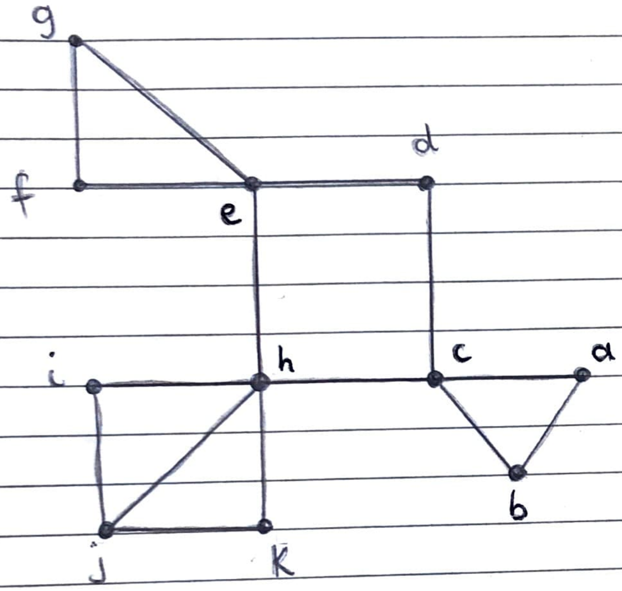
1

a)



b) If we keep the same order of the vertices and edges given in the statement of the problem, the incident table is :

	ab	bc	ac	cd	ch	de	eh	ef	eg	fg	hi	ij	hk	jk	jh
a	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
b	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
c	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
d	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
e	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
f	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
g	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
h	0	0	0	0	1	0	1	0	0	0	1	0	1	0	1
i	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
j	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
K	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0

b) The corresponding incidence matrix is :

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

c)

	a	b	c	d	e	f	g	h	i	j	K
a	0	1	1	0	0	0	0	0	0	0	0
b	1	0	1	0	0	0	0	0	0	0	0
c	1	1	0	1	0	0	0	1	0	0	0
d	0	0	1	0	1	0	0	0	0	0	0
e	0	0	0	1	0	1	1	1	0	0	0
f	0	0	0	0	1	0	1	0	0	0	0
g	0	0	0	0	1	1	0	0	0	0	0
h	0	0	1	0	1	0	0	0	1	1	1
i	0	0	0	0	0	0	0	1	0	1	0
j	0	0	0	0	0	0	0	1	0	1	1
K	0	0	0	0	0	0	0	1	0	1	0

c) The corresponding adjacency matrix is:

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

d) No, as for example ad does not belong to the graph, so not every vertex is connected to every other vertex.

e) No, as for example the graph contains the complete subgraph $V' = \{h, K, j\}$ and $E' = \{hK, Kj, jh\}$, which cannot be partitioned.

f) No, as for example a and b have degree 2 and h and e have degree 4.

g) Any two vertices that have an edge between them taken with that edge form a regular subgraph (1-regular) as do $\{a, b, c\}$ and the edges between them (2-regular) and $\{g, f, e\}$ and the edges between them

g) (2-regular) and $\{e, d, c, h\}$ and the edges between them (2-regular) and $\{h, k, j\}$ and the edges between them and $\{h, i, j\}$ and the edges between them.

h) Isomorphisms are maps from V to V for a graph (V, E)

$\varphi : V \to V$

$$\begin{aligned}
\varphi(a) &= b \\
\varphi(b) &= a \\
\varphi(c) &= c \\
\varphi(d) &= d \\
\varphi(e) &= e \\
\varphi(f) &= f \\
\varphi(g) &= g \\
\varphi(h) &= h \\
\varphi(i) &= i \\
\varphi(j) &= j \\
\varphi(k) &= k
\end{aligned}$$

i) No, it's not unique because as for example we could have

$$\begin{aligned}
\psi(a) &= a \\
\psi(b) &= b \\
\psi(c) &= c \\
\psi(d) &= d \\
\psi(e) &= e \\
\psi(f) &= g \\
\psi(g) &= f \\
\psi(h) &= h \\
\psi(i) &= i \\
\psi(j) &= j \\
\psi(k) &= k
\end{aligned}$$

j) The graph is connected as there is a walk from every vertex to every other vertex.

K) deg h = 5 and deg j = 3, so we have two vertices of odd degree, whereas the rest of the vertices have even degrees deg a = deg b = deg g = deg f = deg d = deg K = deg i = 2 and deg e = deg c = 4. By collary in lecture 37, this graph must have Eulerian trail.

l) Since not all vertices have even degrees, which is a nessary condition for the existance of an Eulerian circuit (Corollary 2 in lecture 35), this graph does not have an Eulerian circuit.

m) No, as we would have to pass through vertexs h, e, and c twice.

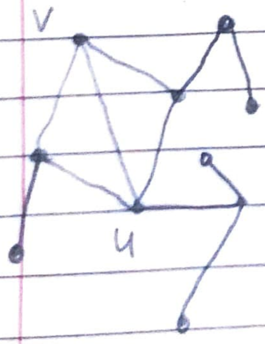
n) It is not a tree as it contains circuits as for example hkjh, abca and gfeg.

2) We have to prove an equivalence, which we will do by proving each implication in turn.

"$\Rightarrow$" The graph (V, E) is a tree. In lecture 39 we proved for any two vertices in the graph (V, E), there is exactly one path between them. $\forall u, v \in V$ s.t. $u \neq v$ and u, v are nonadjacent, adding the edge uv creates a second unique path from u to v. $uv \in E_1$ so we denote the graph (V, E_1). In lecture 34, we proved, for any graph (V, E) if there is more then one unique path between any two vertices in the graph (V, E) then the graph contains a simple circuit. For the graph (V, E_1), there are two unique paths from u to v, so the graph contains at least one simple circuit. Assumn adding uv to (V, E) creates two circuits, for this to be true (V, E) must contain a circuit before uv is added $\Rightarrow\Leftarrow$ This is a contradiction because (V, E) is a tree so it is acyclic. This means adding an edge between any two vertices in (V, E) creates exactly one circuit.

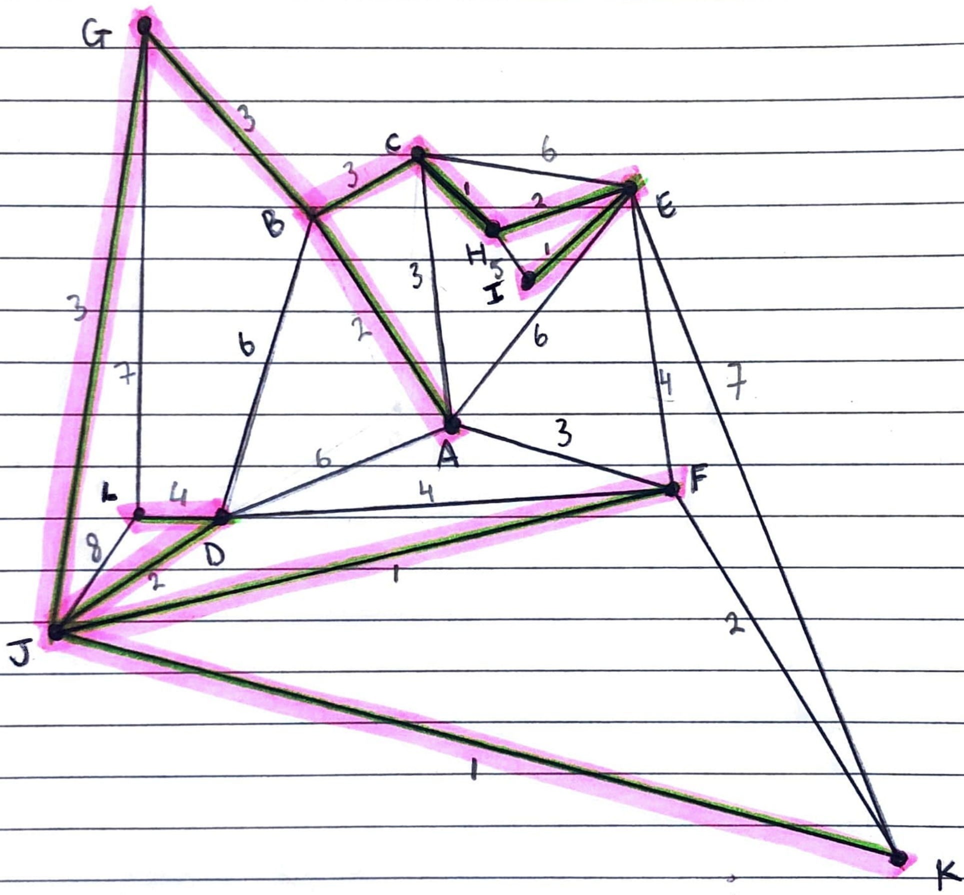
"$\Leftarrow$" Adding an edge between any two vertices in (V, E) creates exactly one circuit. Assume (V, E) has two separate connected components. We have two sets V_1 and V_2 with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. $\forall u, v \in V$ s.t. $u \neq v$ and $u \in V_1$, $v \in V_2$. There dosen't $\exists$ a path between u and v because they are in separate connected components. Adding the edge uv creates one path between u and v. For a circuit to be created, there needs to be at least two unique paths from u to v $\Rightarrow\Leftarrow$ This is a contradiction because adding an edge creates a circuit. Thus (V,E) is a connected component. Assumn (V,E) has a circuit. $\forall u,v \in V$ s.t u,v are nonadjacent and have two unique paths u to v. Adding an edge uv creates 3 unique paths from u to v and 2 circuits $\Rightarrow\Leftarrow$ because adding an edge between any two vertices in (V,E) creates exactly one circuit. Thus (V,E) is acyclic. (V,E) is connected and acyclic so (V,E) is a tree.

q.e.d



3)

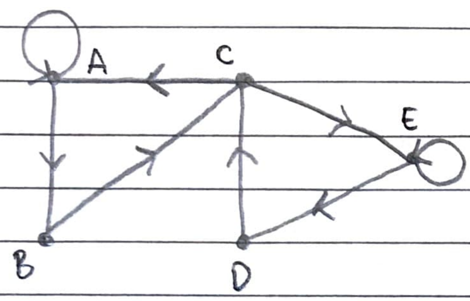
a)



b) The edges are added in the following order: FJ, JK, CH, EI, DJ, AB, EH, BG, GJ, BC, DL (Showed in pink)

c) The edges are added in the following order: DJ, FJ, JK, GJ, BG, AB, BC, CH, EH, EI, DL. (Showed in green)

4 a)



b)

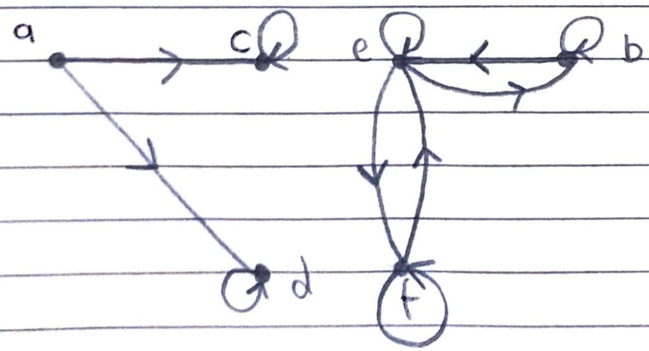
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

c) Isomorphisms are maps ~~from~~ V to V from a graph (V,E)

$\varphi : V \to V$

$\varphi(A) = E$
$\varphi(B) = D$
$\varphi(C) = C$
$\varphi(D) = B$
$\varphi(E) = A$

5a)



5 b) No, for R to be an equivalence relation it must be reflexive, symmetric and transitive. a is not related to a so R is not reflexive. Also it is not symmetric because a relates d but d dosen't relate a. Also it is not transitive because f relates e and e relates b but f dose not relate b.

c) (a, a) (d, a) (c, a) (f, b) (b, f) (d, c) (c, d)