M311S24 Problem Set 1 *Franchi-Pereira, Philip*

1. Problem: Let $f : X \to Y$ and $B_0, B_1 \subseteq Y$ then prove:

(a) $f^*(B_0 \cup B_1) = f^*(B_0) \cup f^*(B_1)$

(b) $f^*(B_0 \cap B_1) = f^*(B_0) \cap f^*(B_1)$

(c) $f^*(\overline{B_0}) = \overline{f^*(B_0)}$

(d) $f^*(B_0 - B_1) = f^*(B_0) - f^*(B_1)$

(e) $f^*(B_0 + B_1) = f^*(B_0) + f^*(B_1)$

1.a $f^*(B_0 \cup B_1) = f^*(B_0) \cup f^*(B_1)$

It is first neccesary to show that $f^*(B_0 \cup B_1) \subseteq f^*(B_0) \cup f^*(B_1)$, and then that $f^*(B_0) \cup f^*(B_1) \subseteq f^*(B_0 \cup B_1)$. Let $a \in f^*(B_0 \cup B_1)$. By definition of a preimage, $f(a) \in B_0 \cup B_1$ and $a \in X$. So, either $f(a) \in B_0$ or $f(a) \in B_1$. To take B_0 first, since $a \in X$ and $f(a) \in B_0$, then $a \in f^*(B_0)$ and so $a \in f^*(B_0) \cup f^*(B_1)$. Similarly, since $a \in X$ and $f(a) \in B_1$, then $a \in f^*(B_1)$ and so $a \in f^*(B_0) \cup f^*(B_1)$. Therefore, $f^*(B_0 \cup B_1) \subseteq f^*(B_0) \cup f^*(B_1)$.

Next, let $a \in f^*(B_0)$. By definition, $a \in X$ and $f(a) \in B_0$. Since $f(a) \in B_0$, that implies that $f(a) \in B_0 \cup B_1$, which means that $a \in f^*(B_0 \cup B_1)$, and therefore $f^*(B_0) \subseteq f^*(B_0 \cup B_1)$. Similarly for B_1, $a \in X$ and $f(a) \in B_1$. Since $f(a) \in B_1$, then $f(a) \in B_0 \cup B_1$. Therefore $a \in f^*(B_0 \cup B_1)$ and so $f^*(B_1) \in f^*(B_0 \cup B_1)$. Since $f^*(B_0)$ and $f^*(B_1)$ are both subsets of $f^*(B_0 \cup B_1)$, $f^*(B_0) \cup f^*(B_1) \subseteq f^*(B_0 \cup B_1)$. Finally, since $f^*(B_0 \cup B_1) \subseteq f^*(B_0) \cup f^*(B_1)$ and $f^*(B_0) \cup f^*(B_1) \subseteq f^*(B_0 \cup B_1)$, $f^*(B_0 \cup B_1) = f^*(B_0) \cup f^*(B_1)$.

1.b $f^*(B_0 \cap B_1) = f^*(B_0) \cap f^*(B_1)$

First, let $a \in f^*(B_0 \cap B_1)$. By definition, $f(a) \in B_0 \cap B_1$, which implies that $f(a) \in B_0$ and $f(a) \in B_1$. Since $a \in X$ and $f(a) \in B_0$, then by definition $a \in f^*(B_0)$. Similarly since $a \in X$ and $f(a) \in B_1$, then $a \in f^*(B_1)$. Therefore, $a \in f^*(B_0) \cap f^*(B_1)$ and so $f^*(B_0 \cap B_1) \subseteq f^*(B_0) \cap f^*(B_1)$.

Next, let $a \in f^*(B_0) \cap f^*(B_1)$. Therefore $f(a)$is in both B_0 and B_1, which implies that $f(a) \in B_0 \cap B_1$. Since $a \in X$, then by definition $a \in f^*(B_0 \cap B_1)$

and so $f^*(B_0) \cap f^*(B_1) \subseteq f^*(B_0 \cap B_1)$. Finally since both $f^*(B_0 \cap B_1) \subseteq f^*(B_0) \cap f^*(B_1)$ and $f^*(B_0) \cap f^*(B_1) \subseteq f^*(B_0 \cap B_1)$, $f^*(B_0 \cap B_1) = f^*(B_0) \cap f^*(B_1)$.

1.c $f^*(\overline{B_0}) = \overline{f^*(B_0)}$

Let $a \in f^*(\overline{B_0})$. Then $f(a) \in \overline{B_0}$, which by definition means that $f(a) \notin B_0$. Since $a \in X$ but $f(a) \notin B_0$, by definition $a \in \overline{f^*(B_0)}$ and therefore $f^*(\overline{B_0}) \subseteq \overline{f^*(B_0)}$.

Next let $a \in \overline{f^*(B_0)}$. Then $a \in X$ but $a \notin f^*(B_0)$. For $a \notin f^*(B_0)$ to hold true, either $a \notin X$ or $f(a) \notin B_0$. Since we know $a \in X$, then $f(a) \notin B_0$. Finally, since $f(a) \notin B_0$, $f(a) \in \overline{B_0}$, and so $a \in f^*(\overline{B_0})$ and $\overline{f^*(B_0)} \subseteq f^*(\overline{B_0})$. Finally, since $f^*(\overline{B_0}) \subseteq \overline{f^*(B_0)}$, and $\overline{f^*(B_0)} \subseteq f^*(\overline{B_0}$, $f^*(\overline{B_0}) = \overline{f^*(B_0)}$.

1.d $f^*(B_0 - B_1) = f^*(B_0) - f^*(B_1)$

First we will show that $f^*(B_0 - B_1) \subseteq f^*(B_0) - f^*(B_1)$. Let $a \in f^*(B_0 - B_1)$. Then $f(a) \in (B_0 - B_1)$, and so $f(a) \in B_0$ and $f(a) \notin B_1$. Since $f(a) \in B_0, a \in f^*(B_0)$, and since $f(a) \notin B_1, a \notin f^*(B_1)$, therefore $a \in f^*(B_0) - f^*(B_1)$, and so $f^*(B_0 - B_1) \subseteq f^*(B_0) - f^*(B_1)$.

Next, we will show that $f^*(B_0) - f^*(B_1) \subseteq f^*(B_0 - B_1)$. Let $a \in f^*(B_0) - f^*(B_1)$. Then, $a \in f^*(B_0)$ but $a \notin f^*(B_1)$, which means $f(a) \in B_0$ but $f(a) \notin B_1$. Therefore $f(a) \in (B_0 - B1)$, and so $a \in f^*(B_0 - B_1)$, and so $f^*(B_0 - B_1) \subseteq f^*(B_0) - f^*(B_1)$.

Since $f^*(B_0 - B_1) \subseteq f^*(B_0) - f^*(B_1)$ and $f^*(B_0 - B_1) \subseteq f^*(B_0) - f^*(B_1)$, $f^*(B_0 - B_1) = f^*(B_0) - f^*(B_1)$

1.e $f^*(B_0 + B_1) = f^*(B_0) + f^*(B_1)$

Following from the proofs of Problems 1.a-1.d, and the definition of the symmetric difference, it is clear that $f^*(B_0 + B_1) = f^*((B_0 \cup B_1) - (B_0 \cap B_1)) = f^*(B_0 \cup B_1) - f^*(B_0 \cap B_1)$ which by definition is $f^*(B_0) + f^*(B_1)$.

2. Problem: Let $A_0, A_1 \subseteq X$. Prove or give counter-examples for the following statements.

(a) $f_*(A_0 \cup A_1) = f_*(A_0) \cup f_*(A_1)$

(b) $f_*(A_0 \cap A_1) = f_*(A_0) \cap f_*(A_1)$

(c) $f_*(\overline{A_0}) = \overline{f_*(A_0)}$

(d) $f_*(A_0 - A_1) = f_*(A_0) - f_*(A_1)$

(e) $f_*(A_0 + A_1) = f_*(A_0) + f_*(A_1)$

2.a $f_*(A_0 \cup A_1) = f_*(A_0) \cup f_*(A_1)$

It is first neccesary to show that $f_*(A_0 \cup A_1) \subseteq f_*(A_0) \cup f_*(A_1)$, and then that $f_*(A_0) \cup f_*(A_1) \subseteq f^*(A_0 \cup A_1)$. First let $b \in f_*(A_0 \cup A_1)$. Then there must exist some $a \in A_0 \cup A_1$ such that $f(a) = b$. Either $a \in A_0$ or $a \in A_1$. If $a \in A_0$, then by definition of the image $f_*(A_0 \cup A_1)$, $b \in Y$ and $f(a) = b$, and so by definition $b \in f_*(A_0)$. Similarly, if $a \in A_1$, then there must be a $b \in Y$ such that $f(a) = b$, and therefore $b \in f_*(A_1)$. It is clear then that if $b \in f_*(A_0)$ or $b \in f_*(A_1)$, then $b \in f_*(A_0 \cup A_1)$, and therefore $f_*(A_0 \cup A_1) \subseteq f_*(A_0) \cup f_*(A_1)$.

Next, let $b \in f_*(A_0) \cup f_*(A_1)$. Either $b \in f_*(A_0)$ or $b \in f_*(A_1)$. To take the case of $b \in f_*(A_0)$ first, then $b \in Y$ and there exists an $a \in A_0$ such that $f(a) = b$. But if $a \in A_0$, then $a \in A_0 \cup A_1$, and so by definition $b \in f_*(A_0 \cup A_1)$. If instead $b \in f_*(A_1)$, then similarly there exists an $a \in A_1$ such that $f(a) = b$, and so $a \in f_*(A_0 \cup A_1)$, and so by definition $b \in f_*(A_0 \cup A_1)$. Therefore, $f_*(A_0) \cup f_*(A_1) \subseteq f^*(A_0 \cup A_1)$. Finally, since $f_*(A_0 \cup A_1) \subseteq f_*(A_0) \cup f_*(A_1)$ and $f_*(A_0) \cup f_*(A_1) \subseteq f^*(A_0 \cup A_1)$, $f_*(A_0 \cup A_1) = f_*(A_0) \cup f_*(A_1)$.

2.b $f_*(A_0 \cap A_1) = f_*(A_0) \cap f_*(A_1)$

This statement is false. As an example, let $X = \{1, 2, 3\}, Y = \{1, 2\}$, and a function $f : X \to Y$ between them such that $f = \{(1, 1), (2, 1), (3, 2)\}$. Let $A_0 \subset A = \{1\}$ and $A_1 \subset A = \{2\}$. Then, $f^*(A_0 \cap A_1) = \emptyset$ since $A_0 \cap A_1 = \emptyset$, but $f^*(A_0) \cap f^*(A_1) = \{1\}$

However, it is the case that $f_*(A_0 \cap A_1) \subseteq f_*(A_0) \cap f_*(A_1)$. Let $b \in f_*(A_0 \cap A_1)$. By definition, $b \in Y$ and there is an $a \in A_0 \cap A_1$ such that $f(a) = b$. Since $a \in A_0 \cap A_1$, $a \in A_0$ and $a \in A_1$, so by definition of an image $b \in f_*(A_0)$ and $b \in f_*(A_1)$, and so $b \in f_*(A_0) \cap f_*(A_1)$. Therefore, $f_*(A_0 \cap A_1) \subseteq f_*(A_0) \cap f_*(A_1)$.

Note that the original statement is true when f is *injective*. To show this, let $b \in f_*(A_0) \cap f_*(A_1)$. By definition, $b \in Y$, and there is an $a_0 \in A_0$ such that $f(a_0) = b$, and an $a_1 \in A_1$ such that $f(a_1) = b$. Since f is injective, then

$f(a_0) = f(a_1) = b$, and therefore $a_1 = a_2$, which will simply be reffered to as a for the remainder of the proof. Since $a \in A_0$ and $a \in A_1$, then $a \in A_0 \cap A_1$, and so by definition, $b \in f_*(A_0 \cap A_1)$ and $f_*(A_0) \cap f_*(A_1) \subseteq f_*(A_0 \cap A_1)$.

Therefore, only when f is injective does $f_*(A_0 \cap A_1) = f_*(A_0) \cap f_*(A_1)$.

2.c $f_*(\overline{A_0}) = \overline{f_*(A_0)}$

This statement is false. As an example, let $X = \{1, 2, 3\}, Y = \{1, 2\}$, and a function $f : X \to Y$ between them such that $f = \{(1, 1), (2, 2), (3, 2)\}$. Let $A_0 \subset A = \{1, 2\}$. Therefore, $\overline{A_0} = \{3\}$ and $f_*(\overline{A_0}) = 2$. However, since $f_*(A_0) = 1, 2$, then $\overline{f_*(A_0)} = \emptyset$, and so $f_*(\overline{A_0}) \neq \overline{f_*(A_0)}$.

2.d $f_*(A_0 - A_1) = f_*(A_0) - f_*(A_1)$

This statement is false. As an example, let $X = \{1, 2, 3\}, Y = \{1, 2\}$, and a function $f : X \to Y$ between them such that $f = \{(1, 1), (2, 2), (3, 2)\}$. Let $A_0 \subset A = \{2, 3\}$ and $A_1 \subset A = \{2\}$. Then $A_0 - A_1 = \{3\}$, and $f^*(A_0 - A_1) = \{2\}$. However, $f_*(A_0) = 2$ and $f_*(A_1) = \{2\}$, so $f_*(A_0) - f_*(A_1) = \emptyset$ and so $f_*(A_0 - A_1) \neq f_*(A_0) - f_*(A_1)$.

2.e $f_*(A_0 + A_1) = f_*(A_0) + f_*(A_1)$

This statement is false. As an example, let $X = \{1, 2, 3\}, Y = \{1, 2\}$, and a function $f : X \to Y$ between them such that $f = \{(1, 1), (2, 2), (3, 2)\}$. Let $A_0 \subset A = \{1, 2\}$ and $A_1 \subset A = \{1, 3\}$. Then $A_0 + A_1 = \{2, 3\}$, and $f^*(A_0 + A_1) = \{2\}$. However, $f_*(A_0) = \{1, 2\}$ and $f_*(A_1) = \{1, 2\}$, so $f_*(A_0) - f_*(A_1) = \emptyset$ and so $f_*(A_0 - A_1) \neq f_*(A_0) - f_*(A_1)$.

Problem 3 Bounding Axioms and the Well Ordered Principle

- **Definition** Let T be a non-empty subset of the integers. An integer l is a lower bound for T if for all $t \in T, l \leq t$. If the set T has some lower bound then we say the set is bounded below.
- **Bounded Below Axiom** Every non-empty set of integers which is bounded below has a smallest element.
- **Well Ordering Principle** Every non-empty set of natural numbers has a smallest element.

Problem 3.a Show that the bounded below axiom implies the Well Ordering Principle.

Choose a non-empty set X such that $X \subseteq \mathbb{N}$. Then, for all elements $x \in X, x \in \mathbb{Z}$, and so $X \subseteq \mathbb{Z}$. Since for all $x \in X, 0 \leq x$, then by definition 0 is a lower bound for X, and so X is bounded below. Then by the Bounding Axiom, X has a smallest element and so every set in the natural numbers has a smallest element.

Problem 3.b Prove that the Well Ordering Principle implies the Bounded Below Axiom.

Let T be a non-empty set of integers with a lower bound l. If $0 \leq l, T \subseteq \mathbb{N}$, and so by the Well Ordering Principle T has a smallest element.

In the case where $l < 0$, then define a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(t) = t - l$, and a set $S = \{f(t) : t \in T\}$. First, note that f is bijective, as it has an inverse $g : \mathbb{Z} \to \mathbb{Z}, g(t) = t + l, g \circ f(t) = g(f(t)) = g(t - l) = t - l + l = t$ and $f \circ g(t) = f(g(t)) = f(t + l) = t + l - l = t$. Also note that if $f(a) \leq f(b)$ then $a \leq b$, as $f(a) = a - l, f(b) = b - l$, and so $a - l \leq b - l = a \leq b$.

Next, we will show that $S \subseteq \mathbb{N}$. Since l is negative, then for all $s \in S, s = f(t) = t - l = t + |l|$, for some $t \in T$. If $t \geq 0, t + |l| \geq 0$ and if $t \leq 0$, since $l \leq t$, then the inequality $l \leq t \leq 0 = 0 \leq t - l \leq -l = 0 \leq t + |l| \leq |l|$, and so for every $s \in S, s \geq 0$.

Since for all $s \in S, 0 \leq s$, then $S \subseteq N$, and therefore by the Well Ordering Principle has a smallest element $j_S \in S$. Since f is bijective, there must must then be a smallest element $j_T \in T$ such that $f^{-1}(j_T) = j_S$. It is clear that since $j_S \leq s$ for all elements $s \in S$, then $j_T \leq t$ for all elements $t \in T$, and so T has a smallest element as well. Therefore, the Well Ordering Principle implies the Bounded Below Axiom.

Problem 3.c Show that the bounded below axiom implies the bounded above axiom.

For completeness, it will first be shown that if $a \geq b, -b \geq -a$. Then, that the function $f : \mathbb{Z} \to \mathbb{Z}, f(t) = -t$ is bijective, and finally that if a non empty set of integers has an upper bound, then that a Bounded Below axiom implies the Bounded Above axiom as well.

To show that if $a \geq b, -b \geq -a$, note that $a \geq b, a - b \geq 0, -b \geq -a$. Second, take a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(t) = -t$. It is clear that f is bijective since it has an inverse, itself. $f \circ f(t) = f(f(t) = f(-t) = -(-t) = t$.

Finally, let $T \subseteq \mathbb{Z}$ with an upper bound u such that $T = \{t \in \mathbb{Z} : u \geq t, u \in \mathbb{Z}\}$ Then construct a new set $S = \{f(t) : t \in T\}$. Since for all $t \in T, u \geq t$, then $f(t) \geq f(u)$. Therefore S has a lower bound, and so by the Bounding Axiom has a smallest element l. However, since f is bijective, there must exist an element $b \in T$ such that $f^{-1}(l) = b$.

Since for all elements $s \in S, s \geq l$, then by the bijectivity of f for all elements $t \in T, b \geq t$, and so T has a largest element. So, if every non-empty set of integers with a lower bound has a smallest element, then every non-empty set of integers with an upper bound has a largest element.

Problem 4 Prove that if X is finite, then $\mathcal{P}(X)$ is finite and $|\mathcal{P}(X)| = 2^{|X|}$.

Base Case Let $X = \emptyset$. Then the $\mathcal{P}(X)$ is equal to $\{\emptyset\}$, $|X| = 0$ and $|\mathcal{P}(X)| = 1$ which is equal to $2^{|X|} = 2^0 = 1$.

Base Case Let $X = \{x\}$ for some element x. Then the $\mathcal{P}(X)$ is equal to $\{\emptyset, \{x\}\}$, $|X| = 1$ and $|\mathcal{P}(X)| = 2$, which is equal to $2^{|X|} = 2^1 = 2$.

Inductive Proposition Assume $|A| = n$ implies $|\mathcal{P}(A)| = 2^{|A|}$, then for some non-empty set X, $|X| = n + 1$ implies $|\mathcal{P}(X)| = 2^{n+1}$

Proof of Inductive Proposition **A.** Assume $|A| = n$ implies $|\mathcal{P}(A)| = 2^{|A|}$ and let $|X| = n + 1$ for some non empty X. Choose an x in X and let $A = X - \{x\}$. Then let $\mathcal{Q} = \mathcal{P}(X) - \mathcal{P}(A)$, or $\mathcal{Q} = \{S \subseteq \mathcal{P}(X) : x \in S\}$.

B. Next, let there be a function $\gamma : \mathcal{P}(A) \to \mathcal{Q}$, such that for some element $a \in A, \gamma(a) = a \cup x$. This function has an inverse $\epsilon : \mathcal{Q} \to \mathcal{P}(A), \epsilon(q) = q - \{x\}, q \in Q$, since $\epsilon \circ \gamma(S) = \epsilon(\gamma(S)) = \epsilon(S \cup \{x\}) = (S \cup \{x\}) - \{x\} = S - \{x\}$, and since $x \notin T, T \subseteq \mathcal{P}(A)$, then $S - \{x\} = S$for some $S \in A$. and $\gamma \circ \epsilon(T) = \gamma(\epsilon(T)) = \gamma(T - \{x\}) = (T - \{x\}) \cup \{x\} = (T \cup \{x\}) - (\{x\} - \{x\}) = (T \cup \{x\}) - \emptyset = T \cup \{x\}$, but since $T \subseteq \mathcal{Q}$, then by the definition of $\mathcal{Q}, x \in T$, and so $T \cup \{x\} = T$ and $\gamma \circ \epsilon(T) = T$. Since γ has an inverse, it is bijective. Since the cardinality of $\mathcal{P}(A)$ is assumed to be 2^n, by definition (found in section 2.2.9 of the class notes) there exists a bijection $h : \underline{2^n} \to \mathcal{P}(A)$. Define a new function $h' : \underline{2^n} \to \mathcal{Q}$ such that $h'(n) = \gamma \circ h(n)$. Since h and γ are bijective, h' is bijective, $|\mathcal{Q}| = 2^n$.

C. By the inductive hypothesis, $|\mathcal{P}(A)| = 2^{|A|} = 2^n$. Therefore, $|\mathcal{Q}| = |\mathcal{P}(A)| = 2^{|A|} = 2^n$. As was proven by the proposition in section 2.2.12 of the class notes, $|\mathcal{Q} \cup \mathcal{P}(A)| = |\mathcal{Q}| + |\mathcal{P}(A)| = 2^n + 2^n = 2^{n+1}$. It is important to note here that this proposition applies because $\mathcal{P}(A)$ and $\mathcal{Q}$ are disjoint, since every subset of $\mathcal{Q}$ contains x, and every subset of $\mathcal{P}(A)$ does not. And, since $\mathcal{Q} \cup \mathcal{P}(A) = (\mathcal{P}(X) - \mathcal{P}(A)) + \mathcal{P}(A) = \mathcal{P}(X)$, then $|\mathcal{P}(X)| = 2^{n+1}$, and the proof is complete.