General Relativity Homework 4

Emer Keeling

3 April 2024

See answers in blue.

1 Question 1

Entropy, Temperature, 1st Law

Area of a 3-sphere is given by

$$\begin{aligned} A &= 2\pi^2 R_s^3 \\ &= 2\pi^2 \left(\frac{8GM}{3\pi}\right)^{\frac{3}{2}} \end{aligned}$$

Entropy of a black hole is given by

$$\begin{aligned} S_{BH} &= \frac{k_B}{4\hbar G} A \\ &= \frac{k_B}{4\hbar G} 2\pi^2 \left(\frac{8GM}{3\pi}\right)^{\frac{3}{2}} \end{aligned}$$

$$S_{BH} = \frac{k_B}{4\hbar G} 2\pi^2 \left(\frac{8G}{3\pi}\right)^{\frac{3}{2}} M^{\frac{3}{2}}$$

$$\Rightarrow dS_{BH} = \frac{k_B}{4\hbar G} 2\pi^2 \left(\frac{8G}{3\pi}\right)^{\frac{3}{2}} \left(\frac{3}{2}\right) M^{\frac{1}{2}} dM$$

The temperature of a black hole is given by

$$T_{BH} = \frac{\kappa}{2\pi}$$

where κ is the surface gravity of the black hole. The surface gravity of a black hole is given by

$$\begin{aligned} \kappa &= \frac{1}{2}\frac{dg_{00}}{dr}|_{r=R_S} \\ g_{00} &= \left(1 - \frac{R_S}{r^2}\right) \\ \frac{dg_{00}}{dr} &= -R_S^2 \frac{d}{dr}(r^{-2}) \\ &= 2R_S^2 r^{-3} \\ \frac{dg_{00}}{dr} &= -R_S^2 \frac{d}{dr}(r^{-2}) \\ &= 2R_S^2 r^{-3} \\ \kappa = \frac{1}{2}\frac{dg_{00}}{dr}|_{r=R_S} &= R_S^2 R_S^{-3} \\ &= R_S^{-1} \\ &= \left(\frac{3\pi}{8GM}\right)^{\frac{1}{2}} \end{aligned}$$

$$T = \frac{1}{2\pi}\left(\frac{3\pi}{8GM}\right)^{\frac{1}{2}}$$

The first law of thermodynamics in this case is given by

$$\begin{aligned} dM &= TdS \\ &= \frac{1}{2\pi}\left(\frac{3\pi}{8GM}\right)^{\frac{1}{2}}\frac{k_B}{4\hbar G}2\pi^2\left(\frac{8G}{3\pi}\right)^{\frac{3}{2}}\left(\frac{3}{2}\right)M^{\frac{1}{2}}dM \\ &= dM, \end{aligned}$$

in natural units. This shows that the first law of thermodynamics holds in this case.

2 Question 2

To find solutions to Freedman's equations, consider them as a system of equations,

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho+3p) \\ \frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{2k}{a^2} &= 4\pi G(\rho-p) \\ p &= -\rho \\ \Rightarrow \dot{a}=\sqrt{\frac{2}{3}4\pi G\rho a^2-k} &= \frac{da}{dt} \\ \Rightarrow t &= \int\frac{da}{\sqrt{\frac{3}{2}4\pi G\rho a^2-k}} \end{aligned}$$

Let $\xi = \frac{2}{3}4\pi G\rho$.

$$\Rightarrow t = \int\frac{da}{\sqrt{\xi a^2-k}}$$

Let $x=\sqrt{\xi}a \Rightarrow da = \frac{1}{\sqrt{\xi}}dx$

$$\begin{aligned} \Rightarrow t &= \int\frac{dx}{\sqrt{\xi}\sqrt{x^2-k}} \\ &= \frac{1}{\sqrt{\xi}}\int\frac{dx}{\sqrt{x^2-k}} \end{aligned}$$

k=1

$$\begin{aligned} t &= \frac{1}{\sqrt{\xi}}\int\frac{dx}{\sqrt{x}^2-1} \\ &= \frac{1}{\sqrt{\xi}}arcosh(\sqrt{\xi}a) \end{aligned}$$

$$\Rightarrow a = \frac{1}{\sqrt{\xi}}\cosh(\sqrt{\xi}t)$$

where $\xi = \frac{2}{3}4\pi G\rho$.

k=-1

$$\begin{aligned} t &= \frac{1}{\sqrt{\xi}}\int\frac{da}{\sqrt{x^2+1}} \\ &= \frac{1}{\sqrt{\xi}}arcsinh(x) \\ &= \frac{1}{\sqrt{\xi}}arcsinh(\sqrt{\xi}a) \end{aligned}$$

$$\Rightarrow a = \frac{1}{\sqrt{\xi}}\sinh(\sqrt{\xi}t)$$

where $\xi = \frac{2}{3}4\pi G\rho$.

k=0

$$\begin{aligned} t &= \frac{1}{\sqrt{\xi}}\int\frac{dx}{\sqrt{a^2}} \\ &= \frac{1}{\sqrt{\xi}}\int\frac{da}{a} \\ &= \frac{1}{\sqrt{\xi}}\ln(a) \\ \Rightarrow \ln a &= \sqrt{\xi}t \end{aligned}$$

$$\Rightarrow a = e^{\sqrt{\xi}t}$$

where $\xi = \frac{2}{3}4\pi G\rho$.
When $\omega = -1, -P = \rho = \frac{\Lambda}{8\pi G}$ therefore $\xi = \frac{\Lambda}{3}$.
These are the solutions to Freedman's equations for each case.

Finding the components of the Ricci Tensor

Einstein's field equations (EFE)

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 8\pi GT_{\mu\nu} \\ T_{\mu\nu} &= -\frac{\Lambda}{2\pi G}g_{\mu\nu} \\ \omega = -1, -P = \rho &= \frac{\Lambda}{8\pi G} \\ R &= 2R + 8\pi GT \\ &= -8\pi GT \\ &= -8\pi G(-4\rho) \\ &= 4\Lambda \end{aligned}$$

Substitute back into EFE

$$\begin{aligned} R_{\mu\nu} &= 2\Lambda g_{\mu\nu} + 8\pi GT_{\mu\nu} \\ &= 2\Lambda g_{\mu\nu} + 8\pi G(-\frac{\Lambda}{8\pi G}g_{\mu\nu}) \\ &= \Lambda g_{\mu\nu} \end{aligned}$$

Therefore

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Therefore they are proportional to each other.
Consider a more general argument without using $\omega = -1, -P = \rho = \frac{\Lambda}{8\pi G}$: The Robertson-Walker metric is given by:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right).$$

From mathematica the components of the Ricci tensor are given by

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{rr} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \\ R_{\theta\theta} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\ R_{\phi\phi} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\ R_{\phi\phi} &= r^2\sin\theta(a\ddot{a} + 2\dot{a}^2 + 2k) \end{aligned}$$

Therefore

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \left[\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + \frac{2k}{a}\right]g_{ij} \end{aligned}$$

where $i, j = 1, 2, 3$.
Therefore the components

$$R_{\mu\nu} \propto g_{\mu\nu}.$$

3 Question 3

Locate the position of the Killing horizon r_K where the Killing vector ∂t is null. Compute the surface gravity there and derive the temperature.
The Killing vector is given by

$$\begin{aligned} \partial t &= K^t\partial_t \\ \Rightarrow K^t &= 1 \\ K^2 &= K^t K_t \\ &= K^t K^t g_{tt} \\ &= g_{tt} \end{aligned}$$

At Killing horizon

$$\begin{aligned} g_{tt} &= 0 \\ \Rightarrow 1 - \frac{\Lambda}{3}r^2 &= 0 \\ \Rightarrow r^2 &= \frac{3}{\Lambda} \\ \Rightarrow r &= \sqrt{\frac{3}{\Lambda}} \end{aligned}$$

This describes the Killing horizon.

$$\Rightarrow r_H = \sqrt{\frac{3}{\Lambda}}.$$

For a static observer

$$\begin{aligned} k^\mu &= V(x)U^\mu \\ U_\mu U^\mu &= -1 \\ \Rightarrow V &= \sqrt{-k_\mu k^\mu} \end{aligned}$$

where V is the redshift factor.

$$\begin{aligned} V &= \sqrt{1 - \frac{\lambda}{3}r^2} \\ \Rightarrow U^\mu &= \left((1 - \frac{\lambda}{3}r^2)^{-\frac{1}{2}}, 0, 0, 0\right) \end{aligned}$$

Surface gravity is given by

$$\begin{aligned} \kappa &= \sqrt{\nabla_\mu V \nabla^\mu V}|_{r_H} \\ &= \left(\frac{\Lambda}{3}\right) r|_{r=\sqrt{\frac{3}{\Lambda}}} \\ &= \left(\frac{\Lambda}{3}\right)\sqrt{\frac{3}{\Lambda}} \\ &= \left(\frac{\Lambda}{3}\right)\left(\frac{\Lambda}{3}\right)^{-\frac{1}{2}} \\ &= \left(\frac{\Lambda}{3}\right)^{\frac{1}{2}} \end{aligned}$$

Therefore the temperature is

$$T = \frac{\kappa}{2\pi} \quad = \quad \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}.$$

4 Question 4

See blue part of answer for the main solution
We are dealing with a closed, radiation-filled FRW universe ($k = 1$). The Robertson-Walker metric is given by:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right).$$

Setting $k = 1$ restricts r to be $r \in [-1, 1]$ - an angular coordinate. Setting $r = \sin\lambda$ this describes the geometry of a 3 sphere. The metric is now given by

$$ds^2 = -dt^2 + a^2(t)(d\lambda^2 + \sin^2(\lambda)d\theta^2 + \sin^2(\lambda)\sin^2(\theta)d\varphi^2).$$

For photon trajectories of fixed λ and θ, these only describe geodesics when $\lambda = \frac{\pi}{2} = \theta$ (unit circle).
We may rewrite the metric as

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{jk}dx^j dx^k,$$

then the components of the Ricci tensor are given by:

$$\begin{aligned} R_{tt} &= 3\frac{\ddot{a}}{a} \\ R_{ij} &= -\tilde{g}_{ij}(a\dot{a} + 2\dot{a}^2 + 2k). \end{aligned}$$

By taking the trace of this, the scalar curvature is given by

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).$$

Einstein's field equations are given by

$$R_{ab} - \frac{1}{2}g_{ab}R = -8\pi G\left(T_{ab} - \frac{\Lambda}{8\pi G}g_{ab}\right).$$

The tt component of Einstein equations is

$$R_{tt} - \frac{1}{2}g_{tt}R = -8\pi G\left(T_{tt} - \frac{\Lambda}{8\pi G}g_{tt}\right).$$

Substituting our values for R into this we obtain the expression

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\rho$$
$$\Rightarrow \dot{a}^2 = \frac{1}{3}8\pi G a^2\rho - k.$$

Where ρ is the density of radiation in the universe. To find the relationship between ρ and a we need to consider the conservation of energy for a perfect fluid.

$$\frac{d}{da}(\rho a^3) = -3pa^2.$$

This is true because of the divergenceless of T_{ab} due to the Bianchi identity for G_{ab}. The equation of state is given by

$$p = \omega\rho.$$

From the conservation of energy argument:

$$\frac{d}{da}(\rho a^3) = 3a^2\rho + a^3\frac{d\rho}{da} = -3\omega\rho a^2$$
$$\Rightarrow \frac{d\rho}{da} = -3(1+\omega)a^{-1}\rho.$$

Intrgrating this

$$\log\rho = -3(1+\omega)\log a + c$$
$$\Rightarrow \rho \propto a^{-3(1+\omega)}.$$

For a radiation dominated universe $\omega = \frac{1}{3}$. Therefore we may write

$$\rho = \frac{3\zeta^2}{2\pi G}a^{-4}$$

where ζ^2 is a constant of integration. Then Einstein's tt equation becomes

$$\begin{aligned}\dot{a}^2 &= \frac{\zeta^2}{a^2} - k \\ &= \frac{\zeta^2}{a^2} - 1.\end{aligned}$$

We know that $a(t)$ has a maximum at $a(t) = \zeta^2$ since at that point $\dot{a}(t) = 0$. Also we know that at our intial time (big bang) and find time (big crunch) $a(t) = 0$. Therefore we are free to reparameterise $a(\eta)$ as

$$\begin{aligned}a &= \frac{\zeta}{2}(1-\cos\eta) \\ \Rightarrow \dot{a}^2 &= \frac{(1+\cos\eta)}{(1-\cos\eta)} \\ = \frac{(1-\cos^2\eta)}{(1-\cos\eta}^2 &= \frac{\sin^2\eta}{(1-\cos\eta)^2} \\ \Rightarrow \frac{da}{dt} &= \frac{\sin\eta}{1-\cos\eta}.\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{dt}{d\eta} &= \frac{dt}{da}\frac{da}{d\eta} \\
&= \frac{1-\cos\eta}{\sin\eta}\frac{\zeta}{2}\sin\eta \\
&= \frac{\zeta}{2}(1-\cos\eta) \\
&= a(\eta)
\end{aligned}$$

(η is known as conformal time).
We consider null geodesics which encircle the universe ($r = 1, \theta = \frac{\pi}{2}$). The null condition is

$$\begin{aligned}
ds^2 &= 0 \\
&= -dt^2 + a^2(t)d\varphi^2 \\
\Rightarrow d\varphi &= \frac{dt}{a(t)}.
\end{aligned}$$

To find the bounds over which to integrate this, consider $a(\eta) = \frac{\zeta}{2}(1-\cos\eta) = 0$. The first solution to this $\eta = 0$ corresponds to the big bang and the second solution $\eta = \pi$ corresponds to the big crunch.

$$\begin{aligned}
\Rightarrow \varphi_{travelled} &= \int d\varphi \\
&= \int_{t=0}^{t=final\ time} \frac{1}{a(t)}dt \\
&= \int_{\eta=0}^{\eta=\pi} \frac{1}{a(\eta)}\frac{dt}{d\eta}d\eta \\
&= \int_0^\pi \frac{a(\eta)}{a(\eta)}d\eta \\
&= \int_0^\pi d\eta \\
&= \pi.
\end{aligned}$$

The photon travels an angle π of the spherical universe in this time, therefore it travels around half the universe by the time of the big crunch.