PNS Assignment 2

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1 Introduction

This is one part of the submission of assignment 2 for Practical Numerical Solutions, module MAU34601, Michaelmas Term 2023/2024, Trinity College Dublin. This aim of this assignment is to demonstrate programming skills in C++ and an understanding of numerical integrators including symplectic integrators. The questions assigned are seen in this document in bold and answered underneath. The source code for this assignment is written in C++ using a text editor and compiler in 'Visual Studio Code 2' version: 1.82.2 (Universal) using Microsoft's C++ extension version: 1.17.5 and the graphs are plotted using the same software with Microsoft's Python Extension version: 2023.18.0. This document is written in LaTeX in Overleaf.
Note: some important results are highlighted in the colour red.

2 Problem 1: Shooting Method

2.1 Background

The equation given in the assignment is a second order, ordinary differential equation (ODE).

$$\frac{d^2x}{dt^2} = -\frac{30x}{2+t^2x^2} \tag{1}$$

It consists of 2 boundary conditions: $x(0) = \frac{3}{4}$ and $x(10) = -1$. The shooting method can be used to reduce this boundary condition problem to an initial value problem. By letting the first condition be the one of the initial values, then by trial and error one can "shoot" with the ODE and analyse which value of x' (a.k.a $\frac{dx}{dt}$) leads to the function "landing" on the other boundary condition. This will give the other value (x'(0)) for the initial solution. Then we have a set of 2 initial values: x and x' at $t = 0$.

2.2 Shooting Method

Make a guess for x'(0). Run that guess through a fourth order Runge Kutta scheme to find the corresponding solution for x and evaluate it at t=10. This value we call $x(10; c)$. When $x(10; c) = -1$ the guess c is "correct". Define an auxiliary function $b(c) = x(10; c) + 1$. When we have a correct guess for c, our auxiliary function will go to zero. Therefore we can use methods such as the Newton-Raphson method or the bisection method to accurately find the roots of b(c), this will give us our iterative guesses for c.

2.3 Bisection Method

In this assignment I used the bisection method. This is a root-finding algorithm. When values of a function straddle the number 0, one can continue to halve the difference between these solutions until it becomes sufficiently close to zero to find a numerical solution to the root. Of course, the lower the tolerance is, the more times the program will bisect and the better the accuracy. This method is simple, but it's main disadvantage is that it is slow.

2.4 Estimating by Observation

To speed up the process I added two extra steps. Firstly I plotted the auxiliary function with a step size of one, over a large range (-100 to 100) to obtain a shape of the plot. I was able to get a better estimate for the range over which there are roots for the auxiliary function and plotted it again over the interval (-25,25). This plot can be seen below in figure 1.

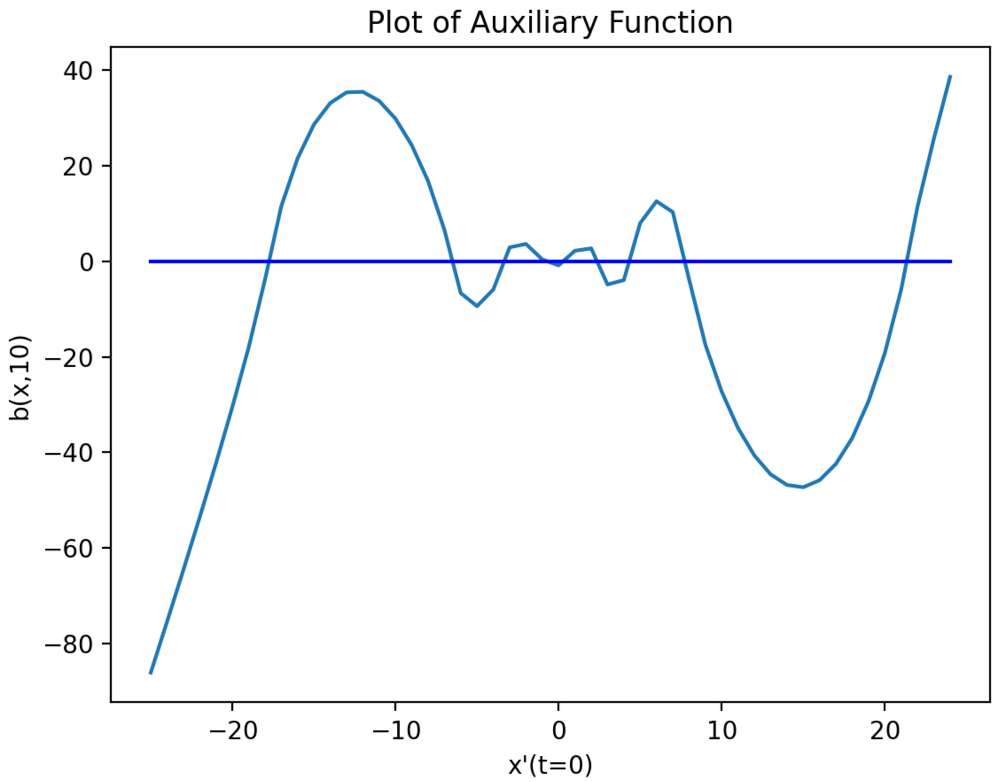

Figure 1: Plot of Auxiliary Function

2.5 Intermediate Value Theorem

Secondly I used the Intermediate Value Theorem which states that any continuous function over an interval obtains all of the values of the interval. An immediate consequence of this theorem is that if a function is in the positive region at one point and in the negative region at another, then it must cross the x-axis at some point between those points (this is also the basis of the theory for the bisection method). Therefore, I was able to define a function which multiplied values of the auxiliary function at various points. Negative solutions of the multiplication of pairs of points are only generated by points straddling the x-axis, therefore with reasonable accuracy (depending on the step size for choosing pairs of points in this function), we can estimate the location of real roots of the auxiliary function. The advantage of this added step is to ensure that the function does cross the x-axis at the points in the graph at which it comes close. For example, around the point $x = 0$ it is unclear from the graph whether or not $b(x, 10)$ crosses the x-axis. This new code showed that it did not cross, therefore there is a complex root and it is not an accepted value of $x'(t = 0)$ for which $b(x, 10)$ hits zero. The approximate location of the roots found with this method (with step size 0.1) were -17.5,-6.3, -3.2, 2.7, 4.6, 8.2 and 21.6 .

2.6 Results

This is a less accurate and slow method to execute over the entire domain, but it is useful for the reasons stated above. After the real roots are located, we use the aforementioned bisection method to continuously bisect the values near the root until the function is sufficiently (to our chosen allowed tolerance) close to zero. Values were chosen on either side of these estimated roots, where it is clear from the graph and the estimations that only one root lies within that interval.

Task: determine the values of x'(t = 0) to 6 significant figures.

After the bisection method was run the values obtained for x'(t=0) to 6 significant figures were -17.8337, -6.51696, -3.40498, 2.43748, 4.26867, 7.85929 and 21.2852.

Task: Plot any solutions you find.

These are plotted on the graph as seen below in figure 2.

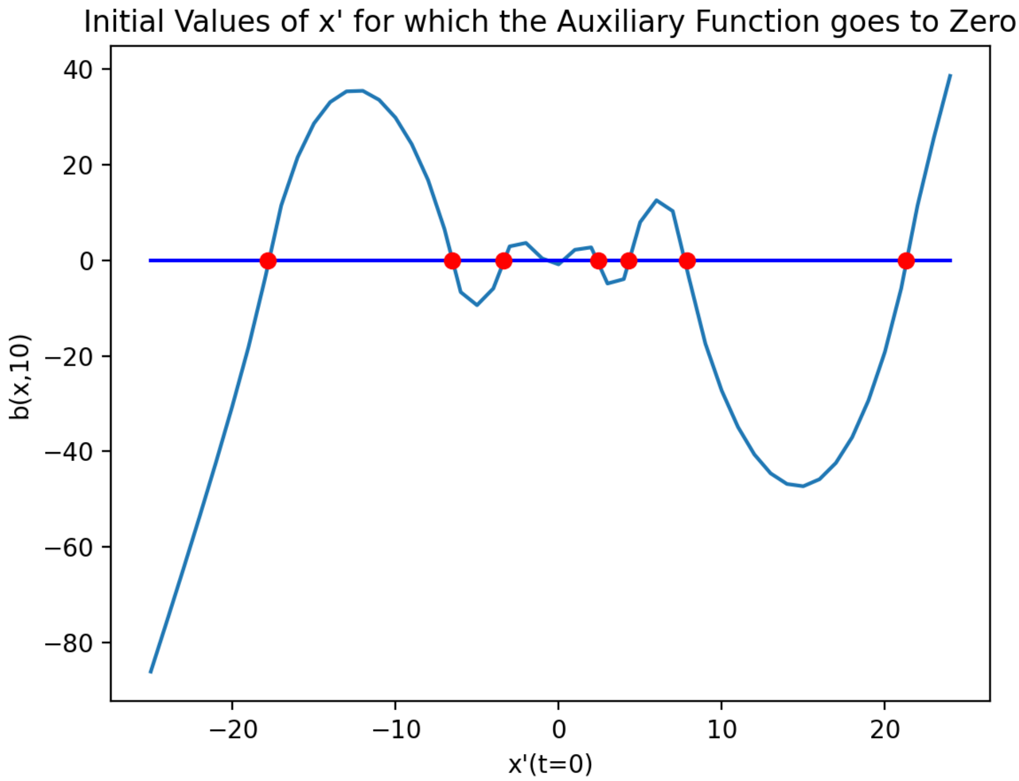

Figure 2: Values of x for which Shooting Method is Successful

3 Newtonian Gravity

3.1 Background

Newton's inverse-square law of gravity describes the force of attraction exerted on planet i by planet j and this force has magnitude:

$$|F_{ij}| = \frac{Gm_i m_j}{r_{ij}^2} \tag{2}$$

where r_{ij} is the distance between those planets. We work in a coordinate system in which $G = 1$. The force is attractive and directed along the line joining the centres of the planets. In this question we are given a set of numerical values for the mass, positions and velocities of planets and we are asked to find the location of these planets at a time $t = 5$ accurate to 3 significant figures using a leap-frog symplectic integrator.

3.2 Symplectic Integrators

The motion of planets in space is an example of a system which does not dissipate energy. There is no atmosphere causing friction (nonconservative force). We require our ODE solver to conserve the energy of the system i.e. a symplectic integrator. A symplectic integrator preserves energy and phase space volumes by "fLiouville's theorem". Symplectic integrators are used to solve Hamilton's equations of motion - they are themselves canonical transformations. One example is the Leapfrog Method.

3.3 Leapfrog Method

The leapfrog scheme is a second order method of numerical integration. We set a certain step size, then using the velocity for each step, we update the equations for the position of the body over a half step size. The velocity is then updated over a full step size and using the acceleration, then using the new velocity the position is again updated pver a half step.

3.4 Solution to Question

The system described is a 2 dimensional solar system of planets, each with 2 coordinates for position, velocity and acceleration. The acceleration is determined by $F = ma$ where F is given by Newton's inverse square law (see equation 1).
The distance r is found using the normal euclidean metric:

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{3}$$

The initial conditions for x and y coordinates of the positions of the planets are given in the question. The acceleration for a planet j is given by Newton's Second Law: $F = ma$ therefore

$$a_j = \frac{F}{m} \tag{4}$$

$$= \frac{m_i m_j}{r_{ij}^2 m_i} \tag{5}$$

However we are working in two dimensions therefore to obtain the acceleration in each direction for a planet we decompose the vector a_j into perpendicular elements: $a_{j,x}$ and $a_{j,y}$.

$$a_{j,x} = a_j \cos\theta \tag{6}$$

$$a_{j,y} = a_j \sin\theta \tag{7}$$

where θ is the angle between the radial vector connecting the planets and x direction. But $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ therefore the acceleration of a planet in each direction is given by

$$a_{j,x} = \frac{m_j x_{ij}}{r_{ij}^3} \tag{8}$$

$$a_{j,y} = \frac{m_j y_{ij}}{r_{ij}^3} \tag{9}$$

where x_j and y_j are the distances between the x and y coordinates between planets i and j, respectively.
By updating the positions and velocities of the planets, using their respective velocities and accelerations in each direction by the leap frog method, one can map their trajectories over time. The trajectories of the four given planets are seen in figure 3 below.

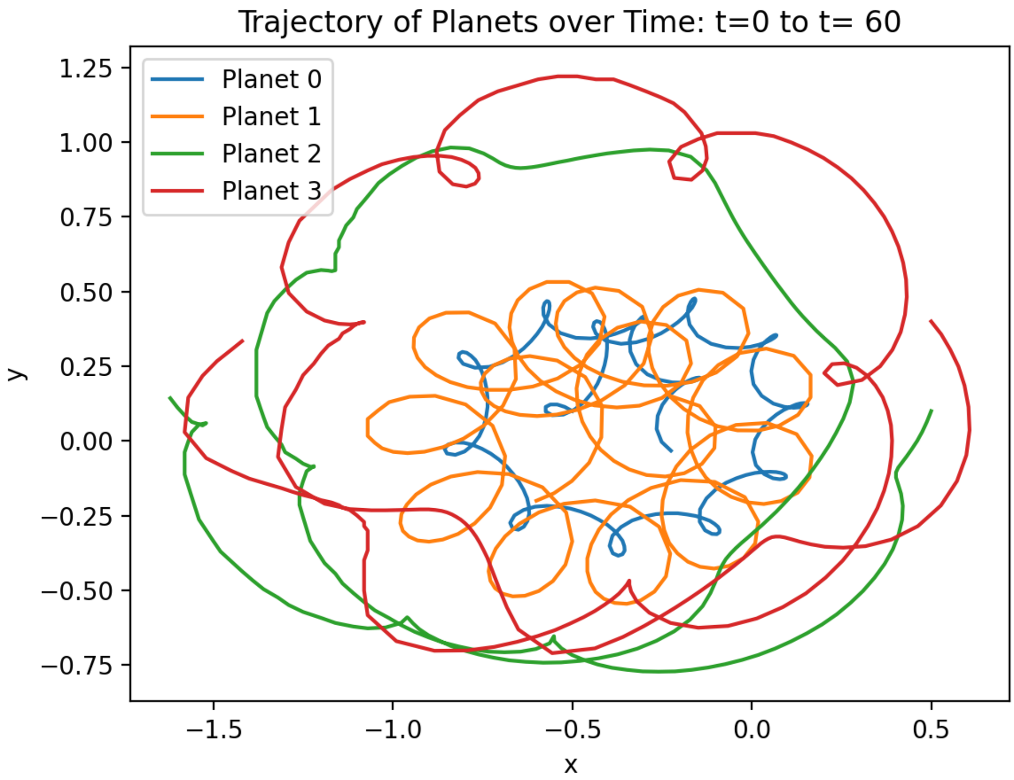

Figure 3: Trajectories of Planets

Task: Find the location of the planets at t = 5, accurate to 3 significant figures.
As required the location of the planets at t = 5 to 3 significant figures is given in the table below:

Final Results:

planet	**x-position**	**y-position**
0	-0.423	0.398
1	-0.483	0.105
2	0.237	-0.580
3	0.0665	-0.320