Домашнее задание на 12.11 (Математический анализ)

Емельянов Владимир, ПМИ гр №247

1. **(a)** $f \in \overline{o}(f)$, при $x \to a \Rightarrow$

$$\Rightarrow \exists \gamma : \begin{cases} f(x) = \gamma(x) f(x) \\ \gamma(x) \to 0 \end{cases}$$

Пусть $f(x) = 2^x \Rightarrow \gamma(x) = \frac{f(x)}{f(x)} = 1$ (т.к. $f(x) \neq 0$) $\Rightarrow$

$$\Rightarrow \begin{cases} \gamma(x) = 1 \\ \gamma(x) \to 0 \end{cases} \quad \varnothing$$

Ответ: неверно

(b) $f \in \underline{O}(f)$, при $x \to a$

$$\Rightarrow \exists \gamma : \begin{cases} f(x) = \gamma(x) f(x) \\ \gamma(x) \in \mathring{U}(a) \end{cases}$$

Пусть $f(x) = 2^x$ и $a = 1$

$$\neg\exists \gamma : \begin{cases} 2^x = \gamma(x) 2^x \Rightarrow \gamma(x) = 0 \\ \gamma(x) \in \mathring{U}(0) \Rightarrow \gamma(x) \neq 0 \end{cases} \quad \varnothing$$

Ответ: неверно

(c) $f \cdot \overline{o}(g) = \overline{o}(f \cdot g)$, при $x \to a$

$$h \in (f \cdot \overline{o}(g)) \Rightarrow \exists \gamma(x) : \begin{cases} h(x) = f(x)\gamma(x)g(x) \\ \gamma(x) \to 0 \end{cases} \quad \Rightarrow$$

$$\Rightarrow \begin{cases} h(x) = \gamma(x) f(x) g(x) \\ \gamma(x) \to 0 \end{cases} \quad \Rightarrow h(x) \in \overline{o}(f \cdot g) \text{ по опр.}$$

Ответ: верно

(d) $\underline{O}(\overline{o}(f)) = \underline{O}(f)$, при $x \to a$

$$g(x) \in (\underline{O}(\overline{o}(f))) \Rightarrow \exists h \exists \gamma_1 : \begin{cases} h(x) \in \overline{o}(f) \\ g(x) = \gamma_1(x)h(x) \\ \gamma_1(x) \in \mathring{U}(a) \end{cases} \Rightarrow$$

$$\Rightarrow \exists h, \gamma_1, \gamma_2 : \begin{cases} h(x) = \gamma_2(x)f(x) \\ \gamma_2(x) \to 0 \\ g(x) = \gamma_1(x)h(x) \\ \gamma_1(x) \in \mathring{U}(a) \end{cases} \Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} g(x) = \gamma_1(x)\gamma_2(x)f(x) \\ \gamma_2(x) \to 0 \\ \gamma_1(x) \in \mathring{U}(a) \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_3 : \begin{cases} g(x) = \gamma_3 f(x) \\ \gamma_3 = \gamma_2(x)\gamma_1(x) \to 0 \end{cases} \Rightarrow g(x) \in \overline{o} \text{ по опр.} \Rightarrow g(x) \in \underline{O}$$

Ответ: верно

(e) $\overline{o}(f) + \underline{O}(f) = \overline{o}(f)$, при $x \to a$

$$g(x) \in (\overline{o}(f)+\underline{O}(f)) \Rightarrow \exists h_1, h_2, \gamma_1, \gamma_2 : \begin{cases} h_1(x) = \gamma_1(x)f(x) \\ \gamma_1(x) \to 0 \\ h_2(x) = \gamma_2(x)f(x) \\ \gamma_2(x) \in \mathring{U}(a) \\ g(x) = h_1(x) + h_2(x) \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} \gamma_1(x) \to 0 \\ \gamma_2(x) \in \mathring{U}(a) \\ g(x) = \gamma_1(x)f(x) + \gamma_2(x)f(x) \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} \gamma_1(x) \to 0 \\ \gamma_2(x) \in \mathring{U}(a) \\ g(x) = (\gamma_1(x) + \gamma_2(x))f(x) \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_3, \gamma_1, \gamma_2 : \begin{cases} \gamma_3(x) = \gamma_1(x) + \gamma_2(x) \to \gamma_2(x) \in \mathring{U}(a) \\ \gamma_1(x) \to 0 \\ \gamma_2(x) \in \mathring{U}(a) \\ g(x) = \gamma_3(x)f(x) \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_3 : \begin{cases} \gamma_3(x) \in \mathring{U}(a) \\ g(x) = \gamma_3(x)f(x) \end{cases} \Rightarrow g(x) \in \underline{O}(f) \Rightarrow \text{ не всегда } g(x) \notin \overline{o}(f)$$

Пусть $f(x) = 2^x$, $g(x) = x \Rightarrow g(x) \in \underline{O}(f)$, так как $\frac{x}{2^x} \in \mathring{U}(a)$, но $\lim\limits_{x\to a} \frac{x}{2^x} \neq 0 \Rightarrow g(x) \notin \overline{o}(a)$

Ответ: неверно

(f) $\overline{o}(f + \underline{O}(f)) = \overline{o}(f)$, при $x \to a$

$$g \in \overline{o}(f + \underline{O}(f)) \Rightarrow \exists \gamma_1 : \begin{cases} g = \gamma_1 \cdot (f + \underline{O}(f)) \\ \gamma_1 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1 : \begin{cases} g = \gamma_1 \cdot f + \gamma_1 \cdot \underline{O}(f) \\ \gamma_1 \to 0 \end{cases} \Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} g = \gamma_1 \cdot f + \gamma_1 \cdot \gamma_2 \cdot f \\ \gamma_2 \in \mathring{U}(a) \\ \gamma_1 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} g = (\gamma_1 + \gamma_1 \cdot \gamma_2) f \\ \gamma_2 \in \mathring{U}(a) \\ \gamma_1 \to 0 \end{cases} \Rightarrow \exists \gamma_1, \gamma_2, \gamma_3 : \begin{cases} g = \gamma_3 \cdot f \\ \gamma_3 = \gamma_1 + \gamma_1 \cdot \gamma_2 \to 0 \\ \gamma_2 \in \mathring{U}(a) \\ \gamma_1 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow g \in \overline{o}(f)$$

Ответ: верно

(g) $\overline{o}(f) + \overline{o}(g) = \overline{o}(f + g)$ при $x \to a$

$$h \in \overline{o}(f) + \overline{o}(g) \Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} h = \gamma_1 \cdot f + \gamma_2 \cdot g \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} h = \gamma_1 \cdot f + \gamma_2 \cdot g \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow \exists \gamma_1, \gamma_2, \gamma_3 : \begin{cases} h = \gamma_3 \cdot f + \gamma_3 \cdot g \\ \gamma_3 = \gamma_1 = \gamma_2 \to 0 \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_3 : \begin{cases} h = \gamma_3(f + g) \\ \gamma_3 \to 0 \end{cases} \Rightarrow h \in \overline{o}(f + g)$$

Ответ: верно

(g) $(x + \overline{o}(x)) \cdot (7x^2 + \overline{o}(x^2)) = 7x^3 + \overline{o}(x^3)$ при $x \to 0$

$$g(x) \in (x + \overline{o}(x)) \cdot (7x^2 + \overline{o}(x^2)) \Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} g(x) = (x + \gamma_1(x) \cdot x)(7x^2 + \gamma_2(x) x^2) \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} g(x) = 7x^3 + 7x^3\gamma_1(x) + x^3\gamma_2(x) + x^3\gamma_1(x)\gamma_2(x) \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2 : \begin{cases} g(x) = 7x^3 + (7\gamma_1(x) + \gamma_2(x) + \gamma_1(x)\gamma_2(x))x^3 \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow$$

$$\Rightarrow \exists \gamma_1, \gamma_2, \gamma_3 : \begin{cases} g(x) = 7x^3 + \gamma_3(x)x^3 \\ \gamma_3 = 7\gamma_1(x) + \gamma_2(x) + \gamma_1(x)\gamma_2(x) \to 0 \\ \gamma_1 \to 0 \\ \gamma_2 \to 0 \end{cases} \Rightarrow g \in 7x^3 + \overline{o}(x^3)$$

Ответ: верно

2. (a) $\lim\limits_{x\to 0} \frac{\sqrt[5]{1+2x}-e^x}{\sqrt[4]{1+x}-\cos x}$

$$\lim_{x\to 0} \frac{\sqrt[5]{1+2x}-e^x}{\sqrt[4]{1+x}-\cos x} = \lim_{x\to 0} \frac{1+\frac{2}{5}x+\overline{o}(2x)-1-x-\overline{o}(x)}{1+\frac{1}{4}x+\overline{o}(x)-1+\frac{x^2}{2}+\overline{o}(x^3)} =$$

$$= \lim_{x\to 0} \frac{-\frac{3}{5}x+\overline{o}(x)-\overline{o}(x)}{\frac{x}{4}+\overline{o}(x)+\frac{x^2}{2}+\overline{o}(x^3)} = \lim_{x\to 0} \frac{-\frac{3}{5}x+\overline{o}(x)}{\frac{x}{4}+\frac{x^2}{2}+\overline{o}(x)} = \lim_{x\to 0} \frac{x(-\frac{3}{5}+\overline{o}(1))}{x(\frac{1}{4}+\frac{x}{2}+\overline{o}(1))} =$$

$$= \lim_{x\to 0} \frac{-\frac{3}{5}+\overline{o}(1)}{\frac{1}{4}+\frac{x}{2}+\overline{o}(1)} = \frac{-\frac{3}{5}+0}{\frac{1}{4}+0+0} = -\frac{12}{5}$$

Ответ: $-\frac{12}{5}$

(b) $\lim\limits_{x\to 0} x\left(\frac{1}{1-\sqrt{1+3x}} - \frac{1}{\sin(x)}\right)$

$$\lim_{x\to 0} x\left(\frac{1}{1-\sqrt{1+3x}} - \frac{1}{\sin(x)}\right) = \lim_{x\to 0} x\left(\frac{1}{1-1-\frac{3}{2}x-\overline{o}(3x)} - \frac{1}{x+\overline{o}(x)}\right) =$$

$$= \lim_{x\to 0} x\left(\frac{1}{x(-\frac{3}{2}-\overline{o}(1))} - \frac{1}{x(1+\overline{o}(1))}\right) = \lim_{x\to 0} x\cdot\frac{1}{x}\left(\frac{1}{-\frac{3}{2}-\overline{o}(1)} - \frac{1}{1+\overline{o}(1)}\right) =$$

$$\lim_{x\to 0} \left(\frac{1}{-\frac{3}{2}-\overline{o}(1)} - \frac{1}{1+\overline{o}(1)}\right) = \frac{1}{-\frac{3}{2}-0} - \frac{1}{1+0} = -\frac{2}{3} - 1 = -\frac{5}{3}$$

Ответ: $-\frac{5}{3}$

(c) $\lim\limits_{x\to0}\frac{(1+3x)^{5x}-1}{x^2}$

$$\lim_{x\to0}\frac{(1+3x)^{5x}-1}{x^2}=\lim_{x\to0}\frac{e^{5x\ln(1+3x)}-1}{x^2}=\lim_{x\to0}\frac{e^{5x\ln(1+3x)}-1}{x^2}=$$
$$=\lim_{x\to0}\frac{1+5x\ln(1+3x)+\overline{o}(5x\ln(1+3x))-1}{x^2}=$$
$$=\lim_{x\to0}\frac{5\ln(1+3x)+\overline{o}(5\ln(1+3x))}{x}=\lim_{x\to0}\frac{\ln(1+3x)(5+\overline{o}(5))}{x}=$$
$$=\lim_{x\to0}\frac{(3x+\overline{o}(3x))(5+\overline{o}(5))}{x}=\lim_{x\to0}(3+\overline{o}(3))(5+\overline{o}(5))=(3+0)(5+0)=15$$

Ответ: 15

(d) $\lim\limits_{x\to0}\frac{\arccos(1-x)}{\sqrt{x}}$

$$\lim_{x\to0}\frac{\arccos(1-x)}{\sqrt{x}}=\begin{pmatrix}1-x=\cos t\\ 1-x\to1\\ \cos t\to1\\ t\to0\end{pmatrix}=\lim_{t\to0}\frac{\arccos(\cos t)}{\sqrt{1-\cos t}}=\lim_{t\to0}\frac{t}{\sqrt{1-\cos t}}=$$
$$=\lim_{t\to0}\frac{t}{\sqrt{1-(1-\frac{t^2}{2}+\overline{o}(t^3))}}=\lim_{t\to0}\frac{t}{\sqrt{\frac{t^2}{2}-t^3\overline{o}(1)}}=$$
$$=\lim_{t\to0}\frac{1}{\sqrt{\frac{1}{2}-t\overline{o}(1)}}=\frac{1}{\sqrt{\frac{1}{2}-0}}=\sqrt{2}$$

Ответ: $\sqrt{2}$

(e) $\lim\limits_{x\to+\infty}\frac{\ln(1+\sqrt[3]{x})}{\ln(2+\sqrt[5]{x})}$

$$\lim_{x\to+\infty}\frac{\ln(1+\sqrt[3]{x})}{\ln(2+\sqrt[5]{x})}=\begin{pmatrix}t^{15}=\frac{1}{x}\\ x\to+\infty\\ t\to0^+\end{pmatrix}=\lim_{t\to0^+}\frac{\ln(1+\sqrt[3]{\frac{1}{t^{15}}})}{\ln(2+\sqrt[5]{\frac{1}{t^{15}}})}=$$
$$=\lim_{t\to0^+}\frac{\ln(1+\frac{1}{t^5})}{\ln(2+\frac{1}{t^3})}=\lim_{t\to0^+}\frac{\ln(\frac{1+t^5}{t^5})}{\ln(\frac{2t^3+1}{t^3})}=\lim_{t\to0^+}\ln(\frac{1+t^5}{t^5}-\frac{2t^3+1}{t^3})=$$
$$=\lim_{t\to0^+}\ln(\frac{1+t^5}{t^5}-\frac{2t^3+1}{t^3})=\lim_{t\to0^+}(\ln(1-\frac{2t^3+1}{t^3}\cdot\frac{t^5}{1+t^5})+\ln\frac{1+t^5}{t^5})=$$
$$=\lim_{t\to0^+}(\ln(1-\frac{2t^3+1}{t^3}\cdot\frac{t^5}{1+t^5})+\ln(\frac{t^5}{1+t^5}))$$