M311/621 S24 Problem Set 1

Reread Section 1.5 of Notes Chapter 1

1. **Problem** Let $f\colon X \to Y$ and $B_0, B_1 \subseteq Y$ then prove:
   (a) $f^*(B_0 \cup B_1) = f^*(B_0) \cup f^*(B_1)$
   (b) $f^*(B_0 \cap B_1) = f^*(B_0) \cap f^*(B_1)$
   (c) $f^*(\overline{B_0}) = \overline{f^*(B_0)}$.
   (d) $f^*(B_0 - B_1) = f^*(B_0) - f^*(B_1)$
   (e) $f^*(B_0 + B_1) = f^*(B_0) + f^*(B_1)$

If you are clever the last two parts have one line proofs by referring to the first three parts

Thus the map f^* is well behaved with respect to unions, intersections, relative complements, and symmetric differences. This is not true for the map f_*

2. **Problem** Let $A_0, A_1 \subseteq X$Prove or give counter-examples for the following statements. For your counter examples you should not need fancy functions from calculus. You should be able to make them with $X = \{1, 2, 3\}, Y = \{1, 2\}$
   (a) $f_*(A_0 \cup A_1) = f_*(A_0) \cup f_*(A_1)$
   (b) $f_*(A_0 \cap A_1) = f_*(A_0) \cap f_*(A_1)$
   (c) $f_*(\overline{A_0}) = \overline{f_*(A_0)}$.
   (d) $f_*(A_0 - A_1) = f_*(A_0) - f_*(A_1)$
   (e) $f_*(A_0 + A_1) = f_*(A_0) + f_*(A_1)$
3. **Problem**
   (a) Show that the bounded below axiom implies the Well Ordering Principal. Hint WOP is a special case of the bounded below axiom.
   (b) Prove that the Well ordering Principal implies the Bounded Below Axiom. Hint: Let T be a non-empty set of integers with lower bound l. Let $S = \{t - l : t \in T\}$ show that $S \subseteq \mathbb{N}$ and relate a smallest element of S to a smallest element of T.
   (c) Show that the bounded below axiom implies the bounded above axiom: Hint let S be a non-empty set bounded above by u. Let $T = \{-s : s \in S\}$

Counting Problems

In the next problem you are going to prove:

Proposition 0.1. If X is finite then $\mathcal{P}(X)$ is finite and $|\mathcal{P}(X)| = 2^{|X|}$.

Obviously we are going to induce on the size of X.

4. **Problem**

   Base Case State and prove the base case.

   Inductive Proposition State the inductive proposition

   Proof of Inductive Proposition You are going to prove the inductive proposition by putting the following steps together

   A Assume $|A| = n$ implies $|\mathcal{P}(A)| = 2^{|A|}$ and let $|X| = n + 1$. the set X is non-empty. Choose an $x \in X$ and let $A = X - \{x\}$. Characterize the elements of $\mathcal{P}(X) - \mathcal{P}(A)$. Denote $\mathcal{P}(X) - \mathcal{P}(A)$ by $\mathcal{Q}$.

   B The is an obvious function $\gamma\colon \mathcal{P}(A) \to \mathcal{Q}$. Define this function.

   The function γ has an inverse ϵ. Define ϵ and show that it is inverse to γ. State the result that shows $|\mathcal{P}(A)| = |\mathcal{Q}|$

   C Now prove the inductive Proposition.

Proposition 0.2 (Size of Function Spaces)**.** If X and Y are finite and Y is non-empty then so is $F(X, Y)$ and

$$|F(X, Y)| = |Y|^{|X|}$$

In the following problem you will prove this proposition.

5. **Problem**

   (a) Define the obvious map $\epsilon\colon F(\{x\}, Y) \to Y$. Hint:evaluation.

   (b) Define the inverse,$\gamma\colon Y \to F(\{x\}$.

   (c) Let $X = A \cup B$ then we have the restriction map $\mathcal{C}\colon F(X, Y) \to F(A, Y) \times F(B, Y)$ given by $C(f) = (f|_A, f|_B)$. Now assume A and B are *disjoint*. Define the inverse function,$\mathcal{D}$ of $\mathcal{C}$. If you don't refer to disjointness your proof is incomplete.

   (d) Use the previous two results, and induction on $|X|$ to prove that, for X finite, and Y non-empty and finite, $|F(X, Y)| = |Y|^{|X|}$.

   Base Case State and prove the base case $|X| = 1$.

   Inductive Proposition State the inductive proposition.

   Proof Prove the inductive proposition.

We give an alternate proof of Proposition 0.1

Notation 0.3. For n a positive integer we denote the set $\{0, 1, 2, \ldots n - 1\}$ by $\mathbb{Z}_n$. Thus $\mathbb{Z}_2 = \{0, 1\}$ We have a map $\Sigma\colon F(X, \mathbb{Z}_2) \to \mathcal{P}(X), f \mapsto f^{-1}(1)$. Given $A \subseteq X$ we define $\chi_A \in F(X, \mathbb{Z}_2)$ by $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$. The function χ_A is called the *characteristic* function of A.

6. **Problem** Define the map $\Xi\colon \mathcal{P}(X) \to F(X, \mathbb{Z}_2)$ by $\Xi(A) = \chi_A$

(a) Show that Σ and Ξ are inverse to each other.

(b) Use this and problem 5 to show that for X finite $|\mathcal{P}(X)| = 2^{|X|}$. It is because of this equality that an alternate notation for $\mathcal{P}(X)$ is 2^X.

We will see that for every prime p there is a field structure on $\mathbb{Z}_p$. We wish to establish the size of the group $Gl(n, \mathbb{Z}_p)$ of invertible $n \times n$ matrices. If $M \in M_{n,n}(Z_p)$ regarding the columns as elements of $\mathbb{Z}_p^n$, M is invertible exactly when the columns of M give an ordered basis of $\mathbb{Z}_p^n$. so we need to count the number of ordered bases of $\mathbb{Z}_p^n$. We can construct an ordered basis $v_1, v_2, \dots v_n$ in the following fashion:

Step 1:Chose $v_1 \in \mathbb{Z}_p^n - 0$
Step 2: Choose v_2 in $\mathbb{Z}_p^n - \text{Span}(v_1)$
Step 3: Chose v_3 in $\mathbb{Z}_p^n - \text{Span}(v_1, v_2)$

Step k: Chose v_k in $\mathbb{Z}_p^n - \text{Span}(v_1, v_2, \dots v_{k-1})$.
This process terminates after n steps

7. **Problem** Show that

$$\#(\text{Gl}(n, \mathbb{Z}_p)) = (p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1}) = p^{n^2}(1 - \frac{1}{p^n})(1 - \frac{1}{p^{n-1}}) \dots (1 - \frac{1}{p})$$