Trinity College Dublin
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SCHOOL OF MATHEMATICS

BLACK HOLE THERMODYNAMICS

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JUNE 27, 2025

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

B.A. THEORETICAL PHYSICS

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Abstract

This thesis includes a proof of Birkhoff's theorem, the calculation of the surface gravity of a Schwarzschild black hole ($\kappa = \frac{1}{4M}$) and a Kerr black hole at each of its horizons, $r_{\pm}$ ($\kappa_{+} = \frac{r_{+}-r_{-}}{2(r_{+}^{2}+a^{2})}$, $\kappa_{-} = \frac{r_{-}-r_{+}}{2(r_{-}^{2}+a^{2})}$). Then follows an overview of the Penrose process and the concept of the irreducible mass of a black hole. Next, Komar integrals are used to calculate conserved quantities. For the Schwarzschild spacetime, this was found to be the mass; for Kerr spacetimes, this was found to be the angular momentum. The first law of black hole thermodynamics was then derived, based on Bekenstein's paper (1) which is given by the expression
$dM = \theta d\alpha + \vec{\Omega} \cdot d\vec{L} + \Phi dQ$. This is followed by a discussion of Bekenstein's work on the generalised second law of black hole thermodynamics $\Delta S_{bh} + \Delta S_{c} = \Delta(S_{bh} + S_{c}) > 0$. This led Bekenstein to calculate a suggested expression for entropy as a function of rationalised area given by $S = \frac{1}{2}\ln 2\hbar^{-1}\alpha$, which is also reproduced in this thesis. The zeroth and third law of black hole thermodynamics are also briefly discussed. An investigation is made into Hawking radiation with the calculation of the mean density of particles created in a Schwarzschild spacetime. This was found to be $n_{\Omega} = \frac{1}{e^{\frac{2\pi\Omega}{\kappa}}-1}$. As a result, the Hawking temperature was then calculated to be $T_{H} = \frac{\hbar c^{3}}{8\pi G M k_{B}}$. Finally, this was used to calculate the lifetime of a Schwarzschild black hole to be given by $t_{life} = \frac{4\pi^{3}k_{B}^{4}}{15c^{2}\hbar^{3}}M_{initial}^{3}$.

Acknowledgement

I thank my supervisor Professor Manya Sahni for her incredible support. I thank my friend Cian Boland for sharing his passion and insights in this area. I thank my parents for always supporting me in my studies.

Contents

1 Introduction

In 1973 Jacob Bekenstein, a student of John Archibald Wheeler, published a paper in which he developed upon the analogies of the known laws black holes at the time and the classical laws of thermodynamics (1). He derived the first law of black hole thermodynamics as well as a generalised second law of black hole thermodynamics. At the time it was not believed that black holes could radiate therefore could not be a thermodynamic system. However, Bekenstein still developed these analogies as they were remarkable in resembling the classical laws. He cleverly employed concepts of information theory to develop an idea of black hole entropy without any awareness of its ability to radiate. Later in 1975, Stephen Hawking published a paper detailing the radiation of black holes due to quantum fluctuations into particle pairs, which would come to be known as Hawking Radiation (2). This was used to make numerous calculations including the temperature of a black hole and a black hole's lifetime. Hawking was also able to revise the exact solutions of Bekenstein's findings by performing a quantum treatment of black holes.

2 | Black Holes

2.1 Introduction to Black Holes

Hawking defines a black hole on a spacelike surface to be a connected component of the region of the surface bounded by the event horizon (3). The event horizon is a boundary in a spacetime through which radial timelike geodesics take infinite coordinate time to escape (see Appendix B for the calculation of the time taken for a radial timelike geodesics to escape the event horizon). Hawking and Penrose define a singularity to have geodesics which cannot be smoothly extended and the point at which the geodesic terminates is known as the singularity (4), (See Appendix A for more details on singularities).
The no-hair theorem states that black holes can only be described by three macroscopic, observable quantities: mass, angular momentum and charge. Therefore the Kerr-Newman metric describes all solutions to the Einstein-Maxwell equations, as follows:

	$Q = 0$	$Q \neq 0$
$\vec{a} = \vec{0}$	Schwarzchild	Kerr
$\vec{a} \neq \vec{0}$	Reissner-Nordström	Kerr-Newman

One may "switch off" angular momentum or charge terms.

2.2 Schwarzschild Black Holes

Based on one of Einsteins's original papers on general relativity (5) Karl Schwarzschild proposed the first black hole metric solution, for what would come to be known as the Schwarzschild black hole (6). Schwarzschild black holes being static have no angular momentum. They also exist in a vacuum, therefore they have no charge (See appendix A for information on static black holes).

Birkhoff's Theorem

For a spherically symmetric object in a vacuum, the unique solution for the metric is given by:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

where r is the radius of the object and M its mass (7). This is the Schwarzschild metric and it describes the area outside of a spherically symmetric and static object in a vacuum of radius $r < 2GM$.

Proof of Birkhoff's Theorem:

Begin with the equation of a general metric in spherical coordinates:

$$ds^2 = -g_{00}(r)dt^2 + g_{\nu\nu}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

For convenience when taking derivatives, let $g_{00} = -e^{2A(r)}$ and $g_{rr} = e^{2B(r)}$. The metric now takes the form:

$$ds^2 = -e^{2A(t)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 \sin^2\theta d\phi^2).$$

This metric will satisfy the vacuum Einstein equation given by:

$$R_{\mu\nu} = 0. \tag{2.2}$$

(See derivation of vacuum Einstein equation in appendix A.)
Therefore

$$R_{tt} = 0,$$

$$\Rightarrow A'' + A'^2 - A'B' + \frac{2A'}{V} = 0,$$

where primed denotes the derivative with respect to r. And

$$R_{rr} = 0,$$

$$\Rightarrow -A'' + A'B' + \frac{2B'}{r} - A'^2 = 0.$$

These give us

$$A = -B.$$

From the Vacuum Einstein equation we again obtain:

$$R_{\theta\theta} = 0,$$

$$\Rightarrow e^{2A} r = 1 + \frac{C}{r}.$$

Now we may substitute these expressions into the equation of the metric in spherical coordinates to obtain:

$$ds^2 = -\left(1 + \frac{C}{r}\right) dt^2 + \left(1 + \frac{C}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

One may take the Newtonian limit (the limit of the weak gravitational field and also the limit of a static system) by taking the Newtonian equations of motion for a body in free fall (see equation A.4):

$$a = -\nabla\Phi,$$

where $\Phi = \frac{-2GM}{r}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

$$\Rightarrow ds^2 = -(1 - 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2).$$

Therefore the unique Solution is given by:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

as needed.

Event Horizon

This metric has a true singularity at $r = 0$ and a coordinate singularity at $r = 2GM$, the latter being the location of the event horizon. The location of the event horizon can be shown to be a coordinate singularity, as under some change of coordinates the singularity does not appear. This is true for the **Kruskal–Szekeres coordinates**, under which the metric does not become singular at the horizon. The line element of the Schwarzschild metric in Kruskal-Szekeres can be seen below:

$$ds^2 = \frac{4{r_s}^3}{r} e^{-\frac{r}{r_s}}(-dT^2 + dX^2) + r^2 d\Omega^2,$$

where $r_s = 2GM$ (8).

2.2.1 Surface Gravity

We will now work in Eddington-Finklestein Coordinates:

$$\begin{aligned}
dv &= dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \\
d\tilde{r} &= dr \\
d\tilde{\theta} &= d\theta \\
d\tilde{\phi} &= d\phi.
\end{aligned}$$

The black hole horizon corresponds to a hypersurface of constant $S(r) = 2M$. The normal vector to this surface is given by $l^{\mu} = fg^{\mu\nu}\partial_{\nu}S = f(1, \left(1 - \frac{2M}{r}\right), 0, 0)$ which is null at $r = 2M$, and it is timelike everywhere outside the horizon. ∂_t is the timelike Killing vector associated with Schwarzschild spacetime which is the same vector field as ∂_v. The **surface gravity** κ is defined by

$$\xi^{\mu}\nabla_{\mu}\xi^{\nu} = \kappa\xi^{\nu}.$$

Let $\xi = \partial_v$ and solve this equation for the $\mu = v$ component.

$$\begin{aligned}
\xi^{\sigma}\nabla_{\sigma}\xi^{v} &= \partial_{\sigma}\xi^{\sigma}\xi^{v} + \xi^{\sigma}\Gamma^{v}_{\sigma\rho}\xi^{\rho} \\
&= \Gamma^{v}_{vv} \\
&= \frac{1}{2}g^{v\sigma}(\partial_v g_{v\sigma} + \partial_{\sigma}g_{\sigma v} - \partial_{\sigma}g_{vv}) \\
&= -\frac{1}{2}g^{vr}\partial_r \\
&= \frac{M}{r^2} \\
\Rightarrow \kappa &= \frac{1}{4M}.
\end{aligned} \tag{2.3}$$

This is the surface gravity of a Schwarzschild black hole.

2.3 Reissner–Nordström Black Holes

Reissner-Nordström black holes have charge and no angular momentum. Due to the accepted view of electro-neutrality of the universe, highly charged black holes cannot exist for long periods of time, and reduce to uncharged black holes. Coulomb's law also ensures that when charged particles enter the black hole, that oppositely charged particles will also enter.

Reissner-Nordström black holes are spherically symmetric, however they cannot exist in a vacuum due to lack of the charge carriers. Therefore, in this spacetime $T_{\mu\nu} \neq 0 \Rightarrow R \neq 0$, hence its spherical symmetry still complies with the uniqueness prescription of Birkhoff's theorem.

In order to find the metric, begin with the general form of a spherically symmetric one:

$$ds^2 = -g_{00}(r)dt^2 + g_{\nu\nu}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

In this case the energy-momentum tensor is given by:

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma},$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor. The spherically symmetric electromagnetic field strength tensor has as its non-zero entries $F_{tr} = -F_{rt}$ and $F_{\theta\phi} = -F_{\phi\theta}$.

The Einstein-Maxwell equations are given by (9):

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.4}$$

$$\nabla^{\mu}F_{\nu\rho} = 0, \tag{2.5}$$

$$\nabla_{[\mu}F_{\nu\rho]} = 0. \tag{2.6}$$

These constitute the **equations of motion** for the Reissner-Nordström metric. They are used to write the metric explicitly:

$$\begin{aligned} ds^2 &= -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2 \\ \Delta &= 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}. \end{aligned} \tag{2.7}$$

Singularities

This metric has a true curvature singularity at $r = 0$ since the Δ term blows up at this point. This can be verified by computing the **Kretschmann scalar** which is given by:

$$\mathcal{K} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{2.8}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann Curvature Tensor. The Kretschmann scalar for the Reissner-Nordström metric is given by (8):

$$\mathcal{K} = 4\frac{3r_s^2r^2 - 12r_s r\rho Q^2 + 14\rho^2 Q^4}{r^8}.$$

The $\frac{1}{r^8}$ term allows us to classify $r = 0$ as a true singularity, since the Kretschmann scalar will not transform under coordinate transformations.

In this coordinate system location of the **event horizon** is at the surface given by: $g^{rr} = 0$ (9), therefore it can be found by:

$$g^{rr}(r) = \Delta(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} = 0.$$

Solving the quadratic gives:

$$r_{\pm} = GM \pm \sqrt{2G^2M^2 - 2G(Q^2 + P^2)}. \tag{2.9}$$

In the case that $GM^2 > Q^2 + P^2$ (which is realistic since the energy in the electromagnetic field is less than the total energy), the metric has coordinate singularities at $r_{\pm}$ removable by a change of coordinates.

In this case that $GM^2 > Q^2 + P^2$ there exists a **naked singularity**. These cases disobey the cosmic censorship conjecture which “forbids the appearance of naked singularities" (see appendix C), however they are a theoretically acceptable occurrence (10).

The **extreme Reissner-Nordström solution** occurs at the case where $GM^2 = Q^2 + P^2$. By solving the quadratic one can see that these extremal black holes have the property $\Delta(r) = 0$ only when $r = GM$.

2.4 Kerr Black Holes

Kerr Black Holes are rotate with respect to time, in a vacuum ($R = 0$). There can no longer be spherical symmetry but we may explore solutions with axial symmetry around a timelike Killing vector.
The metric for a Kerr black hole is given by:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 \\ &\quad + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2, \end{aligned} \tag{2.10}$$

where $\Delta(r) = r^2 - 2GMr + a^2$ and $\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$. $a = \frac{J}{M}$ is the (Komar) angular momentum per unit mass and the mass M is equal to the Komar energy (11) (see section 3.2 ‘Komar Integrals’ for the derivation of these conserved quantities). If one ‘turns off’ the angular momentum, $a = 0$, the metric of the Kerr black hole reduces to:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which is exactly the Schwarzschild metric, as expected.

Singularities

The metric has 1 true singularity and 2 coordinate singularities. The Killing vectors of this metric are given by: $K = \partial_t$ and $R = \partial_r$. In this coordinate system the event horizons occur at the fixed values of r for which $g^{rr} = 0$:

$$g^{rr} = \frac{\Delta}{\rho^2} = 0$$
$$\Rightarrow r^2 - 2GMr + a^2 = 0,$$

at any event horizon locations.
Similarly to before the most realistic case occurs at $GM > a$, in this case solving the quadratic gives:

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2},$$

which are null surfaces and event horizons. The norm of the Killing vector K is given by:

$$K^{\mu}K_{\mu} = \frac{a^2}{\rho^2}\sin^2\theta \geq 0,$$

therefore it is null at the poles ($\theta = 0, \pi$) and timelike everywhere else. The stationary limit surface occurs at $K^{\mu}K_{\mu} = \frac{a^2}{\rho^2}\sin^2\theta = 0$. The stationary limit surface bounds the region in which it is impossible to be a stationary observer. This occurs when

$$(r - GM)^2 = G^2M^2 - a^2\cos^2\theta,$$

and the outer event horizon occurs at

$$(r_+ - GM)^2 = G^2M^2 - a^2.$$

These the outer event horizon and the stationary limit surface bound the region known as the **ergosphere**. See figure 2.1 below for a side view of the horizon structure of this kind of black hole (9).

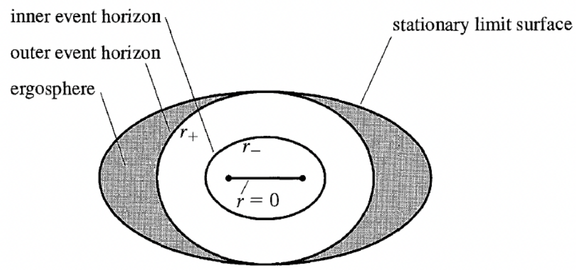

Figure 2.1: Horizons of a Rotating Black Hole

2.5 Kerr–Newman Black Holes

Kerr-Newman black holes have mass, charge and angular momentum. This is the most general equilibrium state black holes solutions of Einstein's equations since the no-hair theorem states that all solutions of the Einstein-Maxwell equations are described by these three properties. The Kerr-Newman metric is given by:

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2GMr - \frac{Q^2}{G}}{\rho^2}\right) dt^2 - \frac{a(2GMr - \frac{Q^2}{G})\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 \\
&\quad + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \qquad (2.11) \\
\Delta &= r^2 - 2Mr + a^2 + Q^2 \\
\rho^2 &= r^2 + a^2\cos\theta \\
\Sigma^2 &= (r^2+a^2)^2 - a^2\Delta\sin^2\theta.
\end{aligned}
$$

Where M is the mass of the black hole, Q is its charge, a is its angular momentum per unit mass as before. The Killing vector field (equation A.5) which describes the rotation of the event horizons is given by a linear combination of the time-translational and rotational Killing vectors:

$$
\begin{aligned}
\Psi^a &= \left(\frac{\partial}{\partial t}\right)^a + \Omega_H\left(\frac{\partial}{\partial \phi}\right)^a \qquad (2.12) \\
\Omega_H &= \frac{a}{r_\pm^2 + a^2},
\end{aligned}
$$

where Ω_H is the (constant) angular velocity at the horizon.

Singularities and Horizons

There are coordinate singularities at $\Delta = 0$. This is expressed in

$$\Delta = (r - r_+)(r - r_-).$$

Therefore, the locations for the event horizons are

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{2.13}$$

2.5.1 Surface Area

The surface area is found by a volume integral over the Killing hypersurface which forms the event horizon. In this case we will integrate over the outer event horizon as we consider this the entire surface of the black hole (12).
The volume element is

$$dV = \sqrt{|g|} d^n x,$$

where g is the determinant of the metric and n is the dimension in question. Consider a time slice with constant radius $r = r_+$. The surface area here is given by

$$\begin{aligned} A &= \int_0^\pi \int_0^{2\pi} \left[\sqrt{g_{\theta\theta} g_{\phi\phi}|} \right]_{r=r_+} d\theta d\phi \\ &= \int_0^\pi \int_0^{2\pi} (r_+^2 + a^2) \sin\theta d\theta d\phi \\ &= 4\pi(r_+^2 + a^2). \end{aligned}$$

Define the rationalised area α as

$$\begin{aligned} \alpha &= \frac{A}{4\pi} \\ &= (r_+^2 + a^2). \end{aligned} \tag{2.14}$$

A black hole may capture particles with angular momentum, charge and mass which will change these parameters for the black hole, however the resultant area will not increase. This is Hawking's area theorem (13).

2.5.2 Surface Gravity

Rewriting the Kerr-Newman metric in Eddington Finklestein coordinates:

$$\begin{aligned} ds^2 &= \frac{\nabla - a^2\sin^2\theta}{\rho^2}d\theta^2 + 2d\theta dr - \frac{2a(r^2+a^2-\Delta)\sin^2\theta}{\rho^2}d\theta d\chi \\ &- 2a\sin^2\theta dr d\chi - \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\rho^2}d\chi^2. \\ d\theta &= dt + \frac{r^2+a^2}{\Delta}dr \\ d\chi &= d\phi + \frac{a}{\Delta}dr. \end{aligned} \tag{2.15}$$

Rewriting the Killing vector from equation 2.12 in these coordinates

$$\Psi = \frac{\partial}{\partial\theta} + \frac{a}{(r_\pm^2 + a^2)}\frac{\partial}{\partial\chi}.$$

As seen before, the surface gravity is defined by the following equation defined at the event horizon

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa\xi^\nu.$$

Therefore from the metric (2.15) one may find (12)

$$\begin{aligned} \kappa_+ &= \frac{r_+ - r_-}{2(r_+^2 + a^2)} \\ \kappa_- &= \frac{r_- - r_+}{2(r_-^2 + a^2)}, \end{aligned} \tag{2.16}$$

the surface gravity at the outer and inner event horizons, respectively.

2.5.3 Extreme Kerr-Newman Black Holes

At $M^2 = a^2 + Q^2$ this corresponds to extreme solutions where they lose their event horizons and the surface gravity is $\kappa = 0$.

3 | Preliminaries to Black Hole Thermodynamics

3.1 The Penrose Process and Irreducible Mass

The Penrose Process is a theoretical approach to extracting energy from a black hole with an ergosphere (i.e. a rotating black hole). The mechanism is as follows: a particle moves under geodesic motion from infinity and enters the ergosphere of the rotating black hole. Let the 4-momentum of this particle be p_0^a. Then the total energy of the particle is given by:

$$E_0 = -p_0^a \xi_a,$$

where ξ_a denotes the Killing field which becomes a time translation asymptotically at infinity, and is spacelike in the ergosphere. As it falls freely towards the black hole, its total energy stays the same. Suppose the particle splits into two fragments after entering the ergosphere. Due to the conservation of 4-momentum we have

$$p_0^a = p_1^a + p_2^a,$$

where $p_i^a, i = 1,2$ denote the 4-momenta of the two pieces. We can contract the 4-momenta with the Killing vector field ξ_a to obtain an equation for the energy of the fragments:

Inside the ergosphere it is not necessary that the particles have positive energy: since k is spacelike in the ergosphere we can have a system with total energy of particle 1 being $E_1 = k_\mu p_1^\mu < 0$, one example of this is if one has the system $k^\mu = (0, x, 0, 0)$ and $p_1^\mu = (1, y, 0, 0)$ where y is small enough for p_1 to remain timelike and where $xy > 0$, then in this case $E_1 < 0$. The other fragment can return to infinity under free geodesic motion - with energy E_2 we have that

$$E_2 = E + |E_1|,$$

and the mass of the black hole is reduced to $M - |E_1|$. The theoretical yield of energy extracted from the black hole is $|E_1|$ (9). See below in figure 3.1 a pictorial representation of the Penrose process (14).

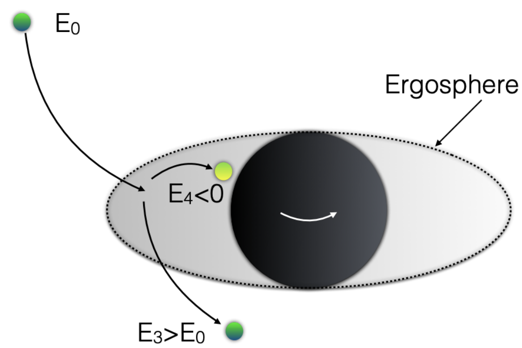

Figure 3.1: Ergoregion of a Kerr black hole in which the Penrose Process takes place

Limit on Energy Extraction

The Killing vector field which describes the rotation of the event horizon is given by a linear combination of the time-translational and rotational Killing vectors:

$$\chi^a = \left(\frac{\partial}{\partial t}\right)^a + \Omega_H \left(\frac{\partial}{\partial \phi}\right)^a,$$

Where Ω_H is the angular velocity given by:

$$\Omega_H = \frac{a}{r_+^2 + a^2}.$$

Hence, for any particle entering the black hole moving forward in time has

$$0 > p^a \chi_a = p^a(\xi_a + \Omega_H \psi_a) = -E + \Omega_H L,$$

where L is the angular momentum of the particle defined by $L = p^a \psi_a$. Therefore

$$L < \frac{E}{\Omega_H}.$$

Thus, any particles entering the black hole with negative energy carry with them negative angular momentum relative to the rotation of the black hole. Therefore, this process is self-limiting, as the more energy extracted from the black hole, the less angular momentum the black hole retains. When angular momentum $J = Ma$ is reduced to zero the black hole is reduced to a non-rotating one, $J = 0$ (while M is still finite). It will no longer have an ergosphere and the Penrose process cannot occur. After the particle enters the black hole, the mass of the black hole should increase by the energy of the particle $\delta M = E_2$ and the angular momentum of the black hole should increase by the angular momentum of the particle $\delta J = L_2$ therefore

$$\delta J < \frac{\delta M}{\Omega_H}. \tag{3.1}$$

It is convenient to work in terms of the irreducible mass of a black hole:

$$M_{irr}^2 = \frac{1}{2}[M^2 + (M^4 - J^2)^{\frac{1}{2}}].$$

Differentiating this gives

$$\delta M_{irr} \propto (\Omega_H^{-1}\delta M - \delta J).$$

Therefore equation 3.1 is equivalent to the condition that

$$\delta M_{irr} > 0. \tag{3.2}$$

This sets the limit on energy extraction from the Penrose process to be $M = M_{irr}$. We can therefore refer to $M_0 - M_{irr}$ as the rotational energy of the black hole (9).

3.2 Komar Integrals

How do Komar integrals describe Conserved Quantities?

An anti-symmetric tensor $K_{\mu\nu}$ has an associated two form given by:

$$K = \frac{1}{2}K_{\mu\nu}dx^\mu \wedge dx^\nu.$$

The curvature tensor is given by:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)k^\sigma = R^\sigma_{\rho\mu\nu}k_\rho.$$

Contract σ and μ to obtain the expression:

$$\nabla_\mu \nabla_\nu k^\mu = R_{\rho\nu}k^\rho \Leftrightarrow \nabla_\mu K^\mu_\nu = R_{\rho\nu}k^\rho.$$

Einstein's equations are given by:

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 8\pi G T_{\mu\nu} \\
\Leftrightarrow R_{\rho\nu} &= 8\pi G (T_{\rho\nu} - \frac{1}{2} g_{\rho\nu} T).
\end{aligned}
$$

In terms of differential forms, Einstein's field equations are given by:

$$
\begin{aligned}
d \star K &= 8\pi G \star \zeta \\
d \star \zeta &= 0,
\end{aligned}
$$

where the co-vector

$$\zeta_\nu \equiv \frac{1}{8\pi G} R_{\rho\nu} k^\rho,$$

and the scalar

$$\zeta = \zeta_\nu dx^\nu.$$

We may choose a volume B such that all matter is inside of it and spacetime is a vacuum outside of it. Define the Komar integral as

$$
\begin{aligned}
Q_k(B) &\equiv \int_B \star\zeta \\
&= \frac{1}{8\pi G} \int_B d \star K \\
&= \frac{1}{8\pi G} \int_{\partial B} \star K,
\end{aligned}
$$

by Stoke's theorem.

Apply the continuity equation to the current $\star\zeta$ over a 4-volume V bounded by B made of surfaces B_1 and B_2:

$$
\begin{aligned}
0 &= \int_V d \star \zeta \\
&= \int_{\partial B} \star\zeta \\
&= \int_{\partial B_1} \star\zeta - \int_{\partial B_2} \star\zeta \\
&= \frac{1}{8\pi G} \int_{\partial B_1} \star K - \frac{1}{8\pi G} \int_{\partial B_2} \star K \\
&= Q_k(B_1) - Q_k(B_2) \\
&\Rightarrow Q_k(B_1) = Q_k(B_2).
\end{aligned}
$$

Therefore, for a region B described above, the value $Q_k(B)$ is conserved.

Conserved Quantities in Schwarzschild Spacetime

We have a Killing vector $k = \partial_t$. In the tangent space with its standard basis of partial derivatives this Killing vector is written as $k = k^\mu e_\mu$, $k^\mu = (1, 0, 0, 0)^T$, $e_t = \frac{\partial}{\partial x^t}$.
Since k is a Killing vector, it satisfies the Killing equation:

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0,$$

i.e. the symmetric part of $\nabla_\mu k_\nu$ vanishes. We define the anti-symmetric tensor:

$$\nabla_\mu k_\nu = \nabla_{[\mu k \nu]} = \frac{1}{2}(\nabla_\mu k_\nu - \nabla_\nu k_\mu) \equiv K_{\mu\nu}.$$

Recall, the definition of the covariant derivative of a vector is as follows:

$$\nabla_\mu k^\nu = \frac{\partial k^\nu}{\partial z^\mu} + \Gamma^\nu_{\rho\mu} k^\rho.$$

Examining our Killing vector $k = (1, 0, 0, 0)^T$ one can easily see that in the first term

$$\frac{\partial k^\nu}{\partial z^\mu} = \frac{\partial}{\partial z^\mu} \delta^\nu_t = 0.$$

In the second term the only terms contributing to the implicit sum over ρ are the terms where $\rho = t \Rightarrow k^\rho = 1$.

$$\Rightarrow \nabla_\mu k^\nu = \Gamma^\nu_{t\mu}.$$

To obtain the covariant derivative of the co-vector k_ν lower the index on k^μ with the metric:

$$k_\mu = g_{\mu\nu} k^\nu,$$

substituting this into the definition for the covariant derivative:

$$\nabla_\nu k_\mu = \nabla g_{\mu\nu} k^\mu + g_{\mu\nu} \nabla_\mu k^\nu.$$

The term $\nabla g_{\mu\nu}$ vanishes for the Levi-Civita connection (by definition).

$$\Rightarrow \nabla_\mu k_\nu = g_{\nu\sigma} \nabla_\mu k^\sigma$$
$$\Rightarrow \nabla_\mu k_\nu = g_{\mu\sigma} \Gamma^\sigma_{t\mu}.$$

Recall the metric for a Schwarzschild spacetime is given by:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This is represented by a diagonal matrix therefore there are no mixed terms $g_{\mu\nu}$, $\mu \neq \nu$. The non-vanishing Christoffel Symbols of the form $\Gamma^{\sigma}_{t\mu}$ are given by (8):

$$\begin{aligned}\Gamma^t_{tr} &= \frac{r_s}{2r(r - r_s)} \\ \Gamma^r_{tt} &= \frac{r_s(r - r_S)}{2r^3},\end{aligned}$$

where r_s denotes the Schwarzschild radius: $r_s = 2GM$. Therefore the non-vanishing terms of $\nabla_\mu k_\nu$ are

$$\begin{aligned}\nabla_r k_t &= g_{tt}\Gamma^t_{tr} = K_{rt} \\ \nabla_t k_r &= g_{rr}\Gamma^r_{tt} = K_{tr}.\end{aligned}$$

Using the coefficients of the Schwarzschild metric above we calculate

$$\begin{aligned}g_{tt}\Gamma^t_{tr} &= -\frac{r_s}{2r^2} = K_{rt} \\ g_{rr}\Gamma^r_{tt} &= \frac{r_s}{2r^2} = K_{tr}.\end{aligned}$$

As expected for an anti-symmetric tensor, $K_{rt} = -K_{tr}$. The associated two-form to an anti-symmetric tensor $K_{\mu\nu}$ is given by:

$$\begin{aligned}K &= \frac{1}{2}K_{\mu\nu}dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}K_{rt}dx^r \wedge dx^t + \frac{1}{2}K_{tr}dx^t \wedge dx^r \\ &= \frac{1}{2}(-\frac{r_s}{2r^2})dx^r \wedge dx^t + \frac{1}{2}(\frac{r_s}{2r^2})dx^t \wedge dx^r \\ &= \frac{rs}{2r^2}dx^t \wedge dx^r.\end{aligned}$$

The definition of the Hodge star of a k-form in d-dimensions is given by:

$$(\star F)_{\mu_1\mu_2\ldots\mu_{n-k}} = \sqrt{-g}\,\epsilon_{\mu_1\mu_2\ldots\mu_{n-k}}{}^{\sigma_1\sigma_2\ldots\sigma_k}F_{\sigma_1\sigma_2\ldots\sigma_k}, \tag{3.3}$$

where n is the dimension of the space and k is the dimension of our tensor. Therefore the Hodge dual of our 2-form $K_{\mu\nu}$ is given by:

$$\begin{aligned}
(\star K)_{\mu\nu} &= \sqrt{-g}\epsilon^{\rho\sigma}_{\mu\nu}K_{\rho\sigma} \\
&= \sqrt{-g}(\epsilon^{01}_{\mu\nu}K_{01} + \epsilon^{10}_{\mu\nu}K_{10}) \\
\Rightarrow (\star K)_{\theta\phi} &= \sqrt{-g}(K_{01} - K_{10}) \\
(\star K)_{\phi\theta} &= \sqrt{-g}(K_{10} - K_{01}).
\end{aligned}$$

Therefore

$$\begin{aligned}
(\star K)_{\theta\phi} &= 2r^2 \sin\theta K_{rt} \\
&= -r\sin\theta \\
&= -(\star K)_{\phi\theta}.
\end{aligned}$$

The associated two-form is given by:

$$\star K = -r_s \sin\theta d\theta \wedge d\phi.$$

We choose a volume B such that all matter is inside of it and spacetime is a vacuum outside of it, in this case a sphere of radius $2GM = r_s$, the boundary of B is then a sphere of radius r_s. This bounds all relevant matter, therefore by the definition of the Komar integral, $Q_k(B)$ is conserved. Now we have derived all of the pieces in order to perform the Komar integral:

$$\begin{aligned}
Q_k(B) &\equiv \frac{1}{8\pi G}\int_{\partial B} \star K \\
&= \frac{1}{8\pi G}\int_{\partial B} -r_s \sin\theta d\theta \wedge d\phi \\
&= -\frac{2GM}{8\pi G}\int \sin\theta(-d\phi d\theta) \\
&= \frac{M}{4\pi}(-\cos\theta)|_0^\pi \phi|_0^{2\pi} \\
&= M.
\end{aligned} \tag{3.4}$$

Thus we have shown that mass is a conserved quantity in Schwarzschild spacetime.

Conserved Quantities in Kerr Spacetime

The Kerr spacetime is characterised by the Killing vector $k = \partial_\phi$. Similarly to before the associated tensor is given by:

$$\begin{aligned} K_{\mu\nu} &= \nabla_\mu k_\nu \\ &= g_{\sigma\nu}\Gamma^\sigma_{\phi\mu}. \end{aligned}$$

Looking at the non-vanishing Christoffel symbols and components of the metric the non-vanishing entries of $K_{\mu\nu}$ are given by:

$$\begin{aligned} K_{tr} &= -K_{rt} \\ &= g_{rr}\Gamma^r_{\phi t}, \\ K_{\phi r} &= -K_{r\phi} \\ &= g_{rr}\Gamma^r_{\phi\phi}, \\ K_{t\theta} &= -K_{\theta t} \\ &= g_{\theta\theta}\Gamma^\theta_{\phi t}, \\ K_{\phi\theta} &= -K_{\theta\phi} \\ &= g_{\theta\theta}\Gamma^\theta_{\phi\phi}. \end{aligned}$$

The associated two-form to $K_{\mu\nu}$ is given by:

$$\begin{aligned} K &= \frac{1}{2}K_{\mu\nu}dx^\mu dx^\nu \\ &= K_{tr}dt \wedge dr + K_{r\phi}dr \wedge d\phi + K_{t\theta}dt \wedge d\theta + K_{\phi\theta}d\phi \wedge d\theta. \end{aligned}$$

Using the definition of the hodge star operation (3.3) on a two-form we find that

$$(\star K)_{\mu\nu} = \sqrt{-g}\epsilon^{\sigma\rho}_{\mu\nu}K_{\sigma\rho}.$$

It's non-vanishing elements are given by:

$$\begin{aligned}
(\star K)_{tr} &= -(\star K)_{rt} \\
&= \sqrt{-g} K_{\phi\theta}, \\
(\star K)_{r_\phi} &= -(\star K)_{\phi r} \\
&= \sqrt{-g} K_{t\theta}, \\
(\star K)_{t\theta} &= -(\star K)_{\theta t} \\
&= \sqrt{-g} K_{r\phi}, \\
(\star K)_{\phi\theta} &= -(\star K)_{\theta\phi} \\
&= \sqrt{-g} K_{tr}.
\end{aligned}$$

Therefore its associated two-form is given by:

$$\begin{aligned}
\star K &= \frac{1}{2}(\star K)_{\mu\nu} dx^\mu \wedge dx^\nu \\
&= \sqrt{-g}(K_{\phi\theta} dt \wedge dr + K_{t\theta} d\phi \wedge dr + K_{r\phi} d\theta \wedge dt + K_{tr} d\phi \wedge d\theta) \\
&= \sqrt{-g}(g_{\theta\theta}\Gamma^\theta_{\phi\phi} dt \wedge dr + g_{\theta\theta}\Gamma^\theta_{\phi t} d\phi \wedge dr + g_{rr}\Gamma^r_{\phi\phi} dt \wedge d\theta + g_{rr}\Gamma^r_{\phi t} d\phi \wedge d\theta).
\end{aligned}$$

The only coefficients of the metric and Christoffel symbols needed are given by (8):

$$\begin{aligned}
g_{\theta\theta} &= \Sigma \\
g_{rr} &= \frac{\Sigma}{\Delta} \\
\Gamma^\theta_{\phi\phi} &= -\frac{\sin\theta}{\Sigma^3}\left[A\Sigma + (r^2+a^2) r_s a^2 r \sin^2\theta\right] \\
\Gamma^\theta_{\phi t} &= \frac{c r_s a r (r^2+a^2)\sin\theta\cos\theta}{\Sigma^3} \\
\Gamma^r_{\phi\phi} &= \frac{\Gamma\sin^2\theta}{2\Sigma^3}\left[-2r\Sigma^2 + r_s a^2 \sin^2\theta(r^2 - a^2\cos^2\theta)\right] \\
\Gamma^2_{\phi t} &= -\frac{c\Delta r_s a \sin^2\theta(r^2 - a^2\cos^2\theta)}{2\Sigma^3}.
\end{aligned}$$

Where

$$\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\theta, \\
\Delta &= r^2 - r_s r + a^2, \\
r_s &= 2GM, \\
a &= \frac{J}{M}.
\end{aligned}$$

The determinant of the Kerr metric is given by:

$$g = -\Sigma^2 \sin^2\theta.$$

Substituting these into the expression for $\star K$ one may calculate the associated Komar integral.

$$\begin{aligned} Q_k(B) &= \frac{1}{8\pi G}\int_{\partial B} \star K \\ &= J. \end{aligned} \tag{3.5}$$

Thus we have shown that the angular momentum is a conserved quantity for the Kerr spacetime.

For a spacetime with both Killing vectors ∂_t and ∂_ϕ (such as the Kerr-Newman Spacetime), there will be two conserved quantities: mass and angular momentum.

4 | Zeroth Law of Black Hole Thermodynamics

The classical zeroth law of thermodynamics is that a system in thermal equilibrium has constant temperature throughout the system. The corresponding zeroth law of black hole thermodynamics is that stationary black holes have constant surface gravity over the entire horizon (9). One can show this easily since the event horizon of a black hole is a Killing horizon therefore equation 2.3 is satisfied. Since the Killing vector is perpendicular to all of the tangent vectors t at the horizon, one can write

$$t^{\mu}\nabla_{\mu}\kappa = 0.$$

Therefore κ is constant over the horizon. This is expected from the definition of surface gravity since it does not depend on the radial or angular coordinates, therefore a stationary solution will have a constant surface gravity (12).

5 | First Law of Black Hole Thermodynamics

From Bekenstein's 1973 paper (1) the first law is given by:

$$dM = \theta d\alpha + \vec{\Omega} \cdot d\vec{L} + \Phi dQ. \tag{5.1}$$

Derivation of First Law: As seen in equation 2.14 the rationalised area of a Kerr-Newman black hole is given by:

$$\alpha = 2Mr_+ - Q^2,$$

where $r_+ = M + (M^2 - Q^2 - a^2)^{\frac{1}{2}}$. Differentiating this we obtain

$$d\alpha = 2r_+ dM + 2M dr_+ - 2Q dQ.$$

From the definition of r_+ we have that

$$dr_+ = MdM + \frac{1}{2}(M^2 - Q^2 - a^2)^{-\frac{1}{2}}(2MdM - 2QdQ - 2\vec{a}d\vec{a}),$$

by the product rule. The formula for $\vec{a}$ is given by: $\vec{a} = \vec{L}M^{-1}$ therefore

$$d\vec{a} = M^{-1}d\vec{L} - \vec{L}M^{-2}dM.$$

Then

$$dr_+ = dM + (M^2 - Q^2 - a^2)^{-\frac{1}{2}}(MdM + L^2M^{-3}dM - QdQ - \vec{L}M^{-2}d\vec{L}).$$

Therefore

$$\begin{aligned} d\alpha = dM[2r_+ &+ 2M + 2(M^2 - Q^2 - a^2)^{-\frac{1}{2}}(M^2 + a^2] + dQ[-2QM(M^2 - Q^2 - a^2)^{-\frac{1}{2}} - 2Q] \\ &- d\vec{L}[-2\vec{a}(M^2 - Q^2 - a^2)^{-\frac{1}{2}}]. \end{aligned} \tag{5.2}$$

We now have our equation in the form:

$$d\alpha = XdM + \vec{Y} \cdot d\vec{L} + ZdQ, \tag{5.3}$$

renaming our coefficients of dM, d$\vec{L}$ and dQ as X,Y and Z respectively. This is equivalent to

$$dM = \frac{1}{X}d\alpha - \frac{\vec{Y}}{X}d\vec{L} - \frac{Z}{X}dQ.$$

From equation 5.2 we have that

$$X = 2(M^2 - Q^2 - a^2)^{-\frac{1}{2}}(2M(M^2 - Q^2 - a^2)^{\frac{1}{2}} + (M^2 - Q^2 - a^2) + (M^2 + a^2)).$$

Therefore

$$X = 2(M^2 - Q^2 - a^2)^{-\frac{1}{2}}(M^2 + 2M(M^2 - Q^2 - a^2)^{\frac{1}{2}} + (M^2 - Q^2 - a^2) + a^2).$$

From the definition of r_+ this gives

$$X = 2(M^2 - Q^2 - a^2)^{-\frac{1}{2}}(r_+^2 + a^2).$$

From the definition of α this gives

$$X = 2\alpha(M^2 - Q^2 - a^2)^{-\frac{1}{2}}.$$

Therefore

$$X = \frac{4\alpha}{(r_+ - r_-)},$$

so

$$\frac{1}{X} = \frac{(r_+ - r_-)}{4\alpha}.$$

Moving onto the second coefficient $\vec{Y}$ we have:

$$\frac{\vec{Y}}{X} = \frac{4\vec{a}}{4\alpha}(M^2 - Q^2 - a^2)^{-\frac{1}{2}}(M^2 - Q^2 - a^2)^{\frac{1}{2}}.$$

Therefore

$$-\frac{\vec{Y}}{X} = \frac{\vec{a}}{\alpha}.$$

Finally, for the third coefficient Z,

$$-\frac{Z}{X} = \frac{Q}{2\alpha}(M(M^2 - Q^2 - a^2)^{-\frac{1}{2}} + 1)(2(M^2 - Q^2 - a^2)^{\frac{1}{2}}).$$

Therefore

$$-\frac{Z}{X} = \frac{Q}{\alpha}(M + (M^2 - Q^2 - a^2)^{\frac{1}{2}}),$$

so

$$-\frac{Z}{X} = \frac{Qr_+}{\alpha}.$$

Making appropriate substitutions into (3.8) we obtain

$$dM = \frac{(r_+ - r_-)}{4\alpha}d\alpha + \frac{\vec{a}}{\alpha}d\vec{L} + \frac{Qr_+}{\alpha}dQ,$$

as required.

Defining new Paramaters from Bekenstein's paper we obtain his first law in its exact form:

$$\begin{aligned}
\theta &\equiv \frac{1}{4}\frac{(r_+ - r_-)}{\alpha} \\
\vec{\Omega} &\equiv \frac{\vec{a}}{\alpha} \\
\Phi &\equiv \frac{Qr_+}{\alpha} \\
\Rightarrow dM &= \theta d\alpha + \vec{\Omega}\cdot d\vec{L} + \Phi dQ.
\end{aligned}$$

To formulate the more commonly used form of the first law of black hole thermodynamics, make further substitutions for the surface gravity of a Kerr black hole (2.16) and the rationalised area:

$$\begin{aligned}
\alpha &= \frac{A}{4\pi} \\
\Rightarrow d\alpha &= \frac{dA}{4\pi} \\
\kappa &= \frac{r_+ - r_-}{2(r_+^2 + a^2)} \\
\Rightarrow dM &= \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ,
\end{aligned}$$

where Ω_H is the angular velocity of the black hole at the horizon and Φ_H is the "electric surface potential".

6 | Generalised Second Law of Black Hole Thermodynamics

6.1 Review of Classical Information Theory

Before looking into Bekenstein's calculation of black hole entropy, it is valuable to review the basic notions of classical information theory as this forms the basis of his approach.

Shannon Entropy

Shannon entropy measures the uncertainty of a system based on its internal configuration and is given by the formula (15)

$$S = -\sum_n p_n \ln p_n,$$

alongside minor requirements which demand that it has corresponding characteristics with entropy. The relation between information about the internal configuration and the Shannon entropy of a system is given by

$$\Delta I = -\Delta S.$$

Therefore, as new information becomes available (less uncertainty), the entropy decreases.
The second law of thermodynamics (that thermal entropy is monotonically increasing) can be understood from the viewpoint of information theory as the washing out of the known initial conditions of the system leading to more uncertainty. The Shannon entropy is maximised for 2 variables of equal probability $p_1 = p_2 = \frac{1}{2}$. This corresponds to $\ln 2$ of information which is 1 bit.

6.2 Black Hole Entropy

Bekenstein's idea of black hole entropy was entirely configurational. Recall the no-hair theorem; at late times black holes are described by three macroscopic parameters, mass, charge and angular momentum. This means that black holes do not differ in traits other than these and since their internal configuration is unknown, due to the event horizon (16) they must have some entropy associated with them. It was not proven that black holes could radiate therefore the entropy in question in this paper was not thermal and did not assume radiation from a black hole. Later on in this document we will perform a quantum treatment of black holes based on Hawking's calculations to revise and compare the form of black hole entropy taking into account a black hole's ability to radiate.

Functional Form of the Entropy

Bekenstein's method for finding an expression for the entropy of a black hole was a basic trial and error process of educated guesses. Bekenstein's first law gave him the hint that the analog of entropy for black hole system would be a function of the area of the black hole. Hawking's area theorem states that the area of a black hole is non-decreasing. Since entropy is a monotonically increasing function with the washing away of initial conditions, it must be an increasing function of area.

Begin with the equation:

$$S_{bh} = f(\alpha).$$

The first guess considered is

$$f(\alpha) = \alpha^{\frac{1}{2}}.$$

However, when two black holes coalesce, their entropy is additive. We also expect that the entropy of the new system will be higher than the sum of the entropy of the original systems since information would be lost about the black hole interior. If $f(\alpha) = \alpha^{\frac{1}{2}}$ then the irreducible mass of the new system is greater than the sum of the irreducible masses of the original systems. In the case of all three being Schwarzchild black holes ($m = m_{irr}$), this would mean an increase in mass from coalescence. However this is not possible - only a decrease in mass could be consistent due to possible gravitational radiation.

The second guess considered by Bekenstein was

$$f(\alpha) = \gamma\alpha,$$

for some constant γ. This form of f does not contradict Hawking's area theorem, therefore it was an acceptable guess. Since the entropy is based on Shannon entropy it is a dimensionless quantity. Therefore γ must be of dimension $length^{-2}$. Through dimensional analysis Bekenstein made the link to the universal constant $\hbar^{-1}$, where $\hbar$ is Planck's constant. Therefore we rewrite the formula for entropy as

$$S_{bh} = \eta \hbar^{-1} \alpha,$$

for a dimensionless constant η.

Expression for η

Bekenstein employed a classical argument to calculate the amount of information lost when a particle falls into a Kerr black hole. The minimum information known before the fall is whether the particle exists or not. As seen previously this yes/no question corresponds to one bit and in our units this is $\ln 2$. Therefore at least this much information is lost when a particle falls into the black hole. We can then compare Christodolou's expression for the minimum increase in black hole area from a particle with radius falling in (17). This was found to be

$$(\Delta\alpha)_{min} = 2\mu b,$$

for a particle with proper radius b and rest mass μ. See appendix C for an outline of the steps Christodolou made to find this.

To set a lower bound on the radius of the particle we consider the lower bound given by both its Compton wavelength $\frac{\hbar}{\mu}$ and its gravitational radius 2μ. Therefore in any case the $2\mu b$ may be no smaller than $2\hbar$. Therefore this is the lower bound on the increase in rationalised area of the black hole (given quantum effects).

We treat the particle as "elementary" meaning that we have no information about its internal structure. Therefore, the entire information lost with its capture by a black hole is whether it exists or not. Since this is a yes/no question, its information is quantised as 1 bit or $\ln 2$, in our units. The decrease in information is equal to the increase in entropy, therefore the minimum increase in entropy is given by $\ln 2$. Substituting this into the hypothesised functional form of entropy:

$$\begin{aligned} (\Delta S_{bh})_{min} &= \eta \hbar^{-1} (\Delta\alpha)_{min} \\ \Rightarrow \ln 2 &= \eta \hbar^{-1} 2\hbar \\ \Rightarrow \eta &= \frac{1}{2} \ln 2. \end{aligned}$$

Therefore the proposed form of entropy now becomes

$$S_{bh} = \frac{1}{2} \ln 2\hbar^{-1}\alpha.$$

Bekenstein argued that even though this prediction is based on the assumption that the particle examined is equal to its Compton wavelength, that any other calculation of entropy that could be performed would still give a very similar result.

6.3 Characteristic Temperature of a Black Hole

Bekenstein then substituted the above expression for black hole entropy into the first law of black hole thermodynamics to obtain a characteristic temperature of a black hole:

$$T = 2\hbar(\ln 2)^{-1}\theta.$$

At this point in time there was no evidence that a black hole could radiate therefore the analogy drawn here for temperature was only formal. However, we will see in the next chapter that this is not the case.

6.4 Generalised Second Law of Black Hole Thermodynamics

From Hawking's area theorem and the expression $S_{bh} = \frac{1}{2} \ln 2\hbar^{-1}\alpha$, we may now argue that the entropy of a black hole is always increasing.
Bekenstein then argued that the second law of black hole thermodynamics would take the form

$$\Delta S_{bh} + \Delta S_c = \Delta(S_{bh} + S_c) > 0, (11)$$

where S_c is the "common entropy" in the black hole exterior and S_{bh} is the black hole entropy described above. This is equivalent to stating that the generalised entropy $(S_{bh} + S_c)$ never decreases.
Again, the differentiation between black hole entropy and common entropy is due to Bekenstein's understanding that black holes could not radiate. Now that it is accepted that black holes do radiate and therefore are thermodynamic bodies, we regard the generalised entropy as the usual thermal entropy.

7 | Third Law of Black Hole Thermodynamics

The classical third law of thermodynamics states that to achieve a temperature $T = 0$ the entropy S must also go to zero. However, as seen in section 2.5.3 the surface gravity is zero for extremal black holes and these do not necessarily have zero area (9). However, there is a third law of black hole thermodynamics. It states that the superficial gravity of a black hole is never annulled.

Recall that $\kappa = 0$ when $M^2 = a^2 + Q^2$. This is an extreme Kerr-Newman black hole. These extremal solutions also have $r_+ = r_- = M$. If the angular momentum and charge is increased beyond this point there will be a black hole without horizon i.e. a naked singularity. However this violates the cosmic censorship conjecture. Therefore to reach $\kappa = 0$ we must be in this extreme situation, and to increase angular momentum or charge any further would result in the violation of the laws of physics (12). Therefore this law is in analogy with the classical third law as to achieve zero entropy a system must be at the extremal case where its temperature is zero.

8 | Particle Creation by Black Holes

8.1 Introduction to Particle Creation

From the analogy between the first law of thermodynamics and Bekenstein's first law of black hole thermodynamics one can make a classical argument for the expressions of black hole temperature and entropy:

$$TdS = \frac{1}{8\pi}\kappa dA.$$

By dimensional arguments

$$T \propto \frac{\hbar\kappa}{ck_B}$$
$$S \propto \frac{c^3 k_B A}{G\hbar},$$

and then still

$$TdS \propto \kappa\frac{c^2 dA}{G},$$

as needed. However this is only a formal argument, therefore this is as much of an analogy as we can make based on our previous work. In order to obtain an equality (with correct separate normalisation for S and T) we must perform a quantum treatment by treating particles as excitations of quantum fields. The aim of this chapter is to calculate the density of particles created by a static black hole.

See section D.1 for a review of quantum field theory in flat spacetime.

8.2 Quantum Field Theory in Curved Spacetime

For simplicity, we will focus on a non-rotating, uncharged black hole in one time and one spatial dimensional (1+1). This is described by the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2.$$

Tortoise Coordinates

$$\begin{aligned} r^* &= r - 2GM + 2GM \ln\left(\frac{r}{2GM} - 1\right) \\ \Rightarrow dr^* &= dr \left(\frac{1}{1 - \frac{2GM}{r}}\right). \end{aligned}$$

In the limit $r \to \infty$ the Schwarzschild metric approaches the Minkowski metric therefore the time coordinates t tends towards the proper time. Therefore, far from the black hole tortoise coordinates are an appropriate choice.
We can use this to rewrite the line element:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(dt^2 - dr^{*^2}).$$

We want to compare vacuum states in different reference frames. An easy choice of frames is the **Schwarzschild** frame and the **Kruskal** frame. (It is noted that in Kruskal coordinates there is no coordinate singularity at the horizon).

Schwarzchild Frame

Let $|0_S\rangle$ denote the vacuum state. Expanding the field ϕ in 2 dimensions in terms of the tortoise coordinates t and r^*:

$$\phi(t, r^*) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2|k|}} \left(e^{-i|k|t + ikr^*} b_k + e^{i|k|t - ikr^*} b_k^\dagger\right),$$

where k denotes the momentum of the described particle and $b_k^\dagger$, b_k denote the creation and annihilation operators associated with the frame respectively, as above, such that $b_k|0_S\rangle = 0 \,\forall\, k$. Let $|k| = \Omega$ be the frequency of the particle in the Schwarzschild frame. This expansion of the field can be rewritten in terms of the Eddington-Finkelstein coordinates, $u = t - r^*, v = t + r^*$

to indicate incoming and outgoing waves:

$$\phi(u,v) = \int_0^\infty \frac{d\Omega}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega u} b_\Omega + e^{i\Omega u} b_\Omega^\dagger + e^{-i\Omega v} b_{-\Omega} + e^{i\Omega v} b_{-\Omega}^\dagger \right).$$

Kruskal Frame

Let $|0_K\rangle$ denote the vacuum state. The Kruskal coordinates are given by U and V. We can expand a field ϕ with the timelike coordinate $T = \frac{1}{2}(U+V)$ and the spacelike coordinate $X = \frac{1}{2}(V-U)$ in the usual way for 2 dimensions:

$$\phi(T,X) = \int_{-\infty}^\infty \frac{dk}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2|k|}} (e^{-i|k|T+ikX} a_k^\dagger + e^{i|k|T-ikX} a_k),$$

where k denotes the momentum of the described particle and $a_k^\dagger, a_k$ denote the creation and annihilation operators associated with the frame respectively, such that the annihilation operator satisfies $a_k|0_K\rangle = 0 \; \forall \; k$. Let $|k| = \omega$ be the frequency of particles in the Kruskal frame.
Similarly to the Schwarzschild frame, rewriting the expansion above in terms of the Kruskal coordinates $U = T+X, V = T-X$:

$$\phi(U,V) = \int_0^\infty \frac{d\omega}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega U} a_\omega + e^{i\omega U} a_\omega^\dagger + e^{-i\omega V} a_{-\omega} + e^{i\omega V} a_{-\omega}^\dagger \right).$$

We will show that the vacuum state in each frame is distinct and therefore is only destroyed by its distinctive associated annihilation operator. This means that there could be particles observed in the state $|0_S\rangle$ with frequency ω.

Bogoliubov Transformation

The Bogoliubov transformation is how one expresses one set of creation and annihilation operators in terms of another. Their solutions are of the form:

$$b_\Omega = \int_0^\infty d\omega (\alpha_{\omega\Omega} \hat{a}_\omega + \beta_{\omega\Omega} \hat{a}_\omega^\dagger).$$

Noticing that in the expansions of ϕ in each coordinate frame can be written as $\phi(u,v) = \psi_1(u) + \psi_2(v)$ and $\phi(U,V) = \Psi_1(U) + \Psi_2(V)$, and also that $u = u(U)$ and $v = v(V)$. Therefore, allowing $\phi(u,v) = \phi(U,V)$ is equivalent to allowing $\psi_1(u) = \Psi_1(U)$ and $\psi_2(v) = \Psi_2(V)$. Using the first of these relations:

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} (e^{-i\omega U} a_\omega + e^{i\omega U} a_\omega^\dagger) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}} (e^{-i\Omega u} b_\Omega + e^{i\Omega u} b_\Omega^\dagger).$$

Substituting the form of b_Ω seen above:

$$RHS = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}} \left(e^{-i\Omega u} \int_0^\infty d\omega (\alpha_{\omega\Omega} \hat{a}_\omega + \beta_{\omega\Omega} \hat{a}^\dagger_\omega) + e^{i\Omega u} \int_0^\infty d\omega (\alpha^*_{\omega\Omega} \hat{a}^\dagger_\omega + \beta^*_{\omega\Omega} \hat{a}_\omega) \right)$$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{\omega}} e^{-i\omega U} a_\omega &= \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \left(e^{-i\Omega u} \alpha_{\omega\Omega} + e^{i\Omega u} \beta^*_{\omega\Omega} \right) a_\omega \\ \Rightarrow \frac{1}{\sqrt{\omega}} e^{-i\omega U} &= \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \left(e^{-i\Omega' u} \alpha_{\omega\Omega} + e^{i\Omega u} \beta^*_{\omega\Omega} \right). \end{aligned}$$

Taking the Fourier transform of both sides of this equation with respect to u:

$$\begin{aligned} \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega' u} e^{-\omega U} &= \int_{-\infty}^\infty \frac{du}{2\pi} \int_0^\infty d\Omega \sqrt{\frac{\omega}{\Omega}} \left(\alpha_{\omega\Omega} e^{iu(\Omega - \Omega')} + \beta^*_{\omega\Omega} e^{iu(\Omega + \Omega')} \right) \\ &= \int_0^\infty d\Omega \sqrt{\frac{\omega}{\Omega}} \left(\alpha_{\omega\Omega} \delta(\Omega - \Omega') + \beta^*_{\omega\Omega} \delta(\Omega + \Omega') \right). \end{aligned}$$

Case 1: if $\Omega' > 0$ then $\Omega' = \Omega$ and

$$\sqrt{\frac{\omega}{\Omega}} \alpha_{\omega\Omega} = \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u} e^{-i\omega U}.$$

Case 2: if $\Omega' < 0$ then $\Omega' = -\Omega$ and

$$\sqrt{\frac{\omega}{\Omega}} \beta_{\omega\Omega} = \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u} e^{i\omega U}.$$

The Kruskal coordinates are defined as $U = \frac{-e^{-\kappa u}}{\kappa}$ and $V = \frac{e^{\kappa v}}{\kappa}$ specifically for this case, then:

$$\begin{Bmatrix} \alpha_{\omega\Omega} \\ \beta_{\omega\Omega} \end{Bmatrix} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u \pm i \frac{\omega}{\kappa} e^{-\kappa u}}.$$

8.3 Density of Particles Created

The number operator measures the number of particles in a quantum system. It is given by:

$$\begin{aligned}
N_\Omega &= b^\dagger_\Omega b_\Omega \\
&= \int_0^\infty \int_0^\infty d\omega d\omega' ((\alpha^*_{\omega\Omega} a^\dagger_\omega + \beta^*_{\omega\Omega} a_\omega) \cdot (\alpha_{\omega'\Omega} a_{\omega'} + \beta_{\omega'\Omega} a^\dagger_{\omega'})) \\
&= \int_0^\infty \int_0^\infty d\omega d\omega' (\alpha^*_{\omega\Omega}\alpha_{\omega'\Omega} a^\dagger_\omega a_{\omega'} + \beta^*_{\omega\Omega}\alpha_{\omega'\Omega} a_\omega a_{\omega'} + \alpha^*_{\omega\Omega}\alpha_{\omega'\Omega} a^\dagger_\omega a_{\omega'} + \beta^*_{\omega\Omega}\beta_{\omega'\Omega} a_\omega a^\dagger_\omega)
\end{aligned}$$

Number of Particles Created:

$$\begin{aligned}
\langle N_\Omega \rangle &= \langle 0_K | b^\dagger_\Omega b_\Omega | 0_K \rangle \\
&= \int_0^\infty \int_0^\infty d\omega d\omega' (\langle 0 | \alpha^*_{\omega\Omega}\alpha_{\omega'\Omega} a^\dagger_\omega a_{\omega'} | 0 \rangle + \langle 0 | \beta^*_{\omega\Omega}\alpha_{\omega'\Omega} a_\omega a_{\omega'} | 0 \rangle \\
&\qquad + \langle 0 | \alpha^*_{\omega\Omega}\alpha_{\omega'\Omega} a^\dagger_\omega a_{\omega'} | 0 \rangle + \langle 0 | \beta^*_{\omega\Omega}\beta_{\omega'\Omega} a_\omega a^\dagger_\omega | 0 \rangle) \\
&= \int_0^\infty \int_0^\infty d\omega d\omega' \beta^*_{\omega\Omega}\beta_{\omega'\Omega} \langle 0 | a_\omega a^\dagger_\omega | 0 \rangle \\
&= \int_0^\infty \int_0^\infty d\omega d\omega' \beta^*_{\omega\Omega}\beta_{\omega'\Omega} \langle 0 | a^\dagger_\omega a_\omega + 1 | 0 \rangle \\
&= \int_0^\infty d\omega |\beta_{\omega\Omega}|^2.
\end{aligned}$$

In appendix D there is a derivation of the following relation: $|\alpha_{\omega\Omega}|^2 = e^{2\pi\frac{\Omega}{\kappa}} |\beta_{\omega\Omega}|^2$, which will be used to solved the above integral.

$$\begin{aligned}
|\alpha_{\omega\Omega}|^2 &= e^{2\pi\frac{\Omega}{\kappa}} |\beta_{\omega\Omega}|^2 \\
\Rightarrow &\quad \int_0^\infty d\omega (|\alpha_{\omega\Omega}|^2 - |\beta_{\omega\Omega}|^2) = \delta(0) \\
\Rightarrow &\quad \int_0^\infty d\omega (e^{2\pi\frac{\Omega}{\kappa}} |\beta\omega\Omega|^2 - |\beta_{\omega\Omega}|^2) = \delta(0) \\
\Rightarrow &\quad \int_0^\infty d\omega |\beta_{\omega\Omega}|^2 = \delta(0) \frac{1}{(e^{\frac{2\pi\Omega}{k}} - 1)}.
\end{aligned}$$

Therefore, the total number of particles in all space of the mode Ω is calculated to be:

$$\langle N_\Omega \rangle = \delta(0) \frac{1}{e^{\frac{2\pi\Omega}{\kappa}} - 1}.$$

Therefore, the mean particle density is

$$n_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{\kappa}} - 1}.$$

8.4 Hawking Temperature

Planck's law of black body radiation in natural units gives the energy density as

$$n = \frac{1}{\exp(\frac{\Omega}{T}) - 1},$$

Therefore, in natural units the Hawking temperature of a black hole (contribution to temperature from Hawking radiation) is given by:

$$T_H = \frac{\kappa}{2\pi}.$$

In SI units this is:

$$T_H = \frac{\hbar\kappa}{2\pi c k_B},$$

where k_B is the Boltzmann constant. From before surface gravity of a Schwarzschild black hole is given by:

$$\begin{aligned} \kappa &= \frac{c^4}{4GM} \\ \Rightarrow T_H &= \frac{\hbar c^3}{8\pi GMk_B} \\ \Rightarrow T_H &\propto \frac{1}{M}. \end{aligned}$$

8.5 Lifetime of a Black Hole

The mass of a radiating black hole is decreasing, so to estimate the lifetime of a black hole we estimate it to be a black body which obeys the Stefan-Boltzmann law:

$$\begin{aligned} P &= \frac{dM}{dt} \\ &= -4\pi R^2 \sigma T_H^4 \\ &= -A\sigma T_H^4 \\ &= -16\pi\sigma M^2 \cdot M^{-4} \\ &= \frac{-16\pi\sigma}{M^2} \\ &= \frac{-4\pi^3 k_B^4}{15c^2\hbar^3 M^2} \end{aligned}$$

This is a separable ordinary differential equation:

$$\begin{aligned}
\int dt &= -\frac{4\pi^3 k_B^4}{15c^2\hbar^3}\int M^2 dM \\
\Rightarrow t_{life} &= \frac{4\pi^3 k_B^4}{15c^2\hbar^3} M_{initial}^3. \\
\Rightarrow t_{life} &\propto M_{initial}^3
\end{aligned}$$

This is under the assumption of a pure photon emission from a black hole's horizon.

9 | Later Research

There is a large amount of research into black holes and their thermodynamic properties following Bekenstein and Hawking of which a small number are described below.

A large area of current research is in observing black holes. For example, relatively recently in 2016 black holes were directly detected at LIGO (18). Gravitational waves were detected, these were formed by the merger of binary black holes.

In 1996 Andrew Strominger and Cumrun Vafa published a paper on the microscopic origin of the Bekenstein-Hawking entropy (19). Their research is based on five-dimensional extremal black holes in string theory.

In 2010 F. Belgiorno and his colleagues (20), created an optical analogue of the event horizon of a black hole using ultrashort laser pulse filaments and their measurements demonstrate a spontaneous emission of photons coinciding with Stephen Hawking's quantum predictions for radiation emitted by a black hole. If their results are confirmed, it would be the first observation of Hawking radiation (12).

Active research includes observation, the black hole information paradox and approaches to quantum gravity including quantum loop gravity and string theory, which could lead to better understanding of black hole entropy.

10 | Conclusion and Summary

In the early 1970s it was not believed that black holes could radiate therefore it was thought meaningless to try and study their thermodynamic properties. Bekenstein revolutionised the field of black hole mechanics with his 1973 paper (1). His analogies led Bardeen, Carter and Hawking to later reproduce the laws (21).
These purely formal analogies led Hawking to be convinced that that black holes were thermodynamic bodies and argued that due to quantum fluctuations near the event horizon of a black hole, it could in fact radiate (2). This gave rise to our current understanding of Hawking radiation and the thermodynamic properties of black holes.

A Formulation of General Relativity

Speed of light: $c = 1$ in our coordinate system.
Spacetime is curved (depending on presence of gravitational fields) but locally flat.

$$ds'^2 = ds^2$$
$$\Rightarrow g'_{\mu\nu} = g_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}$$

A (p,q) tensor is an object that transforms under a diffeomorphism as:

$$T'^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_p} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_p}}{\partial x^{\alpha_p}}\right)\left(\frac{\partial x'^{\beta_1}}{\partial x^{\nu_1}}\cdots\frac{\partial x'^{\beta_q}}{\partial x^{\nu_q}}\right)T^{\alpha_1,\ldots,\alpha_p}{}_{\beta_1,\ldots,\beta_1}.$$

The metric g is a (0,2) tensor that is symmetric:

$$g_{\mu\nu} = g_{\nu\mu},$$

and is a Pseudo-Riemannian metric:

$$g_{\mu\nu}u^\mu v^\nu = 0 \ \forall \ v^\nu \Rightarrow u^\mu = 0.$$

If the metric $g_{\mu\nu}$ is non-singular, $det(g_{\mu\nu}) \neq 0$, on a chart U of the manifold M under consideration, then there is a unique torsion-free (symmetric) connection which is compatible with the metric. It is given in any system of coordinates $x_1, \ldots, xn$ by Christoffel's formula: The **Levi-Civita connection (Christoffel symbols)** is defined by:

$$\nabla_\mu g_{\mu\nu} = 0,$$

and must be given by:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}).$$

Let

$$\Gamma_{j,ik} \equiv g_{pj}\Gamma^p_{ik} = g_{jp}\Gamma^p_{ik}.$$

Then

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{j,ik} - \Gamma_{i,jk} = 0 \tag{A.1}$$

$$\partial_j g_{ki} - \Gamma_{i,kj} - \Gamma_{k,ij} = 0 \tag{A.2}$$

$$\partial_i g_{jk} - \Gamma_{k,ji} - \Gamma_{j,ki} = 0 \tag{A.3}$$

Recall these symbols are symmetric under exchange of the lower indices. Sum A.1 and A.2 and subtract A.3:

$$\partial_k g_{ki} + \partial_i g_{jk} - \partial_k g_{ij} - 2\Gamma_{k,ij} = 0.$$

Therefore

$$\Gamma^k_{ij} = \frac{1}{2} g^{kp}(\partial_i g_{pj} + \partial_j g_{ip} - \partial_p g_{ij}),$$

as required.

A **geodesic** is a path which minimises the proper distance between the 2 nearby points. The Riemann Curvature Tensor is given by:

$$R^\alpha_{\mu\nu\lambda} = \partial_\nu \Gamma^\alpha_{\mu\lambda} - \partial_\mu \Gamma^\alpha_{\nu\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\nu\lambda}\Gamma^\alpha_{\mu\beta}.$$

The **Ricci Tensor** is defined using a contraction of the Riemann tensor:

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}.$$

The **Scalar Curvature** is defined as:

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu.$$

Newton's Laws for a flat metric are given by:

$$F = m\ddot{x} = \frac{GmM}{R^2}. \tag{A.4}$$

Einstein's Tensor is given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{2}{3} g_{\mu\nu} R.$$

Einstein's Equation is given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = K^2 T_{\mu\nu}.$$

In a vacuum we have the Energy Momentum Tensor is equal to zero, therefore, by simply contracting the above equation equation with $g^{\mu\nu}$ we can write:

$$R = 0,$$

hence, the Einstein equation becomes:

$$R_{\mu\nu} = 0,$$

this is the **vacuum Einstein equation** and the derivation of equation 2.2.

Killing Vectors are vector fields V which (in local coordinates) satisfy

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0. \tag{A.5}$$

A **true singularity** is invariant under coordinate transformations (exist in all coordinate transformations), a **coordinate singularity** does not possess this property (can vanish in certain coordinate systems). One can check a true singularity by computing (curvature) scalars which are proportional to that singularity, such as the Kretschmann scalar.

B | Geometry of Spacetimes and Black Holes

Given a manifold M of dimension N, a **hypersurface** Σ of M is an $N-1$ dimensional submanifold.
A hypersurface is called **timelike/null/spacelike** if all its tangent vectors are timelike/null/spacelike.
A vector V^μ is the timelike/null/spacelike if

$$||V||^2 = g_{\mu\nu}V^\mu V^\nu \begin{cases} < 0 & \textit{Timelike} \\ = 0 & \textit{Null} \\ > 0 & \textit{Spacelike}. \end{cases}$$

Two vectors V^μ and U^ν are **orthogonal** if

$$g_{\mu\nu}V^\mu U^\nu = 0.$$

A vector N^μ is said to be **normal** to hypersurface Σ if it is orthogonal to all of the tangent vectors of Σ.

A spacetime is said to be **stationary** if it possesses a timelike Killing vector.
A spacetime is said to be **static** if it is irrotational and does not evolve with time.

Time Taken to Escape the Event Horizon

Looking at radial timelike geodesics, governed by the Schwarzschild metric, through the eyes of an observer at infinity. Radial geodesics follow

$$\dot{\phi} = \dot{\theta} = 0.$$

Timelike geodesics follow

$$|V|^2 = g_{\mu\nu}V^\mu V^\nu < 0,$$

for velocity vectors V^{μ},

$$\Rightarrow g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} < 0.$$

But from before

$$\frac{dx^{\phi}}{d\tau} = \frac{dx^{\theta}}{d\tau} = 0.$$

From the Schwarzschild metric:

$$\begin{aligned}
g_{tt} &= -\left(1 - \frac{2GM}{r}\right) \\
g_{rr} &= \left(1 - \frac{2GM}{r}\right)^{-1} \\
\Rightarrow -\left(1 - \frac{2GM}{r}\right)&\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 < 0.
\end{aligned}$$

Restricting our attention to $r > 2GM$ then $1 - \frac{2GM}{r} > 0$. Then

$$\begin{aligned}
\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 &< \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 \\
\Rightarrow \left(\frac{dt}{d\tau}\frac{d\tau}{dr}\right)^2 &> \left(1 - \frac{2GM}{r}\right)^{-2} \\
\Rightarrow \left|\frac{dt}{dr}\right| &> \left(1 - \frac{2GM}{r}\right)^{-1}.
\end{aligned}$$

The time taken for the geodesic to travel between r_0 and r is given by:

$$\begin{aligned}
\Delta t &= \int dt \\
&= \int_r^{r_0} \left|\frac{dt}{dr'}\right| dr' \\
\Rightarrow \Delta t &> \int_r^{r_0} \left(1 - \frac{2GM}{r'}\right)^{-1} dr' \\
&= \int_r^{r_0} \frac{r'}{r' - 2GM} dr'.
\end{aligned}$$

In order to solve the integral let

$$I = \int_r^{r_0} \frac{r'}{r' - 2GM} dr'.$$

Make a substitution

$$u = r' - 2GM \ \Rightarrow r' = u + 2GM \Rightarrow dr' = du.$$

$$
\begin{aligned}
\Rightarrow I &= \int_r^{r_0} \frac{u+2GM}{u} du \\
&= \int_r^{r_0} \frac{u}{u} du + \int_0^{r_0} \frac{2GM}{u} du \\
&= \int_r^{r_o} dr' + 2GM \int_r^{r_0} \frac{1}{u} du \\
&= r_0 - r + 2GM (\ln u)|_r^{r_0} \\
&= r_0 + r_2 GM \ln(r' - 2GM)|_r^{r_0} \\
&= r_0 - r + 2GM \ln(r_0 - 2GM) - 2GM \ln(r - 2GM) \\
&= (r_0 - r) + 2GM \ln \left(\frac{r_0 - 2GM}{r - 2GM} \right) < 0.
\end{aligned}
$$

As $r \to 2GM$, the denominator in the logarithmic term approaches zero and Δt diverges. (Note that an observer at infinity has proper time as coordinate time.)
An observer far away does not see someone falling towards $r = 2GM$ in a finite amount of time. Therefore one can say that $r \leq 2GM$ is **causally disconnected**.

C | Formulation of Black Hole Thermodynamics

Black Hole Area Theorem

In his 1971 paper Hawking stated that black holes do not increase in area over time (13). The black hole area theorem was an early rule which provided the basis for the concept of black hole entropy and thus the field of black hole thermodynamics. [The black hole area theorem also provides an upper limit for energy radiated away in hawking radiation - see chapter 8.]

Superradiant Scattering

Superradiance is the amplification of wavemodes in the ergoregion, mainly due to the fact that the stationary timelike Killing vector $\frac{\partial}{\partial t}$ becomes spacelike in this region. It is known as the wave analog of the Penrose process. The study of superradiant scattering was one clue to the fact that spontaneous particle creation should occur near a Kerr black hole (22).

Surface Gravity κ

κ represents the extent to which the time coordinate t is not an affine parameter along the generators of the horizon. One can think of κ as the "surface gravity" of the black hole in the following sense: a particle outside the horizon which rigidly corotates with the black hole has an angular velocity Ω_H, a four velocity $v^a = v^t(K^a + \Omega_H \tilde{K}^a)$ and an acceleration four-vector $v^a_{;b}v^b$. The magnitude of the acceleration, multiplied by a factor $\frac{1}{v^t}$ to convert from change in velocity per unit proper time to change in velocity per unit coordinate time t, tends to κ when the particle is infinitesimally close to the event horizon (23).

Carter Israel Conjecture

This states that the end-state of the gravitational collapse of matter is a Kerr-Newman black hole (24).

Cosmic Censorship Conjecture

Roger Penrose argued the cosmic censorship conjecture which asserts that all singularities are hidden by an event horizon i.e. there are no naked singularities.

Christodoulou's Calculation for Minimum Increase in Area

Point Particle

We will calculate the minimum increase in area of a black hole when a particle falls in, ignoring incidental effects of how the particle falls in such as gravitational radiation.
Assume a neutral particle captured by a Kerr black hole. From Carter's first integrals for geodesic motion in Kerr background (25)

$$E^2[r^4 + a^2(r^2 + 2Mr - Q^2)] - 2E(2Mr - Q^2)ap_\phi - (r^2 - 2Mr + Q^2)p_\phi^2 - (\mu^2 r^2 + 1)\Delta = (p_r\Delta)^2,$$

where $E = -p_t$ is the conserved energy, p_ϕ is the conserved component of angular momentum in the direction of axis of symmetry, q is Carter's fourth constant of the motion, μ is the rest mass of the particle, p_r is the covariant radial momentum of the particle, $\Delta = r^2 - 2Mr + a^2 + Q^2$ as seen in the Kerr metric. Solving this for quadratic using the quadratic formula (using the positive energy solution) we obtain an expression for E:

$$\begin{aligned} E &= \left[\frac{1}{2(r^2 + a^2(r^2 + 2Mr - Q^2))}\right][2(2Mr - Q^2)ap_\phi + \\ &+ [4(2Mr - Q^2)^2 a^2 p_\phi^2 + 4(r^4 + a^2(r^2 + 2Mr - Q^2))((r^2 - 2Mr + Q^2)p_\phi^2 + (\mu^2 r^2 + q)\Delta + (p_r\Delta)^2]]^{\frac{1}{2}}. \end{aligned}$$

This can be rewritten in terms of $A \equiv r^4 + a^2(r^2 + 2Mr - Q^2)$ and $B \equiv (2mr - Q^2)A^{-1}$ as

$$E = Bap_\phi + ([B^2a^2 + A^{-1}(r^2 - 2Mr + Q^2)]p_\phi^2 + A^{-1}[(\mu^2 r^2 + q)\Delta + (p_r\Delta)^2])^{\frac{1}{2}}.$$

At the horizon $\Delta = 0$ therefore

$$\begin{aligned} & r^2 - 2Mr + a^2 + Q^2 = 0 \\ \Rightarrow\ & a^2 = -r^2 + 2Mr - Q^2 \\ \Rightarrow\ & A = (r^2 + a^2)^2, \end{aligned}$$

at $r = r_+$. Also at this radius

$$\begin{aligned} & \Delta = 0 \\ \Rightarrow \quad & r_+^2 + a^2 = 2Mr - Q^2 \\ \Rightarrow \quad & B = \frac{2Mr - Q^2}{A} = \frac{(r_+^2 + a^2)}{(r_+^2 + a^2)^2} = (r_+^2 + a^2)^{-1}, \end{aligned}$$

at $r = r_+$. Finally at this radius

$$\begin{aligned} \Omega &= \frac{a}{\alpha} \\ \alpha &= r_+^2 + a^2 \\ B &= B_+ = (r_+^2 + a^2)^{-1} \\ \Rightarrow \Omega &= Ba, \end{aligned}$$

at $r = r_+$. Therefore at this horizon our expression becomes

$$E = \Omega p_\phi + (r_+^2 + a^2)^{-1} |p_r \Delta|_{r=r_+}.$$

By the first law of thermodynamics the increase in the rationalised area of the black hole is given by

$$\Delta\alpha = \theta^{-1}(r_+^2 + a^2)^{-1} |p_r \Delta|_{r=r_+}.$$

Christodoulou argued that there is no increase in area if and only if the particle is captured at a turning point in the orbit. In this case the black hole mass would be increased by

$$E = \Omega p_\phi.$$

Therefore he proved that a point particle could be captured by a black hole without the black hole's area increasing.

Particle with Radius

Christodoulou then went on to examine the change in the rationalised area of a black hole when a particle with radius is captured. Let a particle with radius b fall freely into a Kerr black hole. To generalise the argument from early we must examine the particle with radius as it passes the event horizon r_+ therefore we write

$$\int_{r_+}^{r_+ + \delta} (g_{rr})^{\frac{1}{2}} dr = b,$$

where δ is a proper distance b outside of the horizon r_+.

$$\Rightarrow b = \int_{r_+}^{r_+ + \delta} \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2}.$$

By performing this integral and assuming that $r_+ - r_- >> \delta$ Christodoulou obtained the expression for the radius in terms of the value δ:

$$b = 2\delta^{\frac{1}{2}}(r_+^2 + a^2 \cos^2 \theta)^{\frac{1}{2}}(r_+ - r_-)^{-\frac{1}{2}}$$
$$\Rightarrow \delta = \frac{1}{4} b^2 (r_+^2 + a^2 \cos \theta)^{-1}(r_+ - r-).$$

Christodolou then performed a Taylor expansion of $[4(2Mr - Q^2)^2 a^2 p_\phi^2 + 4(r^4 + a^2(r^2 + 2Mr - Q^2))((r^2 - 2Mr + Q^2)p_\phi^2 + (\mu^2 r^2 + q)\Delta + (p_r \Delta)^2]]^{\frac{1}{2}}$ from the expression for energy. To order b the expression for energy for a particle with radius which is captured at a turning point of the orbit (to minimise the energy) is given by:

$$\begin{aligned} E &= \Omega p_\phi + [(r_+^2 - a^2)(r_+^2 + a^2)^{-1} p_\phi^2 + \\ &+ \mu^2 r_+^2 + 1]^{\frac{1}{2}} (\frac{1}{2}) b (r_+ - r_-)(r_+^2 + a^2)^{-1}(r_+^2 + a^2 \cos^2 \theta)^{-\frac{1}{2}}. \end{aligned}$$

In order to have the momentum p_θ be a real value, q must have a lower bound (25):

$$q \geq \cos^2 \theta \left[a^2(\mu^2 - E^2) + \left(\frac{p_\phi}{\sin \theta} \right)^2 \right].$$

As in the point particle case, let $E = \Omega p_\phi$ since the other coefficients vanish and knowing that there exists an upper bound $a^2 \Omega^2 \leq \frac{1}{4}$ in a Kerr black hole the inequality becomes

$$\begin{aligned} q &\geq \Omega p_\phi + \frac{1}{2} \mu b (r_+ - r_-)(r_+^2 + a^2)^{-1} \\ \Rightarrow E &\geq \Omega p_\phi + \frac{1}{2} \mu b (r_+ - r_-)(r_+^2 + a^2)^{-1}. \end{aligned}$$

$E = \Omega p_\phi + \frac{1}{2} \mu b (r_+ - r_-)(r_+^2 + a^2)^{-1}$ when $p_\phi = p_\theta = p^r = 0$ when the particle is captured. As before the increase in rationalised area is given by the first law of black hole thermodynamics. The (minimum) increase in area of a black hole by the capture of a particle of radius b is therefore given by

$$\Delta \alpha \geq 2\mu b.$$

D | The Hawking Effect

D.1 Review of Free Scalar QFT in Flat Spacetime

The Klein-Gordon equation is the equation of motion for a free scalar field. For a field ϕ

$$(\Box + m^2)\phi(x) = 0,$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \nabla^2$. Positive frequency ($p > 0$) plane waves solutions along with their complex conjugates make up a complete set of solutions to the Klein Gordon equation. Therefore we may decompose the real field ϕ as

$$\phi(x) = \int d^3p(a_p\psi_p(x) + a_p^*\psi_p^*(x)),$$

with expansion coefficients a_p and a_p^*. Substituting plane wave solutions with frequency $\omega_p = \sqrt{p^2+m^2}$ this takes the form:

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3p}{2\omega_p}(a_p e^{ipx-i\omega_p t} + a_p^* e^{-ipx+i\omega_p t}).$$

Upon second quantisation this field becomes an operator

$$\hat{\phi}(t,x) = \int_{-\infty}^{\infty}\frac{dp}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2p}}(e^{ipx}\hat{a}_p^\dagger + e^{-ipx}\hat{a}_p),$$

where $\hat{a}$, $\hat{a}^\dagger$ are the associated creation and annihilation operators, respectively. It follows from the canonical commutation relations:

$$\begin{aligned}[\hat{\phi}(t,\vec{x}), \tilde{\hat{\phi}}(t,\vec{y})] &= i\delta(\vec{x}-\vec{y}) \\ [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] &= 0 \qquad .\end{aligned}$$

The annihilation operator must obey

$$[\hat{a}_p, \hat{a}_q^\dagger] = \delta(\vec{p} - \vec{q})$$
$$[\hat{a}_p, \hat{a}_q] = 0 \qquad .$$

The vacuum is annihilated as follows: $\hat{a}_p|0_p\rangle = 0$.
From this point onwards we will drop hat symbols but remembering that the field and operators defined above remain operators moving forward.
In flat spacetime the vacuum state in all inertial reference frames, $a_p|0\rangle = 0, \tilde{a}_p|\tilde{0}\rangle = 0$, are the same. To prove this one must show that:

$$a_p|0\rangle = 0 \;\forall\; p \Rightarrow \tilde{a}_p|0\rangle = 0 \;\forall p.$$

When dealing with an inertial reference frame, the coordinates of the new frame are related to those of the old frame with a Lorentz transformation:

$$\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu.$$

Consider the set of solutions that are positive frequency plane waves:

$$\tilde{\psi}_p = N_p e^{ip_\mu \tilde{x}_\mu},$$

And fix the normalisation factor as $N_p = \frac{1}{\sqrt{2p^0}}(2\pi)^{\frac{3}{2}}$, then the field expansion is simplified to be

$$\phi(\vec{x}) = \int d^3p(\tilde{a}_p\tilde{\psi}_p + \tilde{a}_p^\dagger\tilde{\psi}_p^*).$$

Momenta transform under Lorentz transformations as $\tilde{p}_\mu = p_\nu\Lambda^\nu_\mu$ therefore the positive frequency plane wave solutions transform from frame x^μ to $\tilde{x}^\mu$ as

$$\begin{aligned}
\tilde{\psi}_p &= \frac{1}{\sqrt{2p^0}(2\pi)^{\frac{3}{2}}}e^{ip_\mu\tilde{x}^\mu} \\
\psi_{\tilde{p}} &= \frac{1}{\sqrt{2\tilde{p}^0}(2\pi)^{\frac{3}{2}}}e^{i\tilde{p}_\mu x^\mu} \\
\Rightarrow \tilde{\psi}_p &= \sqrt{\frac{\tilde{p}^0}{p^0}}\psi_{\tilde{p}},
\end{aligned}$$

and the transformed solutions are still positive solution plane waves. Therefore we can say that $\tilde{a}_p = a_{\tilde{p}}$. Therefore the vacuum states are annihilated by the same operator. A vacuum state is defined by its associated annihilation operator, therefore we can say that **vacuum states in distinct inertial frames are identical.**

D.2 Calculation of Bogoliubov Coefficients

Firstly, make a substitution to find a more applicable form of the Gamma function definition for our coefficients:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

Make substitution $t = e^x \Rightarrow dt = tdx$:

$$\Gamma(z) = \int_{-\infty}^{\infty} e^{xz-e^x} dx. \tag{D.1}$$

$$\tag{D.2}$$

$$\begin{aligned}
\Rightarrow \alpha_{\omega\Omega} &= -\frac{1}{\kappa}\frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}}\Gamma\left[\frac{i\Omega u}{\ln|\frac{\omega}{\kappa}| - i\frac{\pi}{2} - u\kappa}\right] \\
\beta_{\omega\Omega} &= -\frac{1}{\kappa}\frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}}\Gamma\left[\frac{i\Omega u}{\ln|\frac{\omega}{\kappa}| + i\frac{\pi}{2} - u\kappa}\right]
\end{aligned}$$

Similarly to what we found earlier let

$$\begin{aligned}
\alpha_{\omega\Omega} &= \sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty}\frac{du_1'}{2\pi}e^{i\Omega u_1' + i\frac{\omega}{\kappa}e^{-\kappa u_1'}} \\
\beta_{\omega\Omega} &= \sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty}\frac{du_2'}{2\pi}e^{i\Omega u_2' - i\frac{\omega}{\kappa}e^{-\kappa u_2'}}.
\end{aligned}$$

Making the substitutions

$$\begin{aligned}
u_1 &= \kappa u_1' \\
u_2 &= \kappa u_2' \\
e^{-t_1} &= \frac{i\omega}{\kappa}e^{-u_1} \Rightarrow du_1 = -dt_1 \\
e^{-t_2} &= \frac{i\omega}{\kappa}e^{-u_2} \Rightarrow du_2 = -dt_2,
\end{aligned}$$

one gets $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ in the form of a gamma function as above:

$$\begin{aligned}
\alpha_{\omega\Omega} &= -\left(e^{\frac{\pi\Omega}{2\kappa}}\right)\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} dt_1 e^{\frac{i\Omega t_1}{\kappa} + \frac{i\Omega}{\kappa}\ln\frac{\omega}{\kappa} - e^{t_1}} \\
\beta_{\omega\Omega} &= -\left(e^{-\frac{\pi\Omega}{2\kappa}}\right)\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^{\infty} dt_2 e^{\frac{i\Omega t_2}{\kappa} + \frac{i\Omega}{\kappa}\ln\frac{\omega}{\kappa} - e^{t_2}}.
\end{aligned}$$

Therefore without having to substitute them into their respective gamma function forms, one can note the relation:

$$\begin{aligned} \alpha_{\omega\Omega} &= e^{\frac{\pi\Omega}{\kappa}} \beta_{\omega\Omega} \\ \Rightarrow |\alpha_{\omega\Omega}|^2 &= e^{\frac{2\pi\Omega}{\kappa}} |\beta_{\omega\Omega}|^2. \end{aligned}$$

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