Math 156, Sec 4, H.W. 4 *Franchi-Pereira, Philip*

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Problem 1 Suppose $n \in Integers$. If n^2 is odd then n is odd.

Proof by contrapositive: Suppose n is even, then there exists a $k \in \mathbb{Z}$ such that $n = 2k$. Therefore, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, and since $2k^2$ is also an integer, $2(2k^2)$ is even, and so n^2 is even.

Problem 2 Suppose $a, b, c \in Integers$. If a does not divide bc then a does not divide b.

Proof by Contrapositive: Suppose a does divide b. Then there exists a $q \in \mathbb{Z}$ such that $b = aq$, and so $bc = aqc$. It is clear that $a|aqc$. Therefore $a|bc$.

Problem 3 For any $a, b \in \mathbb{Z}$, it follows that $(a+b)^3 = a^3 + b^3 \pmod 3$.

Direct Proof: Expanding the terms, we see that $(a+b)^3 = (a^3 + b^3) + 3(a^2b + ab^2)$, and so $(a+b)^3 - (a^3+b^3) = (a^3+b^3) + 3(a^2b+ab^2) - (a^3+b^3) = 3(a^2b+ab^2)$. Since $3|(a+b)^3 - (a^3+b^3)$, by the definition of modulus it follows that $(a+b)^3 = a^3 + b^3 \pmod 3$.

Problem 4 Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

Proof by Contrapositive Suppose instead $x \leq -1$. Then $x^2 \geq 1$. Next multiply both sides by x, and since x is known to be negative, we flip the inequality to get $x^3 \leq x$ and finally $x^3 - x \leq 0$.

Note for the HW. For the following proofs, the logic statement required by the assignment is the first line of the proof.

Problem 5 There exist no integers a and b for which $21a + 30b = 1$. **Proof by contradiction**: Suppose there exists two integers a and b such that $21a + 30b = 1$. That would mean that $3(7a + 10b) = 1$. Letting $k = 7a + 10b$, the equation becomes $3k = 1$. This is a contradiction, as there is no such integer.

Problem 6 If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Proof by contradiction: Suppose instead that if a and b are positive real numbers, then $a + b < 2\sqrt{ab}$. Without loss of generality, say $a \geq b$. Then $a = b + r$, for some $r \in \mathbb{R}$ and so $a + b = 2b + r$. Meanwhile $2\sqrt{ab} = 2\sqrt{b^2 + br}$. Since both sides of the inequality are clearly positive, we may square both sides, showing

$$\begin{aligned} (2b + r)^2 &< 2\sqrt{b^2 + br} \\ 4b^2 + 4br + r^2 &< 4b^2 + 4br \\ r^2 &< 0 \end{aligned}$$

However, for any value of $r \in \mathbb{R}$, r^2 is clearly greater than or equal to 0 and so this is a contradiction.

Problem 7 Prove by contradiction that no odd integer can be expressed as the sum of three. even integers.

Proof by contradiction: Assume there exists an odd integer x, and even integers a, b, c such that $x = a + b + c$. Since a, b, c are even, there exists some $k_1, k_2, k_3 \in \mathbb{Z}$ such that $a = 2k_1,\ b = 2k_2,\ c = 2k_3$. The sum of these integers is then $x = a+b+c = 2k_1+2k_2+2k_3 = 2(k_1+k_2+k_3)$. Since $k_1 + k_2 + k_3$ is an integer, $2(k_1 + k_2 + k_3)$ is even. But since x was assumed to be odd, that is a contradiction.