Black Hole Thermodynamics

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Overview

The aim of this capstone project is to develop a comprehensive understanding of black hole thermodynamics and its development as a field in the past 60 years. This includes an investigation into the Penrose process for extracting energy from a black hole, and how this idea gave rise to the field of black hole thermodynamics, followed by focusing on the 1973 paper entitled *Black Holes and Entropy* by Jacob Bekenstein in which he made progress on the definitions and limitations of the first and second laws of black hole mechanics and their relation to the classical laws of thermodynamics. The final section is based on Hawking's 1975 paper *Particle Creation by Black Holes* in which he confirmed some of Bekenstein's findings and revised their exact solutions based on ideas from quantum field theory. He also corrected the physical meaning of the thermodynamic variables seen in black hole mechanics.

The Penrose Process

The Penrose Process is a theoretical approach to extracting energy from a rotating black hole. Suppose a particle is captured by a black hole, its energy is given by

$$E_0 = -p_0^a \xi_a. \tag{1}$$

Suppose this particle splits in two inside of the ergosphere, the conservation of momentum ensures

$$p_0^a = p_1^a + p_2^a. \tag{2}$$

Contracting with the Killing vector one can obtain an expression for the energy of the particles.

$$E_0 = E_1 + E_2. \tag{3}$$

Since the Killing vector is Spacelike in the ergoregion, one fragment may have negative energy, the other may return to infinity. The latter bears more energy than the original particle, given by

$$E_2 = E + |E_1|. \tag{4}$$

Therefore, the mass of the black hole is reduced to $M - |E_1|$. An amount $|E_1|$ of energy has been extracted from the black hole. The limit of energy extraction occurs when the angular momentum of the black hole is reduced to zero, at this point the mass of the black hole is its irreducible mass given by

$$M_{irr} = \frac{1}{2}[M^2 + (M^4 - J^2)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{5}$$

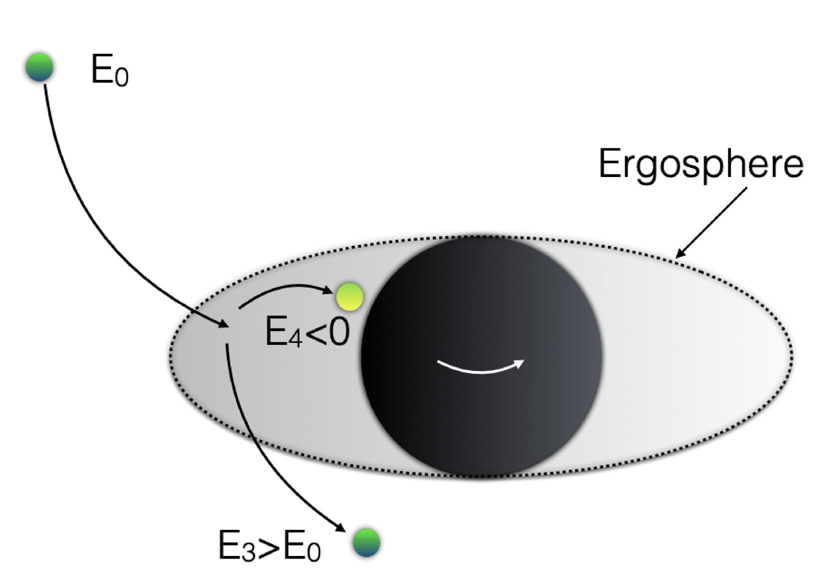

Figure 1. Ergoregion of a Kerr black hole in which the Penrose Process takes place.
Elly Leiderschneider and Tsvi Piran. Maximal efficiency of the collisional penrose process. Physical Review D, 93(4), February 2016

First Law of Black Hole Mechanics

The rationalised area of a Kerr black hole is given by

$$\alpha = 2Mr_+ - Q^2. \tag{6}$$

Bekestein differentiated this to derive this first law.

$$dM = \theta d\alpha + \vec{\Omega} \cdot d\vec{L} + \Phi dQ. \tag{7}$$

He drew an analogy between this equation and the first law of thermodynamics. However, at this point in time there was no evidence that a black hole could radiate therefore the obvious analogy drawn between temperature and the quantity θ was only formal.

Second Law of Black Hole Mechanics

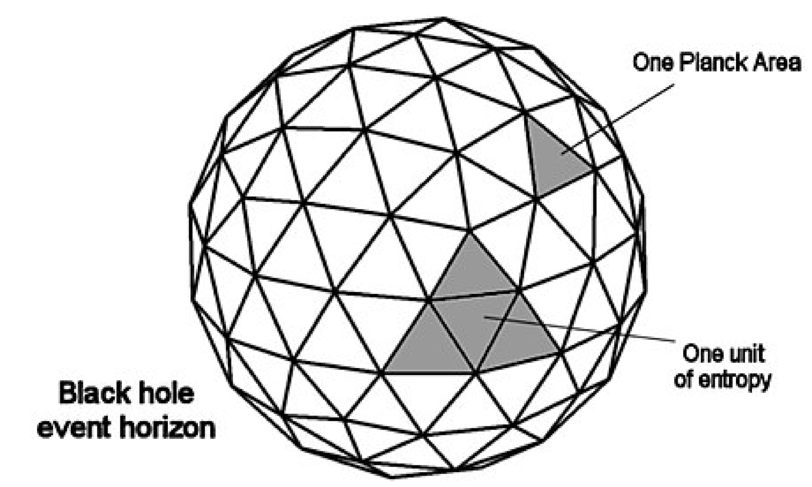

Figure 2. According to the Bekenstein bound, the entropy of a black hole is proportional to the number of Planck areas that it would take to cover the black hole's event horizon.

Bekenstein then drew another analogy from the first law to determine an expression for the entropy of a black hole, in order to formalise his generalised second law. However, once again it was not believed that black holes could radiate therefore he looked to Shannon entropy to develop an expression that would be purely based on uncertainty of the internal configuration. It was clear from the first law that entropy would be related to black hole area. Bekenstein's idea was to find the minimum information lost when a black hole captures a particle and then relate it to the previously calculated expression for the minimum increase in area of a black hole when a particle is captured by a black hole.

He began with a method trial and error of guessing the function form of the entropy. By employing known laws associated with black holes such as the black hole area theorem and through dimensional analysis, he suggested the form

$$S_{bh} = \eta \hbar^{-1} \alpha, \tag{8}$$

with η a dimensionless constant. Christodolou's expression for the minimum increase in black hole area was found to be

$$(\Delta\alpha)_{min} = 2\mu b. \tag{9}$$

The minimum information lost in the capture of a particle is whether or not the particle exists. Since this is a yes/no question the quantity of information is 1 bit, or $\ln 2$ in our units. Comparing the minimum change in entropy ($\Delta I = -\Delta S$) and the minimum increase in area, Bekenstein proposes that the expression for black hole entropy be

$$S_{bh} = (\frac{1}{2}\ln 2)\hbar^{-1}\alpha. \tag{10}$$

Bekenstein then argued that the second law of black hole thermodynamics would take the form

$$\Delta S_{bh} + \Delta S_c = \Delta(S_{bh} + S_c) > 0, \tag{11}$$

where S_c is the common entropy in the black hole exterior and S_{bh} is the black hole entropy described above. This is equivalent to stating that the generalised entropy ($S_{bh} + S_c$) never decreases.

Black Hole Radiation

In 1975 Stephen Hawking argued that particles are created and radiated at the horizon of a black hole using quantum field theory. In this section I calculated the density of particles created by a static black hole using the transformation between the Kruskal and Schwarzschild reference frames,

The Bogoliubov transformation is how one expresses one set of creation and annihilation operators in terms of another. Their solutions are of the form $b_\Omega = \int_0^\infty d\omega(\alpha_{\omega\Omega}\hat{a}_\omega + \beta_{\omega\Omega}\hat{a}^\dagger_\omega)$. The number operator is given by $N_\Omega = b^\dagger_\Omega b_\Omega$. This was used to calculate the mean particle density:

$$n_\Omega = \frac{1}{e^{\frac{2\pi\Omega}{\kappa}-1}}. \tag{12}$$

By approximating a black hole as a black body, the Hawking temperature of a black hole is given by:

$$T_H = \frac{\hbar\kappa}{2\pi c k_B}. \tag{13}$$

By using the Stephan-Boltzmann law the lifetime of a black hole is given by:

$$t_{life} = \frac{4\pi^3 k_B^4}{15c^2\hbar^3} M^3_{initial}. \tag{14}$$

Black Hole Lifetime

We will classify black holes in the following way by an approximate mass and radius for each classification:

Class	Mass	Radius
Ultramassive	$10^{10}M_\odot$	10^{11} km
Supermassive	$10^7 M_\odot$	10^9 km
Intermediate	$10^3 M_\odot$	10^3 km
Stellar	$10 M_\odot$	30 km
Micro	$10^{-8}M_\odot$	10^{-8} km

Table 1. Classification of Black Holes by Mass.

From equation 14, the initial mass of a primordial black hole (formed in the first second after the big bang) which dies within our decamillenium ($\sim 10^{17}s$ after big bang) is 5.3486×10^7 solar masses (seen in green in figure 3).

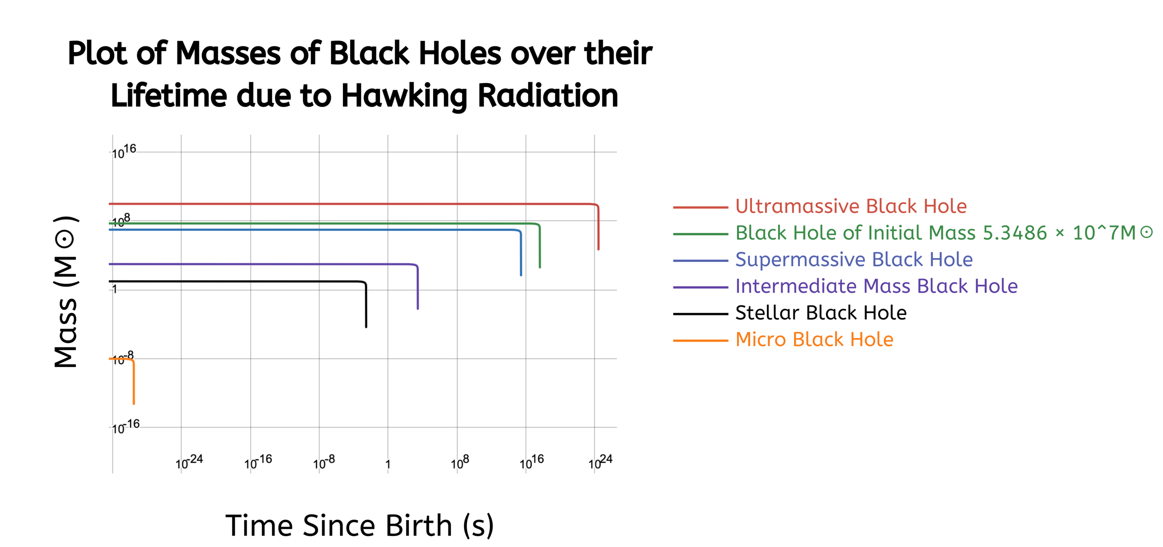

Figure 3. Lifetimes of Black Holes of Various Masses.

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