Lewis Expansion for the Smile In a Stochastic Volatility Jump Diffusion Model

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The Black and Scholes Formula

- $\varphi_m[u] = e^m N\left[\frac{m}{u} + \frac{u}{2}\right] - N\left[\frac{m}{u} - \frac{u}{2}\right] \equiv e^m\left(1 + Erf\left[\frac{\frac{m}{u} + \frac{u}{2}}{\sqrt{2}}\right]\right) - 1 - Erf\left[\frac{\frac{m}{u} - \frac{u}{2}}{\sqrt{2}}\right]$

- then $\Phi[F, K, \sigma_F] = K\varphi_{Log\left[\frac{F}{K}\right]}[\sigma_F]$

- and $BSCall[S, K, T, r, \delta, v] = e^{-rT}\Phi[Se^{-(r-\delta)T}, K, v\sqrt{T}] = e^{-rT}K\varphi_{Log\left[\frac{S}{K}\right] + (r-\delta)T}[v\sqrt{T}]$

- therfore $v = \frac{1}{\sqrt{T}}\varphi^{-1}_{Log\left[\frac{S}{K}\right] + (r-\delta)T}\left[\frac{x}{K}e^{rT}\right]$ gives the most efficient way to compute the implicit vol

The Transform of the process-1

- The process is : $\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}dB_t$ with $dB_t \cdot dW_t = \rho(V_t)dt$

  $dV_t = b(V_t)dt + a(V_t)dW_t$

- So the differential equation followed by a derivative $f(S, V, t)$ is: $-\frac{\partial f}{\partial t} = -rf + \mathcal{A} \cdot f$ where

$$\mathcal{A} \cdot f = (r-\delta)S\frac{\partial f}{\partial S} + \frac{1}{2}VS^2\frac{\partial^2 f}{\partial S^2} + b(V)\frac{\partial f}{\partial V} + \frac{1}{2}(a(V))^2\frac{\partial^2 f}{\partial V^2} + S\rho(V)a(V)\sqrt{V}\frac{\partial^2 f}{\partial V \partial S}$$

- The call can be written as:

$$Call[S, K, \tau, r, \delta, \sigma] = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i - \infty}^{ik_i + \infty} \frac{e^{-ikX}}{k^2 - ik} H(k, V, \tau)dk$$

  - where H is the fundamental transform of the process

Diff.Eq. of the Expansion-1

- The process followed by H is : $\frac{1}{2}\xi^2\eta^2\frac{\partial^2 H}{\partial V^2} + (b + \xi d\chi)\frac{\partial H}{\partial V} - cVH - \frac{\partial H}{\partial \tau} = 0$ where

$$c = \frac{k^2 - ik}{2} \qquad d = -ik \qquad \chi = \rho(V)\eta(V)V^{1/2} \qquad a(V) = \xi(\eta(V))$$

- we assume $H = H_0(1 + \xi h_1 + \xi^2 h_2 + \xi^3 h_3 + \ldots)$

- By injecting the last one into the preceding

$$\frac{1}{2}\xi^2\eta^2\left(\frac{\partial^2 H_0}{\partial V^2}(1 + \xi h_1 + \xi^2 h_2 + \xi^3 h_3 + \ldots) + 2\frac{\partial H_0}{\partial V}(\xi h_1^{(V)} + \xi^2 h_2^{(V)} + \xi^3 h_3^{(V)} + \ldots) + H_0(\xi h_1^{(VV)} + \xi^2 h_2^{(VV)} + \xi^3 h_3^{(VV)} + \ldots)\right)$$

$$+ (b + \xi d\chi)\left(\frac{\partial H_0}{\partial V}(1 + \xi h_1 + \xi^2 h_2 + \xi^3 h_3 + \ldots) + H_0(\xi h_1^{(V)} + \xi^2 h_2^{(V)} + \xi^3 h_3^{(V)} + \ldots)\right)$$

$$-cVH_0(1 + \xi h_1 + \xi^2 h_2 + \xi^3 h_3 + \ldots)$$

$$-\left(\frac{\partial H_0}{\partial \tau}(1 + \xi h_1 + \xi^2 h_2 + \xi^3 h_3 + \ldots) + H_0(\xi h_1^{(\tau)} + \xi^2 h_2^{(\tau)} + \xi^3 h_3^{(\tau)} + \ldots)\right) = 0$$

Diff.Eq. of the Expansion-2

- We split the terms by power of ξ

$$\frac{1}{2}\xi^2\eta^2\left(\frac{\partial^2 H_0}{\partial V^2}(1+\xi h_1+\xi^2 h_2+\xi^3 h_3+\ldots)\right) = \frac{1}{2}\xi^2\eta^2\frac{\partial^2 H_0}{\partial V^2}+\frac{1}{2}\xi^3\eta^2\frac{\partial^2 H_0}{\partial V^2}h_1+\frac{1}{2}\xi^4\eta^2\frac{\partial^2 H_0}{\partial V^2}h_2+\frac{1}{2}\xi^5\eta^2\frac{\partial^2 H_0}{\partial V^2}h_3+..$$

$$+\frac{1}{2}\xi^2\eta^2\left(2\frac{\partial H_0}{\partial V}(\xi h_1^{(V)}+\xi^2 h_2^{(V)}+\xi^3 h_3^{(V)}+\ldots)\right) = \xi^3\eta^2\frac{\partial H_0}{\partial V}h_1^{(V)}+\xi^4\eta^2\frac{\partial H_0}{\partial V}h_2^{(V)}+\xi^5\eta^2\frac{\partial H_0}{\partial V}h_3^{(V)}+..$$

$$+\frac{1}{2}\xi^2\eta^2(H_0(\xi h_1^{(VV)}+\xi^2 h_2^{(VV)}+\xi^3 h_3^{(VV)}+\ldots)) = \frac{1}{2}\xi^3\eta^2 H_0 h_1^{(VV)}+\frac{1}{2}\xi^4\eta^2 H_0 h_2^{(VV)}+\frac{1}{2}\xi^5\eta^2 H_0 h_3^{(VV)}+..$$

$$+(b+\xi d\chi)\left(\frac{\partial H_0}{\partial V}(1+\xi h_1+\xi^2 h_2+\xi^3 h_3+\ldots)\right) = b\frac{\partial H_0}{\partial V}+\xi\frac{\partial H_0}{\partial V}(bh_1+d\chi)+\xi^2\frac{\partial H_0}{\partial V}(bh_2+d\chi h_1)+\xi^3\frac{\partial H_0}{\partial V}(bh_3+d\chi h_2)+ ..$$

$$+(b+\xi d\chi)(H_0(\xi h_1^{(V)}+\xi^2 h_2^{(V)}+\xi^3 h_3^{(V)}+\ldots)) = \xi H_0 b h_1^{(V)}+\xi^2 H_0(bh_2^{(V)}+d\chi h_1^{(V)})+\xi^3 H_0(bh_3^{(V)}+d\chi h_2^{(V)})+..$$

$$-cVH_0(1+\xi h_1+\xi^2 h_2+\xi^3 h_3+\ldots) = -\left(cVH_0+\xi cVH_0h_1+\xi^2 cVH_0h_2+\xi^3 cVH_0h_3+\ldots\right)$$

$$-\left(\frac{\partial H_0}{\partial \tau}(1+\xi h_1+\xi^2 h_2+\xi^3 h_3+\ldots)\right) = -\left(\frac{\partial H_0}{\partial \tau}+\xi\frac{\partial H_0}{\partial \tau}h_1+\xi^2\frac{\partial H_0}{\partial \tau}h_2+\xi^3\frac{\partial H_0}{\partial \tau}h_3+\ldots\right)$$

$$-(H_0(\xi h_1^{(\tau)}+\xi^2 h_2^{(\tau)}+\xi^3 h_3^{(\tau)}+\ldots)) = -\left(\xi H_0 h_1^{(\tau)}+\xi^2 H_0 h_2^{(\tau)}+\xi^3 H_0 h_3^{(\tau)}+\ldots\right)$$

Diff.Eq. of the Expansion-3

- Terms in ξ^0 : $b\frac{\partial H_0}{\partial V} - cVH_0 - \frac{\partial H_0}{\partial \tau} = 0$

- Terms in ξ^1 : $\frac{\partial H_0}{\partial V}(bh_1 + d\chi) + H_0 b h_1^{(V)} - cVH_0 h_1 - \frac{\partial H_0}{\partial \tau} h_1 - H_0 h_1^{(\tau)} = 0$

- Terms in ξ^2 : $\frac{1}{2}\eta^2 \frac{\partial^2 H_0}{\partial V^2} + \frac{\partial H_0}{\partial V}(bh_2 + d\chi h_1) + H_0(bh_2^{(V)} + d\chi h_1^{(V)}) - cVH_0 h_2 - \frac{\partial H_0}{\partial \tau} h_2 - H_0 h_2^{(\tau)} = 0$

- Terms in ξ^3 :

$$\frac{1}{2}\eta^2\left(\frac{\partial^2 H_0}{\partial V^2} h_1 + 2\frac{\partial H_0}{\partial V} h_1^{(V)} + H_0 h_1^{(VV)}\right) + \frac{\partial H_0}{\partial V}(bh_3 + d\chi h_2) + H_0(bh_3^{(V)} + d\chi h_2^{(V)}) - cVH_0 h_3 - \frac{\partial H_0}{\partial \tau} h_3 - H_0 h_3^{(\tau)} = 0$$

- Terms in ξ^n :

$$\frac{1}{2}\eta^2\left(\frac{\partial^2 H_0}{\partial V^2} h_{n-2} + 2\frac{\partial}{\partial V}(H_0) h_{n-2}^{(V)} + H_0 h_{n-2}^{(VV)}\right) + \frac{\partial H_0}{\partial V}(bh_n + d\chi h_{n-1}) + H_0(bh_n^{(V)} + d\chi h_{n-1}^{(V)}) - cVH_0 h_n - \frac{\partial H_0}{\partial \tau} h_n - H_0 h_n^{(\tau)} = 0$$

Diff.Eq. of the Expansion-4

- By introducing: $H_0 \equiv e^{-cU}$ with $\zeta \equiv \frac{\partial U}{\partial V}$, we have $\frac{\partial H_0}{\partial V} = -c\zeta H_0$, $\frac{\partial^2 H_0}{\partial V^2} = H_0\left(c^2\zeta^2 - c\frac{\partial \zeta}{\partial V}\right)$

- and $\frac{\partial H_0}{\partial \tau} = -c\frac{\partial U}{\partial \tau} H_0 = -cYH_0 = -c(\zeta b + V)H_0$ comes from the first differential equation, or from the definition : $\frac{dY}{du} = b[Y]$ with $Y[u = 0] = V$, solution of $\xi = 0$ (determinist eq.)

- because by introducing these results into our expansion we get :

$$\frac{1}{2}\eta^2\left(\left(c^2\zeta^2 - c\frac{\partial \zeta}{\partial V}\right)h_{n-2} - 2c\zeta h^{(V)}_{n-2} + h^{(VV)}_{n-2}\right) - c\zeta(bh_n + d\chi h_{n-1}) + bh^{(V)}_n + d\chi h^{(V)}_{n-1} - cVh_n + c(\zeta b + V)h_n - h^{(\tau)}_n = 0$$

- so

$$h^{(\tau)}_n - bh^{(V)}_n = \frac{1}{2}\eta^2\left(\left(c^2\zeta^2 - c\frac{\partial \zeta}{\partial V}\right)h_{n-2} - 2c\zeta h^{(V)}_{n-2} + h^{(VV)}_{n-2}\right) - c\zeta d\chi h_{n-1} + d\chi h^{(V)}_{n-1}$$

Diff.Eq. of the Expansion-5

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$$h_n^{(\tau)} - bh_n^{(V)} = \frac{1}{2}\eta^2\left(c^2\zeta^2 h_{n-2} - c\frac{\partial}{\partial V}(\zeta h_{n-2}) + h_{n-2}^{(VV)} - c\zeta h_{n-2}^{(V)}\right) + d\chi\left(\frac{\partial}{\partial V} - c\zeta\right)h_{n-1}$$

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$$\left(\frac{\partial}{\partial V} - c\zeta\right)^2 h_{n-2} = \left(\frac{\partial}{\partial V} - c\zeta\right)(h_{n-2}^{(V)} - c\zeta h_{n-2}) = h_{n-2}^{(VV)} - c\frac{\partial}{\partial V}(\zeta h_{n-2}) - c\zeta h_{n-2}^{(V)} + c^2\zeta^2 h_{n-2}$$

- So the Expansion summarizes into : $h_n^{(\tau)} - bh_n^{(V)} = \frac{1}{2}\eta^2\left(\frac{\partial}{\partial V} - c\zeta\right)^2 h_{n-2} + d\chi\left(\frac{\partial}{\partial V} - c\zeta\right)h_{n-1}$

Solution of the Diff.Eq. of the Expansion-1

- We have to solve :

$$\begin{cases} \dfrac{\partial H_0}{\partial \tau} - b\dfrac{\partial H_0}{\partial V} = -cVH_0 \\ h_1^{(\tau)} - bh_1^{(V)} = -c\zeta d\chi \\ h_2^{(\tau)} - bh_2^{(V)} = \dfrac{1}{2}\eta^2\left(c^2\zeta^2 - c\dfrac{\partial \zeta}{\partial V}\right) + d\chi\left(\dfrac{\partial}{\partial V} - c\zeta\right)h_1 \\ h_n^{(\tau)} - bh_n^{(V)} = \dfrac{1}{2}\eta^2\left(\dfrac{\partial}{\partial V} - c\zeta\right)^2 h_{n-2} + d\chi\left(\dfrac{\partial}{\partial V} - c\zeta\right)h_{n-1} \end{cases}$$

- with the boundary condition : $h_m(\tau = 0) = 0$

- Let’s define :

$$f_1(k, V, \tau) = -cd\chi\zeta$$

$$f_2(k, V, \tau) = \frac{1}{2}\eta^2\left(c^2\zeta^2 - c\frac{\partial \zeta}{\partial V}\right) + d\chi\left(\frac{\partial}{\partial V} - c\zeta\right)h_1$$

$$f_n(k, V, \tau) = \frac{1}{2}\eta^2\left(\frac{\partial}{\partial V} - c\zeta\right)^2 h_{n-2} + d\chi\left(\frac{\partial}{\partial V} - c\zeta\right)h_{n-1}$$

Solution of the Diff.Eq. of the Expansion-2

- Theorem : Method of carateristics for $-\frac{\partial g}{\partial \tau} + b[x]\frac{\partial g}{\partial x} - c[x]g + k[\tau, x] = 0$ with $g[0, x] = \varphi[x]$

  The solution is given by : be Y[s,x] the solution to $\frac{dY[s]}{ds} = b[Y[s]]$ with $Y[0] = x$

  Then $g[\tau, x] = \varphi[Y[s, x]]e^{-\int_0^\tau c[Y[\lambda, x]]d\lambda} + \int_0^\tau \left(k[\tau - s, Y[s, x]]e^{-\int_0^s c[Y[\lambda, x]]d\lambda} \right) ds$

- We apply it to $h_n^{(\tau)} - bh_n^{(V)} = f_n(k, V, \tau)$ to get : $h_n[k, V, \tau] = \int_0^\tau (f_n[k, Y[s, x], \tau - s])ds$

Solution of the Diff.Eq. of the Expansion-3

- So it gives us for the first orders :

$$h_1[k,V,\tau] = -\int_0^\tau c d\chi[x]\zeta[x,t]ds$$

$$h_2[V,\tau] = \int_0^\tau \left\{\frac{1}{2}(\eta[x])^2 c\left(c(\zeta[x,t])^2 - \frac{\partial\zeta}{\partial V}[x,t]\right) + d\chi[x]\left(\frac{\partial h_1}{\partial V}[x,t] - c\zeta[x,t]h_1[x,t]\right)\right\}ds$$

$$h_3[k,V,\tau] = \int_0^\tau \left(\frac{1}{2}(\eta[x])^2\left(c^2\frac{\partial^2}{\partial V^2}(h_1)[x,t] - 2c\zeta[x,t]\frac{\partial h_1}{\partial V}[x,t] - c\left(\frac{\partial\zeta}{\partial V}[x,t] - c(\zeta[x,t])^2\right)h_1[x,t]\right) + d\chi[x]\left(\frac{\partial h_2}{\partial V}[x,t] - c\zeta[x,t]h_2[x,t]\right)\right)ds$$

$$h_4[k,V,\tau] = \int_0^\tau \left(\frac{1}{2}(\eta[x])^2\left(c^2\frac{\partial^2}{\partial V^2}(h_2)[x,t] - 2c\zeta[x,t]\frac{\partial h_2}{\partial V}[x,t] - c\left(\frac{\partial\zeta}{\partial V}[x,t] - c(\zeta[x,t])^2\right)h_2[x,t]\right) + d\chi[x]\left(\frac{\partial h_3}{\partial V}[x,t] - c\zeta[x,t]h_3[x,t]\right)\right)ds$$

- where $x = Y[s,V]$ and $t = \tau - s$

- The presentation of these equations is such that the dependency in k is contained in polynomials of c[k] and d[k] with coefficients that are functions (multiple integrales) of V and τ only

Vega

- The Classical Vega is

$$\frac{\partial}{\partial\sigma}BSCall = Se^{-\delta T}\sqrt{T}\frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} = Se^{-\delta T}\sqrt{\frac{T}{2\pi}}e^{-\frac{\left(\frac{X}{U}+\frac{U}{2}\right)^2}{2}} = Se^{-\delta T}\sqrt{\frac{T}{2\pi}}e^{-\frac{\left(\frac{X}{U}-\frac{U}{2}\right)^2}{2}-X} = Ke^{-rT}\sqrt{\frac{T}{2\pi}}e^{-\frac{\left(\frac{X}{U}-\frac{U}{2}\right)^2}{2}}$$

- But

$$\frac{\partial}{\partial V}BSCall = \frac{\partial}{\partial\sigma}BSCall\times\frac{\partial\sigma}{\partial V} = \frac{\partial}{\partial\sigma}BSCall\times\frac{\partial}{\partial V}\sqrt{V} = \frac{\partial}{\partial\sigma}BSCall\times\frac{1}{2\sqrt{V}} = \frac{\partial}{\partial\sigma}BSCall\times\frac{\sqrt{T}}{2U}$$

- So $\frac{\partial}{\partial V}BSCall = Ke^{-rT}\frac{T}{\sqrt{8\pi}}\frac{e^{-\frac{\left(\frac{X}{U}-\frac{U}{2}\right)^2}{2}}}{U}$ and $BSCall = e^{-rT}\Phi[F, K, \sqrt{VT}]$ so

$$\frac{\partial}{\partial V}\Phi[F, K, \sqrt{VT}] = K\sqrt{\frac{T}{8\pi V}}e^{-\frac{\left(\frac{X}{\sqrt{VT}}-\frac{\sqrt{VT}}{2}\right)^2}{2}} \quad \frac{\partial^2}{\partial V^2}\Phi[F, K, \sqrt{VT}] = \frac{K}{\sqrt{512\pi TV^5}}e^{-\frac{\left(\frac{X}{\sqrt{VT}}-\frac{\sqrt{VT}}{2}\right)^2}{2}}(4X^2-T^2V^2-4TV)$$

BS Derivatives

- An ordinary Black and Sholes Call can be computed by :

$$f(X,U) = e^X - \frac{1}{2\pi}\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} \frac{e^{-ikX-\left(\frac{k^2-ik}{2}\right)U}}{k^2-ik}dk = e^X\Phi\left[\frac{X}{\sqrt{U}}+\frac{1}{2}\sqrt{U}\right]-\Phi\left[\frac{X}{\sqrt{U}}-\frac{1}{2}\sqrt{U}\right]$$

- $BSCall[S,K,t,r,\delta,\sigma] = Ke^{-rt}f\left[Log\left[\frac{Se^{-\delta t}}{Ke^{-rt}}\right],\sigma^2 t\right]$

- The derivatives can be represented as :

$$\frac{\partial^{(p+q)}}{\partial U^p \partial X^q}f(X,U) = \frac{-1}{2\pi}\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} \frac{e^{-ikX-\left(\frac{k^2-ik}{2}\right)U}(-ik)^q\left(\frac{k^2-ik}{2}\right)^p}{k^2-ik}dk + \delta_{p,0}e^X$$

- therefore, if $c[k] = (k^2-ik)/2$ and $d[k] = -ik$:

$$(-1)^p\frac{Ke^{-r\tau}}{2\pi}\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty}\frac{e^{d[k]X-c[k]U}(d[k])^q(c[k])^p}{k^2-ik}(dk) \equiv I[p,q] = Ke^{-r\tau}(-1)^p\frac{\partial^{(p+q)}}{\partial U^p\partial X^q}f(X,U)$$

BS Derivatives (2)

- But $\frac{\partial f}{\partial U}\left[Log\left[\frac{Se^{-\delta t}}{Ke^{-rt}}\right], Vt\right] = \frac{1}{t}\frac{\partial}{\partial V}BSCall[S, V, t]$ $\qquad$ $\frac{\partial f}{\partial X}\left[Log\left[\frac{Se^{-\delta t}}{Ke^{-rt}}\right], Vt\right] = S\frac{\partial}{\partial S}BSCall[S, V, t]$

- so $\quad I[p, q] = (-\tau)^{-p}\left(\frac{\partial}{\partial V}\right)^{p}\left(S\frac{\partial}{\partial S}\right)^{n}BSCall[S, V, \tau]$ $\quad$ but $U = \sqrt{VT}$

- and because $\frac{\partial f}{\partial U} = \frac{e^{-\frac{(2X-U)^2}{8U}}}{2\sqrt{2\pi U}}$ all derivatives I[p,q] with $p \geq 1$ will have $\frac{\partial f}{\partial U}$ as factor, so $\frac{I[p, q]}{\frac{\partial}{\partial V}BSCall[S, V, t]}$ is a generalized (laurent serie) polynom in X and V and t. We define :

- $[p, q] = (-\tau)^{p}R[p, q]\frac{\partial}{\partial V}BSCall[S, V, \tau]$ and for exemple :

$$R[1, 1] = \left[-\frac{X}{U} + \frac{1}{2}\right] \qquad R[1, 2] = \left[\frac{X^2}{U^2} - \left(\frac{X}{U} - \frac{1}{4U}(4 - U)\right)\right] \qquad R[2, 1] = \tau\left[-\frac{1}{2}\frac{X^3}{U^3} + \frac{1}{4}\frac{X^2}{U^2} + \frac{1}{8}\frac{X}{U^2}(12 + U) - \frac{1}{16}\frac{1}{U}(4 + U)\right]$$

$$R[2, 0] = \tau\left[\frac{1}{2}\frac{X^2}{U^2} - \frac{1}{2U} - \frac{1}{8}\right] \qquad R[2, 2] = \tau\left[\frac{1}{2}\frac{X^4}{U^4} - \frac{1}{2}\frac{X^3}{U^3} - 3\frac{X^2}{U^3} + \frac{1}{8}\frac{X}{U^2}(12 + U) + \frac{1}{32}\frac{1}{U^2}(48 - U^2)\right]$$

Expansion of the call price for stochastic vol

- By expressing the expansion:

$$Call[S,K,\tau,r,\delta,\sigma] = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-ikX}}{k^2-ik}H_0(1+\xi h_1+\xi^2 h_2+\xi^3 h_3+\ldots)dk = Se^{-\delta\tau} -$$

$$Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-ikX}}{k^2-ik}H_0\left(1+\xi\left(-\int_0^\tau cd\chi[x]\zeta[x,t]ds\right)+\xi^2\int_0^\tau\left\{\frac{1}{2}(\eta[x])^2 c\left(c(\zeta[x,t])^2-\frac{\partial\zeta}{\partial V}[x,t]\right)+d\chi[x]\left(\frac{\partial h_1}{\partial V}[x,t]-c\zeta[x,t]h_1[x,t]\right)\right\}ds+\ldots\right)dk$$

- separating the problem :

$$Call[S,K,\tau,r,\delta,\sigma] = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-ikX}}{k^2-ik}H_0(1+\xi(-cdJ_1)+\xi^2(-cJ_2+c^2J_3-cd^2J_4+c^2d^2J_5)+\ldots)dk$$

d

- with

$$J_1 = \int_0^\tau \chi[Y[s,V],\tau-s]\zeta[Y[s,V],\tau-s]ds$$

$$J_2 = \frac{1}{2}\int_0^\tau \eta^2[Y[s,V],\tau-s]\zeta_V[Y[s,V],\tau-s]ds$$

$$J_3 = \frac{1}{2}\int_0^\tau \eta^2[Y[s,V],\tau-s]\zeta^2[Y[s,V],\tau-s]ds$$

$$J_4 = \int_0^\tau \chi[Y[s,V],\tau-s]J_1[Y[s,V],\tau-s]ds$$

$$J_5 = \frac{1}{2}\int_0^\tau \chi[Y[s,V],\tau-s]\zeta[Y[s,V],\tau-s]J_1[Y[s,V],\tau-s]ds$$

Series

- $Call[S, V, \tau] = BSCall\left[S, \frac{U[V, \tau]}{\tau}, \tau\right] + (\xi\tau^{-1}J_1 R[1, 1] + \xi^2(\tau^{-1}J_2 + \tau^{-2}J_3 R[2, 0] + \tau^{-1}J_4 R[1, 2] + \tau^{-2}J_5 R[2, 2]))C_V$

  - where $C_V = \frac{\partial}{\partial V}BSCall\left[S, \frac{U[V, \tau]}{\tau}, \tau\right] = Ke^{-r\tau}\frac{\tau}{\sqrt{8\pi}}\frac{e^{-\frac{\left(\frac{X}{U} - \frac{U}{2}\right)^2}{2}}}{U}$

- Then if we use $V_{imp} = \frac{U[V, \tau]}{\tau} + \xi g_1 + \xi^2 g_2 + \ldots$ and

$$C[\frac{U[V, \tau]}{\tau} + \xi g_1 + \xi^2 g_2 + \ldots] = C[\frac{U[V, \tau]}{\tau}] + (\xi g_1 + \xi^2 g_2 + \ldots)C_V + \frac{1}{2}\xi^2 g_1^2 C_{VV} + \ldots$$

- if we use $C_{VV} = C_V R[2, 0]$ we get :

$$V_{imp}[S, V, \tau] = \frac{U[V, \tau]}{\tau} + (\xi\tau^{-1}J_1 R[1, 1] + \xi^2(\tau^{-1}J_2 + \tau^{-2}J_3 R[2, 0] + \tau^{-1}J_4 R[1, 2] + \tau^{-2}J_5 R[2, 2] - \tau^{-2}R[1, 1]^2 R[2, 0]))$$

The parametrized model

- $dV_t = (\omega - \theta V)dt + \xi V^{\varphi} dW_t$
- then $Y[s, V] = \frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)$ and $\zeta[Y[s, V], \tau - s] = \frac{1}{\theta}(1 - e^{-\theta(\tau - s)})$, $U[s, V] = \frac{\omega}{\theta}s + \left(\frac{1 - e^{-\theta s}}{\theta}\right)\left(V - \frac{\omega}{\theta}\right)$
- and $h_1 = -cdJ_1$ $\qquad h_2 = -cJ_2 + c^2 J_3 - cd^2 J_4 + c^2 d^2 J_5$

$$J_1 = \frac{\rho}{\theta}\int_0^{\tau}(1 - e^{-\theta(\tau - s)})\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi + \frac{1}{2}} ds$$

$$J_2 = 0$$

$$J_3 = \frac{1}{2\theta^2}\int_0^{\tau}(1 - e^{-\theta(\tau - s)})^2\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{2\varphi} ds$$

\- where

$$J_4 = \left(\varphi + \frac{1}{2}\right)\frac{\rho^2}{\theta}\int_0^{\tau}\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi + \frac{1}{2}} j_6[V, \tau, s] ds$$

$$J_5 = \frac{1}{2}J_1^2$$

$$J_6 = \int_0^{\tau}(e^{-\theta(s - u)} - e^{-\theta s})\left(\frac{\omega}{\theta} + e^{-\theta(\tau - u)}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi - \frac{1}{2}} du$$

Implicit Vol Expansion

- We write $V_{imp} = v + \xi g_1 + \xi^2 g_2 + \ldots$

- So by expanding/solving $Call[V_{imp}] = Call[v] + \xi c_1 + \xi^2 c_2 + \ldots$, we get

$$\begin{cases} g_1 = \dfrac{c_1}{c'} \\ g_2 = \dfrac{2c_2(c')^2 - c_1^2 c''}{2(c')^3} \\ g_3 = \dfrac{6c_3(c')^4 - 6c_1c_2(c')^2c'' + 3c_1^3(c'')^2 - c_1^3c'c^{(3)}}{6(c')^5} \\ g_4 = \dfrac{24c_4(c')^6 - 12c_2^2(c')^4c'' - 24c_1c_3(c')^4c'' + 36c_1^2c_2c'^2(c'')^2 - 15c_1^2(c'')^3 - 12c_1^2c_2(c')^3c^{(3)} + 10c_1^4c'c''c^{(3)} - c_1^4(c')^2c^{(4)}}{24(c')^7} \end{cases}$$

$c' \equiv C_V$

$c'' = R[2,0]c'$ for stoch vol only

- by applying it at the second order to

$$Call[V_{imp}] = BSCall[v] + (\xi T^{-1}J_1R[1,1] + \xi^2(T^{-1}J_2 + T^{-2}J_3R[2,0] + T^{-1}J_4R[1,2] + T^{-2}J_5R[2,2]))C_V ,$$

we get

$$V_{imp}[S,V,T] = V + \xi T^{-1}J_1R[1,1] + \xi^2(T^{-1}J_2 + T^{-2}J_3R[2,0] + T^{-1}J_4R[1,2] + T^{-2}J_5(R[2,2] - (R[1,1])^2R[2,0]))$$

Summary of the model Call Value (Stoch Vol)

- The process is :
$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}\,dB_t$$
$$dV_t = (\omega - \theta V)dt + \xi V_t^{\varphi} dW_t$$
with $dB_t \cdot dW_t = \rho dt$

- $$Call[S, V, T] = BSCall\left[S, K, T, r, \sqrt{\frac{\omega}{\theta} + \left(\frac{1-e^{-\theta T}}{\theta T}\right)\left(V_0 - \frac{\omega}{\theta}\right)}\right] + \xi J_1 I[1,1] + \xi^2 (J_2 + J_3 I[2,0] + J_4 I[1,2] + J_5 I[2,2])$$

- $$R[1,1] = \left[-\frac{X}{U} + \frac{1}{2}\right] \quad R[1,2] = \left[\frac{X^2}{U^2} - \left(\frac{X}{U} - \frac{1}{4U}(4-U)\right)\right] \quad R[2,1] = T\left[-\frac{1}{2}\frac{X^3}{U^3} + \frac{1}{4}\frac{X^2}{U^2} + \frac{1}{8}\frac{X}{U^2}(12+U) - \frac{1}{16}\frac{1}{U}(4+U)\right]$$
$$R[2,0] = T\left[\frac{1}{2}\frac{X^2}{U^2} - \frac{1}{2U} - \frac{1}{8}\right] \qquad R[2,2] = T\left[\frac{1}{2}\frac{X^4}{U^4} - \frac{1}{2}\frac{X^3}{U^3} - 3\frac{X^2}{U^3} + \frac{1}{8}\frac{X}{U^2}(12+U) + \frac{1}{32}\frac{1}{U^2}(48-U^2)\right]$$

$$I[n,p] = T^{1-n} K e^{-rT} R[n,p](X,U)\left(\frac{e^{\frac{-(U-2X)^2}{8U}}}{2\sqrt{2\pi U}}\right) \qquad \begin{cases} X = Log\left[\frac{S}{K}\right] + r - \delta \\ U = \frac{\omega}{\theta}T + \left(\frac{1-e^{-\theta T}}{\theta}\right)\left(V_0 - \frac{\omega}{\theta}\right) \end{cases}$$

$$J_1 = \frac{\rho}{\theta}\int_0^{\tau}(1-e^{-\theta(\tau-s)})\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi+\frac{1}{2}} ds \qquad J_4 = \left(\varphi + \frac{1}{2}\right)\frac{\rho^2}{\theta}\int_0^{\tau}\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi+\frac{1}{2}} j_6[V,\tau,s]\,ds$$
$$J_2 = 0 \qquad J_5 = \frac{1}{2}J_1^2$$
$$J_3 = \frac{1}{2\theta^2}\int_0^{\tau}(1-e^{-\theta(\tau-s)})^2\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{2\varphi} ds \qquad J_6 = \int_0^{\tau}(e^{-\theta(s-u)} - e^{-\theta s})\left(\frac{\omega}{\theta} + e^{-\theta(\tau-u)}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi-\frac{1}{2}} du$$

Summary of the model Implicit Vol (Stoch Vol)

- The process is : $$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}dB_t \qquad dV_t = (\omega-\theta V)dt + \xi V_t^{\varphi} dW_t$$ with $dB_t \cdot dW_t = \rho dt$

- $$V_{imp}[S,V,T] = \left(\frac{\omega}{\theta} + \left(\frac{1-e^{-\theta T}}{\theta T}\right)\left(V_0 - \frac{\omega}{\theta}\right)\right) + \xi J_1 I_V[1,1] + \xi^2(T^{-1}J_2 + J_3 I_V[2,0] + J_4 I_V[1,2] + J_5(I_V[2,2] - (I_V[1,1])^2 I_V[2,0]))$$

- $$R[1,1] = \left[-\frac{X}{U} + \frac{1}{2}\right] \quad R[1,2] = \left[\frac{X^2}{U^2} - \left(\frac{X}{U} - \frac{1}{4U}(4-U)\right)\right] \quad R[2,1] = T\left[-\frac{1}{2}\frac{X^3}{U^3} + \frac{1}{4}\frac{X^2}{U^2} + \frac{1}{8}\frac{X}{U^2}(12+U) - \frac{1}{16}\frac{1}{U}(4+U)\right]$$
$$R[2,0] = T\left[\frac{1}{2}\frac{X^2}{U^2} - \frac{1}{2U} - \frac{1}{8}\right] \qquad R[2,2] = T\left[\frac{1}{2}\frac{X^4}{U^4} - \frac{1}{2}\frac{X^3}{U^3} - 3\frac{X^2}{U^3} + \frac{1}{8}\frac{X}{U^2}(12+U) + \frac{1}{32}\frac{1}{U^2}(48-U^2)\right]$$

$$I_V[n,p] = (T)^{-n} R[n,p](X,U) \qquad n \le 2 \wedge p \le 2$$
$$I_V[2,2] = (T)^{-2}(R[2,2](X,U) - (R[1,1](X,U))^2 R[2,0](X,U))$$
$$\begin{cases} X = Log\left[\frac{S}{K}\right] + r - \delta \\ U = \frac{\omega}{\theta}T + \left(\frac{1-e^{-\theta T}}{\theta}\right)\left(V_0 - \frac{\omega}{\theta}\right) \end{cases}$$

$$J_1 = \frac{\rho}{\theta}\int_0^T (1-e^{-\theta(T-s)})\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi+\frac{1}{2}} ds$$
$$J_2 = 0$$
$$J_3 = \frac{1}{2\theta^2}\int_0^T (1-e^{-\theta(T-s)})^2\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{2\varphi} ds$$
$$J_4 = \left(\varphi + \frac{1}{2}\right)\frac{\rho^2}{\theta}\int_0^T \left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi+\frac{1}{2}} j_6[V,T,s] ds$$
$$J_5 = \frac{1}{2}J_1^2$$
$$J_6 = \int_0^T (e^{-\theta(s-u)} - e^{-\theta s})\left(\frac{\omega}{\theta} + e^{-\theta(T-u)}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi-\frac{1}{2}} du$$

Simple case, $V = \frac{\omega}{\theta}$

- Then

$$J_1 = \frac{\rho}{\theta}\left(\frac{\omega}{\theta}\right)^{\varphi+\frac{1}{2}}\int_0^T (1-e^{-\theta(T-s)})ds = \frac{\rho}{\theta}\left(\frac{\omega}{\theta}\right)^{\varphi+\frac{1}{2}}\left(T-\left(\frac{1-e^{-\theta T}}{\theta}\right)\right)$$

$$J_3 = \frac{1}{2\theta^2}\left(\frac{\omega}{\theta}\right)^{2\varphi}\int_0^T (1-e^{-\theta(T-s)})^2 ds = \frac{1}{2\theta^2}\left(\frac{\omega}{\theta}\right)^{2\varphi}\left(T-\frac{3+e^{-2\theta T}-4e^{-\theta T}}{2\theta}\right)$$

$$J_6 = \left(\frac{\omega}{\theta}\right)^{\varphi-\frac{1}{2}}\int_0^s (e^{-\theta(s-u)}-e^{-\theta s})du = \left(\frac{\omega}{\theta}\right)^{\varphi-\frac{1}{2}} e^{-\theta s}\frac{(-1+e^{\theta s}-\theta s)}{\theta}$$

$$J_4 = \left(\varphi+\frac{1}{2}\right)\frac{\rho^2}{\theta}\left(\frac{\omega}{\theta}\right)^{2\varphi}\int_0^T e^{-\theta s}\frac{(-1+e^{\theta s}-\theta s)}{\theta}ds = \left(\varphi+\frac{1}{2}\right)\frac{\rho^2}{\theta^2}\left(\frac{\omega}{\theta}\right)^{2\varphi}\left(T-\frac{(2-2e^{-\theta T}-\theta T e^{-\theta T})}{\theta}\right)$$

- So $Call[S, V, T] = BSCall[\sqrt{U}] + \xi\frac{J_1}{T}\left(-\frac{X}{U}+\frac{1}{2}\right)\frac{\partial}{\partial V}BSCall$

- $Call[S, \sigma, T] = BSCall[\sqrt{U}] + \rho\frac{\xi V^{\varphi+\frac{1}{2}}}{2\theta T}\left(T-\left(\frac{1-e^{-\theta T}}{\theta}\right)\right)\left(-\frac{X}{U}+\frac{1}{2}\right)\frac{\partial}{\partial V}BSCall[\sqrt{U}]$

The Jump-diffusion model

- $\frac{dS_t}{S_{t^-}} = (r - \delta - \lambda E[\gamma_0])dt + \sigma dW_t + \gamma_t dN_t$ where W_t, γ_t and N_t are independant and

  - W_t is a brownian motion

  - N_t is a poisson process with intensity λ

  - γ_t are i.i.d. random variables with value in $]-1, \infty[$

- There is a strong solution : $S_t = S_0 e^{\left(r - \delta - \lambda E[\gamma_0] - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{j=1}^{N_t} Log[1 + \gamma_{t_j}]}$

Ito formula for Ito processes with jumps

- Starting process

$$dX = \mu dt + \sigma dW + J dq$$

- Image through f

$$f = F(X, t) \qquad df = \left(\frac{\partial F}{\partial X}\mu + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2\right)dt + \frac{\partial F}{\partial X}\sigma dW + (F(X^- + J) - F(X^-))dq$$

Ito Formula (integrated)

- The Formula (One Dimension)

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0 < s \le t} \{f(X_s) - (f(X_{s-}) - f'(X_{s-})\Delta X_s)\} + \frac{1}{2}\int_0^t f''(X_{s-})\sigma_s^2 ds$$

- The Formula (N Dimensions)

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0 < s \le t} \left\{ f(X_s) - \left(f(X_{s-}) - \sum_{1 \le j \le N} D_j f(X_{s-})\Delta X^j_s \right) \right\} + \frac{1}{2}\int_0^t \sum_{\substack{1 \le i \le N \\ 1 \le j \le N}} D_{ij} f(X_{s-})\rho_{i,j}\sigma_i\sigma_j ds$$

Exemple of application

- Let assume $dS_t = S_t \mu ds + S_t \sigma dW_t + S_t (J_t - 1) dq_t$ where $dq_t = \begin{cases} 0 \text{ with probability} (1-\lambda)dt \\ 1 \text{ with probability } \lambda dt \end{cases}$

- Let apply Ito to Log[S] , this is equivalent to

$$d(Log[S_t]) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t + (Log[S_t + (J_t - 1)S_t] - Log[S_t])dq_t$$

- with that we simplify :

$$d(Log[S_t]) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t + Log[J_t]dq_t$$

Another Exemple

- Let assume $dS_t = S_t\mu dt + S_t\sigma dW_t + S_t(J_t - 1)dq_t$ where $dq_s = \begin{cases} 0 \text{ with probability}(1-\lambda)ds \\ 1 \text{ with probability } \lambda ds \end{cases}$

- Let apply Ito to f[S,t] , we have

$$df(S_t) = \left(f_x S_t\mu + f_t - \frac{1}{2}f_{xx}(S_t\sigma)^2\right)dt + f_x S_t\sigma dW_t + (f(S_{t-}J_t) - (f(S_{t-})))dq_t$$

- which is equivalent to :

$$f(S_t) = f(S_0) + \int_0^t\left(\frac{\partial f}{\partial S}S_s\mu + \frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}S_s^2\sigma^2\right)ds + \int_0^t\frac{\partial f}{\partial S}S_s\sigma dW_s + \int_0^t(f(S_{s-}J_s) - f(S_{s-}))dq_s$$

-

Warning ! The process $\int_0^t\frac{\partial f}{\partial S}S_s\sigma dW_s$ is a martingale, but $\int_0^t(f(S_{s-}J_s) - f(S_{s-}))dq_s$ is not !!

The relative jump Size γ_t

- Log Normal distribution : $Log[1+\gamma_t]$ is normal $N[\mu_J, \sigma_J]$

  - then $E[\gamma_0] = e^{\mu_J + \frac{\sigma_J^2}{2}} - 1$

- Double Exponential distribution : $Log[1+\gamma_t]$ has the following density :

$$f_{ded}(x) = p\eta_1 e^{-\eta_1 x} 1_{\{x \geq 0\}} + (1-p)\eta_2 e^{-\eta_2 x} 1_{\{x < 0\}}$$

  - then $E[\gamma_0] = p\frac{\eta_1}{\eta_1 - 1} + (1-p)\frac{\eta_2}{\eta_2 + 1} - 1$
  - the ded has the memoryless property that the log normal does not have :

$$P[x > s + t] = P[x > s]P[x > t]$$

The Pricing Equation

- A hedged porfolio has value $\Pi = V(S_t, t) - \Delta S_t$
- If we apply ito lemma (for semi-martingales) : we note $\gamma_t = J_t - 1$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS + (V(JS, t) - V(S, t) - \Delta(J-1)S)dN_t$$

- we decide to hedge the diffusion risk : $\left(\frac{\partial V}{\partial S} - \Delta\right) = 0$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(V(JS, t) - V(S, t) - (J-1)S\frac{\partial V}{\partial S}\right)dN_t$$

- we apply a minimal form of no arbitrage :$E[d\Pi] = r\Pi dt$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \lambda E[V(JS, t) - V(S, t)] - \lambda S\frac{\partial V}{\partial S}E[J-1] = 0$$

Derivation of the pricing formula

- Black and Sholes : $Log[S_T] \sim N\left[Log[F] - \frac{\sigma_F^2}{2}, \sigma_F\right]$ then $Call = e^{-rT} K \Phi[F, K, \sigma_F] \equiv e^{-rT} K f[F, K, \sigma_F^2]$

  - $Call = e^{-rT}(FN[d_1] - KN[d_2])$ where $d_1 = \frac{Log[F/K]}{\sigma_F} + \frac{\sigma_F}{2}$ and $d_1 = d_2 - \sigma_F$

- $\frac{dS}{S} = \{r - \lambda E[e^J - 1]\}dt + \sigma dW + (e^J - 1)dq(\lambda)$

  - So : $d(Log[S]) = \left\{r - \frac{\sigma^2}{2} - \lambda E[e^J - 1]\right\}dt + \sigma dW + J dq(\lambda)$

- $J \sim N[\mu_J, \sigma_J]$ => $Log\left[\frac{S_T}{S_0}\right]\Bigg| j \sim N\left[\left(r - \frac{\sigma^2}{2} - \lambda\left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right)\right)T + j\mu_J, \sqrt{\sigma^2 T + j\sigma_J^2}\right]$

  - $Log[S_T]\big| j \sim N\left[Log[S_0] + \left(r - \lambda\left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right)\right)T + j\left(\mu_J + \frac{\sigma_J^2}{2}\right) - \frac{(\sigma^2 T + j\sigma_J^2)}{2}, \sqrt{\sigma^2 T + j\sigma_J^2}\right]$

Derivation of the pricing formula (2)

- So this implies $Log[S_T] \mid j \sim N\left[Log[F_j] - \frac{\sigma_j^2}{2}, \sigma_j\right]$ where

$$F_j = Log[S_0] + \left(r - \lambda\left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right)\right)T + j\left(\mu_J + \frac{\sigma_J^2}{2}\right) \qquad \sigma_j = \sqrt{\sigma^2 T + j\sigma_J^2}$$

- $E[(S_T - K)^+ \mid j] = BS[F_j, \sigma_j, rT]$ and $E[(S_T - K)^+] = E[(S_T - K)^+ \mid j] = \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!} E[(S_T - K)^+ \mid j]$

- So $E[(S_T - K)^+] = e^{-rT} K \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!}\Phi[F_j, K, \sigma_j]$

- If we rewrite using the ordinary Black and scholes function (such that $F = Se^{rT}$) , we get an additional factor that we include in the λ ($\lambda' = \lambda e^{\mu_J + \frac{\sigma_J^2}{2}}$) by redefining it.

The Solution of the pricing equation for a lognormal jump

- $$MertonCall[S, K, t, r, \delta, \sigma, \lambda, \mu_J, \sigma_J] = \sum_{i=0}^{\infty} \frac{e^{-\lambda' t}(\lambda' t)^i}{i!} BScall[S, K, t, r_i, \delta, \sigma_i]$$ where

  - $\lambda' = \lambda e^{\mu_J + \frac{\sigma_J^2}{2}}$
  - $r_i = r - \lambda\left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right) + i\frac{\mu_J + \frac{\sigma_J^2}{2}}{t}$
  - $\sigma_i = \sqrt{\sigma^2 + i\frac{\sigma_J^2}{t}}$

- To compare with an ordinary call, we can use alternative parameters (instead of σ, σ_J) :
  - The total volatility : $\sigma_T = \sqrt{\sigma^2 + \lambda\sigma_J^2}$
  - The percentage of volatility associated with the jump : $\kappa = 1 - \frac{\sigma}{\sigma_T}$

Model with two jumps

- $\frac{dS}{S} = \{r - \lambda_1 E[e^{J_1} - 1] - \lambda_2 E[e^{J_2} - 1]\}dt + \sigma dW + (e^{J_1} - 1)dq_1(\lambda_1) + (e^{J_2} - 1)dq_2(\lambda_2)$

- $Log\left[\frac{S_T}{S_0}\right]\Big|(j_1, j_2) \sim N\left[\left(r - \frac{\sigma^2}{2} - \lambda_1\left(e^{\mu_{J,1} + \frac{\sigma_{J,1}^2}{2}} - 1\right) - \lambda_2\left(e^{\mu_{J,2} + \frac{\sigma_{J,2}^2}{2}} - 1\right)\right)T + j_1\mu_{J,1} + j_2\mu_{J,2}, \sqrt{\sigma^2 T + j_1\sigma_{J,1}^2 + j_2\sigma_{J,2}^2}\right]$

- $Log[S_T]\Big|(j_1, j_2) \sim N\left[Log[F_{j_1,j_2}] - \frac{\sigma_{j_1,j_2}^2}{2}, \sigma_{j_1,j_2}\right]$ where $\sigma_{j_1,j_2} = \sqrt{\sigma^2 T + j_1\sigma_{J,1}^2 + j_2\sigma_{J,2}^2}$ and

$$F_{j_1,j_2} = Log[S_0] + \left(r - \lambda_1\left(e^{\mu_{J,1} + \frac{\sigma_{J,1}^2}{2}} - 1\right) - \lambda_2\left(e^{\mu_{J,2} + \frac{\sigma_{J,2}^2}{2}} - 1\right)\right)T + j_1\left(\mu_{J,1} + \frac{\sigma_{J,1}^2}{2}\right) + j_2\left(\mu_{J,2} + \frac{\sigma_{J,2}^2}{2}\right)$$

- $E[(S_T - K)^+] = e^{-rT}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} e^{-(\lambda_1 T + \lambda_2 T)}\frac{(\lambda_1 T)^{j_1}(\lambda_2 T)^{j_2}}{j_1! j_2!}\Phi\left[F_{j_1,j_2}, K, \sigma_{j_1,j_2}\right]$

Derivatives of the price with lognormal jump

- $$MertonCall[S, K, t, r, \delta, \sigma, \lambda, \mu_J, \sigma_J] = \sum_{i=0}^{\infty} \frac{e^{-\lambda' t}(\lambda' t)^i}{i!} K e^{-r_i t} f\left[Log\left[\frac{S e^{-\delta t}}{K e^{-r_i t}}\right], \left(\sigma^2 + i \frac{\sigma_J^2}{t}\right) t \right]$$

FeynmanKac for a Jump Model

- X follows $X_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + J(X_t)dN_t(\lambda(t, X_t))$

- F is defined by $F(X_t) = E_t\left[e^{-\int_t^T g(X_t)ds} f(X_T)\right]$

- then

$$\frac{\partial}{\partial t}F(t, X) + DF(X_t) = g(X_t)F$$

  - where

$$DF(X_t) = \sum_i \mu_i(t, X_t)\frac{\partial}{\partial X_i}F(t, X) + \frac{1}{2}\sum_{i,j}\Lambda_{i,j}(t, X_t)\frac{\partial^2}{\partial X_i \partial X_j}F(t, X) + \sum_i \lambda_i(t, X_t)E_t[F[X_i + J_i] - F[X_i]]$$

PIDE for an option in a model with SV and Jumps

- The undelying is following : $\begin{cases} dS_t/S_t = (r-\delta-\lambda m)dt + \sqrt{V_t}dW_{s,t} + (e^{J_t}-1)dN_t(\lambda) \\ dV_t = b(V_t)dt + a(V_t)dW_{v,t} \qquad dW_{s,t}dW_{v,t} = \rho(V_t)dt \end{cases}$

- then using arbitrage arguments, the option is following :

$$\frac{\partial}{\partial t}F(t,X,V) + (r-\delta-\lambda m)S\frac{\partial F}{\partial S} + \frac{1}{2}VS^2\frac{\partial^2 F}{\partial S\partial S} + b(V)\frac{\partial F}{\partial V} + \frac{1}{2}(a(V))^2\frac{\partial^2 F}{\partial V^2} + \rho(V)a(V)\sqrt{V}S\frac{\partial^2 F}{\partial S\partial V} + \lambda E[F(e^J S) - F(S)] - rF = 0$$

- Using Parity, the put and the call satisfies :

$$call = e^{-\delta\tau}S - E_\tau[Min[K,S]] \qquad put = e^{-k\tau}K - E_\tau[Min[K,S]]$$

- And we have the transforms $\int_{z_i-\infty}^{z_i+\infty} e^{-izx}Min[K,e^x]dx = \frac{K^{1+iz}}{z^2-iz}$ for $0 < Im[z_i] < 1$

The Fundamental Transform

- By taking the fourier transform of the PIDE and $\tau = T - t$:

$$\frac{\partial}{\partial\tau}\hat{F} = (-r - ik(r-\delta) - \lambda\Lambda(k))\hat{F} - \frac{1}{2}c(k)V\hat{F} + (b(V) - ik\rho(V)a(V)\sqrt{V})\frac{\partial}{\partial V}\hat{F} + \frac{1}{2}(a(V))^2\frac{\partial^2}{\partial V^2}\hat{F}$$

- with $\hat{F}(\tau = 0) = 1$ and where $\Lambda(k) = -1 + ikm + \int_{z_i - \infty}^{z_i + \infty} e^{-ikx - ikJ} dx$

- let's define the transform H by $\hat{F}(\tau, k, V) = e^{-r\tau + d[k](r-\delta)\tau - \lambda\tau\Lambda(k)} H(\tau, k, V)$ so the equation of H is $\frac{\partial H}{\partial\tau} = -\frac{1}{2}c(k)VH + (b(V) - ik\rho(V)a(V)\sqrt{V})\frac{\partial H}{\partial V} + \frac{1}{2}(a(V))^2\frac{\partial^2 H}{\partial V^2}$, then we solve it perturbatively

Expansion of the call price for stochastic vol and jump diffusion

- By expressing the expansion:

$$Call[S,K,\tau,r,\delta,\sigma] = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-ikX-\lambda\tau\Lambda(k)}}{k^2-ik}H_0(1+\xi h_1+\xi^2 h_2+\xi^3 h_3+\ldots)dk = Se^{-\delta\tau} -$$

$$Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-ikX}}{k^2-ik}H_0\left(1+\xi\left(-\int_0^\tau cd\chi[x]\zeta[x,t]ds\right)+\xi^2\int_0^\tau\left\{\frac{1}{2}(\eta[x])^2 c\left(c(\zeta[x,t])^2-\frac{\partial\zeta}{\partial V}[x,t]\right)+d\chi[x]\left(\frac{\partial h_1}{\partial V}[x,t]-c\zeta[x,t]h_1[x,t]\right)\right\}ds+\ldots\right)dk$$

- separating the problem :

$$Call[S,K,\tau,r,\delta,\sigma] = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{-ikX-\lambda\tau\Lambda(k)}}{k^2-ik}H_0(1+\xi(-cdJ_1)+\xi^2(-cJ_2+c^2J_3-cd^2J_4+c^2d^2J_5)+\ldots)dk$$

The Jump Diffusion Derivatives

- so we need to compute :

$$I_J[n,p] = \frac{Ke^{-r\tau}}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{d[k](Log[S]+(r-\delta)\tau)-\lambda\tau\Lambda(k)}}{k^2-ik}e^{-c[k]U}c[k]^n d[k]^p dk$$

- so we create the function $G[X,U,L] = e^X - \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{e^{Xd[k]-L\Lambda(k)}}{k^2-ik}e^{-c[k]U}dk$,

- we have $I_J[n,p] = (-1)^{p+1}\frac{\partial^{(n+p)}}{\partial X^n \partial U^p}(G[X,U,L])$ for $p \geq 1$

The Jump Diffusion Derivatives for a lognormal jump

- and we observe that for a lognormal jump : $G[X, U, L; \Theta, W] = \sum_{i=0}^{\infty} \frac{e^{-L\Theta}(L\Theta)^i}{i!} f[X - (L(\Theta - 1) + iLog[\Theta]), U + iW]$

$$MertonCall[S, K, \tau, r, \delta, \sigma, \lambda, \mu_J, \sigma_J] =$$

$$Ke^{-r\tau} G\left[Log\left[\frac{S}{K}\right] + \tau(r - \delta), \sigma\sqrt{\tau}, \lambda\tau; e^{\mu_J + \frac{\sigma_J^2}{2}}, \sigma_J^2 \right] = e^{-r\tau} \sum_{j=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} \Phi[X - (L(\Theta - 1) + jLog[\Theta]), K, \sqrt{U + jW}]$$

- We therefore have

$$I_J[n, p] = Ke^{-r\tau}(-1)^p \sum_{i=0}^{\infty} \frac{e^{-L\Theta}(L\Theta)^i}{i!} \frac{\partial^{(n+p)} f}{\partial X^n \partial U^p}[X - (L(\Theta - 1) + iLog[\Theta]), U + iW]$$

- As for the BS model, we can pursue

$$I_J[n, p] = Ke^{-rT}(-\tau)^{-p} \sum_{i=0}^{\infty} \frac{e^{-L}(L)^i}{i!} \left(\frac{\partial}{\partial V}\right)^p \left(S\frac{\partial}{\partial S}\right)^n \Phi[Ke^{r - (L\Theta - 1 + iLog[\Theta])}, K, \sqrt{U + iW}]$$

- but we cannot factorize c_V and we have to keep the exponentials this time

The Series for a lognormal jump (2)

- $Call[S, V, \tau] = MertonCall\left[S, \frac{U[V,\tau]}{\tau}, \tau\right] + \xi J_1 I_J[1, 1] + \xi^2 (J_2 + J_3 I_J[2, 0] + J_4 I_J[1, 2] + J_5 I_J[2, 2])$

- where $U[V, \tau] = \int_0^{\tau} Y[V, s] ds$ and $Y[V, s]$ being the solution of the the equation $dY = b[Y] ds$

  - in the case of the parametric model : $\frac{U[V,\tau]}{\tau} = \frac{\omega}{\theta} + \left(V - \frac{\omega}{\theta}\right)\left(\frac{1 - e^{-\theta\tau}}{\theta\tau}\right)$

- with

$$I_J[p, q] = (-\tau)^p \sum_{i=0}^{\infty} \frac{e^{-L\Theta}(L\Theta)^i}{i!} D^{(p,q)}\left[S, K, r - (L\Theta - 1 + iLog[\Theta]), \delta, \tau, \sqrt{\frac{U + iW}{\tau}}\right]$$

  - with $\Theta = e^{\mu_J + \frac{1}{2}\sigma_J^2}$ $\quad W = \sigma_J^2$ $\quad L = \lambda\tau$

  and $D^{(p,q)}[S, V, \tau] = \left(\frac{\partial}{\partial V}\right)^p \left(S\frac{\partial}{\partial S}\right)^n BSCall[S, V, \tau] = R[p, q] D^{(1,0)}$

Implicit Vol Expansion

- We write $V_{imp} = v + \xi g_1 + \xi^2 g_2 + \ldots$

- So by expanding/solving $Call[V_{imp}] = Call[v] + \xi c_1 + \xi^2 c_2 + \ldots$, we get

$$\begin{cases} g_1 = \dfrac{c_1}{c'} \\ g_2 = \dfrac{c_2(c')^2 - \dfrac{c_1^2 c''}{2}}{(c')^3} = \dfrac{c_2}{c'} - \dfrac{c_1^2 c''}{2(c')^3} \end{cases} \qquad c' \equiv C_V$$

- by applying it at the second order to

$$Call[v_{imp}] = MertonCall[S, v_{imp}, \tau] + \xi J_1 I_J[1,1] + \xi^2 (J_2 + J_3 I_J[2,0] + J_4 I_J[1,2] + J_5 I_J[2,2]) \quad ,$$

we get $g_1 = J_1 I_J[1,1] / MertonCall_V[v_{imp}]$ and

$$g_2 = \frac{(J_2 + J_3 I_J[2,0] + J_4 I_J[1,2] + J_5 I_J[2,2])}{MertonCall_V[v_{imp}]} - \frac{MertonCall_{VV}[v_{imp}]}{2MertonCall_V[v_{imp}]^3}(J_1 I_J[1,1])^2$$

Case of Square root process

- The solution of the PIDE is given by :

$$\hat{F} = e^{(-r - ik(r-\delta))\tau + A(k,\tau) + B(k,\tau)V + \lambda\tau\Lambda(k)}$$

- where $A(k,\tau) = \left(-k\frac{\theta}{\varepsilon^2}\right)\left(\psi_+\tau + 2Log\left[\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right]\right)$, $B(k,\tau) = -(z^2 - iz)\frac{1 - e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}}$

- and $\psi_\pm = \mp(\kappa + ik\rho\varepsilon) + \zeta$ and $\zeta = \sqrt{(\kappa + ik\rho\varepsilon)^2 + \varepsilon^2(z^2 - iz)}$

- the Call Value is : $Call = Se^{-\delta\tau} - \frac{Ke^{-(r-\delta-\lambda m)\tau}}{2\pi}\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-ikx}\frac{H(k,V,\tau)}{k^2 - ik}dk$

The Jump Diffusion Model

- we just have $\frac{\partial H}{\partial \tau} = (-ik(r-\delta-\lambda m))H - \frac{1}{2}c(k)VH + \lambda E[H(e^{-ikJ}-1)]$

- $E[H(e^{-ikJ}-1)] = HE[(e^{-ikJ}-1)] \equiv H\Lambda(k)$ so we can integrate the preceding equation to get :

$$H = e^{-ik(r-\delta)-\frac{1}{2}(k^2-ik)V+\lambda\tau\Lambda(k))}$$

  - In the lognormal case : $\Lambda(k) = e^{-ik\mu_J - \frac{1}{2}\sigma_J^2 k^2} + ik\left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\right) - 1$

  - In the Double exponential case : $\Lambda(k) = \frac{p}{1+ik\eta_u} + \frac{(1-p)}{1-ik\eta_d} - 1 + iz\left(\frac{p}{1-\eta_u} + \frac{(1-p)}{1+\eta_d} - 1\right)$

Summary of the model Call Value (Stoch Vol+ Jumps)

- The process is :

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}dB_t + \left(e^{J_t} - 1\right)dN_t(\lambda)$$

$$dV_t = (\omega - \theta V)dt + \xi V_t^{\varphi} dW_t$$

with

$$dB_t \cdot dW_t = \rho dt$$

$$J_t \sim N[\mu_J, \sigma_J]$$

- $$Call[S, V, T] = MertonCall\left[S, K, T, r, \sqrt{\frac{\omega}{\theta} + \left(\frac{1-e^{-\theta T}}{\theta T}\right)\left(V_0 - \frac{\omega}{\theta}\right)}\right] + \xi J_1 I_J[1, 1] + \xi^2 (J_2 + J_3 I_J[2, 0] + J_4 I_J[1, 2] + J_5 I_J[2, 2])$$

- $$R[1, 1] = \left[-\frac{X}{U} + \frac{1}{2}\right] \quad R[1, 2] = \left[\frac{X^2}{U^2} - \left(\frac{X}{U} - \frac{1}{4U}(4-U)\right)\right] \quad R[2, 1] = \tau\left[-\frac{1}{2}\frac{X^3}{U^3} + \frac{1}{4}\frac{X^2}{U^2} + \frac{1}{8}\frac{X}{U^2}(12+U) - \frac{1}{16}\frac{1}{U}(4+U)\right]$$

$$R[2, 0] = \tau\left[\frac{1}{2}\frac{X^2}{U^2} - \frac{1}{2U} - \frac{1}{8}\right] \qquad R[2, 2] = \tau\left[\frac{1}{2}\frac{X^4}{U^4} - \frac{1}{2}\frac{X^3}{U^3} - 3\frac{X^2}{U^3} + \frac{1}{8}\frac{X}{U^2}(12+U) + \frac{1}{32}\frac{1}{U^2}(48-U^2)\right]$$

$$I_J[n, p] = Ke^{-rT}T^{1-p}\sum_{i=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^i}{i!}R[n, p](X_i, U_i)\left(\frac{e^{\frac{-(U_i - 2X_i)^2}{8U}}}{2\sqrt{2\pi U_i}}\right)$$

$$\begin{cases} X_i = Log\left[\frac{S}{K}\right] + r - \delta - \lambda\left(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\right) + i\left(\mu_J + \frac{\sigma_J^2}{2}\right) \\ U_i = \frac{\omega}{\theta}T + \left(\frac{1-e^{-\theta T}}{\theta}\right)\left(V_0 - \frac{\omega}{\theta}\right) + i\sigma_J^2 \end{cases}$$

$$J_1 = \frac{\rho}{\theta}\int_0^{\tau}(1-e^{-\theta(\tau-s)})\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi + \frac{1}{2}}ds$$

$$J_2 = 0$$

$$J_3 = \frac{1}{2\theta^2}\int_0^{\tau}(1-e^{-\theta(\tau-s)})^2\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{2\varphi}ds$$

$$J_4 = \left(\varphi + \frac{1}{2}\right)\frac{\rho^2}{\theta}\int_0^{\tau}\left(\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi + \frac{1}{2}} j_6[V, \tau, s]ds$$

$$J_5 = \frac{1}{2}J_1^2$$

$$J_6 = \int_0^{\tau}(e^{-\theta(s-u)} - e^{-\theta s})\left(\frac{\omega}{\theta} + e^{-\theta(\tau-u)}\left(V - \frac{\omega}{\theta}\right)\right)^{\varphi - \frac{1}{2}}du$$

Summary of the model Implicit Vol (Stoch Vol+ Jumps)

- $V_{imp} = v + \xi g_1 + \xi^2 g_2 + \ldots$ with $g_1 = J_1 I_J[1,1] / MertonCall_V[v]$ and $v = \frac{\omega}{\theta} + \left(\frac{1-e^{-\theta T}}{\theta T}\right)\left(V_0 - \frac{\omega}{\theta}\right)$

$$g_2 = \frac{(J_2 + J_3 I_J[2,0] + J_4 I_J[1,2] + J_5 I_J[2,2])}{MertonCall_V[v]} - \frac{MertonCall_{VV}[v]}{2 MertonCall_V[v]^3}(J_1 I_J[1,1])^2$$

- $\begin{cases} \Theta = e^{\mu_J + \frac{1}{2}\sigma_J^2} \\ W = \sigma_J^2 \\ L = \lambda\tau \end{cases}$ $\quad \frac{\partial}{\partial V}\Phi[F, K, \sqrt{VT}] = K\sqrt{\frac{T}{8\pi V}}\, e^{-\frac{\left(\frac{X}{\sqrt{VT}} - \frac{\sqrt{VT}}{2}\right)^2}{2}} \quad \frac{\partial^2}{\partial V^2}\Phi[F, K, \sqrt{VT}] = \frac{K}{\sqrt{512\pi T V^5}}\, e^{-\frac{\left(\frac{X}{\sqrt{VT}} - \frac{\sqrt{VT}}{2}\right)^2}{2}}(4X^2 - T^2V^2 - 4TV)$

$$MertonCall_V[S, K, \tau, r, \delta, \sigma, \lambda, \mu_J, \sigma_J] = \frac{Ke^{-rT}T}{\sqrt{8\pi}} \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!} \frac{e^{-\frac{\left(\frac{X_j}{V_j} - \frac{V_j}{2}\right)^2}{2}}}{\sqrt{V_j}}$$

$$\begin{cases} X_j = X - (L(\Theta - 1) + jLog[\Theta]) \\ V_j = \sqrt{VT + jW} \end{cases}$$

$$MertonCall_{VV}[S, K, \tau, r, \delta, \sigma, \lambda, \mu_J, \sigma_J] = \frac{Ke^{-rT}T^2}{\sqrt{512\pi}} \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!} \frac{e^{-\frac{\left(\frac{X_j}{V_j} - \frac{V_j}{2}\right)^2}{2}}}{V_j^5}(-4V_j^2 - V_j^4 + 4X_j^2)$$

Ordinary Call Pricing Via Fourier

- $z^2 - iz = k^2 + \frac{1}{4}$ when $z = \frac{i}{2} + k$

- we have

$$Call = Se^{-\delta T} - \frac{Ke^{-rT}}{\pi}\int_0^{\infty} \frac{e^{\left(-ik + \frac{1}{2}\right)X - \left(k^2 + \frac{1}{4}\right)\frac{V}{2}}}{k^2 + \frac{1}{4}} dk$$

- where $X = Log\left[\frac{S}{K}\right] + (r - \delta)T$ and $V = \sigma^2 T$

Case where $\varphi = \frac{1}{2}$

- We can Compute price via the fourier transform :

$$e^{-r(T-t)}E_T[f(x_t)] = e^{-r(T-t)}E_T\left[\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} e^{-izx}\hat{f}(z)dz\right] = e^{-r(T-t)}\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} E_T[e^{-izx}]\hat{f}(z)dz = e^{-r(T-t)}\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} \phi_T[-z]\hat{f}(z)dz$$

- where $\phi_T[z] = E[e^{izx_T}]$

- We can show that : $\phi_T[z] = e^{(x+(r-\delta)\tau)iz + A[iz,\tau] + B[iz,\tau]V + \lambda\tau\Lambda(iz)}$ where

$$A[z,\tau] = -\frac{\omega}{\xi^2}\left\{\psi_+\tau + 2Log\left[\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right]\right\}$$

$$B[z,\tau] = -(iz+z^2)\left(\frac{1-e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}}\right)$$

$$\psi_+ = -(\theta - iz\rho\xi) + \zeta \qquad \psi_- = +(\theta - iz\rho\xi) + \zeta \qquad \zeta = \sqrt{((\theta - iz\rho\xi)^2 + \xi^2(iz+z^2))}$$

Case where $\varphi = \frac{1}{2}$ (next)

- that we do at $z_i = \frac{1}{2}$

$$Call = Se^{\delta\tau} - \frac{Ke^{-r\tau}}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\left(k+\frac{i}{2}\right)\left(Log\left[\frac{S}{K}\right]+(r-\delta)\tau\right)}}{\left(k^2+\frac{1}{4}\right)}H\left[k+\frac{i}{2}\right]\right]dk$$

- $H[z] = e^{A[z] + B[z]V + \lambda\tau L[z]}$ where

$$A[z] = -\frac{\omega}{\xi^2}\left(\psi_+\tau + 2Log\left[\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right]\right) \qquad B[z] = -(iz+z^2)\left(\frac{1-e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}}\right) \qquad L[z] = e^{i\mu_J z - \frac{1}{2}\sigma_J^2 z^2} - 1 - iz\left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\right)$$

$$\zeta = \sqrt{(\theta - iz\rho\xi)^2 + \xi^2(iz+z^2)} \qquad \psi_+ = -(\theta - iz\rho\xi) + \zeta \qquad \psi_- = +(\theta - iz\rho\xi) + \zeta$$

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