Math 156, Sec 4, HW 3 *Franchi-Pereira, Philip*

Problem 1 If x is an odd integer, x^3 is odd.

If x is odd, then $\exists n \in \mathbb{Z}, x = 2n + 1$. Therefore

$$\begin{aligned} x^3 &= (2n+1)^3 = (2n+1)\cdot(4n^2+4n+1) \\ &= (6n^3+8n^2+2n)+(4n^2+4n+1) \\ &= 8n^3+12n^2+6n+1 \\ &= 2(4n^3+6n^2+3n)+1 \end{aligned}$$

Letting $u = 4n^3 + 6n^2 + 3n$, it is clear that $u \in \mathbb{Z}$, since $\mathbb{Z}$ is closed under addition and multiplication. So, $x^3 = 2u + 1$ and therefore by definition x^3 is odd.

Problem 2 Suppose $a, b, c, d \in \mathbb{Z}$. If $a|b$ and $a|c$, then $a|(b + c)$.

If $a|b$ then $b = aq$ for some $q \in \mathbb{Z}$. If $a|c$ then $c = ap$ for some $p \in \mathbb{Z}$. Therefore $b + c = aq + ap = (q + p)a$, and since $(q + p) \in \mathbb{Z}$, $a|(q + p)a$ and therefore $a|(b + c)$.

Problem 3 If $n \in \mathbb{Z}$, then $5n^2 + 3n + 17$ is odd.

Since, $\forall n \in \mathbb{Z}, n$ is either even or odd, the proof can be demonstrated by two cases:

Case 1. n is even Let $n \in \mathbb{Z}$ be even. Then $\exists k, k \in \mathbb{Z}, n = 2 \cdot k$. Then

$$\begin{aligned} 5n^2+3n+17 &= 5(2k)^2+3(2k)+17 \\ &= 20k^2+6k+17 \\ &= 2(10k^2+3k+8)+1 \end{aligned}$$

Let $u = 10k^2 + 3k + 8$. Since addition and multiplication are closed under $\mathbb{Z}, u \in \mathbb{Z}$ and so by definition $2u+1$ is odd. Therefore if n is even, $5n^2+3n+17$

is odd.
Case 2. n **is odd** Let $n \in \mathbb{Z}$ be odd instead. Then $\exists k, k \in \mathbb{Z}, n = 2 \cdot k + 1$. Then

$$\begin{aligned} 5n^2 + 3n + 17 &= 5(2k+1)^2 + 3(2k+1) + 17 \\ &= (20k^2 + 20k + 1) + (6k + 3) + 17 \\ &= 20k^2 + 26k + 21 \\ &= 2(10k^2 + 13k + 10) + 1 \end{aligned}$$

Letting $u = 10k^2 + 13k + 10$, it is clear that $u \in \mathbb{Z}$ and so by definition $2u + 1$ is odd. Therefore if n is odd, $5n^2 + 3n + 17$ is odd.

Therefore $\forall n \in \mathbb{Z}, 5n^2 + 3n + 17$ is odd.

Problem 4 If two integers have the same parity, then their sum is even.

All integers are either even or odd, therefore the proof can be shown by two cases.

Case 1 a, b are odd.

Let $a, b \in \mathbb{Z}$ such that $a = 2q + 1, b = 2p + 1$, for some $q, p \in \mathbb{Z}$. Then

$$a + b = (2q + 1) + (2p + 1) = 2q + 2p + 2 = 2(q + p) + 2 = 2((q + p) + 1)$$

Letting $u = (q + p) + 1$, since addition is closed under $\mathbb{Z}, u \in \mathbb{Z}$ and so by definition $2u$ is even. Therefore $a + b$ is even.

Case 2 a, b are even.

Let $a, b \in \mathbb{Z}$ such that $a = 2q, b = 2p$, for some $q, p \in \mathbb{Z}$. Then

$$a + b = 2q + 2p = 2(q + p)$$

Since $(q + p) \in \mathbb{Z}$, then by the definition of even $2 \cdot (q + p)$ is even. Therefore $a + b$ is even.

Therefore, if any two have the same parity, then their sum is even.

Problem 5 Suppose x and y are positive real numbers. If $x < y$ then $x^2 < y^2$.

If $x < y$, then by multiplying both sides by x it is clear that $x^2 < x \cdot y$. Similarly, multiplying both sides by y, $x \cdot y < y^2$. Combining both inequalities, we have $x^2 < x \cdot y < y^2$ and therefore $x^2 < y^2$.

Problem 6 Every odd integer is a difference of two (consecutive) squares.

$\forall n, k \in \mathbb{Z}, n = 2k + 1$. To show that n is the difference of squares between two consecutive integers, take $k, k + 1$ as the first and the second consectuive integers. The difference of squares between $k + 1$ and k is $(k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = n$. Since the choice of n was arbitrary, then every odd integer is a difference of two consecutive squares.