Домашнее задание на 13.11 (Линейная алгебра)

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1. $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \cdot \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$, A_1, B_1, C_1, D_1?

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \cdot \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} AA_1 + BC_1 & AB_1 + BD_1 \\ CC_1 & CD_1 \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} AA_1 + BC_1 = E \\ AB_1 + BD_1 = 0 \\ CC_1 = 0 \Rightarrow C_1 = 0 \\ CD_1 = E \Rightarrow D_1 = C^{-1}E \end{cases} \Rightarrow \begin{cases} AA_1 = E \\ AB_1 + BC^{-1} = 0 \\ C_1 = 0 \\ D_1 = C^{-1} \end{cases} \Rightarrow \begin{cases} A_1 = A^{-1} \\ B_1 = -A^{-1}BC^{-1} \\ C_1 = 0 \\ D_1 = C^{-1} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

Ответ: $\begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$

2. $\frac{(5+i)(7-6i)}{3+i}$

$$\frac{(5+i)(7-6i)}{3+i} = \frac{35+7i-30i-6i^2}{3+i} = \frac{35-23i+6}{3+i} = \frac{41-23i}{3+i} = \frac{(41-23i)(3-i)}{9+1} =$$

$$= \frac{123-69i-41i-23}{10} = \frac{100-110i}{10} = 10-11i$$

Ответ: $10-11i$

3. 1)

$$\begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix} = (a+bi)(a-bi)-(c+di)(-c+di) = a^2+b^2-(-d^2-c^2) =$$

$$= a^2+b^2+d^2+c^2$$

2)

$$\begin{vmatrix} \cos\alpha + i\sin\alpha & 1 \\ 1 & \cos\alpha - i\sin\alpha \end{vmatrix} = (\cos\alpha+i\sin\alpha)(\cos\alpha-i\sin\alpha)-1 =$$

$$= \cos^2\alpha + \sin^2\alpha - 1 = 1 - 1 = 0$$

3)

$$\begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 0 & -i & 1-(1-i^2) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 0 & -i & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 0 & 0 & i^2-1 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 0 & 0 & -2 \end{vmatrix} = -2$$

4.

$$\begin{cases} (1+i)z_1 + (1-i)z_2 = 1+i \\ (1-i)z_1 + (1+i)z_2 = 1+3i \end{cases}$$

Запишем в виде матрицы СЛУ:

$$A = \begin{pmatrix} 1+i & 1-i & | & 1+i \\ 1-i & 1+i & | & 1+3i \end{pmatrix}$$

$$A_{(1)} = A_{(1)} + A_{(2)} \to \begin{pmatrix} 2 & 2 & | & 2+4i \\ 1-i & 1+i & | & 1+3i \end{pmatrix}$$

$$A_{(1)} = A_{(1)}/2 \to \begin{pmatrix} 1 & 1 & | & 1+2i \\ 1-i & 1+i & | & 1+3i \end{pmatrix}$$

$$A_{(2)} = A_{(2)} - A_{(1)} * (1-i) \to \begin{pmatrix} 1 & 1 & | & 1+2i \\ 0 & 2i & | & -2+2i \end{pmatrix}$$

$$A_{(2)} = A_{(2)}/(2i) \to \begin{pmatrix} 1 & 1 & | & 1+2i \\ 0 & 1 & | & 1+i \end{pmatrix}$$

$$A_{(1)} = A_{(1)} - A_{(2)} \to \begin{pmatrix} 1 & 0 & | & i \\ 0 & 1 & | & 1+i \end{pmatrix}$$

Ответ: $\begin{cases} z_1 = i \\ z_2 = 1+i \end{cases}$

5.

$$A = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \Rightarrow detA = (1+i)^2 - (1-i)^2 = 4i$$

$$A_{z_1} = \begin{pmatrix} 1+i & 1-i \\ 1+3i & 1+i \end{pmatrix} \Rightarrow detA_{z_1} = (1+i)^2 - (1-i)(1+3i) = -4$$

$$A_{z_2} = \begin{pmatrix} 1+i & 1+i \\ 1-i & 1+3i \end{pmatrix} \Rightarrow detA_{z_2} = (1+i)(1+3i) - (1-i^2) = -4+4i$$

$$z_1 = \frac{detA_{z_1}}{detA} = \frac{-4}{4i} = -\frac{1}{i} = -\frac{i}{-1} = i$$

$$z_2 = \frac{detA_{z_2}}{detA} = \frac{-4+4i}{4i} = -\frac{1}{i} + 1 = i+1$$

Ответ: $\begin{cases} z_1 = i \\ z_2 = 1+i \end{cases}$

6. 1)

$$-3i = 3(0-i) = 3(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}))$$

2)

$$1+\frac{\sqrt{3}}{3}i=\frac{2}{\sqrt{3}}(\frac{\sqrt{3}}{2}+\frac{1}{2}i)=\frac{2}{\sqrt{3}}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$$

3)

$$\frac{\cos\varphi+i\sin\varphi}{\cos\psi+i\sin\psi}=\cos(\varphi-\psi)+i\sin(\varphi-\psi)$$

7. $(\sqrt{3}-i)^{32}$

$$(\sqrt{3}-i)^{32}=(2(\frac{\sqrt{3}}{2}-\frac{1}{2}i))^{32}=(2(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}))^{32}=2^{32}(\cos\frac{32\pi}{6}-i\sin\frac{32\pi}{6})=$$

$$=2^{32}(\cos(5\pi+\frac{\pi}{3})-i\sin(5\pi+\frac{\pi}{3}))=-2^{32}(\cos(\frac{\pi}{3})-i\sin(\frac{\pi}{3}))=$$

$$=-2^{32}(\frac{1}{2}-i\frac{\sqrt{3}}{2})=-2^{31}+i2^{31}\sqrt{3}$$

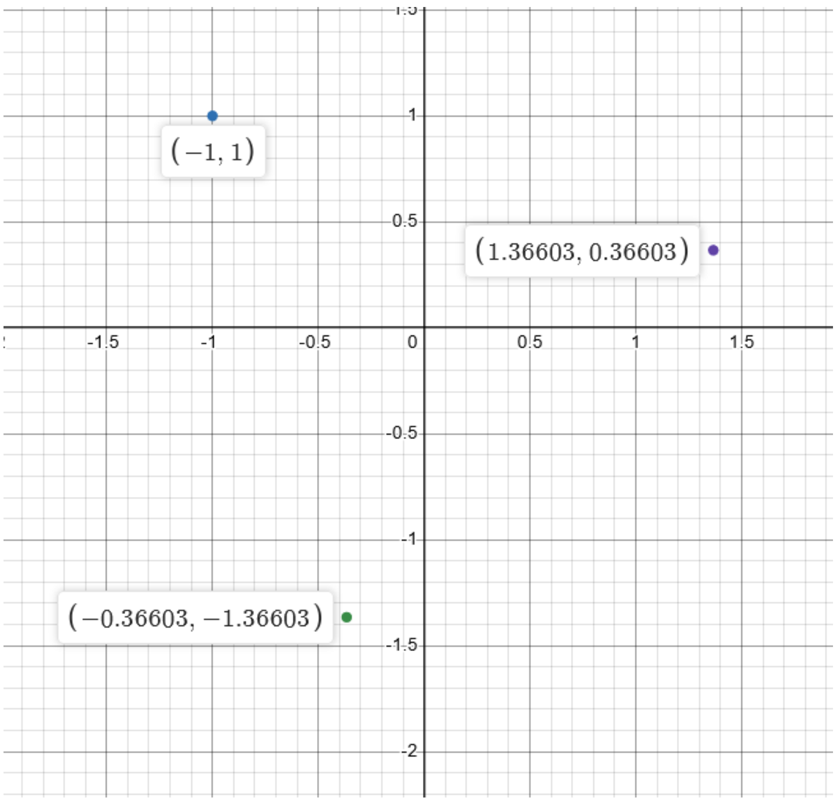
Ответ: $-2^{32}(\cos(\frac{\pi}{3})-i\sin(\frac{\pi}{3}))$ и $-2^{31}+i2^{31}\sqrt{3}$

8. z_1

$$z_1^3=2-2i=2\sqrt{2}(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)=2\sqrt{2}(\cos(\frac{\pi}{4}+2\pi k)-i\sin(\frac{\pi}{4}+2\pi k)),k\in\mathbb{Z}\Rightarrow$$

$$\Rightarrow z_1=\sqrt[3]{2\sqrt{2}}(\cos(\frac{\pi}{12}+\frac{2\pi k}{3})-i\sin(\frac{\pi}{12}+\frac{2\pi k}{3}))=$$

$$=\sqrt{2}(\cos(\frac{\pi}{12}+\frac{2\pi k}{3})-i\sin(\frac{\pi}{12}+\frac{2\pi k}{3})),k=0,1,2$$

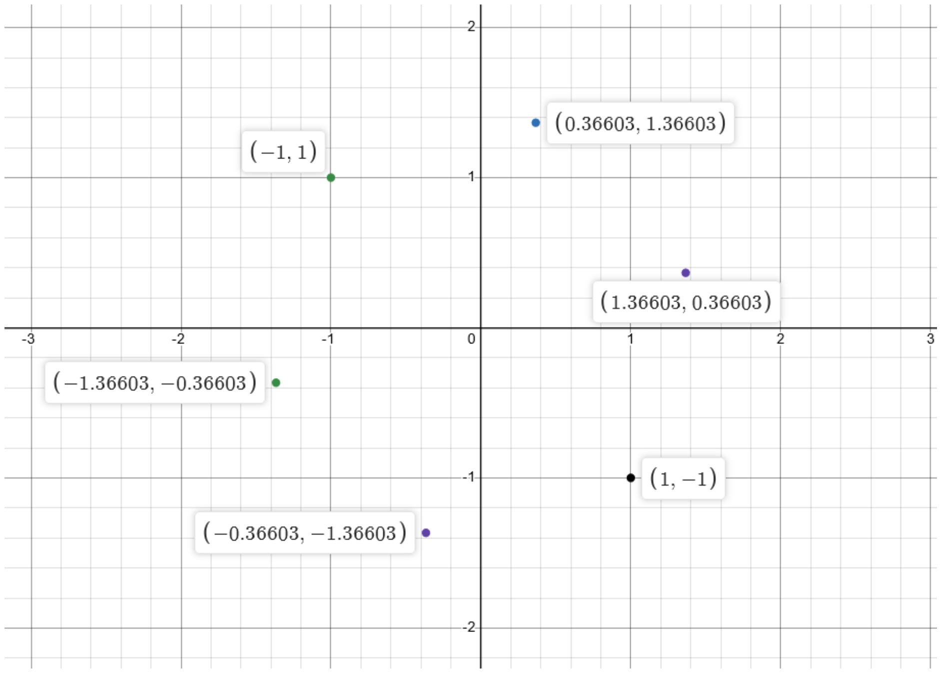


z_2

$$z_2^6 = (2-2i)^2 = 4-8i-4 = -8i = 8(0-i) = 8(\cos(\frac{\pi}{2}+2\pi n)-i\sin(\frac{\pi}{2}+2\pi n)), n \in Z \Rightarrow$$

$$\Rightarrow z_2 = \sqrt[6]{2^3}(\cos(\frac{\pi}{12}+\frac{\pi n}{3}) - i\sin(\frac{\pi}{12}+\frac{\pi n}{3})) =$$

$$= \sqrt{2}(\cos(\frac{\pi}{12}+\frac{\pi n}{3}) - i\sin(\frac{\pi}{12}+\frac{\pi n}{3})), n = 0, 1, 2, 3, 4, 5$$



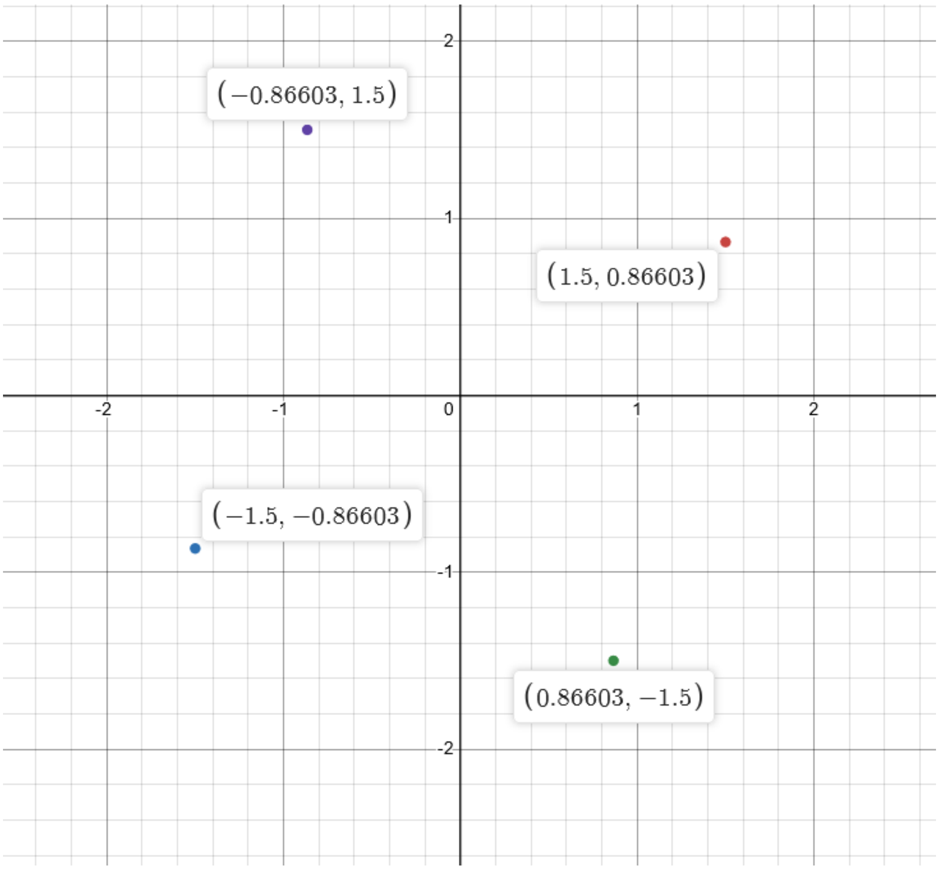
9. $\sqrt[4]{\frac{-18}{1+i\sqrt{3}}} = z$

$$z^4 = \frac{-18}{1+i\sqrt{3}} = \frac{-18(1-i\sqrt{3})}{4} = \frac{-18+18i\sqrt{3}}{4} = \frac{-9+9i\sqrt{3}}{2} = 9(-\frac{1}{2}+i\frac{\sqrt{3}}{2}) =$$

$$= 9(cos(\frac{2\pi}{3}+2\pi k) + i\sin(\frac{2\pi}{3}+2\pi k)), k \in \mathbb{Z}$$

$$z = \sqrt{3}(cos(\frac{\pi}{6}+\frac{\pi k}{2}) + i\sin(\frac{\pi}{6}+\frac{\pi k}{2})), k = 0, 1, 2, 3$$

$$\begin{cases} k=0: z=\sqrt{3}(cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})) = \frac{3}{2}+i\frac{\sqrt{3}}{2} \\ k=1: z=\sqrt{3}(cos(\frac{\pi}{6}+\frac{\pi}{2})+i\sin(\frac{\pi}{6}+\frac{\pi}{2})) = -\frac{\sqrt{3}}{2}+i\frac{3}{2} \\ k=2: z=\sqrt{3}(cos(\frac{\pi}{6}+\pi)+i\sin(\frac{\pi}{6}+\pi)) = -\frac{3}{2}-i\frac{\sqrt{3}}{2} \\ k=3: z=\sqrt{3}(cos(\frac{\pi}{6}+\frac{3\pi}{2})+i\sin(\frac{\pi}{6}+\frac{3\pi}{2})) = \frac{\sqrt{3}}{2}-i\frac{3}{2} \end{cases}$$



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1) $(2\sqrt{3}-i)=0:$

$$0 = 10-6\sqrt{3}i \ \ \varnothing$$

2) $(2\sqrt{3}-i)\neq 0:$

$$z^4 = \frac{10-6\sqrt{3}i}{2\sqrt{3}-i} = \frac{(10-6\sqrt{3}i)(2\sqrt{3}+i)}{13} = \frac{26\sqrt{3}-26i}{13} = 2\sqrt{3}-2i =$$

$$= 4\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) = 4\left(\cos\left(\frac{\pi}{6}+2\pi k\right) - i\sin\left(\frac{\pi}{6}+2\pi k\right)\right), k\in\mathbb{Z} \Rightarrow$$

$$\Rightarrow z = \sqrt{2}(\cos(\frac{\pi}{24} + \frac{\pi k}{2}) - i\sin(\frac{\pi}{24} + \frac{\pi k}{2}))$$

Найдём k при которых аргумент решения принадлежит $(\frac{5\pi}{2}, 3\pi)$:

$$\frac{5\pi}{2} < \frac{\pi}{24} + \frac{\pi k}{2} < 3\pi$$

$$\frac{5\pi}{2} - \frac{\pi}{24} < \frac{\pi k}{2} < 3\pi - \frac{\pi}{24}$$

$$\frac{59\pi}{24} < \frac{\pi k}{2} < \frac{71\pi}{24}$$

$$\frac{59}{12} < k < \frac{71}{12}$$

$$4\frac{11}{12} < k < 5\frac{11}{12} \Rightarrow k = 5, \text{ т.к. } k \in \mathbb{Z}$$

$$k = 5 : z = \sqrt{2}(\cos(\frac{\pi}{24} + \frac{5\pi}{2}) - i\sin(\frac{\pi}{24} + \frac{5\pi}{2})) = \sqrt{2}(\cos(\frac{61\pi}{24}) - i\sin(\frac{61\pi}{24}))$$

Ответ: $\sqrt{2}(\cos(\frac{61\pi}{24}) - i\sin(\frac{61\pi}{24}))$

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$$(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k = \binom{n}{0} + \binom{n}{1} i - \binom{n}{2} - \binom{n}{3} i + \dots$$

$$i\sum_{k=1}^{[\frac{n}{2}]} (-1)^{k-1} \binom{n}{2k-1} = \binom{n}{1} i - \binom{n}{3} i + \binom{n}{5} i$$

$$C_n^0 - C_n^2$$