BAN402: Decision Modelling in Business

Project 1 Report

Norwegian School of Economics, Fall 24

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24/09/2024

Contents

Part A:

Task A1:

The optimization problem can be formulated as follows:

Sets:

L: set of locations ($L = \{L1, L2, L3\}$)

P: set of pollutants ($P = \{P1, P2\}$)

Parameters:

$cost_l =$ # of $ cost per km² with fertilizer at location l, $l \in L$

$abat_{p,l} =$ # of tons of pollutant p abated per km² with fertilizer at location l, $p \in P, l \in L$

$req_p =$ # of tons of pollutant p abated required by government, $p \in P$

Decision variables:

$x_l =$# of km² with new fertilizer at location l, $l \in L$

Objective:

$$\min \sum_{l \in L} x_l cost_l$$

Subject to:

Reduction:

$$\sum_{p \in P} \sum_{l \in L} abat_{p,l} * x_l \geq req_p$$

Non-negativity:

$$x_l \geq 0, \qquad \forall l \in L$$

Solution:

When implementing the model in AMPL, we find that the *optimal objective value* is equal to $4,146.15 with the *optimal solution* presented below:

Location	Km² with new fertilizer
L_1	161.538
L_2	0
L_3	30.769

Sensitivity report:

```
total_cost = 4146.15

:      x          x.rc         x.down      x.up       :=
L1   161.538    1.77636e-15   15          20.3529
L2     0        5.30769       20.6923   1e+20
L3    30.7692   5.32907e-15   31.55       44.3333
;

:  reduction  reduction.down  reduction.current  reduction.up    :=
P1   69.2308       30               35              56
P2   43.0769       25               40              46.6667
;
```

Task A2:

First, we control the pollutants removed with the proposed optimal solution:

The number of tons of pollutant P1 removed is:

$$0.15 * 161.538 + 0.35 * 30.769 \approx 35$$

The number of tons of pollutant P2 removed is:

$$0.20 * 161.538 + \ 0.25 * 30.769 \approx 40$$

Hence, the government requirements for pollution reduction are satisfied.

Additionally, all decision variables are greater than zero, satisfying the non-negativity constraint.

Task A3:

To investigate how sensitive the optimal cost is to changes in government requirements, we look at the sensitivity report for our *reduction* constraint (see Task A1)

The shadow price for *reduction* for P1 is \$69.231. This means that for every unit increase in the RHS (the government requirement), the optimal cost will increase by \$69.231. This is true for RHS values between 30 and 56.

Since the new target of 45 is within the allowable range, we can conclude that the new optimal cost will be:

$$\Delta Cost = Shadowprice * \Delta RHS \ = 69.231 * 10$$

$$Cost^{*} = Cost + \Delta Cost = 4{,}146.15 + 692.31 = 4{,}838.46$$

Intuitively, it makes sense that the optimal cost has increased, as the feasible region was further restricted.

Task A4:

From the sensitivity report, we read that the allowable increase for the objective coefficient of L3 is \$44.333.

This means that a new coefficient value of \$40 is still within the range of optimality and that the optimal decision will remain the same.

The new total cost will be:

$$19 * 161.538 + 40 * 30.769 = 4{,}300$$

In general, the problem is not very sensitive to changes in coefficients. However, if the farmer suspects that the true coefficients may lie outside the range of optimality outlined in the sensitivity report, it may be wise to further investigate, as this would mean that another decision is optimal.

Part B:

Task B1:

The optimal blending model for *HappyCattle* can be formulated as follows:

Sets:

P: set of products ($P = \{Standard, Special, Ultra\}$)

M: set of materials ($M = \{wheat, rye, grain, oats, corn\}$)

N: set of nutrients ($N = \{protein, carbohydrate, vitamin\}$)

Parameters:

SP_p = The selling price in NOK per ton for product $p, p \in P$

MC_m= The material cost in NOK per ton of material $m, m \in M$

PC_p = Production cost in NOK per ton for product $p, p \in P$

$MaxProd$ = Maximum production capacity in tons.

D_p= Demand for product $p, p \in P$

AS_m= Available supply (# of available tons) of material $m, m \in M$

$NV_{m,n}$= Nutritional value; Weight percent of nutrient n in material $m, m \in M, n \in N$

$MinNV_{n,p}$ = The minimum nutritional value of nutrient n in product $p, n \in N, p \in P$

$MaxCarb_p$ = The maximum carbohydrates in product $p,\ p \in P$

Decision Variables:

$b_{m,p}$ = # of tons of material m used in the total production of $p,\ m \in M, p \in P$

X_p = # of tons of product p produced, $p \in P$

Objective function:

$$\max\ (\pi = Revenue - production\ cost - material\ cost)$$

$$= \sum_{p \in P} SP_p X_p - \sum_{p \in P} PC_p X_p - \sum_{m \in M} \sum_{p \in P} b_{m,p} MC_m$$

Subject to:

The production of Standard and Special must meet demand exactly:

$$X_p = D_p, \qquad \forall p \in [Standard, Special]$$

The production of Ultra must meet or exceed demand:

$$X_{Ultra} \geq D_{Ultra}$$

Total production cannot exceed capacity:

$$\sum_{p \in P} X_p \leq MaxProd$$

The total # of tons of each material used cannot exceed the available supply:

$$\sum_{p \in P} b_{m,p} \leq AS_m, \qquad \forall m \in M$$

The nutritional value cannot exceed the maximum carbohydrate restriction for each product:

$$\sum_{m \in M} b_{m,p} NV_{m,n} \leq MaxCarb_p X_p, \qquad \forall p \in P, \qquad n = 'Carbohydrates'$$

The nutritional value must exceed or be equal to the minimum for each product and for each nutrient:

$$\sum_{m \in M} b_{m,p} NV_{m,n} \geq MinNV_{n,p} X_p, \qquad \forall p \in P, \qquad \forall n \in N$$

The production of each product must weigh the same as the sum of its materials:

$$\sum_{m \in M} b_{m,p} = X_p, \qquad \forall p \in P$$

Each blend cannot be negative:

$$b_{m,p} \geq 0, \qquad \forall m \in M, \qquad \forall p \in P$$

Production cannot be negative:

$$X_p \geq 0, \qquad \forall p \in P$$

Solution:

When implementing the model in AMPL, we find that the *optimal objective value* is equal to NOK 9,680,000 with the *optimal solution* presented below:

The optimal production plan for *HappyCattle* is:

Product	**Production output in tons**
Standard	400
Special	400
Ultra	500

With the following optimal blending plan (in tons):

	Standard	Special	Ultra	**Sum**:
Wheat	0	133.33	366.67	500
Rye	66.67	0	133.33	200
Grain	333.33	266.67	0	600
Oats	0	0	0	0
Corn	0	0	0	0
Sum:	400	400	500	**1300**

Sensitivity of material supply:

```
:     max_supply max_supply.down max_supply.current max_supply.up    :=
corn       0            0               500          1e+20
grain    600          266.667           600            666.667
oats       0            0              1000          1e+20
rye        0        Infinity       Infinity          1e+20
wheat    100          300               500            633.333
;
```

Grain is the most used raw material, with 600 tons purchased from the supplier.

We notice that the solution uses all the available Grain (the cheapest material) and Wheat (the second cheapest material). As a combination of these two materials could satisfy all the nutritional constraints for all products, the farmer ideally would have liked to use more if there were more supply (which can be gathered from the positive shadow prices). As there is still vacant production capacity, and the contribution margin is positive, the remaining capacity of 200 is filled with Rye (the third cheapest option).

Task B2:

We make the following changes to the model formulation (changes marked in red):

Introduce new set of suppliers:

S: Set of suppliers ($S = \{1,2\}$)

Introduce new decision variable:

$PQ_{m,s}$= purchase quantity; # of tons of material m to purchase from supplier s, $m \in M, s \in S$

Modify material cost to depend on supplier (and add corresponding parameters):

$MC_{m,s}$= The material cost in NOK per ton of material m purchased from supplier s, $m \in M, s \in S$

Modify available supply to depend on supplier (and add corresponding parameters):

$AS_{m,s}$= Available supply (# of available tons) of material m from supplier s, $m \in M$, $s \in S$

Modify supply constraint:

The total # of tons of each material purchased from each supplier cannot exceed available supply:

$$PQ_{m,s} \leq AS_{m,s}, \qquad \forall m \in M, \qquad \forall s \in S$$

Link purchase quantity from supplier and available material for production:

$$\sum_{s \in S} PQ_{m,s} = \sum_{p \in P} b_{p,m}, \qquad \forall m \in M$$

Modify objective function:

$$\max \pi = \sum_{p \in P} SP_p X_p - \sum_{p \in P} PC_p X_p - \sum_{m \in M} \sum_{s \in S} PQ_{m,s}\, MC_{m,s}$$

The new optimal solution is:

When implementing the model in AMPL, we find that the *optimal objective value* equals NOK 9,692,000. This is an improvement of NOK 12,000 compared to the profit in B1 which makes sense as the feasibility region has increased with a new supplier of wheat.

The optimal production plan for *HappyCattle* is unchanged.

The new optimal blending plan (in tons. Changes marked in red/green) is:

	Standard	**Special**	**Ultra**	**Sum:**
Wheat	0	200 (+66.67)	500 (+133.33)	700 (+200)
Rye	0 (-66.67)	0	0 (-133.33)	0 (-200)
Grain	400 (+66.67)	200 (-66.67)	0	600
Oats	0	0	0	0
Corn	0	0	0	0
Sum:	400	400	500	1300

And the new purchasing plan is:

	S1	**S2**	**Sum**
Wheat	500	200	700
Rye	0	0	0
Grain	600	0	600
Oats	0	0	0
Corn	0	0	0
Sum:	1100	200	1300

HappyCattle should purchase 200 tons of wheat from the new supplier.

We notice, that with an increased supply of Wheat, Rye is no longer in the optimal blending mix. This is because a mix of only Wheat and Grain can satisfy the nutritional requirements at lower cost, whilst having enough supply to meet production capacity.

This is expected, as we saw in B1 that the shadow price of Wheat was 100, and the increase in price from the new supplier is only 40.

Task B3:

We update the following parameter values:

$$D_{Standard} = 500\ (+100)$$

$$SP_{standard} = 8750\ (+250)$$

$$MC_{oats} = 1400\ (-300)$$

The new solution is:

Optimal production plan:

Product:	Tons:
Standard	500 (+100)
Special	400
Ultra	400 (-100)
Sum:	1300

Optimal blending plan (in tons):

	Standard	Special	Ultra	Sum:
Wheat	0	0 (-133.33)	0 (-366.67)	0 (-500)
Rye	0 (-66.67)	0	0 (-133.33)	0 (-200)
Grain	317.647 (-15.68)	282.353 (+15.68)	0	600
Oats	182.353 (+182.35)	117.647 (+117.65)	400 (+400)	700 (+700)
Corn	0	0	0	0
Sum:	500 (+100)	400	400 (-100)	1300

The new total profit is NOK 9,745,000 (+65,000 compared to B1)

We first notice that the production plan has substituted 100 Ultra for Standard, in order to satisfy the increased demand. The production of Ultra is still maximized within the total production constraint of 1300.

Secondly, we observe that Wheat and Rye have been replaced by Oats, as this is now the second cheapest material and can satisfy all nutritional requirements in combination with Grain, with enough combined supply to meet production capacity.

Part C:

Task C1:

The optimal shipment plan may be modelled as follows:

Sets:

R: set of regions ($R = \{1,2\}$)

P: set of ports ($P = \{1,2\}$)

K: set of markets ($K = \{1, \dots, 20\}$)

Parameters:

S_r = # of tons that can be obtained from region r per week, $r \in R$

D_k= # of tons demanded weekly in market $k, k \in \ K$

C_{rp}= # $ to ship 1 ton from region r to port $p, r \in R, p \in P$

C_{rk}= # $ to ship 1 ton from region r to market $k, r \in R, k \in K$

C_{pk}= # $ to ship 1 ton from port p to market $k, , p \in P, k \in K$

Decision variables:

x_{rp}= # of tons to transport from region r to port $p, r \in R, p \in P$

x_{rk}= # of tons to transport from region r to market $k, r \in R, k \in K$

x_{pk}= # of tons to transport from port p to market $k\ , p \in P, k \in K$

(We assume that it is possible to send fractional tons of bananas)

Objective function:

$$\min \textit{transport costs } (tc)$$

$$tc = \sum_{r \in R} \sum_{p \in P} x_{rp} C_{rp} + \sum_{r \in R} \sum_{k \in K} x_{rk} C_{rk} + \sum_{p \in P} \sum_{k \in K} x_{pk} C_{pk}$$

Subject to:

Outflow from each region cannot exceed supply:

$$\sum_{k \in K} \sum_{p \in P} (x_{rp} + x_{rk}) \leq S_r, \qquad \forall r \in R$$

Inflow to each market must equal demand in each market:

$$\sum_{r \in R} \sum_{p \in P} (x_{pk} + x_{rk}) = D_k, \qquad \forall k \in K$$

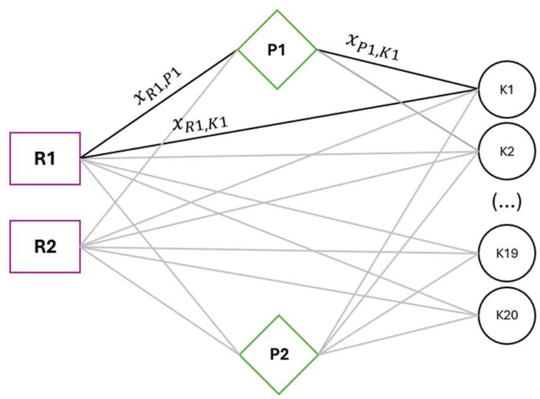
Outflow from a port must be equals to inflow (no storage option):

$$\sum_{r \in R} x_{rp} = \sum_{k \in K} x_{pk}\,, \qquad \forall p \in P$$

Non-negativity:

$$x_{rp}, x_{rk}, x_{pk} \geq 0, \qquad \forall p \in P, \qquad \forall r \in R, \qquad \forall k \in K$$

Graphical illustration:



The highlighted arcs show an example of the three types of decision variables: x_{rp}, x_{pk} and x_{rk}. In this instance: $x_{R1,P1}$, $x_{P1,K1}$ and $x_{R1,K1}$, respectively.

Solution:

Implementing the model in AMPL gives to following cost minimizing solution:

#	K 1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15	K16	K17	K18	K19	K20	P1	P2	Sum:
R1	-	-	-	-	-	-	-	-	-	19	-	-	-	-	26	-	-	-	-	13	142	-	200
R2	-	-	-	-	-	13	-	-	-	-	20	-	27	-	-	32	-	-	-	-	-	132	224
P1	15	23	-	16	-	-	-	-	16	-	-	30	-	25	-	-	-	-	17	-	x	x	142
P2	-	-	19	-	26	-	21	14	-	-	-	-	-	-	-	-	25	27	-	-	x	x	132
Sum:	15	23	19	16	26	13	21	14	16	19	20	30	27	25	26	32	25	27	17	13	142	132	

The total transport cost is $23,640.

Task C2:

To find the optimal shipping plan while port P2 is closed for renovation, we remove P2 from set P and its corresponding parameters, before solving the model again.

The new optimal transport plan is:

#	K 1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15	K16	K17	K18	K19	K20	P1	P2	Sum:
R1	-	-	-	-	-	-	-	-	-	19	-	-	-	-	26	-	-	-	-	13	142	-	200
R2	-	-	-	-	26	13	21	14	-	-	20	-	27	-	-	32	25	27	-	-	19	-	224
P1	15	23	19	16	-	-	-	-	16	-	-	30	-	25	-	-	-	-	17	-	x	x	161
P2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	x	x	-
Sum:	15	23	19	16	26	13	21	14	16	19	20	30	27	25	26	32	25	27	17	13	161	-	

With a total transport cost of $28.663.

By closing P2, the problem is further restricted. Consequently, the 132 tons of bananas that previously were shipped through P2 are now distributed directly though the more expensive routes from R2 to the markets (except for 19 tons going to K3 via P1) and the total cost has increased by $5.023.

Only the bananas previously sent from R2 is affected, because the original solution did not send any bananas from R1 to P2.

Task C3:

C3.a:

We modify the original model as follows:

We remove the decision variable x_{rk} and parameter C_{rk} from the objective function and constraints, as shipments directly from regions to markets are no longer possible.

The inflow to each port now must equal 1.01 times the outflow, to account for losses due to failed quality control:

$$\sum_{r \in R} x_{rp} = 1.01 \sum_{k \in K} x_{pk}, \qquad \forall p \in P$$

Solving the new model gives the following optimal transport plan:

#	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15	K16	K17	K18	K19	K20	P1	P2	Sum:
R1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	200	0	200
R2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2	226.24	228
P1	15	23	-	16	-	-	-	-	16	19	-	30	-	25	26	-	-	-	17	13	x	x	200
P2	-	-	19	-	26	13	21	14	-	-	20	-	27	-	-	32	25	27	-	-	x	x	224
Sum:	15	23	19	16	26	13	21	14	16	19	20	30	27	25	26	32	25	27	17	13	202	226	

The new total transport cost is \$25.351,4, an increase of \$1.711,4.

An increase in total cost is expected as the problem is further restricted, and the total shipment volume from the regions increases by 1%.

C3.b:

We add a new constraint, specifying that the inflow to each port cannot exceed the inspection capacity for each port:

$$\sum_{r \in R} x_{rp} \leq IC_p, \qquad \forall p \in P$$

Where IC_p = the inspection capacity at port p, $p \in P$, is a new parameter added to the model. (175 for P1 and 275 for P2).

Solving the problem gives the following new cost-minimizing transportation plan:

#	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15	K16	K17	K18	K19	K20	P1	P2	Sum:
R1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	175	3.24	178
R2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	250	250
P1	15	23	-	16	-	-	-	-	16	19	-	30	-	-	26	-	-	-	17	11.3	x	x	173
P2	-	-	19	-	26	13	21	14	-	-	20	-	27	25	-	32	25	27	-	1.7	x	x	251
Sum:	15	23	19	16	26	13	21	14	16	19	20	30	27	25	26	32	25	27	17	13	175	253	

The total transport cost has increased from \$25.351,4 to \$26.307,2, which is as expected as the problem was further restricted.

The new binding constraints are the inspection capacity at P1 and supply of bananas from R2. Market K14 is now fully supplied by P2 instead of P1. Market K20 is now partly supplied by P2 instead of just P1.

Part D:

For improved readability, we will use the following abbreviations:

$$WI = sumproduct\ of\ weighted\ inputs = \sum_{m \in M} w_m\, x_{m\, j_0}$$

$$WO = \ sumproduct\ of\ weighted\ inputs = \sum_{n \in N} u_n\, y_{n\, j_0}$$

Task D1:

The recommendations for LJI are based on a *data envelopment analysis* (DEA), which compares efficiency across stations. In this instance, the efficiency is based on each station's ability to produce output (IRF and VIF metrics) from their input (staff count, staff hours and test scores) and is measured by the ratio of *scaled* WI to WO: [1]

$$eff. = \frac{WO}{WI - v} = \frac{1}{objective\ function\ value}$$

Where $WI - v$ (the objective function in the LP model presented in Table A.1) is minimized for each station, subject to the constraints that the stations WO equals 1, that $WI - v$ cannot be greater than the WO when using the chosen weights for every other station, and that all weights cannot be negative. By doing this, we are effectively using linear programming to maximize the output-to-input ratio for each station, to enable a relative comparison. If a station gets an efficiency score of less than one, it implies that it is *inefficient* compared to its peers.

By conducting a DEA, the authors can give LJI recommendations for which stations are inefficient, what outputs have the biggest room for improvement, and how many additional percentage points the output needs to reach the "best-practice frontier" for their relevant input level.

Task D2:

A model is infeasible if no solution simultaneously satisfies all constraints. We will investigate infeasibility by discussing each constraint in turn:

The first constraint states that WO for the station in question must equal one. This can always be achieved with an appropriate choice of u_n, as long as at least one $y_{n_{j_0}}$ is positive and different from zero.

[1] The formula is gathered from looking at the R-code provided by the authors (link) and deaR documentation

The second constraint states that $WI - v$ must be larger than WO, and is further restrictive as the chosen u_n, w_m and v must satisfy the constraint not only for the station in question, j_0, but for all stations in J. Mathematically, this constraint can always be satisfied as the scale factor v is free (it can be set sufficiently negative to balance out the other terms in the LHS regardless of their values, including if all input parameters are negative)

To summarize, the model can only be infeasible if all outputs are exactly zero or negative. Hence infeasibility is possible, but practically unlikely, as long as LJI ensures proper training to avoid severe human error in data entry.

Task D3:

A model is unbounded if the objective function value can be improved without limit.

With the scale factor v as a negative term in the second constraint, it can only be set as high as the difference between WI and WO. However, if v is changed into a positive term, the model is no longer bounded, as the value of v can be set infinitely high, giving the objective function an infinitely negative value.

Task D4:

To ensure that all output weights fall within a constant α, we add the following new constraint to the model:

$$u_n - u_{n^*} \leq \alpha, \qquad \forall n, n^* \in N, \qquad n \neq n^*$$

This constraint iterates through all existing weight pairs in the set, checking that no difference between them can exceed α.