Chapter 8

Transport of light with an optimized wavefront

In our experiments we construct a wavefront that maximizes the intensity at a given target position. In this chapter, we investigate the transport properties of light with such an optimized wavefront. We introduce the concept of 'active' measurements to describe a measurement where the wavefront is tailored to the specific realization of disorder in the sample. We show that this new type of experiments is very promising as it could be used to find signatures of absorption and possibly Anderson localization of light, or to test mesoscopic transport theories and selectively study extended modes in a sample.

So far, we have used a simple *ad hoc* matrix model (Eq. (3.1)) to describe scattering and focusing through a disordered medium. This model is based on the assumption of uncorrelated transmission coefficients (UTC). In reality, the constraint on energy conservation makes that the matrix elements are correlated. Such correlations can be described by random matrix theory (RMT). RMT predicts that it is possible to generate an incident wavefront that is fully transmitted through an opaque object, regardless of the thickness of the object.

In Section 8.1 we give a brief introduction to RMT and discuss how it can be applied to describe scattering in a disordered optical waveguide. In Section 8.2 we use RMT to describe the transport of light in an active measurement. We calculate experimentally observable quantities and compare the situations with the matched and unmatched incident wavefronts. Furthermore, we discuss the physical relevance of these 'actively measured' quantities and compare the results with predictions made with the UTC model. The total transmission of an optimized wavefront is found to be an observable that characterizes the correlations in the scattering matrix. Finally, in Section 8.3, we extend the RMT model so that it applies to a slab geometry and accounts for other experimental parameters, such as the resolution of the light modulator. We use the extended model to calculate how experimental limitations affect the measurements and we show how they can be compensated for.

8.1 Random matrix theory

Random matrix theory is a very general mathematical theory for statistically describing extremely complex systems. Some excellent reviews on RMT can be found in Refs. 1, 2 and 3. The theory has been developed by Wigner[4] to describe statistics of eigenvalues of many-body systems. It was originally employed to describe the properties of excited states of atomic nuclei and also proved very successful in calculating the statistics of electron scattering in mesoscopic conductors and quan-

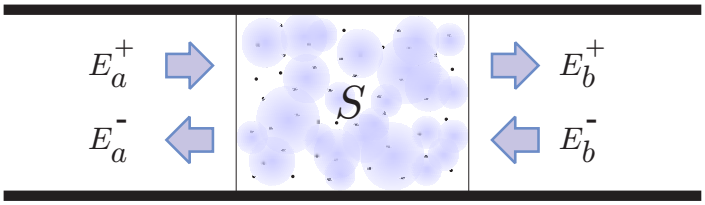

Figure 8.1: Scattering in a waveguide geometry. The scattering matrix S couples incident waveguide modes E_a^+ and E_b^- to outgoing waveguide modes E_a^-, E_b^+. In a transmission experiment, $E_b^- = 0$.

tum dots[5–8]. Weaver[9] was the first to study the application of RMT to acoustical waves instead of quantum mechanical wave functions. In pioneering experiments in acoustics[9, 10] and microwaves[11, 12] the distribution of eigenfrequencies in irregular cavities was measured. RMT proved very successful in predicting the statistical properties of eigenfrequency spacing in these systems.[3] Although most results of RMT are for monochromatic waves and, therefore, do not describe the dynamics of scattering, methods for extending the matrix treatment to describe time dependence are being developed[13–16].

More recently, RMT has been used to calculate correlations of light scattering in disordered optical waveguides (see e.g. [2]) and random lasers[17]. Because only very general concepts are used, it is very easy to translate results for one type of waves to a different system. This also means that experimental observations in one field of physics are directly relevant for many other disciplines.

The basic assumption of RMT is that the system is so complex that you might as well use a completely random Hamiltonian matrix to describe it. In Wigner's original approach[4], the 'randomness' of the matrix is invariant under unitary transformations of Hilbert space. In terms of light scattering, this invariance means that the scattered light has no spatial or angular correlation with the incident light whatsoever. The only other constraint that Wigner imposed on the random matrices is that there is a conserved quantum number that imposes a symmetry condition on the matrix. In the case of scattering this condition is that the energy is conserved (no absorption). Usually, an optical system also has time-reversal symmetry, which puts an extra constraint on the random matrix. Due to these constraints the elements of the matrix have to be statistically correlated. Typically, the number of scattering channels is large and the correlation between any pair of channels is negligible. Only when many scattering channels are involved, the effects of the correlations become visible.

8.1.1 Random matrix theory in a waveguide geometry

We first explain how RMT is applied to a simple waveguide geometry as is shown in Fig. 8.1. To both sides of a strongly scattering disordered sample a waveguide supporting exactly N propagating waveguide modes (channels) is connected. Evanescent fields in the waveguide are not taken into consideration. The sample scatters incident light into reflected and transmitted waveguide channels.

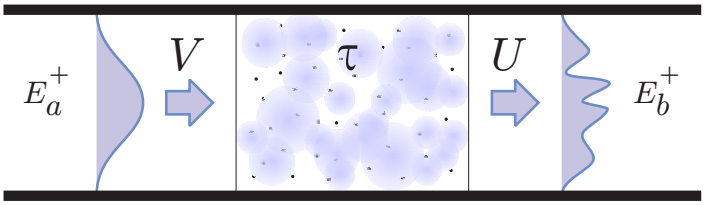

Figure 8.2: Transmission experiment in a waveguide geometry. The transmission matrix of the system is decomposed in three matrices: V maps the incident light to eigenmodes of the sample, τ gives amplitude transmission coefficients for the eigenmodes. U maps the eigenmodes back to real space on the other side of the sample.

Scattering by the sample is represented by the scattering matrix S that connects the incident field E_a^+, E_b^- to the channels leaving the sample (E_a^- and E_b^+). Channels labelled a are channels on the left side of the sample and Channels labelled b are channels on the right side of the sample. The scattering matrix S is defined as

$$\begin{bmatrix} E_a^- \\ E_b^+ \end{bmatrix} = S \begin{bmatrix} E_a^+ \\ E_b^- \end{bmatrix}. \tag{8.1}$$

We assume that the sample does not absorb any light and, therefore, S is unitary. When, in addition, scattering in the sample is reciprocal, time-reversal symmetry is obeyed and S is both unitary and symmetric. Although all our experimental systems have time-reversal symmetry, unitarity alone is sufficient for the derivations made in this chapter to be valid. The scattering matrix can be split into four matrices describing reflection and transmission

$$\begin{bmatrix} E_a^- \\ E_b^+ \end{bmatrix} = \begin{bmatrix} r_{aa}^{-+} & t_{ab}^{--} \\ t_{ba}^{++} & r_{bb}^{+-} \end{bmatrix} \begin{bmatrix} E_a^+ \\ E_b^- \end{bmatrix}. \tag{8.2}$$

From now on, we only consider transmission from left (channels a) to right (channels b) as described by the matrix t_{ba}^{++}, in short t. RMT tells us that the restriction of energy conservation gives rise to a statistical structure in the transmission matrix. This structure is visible in the singular value decomposition (SVD) of the transmission matrix. Like any matrix, the transmission matrix t can be decomposed as

$$t = U\tau V, \tag{8.3}$$

where U and V are unitary matrices and τ is a diagonal matrix with real, non-negative elements. The physical interpretation of Eq. (8.3) is illustrated in Fig. 8.2. Matrix V maps the incident light from real space coordinates onto transmission eigenmodes of the sample. The intensity transmission coefficients for each of these eigenmodes are given by the diagonal elements of τ^2. Matrix U maps the eigenmodes back to spatial coordinates at the back of the sample.

In RMT U and V are random unitary matrices with elements that have a circular Gaussian distribution. The statistical properties of these matrices are invariant to any

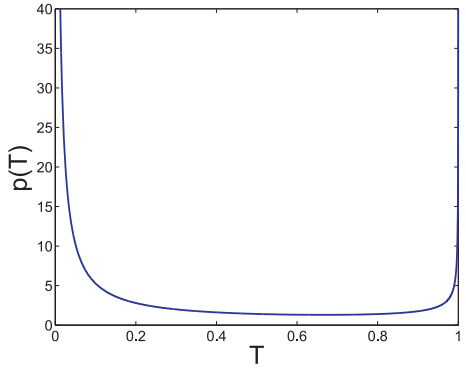

Figure 8.3: Probability density of the intensity transmission eigenvalues $T \equiv \tau^2$ as derived in Ref. 5.

unitary transform of the Hilbert space.[4] In other words, when we replace U (or V) by $U' = BU$ ($V' = BV$), with B any unitary matrix, the ensemble properties do not change. It, therefore, does not matter whether a channel is defined as a waveguide mode, an incident plane wave or a position on the sample surface. Moreover, these constraints on U and V make that these matrixes are statistically independent of τ.

8.1.2 Distribution of transmission eigenvalues

Intuitively, one might expect that all modes in a random system are roughly equivalent and the distribution of $T \equiv \tau^2$ is peaked around the total transmission of the sample. However, as was shown first by Dorokhov[5], the eigenvalues have a probability density that is given by

$$p(T) = \langle T \rangle \frac{1}{2T\sqrt{1-T}}. \tag{8.4}$$

This probability density is plotted in Fig. 8.3. The eigenvalue distribution is bimodal: almost all eigenvalues are either 1 or 0. A common terminology is to speak of 'open' and 'closed' channels. Light in an open channel is almost fully transmitted ($T \approx 1$), whereas light in closed channels is almost completely reflected ($T \approx 0$). The prediction that all channels are either open or closed is also known as the maximal fluctuation theorem[18, 19].

The distribution in Eq. (8.4) is the result of the restriction of energy conservation that is imposed on the scattering matrix. Due to this restriction, a mechanism called eigenvalue repulsion spreads out the values of $1/\xi$ to a uniform distribution.[2, 18, 20] Here, ξ is the localization length that is associated with a mode in the sample. The localization length relates to the transmission of a sample by $T = \cosh^{-2}(L/\xi)$. The uniform distribution of $1/\xi$ transforms into the distribution of T as is given by Eq. (8.4). Originally, RMT was used to derive Eq. (8.4) for a waveguide. However, later it was shown that this probability density function is more generally valid and also applies to a slab geometry.[21]

Equation (8.4) has divergencies at 0 and at 1. The divergency at 0 is not integrable. In fact, there is a minimum value for the transmission coefficient of a mode[2, 22], which is given by $T = \cosh^{-2}(1/\langle T\rangle)$. This cutoff normalizes the probability distribution of T. The physical interpretation of the cutoff is that with $\langle T\rangle \approx \ell/L$, the localization length of a mode cannot be shorter than the mean free path ℓ. In practice, the cutoff for T is often ignored since most of the time one is only interested in the moments of T. These moments depend very little on the cutoff.[22]

8.1.3 The effect of refractive indices

Random matrix theory is most conveniently formulated in a waveguide geometry with an equal number of waveguide channels on both side of the sample. In an experiment, the refractive index in front of the sample is often different from the refractive index at the back. Moreover, the sample itself has an effective refractive index that may differ from that of its environment. The number of channels scales with the refractive index squared. Therefore, the number of channels in the sample is not the same as the number of channels in the connected waveguides.

To account for the different number of scattering channels on both sides of the medium we assume that all channels are statistically equivalent. The number of channels is determined by the refractive index in each of the media. At an interface, scattering randomly redistributes the light over all channels. When the light enters a medium with a lower refractive index, some linear combinations of channels are completely reflected. This reflection is the random equivalent of total internal reflection.

In general the transmission coefficient at the surface of a disordered medium is angle dependent[23–25]. If we define channels as incident angles, different channels are not statistically equivalent. Therefore, we chose to define channels in real space coordinates. Since angular effects are equal for all channels, the channels are still statistically equivalent. Furthermore, we assume that the refractive index contrasts are low and that scattering is strong, so that neighboring channels can be considered statistically independent.

We now consider three possible combinations of refractive indices and analyze their effect on the RMT formalism.

Matrix description of a sample in air

When a sample is surrounded by air, its effective refractive index is higher than that of the surrounding medium. This means that the sample supports more channels than can possibly couple to the outside world. The difference in supported channels is shown schematically in Fig. 8.4. Since no light can couple to these bound modes, we ignore them and just use the formalism in Eq. (8.3), with τ a random subset of the actual full τ matrix of the sample.

Matrix description of a sample in a dense medium

In the situation where the refractive index of the sample is lower than that of its environment not all incident channels can couple into the sample. Or, more correctly,

The prefactor A_0 was defined to normalize the incident wavefront. Using Eq. (8.14) we find

$$A_0^{-2} = \sum_a^N |t_{\beta a}|^2 = \sum_k^K \left|U_{\beta k}\right|^2 \tau_{kk}^2 , \tag{8.24}$$

where the equality of Eqs. (8.15) and (8.18) was used in the last step. Finally, we have

$$\widetilde{T}_{\text{tot}} = \frac{\sum_k^K \left|U_{\beta k}\right|^2 \tau_{kk}^4}{\sum_{k'}^K \left|U_{\beta k'}\right|^2 \tau_{k'k'}^2}. \tag{8.25}$$

Here we see that the numerator is a weighted average over τ^4 and the denominator is a weighted average over τ^2 with equal weights. Averaging over U gives

$$\left\langle \widetilde{T}_{\text{tot}} \right\rangle = \frac{\langle \tau^4 \rangle}{\langle \tau^2 \rangle} \equiv C_{4,2}. \tag{8.26}$$

We see that the total transmission after optimizing the incident wavefront is equal to $C_{4,2}$, which is defined as the ratio between the fourth and the second moment of τ. $C_{4,2}$ does not depend on the refractive indices outside the sample. Using the distribution $p(T)$ that is predicted by RMT (Eq. (8.4)), $C_{4,2}$ can be calculated

$$C_{4,2} = \frac{2}{3}. \tag{8.27}$$

Surprisingly, $C_{4,2}$ turns out to be a universal constant. A fraction of 2/3 of the power of an ideally optimized wavefront is transmitted, regardless of the thickness of the sample.

8.2.3 Comparison with the uncorrelated model

The RMT model is very similar to the uncorrelated transmission coefficients (UTC) model in the sense that both models use a matrix with statistically equivalent random elements. The singular value decomposition that formed the start of our RMT calculations can also be performed on the UTC matrix and all observables can be calculated in exactly the same way. In effect, the only difference between the two models is the distribution of transmission eigenvalues. When we choose $\langle \tau^2 \rangle$ to be equal in both models, the only observable difference is the value of $C_{4,2}$.

RMT takes into account the correlations arising from the constraint of energy conservation and gives the universal result of $C_{4,2} = 2/3$. In the UTC model the matrix elements are uncorrelated. For such a matrix the eigenvalues are expected to have an exponential distribution.[33] In that case, $C_{4,2} = 2\langle \tau^2 \rangle$. The power in the optimized channel increases to $\langle \tau^2 \rangle$, whereas the background transmission remains constant at $\langle \tau^2 \rangle$. The optimized transmission is twice the original transmission and, therefore, it is not a universal constant in this model.

Apparently, $C_{4,2}$ is an accessible observable that depends on the symmetry of the disordered system, which makes it very interesting to determine it experimentally.