The goal is to calculate the difference between the sum of squares of the first hundred natural numbers and the square of the sum. More generally, we wish to evaluate

$$f = S_N^{(2)} - \left(S_N^{(1)}\right)^2 \tag{1}$$

where

$$S_N^{(\alpha)} = \sum_{n=1}^{N} n^\alpha. \tag{2}$$

The squared sum may be calculated via a popular method due to Gauss,

$$\begin{aligned} S_N^{(1)} &= 1 + 2 + \cdots + (N-1) + N \\ &= N + (N-1) + \cdots + 2 + 1 \\ &= \frac{1}{2}N(N+1), \end{aligned} \tag{3}$$

so that

$$\left(S_N^{(1)}\right)^2 = \frac{1}{4}N^2(N+1)^2. \tag{4}$$

A similar expression may be arrived at for the other sum by noting

$$(n-1)^3 = n^3 - 3n^2 + 3n - 1, \tag{5}$$

so that

$$\sum_{n=1}^{N} \left[n^3 - (n-1)^3\right] = \sum_{n=1}^{N} \left(3n^2 - 3n + 1\right). \tag{6}$$

Evaluating the sum on the left-hand side and the sum over $3n + 1$ on the right-hand side yields

$$N^3 = 3\sum_{n=1}^{N} n^2 - \frac{3}{2}N(N+1) + N. \tag{7}$$

Hence

$$\sum_{n=1}^{N} n^2 = \frac{1}{6}N\left(2N+1\right)\left(N+1\right). \tag{8}$$

Therefore $S_{100}^{(2)} = 338350$ and $S_{100}^{(1)} = 5050$ and so

$$f = -25164150. \tag{9}$$