}{60a} \times \frac{60\omega}{2\pi} \quad \left( \because L = 2r \text{ and } N = \frac{60\omega}{2\pi} \right) \\ &= \frac{2A}{a} \times \frac{\omega r}{\pi} \end{aligned}$$

$\therefore$  Loss of head due to friction for double-acting

$$= \frac{4fl \times \bar{V}^2}{d \times 2g} = \frac{4fl}{d \times 2g} \times \left(\frac{2A}{a} \times \frac{\omega r}{\pi}\right)^2$$

$\therefore$  Work lost due to friction per stroke

$$\begin{aligned} W_2 &= \text{Area of the rectangle} \\ &= L \times \frac{4fl}{d \times 2g} \times \left(\frac{2A}{a} \times \frac{\omega r}{\pi}\right)^2 \\ &= \frac{4}{\pi^2} \times L \times \frac{4fl}{d \times 2g} \times \left(\frac{A}{a} \times \omega r\right)^2 \quad \dots(20.34A) \end{aligned}$$

$\therefore$  Saving in work done per stroke =  $\frac{W_1 - W_2}{W_1}$

$$\frac{\frac{2}{3} \times L \times \frac{4fl}{d \times 2g} \times \left(\frac{A}{a} \times \omega r\right)^2 - \frac{4}{\pi^2} \times L \times \frac{4fl}{d \times 2g} \times \left(\frac{A}{a} \times \omega r\right)^2}{\frac{2}{3} \times L \times \frac{4fl}{d \times 2g} \times \left(\frac{A}{a} \times \omega r\right)^2} = \frac{\left(\frac{2}{3}\right) - \left(\frac{4}{\pi^2}\right)}{\left(\frac{2}{3}\right)} = 0.392 = 39.2\%.$$

(e) **Discharge of liquid into and from the air vessel.** 1. Let the air vessel is fitted to both suction and delivery pipes of a *single-acting* reciprocating pump. Due to air vessel, the liquid in suction and delivery pipes beyond air vessel will be moving with a constant mean velocity. This mean velocity in the pipes is given by equation (20.24) as

$$\bar{V} = \frac{A}{a} \times \frac{\omega r}{\pi}$$

The mean discharge ( $\bar{Q}$ ) in the pipes (suction or delivery) will be equal to,

$$\begin{aligned}\bar{Q} &= \bar{V} \times \text{area of pipe} \\ &= \left( \frac{A}{a} \times \frac{\omega r}{\pi} \right) \times a \quad (\because a = \text{area of pipe}) \\ &= \frac{A \times \omega \times r}{\pi} \quad \dots(20.35)\end{aligned}$$

The velocity of piston in the cylinder at any instant is given by equation (20.10) as

$$V = \omega r \times \sin(\omega t) = \omega r \times \sin \theta \quad (\because \omega t = \theta)$$

Hence instantaneous discharge to or from the cylinder of the pump will be as

$$\begin{aligned}Q_i &= \text{Velocity of piston} \times \text{Area of piston} \\ &= (\omega r \sin \theta) \times A \quad (\because A = \text{Area of piston}) \\ &= A \omega r \sin \theta \quad \dots(20.36)\end{aligned}$$

The difference of the two discharges given by equations (20.36) and (20.35) will be equal to the rate of flow of liquid into or from the air vessel.

$\therefore$  Rate of flow of liquid into the air vessel

$$= \left( A \omega r \sin \theta - \frac{A \omega r}{\pi} \right) = A \omega r \left( \sin \theta - \frac{1}{\pi} \right) \quad \dots(20.37)$$

(i) *For air vessel fitted to delivery pipe.* The liquid will be flowing into the air vessel if equation (20.37) is positive. But if equation (20.37) is negative, then liquid will flow from the air vessel. And if equation (20.37) is zero, then no flow is taking place from or to the air vessel.

(ii) *For air vessel fitted to suction pipe.* If equation (20.37) is positive, then liquid is flowing from the air vessel. If equation (20.37) is negative, then liquid is flowing into the air vessel.

For no flow of liquid into or from the air vessel, the equation (20.37) should be zero.

$$\therefore A \omega r \left( \sin \theta - \frac{1}{\pi} \right) = 0 \text{ or } \sin \theta - \frac{1}{\pi} = 0$$

$$\text{or} \quad \sin \theta = \frac{1}{\pi} = 0.3183$$

$$\therefore \theta = 18^\circ 34' \text{ or } 161^\circ 26'$$

**2. For double-acting pump.** The discharge for double-acting is given by equation (20.5) as,

$$\begin{aligned}Q &= \frac{2ALN}{60} \quad \left( \omega = \frac{2\pi N}{60} \text{ or } N = \frac{60\omega}{2\pi} \right) \\ &= \frac{2AL}{60} \times \frac{60\omega}{2\pi} = \frac{AL\omega}{\pi} = \frac{A \times 2r \times \omega}{\pi} \quad (\because L = 2r) \\ &= \frac{2A\omega r}{\pi}\end{aligned}$$

The above discharge is the mean discharge. Hence for double-acting mean discharge ( $Q$ ) is given by

$$Q = \frac{2A\omega r}{\pi}$$

But the instantaneous discharge ( $Q_i$ ) to or from the cylinder of the pump will be

$$Q_i = \text{Velocity of piston} \times \text{Area of piston}$$

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$$= (\omega r \sin \theta) \times A = A\omega r \sin \theta$$

Hence, rate of flow of liquid into air vessel

$$= Q_i - Q = A\omega r \sin \theta - \frac{2A\omega r}{\pi} = A\omega r \left( \sin \theta - \frac{2}{\pi} \right) \quad \dots(20.38)$$

(i) For air vessel fitted to delivery pipe. If equation (20.38) is positive, the liquid is flowing into the air vessel fitted to delivery pipe. If equation (20.38) is negative, then liquid is flowing from the air vessel. And if equation (20.38) is zero then no flow is taking place into or from the air vessel.

(ii) For air vessel fitted to suction pipe. If equation (20.38) is positive, the liquid is flowing from the air vessel fitted to suction pipe. If equation (20.38) is negative, the liquid is flowing into the air vessel.

For no flow of liquid into or from the air vessel, equation (20.38) should be zero.

$$\therefore A\omega r \left( \sin \theta - \frac{2}{\pi} \right) = 0 \text{ or } \sin \theta = \frac{2}{\pi} = 0.6366$$

$$\therefore \theta = 39^\circ 32' \text{ or } 140^\circ 28'.$$

**Problem 20.14** The cylinder of a single-acting reciprocating pump is 15 cm in diameter and 30 cm in stroke. The pump is running at 30 r.p.m. and discharge water to a height of 12 m. The diameter and length of the delivery pipe are 10 cm and 30 m respectively. If a large air vessel is fitted in the delivery pipe at a distance of 2 m from the centre of the pump, find the pressure head in the cylinder.

(i) At the beginning of the delivery stroke, and

(ii) In the middle of the delivery stroke. Take  $f = .01$

**Solution.** Given :

$$\text{Diameter of cylinder, } D = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} \times .15^2 = 0.01767 \text{ m}^2$$

$$\text{Stroke length, } L = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{0.30}{2} = 0.15 \text{ m}$$

$$\text{Speed of pump, } N = 30 \text{ r.p.m.}$$

$$\therefore \text{Angular speed } \omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 30}{60} = \pi \text{ rad/s.}$$

$$\text{Delivery head, } h_d = 12 \text{ m}$$

$$\text{Diameter of delivery pipe, } d_d = 10 \text{ cm} = 0.10 \text{ m}$$

$$\therefore \text{Area, } a_d = (\pi/4) (.1)^2 = .007854$$

$$\text{Length of delivery pipe, } l = 30 \text{ m}$$

$$\text{Length of air vessel from the centre of the cylinder, } l_d' = 2 \text{ m}$$

$$\therefore \text{Length of delivery pipe above the air vessel, } l_d = l - l_d' = 30 - 2 = 28 \text{ m}$$

$$\text{Co-efficient of friction, } f = 0.01.$$

(i) The pressure head in the cylinder at the beginning of the delivery stroke is given by equation (20.25) as

$$\begin{aligned}
 &= h_d + \frac{l_d'}{g} \times \frac{A}{a_d} \omega^2 r + \frac{4fl_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 \\
 &= 12 + \frac{2}{9.81} \times \frac{.01767}{.007854} \times \pi^2 \times .15 + \frac{4 \times .01 \times 28}{0.1 \times 2 \times 9.81} \left( \frac{.01767}{.007854} \times \frac{\pi \times .15}{\pi} \right)^2 \\
 &\quad + \frac{1}{2 \times 9.81} \left( \frac{.01767}{.007854} \times \frac{\pi \times .15}{\pi} \right)^2 \\
 &= 12 + .6709 + 0.065 + .0058 = \mathbf{12.75 \text{ m. Ans.}}
 \end{aligned}$$

(ii) The pressure head in the cylinder in the middle of the delivery stroke is given by equation (20.26) as

$$\begin{aligned}
 &= h_d + \frac{4f \times l_d'}{d_d \times 2g} \times \left( \frac{A}{a_d} \omega r \right)^2 + \frac{4fl_d}{d_d \times 2g} \times \left( \frac{A}{a_d} \times \frac{\omega r}{\pi} \right)^2 + \frac{1}{2g} \left( \frac{A}{a_d} \frac{\omega r}{\pi} \right)^2 \\
 &= 12 + \frac{4 \times .01 \times 2}{0.1 \times 2 \times 9.81} \times \left( \frac{.01767}{.007854} \times \pi \times .15 \right)^2 + .065 + .0058 \\
 &= 12 + .0458 + .065 + .0058 = \mathbf{12.116 \text{ m. Ans.}}
 \end{aligned}$$

**Problem 20.15** A single-acting reciprocating pump is to raise a liquid of density 1200 kg per cubic metre through a vertical height of 11.5 metres, from 2.5 metres below pump axis to 9 metres above it. The plunger, which moves with S.H.M., has diameter 125 mm and stroke 225 mm. The suction and delivery pipes are 75 mm diameter and 3.5 metres and 13.5 metres long respectively. There is a large air vessel placed on the delivery pipe near the pump axis. But there is no air vessel on the suction pipe. If separation takes place at 8.829 N/cm<sup>2</sup> below atmospheric pressure, find :

- (i) maximum speed, with which the pump can run without separation taking place, and
- (ii) power required to drive the pump, if  $f = 0.02$ .

Neglect slip for the pump.

**Solution.** Given :

Liquid density,	$\rho = 1200 \text{ kg/m}^3$
Total vertical height	$= 11.5 \text{ m}$
Suction head,	$h_s = 2.5 \text{ m}$
Delivery head,	$h_d = 9 \text{ m}$
Dia. of plunger,	$D = 125 \text{ mm} = 0.125 \text{ m}$

$$\text{Area of plunger, } A = \frac{\pi}{4} \times .125^2 = 0.0123 \text{ m}^2$$

$$\text{Stroke length, } L = 225 \text{ mm} = 0.225 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{.225}{2} = .1125$$

$$\text{Dia. of suction and delivery pipe, } d = 75 \text{ mm} = 0.075 \text{ m}$$

$$\text{Area of suction and delivery pipe, } a = \frac{\pi}{4} (0.075)^2 = 0.00442 \text{ m}^2$$

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Length of suction pipe,  $l_s = 3.5 \text{ m}$   
 Length of delivery pipe,  $l_d = 13.5 \text{ m}$ .

Air vessel is placed on the delivery side only. Hence, the velocity in the delivery pipe will be uniform. And there will be no accelerating head on delivery side.

$$\begin{aligned}\text{Separation pressure} &= 8.829 \frac{\text{N}}{\text{cm}^2} \text{ below atmospheric pressure} \\ &= 8.829 \times 10^4 \frac{\text{N}}{\text{m}^2} \text{ below atmospheric pressure}\end{aligned}$$

$$\begin{aligned}\therefore \text{Separation pressure head, } h_{sep} &= \frac{\text{Separation pressure}}{\rho \times g} \\ &= \frac{8.829 \times 10^4}{1200 \times 9.81} \text{ m below atmosphere} \\ &= 7.5 \text{ m below atmosphere} \quad \dots(i)\end{aligned}$$

(i) Maximum speed, with which pump can run without separation taking place.

Let  $N = \text{Max. speed with which pump can run without separation taking place.}$

The separation can take place only at the beginning of suction stroke. As air vessel is not fitted on the suction pipe, there will be accelerating head acting on suction side.

Pressure head at the beginning of suction stroke

$$= h_s + h_{as} \text{ below atmosphere}$$

This pressure should be equal to  $h_{sep}$  in the limiting case

$$\begin{aligned}\therefore \quad 7.5 &= h_s + h_{as} = 2.5 + h_{as} \\ \therefore \quad h_{as} &= 7.5 - 2.5 = 5.0 \text{ m}\end{aligned}$$

But ' $h_{as}$ ' at the beginning of suction stroke

$$\begin{aligned}&= \frac{l_s}{g} \times \frac{A}{a} \omega^2 r \\ \therefore \quad 5.0 &= \frac{3.5}{9.81} \times \frac{0.0123}{.00442} \times \omega^2 \times .1125 \\ \therefore \quad \omega &= \sqrt{\frac{5.0 \times 9.81 \times .00442}{3.5 \times .0123 \times .1125}} = 6.69 \text{ radians/s.}\end{aligned}$$

$$\text{But } \omega = \frac{2\pi N}{60}$$

$$\therefore N = \frac{60 \times \omega}{2\pi} = \frac{60 \times 6.69}{2\pi} = 63.88 \text{ r.p.m. Ans.}$$

$\therefore$  Maximum speed with which the pump can run without separation taking place is 63.88 r.p.m.

(ii) Power required to drive the pump.

New discharge ( $Q$ ) of the single-acting pump is given by equation (20.1) as

$$Q = \frac{ALN}{60} = \frac{0.0123 \times 0.225 \times 63.88}{60} = 0.00294 \text{ m}^3/\text{s.}$$

Velocity of liquid in delivery pipe will be uniform.

$$\therefore Q = \text{Area of delivery pipe} \times \text{Velocity} = a \times v$$

$$\therefore v = \frac{Q}{a} = \frac{0.00294}{0.00442} = 0.665 \text{ m/s.}$$

$\therefore$  Head loss due to friction in delivery pipe,

$$h_{fd} = \frac{4f \times l_d \times v^2}{d \times 2g} = \frac{4 \times .02 \times 13.5 \times (.665)^2}{.075 \times 2 \times 9.81} = 0.324 \text{ m.}$$

During suction stroke, the value of maximum  $h_{fs}$  is given by

$$h_{fs} = \frac{4 \times f \times l_s}{d \times 2g} \times \left( \frac{A}{a} \omega r \right)^2 = \frac{4 \times .02 \times 3.5}{.075 \times 2 \times 9.81} \left( \frac{.0123}{.00442} \times 6.69 \times .1125 \right)^2 \\ = 0.834 \text{ m}$$

Now power required to drive the pump in kW

$$= \frac{\text{Work done/s}}{1000} = \frac{\rho \times g \times Q}{1000} \times \left[ h_s + h_d + \frac{2}{3} h_{fs} + h_{fd} \right] \\ = \frac{1200 \times 9.81 \times .00294}{1000} \times \left[ 2.5 + 9.0 + \frac{2}{3} \times .834 + .324 \right] \\ = 0.428 \text{ kW. Ans.}$$

**Problem 20.16** A double-acting reciprocating piston pump is pumping water (diameter of the piston 250 mm, diameter of piston rod, which is on one side of the piston 50 mm, piston stroke 380 mm). The suction and discharge heads are 4.5 m and 18.6 m respectively. Find the work done by the piston during outward stroke. Would the work done change for the inward stroke?

**Solution.** Given :

$$\text{Dia. of piston, } D = 250 \text{ mm} = 0.25 \text{ m}$$

$$\therefore \text{Area of piston, } A = \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times 0.25^2 \text{ m}^2 = 0.0491 \text{ m}^2$$

$$\text{Dia. of piston rod, } d = 50 \text{ mm} = 0.05 \text{ m}$$

$$\therefore \text{Area of piston rod, } a = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 0.05^2 = 0.001963 \text{ m}^2$$

$$\text{Stroke length, } L = 380 \text{ mm} = 0.380 \text{ m}$$

$$\text{Suction head, } h_s = 4.5 \text{ m}$$

$$\text{Discharge or delivery head, } h_d = 18.6 \text{ m}$$

Find work done during outward stroke

In a double-acting pump for outward stroke, suction side will be towards the piston and delivery side will be towards the piston rod.

Hence, total work done during outward stroke

$$= \text{Weight of water lifted} \times \text{height through which water is lifted} \\ + \text{Weight of water delivered} \times \text{height through which water is delivered} \\ = \rho \times g \times Q_1 \times h_s + \rho \times g \times Q_2 \times h_d$$

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where during outward stroke,  $Q_1 = A \times L = 0.0491 \times 0.380 = 0.01865 \text{ m}^3$

$$\begin{aligned} Q_2 &= (A - a) \times L \\ &= (0.0491 - 0.001963) \times 0.380 = 0.01791 \text{ m}^3 \end{aligned}$$

and

$$\rho \times g = 1000 \times 9.81 \text{ N/m}^3$$

$\therefore$  Total work done during outward stroke

$$\begin{aligned} &= (1000 \times 9.81 \times 0.01865 \times 4.5 + 1000 \times 9.81 \times 0.01791 \times 18.6) \text{ Nm} \\ &= 9.81 \times 0.01865 \times 4.5 + 9.81 \times 0.01791 \times 18.6 (\text{kJ}) \\ &= (0.8233 + 3.268) \text{ kJ} \quad (\because \text{ J} = \text{Nm}) \\ &= \mathbf{4.0913 \text{ kJ. Ans.}} \end{aligned}$$

For inward stroke, suction side will be towards the piston rod whereas the delivery side will be towards the piston.

$\therefore$  Total work done during inward stroke

$$\begin{aligned} &= \rho \times g \times Q_2 \times h_s + \rho \times g \times Q_1 \times h_d = \rho \times g (Q_2 \times h_s + Q_1 \times h_d) \\ &= 1000 \times 9.81 (0.01791 \times 4.5 + 0.01865 \times 18.6) \text{ Nm} \\ &= 1000 \times 9.81 (0.0806 + 0.3468) \text{ J} \\ &= 9.81 (0.0806 + 0.3468) \text{ kJ} = 4.192 \text{ kJ} \end{aligned}$$

Hence, work done during inward stroke will be different. **Ans.**

**Problem 20.17** A single-acting reciprocating pump has a plunger diameter of 250 mm and stroke of 450 mm and it is driven with S.H.M. at 60 r.p.m. The length and diameter of delivery pipe are 60 m and 100 mm respectively. Determine the power saved in overcoming friction in the delivery pipe by fitting an air vessel on the delivery side of the pump. Assume friction factor = 0.01.

**Solution.** Given :

$$\text{Dia. of plunger, } D = 250 \text{ mm} = 0.25 \text{ m}$$

$$\therefore \text{Area, } A = \frac{\pi}{4} \times 0.25^2$$

$$\text{Stroke length, } L = 450 \text{ mm} = 0.45 \text{ m}$$

$$\therefore \text{Crank radius, } r = \frac{L}{2} = \frac{0.45}{2} = 0.225 \text{ m}$$

$$\text{Speed, } N = 60 \text{ r.p.m.}$$

$$\therefore \text{Angular speed, } \omega = 2\pi N/60 = 2\pi \times 60/60 = 2\pi \text{ rad/s.}$$

$$\text{Length of delivery pipe, } L = 60 \text{ m}$$

$$\text{Dia. of delivery pipe, } d = 100 \text{ mm} = 0.1 \text{ m}$$

$$\therefore \text{Area of pipe, } a = \frac{\pi}{4} \times 0.1^2$$

$$\text{Friction factor, } f = 0.01$$

Power saved is given by,

$$\text{Power saved} = \rho \times g \times Q \times \left[ \frac{2}{3} (h_f)_{\text{without air vessel}} - (h_f)_{\text{with air vessel}} \right]$$

where  $\rho \times g = 1000 \times 9.81 \frac{\text{N}}{\text{m}^3}$

$$Q = \frac{ALN}{60} = \frac{\pi}{4} \times \frac{0.25^2 \times 0.45 \times 60}{60} = 0.02209 \text{ m}^3/\text{s}$$

Without air vessel,  $h_f = \frac{f^* \times L \times v^2}{d \times 2g} = \frac{f \times L}{d \times 2g} \times \left( \frac{A}{a} \omega \times r \right)^2 \quad \left[ \because v = \frac{A}{a} \times \omega \times r \right]$

$$= \frac{0.01 \times 60}{0.1 \times 2 \times 9.81} \times \frac{\left( \frac{\pi}{4} \times 0.25^2 \times 2\pi \times 0.225 \right)^2}{\frac{\pi}{4} \times 0.1^2} = 23.87 \text{ m}$$

With air vessel,  $h_f = \frac{f^* \times L \times \bar{V}^2}{d \times 2g}$

$$= \frac{f^* \times L}{d \times 2g} \times \left( \frac{A}{a} \times \frac{\omega r}{\pi} \right)^2, \text{ where } \bar{V}^2 = \frac{A}{a} \times \frac{\omega \times r}{\pi}$$

$$= \frac{0.01 \times 60}{0.1 \times 2 \times 9.81} \times \left( \frac{\frac{\pi}{4} \times 0.25^2}{\frac{\pi}{4} \times 0.1^2} \times \frac{2\pi \times 0.225}{\pi} \right)^2 = 2.419 \text{ m.}$$

$\therefore$  Power saved  $= \rho \times g \times Q \times \left[ \frac{2}{3} (h_f)_{\text{without air vessel}} - (h_f)_{\text{with air vessel}} \right]$

$$= 1000 \times 9.81 \times 0.02209 \left[ \frac{2}{3} \times 23.87 - 2.419 \right] \text{ W}$$

$$= 9.81 \times 0.02209 \left[ \frac{2}{3} \times 23.87 - 2.419 \right] \text{ kW} = 2.924 \text{ kW. Ans.}$$

**Problem 20.18** A double-acting reciprocating pump runs at 120 r.p.m. When its suction pipe of 100 mm diameter is fitted with an air vessel on its suction side. The diameter of cylinder and stroke are 150 mm and 450 mm respectively. If piston is to be driven with S.H.M., find the rate of flow from or into the air vessel when the crank makes angles of  $30^\circ$ ,  $90^\circ$  and  $120^\circ$  with the inner dead centre. Find also the crank angles at which there is no flow into or from the air vessel.

**Solution.** Given :

Speed,  $N = 120 \text{ r.p.m.}$

$\therefore$  Angular speed,  $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 120}{60} = 4\pi \text{ rad/s}$

\* Here friction factor is given and hence the formula is  $h_f = \frac{f \times L \times V^2}{d \times 2g}$  and not  $\frac{4f \times L \times V^2}{d \times 2g}$ .

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$$\begin{aligned}
 \text{Dia. of suction pipe} &= 100 \text{ mm} = 0.1 \text{ m} \\
 \therefore \text{Area of suction pipe, } a &= \frac{\pi}{4} (0.1)^2 = 0.007854 \text{ m}^2 \\
 \text{Dia. of cylinder} &= 150 \text{ mm} = 0.15 \text{ m} \\
 \therefore \text{Area of cylinder, } A &= \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2 \\
 \text{Stroke length, } L &= 450 \text{ mm} = 0.45 \text{ m} \\
 \therefore \text{Crank radius, } r &= \frac{L}{2} = \frac{0.45}{2} = 0.225 \text{ m}
 \end{aligned}$$

Find rate of flow from or into air vessel when  $\theta = 30^\circ, 90^\circ$  and  $120^\circ$

The rate of flow of liquid for double-acting pump into the air vessel is given by equation (20.38).

$\therefore$  Rate of flow of liquid into air vessel

$$= A\omega r \left( \sin \theta - \frac{2}{\pi} \right) = 0.01767 \times 4\pi \times 0.225 \left( \sin \theta - \frac{2}{\pi} \right) = 0.04996 \left( \sin \theta - \frac{2}{\pi} \right)$$

In this problem, air vessel is fitted to the suction pipe. Hence if the above rate of flow is positive, the liquid will be flowing from the air vessel. And if the above rate of flow is negative, the liquid will be flowing into the air vessel.

(i) For  $\theta = 30^\circ$ .

$$\text{The above rate of flow} = 0.04996 \left( \sin 30^\circ - \frac{2}{\pi} \right) = 0.04996(0.5 - 0.6366) = -0.00682 \text{ m}^3/\text{s. Ans.}$$

Since the rate of flow is negative, hence the flow is taking place into the air vessel.

(ii) For  $\theta = 90^\circ$ .

$$\text{The rate of flow becomes} = 0.04996 \left( \sin 90^\circ - \frac{2}{\pi} \right) = 0.04996(1 - 0.6366) = 0.0181 \text{ m}^3/\text{s. Ans.}$$

As it is positive, hence rate of flow is taking place from the air vessel.

(iii) For  $\theta = 120^\circ$ .

$$\text{The rate of flow becomes} = 0.04996 \left( \sin 120^\circ - \frac{2}{\pi} \right) = 0.04996(0.866 - 0.6366) = 0.01146 \text{ m}^3/\text{s. Ans.}$$

As it is positive, hence rate of flow is taking place from the air vessel.

(iv) Crank angle at which there is no flow into or from air vessel

Let  $\theta$  = angle at which there is no flow. But rate of flow

$$= 0.04996 \left( \sin \theta - \frac{2}{\pi} \right)$$

For no flow from or into air vessel

$$0.04996 \left( \sin \theta - \frac{2}{\pi} \right) = 0 \text{ or } \sin \theta = \frac{2}{\pi} = 0.6366$$

$$\therefore \theta = \sin^{-1} 0.6366 = 39^\circ 32' \text{ and } 140^\circ 28'. \text{ Ans.}$$

## ► 20.10 COMPARISON BETWEEN CENTRIFUGAL PUMPS AND RECIPROCATING PUMPS

<i>Centrifugal pumps</i>	<i>Reciprocating pumps</i>
<ol style="list-style-type: none"> <li>1. The discharge is continuous and smooth.</li> <li>2. It can handle large quantity of liquid.</li> <li>3. It can be used for lifting highly viscous liquids.</li> <li>4. It is used for large discharge through smaller heads.</li> <li>5. Cost of centrifugal pump is less as compared to reciprocating pump.</li> <li>6. Centrifugal pump runs at high speed. They can be coupled to electric motor.</li> <li>7. The operation of centrifugal pump is smooth and without much noise. The maintenance cost is low.</li> <li>8. Centrifugal pump needs smaller floor area and installation cost is low.</li> <li>9. Efficiency is high.</li> </ol>	<ol style="list-style-type: none"> <li>1. The discharge is fluctuating and pulsating.</li> <li>2. It handles small quantity of liquid only.</li> <li>3. It is used only for lifting pure water or less viscous liquids.</li> <li>4. It is meant for small discharge and high heads.</li> <li>5. Cost of reciprocating pump is approximately four times the cost of centrifugal pump.</li> <li>6. Reciprocating pump runs at low speed. Speed is limited due to consideration of separation and cavitation.</li> <li>7. The operation of reciprocating pump is complicated and with much noise. The maintenance cost is high.</li> <li>8. Reciprocating pump requires large floor area and installation cost is high.</li> <li>9. Efficiency is low.</li> </ol>

### HIGHLIGHTS

1. A reciprocating pump consists of a cylinder with a piston, a suction pipe, a delivery pipe, a suction valve and a delivery valve.
2. Discharge through a pump per second is given as

$$Q = \frac{ALN}{60} \quad \text{...For a single-acting}$$

$$= \frac{2ALN}{60} \quad \text{...For a double-acting.}$$

3. Work done by reciprocating pump per second

$$= \frac{\rho g ALN}{60} (h_s + h_d) \quad \text{...For a single-acting}$$

$$= \frac{2\rho g ALN}{60} (h_s + h_d) \quad \text{...For a double-acting.}$$

4. Slip of a pump is defined as the difference between the theoretical discharge and actual discharge of the pump.
5. The pressure head ( $h_a$ ) due to acceleration in the suction and delivery pipes is given as

$$h_{as} = \frac{l_s}{g} \times \frac{A}{a_s} \omega^2 r \cos \theta \quad \text{...For suction pipe}$$

$$h_{ad} = \frac{l_d}{g} \times \frac{A}{a_d} \omega^2 r \cos \theta \quad \text{... For delivery pipe.}$$

6. The loss of head due to friction in suction and delivery pipes is obtained from

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$$h_f = \frac{4fl}{d \times 2g} \times \left( \frac{A}{a} \omega r \sin \theta \right)^2.$$

7. Indicator diagram is a graph between the pressure head in the cylinder and the distance travelled by the piston from inner dead centre for one complete revolution of the crank.
8. Work done by the pump is proportional to the area of the indicator diagram. Area of ideal indicator diagram is the same as the area of indicator diagram due to acceleration in suction and delivery pipes.
9. Work done by the pump per second due to acceleration and friction in suction and delivery pipes

$$= \frac{\rho g A L N}{60} \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) \quad \dots \text{For a single-acting}$$

$$= \frac{2\rho g A L N}{60} \left( h_s + h_d + \frac{2}{3} h_{fs} + \frac{2}{3} h_{fd} \right) \quad \dots \text{For a double-acting.}$$

10. Air vessel is used to obtain a continuous supply of water at uniform rate, to save a considerable amount of work and to run the pump at a high speed without separation.
  11. Mean velocity ( $\bar{V}$ ) for a single-acting pump is given as
- $$\bar{V} = \frac{A \omega r}{a \pi}.$$
12. Work done by reciprocating with air vessels fitted to suction and delivery pipes
- $$\approx \frac{\rho g A L N}{60} [h_s + h_d + h_{fs} + h_{fd}].$$
13. Work saved by fitting air vessels in a single-acting reciprocating pump is 84.8% while in a double-acting reciprocating pump, the work saved is 39.2%.

## EXERCISE

### (A) THEORETICAL PROBLEMS

1. What is a reciprocating pump ? Describe the principle and working of a reciprocating pump with a neat sketch. Why is a reciprocating pump not coupled directly to the motor ? Discuss the reason in detail.
2. Differentiate : (i) Between a single-acting and double-acting reciprocating pump, (ii) Between a single cylinder and double cylinder reciprocating pump.
3. Define slip, percentage slip and negative slip of a reciprocating pump.
4. How will you classify the reciprocating pumps ?
5. What is the effect of acceleration of the piston on the velocity and acceleration of the water in suction and delivery pipes? Obtain an expression for the pressure head due to acceleration in the suction and delivery pipes.
6. Find an expression for the head lost due to friction in suction and delivery pipes.
7. Define indicator diagram. How will you prove that area of indicator diagram is proportional to the work done by the reciprocating pump?
8. What is the effect of acceleration in suction and delivery pipes on indicator diagram ? Does the area of the indicator diagram change as compared to the area of ideal indicator diagram ?
9. Draw an indicator diagram, considering the effect of acceleration and friction in suction and delivery pipes. Find an expression for the work done per second in case of single-acting reciprocating pump.
10. What is an air vessel ? Describe the function of the air vessel for reciprocating pumps.

11. Show from first principle that the work saved, against friction in the delivery pipe of a single-acting reciprocating pump, by fitting an air vessel is 84.8% while for a double-acting reciprocating pump the work saved is only 39.20%.
12. What is negative slip in a reciprocating pump ? Explain with neat sketches the function of air vessels in a reciprocating pump.
13. Differentiate, with examples between :
 

(i) Turbines and pumps,	(ii) Impulse and reaction turbines,
(iii) Radial and axial flow turbines, flow turbines,	(iv) Inward and outward radial, and
(v) Kaplan and propeller turbines.	
14. Explain in brief how and when separation of flow takes place in a reciprocating pump. Discuss the preventive measures usually adopted for effective reduction of separation in such a pump.
15. Why is it that the speed of a reciprocating pump without air vessels is not high ? Explain with sketches.
16. Derive an expression for the head lost due to friction in the delivery pipe of a reciprocating pump with and without an air vessel.

### (B) NUMERICAL PROBLEMS

1. A single-acting reciprocating pump running at 30 r.p.m., delivers  $0.012 \text{ m}^3/\text{s}$  of water. The diameter of the piston is 25 cm and stroke length is 50 cm. Determine : (i) The theoretical discharge of the pump, (ii) Co-efficient of discharge, and (iii) Slip and percentage slip of the pump.  
 [Ans. (i)  $0.01227 \text{ m}^3/\text{s}$ , (ii) 0.978, (iii)  $.00027 \text{ m}^2/\text{s}$  and 2.20%]
2. A double-acting reciprocating pump, running at 50 r.p.m. is discharging 900 litres of water per minute. The pump has stroke of 400 mm. The diameter of piston is 250 mm. The delivery and suction heads are 25 m and 4 m respectively. Find the slip of the pump and power required to drive the pump.  
 [Ans.  $.0027 \text{ m}^3/\text{s}$ , 9.3 kW]
3. A single-acting reciprocating pump has a cylinder of a diameter 150 mm and of stroke length 300 mm. The centre of the pump is 4 m above the water surface in the sump. The atmospheric pressure head is 10.3 m of water and pump is running at 40 r.p.m. If the length and diameter of the suction pipe are 5 m and 10 cm respectively, determine the pressure head due to acceleration in the cylinder : (i) At the beginning of the suction stroke, and (ii) In the middle of suction stroke.  
 [Ans. (i) 3.018 m, (ii) 0]
4. If in Problem 3, the length and diameter of delivery pipe are 35 m and 100 mm respectively and water is delivered by the pump to a tank which is 25 m above the centre of the pump, determine the pressure head in the cylinder : (i) At the beginning of the delivery stroke, (ii) In the middle of the stroke, and (iii) At the end of the delivery stroke.  
 [Ans. (i) 5.426 m (abs.), (ii) 35.3 m (abs.), (iii) 14.174 m (abs.)]
5. A single-acting reciprocating pump has piston diameter 15 cm and stroke length 30 cm. The centre of the pump is 5 m above the water level in the sump. The diameter and length of the suction pipe are 10 cm and 8 m respectively. The separation occurs if the absolute pressure head in the cylinder during suction stroke falls below 2.5 m of water. Calculate the maximum speed at which the pump can run without separation. Take atmospheric pressure head = 10.3 m of water.  
 [Ans. 30.45 r.p.m.]
6. A single-acting reciprocating pump has a plunger of 100 mm diameter and a stroke length 200 mm. The centre of the pump is 3 m above the water level in the sump and 20 m below the water level in a tank to which water is delivered by the pump. The diameter and length of suction pipe are 50 mm and 5 m while of the delivery pipe are 40 mm and 30 m respectively. Determine the maximum speed at which the pump may be run without separation, if separation occurs at  $7.3575 \text{ N/cm}^2$  below the atmospheric pressure. Take atmospheric pressure head = 10.3 m of water.  
 [Ans. 36.22 r.p.m.]
7. The diameter and stroke length of a single-acting reciprocating pump are 100 mm and 200 mm respectively. The lengths of suction and delivery pipes are 10 m and 30 m respectively and their diameters are 50 mm. If the pump is running at 30 r.p.m. and suction and delivery heads are 3.5 m and 20 m respectively,

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find the pressure head in the cylinder : (i) at the beginning of the suction and delivery stroke, (ii) in the middle of suction and delivery stroke, and (iii) at the end of the suction and delivery stroke. Take atmospheric pressure head = 10.3 m of water and co-efficient of friction = .009 for both pipes.

[Ans. (i) 2.776 m (abs.), 42.373 m (abs.), (ii) 6.22 m, 43.34 m, (iii) Not possible, 18.225 m]

8. For Problem 7, find the power required to drive the pump, if the liquid flowing through the pump is water. [Ans. 0.25 kW]

9. The cylinder of a single-acting reciprocating pump is 125 mm in diameter and 250 mm in stroke. The pump is running at 40 r.p.m. and discharge water to a height of 15 m. The diameter and length of the delivery pipe are 100 mm and 30 m respectively. If a large air vessel is fitted in the delivery pipe at a distance of 1.5 m from the centre of the pump, find the pressure head in the cylinder : (i) At the beginning of the delivery stroke, and (ii) In the middle of the delivery stroke. Take the efficiency of friction = .01.

[Ans. (i) 15.566 m, (ii) 15.07 m]

# 21

## CHAPTER



### ► 21.1 INTRODUCTION

Fluid system is defined as the device in which power is transmitted with the help of a fluid which may be liquid (water or oil) or a gas (air) under pressure. Most of these devices are based on the principles of fluid statics and fluid kinematics. In this chapter, the following devices will be discussed:

1. The hydraulic press,
2. The hydraulic accumulator,
3. The hydraulic intensifier,
4. The hydraulic ram,
5. The hydraulic lift,
6. The hydraulic crane,
7. The fluid or hydraulic coupling,
8. The fluid or hydraulic torque converter,
9. The air lift pump, and
10. The gear-wheel pump.

### ► 21.2 THE HYDRAULIC PRESS

The hydraulic press is a device used for lifting heavy weights by the application of a much smaller force. It is based on Pascal's law, which states that the intensity of pressure in a static fluid is transmitted equally in all directions.

The hydraulic press consists of two cylinders of different diameters. One of the cylinder is of large diameter and contains a ram, while the other cylinder is of smaller diameter and contains a plunger as shown in Fig. 21.1. The two cylinders are connected by a pipe. The cylinders and pipe contain a liquid through which pressure is transmitted.

When a small force  $F$  is applied on the plunger in the downward direction, a pressure is produced on the liquid in contact with the plunger. This pressure is transmitted equally in all directions and acts on the ram in the upward direction as shown in Fig. 21.1. The heavier weight placed on the ram is then lifted up.

Let

$W$  = Weight to be lifted,

$F$  = Force applied on the plunger,

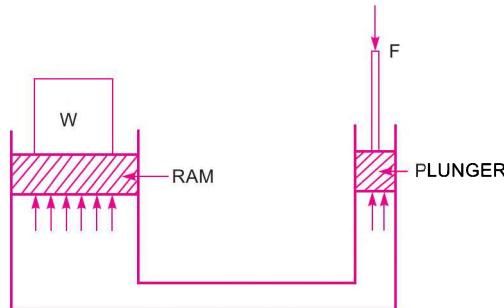


Fig. 21.1 *The hydraulic press.*

$A$  = Area of ram,  
 $a$  = Area of plunger, and  
 $p$  = Pressure intensity produced by force  $F$ .

$$= \frac{\text{Force } F}{\text{Area of plunger}} = \frac{F}{a}$$

Due to Pascal's law, the above intensity of pressure will be equally transmitted in all directions.

Hence, the pressure intensity at the ram will be  $= p = \frac{F}{a}$ .

But the pressure intensity on ram is also  $= \frac{\text{Weight}}{\text{Area of ram}} = \frac{W}{A}$ .

Equating the pressure intensity on ram,  $\frac{F}{a} = \frac{W}{A}$

$$\therefore W = \frac{F}{a} \times A. \quad \dots(21.1)$$

**21.2.1 Mechanical Advantage.** The ratio of weight lifted to the force applied on the plunger is defined as the mechanical advantage. Mathematically, mechanical advantage is written as

$$\text{M. A.} = \frac{W}{F}. \quad \dots(21.2)$$

**21.2.2 Leverage of the Hydraulic Press.** If a lever is used for applying force on the plunger, then a force  $F'$  smaller than  $F$  can lift the weight  $W$  as shown in Fig. 21.2. The ratio of  $L/l$  is called the leverage of the hydraulic press.

Taking moments about  $Q$ ,  $F' \times L = F \times l$

$$\therefore F = F' \times \frac{L}{l} \quad \dots(21.3)$$

Substituting the value of  $F$  in equation (21.1), we get the expression for weight lifted as

$$W = \left( F' \times \frac{L}{l} \right) \times \frac{A}{a} = F' \times \frac{L}{l} \times \frac{A}{a} \quad \dots(21.4)$$

### 21.2.3 Actual Heavy Hydraulic Press.

Based on the nature of the work required, actual hydraulic press is different in shape. But all actual hydraulic press consist of a ram sliding in a cylinder to which high-pressure liquid is forced.

Fig. 21.3 shows one of the actual hydraulic press. It consists of a fixed cylinder in which a ram is sliding. To the lower end of the ram, movable plate is attached. As the ram moves up and down, the movable plate attached to the ram also moves up and down between two fixed plates. When any liquid under high pressure is supplied into the cylinder, the ram

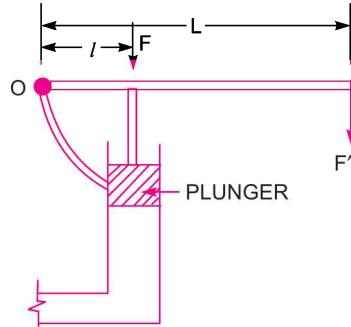


Fig. 21.2

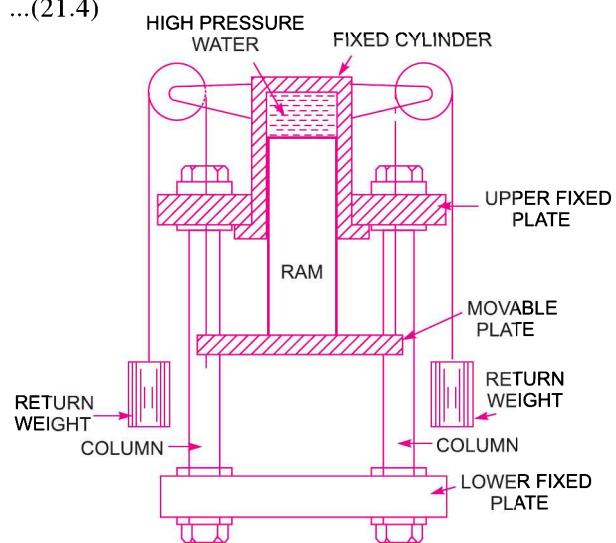


Fig. 21.3 Actual hydraulic press.

moves in the downward direction and exerts a force equal to the product of intensity of pressure supplied and area of the ram, on any material placed between the lower fixed plate and the movable plate. Thus the material gets pressed.

To bring back the ram in the upward position, the liquid from the cylinder is taken out. Then by the action of the return weights, the ram along with the movable plate will move up.

**Problem 21.1** A hydraulic press has a ram of 300 mm diameter and a plunger of 45 mm diameter. Find the weight lifted by the hydraulic press when the force applied at the plunger is 50 N.

**Solution.** Given :

$$\text{Diameter of ram, } D = 300 \text{ mm} = 0.30 \text{ m}$$

$$\text{Diameter of plunger, } d = 45 \text{ mm} = 0.045 \text{ m}$$

$$\text{Force on plunger, } F = 50 \text{ N}$$

$$\text{Let weight lifted } = W \text{ N}$$

$$\text{Area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (0.30)^2 = 0.07068 \text{ m}^2$$

$$\text{Area of plunger, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.045)^2 = .00159 \text{ m}^2$$

The weight lifted ( $W$ ) is given by equation (21.1) as

$$W = \frac{F \times A}{a} = \frac{50 \times 0.07068}{.00159} = 2222.64 \text{ N. Ans.}$$

**Problem 21.2** A hydraulic press has a ram of 200 mm diameter and a plunger of 30 mm diameter. It is used for lifting a weight of 3 kN. Find the force required at the plunger.

**Solution.** Given :

$$\text{Diameter of ram, } D = 200 \text{ mm} = 0.20 \text{ m}$$

$$\therefore \text{Area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times (0.20)^2 = 0.0314 \text{ m}^2$$

$$\text{Diameter of plunger, } d = 30 \text{ mm} = .03 \text{ m}$$

$$\therefore \text{Area of plunger, } a = \frac{\pi}{4} (.03)^2 = 7.068 \times 10^{-4} \text{ m}^2$$

$$\text{Weight lifted, } W = 3 \text{ kN} = 3 \times 1000 = 3000 \text{ N.}$$

$$\text{Let the force on plunger } = F.$$

Using relation given equation (21.1),

$$W = \frac{F}{a} \times A$$

$$F = \frac{W \times a}{A} = \frac{3000 \times 7.068 \times 10^{-4}}{.0314} = 67.52 \text{ N. Ans.}$$

**Problem 21.3** If in the problem 21.2, a lever is used for applying force on the plunger, find the force required at the end of the lever if the ratio  $l/L$  is 1/10.

**Solution.** Given :

$$D = 0.20 \text{ m}, A = 0.0314 \text{ m}^2$$

$$d = 0.03 \text{ m}, a = 7.068 \times 10^{-4} \text{ m}^2$$

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$$W = 3000 \text{ N}, \frac{l}{L} = \frac{1}{10}$$

Let  $F'$  = Force required at the end of the lever.

Using equation (21.4),  $W = F' \times \frac{L}{l} \times \frac{A}{a}$

$$\therefore F' = W \times \frac{l}{L} \times \frac{a}{A} = 3000 \times \frac{1}{10} \times \frac{7.068 \times 10^{-4}}{0.0314} = 6.752 \text{ N. Ans.}$$

**Problem 21.4** If in the problem 21.1, the stroke of the plunger is 100 mm, find the distance travelled by the weight in 100 strokes. Determine the work done during 100 strokes.

**Solution.** The data given in problem 21.1 :

$$D = 0.30 \text{ m}, A = 0.07068 \text{ m}^2, d = 0.045 \text{ m}, a = .00159 \text{ m}^2$$

$$F = 50 \text{ N and } W (\text{calculated}) = 2222.64 \text{ N}$$

Stroke of plunger = 100 mm = 0.10 m

Number of strokes = 100

Volume of liquid displaced by plunger in one stroke

$$= \text{Area of plunger} \times \text{Stroke of plunger}$$

$$= a \times 0.10 \text{ m}^3 = .00159 \times 0.10 = .000159 \text{ m}^3.$$

The liquid displaced by plunger will enter the cylinder in which ram is fitted and this liquid will move the ram in the upward direction.

Let the distance moved by the ram or weight in one stroke

$$= x \text{ m}$$

Then volume displaced by ram in one stroke

$$= \text{Area of ram} \times x = A \times x = 0.07068 \times x \text{ m}^3$$

As volume displaced by plunger and ram is the same,

$$\therefore .000159 = .07068 \times x$$

$$\therefore x = \frac{.000159}{.07068} = .00225 \text{ m}$$

∴ Distance moved by weight in 100 strokes

$$= x \times 100 = .00225 \times 100 = 0.225 \text{ m. Ans.}$$

Work done during 100 strokes = Weight lifted × Distance moved

$$= W \times 0.225 = 2222.64 \times 0.225 \text{ Nm} = 500.094 \text{ Nm. Ans.}$$

**Problem 21.5** A hydraulic press has a ram of 150 mm diameter, plunger of 20 mm diameter. The stroke of the plunger is 200 mm and weight lifted is 800 N. If the distance moved by the weight is 1.0 m in 20 minutes determine :

- (i) The force applied on the plunger,      (ii) Power required to drive the plunger, and
- (iii) Number of strokes performed by the plunger.

**Solution.** Given :

Diameter of ram,  $D = 150 \text{ mm} = 0.15 \text{ m}$

Diameter of plunger,  $d = 20 \text{ mm} = 0.02 \text{ m}$

Stroke of plunger = 200 mm = 0.20 m

Weight lifted,  $W = 800 \text{ N}$

Distance moved by weight = 1.0 m

Time taken by weight = 20 minutes

$$\text{Now, area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (.15)^2 = 0.01767 \text{ m}^2$$

$$\text{Area of plunger, } a = \frac{\pi}{4} (.02)^2 = .00031416 \text{ m}^2.$$

(i) Let the force applied on the plunger =  $F$ .

$$\text{Using equation (21.1), we have } W = \frac{F}{a} \times A$$

$$\therefore F = \frac{W \times a}{A} = \frac{800 \times .00031416}{.01767} = 14.22 \text{ N. Ans.}$$

(ii) Work done by the press per second

$$\begin{aligned} &= \frac{\text{Weight lifted} \times \text{Distance travelled}}{\text{Time}} \\ &= \frac{800 \times 1.0}{20 \times 60} = 0.6667 \text{ Nm/s} \end{aligned}$$

$\therefore$  Power required to drive the plunger

$$= \frac{\text{Work done per sec}}{1000} = \frac{0.6667}{1000} = 0.000666 \text{ kW. Ans.}$$

(iii) Volume of liquid displaced by plunger in one stroke

$$\begin{aligned} &= \text{Area of plunger} \times \text{Stroke length} \\ &= .00031416 \times 0.20 = .000062832 \text{ m}^3 \end{aligned}$$

Total volume of liquid displaced in cylinder

$$\begin{aligned} &= \text{Area of ram} \times \text{Distance moved by weight} \\ &= .01767 \times 1.0 = 0.01767 \text{ m}^3 \end{aligned}$$

$\therefore$  Number of strokes performed by plunger or pump

$$\begin{aligned} &= \frac{\text{Total volume of liquid displaced}}{\text{Volume of liquid displaced per stroke}} \\ &= \frac{0.01767}{.000062832} = 281.22 \approx 281. \text{ Ans.} \end{aligned}$$

### ► 21.3 THE HYDRAULIC ACCUMULATOR

The hydraulic accumulator is a device used for storing the energy of a liquid in the form of pressure energy, which may be supplied for any sudden or intermittent requirement. In case of hydraulic lift or the hydraulic crane, a large amount of energy is required when lift or crane is moving upward. This energy is supplied from hydraulic accumulator. But when the lift is moving in the downward direction, no large external energy is required and at that time, the energy from the pump is stored in the accumulator.

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Fig. 21.4 shows a hydraulic accumulator which consists of a fixed vertical cylinder containing a sliding ram. A heavy weight is placed on the ram. The inlet of the cylinder is connected to the pump, which continuously supplies water under pressure to the cylinder. The outlet of the cylinder is connected to the machine (which may be lift or crane etc.).

The ram is at the lowermost position in the beginning. The pump supplies water under pressure continuously. If the water under pressure is not required by the machine (lift or crane), the water under pressure will be stored in the cylinder. This will raise the ram on which a heavy weight is placed. When the ram is at the uppermost position, the cylinder is full of water and accumulator has stored the maximum amount of pressure energy. When the machine (lift or crane) requires a large amount of energy, the hydraulic accumulator will supply this energy and ram will move in the downward direction.

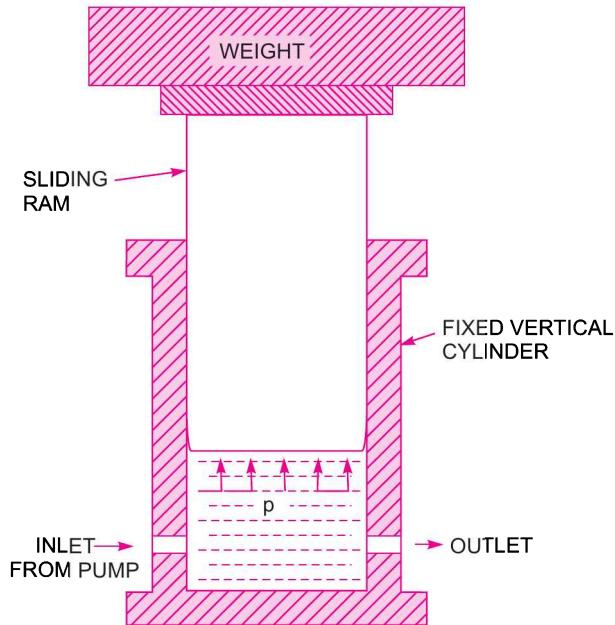


Fig. 21.4 *The hydraulic accumulator.*

**21.3.1 Capacity of Hydraulic Accumulator.** It is defined as the maximum amount of hydraulic energy stored in the accumulator. The expression for the capacity of accumulator is obtained as :

Let

$A$  = Area of the sliding ram,

$L$  = Stroke or lift of the ram,

$p$  = Intensity of water pressure supplied by the pump, and

$W$  = Weight placed on the ram (including the weight of ram),

$W$  = Intensity of pressure  $\times$  Area of ram

Then

$$= p \times A$$

The work done in lifting the ram =  $W \times$  Lift of ram =  $WL$

$$= p \times A \times L \quad (W = p \times A)$$

The work done in lifting the ram is also the energy stored in the accumulator. And energy stored is equal to the capacity of the accumulator.

$$\therefore \text{Capacity of accumulator} = \text{Work done in lifting the ram} \\ = p \times A \times L \quad \dots(21.5)$$

But  $A \times L = \text{Volume of accumulator}$

$$\therefore \text{Capacity of accumulator} = p \times \text{Volume of accumulator.} \quad \dots(21.6)$$

**Problem 21.6** Determine the length of stroke for an accumulator having a displacement of 115 litres. The diameter of the plunger is 350 mm.

**Solution.** Given :

$$\begin{aligned} \text{Displacement} &= 115 \text{ litres} = 0.115 \text{ m}^3 \\ \text{or Volume of accumulator} &= 0.115 \text{ m}^3 \end{aligned}$$

$$\text{Dia. of plunger, } D = 350 \text{ mm} = 0.35 \text{ m}$$

$$\therefore \text{Area of plunger, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.35^2 \text{ m}^2$$

$$\text{But volume of accumulator} = A \times L$$

where  $L$  = length of stroke.

$$\begin{aligned} \therefore \text{Volume} &= \frac{\pi}{4} D^2 \times L \text{ or } 0.115 = \frac{\pi}{4} \times (0.35)^2 \times L \\ \therefore L &= \frac{0.115 \times 4}{\pi \times 0.35^2} = 1.195 \text{ m. Ans.} \end{aligned}$$

**Problem 21.7** The water is supplied at a pressure of  $14 \text{ N/cm}^2$  to an accumulator, having a ram of diameter 1.5 m. If the total lift of the ram is 8 m, determine :

- (i) The capacity of the accumulator, and
- (ii) Total weight placed on the ram (including the weight of ram).

**Solution.** Given :

$$\text{Supply pressure, } p = 14 \text{ N/cm}^2 = 14 \times 10^4 \text{ N/m}^2$$

$$\text{Dia. of ram, } D = 1.5 \text{ m}$$

$$\therefore \text{Area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (1.5)^2 = 1.767 \text{ m}^2$$

$$\text{Lift of ram, } L = 8 \text{ m.}$$

(i) Capacity of accumulator is given by equation (21.5) as

$$\begin{aligned} \text{Capacity} &= p \times A \times L = 14 \times 10^4 \times 1.767 \times 8 \\ &= 1979.04 \times 10^3 \text{ Nm} = 1979.04 \text{ kNm. Ans.} \end{aligned}$$

(ii) Total weight ( $W$ ), placed on the ram is given by

$$W = p \times A = 14 \times 10^4 \times 1.767 \text{ N} = 247380 \text{ N. Ans.}$$

**Problem 21.8** The total weight (including the self-weight of ram) placed on the sliding ram of a hydraulic accumulator is 40 kN. The diameter of the ram is 500 mm. If the frictional resistance against the movement of the ram is 5% of the total weight, determine the intensity of pressure of water when :

- (i) The ram is moving up with a uniform velocity, and
- (ii) The ram is moving down with uniform velocity.

**Solution.** Given :

$$\text{Total weight, } W = 40 \text{ kN} = 40 \times 1000 = 40000 \text{ N}$$

$$\text{Dia. of ram, } D = 500 \text{ mm} = 0.50 \text{ m}$$

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$$\therefore \text{Area of ram, } A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (.50)^2 = 0.1963 \text{ m}^2$$

Frictional resistance against the movement of ram = 5% of total weight =  $\frac{5}{100} \times 40000 = 2000 \text{ N}$ .

(i) *Intensity of pressure of water when ram is moving up with a uniform velocity.* When ram is moving up, the frictional resistance is acting opposite to the direction of movement of the ram, i.e., frictional resistance is acting in the downward direction. Weight is also acting in the downward direction.

$$\begin{aligned}\therefore \text{Total force on the ram} &= \text{Total weight} + \text{Frictional resistance} \\ &= 40000 + 2000 = 42000 \text{ N.}\end{aligned}$$

$$\begin{aligned}\therefore \text{Pressure intensity (p)} &= \frac{\text{Total force on ram}}{\text{Area of ram}} = \frac{42000}{0.1963} \\ &= 213958 \text{ N/m}^2 = 21.3958 \text{ N/cm}^2. \text{ Ans.}\end{aligned}$$

(ii) *Intensity of pressure when ram is moving down with a uniform velocity.* In this case, the frictional resistance is acting in the upward direction.

$$\begin{aligned}\therefore \text{Total force on the ram} &= \text{Total weight} - \text{Frictional resistance} \\ &= 40000 - 2000 = 38000 \text{ N}\end{aligned}$$

$$\begin{aligned}\therefore \text{Pressure intensity (p)} &= \frac{\text{Total force}}{\text{Area}} = \frac{38000}{0.1963} \\ &= 193581 \text{ N/m}^2 = 19.3581 \text{ N/cm}^2. \text{ Ans.}\end{aligned}$$

**Problem 21.9** If in the problem 21.8, the stroke of the ram is 10 m and the ram falls through the full stroke in 4 minutes steadily, find the work done by the accumulator per second. If the pump, connected to the inlet of the accumulator, supplies  $.01 \text{ m}^3/\text{s}$  at the same time, determine the work supplied by the pump per second and also power delivered by the accumulator to the hydraulic machine, connected at the outlet of the hydraulic accumulator, when ram is moving downwards.

**Solution.** The data from problem 21.8, when ram is moving downward :

$$\text{Total weight} = 40000 \text{ N}$$

$$\text{Frictional resistance} = 2000 \text{ N}$$

$$\begin{aligned}\text{Total force on ram} &= \text{Total weight} - \text{Frictional resistance} \\ &= 40000 - 2000 = 38000 \text{ N}\end{aligned}$$

$$\text{Area, } A = 0.1963 \text{ m}^2$$

Pressure intensity of water when ram is moving downward

$$p = 193581 \text{ N/m}^2$$

$$\text{Stroke of ram, } L = 10 \text{ m}$$

$$\text{Time taken by ram to fall through full stroke, } t = 4 \text{ minutes} = 4 \times 60 = 240 \text{ s}$$

$$\text{Discharge supplied } Q = .01 \text{ m}^3/\text{s}$$

$$\text{Distance moved by ram in one second} = \frac{\text{Stroke of ram}}{\text{Time}} = \frac{L}{t} = \frac{10}{240} = \frac{1}{24} \text{ m/s.}$$

(i) Work done by accumulator per second

$$= \text{Total force on ram} \times \text{Distance moved by ram per sec}$$

$$= 38000 \times \frac{1}{24} = 1583.33 \text{ Nm. Ans.}$$

(ii) Work supplied by the pump per sec = Weight of water supplied by pump per second × Head of supply pressure

$$= \rho g \times Q \times H = 1000 \times 9.81 \times .01 \times H \text{ Nm}$$

$$\text{where } H = \text{Pressure head of water supplied} = \frac{p}{\rho \times g} = \frac{193581}{1000 \times 9.81} = 19.733 \text{ m}$$

$$\therefore \text{Work supplied by pump per sec} = 1000 \times 9.81 \times 0.01 \times 19.733 = 1935.81 \text{ Nm. Ans.}$$

(iii) Power delivered by the accumulator to the hydraulic machine connected at the outlet of accumulator

$$\begin{aligned} &= \frac{1}{1000} (\text{Work done by accumulator per second} + \text{Work supplied by pump per second}) \\ &= \frac{1}{1000} (1583.33 + 1935.81) = 3.519 \text{ kW. Ans.} \end{aligned}$$

**Problem 21.10** An accumulator is loaded with 40 kN weight. The ram has a diameter of 30 cm and stroke of 6 m. Its friction may be taken as 5%. It takes two min. to fall through its full stroke. Find the total work supplied and power delivered to the hydraulic appliance by the accumulator, when 7.5 lit/s is being delivered by a pump, while the accumulator descends with the stated velocity.

**Solution.** Given :

$$\text{Total weight} = 40 \text{ kN} = 40 \times 1000 = 40000 \text{ N}$$

$$\text{Dia. of ram, } D = 30 \text{ cm} = 0.3 \text{ m}$$

$$\therefore \text{Area of ram, } A = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}^2$$

$$\text{Stroke of ram, } L = 6 \text{ m}$$

$$\text{Friction} = 5\%$$

$$\therefore \text{Net load on accumulator (when it descends)}$$

$$= 40000 \times 0.95 = 38000 \text{ N}$$

$$\text{Time taken by ram to fall through full stroke, } t = 2 \text{ min} = 2 \times 60 = 120 \text{ sec}$$

$$\therefore \text{Distance moved by ram per sec} = \frac{L}{t} = \frac{6}{120} = \frac{1}{20} \text{ m/s}$$

$$\text{Water supplied by pump} = 7.5 \text{ lit/s} = \frac{7.5}{1000} \text{ m}^3/\text{s} = .0075 \text{ m}^3/\text{s}$$

$$\text{Work supplied by accumulator per second}$$

$$= \text{Net load on ram} \times \text{Distance moved by ram per sec}$$

$$= 38000 \times \frac{1}{20} = 1900 \text{ Nm/s}$$

$$\text{Intensity of pressure of water, } p = \frac{\text{Net load}}{\text{Area}} = \frac{38000}{0.07068} = 542857 \frac{\text{N}}{\text{m}^2}$$

$$\text{Head due to pressure, } H = \frac{p}{\rho g} = \frac{542857}{1000 \times 9.81} = 55.337 \text{ m}$$

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$$\begin{aligned}\text{Work supplied by pump per second} &= \text{Weight of water supplied per second} \times \text{Head of supplied pressure} \\ &= \rho g \times Q \times H = 1000 \times 9.81 \times .0075 \times 55.337 = 4071.35 \text{ Nm/s}\end{aligned}$$

$$\begin{aligned}\therefore \text{Total work supplied per second to hydraulic machine} \\ &= \text{Work supplied by accumulator and by pump} \\ &= 1900 + 4071.35 \text{ Nm/s} = \mathbf{5971.35 \text{ Nm/s. Ans.}}\end{aligned}$$

Power delivered to the hydraulic machine

$$= \frac{\text{Total work supplied per sec}}{1000} = \frac{5971.35}{1000} = \mathbf{5.9713 \text{ kW. Ans.}}$$

**Problem 21.11** An accumulator has a ram of diameter 250 mm and a lift of 8 m. The total weight on accumulator is 70 kN. The packing friction is 5% of the load on the ram. Find the power delivered to the machine if ram falls through the full height in 100 sec and at the same time the pumps are delivering 0.028 m<sup>3</sup>/s through the accumulator.

**Solution.** Given :

$$\text{Dia. of ram, } D = 250 \text{ mm} = 0.25 \text{ m}$$

$$\therefore \text{Area of ram, } A = \frac{\pi}{4} (.25)^2 = \frac{\pi}{64} \text{ m}^2$$

$$\text{Lift of ram, } L = 8 \text{ m}$$

$$\text{Total weight} = 70 \text{ kN} = 70 \times 1000 = 70000 \text{ N}$$

$$\text{Packing friction} = 5\% \text{ of } 70000 = \frac{5}{100} \times 70000 = 3500 \text{ N}$$

$$\begin{aligned}\therefore \text{Net load on accumulator, when the ram is moving downwards} \\ &= 70000 - 3500 = 66500 \text{ N}\end{aligned}$$

Time taken by ram to fall through 8 m,  $t = 100 \text{ sec}$

Water supplied by pump,  $Q = 0.028 \text{ m}^3/\text{s.}$

When ram is moving downwards, the pressure intensity ( $p$ ) is given by,

$$p = \frac{\text{Net load}}{\text{Area}} = \frac{66500}{\left(\frac{\pi}{64}\right)} = \frac{66500 \times 64}{\pi} \text{ N/m}^2$$

Head corresponding to the above pressure intensity,

$$h = \frac{p}{\rho g} = \frac{66500 \times 4}{1000 \times 9.81 \times \pi} = 138.09 \text{ m of water.}$$

$$\text{Power delivered by pump} = \frac{\rho g \cdot Q \cdot H}{1000} = \frac{1000 \times 9.81 \times 0.028 \times 138.09}{1000} = 37.931 \text{ kW}$$

$$\text{Power supplied by accumulator} = \frac{\text{Net load on ram} \times \text{Lift}}{1000 \times \text{Time}} = \frac{66500 \times 8}{1000 \times 100} = 5.32 \text{ kW.}$$

$$\therefore \text{Total power} = 37.931 + 5.32 = \mathbf{43.251 \text{ kW. Ans.}}$$

### 21.3.2 Differential Hydraulic Accumulator.

It is a device in which the liquid is stored at a high pressure by a comparatively small load on the ram. It consists of a fixed vertical cylinder of small diameter as shown in Fig. 21.5. The fixed vertical cylinder is surrounded by closely fitting brass bush, which is surrounded by an inverted moving cylinder, having circular projected collar at the base on which weights are placed.

The liquid from the pump is supplied to the fixed vertical cylinder. The liquid moves up through the small diameter of fixed vertical cylinder and then enters the inverted cylinder. The water exerts an upward pressure force on the internal annular area of the inverted moving cylinder, which is loaded at the base. The internal annular area of the inverted moving cylinder is equal to the sectional area of the brass bush. When the inverted moving cylinder moves up, the hydraulic energy is stored in the accumulator.

Let

$p$  = Intensity of pressure of liquid supplied by pump,

$a$  = Area of brass-bush,

$L$  = Vertical lift of the moving cylinder,

$W$  = Total weight placed on the moving cylinder including the weight of cylinder.

Then

$$W = p \times a$$

$$\therefore P = \frac{W}{a} \quad \dots(21.7)$$

From equation (21.7), it is clear that pressure intensity can be increased with a small load  $W$ , by making area ' $a$ ' small.

Now total energy stored in the accumulator = Total weight  $\times$  Vertical lift

$$= W \times L \text{ Nm.} \quad \dots[21.7 (a)]$$

## ► 21.4 THE HYDRAULIC INTENSIFIER

The device, which is used to increase the intensity of pressure of water by means of hydraulic energy available from a large amount of water at a low pressure, is called the hydraulic intensifier. Such a device is needed when the hydraulic machines such as hydraulic press requires water at very high pressure which cannot be obtained from the main supply directly.

A hydraulic intensifier consists of fixed ram through which the water, under a high pressure, flows to the machine. A hollow inverted sliding cylinder, containing water under high pressure, is mounted over the fixed ram. The inverted sliding cylinder is surrounded by another fixed inverted cylinder which contains water from the main supply at a low pressure as shown in Fig. 21.6

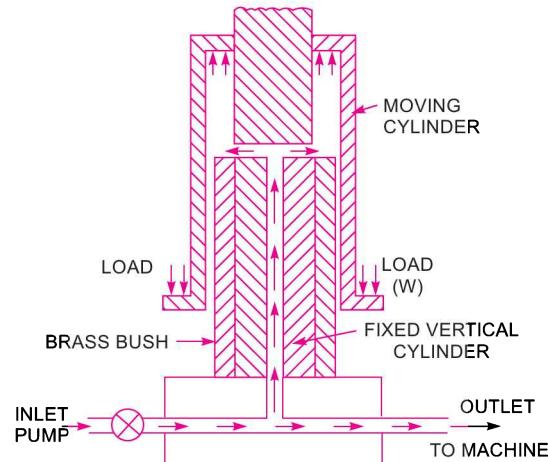


Fig. 21.5 Differential hydraulic accumulator.

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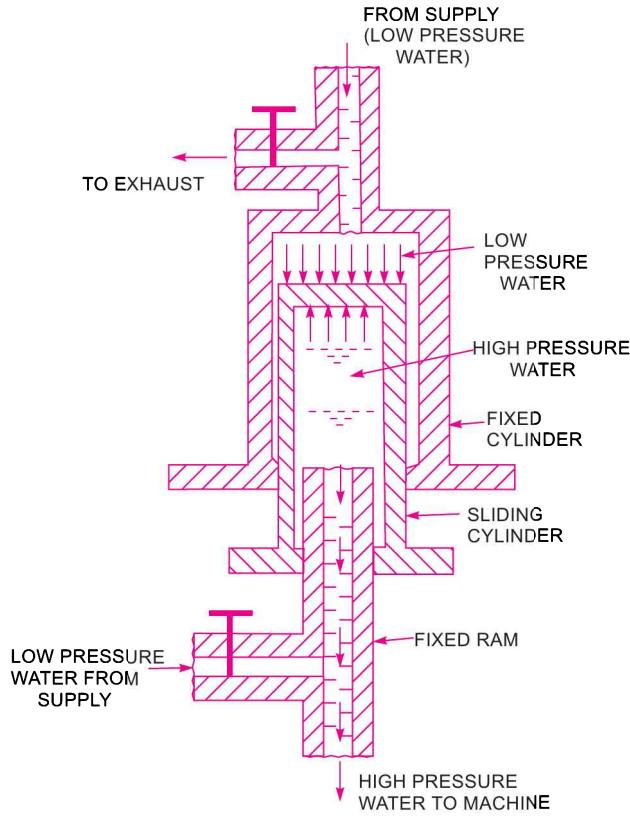


Fig. 21.6 *The hydraulic intensifier.*

A large quantity of water at low pressure from supply enters the inverted fixed cylinder. The weight of this water pressure the sliding cylinder in the downward direction. The water in the sliding cylinder gets compressed due to the downward movement of the sliding cylinder and its pressure is thus increased. The high pressure water is forced out of the sliding cylinder through the fixed ram, to the machine as shown in Fig. 21.6.

Let  $p$  = Intensity of pressure of water from supply to the fixed cylinder (low pressure water),

$A$  = External area of the sliding cylinder,

$a$  = Area of the end of the fixed ram, and

$p^*$  = Intensity of the pressure of water in the sliding cylinder (high pressure water).

The force exerted by low pressure water on the sliding cylinder in the downward direction

$$= p \times A.$$

The force exerted by the high pressure water on the sliding cylinder in the upward direction

$$= p^* \times a.$$

Equating the upward and downward forces,

$$p \times A = p^* \times a.$$

$$p^* = \frac{p \times A}{a}. \quad \dots(21.8)$$

**Problem 21.12** The diameters of fixed ram and fixed cylinder of an intensifier are 8 cm and 20 cm respectively. If the pressure of the water supplied to the fixed cylinder is 300 N/cm<sup>2</sup>, find the pressure of the water flowing through the fixed ram.

**Solution.** Given :

$$\text{Dia. of fixed ram, } d = 8 \text{ cm}$$

$$\therefore \text{Area of fixed ram, } a = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 8^2 = 16\pi \text{ cm}^2$$

$$\text{Dia. of fixed cylinder, } D = 20 \text{ m}$$

$$\therefore \text{Area of fixed cylinder, } A = \frac{\pi}{4} \times 20^2 = 100\pi \text{ cm}^2$$

$$\text{Intensity of supply pressure, } p = 300 \text{ N/cm}^2$$

Let the intensity of pressure of water flowing through fixed ram

$$= p^*$$

$$\text{Using equation (21.8), } p^* = \frac{p \times A}{a} = \frac{300 \times 100\pi}{16\pi} = 1875 \text{ N/cm}^2. \text{ Ans.}$$

**Problem 21.13** The pressure intensity of water supplied to an intensifier is 20 N/cm<sup>2</sup> while the pressure intensity of water leaving the intensifier is 100 N/cm<sup>2</sup>. The external diameter of the sliding cylinder is 20 cm. Find the diameter of the fixed ram of the intensifier.

**Solution.** Given :

$$\text{Supply pressure, } p = 20 \text{ N/cm}^2$$

$$\text{Intensity of pressure leaving the intensifier, } p^* = 100 \text{ N/cm}^2$$

$$\text{External dia. of sliding cylinder, } D = 20 \text{ cm}$$

$$\therefore \text{Area of sliding cylinder, } A = \frac{\pi}{4} \times 20^2 = 100\pi \text{ cm}^2$$

$$\text{Let the dia. of the fixed ram } = d$$

$$\therefore \text{Area of the fixed ram, } a = \frac{\pi}{4} d^2$$

$$\text{Using equation (21.8), } p^* = \frac{p \times A}{a}$$

$$100 = \frac{20 \times 100\pi}{\frac{\pi}{4} d^2} = \frac{20 \times 100 \times 4}{d^2}$$

$$\therefore d = \sqrt{\frac{20 \times 100 \times 4}{100}} = \sqrt{80} = 8.94 \text{ cm. Ans.}$$

## ► 21.5 THE HYDRAULIC RAM

The hydraulic ram is a pump which raises water without any external power for its operation. When large quantity of water is available at a small height, a small quantity of water can be raised to a greater height with the help of hydraulic ram. It works on the principle of water hammer.

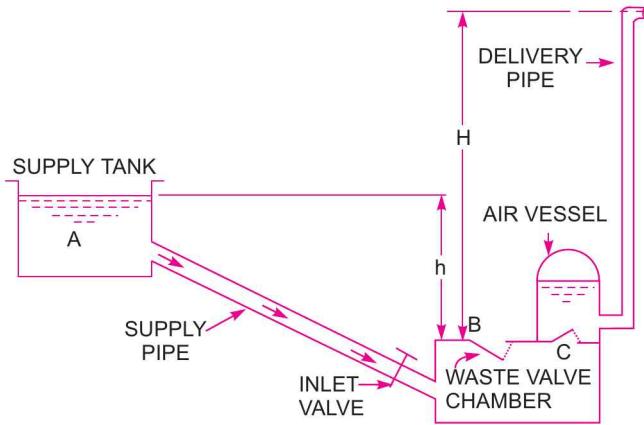


Fig. 21.7 The hydraulic ram.

Fig. 21.7 shows the main components of the hydraulic ram. When the inlet valve fitted to the supply pipe is opened, water starts flowing from the supply tank to the chamber, which has two valves at *B* and *C*. The valve *B* is called waste valve and valve *C* is called delivery valve. The valve *C* is fitted to an air vessel. As the water is coming into the chamber from supply tank, the level of water rises in the chamber and waste valve *B* starts moving upward. A stage comes, when the waste valve *B* suddenly closes. This sudden closure of waste valve creates high pressure inside the chamber. This high pressure force opens the delivery valve *C*. The water from chamber enters the air vessel and compresses the air inside the air vessel. This compressed air exerts force on the water in the air vessel and small quantity of water is raised to a greater height as shown in Fig. 21.7.

When the water in the chamber loses its momentum, the waste valve *B* opens in the downward direction and the flow of water from supply tank starts flowing to the chamber and the cycle will be repeated.

Let

$W$  = Weight of water flowing per second into chamber,

$w$  = Weight of water raised per second,

$h$  = Height of water in supply tank above the chamber,

$H$  = Height of water raised from the chamber.

The energy supplied by the supply tank to ram

$$\begin{aligned} &= \text{Weight of water supplied} \times \text{Height of supply water} \\ &= W \times h \end{aligned} \quad \dots(i)$$

Energy delivered by the ram = Weight of water raised  $\times$  Height through which water is raised

$$= w \times H \quad \dots(ii)$$

$\therefore$  Efficiency of the hydraulic ram,

$$\eta = \frac{\text{Energy delivered by the ram}}{\text{Energy supplied to the ram}} = \frac{w \times H}{W \times h} \quad \dots(21.9)$$

The above expression of efficiency was given by D' Aubuisson and hence known as *D' Aubuisson's efficiency*.

Rankine gave another form of the above efficiency. According to him, the weight of water ( $w$ ) is raised to a height of  $(H - h)$  and not  $H$ . The water is initially at a height of  $h$  from the ram and hence the water is only raised to a height equal to  $(H - h)$ . Hence according to Rankine :

$$\text{Energy delivered by the ram} = w \times (H - h)$$

$$\text{Energy supplied} = (W - w) h$$

$$\therefore \text{Efficiency, } \eta = \frac{w \times (H - h)}{(W - w) \times h} \quad \dots(21.10)$$

Equation (21.10) is known as *Rankine's efficiency*.

The above two efficiencies, in terms of discharge is written as,

$$\text{D' Aubuisson's} \quad \eta = \frac{q \times H}{Q \times h} \quad \dots(21.11)$$

$$\text{Rankine's} \quad \eta = \frac{q(H - h)}{(Q - q) \times h} \quad \dots(21.12)$$

where  $q$  = Discharge of delivery pipe,

$Q$  = Discharge through supply pipe.

**Problem 21.14** The water is supplied at the rate of  $0.02 \text{ m}^3/\text{s}$  from a height of  $3 \text{ m}$  to a hydraulic ram, which raises  $0.002 \text{ m}^3/\text{s}$  to a height of  $20 \text{ m}$  from the ram. Determine D' Aubuisson's and Rankine's efficiencies of the hydraulic ram.

**Solution.** Given :

Discharge through supply pipe,  $Q = 0.02 \text{ m}^3/\text{s}$

Supply head,  $h = 3 \text{ m}$

Discharge raised,  $q = 0.002 \text{ m}^3/\text{s}$

Height of water raised from hydraulic ram,  $H = 20 \text{ m}$

Using equation (21.11),

$$\text{D' Aubuisson's} \quad \eta = \frac{q \times H}{Q \times h} = \frac{.002 \times 20}{.02 \times 3} = .6667 = \mathbf{66.67\% \text{ Ans.}}$$

Rankine's efficiency is given by equation (21.12) as

$$\begin{aligned} \text{Rankine's} \quad \eta &= \frac{q(H - h)}{(Q - q) \times h} \\ &= \frac{0.002 \times (20 - 3)}{(0.020 - .0002) \times 3} = \frac{0.002 \times 17}{.018 \times 3} = 0.6296 = \mathbf{62.96\% \text{ Ans.}} \end{aligned}$$

**Problem 21.15** The water is supplied at the rate of  $3000 \text{ litres per minute}$  from a height of  $4 \text{ m}$  to a hydraulic ram, which raises  $300 \text{ litres/minute}$  to a height of  $30 \text{ m}$  from the ram. The length and diameter of the delivery pipe is  $100 \text{ m}$  and  $70 \text{ mm}$  respectively. Calculate the efficiency of the hydraulic ram if the co-efficiency of friction  $f = .009$ .

**Solution.** Given :

Discharge supplied,  $Q = 3000 \text{ litres/minute}$

$$= \frac{3000}{60} \text{ lit/s} = \frac{3000}{60 \times 1000} \text{ m}^3/\text{s} = 0.05 \text{ m}^3/\text{s}$$

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Supply head,  $h = 4 \text{ m}$

Discharge raised,  $q = 300 \text{ lit/min} = \frac{0.3}{60} = .005 \text{ m}^3/\text{s}$  ( $\because 300 \text{ lit} = 0.3 \text{ m}^3$ )

Height of water raised from hydraulic ram,  $H = 30 \text{ m}$

Length of delivery pipe,  $L = 100 \text{ m}$

Dia. of delivery pipe,  $d = 70 \text{ mm} = .07 \text{ m}$

Co-efficient of friction,  $f = .009$

Head lost due to friction in delivery pipe is

$$h_f = \frac{4fLV^2}{d \times 2g} = \frac{4 \times .009 \times 100 \times V^2}{.07 \times 2 \times 9.81} \quad \dots(i)$$

But

$V$  = Velocity of water in delivery pipe

$$\begin{aligned} &= \frac{\text{Discharge in delivery pipe}}{\text{Area}} \\ &= \frac{q}{\frac{\pi d^2}{4}} = \frac{0.005}{\frac{\pi}{4} \times (.07)^2} = 1.299 \approx 1.3 \text{ m/s.} \end{aligned}$$

Substituting this value of  $V$  in equation (i), we get

$$h_f = \frac{4 \times .009 \times 100 \times (1.3)}{.07 \times 2 \times 9.81} = 4.43 \text{ m}$$

$\therefore$  Effective head developed by the ram

$$= H + h_f = 30 + 4.43 = 34.43 \text{ m}$$

D' Aubuisson's efficiency is given by equation (21.11), as

$$\begin{aligned} \eta &= \frac{q \times \text{Effective head}}{Q \times h} \quad (\text{Here } H = \text{Effective head}) \\ &= \frac{.005 \times 34.43}{0.05 \times 4} = 0.8607 = \mathbf{86.07\%}. \text{ Ans.} \end{aligned}$$

Rankine's efficiency is given by equation (21.12) as

$$\begin{aligned} \eta &= \frac{q(\text{Effective head} - h)}{(Q - q) \times h} \\ &= \frac{.005(34.43 - 4.0)}{(.05 - .005) \times 4.0} = \frac{.005 \times 30.43}{.045 \times 4.0} = 0.8453 = \mathbf{84.53\%}. \text{ Ans.} \end{aligned}$$

### ► 21.6 THE HYDRAULIC LIFT

The hydraulic lift is a device used for carrying passenger or goods from one floor to another in multi-storeyed building. The hydraulic lifts are of two types, namely,

1. Direct acting hydraulic lift, and
2. Suspended hydraulic lift.

**21.6.1 Direct Acting Hydraulic Lift.** It consists of a ram, sliding in fixed cylinder as shown in Fig. 21.8. At the top of the sliding ram, a cage (on which the persons may stand or goods may be placed) is fitted. The liquid under pressure flows into the fixed cylinder. This liquid exerts force on the sliding ram, which moves vertically up and thus raises the cage to the required height.

The cage is moved in the downward direction, by removing the liquid from the fixed cylinder.

**21.6.2 Suspended Hydraulic Lift.** Fig. 21.9 shows the suspended hydraulic lift. It is a modified form of the direct acting hydraulic lift. It consists of a cage (on which persons may stand or goods may be placed) which is suspended from a wire rope. A jigger, consisting of a fixed cylinder, a sliding ram and a set of two pulley blocks, is provided at the foot of the hole of the cage. One of the pulley block is movable and the other is a fixed one. The end of the sliding ram is connected to the movable pulley block. A wire rope, one end of which is fixed at A and the other end is taken round all the pulleys of the movable and fixed blocks and finally over the guide pulleys as shown in Fig. 21.9. The cage is suspended from the other end of the rope. The raising or lowering of the cage of the lift is done by the jigger as explained below.

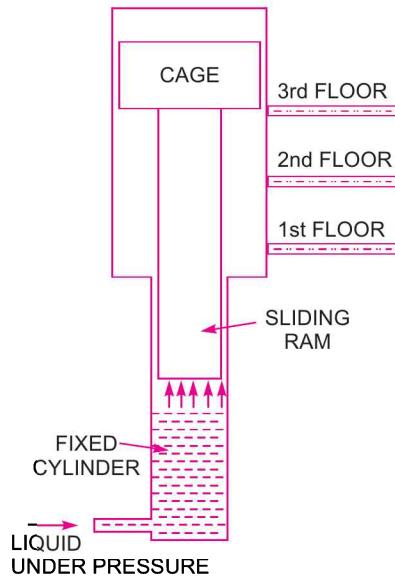


Fig. 21.8 Suspended hydraulic lift.

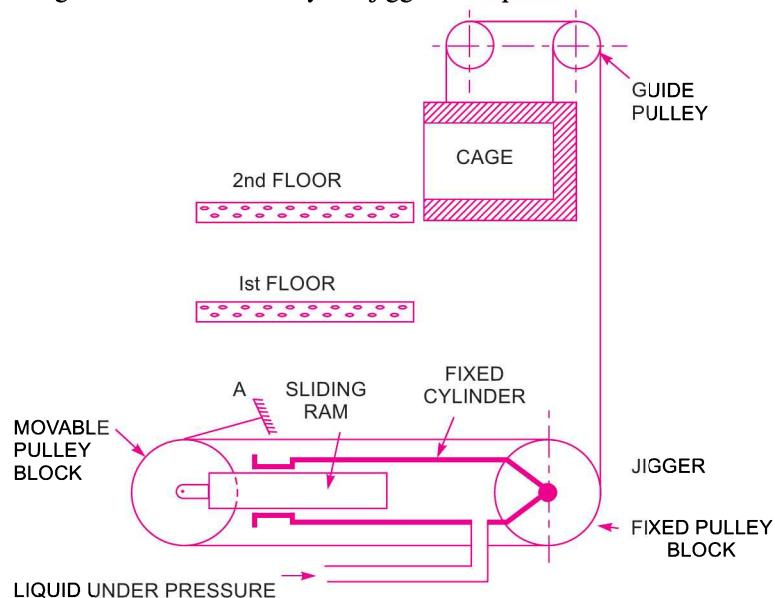


Fig. 21.9 Suspended hydraulic lift.

When water under high pressure is admitted into the fixed cylinder of the jigger, the sliding ram is forced to move towards left. As one end of the sliding ram is connected to the movable pulley block and hence the movable pulley block moves towards the left, thus increasing the distance between two

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pulley blocks. The wire rope connected to the cage is pulled and the cage is lifted. For lowering the cage, water from the fixed cylinder is taken out. The sliding ram moves towards right and hence movable pulley blocks also moves towards right. This decreases the distance between two pulley blocks and the cage is lowered due to increased length of the rope.

**Problem 21.16** A hydraulic lift is required to lift a load of 8 kN through a height of 10 metres, once in every 80 seconds. The speed of the lift is 0.5 m per second. Determine :

- (i) Power required to drive the lift,
- (ii) Working period of lift in seconds, and
- (iii) Idle period of the lift in seconds.

**Solution.** Given :

$$\text{Load lifted, } W = 8 \text{ kN} = 8 \times 1000 = 8000 \text{ N}$$

$$\text{Height, } H = 10 \text{ m}$$

$$\text{Time for one operation, } t = 80 \text{ s}$$

$$\text{Speed of lift, } v = 0.5 \text{ m/s.}$$

(i) Work done in lifting the load in 80 seconds

$$= W \times H = 8000 \times 10 = 80000 \text{ Nm}$$

$$\therefore \text{Work done/s} = \frac{80000}{80} = 1000 \text{ Nm/s}$$

$$\therefore \text{Power required to drive the lift} = \frac{1}{1000} \times \text{Work done/s} = \frac{1}{1000} \times 1000 = 1.0 \text{ kW. Ans.}$$

$$(ii) \text{Working period of the lift} = \frac{\text{Height of the lift}}{\text{Velocity of lift}} = \frac{10}{0.50} = 20 \text{ sec. Ans.}$$

$$(iii) \text{Idle period of the lift} = \text{Total time} - \text{Working period of lift} \\ = 80 - 20 = 60 \text{ sec. Ans.}$$

**Problem 21.17** A hydraulic lift is required to lift a load of 12 kN through a height of 10 m, once in every 1.75 minutes. The speed of the lift is 0.75 m/s. During working stroke of the lift, water from accumulator and the pump at a pressure of  $400 \text{ N/cm}^2$  is supplied to the lift. If the efficiency of the pump is 8% and that of lift is 75%, find the power required to drive the pump and the minimum capacity of the accumulator. Neglect friction losses in the pipe.

**Solution.** Given :

$$\text{Load lifted, } W = 12 \text{ kN} = 12 \times 1000 = 12000 \text{ N}$$

$$\text{Height, } H = 10 \text{ m}$$

$$\text{Total time for one operation, } t = 1.75 \text{ min} = 1.75 \times 60 = 105 \text{ s}$$

$$\text{Speed of lift, } v = 0.75 \text{ m/s}$$

Water pressure from accumulator and pump,

$$p = 400 \text{ N/cm}^2 = 400 \times 10^4 \text{ N/m}^2$$

$$\text{Efficiency of pump, } \eta_p = 80\% = 0.80$$

$$\text{Efficiency of lift, } \eta_l = 75\% = 0.75$$

Work done by water (supplied from accumulator and pump) in raising lift per second

$$= \text{Load lifted} \times \text{Distance travelled per s}$$

$$= W \times \text{Velocity of lift}$$

$$= W \times v = 12000 \times 0.75 = 9000 \text{ Nm/s}$$

$$\therefore \text{Useful power} = \frac{\text{Work done per second}}{1000} = \frac{9000}{1000} = 9 \text{ kW}$$

$$\therefore \text{Actual power supplied to lift} = \frac{\text{Useful horse power}}{\eta_l} = \frac{9.0}{0.75} = 12 \text{ kW.}$$

The power 12 kW has been supplied by the pump and by accumulator to the lift.

Let  $P_1$  = Output of the pump in kW

Then  $(12 - P_1)$  = Output of the accumulator in kW ... (i)

$$\text{Now, working period of the lift} = \frac{\text{Height of lift}}{\text{Velocity of lift}} = \frac{H}{v} = \frac{10}{0.75} = 13.33 \text{ s}$$

$$\begin{aligned}\text{Idle period of lift} &= \text{Total time} - \text{Working period of lift} \\ &= 105 - 13.33 = 91.67 \text{ s}\end{aligned}$$

Thus during idle period of lift, the energy will be stored in the accumulator and during working period of lift of 13.33 s, the energy will be supplied by the accumulator to the lift.

$\therefore$  Energy stored during idle period in accumulator

$$\begin{aligned}&= \text{Output of pump} \times \text{Idle period} \\ &= (P_1 \times 1000) \times 91.67 \text{ Nm/s} \quad (\because 1 \text{ kW} = 1000 \text{ Nm/s}) \quad \dots \text{(ii)}\end{aligned}$$

The above energy is supplied by accumulator in 13.33 seconds.

$\therefore$  Energy supplied by accumulator per second

$$= \frac{(P_1 \times 1000) \times 91.67}{13.33} \text{ Nm/s}$$

$$\begin{aligned}\therefore \text{Power supplied by accumulator} &= \frac{1}{1000} \quad [\text{Energy supplied by accumulator per second}] \\ &= \frac{1}{1000} \left[ \frac{P_1 \times 1000 \times 91.67}{13.33} \right] = 6.877 P_1.\end{aligned}$$

But from equation (i), power supplied by accumulator is also

$$= (12 - P_1)$$

$\therefore$  Equating the two values of power supplied by accumulator,

$$6.877 P_1 = 12 - P_1 \text{ or } 6.877 P_1 + P_1 = 12 \text{ or } 7.877 P_1 = 12$$

$$\therefore P_1 = \frac{12}{7.877} = 1.523.$$

But we have assumed  $P_1$  as the output of the pump.

$$\therefore \text{Input to the pump} = \frac{\text{Output}}{\text{Efficiency of pump}} = \frac{P_1}{\eta_p} = \frac{1.523}{0.80} = 1.90375 \text{ kW.}$$

(i)  $\therefore$  Power required to drive the pump = 1.90375 kW. Ans.

(ii) Minimum capacity of the accumulator.

From equation (ii), the energy stored in the accumulator

$$\begin{aligned}&= P_1 \times 1000 \times 91.67 \text{ Nm} \\ &= 1.523 \times 1000 \times 91.67 \text{ Nm} \quad (\because P_1 = 1.523) \\ &= 139613.4 \text{ Nm.}\end{aligned}$$

The capacity of the accumulator means the amount of hydraulic energy stored in the accumulator.

$\therefore$  Minimum capacity of accumulator = Energy stored = 139613.4 Nm. Ans.

### ► 21.7 THE HYDRAULIC CRANE

Hydraulic crane is a device, used for raising or transferring heavy loads. It is widely used in workshops, warehouses and dock sidings.

A hydraulic crane consists of a mast, tie, jib, guide pulley and a jigger. The jib and tie are attached to the mast. The jib can be raised or lowered in order to decrease or increase the radius of action of the crane. The mast along with the jib can revolve about a vertical axis and thus the load attached to the rope can be transferred to any place within the area of the crane's action. The jigger, which consists of a movable ram sliding in a fixed cylinder, is used for lifting or lowering the heavy loads. One end of the ram is in contact with water and the other end is connected to set of movable pulley block. Another pulley block, called the fixed pulley block is attached to the fixed cylinder. The pulley block, attached to the ram, moves up and down while the pulley block, attached to the fixed cylinder, is not having any movement.

A wire rope, one end of which is fixed to a movable pulley (which is attached to the sliding ram) is taken round all the pulleys of the two sets of the pulleys and finally passes over the guide pulley, attached to the jib as shown in Fig. 21.10. The other end of the rope is provided with a hook, for suspending the load.

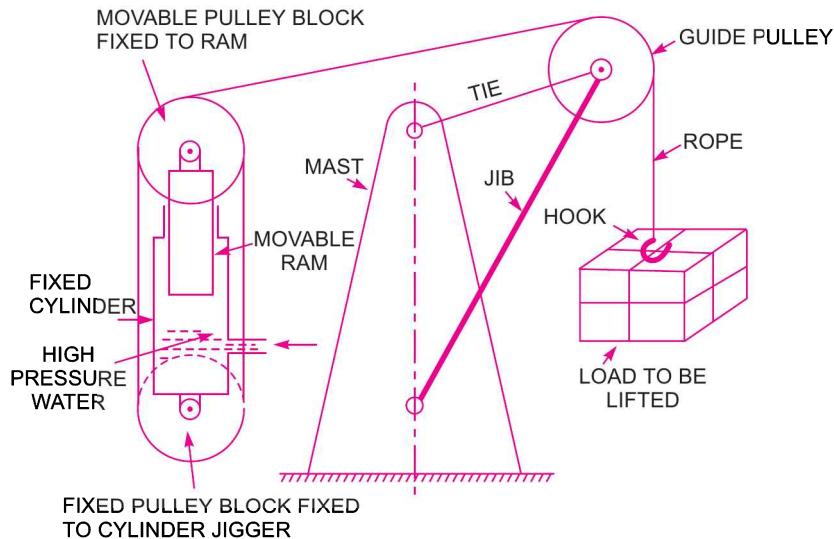


Fig. 21.10 The hydraulic crane.

For lifting the load by the crane, the water under high pressure is admitted into the cylinder of the jigger. This water forces the sliding ram to move vertically up. Due to the movement of the ram in the vertically up direction, the movable pulley block attached to the ram also moves upward. This increases the distance between two pulley blocks and hence the wire passing over the guide pulley is pulled by the jigger. This raises the load attached to the hook.

**Problem 21.18** Find the efficiency of a hydraulic crane, which is supplied 300 litres of water under a pressure of  $60 \text{ N/cm}^2$  for lifting a weight of 12 kN through a height of 11 m.

**Solution.** Given :

Water supplied,

$$Q = 300 \text{ litres} = 0.30 \text{ m}^3$$

Pressure,

$$p = 60 \text{ N/cm}^2 = 60 \times 10^4 \frac{\text{N}}{\text{m}^2}$$

Weight lifted,	$W = 12 \text{ kN} = 12 \times 1000 = 12000 \text{ N}$
Height,	$h = 11 \text{ m}$
Output of the crane	$= \text{Weight lifted} \times \text{Height through which weight is lifted}$ $= W \times h = 12000 \times 11 \text{ Nm}$
Input of the crane	$= \text{Energy supplied by the water}$ $= \text{Work done by water on the ram}$ $= \text{Force on ram} \times \text{Distance moved by ram}$ $= \text{Pressure} \times \text{Area of ram} \times \text{Stroke of ram}$ $= p \times A \times L$ $= 60 \times 10^4 \times \text{Volume displaced}$ $= 60 \times 10^4 \times 0.30$ $= 18 \times 10^4 \text{ Nm}$
. . . Efficiency of the crane	$= \frac{\text{Output}}{\text{Input}} = \frac{12000 \times 11}{18 \times 10^4} = 0.7333 = 73.33\% \text{ Ans.}$ $(\because A \times L = Q)$

**Problem 21.19** The efficiency of a hydraulic crane, which is supplied water under a pressure of  $70 \text{ N/cm}^2$  for lifting a weight through a height of  $10 \text{ m}$ , is  $60\%$ . If the diameter of the ram is  $150 \text{ mm}$  and velocity ratio is  $6$ , find

- (i) the weight lifted by the crane, and
- (ii) the volume of water required in litres to lift the weight.

**Solution.** Given :

Efficiency,	$\eta = 60\% = 0.60$
Pressure of water,	$p = 70 \text{ N/cm}^2 = 70 \times 10^4 \text{ N/m}^2$

Height through which weight is lifted,  $h = 10 \text{ m}$

Diameter of the ram,  $D = 150 \text{ mm} = 0.15 \text{ m}$

. . . Area of ram,  $A = \frac{\pi}{4} (0.15)^2 = 0.01767 \text{ m}^2$

Velocity ratio  $= 6$

Pressure force on ram,  $P = \text{Pressure} \times \text{Area of ram}$   
 $= p \times A = 70 \times 10^4 \times 0.01767 = 12369 \text{ N.}$

(i) We know efficiency of the hydraulic crane is given as

$$\eta = \frac{\text{Output}}{\text{Input}} = \frac{\text{Weight} \times \text{Distance moved by weight}}{\text{Force} \times \text{Distance moved by force}}$$

or  $0.60 = \frac{W \times \text{Distance moved by weight}}{P \times \text{Distance moved by force}}$

But  $\frac{\text{Distance moved by weight}}{\text{Distance moved by force}} = \text{Velocity ratio} = 6$

. . .  $0.60 = \frac{W}{P} \times 6 = \frac{W}{12369} \times 6$

. . .  $W = \frac{0.60 \times 12369}{6} = 1236.9 \text{ N. Ans.}$

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(ii) Volume of water required to lift the weight :

$$\text{Velocity ratio} = \frac{\text{Distance moved by weight}}{\text{Distance moved by force on ram}}$$

$$\text{or } 6 = \frac{h}{\text{Stroke of ram}} \quad (\because \text{Distance moved by ram} = \text{Stroke of ram})$$

$$\therefore \text{Stroke of ram}, \quad L = \frac{h}{6} = \frac{10}{6} = 1.667 \text{ m}$$

$$\therefore \text{Volume of water} = \text{Area of ram} \times \text{Stroke of ram} = A \times L = 0.01767 \times 1.667 \\ = 0.02945 \text{ m}^3 = 0.02945 \times 1000 = \mathbf{29.45 \text{ litres. Ans.}}$$

**Problem 21.20** A hydraulic crane is lifting a weight of 12000 N through a height of 12 m with a speed of 18 m per minute once in every two minutes. The efficiency of the hydraulic crane is 65% and it is working under a pressure of 500 N/cm<sup>2</sup> of water. The crane is fed from an accumulator to which water is supplied by a pump. Find :

- (i) the capacity of the cylinder of the jigger in litres,
- (ii) the capacity of the accumulator in litres, and
- (iii) minimum power required for the pump.

**Solution.** Given :

$$\text{Weight lifted,} \quad W = 12000 \text{ N}$$

$$\text{Height,} \quad h = 12 \text{ m}$$

$$\text{Speed of weight,} \quad V = 18 \text{ m/min}$$

No. of times the weight is lifted = Once in every two minutes

$$\text{Efficiency,} \quad \eta = 65\% = 0.65$$

$$\text{Pressure of water,} \quad p = 500 \text{ N/cm}^2 = 500 \times 10^4 \text{ N/m}^2$$

$$(i) \text{Output of the crane} = \text{Weight lifted} \times \text{Height} \\ = W \times h = 12000 \times 12 = 144000 \text{ Nm.}$$

$$\text{Input of the crane} = \text{Work done by water on ram} \\ = \text{Force on ram} \times \text{Distance moved by ram} \\ = p \times A \times L \\ = p \times \text{Volume of cylinder} \quad (\because A \times L = \text{Volume of cylinder}) \\ = 500 \times 10^4 \times \text{Volume of cylinder}$$

$$\therefore \eta = \frac{\text{Output}}{\text{Input}} = \frac{144000}{500 \times 10^4 \times \text{volume of cylinder}}$$

$$\therefore \text{Volume of cylinder} = \frac{144000}{500 \times 10^4 \times \eta} = \frac{144000}{500 \times 10^4 \times 0.65} = 0.0443 \text{ m}^3 = 44.3 \text{ litres.}$$

$\therefore$  Capacity of the cylinder of the jigger = **44.3 litres. Ans.**

$$(ii) \text{Input of the crane} = p \times \text{Volume of cylinder} = 500 \times 10^4 \times 0.0443 = 221500 \text{ Nm.}$$

This input is given to the crane once in every two minutes.

$$\therefore \text{Input to crane per min.} = \frac{221500}{2} = 110750 \text{ Nm.}$$

The weight 12000 N is lifted to a height of 12 m with a speed of 18 m/min.

$$\text{Time required to lift the weight through height of } 12 \text{ m} = \frac{\text{Height}}{\text{Speed}} = \frac{12}{18} = \frac{2}{3} \text{ min.}$$

$\therefore$  Work done by the pump during lifting

$$= \text{Work done per min.} \times \text{Time required to lift the weight}$$

$$= 110750 \times \frac{2}{3} = 73833.33 \text{ Nm}$$

$$\therefore \text{Energy supplied by accumulator} = \text{Total input energy to the crane} - \text{Work done during lifting}$$

$$= 221500 - 73833.33 = 147666.67 \text{ Nm} \quad \dots(i)$$

But energy supplied by accumulator = Force on the ram of accumulator  $\times$  Lift of ram

$$= p \times A \times H = p \times \text{Capacity of accumulator}$$

$$= 500 \times 10^4 \times \text{Capacity of accumulator} \quad \dots(ii)$$

Equating the two values given by equations (i) and (ii),

$$147666.67 = 500 \times 10^4 \times \text{Capacity of accumulator}$$

$$\therefore \text{Capacity of accumulator} = \frac{147666.67}{500 \times 10^4} = 0.0295 \text{ m}^3 = 29.5 \text{ litres. Ans.}$$

(iii) Minimum power required for the pump

$$= \frac{\text{Work input per minute}}{1000 \times 60} = \frac{110750}{1000 \times 60} = 1.846 \text{ kW. Ans.}$$

## ► 21.8 THE FLUID OR HYDRAULIC COUPLING

The fluid or hydraulic coupling is a device used for transmitting power from driving shaft to driven shaft with the help of fluid (generally oil). There is no mechanical connection between the two shafts. It consists of a radial pump impeller mounted on a driving shaft *A* and a radial flow reaction turbine mounted on the driven shaft *B*. Both the impeller and runner are identical in shape and they together form a casing which is completely enclosed and filled with oil.

In the beginning, both the shafts *A* and *B* are at rest. When the driving shaft *A* is rotated, the oil starts moving from the inner radius to the outer radius of the pump impeller as shown in Fig. 21.11. The pressure energy and kinetic energy of the oil increases at the outer radius of the pump impeller. This oil of increased energy enters the runner of the reaction turbine at the outer radius of the turbine runner and flows inwardly to the inner radius of the turbine runner. The oil, while flowing through the runner, transfers its energy to the blades of the runner and makes the runner to rotate. The oil, from the runner then flows back into the pump impeller, thus having a continuous circulation.

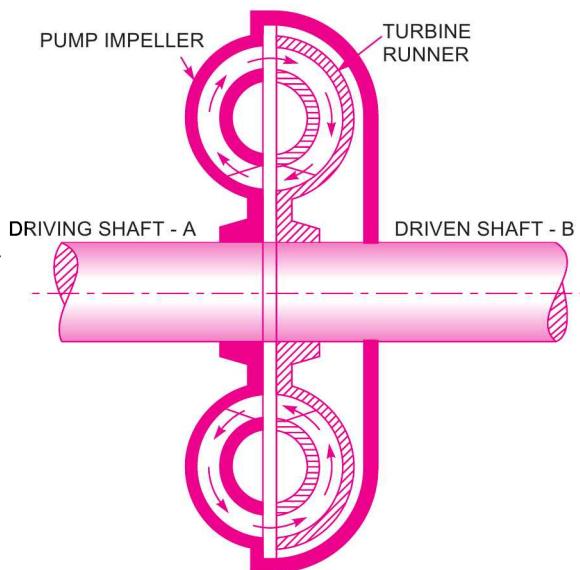


Fig. 21.11 The hydraulic coupling.

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The power is transmitted hydraulically from the driving shaft to driven shaft and the driven shaft is free from engine vibrations. The speed of the driven shaft  $B$  is always less than the speed of the shaft  $A$ , by about 2 per cent. The efficiency of the power transmission by hydraulic coupling is about 98%. This is derived as given below.

$$\text{Efficiency of a fluid coupling} = \frac{\text{Power output}}{\text{Power input}}$$

$$\text{or } \eta = \frac{\text{Power transmitted to shaft } B}{\text{Power available at shaft } A} \quad \dots(i)$$

$$\text{But power at any shaft} = \frac{2\pi NT}{60,000} \propto NT \propto \text{Speed} \times \text{Torque}$$

$$\text{Let } N_A = \text{Speed of shaft } A, \quad \dots(ii)$$

$$T_A = \text{Torque at the shaft } A,$$

$$N_B = \text{Speed of shaft } B,$$

$$T_B = \text{Torque transmitted to shaft } B.$$

From equation (ii), we have

$$\text{Power available to shaft } A \propto (\text{Speed of shaft } A) \times \text{Torque of } A$$

$$\propto N_A \times T_A.$$

$$\text{Similarly, power transmitted to shaft } B \propto N_B \times T_B.$$

Substituting these values of powers in equation (ii),

$$\eta = \frac{N_B \times T_B}{N_A \times T_A}$$

$$\text{But } T_A = T_B \quad (\because \text{Torque transmitted is same})$$

$$\therefore \eta = \frac{N_B}{N_A} \quad \dots(21.13)$$

Slip of fluid coupling is defined as the ratio of the difference of the speeds of the driving and driven shaft to the speed of the driving sharp. Mathematically,

$$\text{Slip, } S = \frac{N_A - N_B}{N_A} = 1 - \frac{N_B}{N_A} = 1 - \eta \quad \left( \because \frac{N_B}{N_A} = \eta \right) \quad \dots(21.14)$$

### ► 21.9 THE HYDRAULIC TORQUE CONVERTER

The hydraulic torque converter is a device used for transmitting increased torque at the driven shaft. The torque transmitted at the driven shaft may be more or less than the torque available at the driving shaft. The torque at the driven shaft may be increased about five times the torque available at the driving shaft with an efficiency of about 90%.

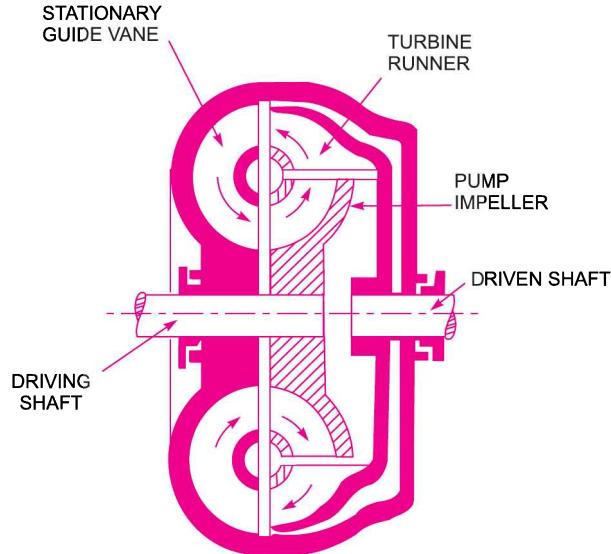


Fig. 21.12 Fluid of hydraulic torque converter.

As mentioned in Art. 21.8, the power at any shaft is proportional to the product of torque and speed of the shaft. Hence, if the torque at the driven shaft is to be increased, the corresponding value of the speed at the same shaft should be decreased. The speed of the driven shaft is decreased by decreasing the velocity of oil, which is allowed to flow from the pump impeller to the turbine runner and then through stationary guide vanes as shown in Fig. 21.12. Due to the decrease in speed at the driven shaft, the torque increases.

#### ► 21.10 THE AIR LIFT PUMP

The air lift pump is a device which is used for lifting water from a well or sump by using compressed air. The compressed air is made to mix with the water. The density of the mixture of air and water is reduced. The density of this mixture is much less than that of pure water. Hence a very small column of pure water can balance a very long column of air water mixture. This is the principle on which the air lift pump works.

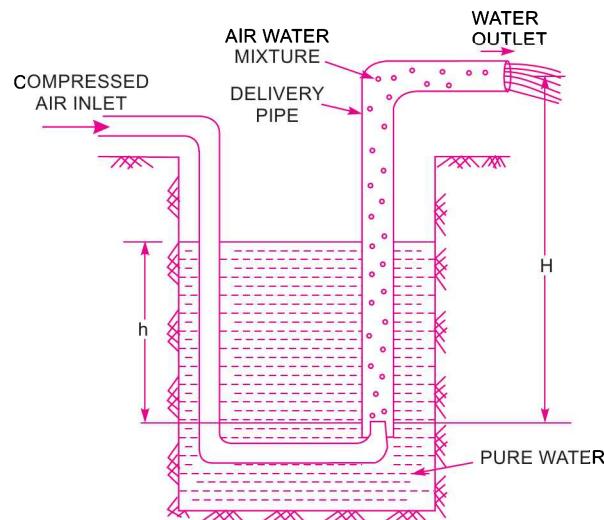


Fig. 21.13 Air lift pump.

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Fig. 21.13 shows the air lift pump. The compressed air is introduced through one or more nozzles at the foot of the delivery pipe, which is fixed in the well from which water is to be lifted. In the delivery pipe, a mixture of air and water is formed. The density of this air water mixture becomes very less as compared to the density of pure water. Hence, a small column of pure water will balance a very long column of air water mixture. This air water mixture will be discharged out of the delivery pipe. The flow will continue as long as there is supply of compressed air.

Let

$h$  = Height of static water level above the tip of the nozzle,

$H$  = Height to which water is lifted above the tip of the nozzle.

The  $(H - h)$  is known as the useful lift. The best results are obtained if the useful lift  $(H - h)$  is less than the height of static water ( $h$ ) above the tip of the nozzle. Hence for best results,  $(H - h)$  should be less than  $h$ .

The ratio  $\left(\frac{h}{H-h}\right)$  generally varies from 4 to 1.

When  $h = 30$  m, the ratio  $\left(\frac{h}{H-h}\right)$  is about 4.

When  $h = 90$  m, the ratio  $\left(\frac{h}{H-h}\right)$  is 1.

For  $h = 30$  m,  $\frac{h}{(H-h)} = 4$  or  $\frac{30}{(H-30)} = 4$  or  $30 = 4H - 120$

or  $30 + 120 = 4H$  or  $H = \frac{150}{4} = 37.5$  m.

The air lift pump is not having any moving parts below water level and hence there are no chances of suspended solid particles damaging the pump. This is the main advantage of this pump. Also this pump can raise more water through a bore hole of given diameter than any other pump. But the efficiency of this pump is low as out of the energy expended in compressing the air, only 20 to 40% energy appears in the form of useful water horse-power.

### ► 21.11 THE GEAR-WHEEL PUMP

The gear pump is a rotary pump in which two gears mesh to provide the pumping action. This type of pump is mostly used for cooling water and pressure oil to be supplied for lubrication to motors, turbines, machine tools etc. Although the gear pump is a rotating machinery, yet its action on liquid to be pumped is not dynamic and it merely displaces the liquid from one side to the other. The flow of liquid to be pumped is continuous and uniform.

Fig. 21.14 shows the gear pump, which consists of two identical intermeshing gears working in a fine clearance inside a casing. One of the gear is keyed to a driving shaft. The other gear revolves due to driving gear. The space between teeth and the casing is filled with oil. The oil is carried round between the gears from the suction pipe to the delivery pipe.

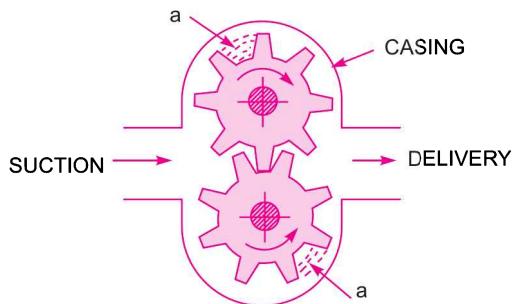


Fig. 21.14 Gear-wheel pump.

The mechanical contact between the gears does not allow the flow from inlet to outlet directly. The outer radial tips of the gears and sides of the gears form a part off moving oil.

The oil pushed into the delivery pipe, cannot back into the suction pipe due to the meshing of the gears. The theoretical oil pumped per second is obtained as :

Let

$N$  = Speed of rotating gear in r.p.m.,

$a$  = Area enclosed between two successive teeth and casing,

$n$  = Total number of teeth in each gear,

$L$  = Axial length of teeth.

$$\text{Volume of oil discharged per revolution} = 2 \times a \times L \times N \text{ m}^3$$

$$\begin{aligned}\therefore \text{Discharge/s} &= \text{Volume of oil per revolution} \times \text{No. of revolution in one second} \\ &= 2aLn \times \frac{N}{60} \text{ m}^3\end{aligned}$$

The actual discharge will be less than the theoretical discharge.

$$\text{Now, volumetric efficiency} = \frac{\text{Actual discharge}}{\text{Theoretical discharge}}.$$

## HIGHLIGHTS

1. The devices, based on the principles of fluid statics and fluid kinematics for the transmission of power with the help of a fluid (oil or air), are called fluid or hydraulic devices.
2. A device, which is used for lifting heavy weights by the application of a much smaller force is known as hydraulic press.
3. The device, used for storing the energy of a liquid in the form of pressure energy which may be supplied for any sudden or intermittent requirement, is known as hydraulic accumulator.
4. Capacity of the hydraulic accumulator is given as

$$= p \times A \times L$$

where  $p$  = Liquid pressure supplied by pump,  $A$  = Area of the sliding ram,  $L$  = Stroke or lift of the ram.

5. Differential hydraulic accumulator is a device in which liquid is stored at a high pressure by comparatively small load on the ram.
6. Hydraulic intensifier is a device, in which the pressure intensity of a liquid is increased by means of hydraulic energy available from a large amount of liquid at a low pressure. The increased pressure intensity ( $p^*$ ) is given by the relation,

$$p^* = \frac{p \times A}{a}$$

where  $p$  = Low pressure intensity of liquid,  $A$  = External area of the sliding ram,

$a$  = Area of the end of the fixed ram.

7. Hydraulic ram is a pump which raises water without any external power (electricity etc.) for its operation. There are two efficiencies of a hydraulic ram namely D' Aubuisson's efficiency and Rankine efficiency.

They are given by the relations, D' Aubuisson's  $\eta = \frac{wh}{Wh}$  or  $\frac{qH}{Qh}$

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$$\text{Rankine's } \eta = \frac{w \times (H - h)}{(W - w) \times h} \text{ or } \frac{q \times (H - h)}{(Q - q) \times h}$$

where  $w$  = Weight of water raised per sec,  $W$  = Weight of water flowing per sec into chamber,

$h$  = Height of water above chamber in supply tank,  $H$  = Height of water raised above chamber.

8. Hydraulic lift is a device used for carrying persons or goods from one floor to another floor in a multi-storeyed building. They are of two types namely direct acting hydraulic lifts and suspended hydraulic lifts.
9. Hydraulic crane is a device used for raising or transferring heavy weights.
10. Fluid coupling is a device, in which power is transmitted from driving shaft to driven shaft without any change of torque while torque converter is a device in which arrangement is provided for getting increased or decreased torque at the driven shaft.

### EXERCISE

#### (A) THEORETICAL PROBLEMS

1. Explain the term, 'Hydraulic devices'. Name any five hydraulic devices.
2. Draw a neat sketch and explain the principle and working of a hydraulic press.
3. Define the term, the hydraulic accumulator. Obtain an expression for the capacity of a hydraulic accumulator. Differentiate between hydraulic accumulator and differential accumulator.
4. What is a hydraulic intensifier ? Explain its principle and working.
5. Differentiate between a hydraulic ram and a centrifugal pump. Obtain an expression for the efficiencies of the hydraulic ram.
6. Explain with the help of a neat sketch, the principle and working of the following hydraulic devices :  
(a) Hydraulic lift, (b) Hydraulic crane,  
(c) Hydraulic coupling, and (d) Hydraulic torque converter.
7. What is a difference between a fluid coupling and fluid torque converter? Explain the torque converter with a sketch.
8. Explain with neat sketch, the working of air lift pump. Mention its advantages.
9. How does a torque converter differ from a fluid coupling ? Explain the working principle of any one of them.

#### (B) NUMERICAL PROBLEMS

1. A hydraulic press has a ram of 300 mm diameter and a plunger of 50 mm diameter. Find the weight lifted by the hydraulic press when the force applied at the plunger is 40 N. [Ans. 1440 N]
2. A hydraulic press has a ram of 150 mm diameter and plunger of 30 mm. The stroke of the plunger is 250 mm and weight lifted is 600 N. If the distance moved by the weight is 1.20 m in 20 minutes, determine :  
(a) the force applied on the plunger, (b) power required to drive the plunger, and (c) number of strokes performed by the plunger. [Ans. (a) 24 N, (b) 0.0006 kW, (c) 120]
3. The water is supplied at a pressure of 15 N/cm<sup>2</sup> to an accumulator, having a ram of diameter 2.0 m. If the total lift of the ram is 10 m, determine :  
(a) the capacity of the accumulator, and  
(b) total weight placed on the ram (including the weight of the ram). [Ans. (a) 4712.4 kNm, (b) 471240 N]

4. The diameters of the fixed ram and fixed cylinder of an intensifier are 100 mm and 250 mm respectively. If the pressure of the water supplied to the fixed cylinder is  $25 \text{ N/cm}^2$ , find the pressure of the water flowing through the fixed ram. [Ans.  $156.25 \text{ N/cm}^2$ ]
5. The water is supplied at the rate of 30 litres per second from a height of 4 m to a hydraulic ram, which raises 3 litres per second to a height of 18 m from the ram. Determine D' Aubuisson's and Rankine's efficiencies of the hydraulic ram. [Ans. 45%, 38.8%]
6. A hydraulic lift is required to lift a load of 98.1 kN through a height of 12 m, once in every 100 seconds. The speed of the lift is 600 mm/s. Determine :
  - (a) power required to drive the lift, (b) working period of lift in seconds, and (c) idle period of the lift in seconds. [Ans. (a) 11.772 kW, (b) 20 sec, (c) 80 sec]
7. Find the efficiency of a hydraulic crane, which is supplied 400 litres of water under a pressure of  $490.5 \text{ N/cm}^2$  for lifting a weight of 98.1 kN through a height of 10 m. [Ans. 50%]
8. In a hydraulic coupling, the speeds of the driving and driven shafts are 800 r.p.m. and 780 r.p.m. respectively. Find :
  - (a) the efficiency of the hydraulic coupling, and (b) the slip of the coupling. [Ans. (a) 97.5%, (b) 2.5%]



## OBJECTIVE TYPE QUESTIONS

Tick mark (✓) the most appropriate statement of the multiple choice answers :

1. An ideal fluid is defined as the fluid which
  - (a) is compressible
  - (b) is incompressible
  - (c) is incompressible and non-viscous (inviscid)
  - (d) has negligible surface tension.
2. Newton's law of viscosity states that
  - (a) shear stress is directly proportional to the velocity
  - (b) shear stress is directly proportional to velocity gradient
  - (c) shear stress is directly proportional to shear strain
  - (d) shear stress is directly proportional to the viscosity.
3. A Newtonian fluid is defined as the fluid which
  - (a) is incompressible and non-viscous
  - (b) obeys Newton's law of viscosity
  - (c) is highly viscous
  - (d) is compressible and non-viscous.
4. Kinematic viscosity is defined as equal to
  - (a) dynamic viscosity  $\times$  density
  - (b) dynamic viscosity/density
  - (c) dynamic viscosity  $\times$  pressure
  - (d) pressure  $\times$  density.
5. Dynamic viscosity ( $\mu$ ) has the dimensions as
  - (a)  $MLT^{-2}$
  - (b)  $ML^{-1} T^{-1}$
  - (c)  $ML^{-1} T^{-2}$
  - (d)  $M^{-1} L^{-1} T^{-1}$
6. Poise is the unit of
  - (a) mass density
  - (b) kinematic viscosity
  - (c) viscosity
  - (d) velocity gradient.
7. The increase of temperature
  - (a) increases the viscosity of a liquid
  - (b) decreases the viscosity of a liquid
  - (c) decreases the viscosity of a gas
  - (d) increases the viscosity of a gas.
8. Stoke is the unit of
  - (a) surface tension
  - (b) viscosity
  - (c) kinematic viscosity
  - (d) none of the above.
9. The dividing factor for converting one poise into MKS unit of dynamic viscosity is
  - (a) 9.81
  - (b) 98.1
  - (c) 981
  - (d) 0.981.
10. Surface tension has the units of
  - (a) force per unit area
  - (b) force per unit length
  - (c) force per unit volume
  - (d) none of the above.
11. The gases are considered incompressible when Mach number
  - (a) is equal to 1.0
  - (b) is equal to 0.50
  - (c) is more than 0.3
  - (d) is less than 0.2.
12. Pascal's law states that pressure at a point is equal in all directions
  - (a) in a liquid at rest
  - (b) in a fluid at rest
  - (c) in a laminar flow
  - (d) in a turbulent flow.
13. The hydrostatic law states that rate of increase of pressure in a vertical direction is equal to
  - (a) density of the fluid
  - (b) specific weight of the fluid
  - (c) weight of the fluid
  - (d) none of the above.
14. Fluid statics deals with
  - (a) viscous and pressure forces
  - (b) viscous and gravity forces
  - (c) gravity and pressure forces
  - (d) surface tension and gravity forces.
15. Gauge pressure at a point is equal to
  - (a) absolute pressure plus atmospheric pressure
  - (b) absolute pressure minus atmospheric pressure
  - (c) vacuum pressure plus absolute pressure
  - (d) none of the above.
16. Atmospheric pressure held in terms of water column is
  - (a) 7.5 m
  - (b) 8.5 m
  - (c) 9.81 m
  - (d) 10.30 m.
17. The hydrostatic pressure on a plane surface is equal to
  - (a)  $wA\bar{h}$
  - (b)  $wA\bar{h} \sin^2 \theta$ .
  - (c)  $\frac{1}{2}wA\bar{h}$
  - (d)  $wA\bar{h} \sin \theta$ .

where  $A$  = Area of plane surface, and  
 $h$  = Depth of centroid of the plane area below the liquid-free surface.

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- 60.** The loss of head due to sudden expansion of a pipe is given by

$$(a) h_L = \frac{V_1^2 - V_2^2}{2g} \quad (b) h_L = \frac{0.5 V_1^2}{2g}$$

$$(c) h_L = \frac{(V_1 - V_2)^2}{2g} \quad (d) \text{none of the above.}$$

- 61.** The loss of head due to sudden contraction of a pipe is equal to

$$(a) \left(\frac{1}{C_c} - 1\right)^2 \frac{V_2}{2g} \quad (b) \left(1 - \frac{1}{C_c}\right)^2 \frac{V_2}{2g}$$

$$(c) \frac{1}{C_c} \left(1 - \frac{V_2^2}{2g}\right) \quad (d) \text{none of the above.}$$

- 62.** Hydraulic gradient line (H.G.L.) represents the sum of

- (a) pressure head and kinetic head
- (b) kinetic head and datum head
- (c) pressure head, kinetic head and datum head
- (d) pressure head and datum head.

- 63.** Total energy line (T.E.L.) represents the sum of

- (a) pressure head and kinetic head
- (b) kinetic head and datum head
- (c) pressure head and datum head
- (d) pressure head, kinetic head and datum head.

- 64.** When the pipes are connected in series, the total rate of flow

- (a) is equal to the sum of the rate of flow in each pipe
- (b) is equal to the reciprocal of the sum of the rate of flow in each pipe
- (c) is the same as flowing through each pipe
- (d) none of the above.

- 65.** Power transmitted through pipes, will be maximum when

- (a) head lost due to friction =  $\frac{1}{2}$  total head at inlet of the pipe
- (b) head lost due to friction =  $\frac{1}{4}$  total head at inlet of the pipe
- (c) head lost due to friction = total head at the inlet of the pipe
- (d) head lost due to friction =  $\frac{1}{3}$  total head at the inlet of the pipe.

- 66.** The valve closure is said to be gradual if the time required to close the valve

$$(a) t = \frac{2L}{C} \quad (b) t \leq \frac{2L}{C}$$

$$(c) t < \frac{4L}{C} \quad (d) t > \frac{2L}{C}$$

where  $L$  = Length of pipe,  $C$  = Velocity of pressure wave.

- 67.** The velocity of pressure wave in terms of bulk modulus ( $K$ ) and density ( $\rho$ ) is given by

$$(a) C = \sqrt{\frac{\rho}{K}} \quad (b) C = \sqrt{K\rho}$$

$$(c) C = \sqrt{\frac{K}{\rho}} \quad (d) \text{none of the above.}$$

- 68.** Reynold's number is defined as the

- (a) ratio of inertia force to gravity force
- (b) ratio of viscous force to gravity force
- (c) ratio of viscous force to elastic force
- (d) ratio of inertia force to viscous force.

- 69.** Froude's number is defined as the ratio of

- (a) inertia force to viscous force
- (b) inertia force to gravity force
- (c) inertia force to elastic force
- (d) inertia force to pressure force.

- 70.** Mach number is defined as the ratio of

- (a) inertia force to viscous force
- (b) viscous force to surface tension force
- (c) viscous force to elastic force
- (d) inertia force to elastic force.

- 71.** Euler's number is the ratio of

- (a) inertia force to pressure force
- (b) inertia force to elastic force
- (c) inertia force to gravity force
- (d) none of the above.

- 72.** Models are known undistorted model if

- (a) the prototype and model are having different scale ratios
- (b) the prototype and model are having same scale ratio
- (c) model and prototype are kinematically similar
- (d) none of the above.

- 73.** Geometric similarity between model and prototype means

- (a) the similarity discharge
- (b) the similarity of linear dimensions
- (c) the similarity of motion
- (d) the similarity of forces.

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74. Kinematic similarity between model and prototype means  
 (a) the similarity of forces  
 (b) the similarity of shape  
 (c) the similarity of motion  
 (d) the similarity of discharge.
75. Dynamic similarity between model and prototype means  
 (a) the similarity of forces  
 (b) the similarity of motion  
 (c) the similarity of shape  
 (d) none of the above.
76. Reynolds number is expressed as  
 (a)  $R_e = \frac{\rho \mu L}{V}$       (b)  $R_e = \frac{V \mu L}{\rho}$   
 (c)  $R_e = \frac{\rho V L}{\mu}$       (d)  $R_e = \frac{V \times d}{\nu}$ .
77. Froude's number ( $F_e$ ) is given by  
 (a)  $F_e = V \sqrt{\frac{L}{g}}$       (b)  $F_e = V \sqrt{\frac{g}{L}}$   
 (c)  $F_e = \frac{V}{\sqrt{Lg}}$       (d) none of the above.
78. Mach number ( $M$ ) is given by  
 (a)  $M = \frac{C}{V}$       (b)  $M = V \times C$   
 (c)  $M = \frac{V}{C}$       (d) none of the above.
79. Boundary layer on a flat plate is called laminar boundary layer if  
 (a) Reynold number is less than 2000  
 (b) Reynold number is less than 4000  
 (c) Reynold number is less than  $5 \times 10^5$   
 (d) None of the above.
80. Boundary layer thickness ( $\delta$ ) is the distance from the surface of the solid body in the direction perpendicular to flow, where the velocity of fluid is equal to  
 (a) free-stream velocity  
 (b) 0.9 times the free-stream velocity  
 (c) 0.99 times the free-stream velocity  
 (d) none of the above.
81. Displacement thickness ( $\delta^*$ ) is given by  
 (a)  $\delta^* = \int_0^\delta \left(1 - \frac{U}{u}\right) dy$   
 (b)  $\delta^* = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$
82. Momentum thickness ( $\Theta$ ) is given by  
 (a)  $\Theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$   
 (b)  $\Theta = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$   
 (c)  $\Theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy$   
 (d) none of the above.
83. Energy thickness ( $\delta^{**}$ ) is equal to  
 (a)  $\int_0^\delta \frac{u}{U} \left[1 - \frac{u}{U}\right] dy$   
 (b)  $\int_0^\delta \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy$   
 (c)  $\int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right)^2 dy$   
 (d) none of the above.
84. Von-Karman momentum integral equation is given as  
 (a)  $\frac{1}{2} \frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x}$       (b)  $\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x}$   
 (c)  $\frac{\tau_0}{2\rho U^2} = \frac{\partial \theta}{\partial x}$       (d) none of the above.
85. The boundary layer separation takes place if  
 (a) pressure gradient is zero  
 (b) pressure gradient is positive  
 (c) pressure gradient is negative  
 (d) none of the above.
86. The condition for boundary layer separation is  
 (a)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = +ve$       (b)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = -ve$   
 (c)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$       (d) none of the above.
87. The boundary layer flow will be attached to the surface if  
 (a)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$       (b)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = +ve$

- (c)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = -ve$  (d) none of the above.
- 88.** The condition for detached flow is  
 (a)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$  (b)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = +ve$   
 (c)  $\left(\frac{\partial u}{\partial y}\right)_{y=0} = -ve$  (d) none of the above.
- 89.** Drag is defined as the force exerted by a flowing fluid on a solid body  
 (a) in the direction of flow  
 (b) perpendicular to the direction of flow  
 (c) in the direction which is at an angle of  $45^\circ$  to the direction of flow  
 (d) none of the above.
- 90.** Lift force is defined as the force exerted by a flowing fluid on a solid body  
 (a) in the direction of flow  
 (b) perpendicular to the direction of flow  
 (c) at an angle of  $45^\circ$  to the direction of flow  
 (d) none of the above.
- 91.** Drag force is expressed mathematically, as  
 (a)  $F_D = \frac{1}{2} \rho U^2 \times C_D \times A$   
 (b)  $F_D = \rho U^2 \times C_D \times A$   
 (c)  $F_D = 2\rho U^2 \times C_D \times A$   
 (d) none of the above.
- 92.** Lift force ( $F_L$ ) is expressed mathematically, as  
 (a)  $F_L = \frac{1}{2} \rho U^2 \times C_L \times A$   
 (b)  $F_L = \frac{1}{2} \rho U^2 \times C_L \times A$   
 (c)  $F_L = 2\rho U^2 \times C_L \times A$   
 (d)  $F_L = \rho U^2 \times C_L \times A$ .
- 93.** Total drag on a body is the sum of  
 (a) pressure drag and velocity drag  
 (b) pressure drag and friction drag  
 (c) friction drag and velocity drag  
 (d) none of the above.
- 94.** A body is called stream lined body when it is placed in a flow and the surface of the body  
 (a) coincides with streamlines  
 (b) does not coincide with the streamlines  
 (c) is perpendicular to the streamlines  
 (d) none of the above.
- 95.** A body is called bluff body if the surface of the body  
 (a) coincides with the streamlines  
 (b) does not coincide with the streamlines  
 (c) is very smooth  
 (d) none of the above.
- 96.** The drag on a sphere ( $F_D$ ) for Reynold's number less than 0.2 is given by  
 (a)  $F_D = 5\pi\mu DU$  (b)  $F_D = 3\pi\mu DU$   
 (c)  $F_D = 2\pi\mu DU$  (d)  $F_D = \pi\mu DU$ .
- 97.** The skin friction drag on a sphere (for Reynold's number less than 0.2) is equal to  
 (a) one-third of the total drag  
 (b) half of the total drag  
 (c) two-third of the total drag  
 (d) none of the above.
- 98.** The pressure drag on a sphere (for Reynold's number less than 0.2) is equal to  
 (a) one-third of the total drag  
 (b) half of the total drag  
 (c) two-third of the total drag  
 (d) none of the above.
- 99.** Terminal velocity of a falling body is equal to  
 (a) a maximum velocity with which body will fall  
 (b) the maximum constant velocity with which body will fall  
 (c) half of the maximum velocity  
 (d) none of the above.
- 100.** When a falling body has attained terminal velocity, the weight of the body is equal to  
 (a) drag force minus buoyant force  
 (b) buoyant force minus drag force  
 (c) drag force plus the buoyant force  
 (d) none of the above.
- 101.** The tangential velocity of ideal fluid at any point on the surface of the cylinder is given by  
 (a)  $u_\theta = \frac{1}{2} U \sin \theta$  (b)  $u_\theta = U \sin \theta$   
 (c)  $u_\theta = 2U \sin \theta$  (d) none of the above.
- 102.** The lift force ( $F_L$ ) produced on a rotating circular cylinder in a uniform flow is given by  
 (a)  $F_L = \frac{LU\Gamma}{\rho}$  (b)  $F_L = \rho L U \Gamma$   
 (c)  $F_L = \frac{\rho U \Gamma}{\rho}$  (d)  $F_L = \frac{\rho L U}{\Gamma}$   
 where  $L$  = Length of the cylinder,  $U$  = Free-stream velocity,  $\Gamma$  = Circulation.

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103. The lift co-efficient ( $C_L$ ) for a rotating cylinder in a uniform flow is given by

$$(a) C_L = \frac{\Gamma U}{R} \quad (b) C_L = \frac{\Gamma R}{U}$$

$$(c) C_L = \frac{\Gamma}{RU} \quad (d) C_L = \frac{RU}{\Gamma}.$$

104. Kinematic viscosity ( $\nu$ ) is equal to

$$(a) \mu \times \rho \quad (b) \frac{\mu}{\rho}$$

$$(c) \frac{\rho}{\mu} \quad (d) \text{none of the above.}$$

105. Compressibility is equal to

$$(a) \frac{\left(\frac{dV}{V}\right)}{dp} \quad (b) -\frac{dp}{\left(\frac{dV}{V}\right)}$$

$$(c) \frac{dp}{dp} \quad (d) \sqrt{\frac{dp}{dp}}.$$

106. Hydrostatic law of pressure is given as

$$(a) \frac{\partial p}{\partial z} = \rho g \quad (b) \frac{\partial p}{\partial z} = 0$$

$$(c) \frac{\partial p}{\partial z} = z \quad (d) \frac{\partial p}{\partial z} = \text{constant.}$$

107. Four curves are shown in Fig. 1 with velocity

gradient  $\left(\frac{\partial u}{\partial y}\right)$  along  $x$ -axis and viscous shear stress ( $\tau$ ) along  $y$ -axis. Curve A corresponds to

- (a) ideal fluid
- (b) newtonian fluid
- (c) non-newtonian fluid
- (d) ideal solid.

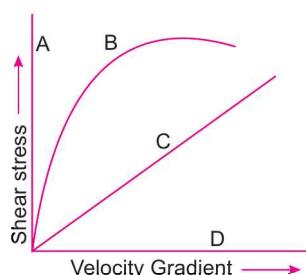


Fig. 1

108. Curve B in Fig. 1 corresponds to
- (a) ideal fluid

- (b) newtonian fluid

- (c) non-newtonian fluid

- (d) ideal solid.

109. Curve C in Fig. 1 corresponds to

- (a) ideal fluid

- (b) newtonian fluid

- (c) non-newtonian fluid

- (d) ideal solid.

110. Curve D in Fig. 1 corresponds to

- (a) ideal fluid

- (b) newtonian fluid

- (c) non-newtonian fluid

- (d) ideal solid.

111. The relation between surface tension ( $\sigma$ ) and difference of pressure ( $\Delta p$ ) between the inside and outside of a liquid droplet is given as

$$(a) \Delta p = \frac{\sigma}{4d} \quad (b) \Delta p = \frac{\sigma}{2d}$$

$$(c) \Delta p = \frac{4\sigma}{d} \quad (d) \Delta p = \frac{\sigma}{d}.$$

112. For a soap bubble, the surface tension ( $\sigma$ ) and difference of pressure ( $\Delta p$ ) are related as

$$(a) \Delta p = \frac{\sigma}{4d} \quad (b) \Delta p = \frac{\sigma}{2d}$$

$$(c) \Delta p = \frac{4\sigma}{d} \quad (d) \Delta p = \frac{8\sigma}{d}.$$

113. For a liquid jet, the surface tension ( $\sigma$ ) and difference of pressure ( $\Delta p$ ) are related as

$$(a) \Delta p = \frac{\sigma}{4d} \quad (b) \Delta p = \frac{\sigma}{2d}$$

$$(c) \Delta p = \frac{4\sigma}{d} \quad (d) \Delta p = \frac{2\sigma}{d}.$$

114. The capillary rise or fall of a liquid is given by

$$(a) h = \frac{\sigma \cos \theta}{4\rho gd} \quad (b) h = \frac{4\sigma \cos \theta}{\rho gd}$$

$$(c) h = \frac{8\sigma \cos \theta}{\rho gd} \quad (d) \text{none of the above.}$$

115. Manometer is a device used for measuring

- (a) velocity at a point in fluid

- (b) pressure at a point in a fluid

- (c) discharge of a fluid

- (d) none of the above.

116. Differential manometers are used for measuring

- (a) velocity at a point in a fluid

- (b) pressure at a point in a fluid

- (c) difference of pressure between two points

- (d) none of the above.

117. The pressure at a height  $Z$  in a static compressible fluid undergoing isothermal compression is given by

$$(a) p = p_0 e^{-\frac{gR}{ZT}} \quad (b) p = p_0 e^{-\frac{gT}{RZ}}$$

$$(c) p = p_0 e^{-\frac{RT}{gZ}} \quad (d) p = p_0 e^{-\frac{gT}{RT}}$$

where  $p_0$  = Pressure at ground level,  $R$  = Gas constant,  $T$  = Absolute temperature.

118. The pressure at a height  $Z$  in a static compressible fluid undergoing adiabatic compression is given by

$$(a) p = p_0 \left[ 1 - \frac{\gamma - 1}{\gamma} \frac{RT_0}{gZ} \right]^{\frac{\gamma}{\gamma-1}}$$

$$(b) p = p_0 \left[ 1 - \frac{\gamma}{\gamma - 1} \frac{RT_0}{gZ} \right]^{\frac{\gamma}{\gamma-1}}$$

$$(c) p = p_0 \left[ 1 - \frac{\gamma - 1}{\gamma} \frac{gZ}{RT_0} \right]^{\frac{\gamma}{\gamma-1}}$$

(d) none of the above.

119. The temperature at a height  $Z$  in a static compressible fluid undergoing adiabatic compression is given as

$$(a) T = T_0 \left[ 1 - \frac{\gamma - 1}{\gamma} \frac{RT_0}{gZ} \right]$$

$$(b) T = T_0 \left[ 1 - \frac{\gamma - 1}{\gamma} \frac{gZ}{RT_0} \right]$$

$$(c) T = T_0 \left[ 1 - \frac{\gamma}{\gamma - 1} \frac{RT_0}{gZ} \right]$$

(d) none of the above.

120. Temperature lapse-rate is given by

$$(a) L = -\frac{R}{g} \left[ \frac{\gamma - 1}{\gamma} \right] \quad (b) L = -\frac{R}{g} \left[ \frac{\gamma}{\gamma - 1} \right]$$

$$(c) L = -\frac{g}{R} \left[ \frac{\gamma - 1}{\gamma} \right] \quad (d) \text{none of the above.}$$

121. When the fluid is at rest, the shear stress is  
 (a) maximum      (b) zero  
 (c) unpredictable      (d) none of the above.

122. The depth of centre of pressure of an inclined immersed surface from free surface of liquid is equal to

$$(a) \frac{I_G}{Ah} + \bar{h} \quad (b) \frac{I_G A \sin^2 \theta}{h} + \bar{h}$$

$$(c) \frac{I_G \sin^2 \theta}{Ah} + \bar{h} \quad (d) \frac{I_G \bar{h}}{A \sin^2 \theta} + \bar{h}.$$

123. The depth of centre of pressure of a vertical immersed surface from free surface of liquid is equal to

$$(a) \frac{I_G}{Ah} + \bar{h} \quad (b) \frac{I_G A}{h} + \bar{h}$$

$$(c) \frac{I_G \bar{h}}{h} + \bar{h} \quad (d) \frac{A \bar{h}}{I_G} + \bar{h}.$$

124. The centre of pressure for a plane vertical surface lies at a depth of

- (a) half the height of the immersed surface
- (b) one-third the height of the immersed surface
- (c) two-third the height of the immersed surface
- (d) none of the above.

125. The inlet length of a venturimeter

- (a) is equal to the outlet length
- (b) is more than the outlet length
- (c) is less than the outlet length
- (d) none of the above.

126. Flow of a fluid in a pipe takes place from

- (a) higher level to lower level
- (b) higher pressure to lower pressure
- (c) higher energy to lower energy
- (d) none of the above.

127. The point, through which the buoyant force is acting, is called

- (a) centre of pressure
- (b) centre of gravity
- (c) centre of buoyancy
- (d) none of the above.

128. The point, through which the weight is acting, is called

- (a) centre of pressure
- (b) centre of gravity
- (c) centre of buoyancy
- (d) none of the above.

129. The point, about which a floating body starts oscillating when the body is tilted, is called

- (a) centre of pressure
- (b) centre of buoyancy
- (c) centre of gravity
- (d) metacentre.

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130. The metacentric height ( $GM$ ) is given by

- (a)  $GM = BG - \frac{I}{V}$
- (b)  $GM = \frac{V}{I} - BG$
- (c)  $GM = \frac{I}{V} - BG$
- (d) none of the above.

131. For a floating body, if the metacentre is above the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

132. For a floating body, if the metacentre is below the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

133. For a floating body, if the metacentre coincides with the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

134. For a floating body, if centre of buoyancy is above the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

135. For a sub-merged body, if the centre of buoyancy is above the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

136. For a sub-merged body, if the centre of buoyancy is below the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

137. For a sub-merged body, if the centre of buoyancy coincides with the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

138. For a sub-merged body, if the metacentre is below the centre of gravity, the equilibrium is called

- (a) stable
- (b) unstable
- (c) neutral
- (d) none of the above.

139. The metacentric height ( $GM$ ) experimentally is given as

- (a)  $GM = \frac{W \tan \theta}{wx}$
- (b)  $GM = \frac{w \tan \theta}{W \times x}$
- (c)  $GM = \frac{wx}{W \tan \theta}$
- (d)  $GM = \frac{Wx}{w \tan \theta}$

where  $w$  = Movable weight,  $W$  = Weight of floating body including  $w$ ,  $\theta$  = Angle of tilt.

140. The time period of oscillation of a floating body is given by

$$(a) T = 2\pi \sqrt{\frac{GM \times g}{k^2}} \quad (b) T = 2\pi \sqrt{\frac{k^2}{GM \times g}}$$

$$(c) T = 2\pi \sqrt{\frac{GM}{gk^2}} \quad (d) T = 2\pi \sqrt{\frac{gk^2}{GM}}$$

where  $k$  = Radius of gyration,  $GM$  = Metacentric height, and  $T$  = Time period.

141. If the velocity, pressure, density etc., do not change at a point with respect to time, flow is called

- (a) uniform
- (b) incompressible
- (c) non-uniform
- (d) steady.

142. If the velocity, pressure, density, etc., change at a point with respect to time, the flow is called

- (a) uniform
- (b) compressible
- (c) unsteady
- (d) incompressible.

143. If the velocity in a fluid flow does not change with respect to length of direction of flow, it is called

- (a) steady flow
- (b) uniform flow
- (c) incompressible flow
- (d) rotational flow.

144. If the velocity in a fluid flow changes with respect to length of direction of flow, it is called

- (a) unsteady flow
- (b) compressible flow
- (c) irrotational flow
- (d) none of the above.

145. If the density of a fluid is constant from point to point in a flow region, it is called

- (a) steady flow
- (b) incompressible flow
- (c) uniform flow
- (d) rotational flow.

146. If the density of a fluid changes from point to point in a flow region, it is called

- (a) steady flow
- (b) unsteady flow
- (c) non-uniform flow
- (d) compressible flow.

147. If the fluid particles move in straight lines and all the lines are parallel to the surface, the flow is called

- (a) steady
- (b) uniform
- (c) compressible
- (d) laminar.

148. If the fluid particles move in a zig-zag way, the flow is called

- (a) unsteady
- (b) non-uniform
- (c) turbulent
- (d) incompressible.

- 149.** The acceleration of a fluid particle in the direction of  $x$  is given by
- $A_x = u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial u}{\partial t}$
  - $A_x = u \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial y} + u \frac{\partial w}{\partial z} + \frac{\partial u}{\partial t}$
  - $A_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$
  - none of the above.
- 150.** The local acceleration in the direction of  $x$  is given by
- $u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$
  - $\frac{\partial u}{\partial t}$
  - $u \frac{\partial u}{\partial x}$
  - none of the above.
- 151.** The convective acceleration in the direction of  $x$  is given by
- $u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} + w \frac{\partial w}{\partial z}$
  - $u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} + u \frac{\partial u}{\partial z}$
  - $u \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial y} + u \frac{\partial w}{\partial z}$
  - $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$ .
- 152.** Shear strain rate is given by
- $\frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$
  - $\frac{1}{2} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$
  - $\frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$
  - $\frac{1}{2} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ .
- 153.** For a two-dimensional fluid element in  $x-y$  plane, the rotational component is given as
- $\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$
  - $\omega_z = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)$
  - $\omega_z = \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$
  - $\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$ .
- 154.** Vorticity is given by
- two times the rotation
  - 1.5 times the rotation
  - three times the rotation
  - equal to the rotation.
- 155.** Study of fluid motion with the forces causing the flow is known as
- kinematics of fluid flow
  - dynamics of fluid flow
  - statics of fluid flow
  - none of the above.
- 156.** Study of fluid motion without considering the forces, causing the flow, is known as
- kinematics of fluid flow
  - dynamics of fluid flow
  - statics of fluid flow
  - none of the above.
- 157.** Study of fluid at rest is known as
- kinematics
  - dynamics
  - statics
  - none of the above.
- 158.** The term  $V^2/2g$  is known as
- kinetic energy
  - pressure energy
  - kinetic energy per unit weight
  - none of the above.
- 159.** The terms  $p/\rho g$  is known as
- kinetic energy per unit weight
  - pressure energy
  - pressure energy per unit weight
  - none of the above.
- 160.** The term  $Z$  is known as
- potential energy
  - pressure energy
  - potential energy per unit weight
  - none of the above.
- 161.** The discharge through a venturimeter is given as
- $Q = \frac{A_1^2 A_2^2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}$
  - $Q = \frac{A_1 A_2}{\sqrt{2A_1^2 - A_2^2}} \times \sqrt{2gh}$
  - $Q = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}$
  - none of the above.
- 162.** The difference of pressure head ( $h$ ) measured by mercury-oil differential manometer is given as

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$$(a) h = x \left[ 1 - \frac{S_g}{S_o} \right] \quad (b) h = x [S_g - S_o]$$

$$(c) h = x [S_o - S_g] \quad (d) h = x \left[ \frac{S_g}{S_o} - 1 \right]$$

where  $x$  = Difference of mercury level,  $S_g$  = Specific gravity of mercury and  $S_o$  = Specific gravity of oil.

163. The difference of pressure head ( $h$ ) measured by a differential manometer containing lighter liquid is

$$(a) h = x \left[ 1 - \frac{S_l}{S_o} \right] \quad (b) h = x \left[ \frac{S_l}{S_o} - 1 \right]$$

$$(c) h = x [S_o - S_l] \quad (d) \text{none of the above.}$$

where  $S_l$  = Specific gravity of lighter liquid in manometer.

$S_o$  = Specific gravity of fluid flowing  
 $x$  = Difference of lighter liquid levels in differential manometer.

164. Pitot-tube is used to measure

- (a) discharge
- (b) average velocity
- (c) velocity at a point
- (d) pressure at a point.

165. Venturimeter is used to measure

- (a) discharge
- (b) average velocity
- (c) velocity at a point
- (d) pressure at a point.

166. Orifice-meter is used to measure

- (a) discharge
- (b) average velocity
- (c) velocity at a point
- (d) pressure at a point.

167. For a sub-merged curved surface, the horizontal component of force due to static liquid is equal to

- (a) weight of liquid supported by the curved surface
- (b) force on a projection of the curved surface on a vertical plane
- (c) area of curved surface  $\times$  pressure at the centroid of the submerged area
- (d) none of the above.

168. For a sub-merged curved surface, the vertical component of force due to static liquid is equal to

(a) weight of the liquid supported by curved surface

(b) force on a projection of the curved surface on a vertical plane

(c) area of curved surface  $\times$  pressure at the centroid of the sub-merged area

(d) none of the above.

169. An oil of specific gravity 0.7 and pressure 0.14 kgf/cm<sup>2</sup> will have the height of oil as

- (a) 70 cm of oil
- (b) 2 m of oil
- (c) 20 cm of oil
- (d) 80 cm of oil.

170. The difference in pressure head, measured by a mercury water differential manometer for a 20 cm difference of mercury level will be

- (a) 2.72 m
- (b) 2.52 m
- (c) 2.0 m
- (d) 0.2 m.

171. The difference in pressure head, measured by a mercury-oil differential manometer for a 20 cm difference of mercury level will be (sp. gravity of oil = 0.8)

- (a) 2.72 m of oil
- (b) 2.52 m of oil
- (c) 3.20 m of oil
- (d) 2.0 m of oil.

172. The rate of flow through a venturimeter varies as

- (a)  $H$
- (b)  $\sqrt{H}$
- (c)  $H^{3/2}$
- (d)  $H^{5/2}$ .

173. The rate of flow through a V-notch varies as

- (a)  $H$
- (b)  $\sqrt{H}$
- (c)  $H^{3/2}$
- (d)  $H^{5/2}$ .

174. Orifices are used to measure

- (a) velocity
- (b) pressure
- (c) rate of flow
- (d) none of the above.

175. Mouthpieces are used to measure

- (a) velocity
- (b) pressure
- (c) viscosity
- (d) rate of flow.

176. The ratio of actual velocity of a jet of water at vena-contracta to the theoretical velocity, is known as

- (a) co-efficient of discharge
- (b) co-efficient of velocity
- (c) co-efficient of contraction
- (d) co-efficient of viscosity.

177. The ratio of actual discharge of a jet of water to its theoretical discharge is known as

- (a) co-efficient of discharge
- (b) co-efficient of velocity
- (c) co-efficient of contraction
- (d) co-efficient of viscosity.

**178.** The ratio of the area of the jet of water at vena-contracta to the area of orifice, is known as

- (a) co-efficient of discharge
- (b) co-efficient of velocity
- (c) co-efficient of contraction
- (d) co-efficient of viscosity.

**179.** The discharge through a large rectangular orifice is

- (a)  $\frac{2}{3} C_d \times b \times \sqrt{2g} (\sqrt{H_2} - \sqrt{H_1})$
- (b)  $\frac{8}{15} C_d \times b \times \sqrt{2g} (H_2^{3/2} - H_1^{3/2})$
- (c)  $\frac{2}{3} C_d \times b \times \sqrt{2g} (H_2^{3/2} - H_1^{3/2})$

(d) none of the above.

where  $b$  = Width of orifice,  $H_1$  = Height of liquid above top edge of the orifice,  
 $H_2$  = Height of liquid above bottom edge of orifice.

**180.** The discharge through fully sub-merged orifice is

- (a)  $C_d \times b \times (H_2 - H_1) \times \sqrt{2g} \times H^{3/2}$
- (b)  $C_d \times b \times (H_2 - H_1) \times \sqrt{2gH}$
- (c)  $C_d \times b \times (H_2^{3/2} - H_1^{3/2}) \times \sqrt{2gH}$

(d) none of the above

where  $H$  = Difference of liquid level on both sides of the orifice,

$H_1$  = Height of liquid above top edge orifice on upstream side,

$H_2$  = Height of liquid above bottom edge of orifice on upstream side.

**181.** Notch is a device used for measuring

- (a) rate of flow through pipes
- (b) rate of flow through a small channel
- (c) velocity through a pipe
- (d) velocity through a small channel.

**182.** The discharge through a rectangular notch is given by

- (a)  $Q = \frac{2}{3} C_d \times L \times H^{5/2}$
- (b)  $Q = \frac{2}{3} C_d \times L \times H^{3/2}$
- (c)  $Q = \frac{8}{15} C_d \times L \times H^{5/2}$
- (d)  $\frac{8}{15} Q = \frac{8}{15} C_d \times L \times H^{3/2}$ .

**183.** The discharge through a triangular notch is given by

- (a)  $Q = \frac{2}{3} C_d \times \tan \frac{\theta}{2} \times \sqrt{2gH}$
- (b)  $Q = \frac{2}{3} C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} \times H^{3/2}$
- (c)  $Q = \frac{2}{15} C_d \times \tan \frac{\theta}{2} \times \sqrt{2g} H^{5/2}$

(d) none of the above.

where  $\theta$  = Total angle of triangular notch,  
 $H$  = Head over notch.

**184.** The discharge through a trapezoidal notch is given as

- (a)  $Q = \frac{2}{3} C_{d_1} \times L \times H^{3/2}$   
 $+ \frac{8}{15} \times C_{d_2} \times \tan \theta / 2 \times \sqrt{2g} \times H^{3/2}$
- (b)  $Q = \frac{2}{3} C_{d_1} \times L \times H^{5/2}$   
 $+ \frac{8}{15} \times C_{d_2} \times \tan \theta / 2 \times \sqrt{2g} H^{3/2}$
- (c)  $Q = \frac{2}{3} C_{d_1} \times L \times H^{3/2}$   
 $+ \frac{8}{15} \times C_{d_2} \times \tan \theta / 2 \times \sqrt{2g} H^{5/2}$

(d) none of the above

where  $\theta/2$  = Slope of the side of the trapezoidal notch.

**185.** The error in discharge due to the error in the measurement of head over a rectangular notch is given by

- (a)  $\frac{dQ}{Q} = \frac{5}{2} \frac{dH}{H}$
- (b)  $\frac{dQ}{Q} = \frac{3}{2} \frac{dH}{H}$
- (c)  $\frac{dQ}{Q} = \frac{7}{2} \frac{dH}{H}$
- (d)  $\frac{dQ}{Q} = \frac{1}{2} \frac{dH}{H}$ .

**186.** The error in discharge due to the error in the measurement of head over a triangular notch is given by

- (a)  $\frac{dQ}{Q} = \frac{5}{2} \frac{dH}{H}$
- (b)  $\frac{dQ}{Q} = \frac{3}{2} \frac{dH}{H}$
- (c)  $\frac{dQ}{Q} = \frac{7}{2} \frac{dH}{H}$
- (d)  $\frac{dQ}{Q} = \frac{1}{2} \frac{dH}{H}$ .

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187. The velocity with which the water approaches a notch is called  
 (a) velocity of flow  
 (b) velocity of approach  
 (c) velocity of whirl  
 (d) none of the above.
188. The discharge over a rectangular notch considering velocity of approach is given as  
 (a)  $Q = \frac{2}{3} C_d L \sqrt{2g} (H^{3/2} - h_a^{3/2})$   
 (b)  $Q = \frac{2}{3} C_d L \sqrt{2g} (H - h_a)^{3/2}$   
 (c)  $Q = \frac{2}{3} C_d L \sqrt{2g} [(H + h_a)^{3/2} - h_a^{3/2}]$   
 (d) none of the above.  
 where  $H$  = Head over notch, and  $h_a$  = Head due to velocity of approach.
189. The velocity of approach ( $V_a$ ) is given by  
 (a)  $V_a = \frac{\text{Discharge over notch}}{\text{Area of notch}}$   
 (b)  $V_a = \frac{\text{Discharge over notch}}{\text{Area of channel}}$   
 (c)  $V_a = \frac{\text{Discharge over notch}}{\text{Head over notch} \times \text{Width of channel}}$   
 (d) none of the above.
190. Francis's formula for a rectangular weir with end contraction suppressed is given as  
 (a)  $Q = 1.84 L H^{5/2}$  (b)  $Q = \frac{2}{3} L \times H^{3/2}$   
 (c)  $Q = 1.84 L H^{3/2}$  (d)  $Q = \frac{2}{3} L \times H^{5/2}$ .
191. Francis's formula for a rectangular weir for two end contractions is given by  
 (a)  $Q = 1.84[L - 0.2H]H^{5/2}$   
 (b)  $Q = 1.84[L - 0.2H]H^{3/2}$   
 (c)  $Q = 1.84[L - 0.2H]H^{5/2}$   
 (d) none of the above.
192. Bazin's formula for discharge over a rectangular weir without velocity of approach is given by  
 (a)  $Q = mL \times \sqrt{2g} H^{5/2}$   
 (b)  $Q = mL \times \sqrt{2g} \times H^{3/2}$   
 (c)  $Q = m \times L \times \sqrt{2gH}$   
 (d) none of the above.  
 where  $m = 0.405 + \frac{0.003}{H}$  and  $H$  = Head over weir.
193. Cipolletti weir is a trapezoidal weir having side slope of  
 (a) 1 horizontal to 2 vertical  
 (b) 4 horizontal to 1 vertical  
 (c) 1 horizontal to 4 vertical  
 (d) 1 horizontal to 3 vertical.
194. The co-efficient of friction in terms of shear stress is given by  
 (a)  $f = \frac{2\rho V^2}{\tau_0}$  (b)  $f = \frac{2\tau_0}{\rho V^2}$   
 (c)  $f = \frac{\tau_0}{2\rho V^2}$  (d)  $f = \frac{\rho V^2}{2\tau_0}$
195. When the pipes are connected in parallel, the total loss of head  
 (a) is equal to the sum of the loss of head in each pipe  
 (b) is same as in each pipe  
 (c) is equal to the reciprocal of the sum of loss of head in each pipe  
 (d) none of the above.
196.  $L_1$ ,  $L_2$ ,  $L_3$  and the length of three pipes, connected in series. If  $d_1$ ,  $d_2$  and  $d_3$  are their diameters, then the equivalent size of the pipe is given by  
 (a)  $\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$   
 (b)  $\frac{d^5}{L} = \frac{d_1^5}{L_1} + \frac{d_2^5}{L_2} + \frac{d_3^5}{L_3}$   
 (c)  $Ld^5 = L_1 d_1^5 + L_2 d_2^5 + L_3 d_3^5$   
 (d) none of the above.
197. The power transmitted through pipe is given by  
 (a)  $\frac{\rho g \times Q \times H}{1000}$   
 (b)  $\frac{\rho g \times Q \times h_f}{1000}$   
 (c)  $\frac{\rho g \times Q \times (H - h_f)}{4500}$   
 (d)  $\frac{\rho g \times Q \times (H - h_f)}{1000}$

where  $H$  = Total head at the inlet of pipe,  $h_f$  = Head lost due to friction in pipe and  $Q$  = discharge per second.

198. Efficiency of power transmission through pipe is given by

$$(a) \frac{H - h_f}{H} \quad (b) \frac{H}{H + h_f}$$

$$(c) \frac{H - h_f}{H + h_f} \quad (d) \text{none of the above}$$

where  $H$  = Total head at inlet,  $h_f$  = Head lost due to friction.

199. Maximum efficiency of power transmission through pipe is

$$(a) 50\% \quad (b) 66.67\%$$

$$(c) 75\% \quad (d) 100\%.$$

200. For a viscous flow through circular pipes, certain curves are shown in Fig. 2, curve A is for

- (a) shear stress distribution
- (b) velocity distribution
- (c) pressure distribution
- (d) none of the above.

201. Curve B in Fig. 2 is for

- (a) shear stress distribution
- (b) velocity distribution
- (c) pressure distribution
- (d) none of the above.

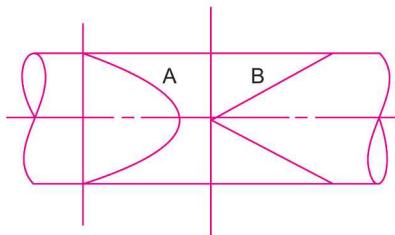


Fig. 2

202. Fig. 3 shows four curves for velocity distribution across a section for Reynolds number equal to 1000, 4000, 6000 and 10000. Curve A corresponds to Reynolds number equal to

- (a) 1000
- (b) 4000
- (c) 6000
- (d) 10000.

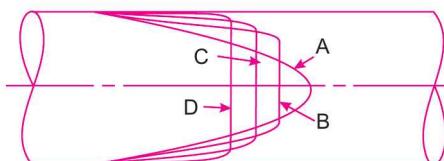


Fig. 3

203. Curve B in Fig. 3 corresponds to Reynolds number

- (a) 1000
- (b) 4000
- (c) 6000
- (d) 10000.

204. Curve C in Fig. 3 corresponds to the Reynolds number

- (a) 1000
- (b) 4000
- (c) 6000
- (d) 10000.

205. Curve D in Fig. 3 corresponds to the Reynolds number

- (a) 1000
- (b) 4000
- (c) 6000
- (d) 10000.

206. The shear stress distribution across a section of a circular pipe, having viscous flow is given by

$$(a) \tau = \frac{\partial p}{\partial x} r^2 \quad (b) \tau = \frac{\partial p}{\partial x} \frac{r}{2}$$

$$(c) \tau = -\frac{\partial p}{\partial x} \frac{r}{2} \quad (d) \tau = -\frac{\partial p}{\partial x} \times 2r.$$

207. The velocity distribution across a section of a circular pipe having viscous flow is given by

$$(a) u = U_{\max} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]$$

$$(b) u = U_{\max} [R^2 - r^2]$$

$$(c) u = U_{\max} \left[ 1 - \frac{r}{R} \right]^2$$

- (d) none of the above.

208. The velocity distribution across a section of two fixed parallel plates having viscous flow is given by

$$(a) u = \frac{1}{2\mu} \left( -\frac{\partial p}{\partial x} \right) (t^2 - y^2)$$

$$(b) u = \frac{1}{2\mu} \frac{\partial p}{\partial x} [t_y - y^2]$$

$$(c) u = \frac{1}{2\mu} \frac{\partial p}{\partial x} [y - ty]$$

$$(d) u = -\frac{1}{2\mu} \frac{\partial p}{\partial x} [t - y^2]$$

where  $t$  = Distance between two plates and  $y$  is measured from the lower plate.

209. The shear stress distribution across a section of two fixed parallel plates having viscous flow is given by

$$(a) \tau = -\frac{1}{2} \frac{\partial p}{\partial x} [t^2 - y^2]$$

$$(b) \tau = -\frac{1}{2} \frac{\partial p}{\partial x} [t - 2y]$$

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(c)  $\tau = -\frac{1}{2} \frac{\partial p}{\partial x} [ty - y^2]$

(d)  $\tau = \frac{1}{2} \frac{\partial p}{\partial x} [y - ty]$

where  $t$  = Distance between two parallel plates and  $y$  is measured from the plate.

210. The ratio of inertia force to viscous force is known as  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler number.
211. The square root of the ratio of inertia force to gravity force is called  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler number.
212. The square root of the ratio of inertia force to force due to compressibility is known as  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler number.
213. The square root of the ratio of inertia force to pressure force is known as  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler's number.
214. Model analysis of pipes flow are based on  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler number.
215. Model analysis of free surface flows are based on  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler number.
216. Model analysis of aeroplanes and projectile moving at supersonic speed based on  
 (a) Reynolds number (b) Froude number  
 (c) Mach number (d) Euler number.
217. The boundary-layer takes place  
 (a) for ideal fluids  
 (b) for pipe-flow only  
 (c) for real fluids  
 (d) for flow over flat plate only.
218. The boundary layer is called turbulent boundary layer if  
 (a) Reynolds number is more than 2000  
 (b) Reynolds number is more than 4000  
 (c) Reynolds number is more than  $5 \times 10^5$   
 (d) none of the above.
219. Laminar sub-layer exists in  
 (a) laminar boundary layer region  
 (b) turbulent boundary layer region  
 (c) transition zone  
 (d) none of the above.

220. The thickness of laminar boundary layer at a distance  $x$  from the leading edge over a flat plate varies as

(a)  $x^{4/5}$  (b)  $x^{1/2}$   
 (c)  $x^{1/5}$  (d)  $x^{3/5}$ .

221. The thickness of turbulent boundary layer at a distance  $x$  from the leading edge over a flat plate varies as

(a)  $x^{4/5}$  (b)  $x^{1/2}$   
 (c)  $x^{1/5}$  (d)  $x^{3/5}$ .

222. The separation of boundary layer takes place in case of

- (a) negative pressure gradient  
 (b) positive pressure gradient  
 (c) zero pressure gradient  
 (d) none of the above.

223. The velocity profile for turbulent boundary layer is

$$(a) \frac{u}{U} = \sin \left( \frac{\pi y}{2 \delta} \right)$$

$$(b) \frac{u}{U} = \left( \frac{y}{\delta} \right)^{4/7}$$

$$(c) \frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2$$

$$(d) \frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3.$$

224. The drag force exerted by a fluid on a body immersed in the fluid is due to

- (a) pressure and viscous force  
 (b) pressure and gravity force  
 (c) pressure and turbulence force  
 (d) none of the above.

225. For supersonic flow, if the area of flow increases then

- (a) velocity decreases  
 (b) velocity increases  
 (c) velocity is constant  
 (d) none of the above.

226. The area velocity relationship for compressible fluids is

$$(a) \frac{dA}{A} = \frac{dV}{A} [1 - M^2] (b) \frac{dA}{A} = \frac{dV}{V} [M^2 - 1]$$

$$(c) \frac{dA}{A} = \frac{dV}{V} [1 - V^2] (d) \frac{dA}{A} = \frac{dV}{V} [C^2 - 1].$$

- 227.** The flow in open channel is laminar if the Reynolds number is  
 (a) 2000 (b) less than 2000  
 (c) less than 500 (d) none of the above.

**228.** The flow in open channel is turbulent if the Reynolds number is  
 (a) 2000 (b) more than 2000  
 (c) more than 4000 (d) 4000.

**229.** If the Froude number in open channel flow is less than 1.0, the flow is called  
 (a) critical flow  
 (b) super-critical flow  
 (c) sub-critical flow  
 (d) none of the above.

**230.** If the Froude number in open channel flow is equal to 1.0, the flow is called  
 (a) critical flow (b) streaming flow  
 (c) shooting (d) none of the above.

**231.** If the Froude number in open channel flow is more than 1.0, the flow is called  
 (a) critical flow (b) streaming flow  
 (c) shooting flow (d) none of the above.

**232.** Chezy's formula is given as  
 (a)  $V = i\sqrt{mC}$  (b)  $V = C\sqrt{mi}$   
 (c)  $V = m\sqrt{Ci}$  (d) none of the above.

**233.** The discharge through a rectangular channel is maximum when  
 (a)  $m = \frac{d}{3}$  (b)  $m = \frac{d}{2}$   
 (c)  $m = 2d$  (d)  $m = \frac{3d}{2}$   
 where  $m$  = Hydraulic mean depth,  $d$  = Depth of flow.

**234.** The discharge through a trapezoidal channel is maximum when  
 (a) half of top width = sloping side  
 (b) top width = half of sloping side  
 (c) top width =  $1.5 \times$  sloping side  
 (d) none of the above.

**235.** The maximum velocity through a circular channel takes place when depth of flow is equal to  
 (a) 0.95 times the diameter  
 (b) 0.5 times the diameter  
 (c) 0.81 times the diameter  
 (d) 0.5 times the diameter.

**236.** The maximum discharge through a circular channel takes place when depth of flow is equal to  
 (a) 0.95 times the diameter  
 (b) 0.3 times the diameter  
 (c) 0.81 times the diameter  
 (d) 0.5 times the diameter.

**237.** Specific energy of a flowing fluid per unit weight is equal to  
 (a)  $\frac{p}{w} + \frac{V^2}{2g}$  (b)  $\frac{p}{w} + h$   
 (c)  $\frac{V^2}{2g} + h$  (d)  $\frac{p}{w} + \frac{V^2}{2g} + h$ .

**238.** The depth of flow after hydraulic jump is  
 (a)  $d_2 = \frac{d_1}{2} [\sqrt{1+8(F_e)_1^2} - 1]$   
 (b)  $d_2 = \frac{d_1}{2} [1 + \sqrt{8(F_e)_1^2} - 1]$   
 (c)  $d_2 = \frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + 8(F_e)_1}$   
 (d) none of the above.

**239.** The depth of flow at which specific energy is minimum is called  
 (a) normal depth (b) critical depth  
 (c) alternate depth (d) none of the above.

**240.** The critical depth ( $h_c$ ) is given by  
 (a)  $\left(\frac{q^2}{g}\right)^{1/2}$  (b)  $\left(\frac{q}{g}\right)^{1/3}$   
 (c)  $\left(\frac{q^2}{g}\right)^{1/3}$  (d)  $\left(\frac{q^2}{g}\right)^{2/3}$

where  $q$  = Rate of flow per unit width of channel.

**241.** For a circular channel, the wetted perimeter is given by  
 (a)  $\frac{R\theta}{2}$  (b)  $3R\theta$   
 (c)  $2R\theta$  (d)  $R\theta$   
 where  $R$  = Radius of circular channel and  $\theta$  = half the angle subtended by the water surface at the centre.

**242.** For a circular channel the area of flow is given by  
 (a)  $R^2 \left(2\theta - \frac{\sin 2\theta}{2}\right)$  (b)  $R^2 \left(\theta - \frac{\sin 2\theta}{2}\right)$   
 (c)  $R^2(\theta - \sin 2\theta)$  (d) none of the above  
 where  $\theta$  = Half the angle subtended by water surface at the centre, and  $R$  = Radius of circular channel.

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243. The hydraulic mean depth is given by

$$\begin{array}{ll} (a) \frac{P}{A} & (b) \frac{P^2}{A} \\ (c) \frac{A}{P} & (d) \sqrt{\frac{A}{P}} \end{array}$$

where  $A$  = Area , and  $P$  = Wetted perimeter.

244. A most economical section is one which for a given cross-sectional area, slope of bed  
(i) and co-efficient of resistance has  
(a) maximum wetted perimeter  
(b) maximum discharge  
(c) maximum depth of flow  
(d) none of the above.
245. Specific speed of a turbine is defined as the speed of the turbine which  
(a) produces unit power at unit head  
(b) produces unit horse power at unit discharge  
(c) delivers unit discharge at unit head  
(d) delivers unit discharge at unit power.
246. A pump is defined as a device which converts  
(a) Hydraulic energy into mechanical energy  
(b) Mechanical energy into hydraulic energy  
(c) Kinetic energy into mechanical energy  
(d) None of the above.
247. A turbine is a device which converts  
(a) Hydraulic energy into mechanical energy  
(b) Mechanical energy into hydraulic energy  
(c) Kinetic energy into mechanical energy  
(d) Electrical energy into mechanical energy.
248. The force exerted by a jet of water on a stationary vertical plate in the direction of jet is given by  
(a)  $F_x = \rho A V^2 \sin^2 \theta$   
(b)  $F_x = \rho A V^2 [1 + \cos \theta]$   
(c)  $F_x = \rho A V^2$   
(d) none of the above.
249. The force exerted by a jet of water on a stationary inclined plate in the direction of jet is given by  
(a)  $F_x = \rho A V^2$   
(b)  $F_x = \rho A V^2 \sin^2 \theta$   
(c)  $F_x = \rho A V^2 [1 + \cos \theta]$   
(d)  $F_x = \rho A V^2 [1 + \sin \theta]$ .

250. The force exerted by a jet of water on a stationary curved plate in the direction of jet is equal to

$$\begin{array}{l} (a) \rho A V^2 \\ (b) \rho A V^2 \sin^2 \theta \\ (c) \rho A V^2 (1 + \cos \theta) \\ (d) \rho A V^2 [1 + \sin \theta]. \end{array}$$

251. The force exerted by a jet of water having velocity  $V$  on a vertical plate, moving with a velocity  $u$  is given by  
(a)  $F_x = \rho A (V - u)^2 \sin^2 \theta$   
(b)  $F_x = \rho A (V - u)^2$   
(c)  $F_x = \rho A (V - u)^2 [1 + \cos \theta]$   
(d) None of the above.
252. The force exerted by a jet of water having velocity  $V$  on a series of vertical plates moving with velocity  $u$  is given by  
(a)  $F_x = \rho A V^2$       (b)  $F_x = \rho A (V - u)^2$   
(c)  $F_x = \rho A V u$       (d) None of the above.
253. Efficiency of the jet of water having velocity  $V$  striking a series of vertical plates moving with a velocity  $u$  is given by  
(a)  $\eta = \frac{2V(V-u)}{u^2}$       (b)  $\eta = \frac{2u(V-u)}{V^2}$   
(c)  $\eta = \frac{u^2}{V^2(V-u)}$       (d) None of the above.
254. Efficiency, of the jet of water having velocity  $V$  and striking a series of vertical plates moving with a velocity  $u$ , is maximum when  
(a)  $u = 2V$       (b)  $u = \frac{V}{2}$   
(c)  $u = \frac{3V}{2}$       (d)  $u = \frac{4V}{3}$ .
255. Maximum efficiency of a series of vertical plates is  
(a) 66.67%      (b) 33.33%  
(c) 50%      (d) 80%.
256. For a series of curved radial vanes, the work done per second per unit weight is equal to  
(a)  $\frac{1}{g} V_{W_1} u_1 + V_{W_2} u_2$       (b)  $\frac{1}{g} [V_1 u_1 + V_2 u_2]$   
(c)  $\frac{1}{g} [V_{W_1} u_1 \pm V_{W_2} u_2]$  (d) none of the above.
257. The net head ( $H$ ) on the turbine is given by  
(a)  $H = \text{Gross Head} + \text{Head lost due to friction}$   
(b)  $H = \text{Gross Head} - \text{Head lost due to friction}$

- (c)  $H = \text{Gross Head} + \frac{V^2}{2g}$  – Head lost due to friction.
- 258.** Hydraulic efficiency of a turbine is defined as the ratio of  
 (a) Power available at the inlet of turbine to power given by water to the runner  
 (b) Power at the shaft of the turbine to power given by water to the runner  
 (c) Power at the shaft of the turbine to the power at the inlet of turbine  
 (d) None of the above.
- 259.** Mechanical efficiency of a turbine is the ratio of  
 (a) Power at the inlet to the power at the shaft of turbine  
 (b) Power at the shaft to the power given to the runner  
 (c) Power at the shaft to the power at the inlet of turbine  
 (d) None of the above.
- 260.** The overall efficiency of a turbine is the ratio of  
 (a) Power at the inlet of turbine to the power at the shaft  
 (b) Power at the shaft to the power given to the runner  
 (c) Power at the shaft to the power at the inlet of turbine  
 (d) None of the above.
- 261.** The relation between hydraulic efficiency ( $\eta_h$ ), mechanical efficiency ( $\eta_m$ ) and overall efficiency ( $\eta_o$ ) is  
 (a)  $\eta_h = \eta_o \times \eta_m$       (b)  $\eta_o = \eta_h \times \eta_m$   
 (c)  $\eta_o = \frac{\eta_m}{\eta_h}$       (d) none of the above.
- 262.** A turbine is called impulse if at the inlet of the turbine  
 (a) total energy is only kinetic energy  
 (b) total energy is only pressure energy  
 (c) total energy is the sum of kinetic energy and pressure energy  
 (d) none of the above.
- 263.** A turbine is called reaction turbine if at the inlet of the turbine the total energy is  
 (a) kinematic energy only  
 (b) kinetic energy and pressure energy  
 (c) pressure energy only  
 (d) none of the above.
- 264.** Which of the following statement is correct?  
 (a) Pelton wheel is a reaction turbine  
 (b) Pelton wheel is a radial flow turbine  
 (c) Pelton wheel is an impulse turbine  
 (d) None of the above.
- 265.** Francis turbine is  
 (a) an impulse turbine  
 (b) a radial flow impulse turbine  
 (c) an axial flow turbine  
 (d) a reaction radial flow turbine.
- 266.** Kaplan turbine is  
 (a) an impulse turbine  
 (b) a radial flow impulse turbine  
 (c) an axial flow reaction turbine  
 (d) a radial flow reaction turbine.
- 267.** Jet ratio ( $m$ ) is defined as the ratio of  
 (a) diameter of jet of water to diameter of Pelton wheel  
 (b) velocity of vane to the velocity of jet of water  
 (c) velocity of flow to the velocity of jet of water  
 (d) diameter of Pelton wheel to diameter of the jet of water.
- 268.** Flow ratio is defined as the ratio of  
 (a) Velocity of flow at inlet to the velocity given by  $\sqrt{2gH}$   
 (b) Velocity of runner at inlet to the velocity of flow at inlet  
 (c) Velocity of runner to the velocity given by  $\sqrt{2gH}$   
 (d) None of the above.
- 269.** Speed ratio is given by  
 (a)  $\frac{u}{\sqrt{2gH}}$       (b)  $\frac{V_f}{\sqrt{2gH}}$   
 (c)  $\frac{\sqrt{2gH}}{V_f}$       (d)  $\frac{V_w}{\sqrt{2gH}}$ .
- 270.** The speed ratio for Pelton wheel varies from  
 (a) 0.45 to 0.50      (b) 0.6 to 0.7  
 (c) 0.3 to 0.4      (d) 0.8 to 0.9.
- 271.** The discharge through Pelton Turbine is given by  
 (a)  $Q = \pi DB V_f$   
 (b)  $Q = \frac{\pi}{4} d^2 \times \sqrt{2gH}$   
 (c)  $Q = \frac{\pi}{4} [D_o^2 - D_b^2] \times V_f$   
 (d) None of the above.

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272. The discharge through Francis Turbine is given by  
 (a)  $Q = \pi DB V_f$   
 (b)  $Q = \frac{\pi}{4} d^2 \times \sqrt{2gH}$   
 (c)  $Q = \frac{\pi}{4} [D_o^2 - D_b^2]$   
 (d) None of the above.
273. The discharge through Kaplan turbine is given by  
 (a)  $Q = \pi DB V_f$       (b)  $Q = \frac{\pi}{4} d^2 \times \sqrt{2gH}$   
 (c)  $Q = \frac{\pi}{4} [D_o^2 - D_b^2]$       (d)  $Q = 0.9 \pi DB V_f$
274. Draft tube is used for discharging water from the exit of  
 (a) an impulse turbine (b) a Francis turbine  
 (c) a Kaplan turbine (d) a Pelton wheel.
275. Specific speed of a turbine is defined as the speed at which the turbine runs when  
 (a) working under unit head and discharging one litre per second  
 (b) working under unit head and develops unit horse power  
 (c) develops unit horse power and discharges one litre per second  
 (d) none of the above.
276. The specific speed ( $N_s$ ) of a turbine is given by  
 (a)  $N_s = \frac{N\sqrt{P}}{H^{3/4}}$       (b)  $N_s = \frac{N\sqrt{Q}}{H^{3/4}}$   
 (c)  $N_s = \frac{N\sqrt{P}}{H^{5/4}}$       (d)  $N_s = \frac{NP^{5/4}}{\sqrt{H}}$ .
277. Unit speed is the speed of a turbine when it is working  
 (a) under unit head and develops unit power  
 (b) under unit head and discharge one  $m^3/sec$   
 (c) under unit head  
 (d) none of the above.
278. Unit discharge is the discharge of a turbine when  
 (a) the head on turbine is unity and it develops unit power  
 (b) the head on turbine is unity and it moves at unit speed  
 (c) the head on the turbine is unity  
 (d) none of the above.
279. Unit power is the power developed by a turbine when  
 (a) head on turbine is unity and discharge is also unity
- (b) head = one metre and speed is unity  
 (c) head on turbine is unity  
 (d) none of the above.
280. The unit speed ( $N_u$ ) is given by the expression  
 (a)  $N_u = \frac{N}{H^{3/2}}$       (b)  $N_u = \frac{N}{H^{3/4}}$   
 (c)  $N_u = \frac{N}{\sqrt{H}}$       (d)  $N_u = \frac{N}{H^{5/4}}$ .
281. The unit discharge ( $Q_u$ ) is given by the expression  
 (a)  $Q_u = \frac{Q}{\sqrt{H}}$       (b)  $Q_u = \frac{Q}{H^{3/2}}$   
 (c)  $Q_u = \frac{Q}{H^{3/4}}$       (d)  $Q_u = \frac{Q}{H^{5/4}}$ .
282. Unit power ( $P_u$ ) is given by the expression  
 (a)  $P_u = \frac{P}{\sqrt{H}}$       (b)  $P_u = \frac{P}{H^{3/2}}$   
 (c)  $P_u = \frac{P}{H^{3/4}}$       (d)  $P_u = \frac{P}{H^{5/4}}$ .
283. The unit discharge ( $Q_u$ ) and unit speed ( $N_u$ ) curves for different turbines are shown in Fig. 4. Curve A is for  
 (a) Francis Turbine  
 (b) Kaplan Turbine  
 (c) Pelton Turbine  
 (d) Propeller Turbine.

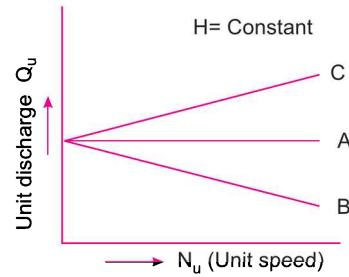


Fig. 4

284. Curve B in Fig. 4 is for  
 (a) Francis Turbine  
 (b) Kaplan Turbine  
 (c) Pelton Turbine  
 (d) Propeller Turbine.
285. Curve D in Fig. 4 is for  
 (a) Francis Turbine  
 (b) Kaplan Turbine  
 (c) Pelton Turbine  
 (d) Propeller Turbine.

- 286.** Tick mark the correct statement  
 (a) curves at constant speed are called main characteristic curves  
 (b) curves at constant head are called main characteristic curves  
 (c) curves at constant efficiency are called operating characteristic curves  
 (d) curves at constant efficiency are called main characteristic curves.
- 287.** Main characteristic curves of a turbine means  
 (a) curves at constant speed  
 (b) curves at constant efficiency  
 (c) curves at constant head  
 (d) none of the above.
- 288.** Operating characteristic curves of a turbine means  
 (a) curves drawn at constant speed  
 (b) curves drawn at constant efficiency  
 (c) curves drawn at constant head  
 (d) none of the above.
- 289.** Muschel curves means  
 (a) curves at constant head  
 (b) curves at constant speed  
 (c) curves at constant efficiency  
 (d) none of the above.
- 290.** Governing of a turbine means  
 (a) the head is kept constant under all condition of working  
 (b) the speed is kept constant under all conditions  
 (c) the discharge is kept constant under all conditions  
 (d) none of the above.
- 291.** The work done by impeller of a centrifugal pump on water per second per unit weight of water is given by  
 (a)  $\frac{1}{g} V_{W_1} u_1$       (b)  $\frac{1}{g} V_{W_2} u_2$   
 (c)  $\frac{1}{g} (V_{W_2} u_2 - V_{W_1} u_1)$       (d) none of the above.
- 292.** The manometer head ( $H_m$ ) of a centrifugal pump is given by  
 (a) Pressure head at outlet of pump – pressure head at inlet  
 (b) Total head at inlet – total head at outlet  
 (c) Total head at outlet – total head at inlet  
 (d) None of the above.
- 293.** The manometric efficiency ( $\eta_{man}$ ) of a centrifugal pump is given by  
 (a)  $\frac{H_m}{g V_{W_2} u_2}$       (b)  $\frac{g H_m}{V_{W_2} u_2}$
- (c)  $\frac{V_{W_2} u_2}{g H_m}$       (d)  $\frac{g \times V_{W_2} u_2}{H_m}$ .
- 294.** Mechanical efficiency ( $\eta_{mech}$ ) of a centrifugal pump is given by  
 (a) (Power at the impeller)/S.H.P.  
 (b) S.H.P./Power at the impeller  
 (c) Power possessed by water/power at the impeller  
 (d) Power possessed by water/S.H.P.
- 295.** To produce a high head by multistage centrifugal pumps, the impellers are connected  
 (a) in parallel  
 (b) in series  
 (c) in parallel and in series both  
 (d) none of the above.
- 296.** To discharge a large quantity of liquid by multi-stage centrifugal pump, the impellers are connected  
 (a) in parallel  
 (b) in series  
 (c) in parallel and in series  
 (d) none of the above.
- 297.** Specific speed of a pump is the speed at which a pump runs when  
 (a) head developed is unity and discharge is one cubic metre  
 (b) head developed is unity and shaft horse power is also unity  
 (c) discharge is one cubic metre and shaft horse power is unity  
 (d) none of the above.
- 298.** The specific speed ( $N_s$ ) of a pump is given by the expression  
 (a)  $N_s = \frac{N\sqrt{Q}}{H_m^{5/4}}$       (b)  $N_s = \frac{N\sqrt{P}}{H_m^{3/4}}$   
 (c)  $N_s = \frac{N\sqrt{Q}}{H_m^{3/4}}$       (d)  $N_s = \frac{N\sqrt{P}}{H_m^{5/4}}$ .
- 299.** The operating characteristic curves of a centrifugal pump are shown in Fig. 5, curve A is for  
 (a) Head      (b) Efficiency  
 (c) Power      (d) None of the above.

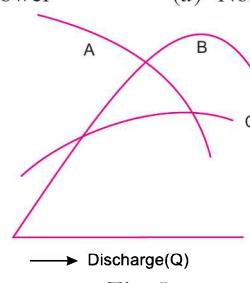


Fig. 5

**1092** Fluid Mechanics

- 316.** Hydraulic ram is pump which works  
 (a) on the principle of water-hammer  
 (b) on the principle of centrifugal action  
 (c) on the principle of reciprocating action  
 (d) none of the above.
- 317.** Hydraulic coupling is a device used for  
 (a) transmitting same torque to the driven shaft  
 (b) transmitting increased torque to the driven shaft  
 (c) transmitting decreased torque to the driven shaft  
 (d) none of the above.
- 318.** Torque converter is a device used for  
 (a) transmitting same torque to the driven shaft  
 (b) transmitting increased torque to the driven shaft  
 (c) transmitting decreased torque to the driven shaft  
 (d) transmitting increased or decreased torque to the driven shaft.
- 319.** Capacity of a hydraulic accumulator is given as equal to  
 (a) pressure of water supplied by pump  $\times$  volume of accumulator  
 (b) pressure of water  $\times$  area of accumulator  
 (c) pressure of water  $\times$  stroke of the ram of accumulator  
 (d) none of the above.
- 320.** Kaplan turbine is a propeller turbine in which the vanes fixed on the hub are  
 (a) non-adjustable (b) adjustable  
 (c) fixed (d) none of the above.
- 321.** If the head on the turbine is more than 300 m, the type of turbine used should be  
 (a) Kaplan (b) Francis  
 (c) Pelton (d) Propeller.
- 322.** If the specific speed of a turbine is more than 300, the type of turbine is  
 (a) Pelton  
 (b) Kaplan  
 (c) Francis  
 (d) Pelton with more jets.
- 323.** Run-away speed of a Pelton wheel means  
 (a) Full load speed  
 (b) No load speed  
 (c) No load speed with no governor mechanism  
 (d) None of the above.
- 324.** Spouting velocity means  
 (a) actual velocity of jet  
 (b) ideal velocity of jet  
 (c) half of ideal velocity of jet  
 (d) none of the above.
- 325.** Surge tank in a pipe line is used to  
 (a) reduce the loss of head due to friction in pipe  
 (b) make the flow uniform in pipe  
 (c) relieve the pressure due to water hammer  
 (d) none of the above.
- 326.** Hydraulic ram is a device used for  
 (a) storing energy of water in the form of pressure energy  
 (b) increasing pressure intensity of water  
 (c) lifting small quantity of water to a greater height by means of large quantity of water falling through small height  
 (d) none of the above.
- 327.** For low head and high discharge, the suitable turbine is  
 (a) Pelton (b) Francis  
 (c) Kaplan (d) None of the above.
- 328.** For high head and low discharge, the suitable turbine is  
 (a) Pelton (b) Francis  
 (c) Kaplan (d) None of the above.
- 329.** The flow of water, leaving the impeller, in a centrifugal pump casing is  
 (a) Forced vortex flow  
 (b) Free vortex flow  
 (c) Centrifugal flow  
 (d) None of the above.
- 330.** Rotameter is used for measuring  
 (a) density of fluids  
 (b) velocity of fluids in pipes  
 (c) discharge of fluids  
 (d) viscosity of fluids.
- 331.** A current meter is a device used for measuring  
 (a) velocity (b) viscosity  
 (c) current (d) pressure.
- 332.** A hot wire anemometer is a device used for measuring  
 (a) viscosity (b) velocity of gases  
 (c) pressure of gases (d) none of the above.

**1094 Fluid Mechanics****ANSWERS**

1. (c)    2. (b)    3. (b)    4. (b)    5. (b)    6. (c)    7. (b), (d)    8. (c)    9. (b)    10. (b)  
11. (d)    12. (b)    13. (b)    14. (c)    15. (b)    16. (d)    17. (a)    18. (c)    19. (c)    20. (b)  
21. (d)    22. (c)    23. (d)    24. (c)    25. (b)    26. (b)    27. (d)    28. (c)    29. (c)    30. (c)  
31. (c)    32. (a)    33. (c)    34. (d)    35. (d)    36. (b),(c) 37. (c)    38. (b)    39. (c)    40. (d)  
41. (b),(c) 42. (d)    43. (c)    44. (b)    45. (c)    46. (b)    47. (c)    48. (c)    49. (b)    50. (b)  
51. (c)    52. (d)    53. (c)    54. (a)    55. (d)    56. (d)    57. (a)    58. (c)    59. (b)    60. (c)  
61. (d)    62. (d)    63. (d)    64. (c)    65. (d)    66. (d)    67. (c)    68. (d)    69. (b)    70. (d)  
71. (a)    72. (b)    73. (b)    74. (c)    75. (a)    76. (c)    77. (c)    78. (c)    79. (c)    80. (c)  
81. (d)    82. (a)    83. (b)    84. (b)    85. (b)    86. (c)    87. (b)    88. (c)    89. (a)    90. (b)  
91. (a)    92. (b)    93. (b)    94. (a)    95. (b)    96. (b)    97. (c)    98. (a)    99. (b)    100. (c)  
101. (c)    102. (b)    103. (c)    104. (b)    105. (a)    106. (a)    107. (d)    108. (c)    109. (b)    110. (a)  
111. (c)    112. (d)    113. (d)    114. (b)    115. (b)    116. (c)    117. (d)    118. (c)    119. (b)    120. (c)  
121. (b)    122. (c)    123. (a)    124. (c)    125. (c)    126. (c)    127. (c)    128. (b)    129. (d)    130. (c)  
131. (a)    132. (b)    133. (c)    134. (d)    135. (a)    136. (b)    137. (c)    138. (d)    139. (c)    140. (b)  
141. (d)    142. (c)    143. (b)    144. (d)    145. (b)    146. (d)    147. (d)    148. (c)    149. (c)    150. (b)  
151. (d)    152. (c)    153. (d)    154. (a)    155. (a)    156. (a)    157. (c)    158. (c)    159. (c)    160. (c)  
161. (c)    162. (d)    163. (a)    164. (c)    165. (a)    166. (a)    167. (b)    168. (a)    169. (b)    170. (b)  
171. (c)    172. (b)    173. (d)    174. (c)    175. (d)    176. (b)    177. (a)    178. (c)    179. (c)    180. (b)  
181. (b)    182. (b)    183. (c)    184. (c)    185. (b)    186. (a)    187. (b)    188. (c)    189. (b)    190. (c)  
191. (b)    192. (b)    193. (c)    194. (b)    195. (b)    196. (a)    197. (d)    198. (a)    199. (b)    200. (b)  
201. (a)    202. (a)    203. (b)    204. (c)    205. (d)    206. (c)    207. (a)    208. (b)    209. (b)    210. (a)  
211. (b)    212. (c)    213. (d)    214. (a)    215. (b)    216. (c)    217. (c)    218. (c)    219. (b)    220. (b)  
221. (a)    222. (b)    223. (b)    224. (a)    225. (b)    226. (b)    227. (c)    228. (b)    229. (c)    230. (a)  
231. (c)    232. (b)    233. (b)    234. (a)    235. (c)    236. (a)    237. (c)    238. (a)    239. (b)    240. (c)  
241. (c)    242. (b)    243. (c)    244. (b)    245. (a)    246. (b)    247. (a)    248. (c)    249. (b)    250. (c)  
251. (b)    252. (a)    253. (b)    254. (b)    255. (c)    256. (c)    257. (b)    258. (d)    259. (b)    260. (c)  
261. (b)    262. (a)    263. (b)    264. (c)    265. (d)    266. (c)    267. (d)    268. (a)    269. (a)    270. (a)  
271. (b)    272. (a)    273. (c)    274. (b), (c)    275. (b)    276. (c)    277. (c)    278. (c)    279. (c)  
280. (c)    281. (a)    282. (b)    283. (c)    284. (a)    285. (b)    286. (b)    287. (c)    288. (a)    289. (c)  
290. (b)    291. (b)    292. (c)    293. (b)    294. (a)    295. (b)    296. (a)    297. (a)    298. (c)    299. (a)  
300. (b)    301. (c)    302. (c)    303. (b), (d)    304. (c)    305. (a)    306. (c)    307. (b)    308. (a)  
309. (b)    310. (a)    311. (a)    312. (c)    313. (a)    314. (b)    315. (b)    316. (a)    317. (a)    318. (d)  
319. (a)    320. (b)    321. (c)    322. (b)    323. (c)    324. (b)    325. (c)    326. (c)    327. (c)    328. (a)  
329. (b)    330. (c)    331. (a)    332. (b).

## APPENDIX

**(A) Base and Derived Units in MKS and SI Units :**

S. No.	Physical Quantity	Unit Symbol in MKS Units	Unit Symbol in SI Units	Name
1.	Mass	kg ( $M$ )	kg( $M$ )	kilogram
2.	Length	m ( $L$ )	m ( $L$ )	metre
3.	Time	Sec ( $t$ )	s ( $t$ )	second
4.	Force	kgf	N	newton
5.	Work, energy	kgf-m	Nm = J	joule
6.	Power	kgf-m/sec	Nm/s = J/s = W	watt

**(B) Conversion Factors from MKS to SI Units :**

$$1 \text{ kgf} = 9.81 \text{ N}$$

$$1 \text{ kgf-m} = 9.81 \text{ N m} = 9.81 \text{ J}$$

$$1 \text{ kgf-m/sec} = 9.81 \text{ N m/s} = 9.81 \text{ J/s} = 9.81 \text{ W}$$

$$\begin{aligned} 1 \text{ metric h.p.} &= 75 \text{ kgf-m/sec} = 75 \times 9.81 \text{ N m/s} \\ &= 735.75 \text{ N m/sec} \approx 736 \text{ W} \end{aligned} \quad (\because \text{ N m/s} = \text{W})$$

**(C) Conversion Factors from MKS to CGS Units :**

$$\begin{aligned} 1 \text{ kgf} &= 1000 \text{ gmf} = 1000 \times \text{gm} \times \text{g} \\ &= 1000 \times \text{gm} \times 981 \text{ cm/sec}^2 \\ &= 1000 \times 981 \times \text{gm-cm/sec}^2 = 1000 \times 981 \text{ dyne} \\ &= 9.81 \times 10^5 \text{ dyne} \quad (\because \text{ gm-cm/sec}^2 = \text{dyne}) \\ 1 \text{ kgf/m}^2 &= 9.81 \times 10^5 \text{ dyne}/10^4 \text{ cm}^2 \\ &= 9.81 \times 10 \text{ dyne/cm}^2 = 98.1 \text{ dyne/cm}^2 \quad (\because \text{ m}^2 = 10^4 \text{ cm}^2) \\ 1 \text{ m}^3 &= 1000 \text{ litre.} \end{aligned}$$

**(D) Conversion Factors for Poise and Stoke in MKS and SI Units :**

$$\text{Poise} = \frac{1}{98.1} \text{ (MKS unit)} = \frac{1}{10} \text{ (SI unit)}$$

$$\text{Stoke} = 10^{-4} \text{ (MKS unit)} = 10^{-4} \text{ (SI unit)}$$

**(E) Important Values in S.I. Units :**

$$\text{Pascal (Pa)} = \text{N/m}^2 = \text{kg/ms}^2$$

$$\text{Newton (N)} = \text{kg m/s}^2$$

$$\text{Joule (J)} = \text{N m} = \text{kg m}^2/\text{s}^2$$

$$\text{Atmospheric pressure} = 101.325 \text{ kN/m}^2$$

$$1 \text{ bar} = 10^5 \text{ Pa} = 10^5 \text{ N/m}^2$$

$$1 \text{ m bar or 1 mb} = 10^{-3} \text{ bar} = 10^2 \text{ Pa} = 10^2 \text{ N/m}^2$$

Pascal and bar are used for pressure.



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