Decrease & Conquer: Decrease-by-constant + Variable size decrease

4.4- 4.5

Decrease and Conquer technique

Decrease or reduce problem instance to smaller instance of the same problem and extend solution.

Conquer the problem by solving smaller instance of the problem.

Extend solution of smaller instance to obtain solution to original problem.

Implementations of decrease and conquer

This approach can be either implemented as top-down or bottom-up.

Top-down approach : It always leads to the recursive implementation of the problem.

Bottom-up approach : It is usually implemented in iterative way, starting with a solution to the smallest instance of the problem.

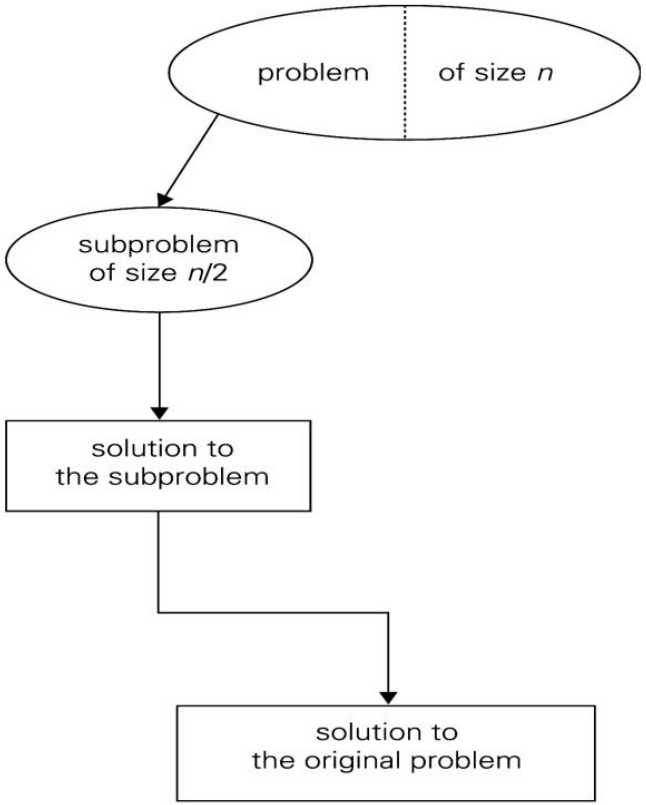
Decrease and Conquer technique

There are three major variations of decrease-and-conquer:

1. Decrease by a constant
    a. Insertion (previous lecture)
    b. Topological (previous lecture)
2. Decrease by a constant factor
    a. Binary
    b. Russian Peasant Multiplication
3. Variable size decrease
    a. Computing a Median and the Selection Problem

Decrease by a constant

This technique suggests reducing a problem instance by the same constant factor on each iteration of the algorithm. In most applications, this constant factor is equal to two. A reduction by a factor other than two is especially rare.



Examples we will discuss today include:

1. Binary Search
2. Russian Peasant Multiplication
3. Computing a Median and the Selection Problem

Example of when we would use Binary Search

Given a sorted array arr[] of n elements, write a function to search a given element x in arr[].

A simple approach is to do a linear search. The worst-case time complexity of the above algorithm is O(n). Another approach to perform the same task is using Binary Search.

Binary Search explanation reminder

Binary Search: Search a sorted array by repeatedly dividing the search interval in half. Begin with an interval covering the whole array. If the value of the search key is less than the item in the middle of the interval, narrow the interval to the lower half. Otherwise, narrow it to the upper half. Repeatedly check until the value is found or the interval is empty.

The worst case time complexity of binary search is O(log n)

Binary Search explanation reminder

We basically ignore half of the elements just after one comparison.

1. Compare x with the middle element.
2. If x matches with the middle element, we return the mid index.
3. Else If x is greater than the mid element, then x can only lie in the right half subarray after the mid element. So we recur for the right half.
4. Else (x is smaller) recur for the left half.

Decrease by a constant factor

It should be fairly straightforward why binary search is an example of decreasing by a constant factor, because each time we are cutting our dataset in half.

Let’s jump to another example that deals with multiplication of a constant factor rather than division

Russian Peasant Multiplication

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- for even values of b: $a \times b = (a \times 2) \times (b/2)$
- for odd values of b :$a \times b = ((a \times 2) \times (b/2) + a)$.

Russian Peasant Multiplication in Words

In the while loop, we keep multiplying ‘a’ with 2 and keep dividing ‘b’ by 2. If ‘b’ becomes odd in loop, we add ‘a’ to ‘res’. When value of ‘b’ becomes 1, the value of ‘res’ + ‘a’, gives us the result.

Russian Peasant Multiplication Pseudocode

Let the two given numbers be 'a' and 'b'

1) Initialize result 'res' as 0.

2) Do following while 'b' is greater than 0

   a) If 'b' is odd, add 'a' to 'res'

   b) Double 'a' and halve 'b'

   3) Return 'res'.

Example of Russian Peasant Multiplication

Think of 2 numbers you need to multiply

We will use 146 and 37, that is, 146 x 37

Example of Russian Peasant Multiplication

Make two columns. Using a piece of paper and pen, divide the piece of paper into two columns by drawing a line down the middle of the paper. Write one of the numbers you want to multiply at the top the each column.

- In this example, write “146” at the top of the left column, and “37” at the top of the right column.

Make two columns

146

37

Example of Russian Peasant Multiplication

Halve the number in the left column repeatedly. Divide the number at the top of the left column by 2 continually until you get to 1. Ignore any remainder each time you halve the number. Write each halved number down in the left column, in order.

Example of Russian Peasant Multiplication

In this example:

- Start by halving 146 (146 ÷ 2 = 73). Write “73” in the left column below “146.”
- Next, halve 73 (73 ÷ 2 = 36 with a remainder of 1). Write “36” in the left column below “73,” ignoring the remainder.
- Next, halve 36 (36 ÷ 2 = 18). Write “18” in the left column below “36.”
- Next, halve 18 (18 ÷ 2 = 9). Write “9” in the left column below “18.”
- Next, halve 9 (9 ÷ 2 = 4 with a remainder of 1). Write “4” in the left column below “18,” ignoring the remainder.
- Next, halve 4 (4 ÷ 2 = 2). Write “2” in the left column below “4.”
- Finally, halve 2 (2÷ 2 = 1). Write “1” in the left column below “2.”

Halve the number in the left column repeatedly

146 ÷ 2	37
73 ÷ 2	
36 ÷ 2	
18 ÷ 2	
9 ÷ 2	
4 ÷ 2	
2 ÷ 2	
1	

Example of Russian Peasant Multiplication

Double the number in the right column until the columns are the same length. Multiply the number in the second column by 2 until there are the same amount of numbers here as there are in the first column.

Example of Russian Peasant Multiplication

In this example:

- Each column should have 8 numbers in it. This is because it took seven steps of dividing the original number in the right column to reach 1.
- In the right column, start by doubling 37 (37 x 2 = 74). Write “74” in the right column below “37.”
- Next, double 74 (74 x 2 = 148). Write “148” in the right column below “74.”
- Next, double 148 (148 x 2 = 296). Write “296” in the right column below “148.”
- Next, double 296 (296 x 2 = 592). Write “592” in the right column below “296.”
- Next, double 592 (592 x 2 = 1184). Write “1184” in the right column below “592.”
- Next, double 1184 (1184 x 2 = 2368). Write “2368” in the right column below “1184.”
- Finally, double 2368 (2368 x 2 = 4736). Write “4736” in the right column below “2368.”

Double the number in the right column until the columns are the same length

146 ÷ 2	37 x 2
73 ÷ 2	74 x 2
36 ÷ 2	148 x 2
18 ÷ 2	298 x 2
9 ÷ 2	592 x 2
4 ÷ 2	1184 x 2
2 ÷ 2	2368 x 2
1	4736 x 2

Example of Russian Peasant Multiplication

Cross out rows with an even number in the left column. Using your pen, strike through those horizontal rows which begin with an even number in the left column.

Example of Russian Peasant Multiplication

In this example:

- There are 8 rows. You will strike through 5 of them.
- Strike through the rows beginning with 146, 36, 18, 4, and 2 in the left column, since these are even numbers. Working left to right, this means striking through the first row (146, 37), the third row (36, 148), the fourth row (18, 296), the sixth row (4, 1184), and the seventh row (2, 2368).

Cross out rows with an even number in the left column

~~146~~	~~37~~
73	74
~~36~~	~~148~~
~~18~~	~~298~~
9	592
~~4~~	~~1184~~
~~2~~	~~2368~~
1	4736

Example of Russian Peasant Multiplication

Find the sum of the remaining numbers in the right column. Add the numbers in the right column that you did not strike through. The sum of these numbers is equal to the product you would get from multiplying the original numbers using the standard method.

Example of Russian Peasant Multiplication

In this example:

- The remaining numbers in the right column are 74, 592, 4736.
- Add these numbers to get the sum of 5402 (74 + 592 + 4736 = 5402).

Example of Russian Peasant Multiplication

In this example:

- The remaining numbers in the right column are 74, 592, 4736.
- Add these numbers to get the sum of 5402 (74 + 592 + 4736 = 5402).
- This number is the product of multiplying the original numbers in this example. 146 x 37 = 5402

Understand why this method works

The Russian peasant method works because it converts the problem into binary (base 2) multiplication, rather than base 10 (which standard multiplication uses). It does this by halving and doubling the numbers you are trying to multiply (since halving and doubling convert all numbers into multiples of 2, or into factors of numbers that are divisible by 2).

Understand why this method works

$A_{10} \times B_{10}$

$a_2 \times b_2$

Time complexity of Russian Peasant Multiplication

$O(Log_2 b)$