# Домашнее задание на 24.09.2024 (Математический анализ)

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**1.** (a)

$$\lim_{n\to\infty}\frac{n^2+6}{n^2-10n+26}=\lim_{n\to\infty}\frac{1+\frac{6}{n^2}}{1-\frac{10}{n}+\frac{26}{n^2}}$$

$$\frac{6}{n^2}\to 0, \frac{10}{n}\to 0, \frac{26}{n^2}\to 0 \Rightarrow$$

$$\lim_{n\to\infty}\frac{1+\frac{6}{n^2}}{1-\frac{10}{n}+\frac{26}{n^2}}=\frac{1+0}{1-0+0}=1$$

Докажем, что:

$$\forall\varepsilon>0, N=\left[\frac{10}{\varepsilon}+9\right], \forall n\in\mathbb{N}, [n\geqslant N: |a_n-a|<\varepsilon]$$

$$\left|\frac{n^2+6}{n^2-10n+26}-1\right|=\left|\frac{10n-20}{n^2-10n+26}\right|\leqslant$$

$$\frac{10n-10}{n^2-10n+9}=10\frac{n-1}{(n-1)(n-9)}=\frac{10}{n-9}\leqslant\frac{10}{N-9}<\varepsilon\Rightarrow$$

$$\Rightarrow N>\frac{10}{\varepsilon}+9\Rightarrow N=\left[\frac{10}{\varepsilon}+9\right]$$

**Ответ:** 1, $N=\left[\frac{10}{\varepsilon}+9\right]$

(b)

$$\lim_{n\to\infty}(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2})$$

$$\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}=\sum_{k=1}^{n-1}\frac{k}{n^2}=\frac{1}{n^2}\cdot\sum_{k=1}^{n-1}k=$$

$$=\frac{1}{n^2}\cdot\frac{1+(n-1)}{2}(n-1)=\frac{1}{n^2}\cdot\frac{n}{2}(n-1)=\frac{n-1}{2n}$$

$$\lim_{n\to\infty}(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2})=\lim_{n\to\infty}(\frac{n-1}{2n})=$$

$$=\lim_{n\to\infty}(\frac{1-\frac{1}{n}}{2})=\frac{1}{2}$$

Докажем, что:

$$\forall\varepsilon>0, N=\left[\frac{1}{2\varepsilon}\right], \forall n\in\mathbb{N}, [n\geqslant N : |a_n-a|<\varepsilon]$$

$$\left|\frac{n-1}{2n}-\frac{1}{2}\right|=\left|-\frac{1}{2n}\right|\leqslant\frac{1}{2n}\leqslant\frac{1}{2N}<\varepsilon\Rightarrow$$

$$\Rightarrow N>\frac{1}{2\varepsilon}\Rightarrow N=\left[\frac{1}{2\varepsilon}\right]$$

**Ответ:** $\frac{1}{2}$, $N=\left[\frac{1}{2\varepsilon}\right]$

**2** (a)

$$\lim_{n\to\infty}\frac{5^n+n3^n+n^{10}}{3^{n+7}+n^{100}+5^{n+1}}=\lim_{n\to\infty}\frac{1+\frac{n3^n}{5^n}+\frac{n^{10}}{5^n}}{\frac{3^{n+7}}{5^n}+\frac{n^{100}}{5^n}+5}=\frac{1+0+0}{0+0+5}=\frac{1}{5}$$

**Ответ:** $\frac{1}{5}$

(b)

$$\lim_{n\to\infty}\frac{2n-\sqrt{4n^2-1}}{\sqrt{n^2+3}-n}=\lim_{n\to\infty}\frac{\sqrt{4n^2}-\sqrt{4n^2-1}}{\sqrt{n^2+3}-\sqrt{n^2}}=$$

$$=\lim_{n\to\infty}\frac{(\sqrt{4n^2}-\sqrt{4n^2-1})(\sqrt{n^2+3}+\sqrt{n^2})}{n^2+3-n^2}=$$

$$=\lim_{n\to\infty}\frac{(4n^2-4n^2+1)(n+\sqrt{n^2+3})}{3(\sqrt{4n^2}+\sqrt{4n^2-1})}=\lim_{n\to\infty}\frac{n+\sqrt{n^2+3}}{3(2n+\sqrt{4n^2-1})}=$$

$$=\lim_{n\to\infty}\frac{n+\sqrt{n^2+3}}{6n+3\sqrt{4n^2-1}}=\lim_{n\to\infty}\frac{1+\sqrt{1+\frac{3}{n^2}}}{6+3\sqrt{4-\frac{1}{n^2}}}=\frac{1+\sqrt{1+0}}{6+3\sqrt{4-0}}=$$

$$=\frac{2}{6+3\cdot 2}=\frac{2}{12}=\frac{1}{6}$$

**Ответ:** $\frac{1}{6}$

(c)

$$\lim_{n\to\infty} sin(\frac{n2^n}{n!+1})$$

$$\frac{n2^n}{n!+1}>0\Rightarrow sin(\frac{n2^n}{n!+1})\leqslant\frac{n2^n}{n!+1}$$

$$0\leqslant sin(\frac{n2^n}{n!+1})\leqslant\frac{n2^n}{n!+1}$$

По теореме о зажатой последовательности:

$$\begin{cases}\lim_{n\to\infty}\frac{n2^n}{n!+1}=\lim_{n\to\infty}\frac{n\frac{2^n}{n!}}{1+\frac{1}{n!}}=\frac{0}{1+0}=0\\ \lim_{n\to\infty}0=0\end{cases}\Rightarrow\lim_{n\to\infty} sin(\frac{n2^n}{n!+1})=0$$

**Ответ:** 0

(d)

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^2+4^n}{n+5^n}}$$

$$\sqrt[n]{\frac{n^2+4^n}{n+5^n}} \leqslant \sqrt[n]{\frac{4^n+4^n}{5^n}} = \sqrt[n]{\frac{2\cdot 4^n}{5^n}} = \frac{4}{5}\sqrt[n]{2}$$

$$\sqrt[n]{\frac{n^2+4^n}{n+5^n}} \geqslant \sqrt[n]{\frac{4^n}{5^n+5^n}} = \sqrt[n]{\frac{4^n}{2\cdot 5^n}} = \frac{4}{5}\sqrt[n]{\frac{1}{2}}$$

$$\frac{4}{5}\sqrt[n]{\frac{1}{2}} \leqslant \sqrt[n]{\frac{n^2+4^n}{n+5^n}} \leqslant \frac{4}{5}\sqrt[n]{2} \Rightarrow$$

$\Rightarrow$ По теореме о зажатой последовательности:

$$\begin{cases} \lim_{n\to\infty} \frac{4}{5}\sqrt[n]{2} = \frac{4}{5} \\ \lim_{n\to\infty} \frac{4}{5}\sqrt[n]{\frac{1}{2}} = \frac{4}{5} \end{cases} \Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{n^2+4^n}{n+5^n}} = \frac{4}{5}$$

**Ответ:** $\frac{4}{5}$

(e)

$$\lim_{n\to\infty} \frac{1!+2!+3!+\cdots+n!}{n!}$$

$$\frac{1!+2!+3!+\cdots+n!}{n!} \leqslant \frac{(n-2)\cdot(n-2)!+(n-1)!+n!}{n!} =$$

$$= \frac{n-2}{n(n-1)} + \frac{1}{n} + 1 = \frac{n-2+n-1+n^2-n}{n^2-n} = \frac{n^2+n-3}{n^2-n} =$$

$$= \frac{1+\frac{1}{n}-\frac{3}{n^2}}{1-\frac{1}{n}}$$

$$1 \leqslant \frac{1!+2!+3!+\cdots+n!}{n!} \leqslant \frac{1+\frac{1}{n}-\frac{3}{n^2}}{1-\frac{1}{n}} \Rightarrow$$

$\Rightarrow$ По теореме о зажатой последовательности:

$$\begin{cases} \lim_{n\to\infty} 1 = 1 \\ \lim_{n\to\infty} \frac{1+\frac{1}{n}-\frac{3}{n^2}}{1-\frac{1}{n}} = \frac{1+0+0}{1-0} = 1 \end{cases} \quad \Rightarrow \lim_{n\to\infty} \frac{1!+2!+3!+\dots+n!}{n!} = 1$$

**Ответ:** 1

(f)

$$\lim_{n\to\infty} \frac{n!\cdot n^n}{(3n)!}$$

$$\frac{n!\cdot n^n}{(3n)!} = \frac{n!\cdot n^n}{n!\cdot(n+1)\dots(2n)\cdot(2n+1)\dots(3n)} =$$

$$= \frac{n^n}{(n+1)\dots(n+n)\cdot(2n+1)\dots(2n+n)} \leqslant \frac{n^n}{n^n\cdot(2n)^n} = \frac{1}{2^n n^n}$$

$$0 \leqslant \frac{n!\cdot n^n}{(3n)!} \leqslant \frac{1}{2^n n^n} \Rightarrow$$

$\Rightarrow$ По теореме о зажатой последовательности:

$$\begin{cases} \lim_{n\to\infty} 0 = 0 \\ \lim_{n\to\infty} \frac{1}{2^n n^n} = 0 \end{cases} \quad \Rightarrow \lim_{n\to\infty} \frac{n!\cdot n^n}{(3n)!} = 0$$

**Ответ:** 0

(g)

$$\lim_{n\to\infty} sin(\pi\sqrt[3]{n^3+1})$$

$$sin(\pi\sqrt[3]{n^3+1}) = (-1)^n sin(\pi\sqrt[3]{n^3+1} - \pi n) =$$

$$= (-1)^n sin(\pi(\sqrt[3]{n^3+1} - n)) = (-1)^n sin(\pi(\sqrt[3]{n^3+1} - \sqrt[3]{n^3})) =$$

$$= (-1)^n sin(\pi \frac{n^3+1-n^3}{\sqrt[3]{n^3+1}^2 + \sqrt[3]{(n^3+1)n^3} + \sqrt[3]{n^3}^2}) =$$

$$= (-1)^n sin(\pi \frac{1}{\sqrt[3]{n^3+1}^2 + n\sqrt[3]{(n^3+1)} + n^2})$$

$$\lim_{n\to\infty} \pi \frac{1}{\sqrt[3]{n^3+1}^2 + n\sqrt[3]{(n^3+1)} + n^2} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} sin(\pi \frac{1}{\sqrt[3]{n^3+1}^2 + n\sqrt[3]{(n^3+1)} + n^2}) = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} (-1)^n sin(\pi \frac{1}{\sqrt[3]{n^3+1}^2 + n\sqrt[3]{(n^3+1)} + n^2}) = 0 =$$

$$= \lim_{n\to\infty} sin(\pi \sqrt[3]{n^3+1})$$

**Ответ:** 0