PNS Assignment 3

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1 Background

A field $\phi(x, y)$ takes values inside the unit square in two dimensions and obeys the Laplace equation:

$$\frac{d^2\phi}{d^2x} + \frac{d^2\phi}{d^2y} = 0 \tag{1}$$

When mapped onto a grid of finite size [i,j] the Laplace equation becomes:

$$\phi(i+1, j) + \phi(i-1, j) + \phi(i, j+1) + \phi(i, j-1) - 4\phi(i, j) \tag{2}$$

The Gauss-Seidel equation follows from the matrix equation $A\phi = 0$, a sparse linear system.

$$\begin{aligned} A\phi &= 0 \\ P\phi &= -(A-P)\phi \\ PP^{-1} &= I \\ A &= L + D + U \\ D &= diag(A) \end{aligned}$$

U is the traceless upper triangular part of A and L and is traceless lower triangular part of A. If we choose $P = D + L$ we obtain the Guass-Seidel Iteration:

$$x^{(k+1)} = (D+L)^{-1}b - (D+L)^{-1}Ux^{(k)} \tag{3}$$

If we choose $P = \frac{1}{\omega}D + L$ we obtain the formula for Successive Over-Relaxed (SOR) iteration:

$$\phi^{(k+1)}_{x,y} = (1-\omega)\phi^{(k)}_{x,y} + \frac{\omega}{4}(\phi^{(k+1)}_{x-1,y} + \phi^{(k+1)}_{x,y-1} + \phi^{(k)}_{x+1,y} + \phi^{(k)}_{x,y+1}) \tag{4}$$

where ω is the choice of over-relaxation parameter. One must tune ω in order to optimise convergence, in this case by numerical experiment. For over-relaxation, use $1 < \omega < 2$ in the Gauss-Seidel Algorithm.

2 Laplace equation with Dirichlet Boundary Conditions

2.1 Dirichlet Boundary Conditions

Dirichlet boundary conditions specify the values themselves that the solution to the ODE will take on the boundary. We are given a region Ω with these boundary conditions.

2.2 Finding the Optimal Value of ω

The field has values given for the boundary points and the other points are found using the equation for the SOR method. In order to tune the relaxation parameter ω, one must find the value for ω which minimises the difference between $\phi^{(k)}_{\omega}$ and $\phi^{(k-1)}_{\omega}$.
Let $\Phi = |\phi_{initial} - \phi_{final}|$, then the optimal value of ω is the one which requires the least number of iterations for the value of ϕ to stabilise. To illustrate this, see the graph below (fig 1). These graphs describe a numerical experiment to derive the value of ω for which Φ stabilises fastest (for the lowest number of iterations). An appropriate choice for

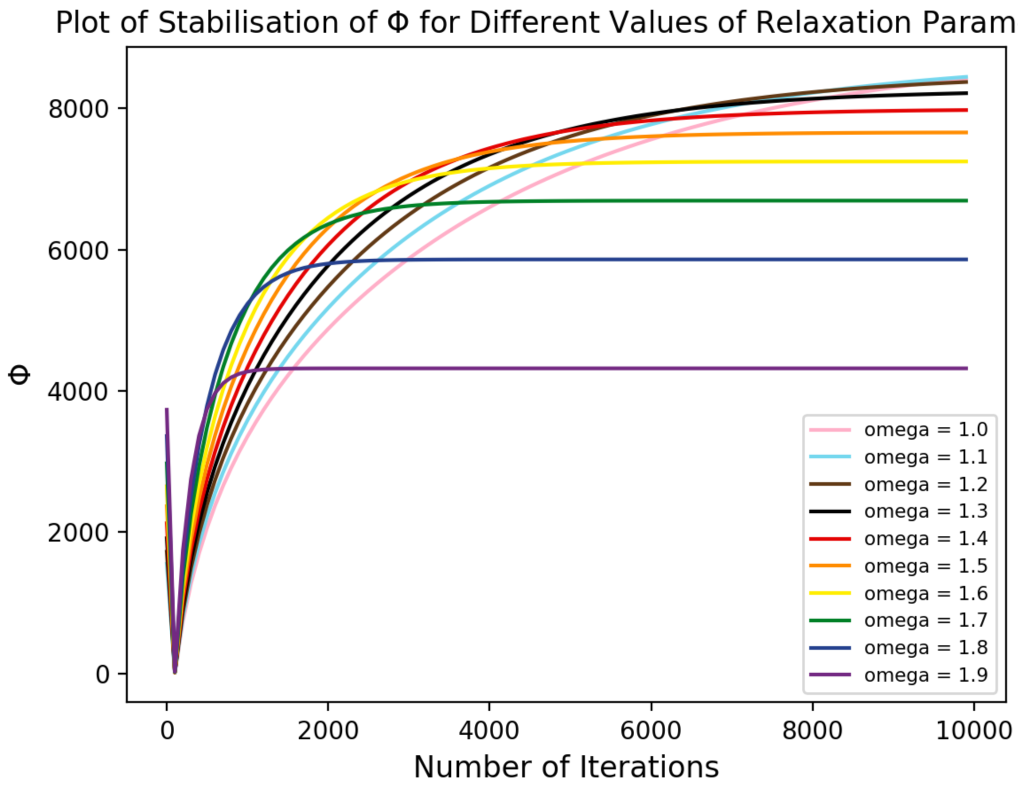

Figure 1: Plot to find an optimal relaxation parameter.

the values of ω to test were values between 1 and 2 as this is the range over which equation (4) describes successive over-relaxation. By observation of the graphs in figure 1, the optimal value of the relaxation parameter was found to be $\omega^* = 1.9$ (2 sf). A suggested improved method for finding the optimal value of omega is to run the code inside a while loop for a certain tolerance, over different values of omega, such that the program continues to iterate until the difference between the initial field $\phi_{initial}$ and final field ϕ_{final} is less than the tolerance. If one plots the number of iterations needed to obtain this difference within the tolerance for different values of omega, the optimal value of omega minimises the number of iterations needed, therefore one could choose the minimum value of the graph as the optimal omega. This is a better method as it requires less computing power to obtain a value for the optimal omega to a higher number of significant figures. If one were to use the method that I used (plot of stabilisation method), it would require significantly more time and computing power to get an optimal omega to a higher number of significant figures.

2.3 Finding the y-derivative

We are then asked to use this value of ω to find the numerical value of $\frac{\partial \phi}{\partial y}$ at the point $x = \frac{1}{2}, y = \frac{2}{5}$. This is found numerically by Euler's method. In this case the number of steps was chosen to be 5000 and a grid size of 100x100. The result was found to 3 significant figures to be: $\frac{\partial \phi}{\partial y}(\frac{1}{2}, \frac{2}{5}) = 1.07$.

3 Laplace equation with Neumann boundary conditions

Neumann boundary conditions specify the value of the derivative $\frac{dy}{dx}$ at a constant point, unlike the Dirichlet boundary conditions which specify the solution x.

3.1 Finding the Optimal Value of ω

As discussed previously, my method to determine omega is to find the omega which corresponds to Φ stabilising the fastest (over the lowest number of iterations). However, using this method is not the most efficient method. A more efficient suggestion for a method would be to perform the one described in section 2.2. To two significant figures the optimal value of the relaxation parameter was found to be $\omega^* = 1.9$.

3.2 Finding the y-derivative

In order to use the Neumann boundary conditions we use the equation:

$$\frac{d\phi}{dx}(0,y) = \frac{4\phi(h,y) - \phi(2h,y) - 3\phi(0,y)}{2h} \tag{5}$$

which is a $O(h^2)$ accurate expression for the derivative found using a Taylor expansion.
Using the given boundary conditions and performing the same computation as for the Dirichlet boundary conditions the numerical value of $\frac{\partial\phi}{\partial y}$ at the point $x = \frac{1}{2}, y = \frac{2}{5}$, using 5000 iterations and a grid size of 100x100, was found to 3 significant figures to be: $\frac{\partial\phi}{\partial y}(\frac{1}{2}, \frac{2}{5}) = 0.892$.