Electricity Spot Simulation

by Olivier Croissant

Map of the Road Ahead

General Presentation

-Introduction to the Power Market Caracteristics
-General Steps of the Methodology
-Parameters of the simulation model
-Maximum of Likelihood Method
-Exemple of Calibration
-Exemple of Simulation
-Work still to be done

Annexes

-Ito Formula for Non Continuous Paths
-Generalization of Variance for Non Continous Paths
-Jump diffusion formula for Options
-Black formula for Mean Reverting Markets
-Reflecting Brownian Motion
-Calibration via Indirect Maximization
-Generalization to Multidimensional Markets and Multi-Jumps

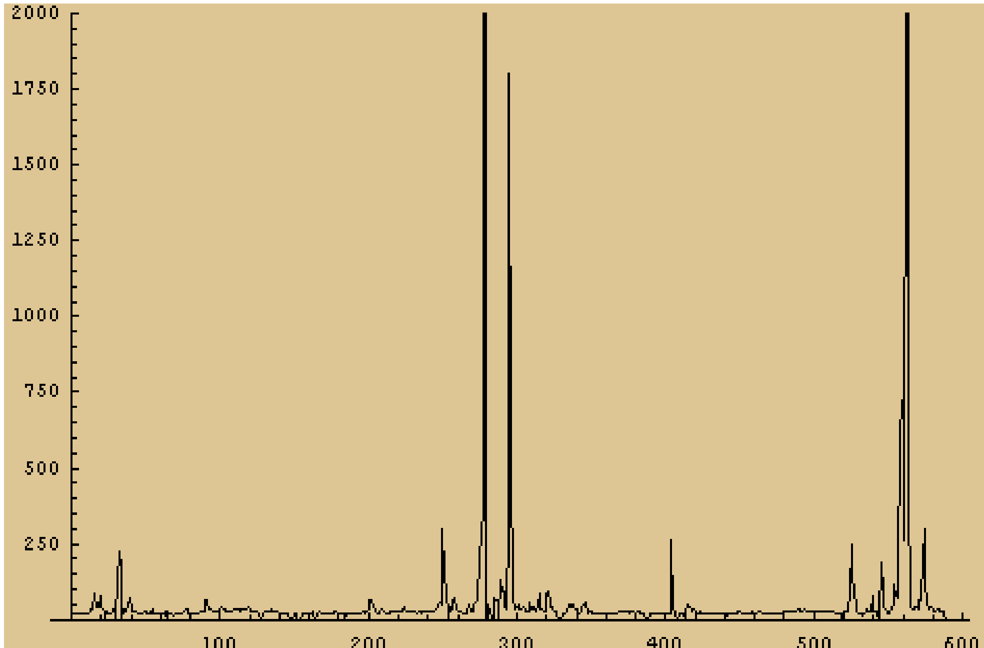
Caracteristics of the Power Markets

- The supply for electricity change slowly (Fixed capital of generation and transmission)
- The demand for electricity is relatively inelastic and will respond only slowly to a change in price pressure
- We cannot store electricity, therefore we cannot hedge even with a “convenient yield”
- Due to transmission limits there are several eletricity markets (regionalisation)
- Monthly contract are easier to price than daily or even hourly contract
- Contracts trend to be more complex than for the money markets
- Events are more frequent and economic drivers are more numerous than for the money markets

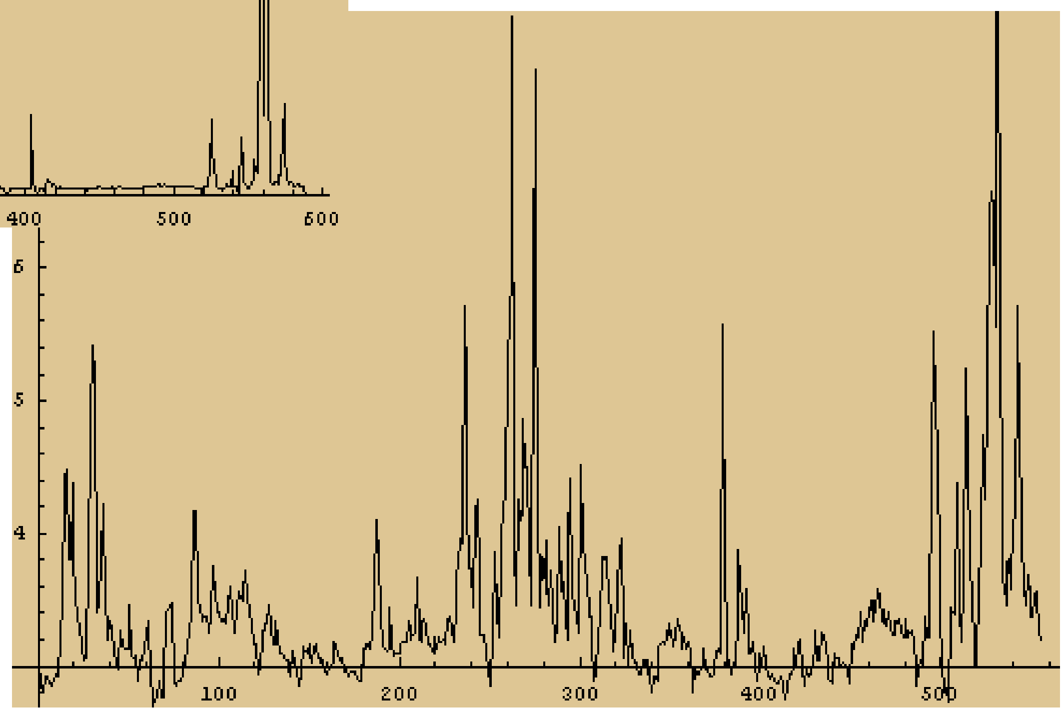
Caracteristics of the Power Market Future Prices

- Current hourly prices are strongly conditioned on the previous hour prices
- Prices have a strong tendency to mean reversion
- When prices rise, price volatility rise also (stochastic volatility) (peaking units : gas-fired)
- Seasonal patterns , weekly patterns and daily patterns
- Volatilities of a future contract decreases when maturity increases, associated with a mean reverting feature : $T \to \bar{\sigma} = \sqrt{\frac{1-e^{-2bT}}{2bT}}\sigma$ decrease like $\frac{1}{\sqrt{T}}$

Exemple of Electricity Spot Price Serie



We
Take the Log



Caracteristics of the Suggested Model

- Jump-Diffusion model $dS_t = S_t\mu dt + S_t\sigma dW_t + S_t(J_t - 1)dq_t$

- Mean reverting feature

$$d(Log[S_t]) = b(Log[S_\infty] - Log[S_t])dt + \sigma dW_t + Log[J]dq_t$$

- 2 reflecting barriers

- Dependency of the parameters on a “term structure” introducing a seasonality effect.

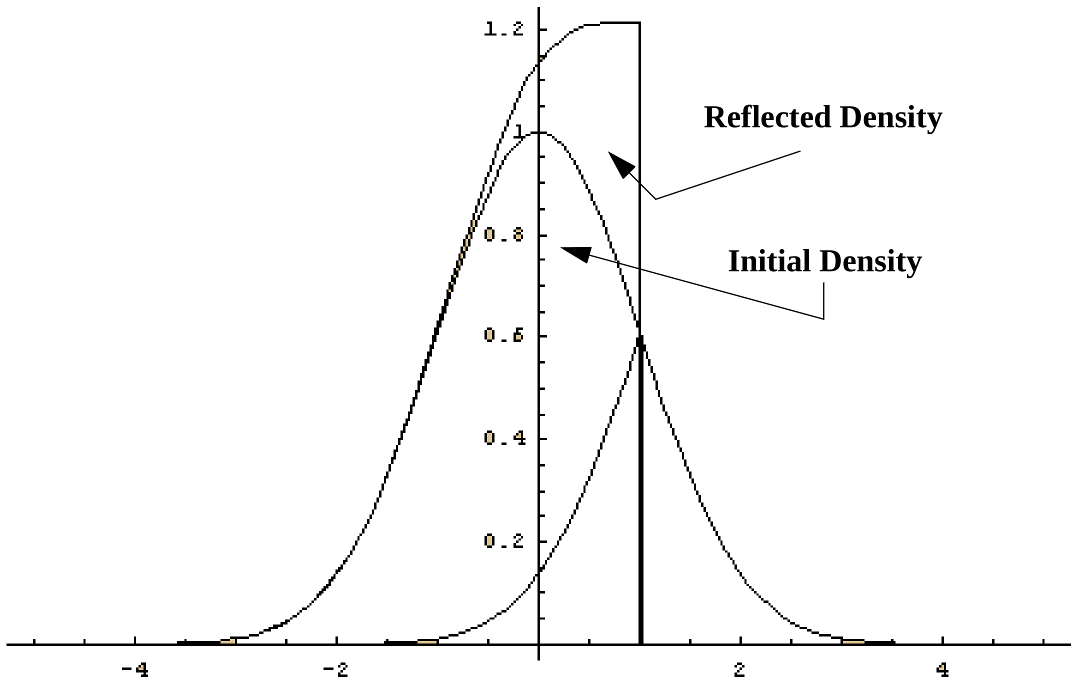
- Split of the mean reverting into two regions (in the price domain)

  - Fast reversion region above a limit-price S_{lim}

  - Slow reversion region below S_{lim}

Reflected Brownian Motion

Picture of the Density of S_T



One Barrier -> Simple Folding

Two Barriers -> Infinit Number of Folding

General Steps of the Methodology

Term Structure Extraction

Separation of the random from the predictable

Simulation and Calibration

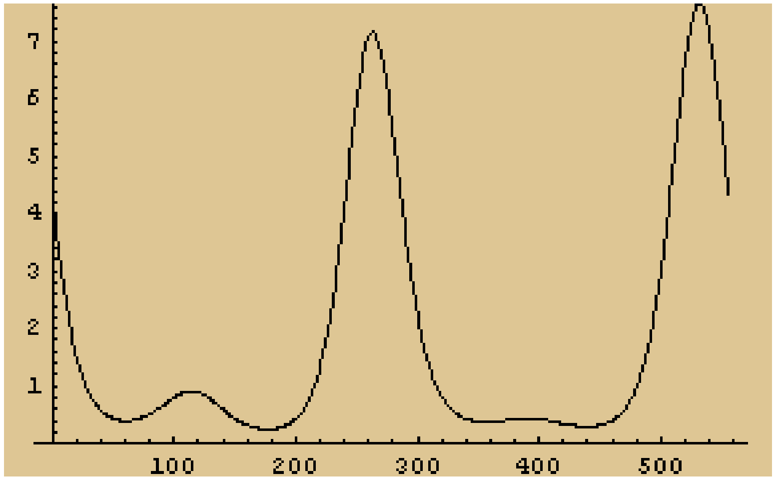
Finding a family of process able to simulate electricity prices

Automated Optimization

Finding the optimal parameters of the model

Term Structure Extraction

- Assumption 1 : Jump probability is linked to the price level
- General methodology : Fourier, truncating, inverse fourier



- Assumtion 2 : The term structure is given by this jump probability

Parameters of the Model (1)

- Brownian motion :

  - Volatility : σ and N_σ - Actual instantaneous volatility : $f(t)^{N_\sigma}\sigma$

- Normal Jumps :

  - Jump probability : λ and N_λ - Actual instantaneous jump prob: $f(t)^{N_\lambda}\lambda$

  - Jumps average size : ε and N_ε - Actual instantaneous jump size: $f(t)^{N_\varepsilon}\varepsilon$

  - Jump standard deviations : χ and N_χ - Actual instantaneous jump SD: $f(t)^{N_\chi}\chi$

Parameters of the Model(2)

- Mean reversion

  - Reverting level : S_∞ and N_{S_∞} -> instantaneous reverting level: $S_\infty(f(t))^{N_{S_\infty}}$

  - Reverting regime limit : X and N_X -> instantaneous regime limit :$X(f(t))^{N_X}$

  - Low Reversion speed : b_{up} and $N_{b_{up}}$ -> instantaneous low speed : $b_{up}(f(t))^{N_{b_{up}}}$

  - High Reversion speed : b_{up} and $N_{b_{up}}$ -> instantaneous high speed : $b_{up}(f(t))^{N_{b_{up}}}$

- Reflecting barriers

  - Up limit : k_{up}

  - Down limit : k_{down}

Maximum Likelihood Method

- In Theory
  - definition of a “probability” on the parameters: $Jdens_P(\{x_0, x_1, x_2, ..., x_n\}) = Dens(P)$
  - Most Likely P<=>Max[density]<=>{derivatives=0}<=>{derivatives[Log]=0}
  - In case of independent processes $\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}$

$$Jdens_P(\{x_0, x_1, x_2, ..., x_n\}) = \prod_{1 \le n \le T} Cdens_{P,n}(x_n | x_{n-1})$$

$$\longrightarrow \quad \frac{\partial}{\partial P} \sum_{1 \le n \le T} Log[Cdens_{P,n}(x_n | x_{n-1})] = 0$$

- In Practice a two steps process
  - simplification : $L[S_1] = L[S_2]$

$$\underbrace{\{x_0, x_1, x_2, ..., x_n\}}_{S_1} \Leftrightarrow \underbrace{\{x_0, y_1, y_2, ..., y_n\}}_{S_2} \qquad y_{n+1} = x_{n+1} - x_n - b_n(S_n - x_n)$$

  - computation of the Likelihood

$$L[S_2] = \sum_{1 \le n \le T} Log[dens(y_n | x_n)]$$

Representative Equations

- Calibration : - The no jumps density φ is defined by

$$\varphi\left[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma, s_\infty, \varepsilon, \delta^2\right]$$

$$= \frac{1}{2\sigma\sqrt{2\pi\Delta_k}} \sum_{-\infty < n < \infty} \exp\left[-\frac{(s_k + b(s_\infty - s_k)\Delta_k + \varepsilon - s_{k+1} - 2n(k_2 - k_1))^2}{2(\sigma^2\Delta_k + \delta^2)}\right] + \exp\left[-\frac{(s_k + b(s_\infty - s_k)\Delta_k + \varepsilon + s_{k+1} - 2k_1 + 2n(k_2 - k_1))^2}{2(\sigma^2\Delta_k + \delta^2)}\right]$$

- The density with jumps $\psi_{\Theta, k}$ is defined by : (under the Bernouilli simplification)

$$\psi_{\Theta, k}(t_{k+1}, s_{k+1} | t_k, s_k) =$$

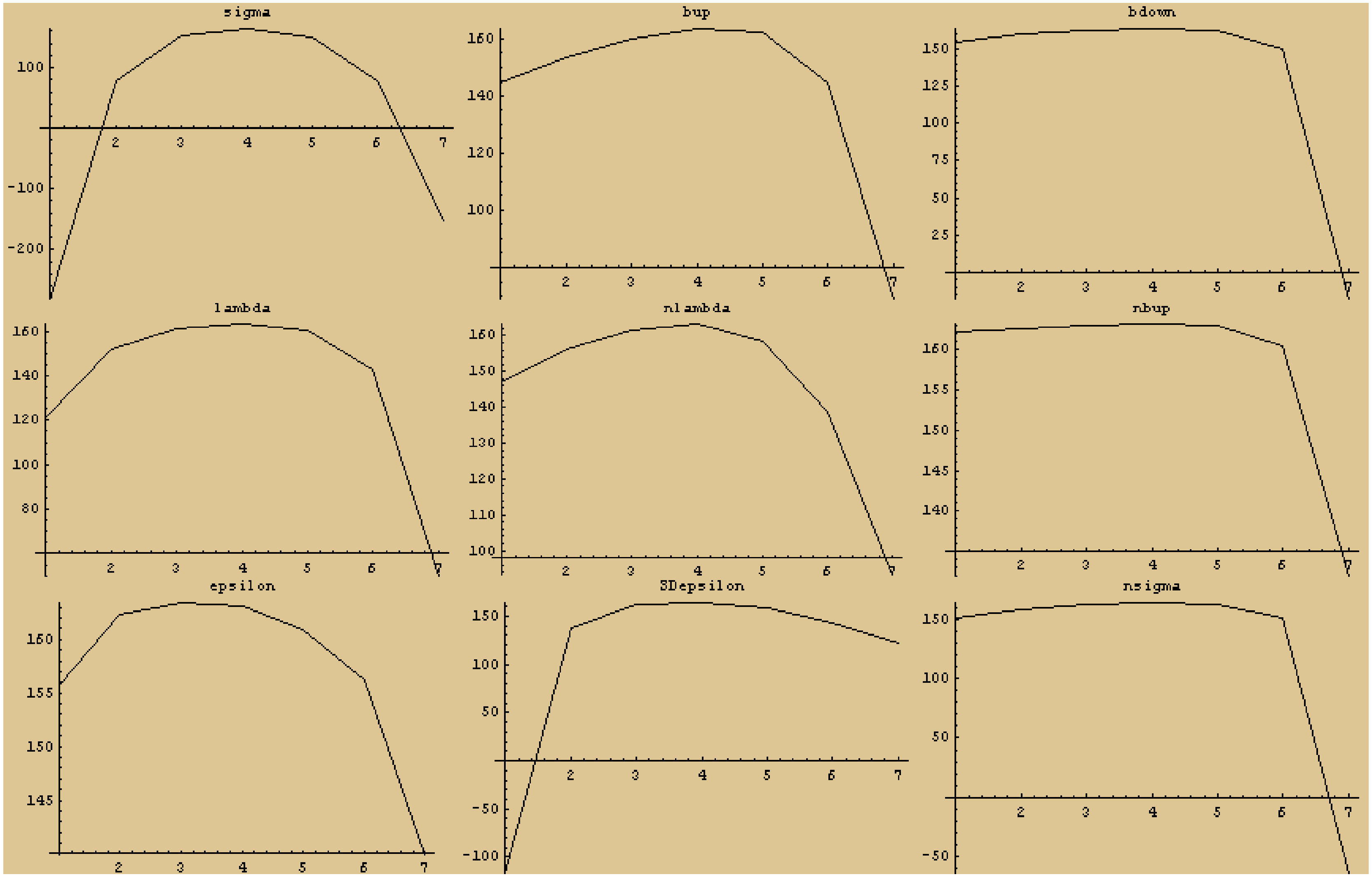
$$\left(1 - \lambda f(t_k)^{n_\lambda}\Delta_k\right)\varphi\left[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma f(t_k)^{n_\sigma}, s_\infty f(t_k)^{n_{s_\infty}}, 0, 0\right] + \lambda f(t_k)^{n_\lambda}\Delta_k \varphi\left[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma f(t_k)^{n_\sigma}, s_\infty f(t_k)^{n_{s_\infty}}, \varepsilon f(t_k)^{n_\varepsilon}, \left(\delta f(t_k)^{n_\delta}\right)^2\right]$$

- Maximization of the likelihood $\displaystyle\sum_{p=1}^{m}\sum_{k=1}^{n} Log[\psi_{\Theta, k}\left(t_k, s_{p,k} | t_{k-1}, s_{p,k-1}\right)]$ for all the parameters described by $\left\{\{f_k\}\big|_{1 \le k \le T}, \Theta\right\}$. (b handling is simplified for the presentation)

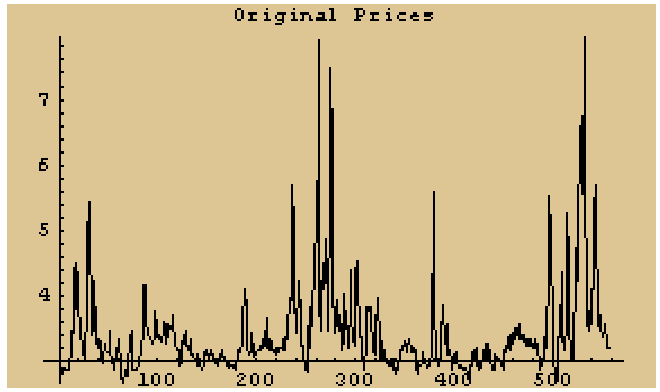
- Simulation: - If only one reflection :(if more than one reflexion the algorithm is a little bit more sophisticated) the path is computed by

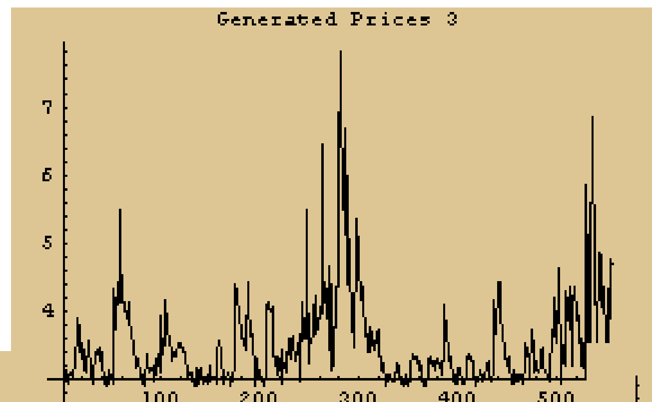
$$s_{k+1} = k_2 - \left|(\left|s_k + bf(t_k)(s_\infty - s_k)\Delta_k + \sigma_k(\beta)\sqrt{\Delta_k}a_k + c_k(\varepsilon + \delta b_k) - k_1\right| + k_1) - k_2\right|$$

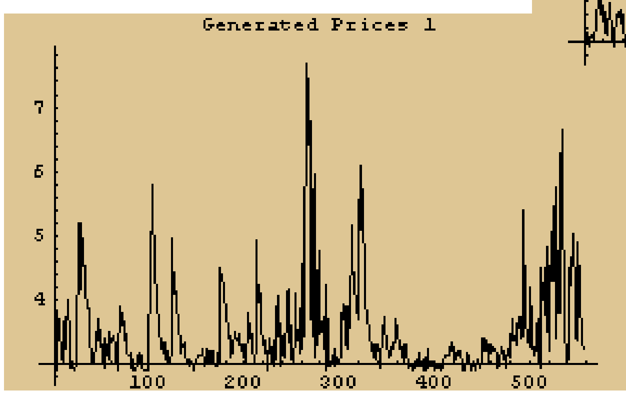
Example of likelihood Maximization

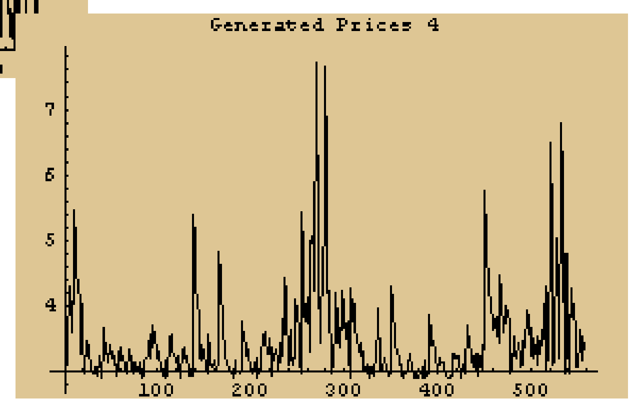


Exemple of Price Simulation









Work Still To Be Done

- Exact Methodology to extract the term structure (seasonality). But the ideas are there
- Built an efficient optimizer to do the calibration automatically
- Implement a multidimensional version. The ideas are there

Conclusion

- We can simulate electricity spot prices using a jump-diffusion model with mean-reversion and reflecting barriers

- The calibration involves a 16-dimension optimization of the likelihood, a very computer-intensive task

- More research are necessary to fill the gaps

- The technology is salable

Annexes

Finite Variation Processes

- Finite variation =difference of two increasing processes

- Finite Variation $A_t = A_t^c + A_t^d$ and $A_t^d = \sum_{0 < s \le t} \Delta A_s$

Predictable Quadratic Variation

- $\langle M \rangle$: Unique predictable (integrable) increasing process such $M^2 - \langle M \rangle \in \mathcal{M}_0$

- $\mathcal{M}_0$ Martingales such M(0)=0 (and uniformly integrable)

- Definition of the Predictable Quadratic Covariation by polarisation:

$$\langle M, N \rangle = \frac{1}{4}(\langle M + N \rangle - \langle M - N \rangle)$$

Total Quadratic Variation

- Defined by : $[M]_t = M_0^2 + \langle M^c \rangle_t + \sum_{s \le t} (\Delta M_s)^2$

- Definition of the Total Quadratic Covariation by Polarisation :

  $[M, N] = \frac{1}{4}([M+N] - [M-N])$

- property 1 : $\Delta[M, N] = \Delta M \Delta N$

- property 2 : $[M, N] \in \mathcal{M}_0 \Leftrightarrow \langle M, N \rangle = 0$

Ito Formula

- The Formula (One Dimension)

$$f(X_t) \;=\; f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0 < s \le t} \{f(X_s) - (f(X_{s-}) - f'(X_{s-})\Delta X_s)\} + \frac{1}{2}\int_0^t f''(X_{s-})d\langle (X)^c \rangle_s$$

- The Formula (N Dimensions)

$$f(X_t) \;=\; f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0 < s \le t} \left\{ f(X_s) - \left(f(X_{s-}) - \sum_{1 \le j \le N} D_j f(X_{s-})\Delta X^j{}_s \right) \right\} + \frac{1}{2}\int_0^t \sum_{\substack{1 \le i \le N \\ 1 \le j \le N}} D_{ij} f(X_{s-}) d\langle (X^i)^c, (X^j)^c \rangle_s$$

Exemple of application

- Let assume $S_t = S_0 + \int_0^t (S_s \mu ds + S_s \sigma dW_s + S_s (J_s - 1) dq_s)$ where $dq_s = \begin{cases} 0 \text{ with probability} (1-\lambda) ds \\ 1 \text{ with probability } \lambda ds \end{cases}$
  - Let apply Ito to Log[S] , this is equivalent to

$$Log[S_t] = Log[S_0] + \int_0^t \frac{1}{S_{s-}} dS_s + \sum_{0 < s \le t} \left\{ Log(S_s) - \left(Log(S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right) \right\} + \frac{1}{2} \int_0^t \left(-\frac{1}{S_{s-}^2} \right) d\langle (S)^c \rangle_s$$

  - with $d\langle (S)^c \rangle_s = \sigma^2 ds$ that we simplify :

$$Log[S_t] = Log[S_0] + \int_0^t \left\{ \left(\mu - \frac{\sigma^2}{2} \right) ds + \sigma dW_s \right\} + \sum_{0 < s \le t} \left\{ Log(S_s) - \left(Log(S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right) \right\} + \int_0^t (J_s - 1) dq_s$$

  - but $\sum_{0 < s \le t} \{ Log(S_s) - Log(S_{s-}) \} = \int_0^t \{ Log(S_{s-} J_s) - Log(S_{s-}) dq_s \}$
  - and $\sum_{0 < s \le t} \left\{ \frac{1}{S_{s-}} \Delta S_s \right\} = \int_0^t \frac{(S_{s-} J_s - S_{s-})}{S_{s-}} dq_s = \int_0^t (J_s - 1) dq_s$
  - then $Log(S_t) = Log(S_0) + \int_0^t \left\{ \left(\mu - \frac{\sigma^2}{2} \right) ds + \sigma dW_s + Log(J_s) dq_s \right\}$

Another Exemple

- Let assume $S_t = S_0 + \int_0^t (S_s \mu ds + S_s \sigma dW_s + S_s(J_s - 1) dq_s)$ where $dq_s = \begin{cases} 0 \text{ with probability} (1-\lambda) ds \\ 1 \text{ with probability } \lambda ds \end{cases}$

- Let apply Ito to f[S,t] , after simplification of $\int_0^t S_s(J_s - 1) dq_s$ we have

$$f(S_t) = f(S_0) + \int_0^t \left(f_x S_s \mu + f_t - \frac{1}{2} f_{xx} (S_s \sigma)^2 \right) ds + \int_0^t f_x S_s \sigma dW_s + \sum_{0 < s \le t} \left\{ f(S_{s-} J_s) - (f(S_{s-})) \right\}$$

- which is equivalent to :

$$f(S_t) = f(S_0) + \int_0^t \left(\frac{\partial f}{\partial S} S_s \mu + \frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} S_s^2 \sigma^2 \right) ds + \int_0^t \frac{\partial f}{\partial S} S_s \sigma dW_s + \int_0^t (f(S_{s-} J_s) - f(S_{s-})) dq_s$$

-

Warning ! The process $\int_0^t \frac{\partial f}{\partial S} S_s \sigma dW_s$ is a martingale, but $\int_0^t (f(S_{s-} J_s) - f(S_{s-})) dq_s$ is not !!

Jump Diffusion Formula For a Brownian Motion

- Let assume that the forward price is following :

$$F_{T,t} = F_{T,0} + \int_0^t (F_{T,s}\mu ds + F_{T,s}\sigma dW_s + F_{T,s}(J_s - 1)dq_s) \text{ where } dq_s = \begin{cases} 0 \text{ with probability}(1-\lambda)ds \\ 1 \text{ with probability } \lambda ds \end{cases}$$

- We also assume that the jump distribution conditional to it to appear is normal.

- We want to compute the value of an European call, maturity T, strike price K with $\varepsilon = E[Y-1]$ and δ^2 is the variance of the size of the jump conditional to this one to occur: $\delta^2 = E[(Y-1)^2 | \Delta S > 0] - \varepsilon^2$

- $Call = e^{-\lambda(1+\varepsilon)T} \sum_n \frac{(\lambda(1+\varepsilon)T)^n}{n!} \left[F_{T,0} e^{b_n T} N(d_{1,n}) - KN(d_{2,n})\right]$ where

  - $b_n = -\lambda + nLog[1+\varepsilon]$, $d_{1,n} = \dfrac{Log\left[\frac{F}{K}\right] + b_n T + \frac{1}{2}(\sigma^2 T + n\delta^2)}{\sqrt{\sigma^2 T + n\delta^2}}$ and $d_{2,n} = d_{1,n} - \sqrt{\sigma^2 T + n\delta^2}$

Jump Diffusion Model For a Brownian Explained

- Structure :

$$Call = \sum_n e^{-\lambda T}\frac{(\lambda T)^n}{n!}(e^{-\lambda\varepsilon T}(1+\varepsilon)^n)[Fe^{b_n T}N(d_{1,n}) - KN(d_{2,n})]$$

Prob(n Jumps)

$$E_N[Max\{F_T - X, 0\} \mid \text{n Jumps}]$$

- Drift coming from non zero expectation of the jumps : $b_n = nLog[1+\varepsilon]$ If no arbitrage

$$b_n = -\lambda + nLog[1+\varepsilon]$$

Drift of the risk neutral

- Volatility spread coming from the jumps : $d_{1,n} = \dfrac{Log\left[\frac{F}{K}\right] + b_n T + \frac{1}{2}(\sigma^2 T + n\delta^2)}{\sqrt{\sigma^2 T + n\delta^2}}$

Volatility spread

Jump Diffusion Model For a Brownian : Arbitrages

- Equilibrium between the spot price and the forward price :

  $$(dS_t = S_t\mu dt + S_t\sigma dW_t + S_t(J_t - 1)dq_t) \Leftrightarrow (dF_{T,t} = F_{T,t}(\mu - r_t + y_t)dt + F_{T,t}\sigma dW_t + F_{T,t}(J_t - 1)dq_t)$$

  because $F_{T,t} = S_t e^{\int_t^T (r_s - y_s)ds}$ by arbitrage

- Risk neutral equilibrium with the bond prices : $\left(\frac{dB_{T,t}}{B_{T,t}} = r_t dt\right) \Leftrightarrow \left(E_{NR}\left[\frac{dS_t}{S_t}\right] = r_t dt\right)$ implies that : $\mu + \lambda E[J_t - 1] = r_t$ or with our preceding notation : $\mu = r_t - \lambda\varepsilon$

- Therefore the risk neutral equations are :

  $$\begin{aligned} dS_t &= S_t(r_t - \lambda\varepsilon)dt + S_t\sigma dW_t + S_t(J_t - 1)dq_t \\ dF_{T,t} &= F_{T,t}(y_t - \lambda\varepsilon)dt + F_{T,t}\sigma dW_t + F_{T,t}(J_t - 1)dq_t \end{aligned}$$

Origin of the Jump Diffusion Formula and Generalisation

- If $dS_t = S_t\mu_t dt + S_t\sigma_t dW_t + S_t(J_t - 1)dq_t$ then $S_t = S_0 Exp\left[\int_0^t\left(\mu_s - \frac{\sigma_s^2}{2}\right)ds\right]Exp\left[\int_0^t \sigma_s dW_s\right]Y_t$

  where the variable Y_t follows $Y_t = Y_{n_t} \equiv \prod_{i=1}^{n_t} Y_i$ with n_t is poisson distributed with a parameter

  equal to $\lambda_Y = \int_0^t \lambda_s ds$ and Y_j is a sequence of independent variable distributed like J

- Then the option formula looks like

$$Call = \sum_n e^{-\lambda_Y}\frac{(\lambda_Y)^n}{n!}E[S_0 Y_{n_t}N(d_1) - KN(d_2)|(n_t = n)]$$

$$\boxed{\text{Assumptions : } \mu_s, \sigma_s, \lambda_s \text{ deterministic}}$$

- with $d_1 = \left(Log\left[\frac{S_0}{K}\right] + \int_0^t\left(\mu_s + \frac{\sigma_s^2}{2}\right)ds\right) \Big/ \int_0^t \frac{\sigma_s^2}{2}ds$ and $d_2 = d_1 - \int_0^t \frac{\sigma_s^2}{2}ds$

Jump Diffusion : Hedging the Option

- Let the hedged portfolio : $\Pi = V(S,t) - \Delta S$ by applying Ito we get :

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS + (V[JS] - V[S] - \Delta(J-1)S)dq$$

- If we hedge only the diffusion, $\Delta = \frac{\partial V}{\partial S}$, we can adjust $E[d\Pi] = rdt$ we get the classical jump diffusion option formula (Merton 1976)

- We can try to find Δ to minimise the variance of $d\Pi$ and then equate the expectation of $d\Pi$ to the risk free rate . We find : $\Delta = \dfrac{\lambda E[(J-1)(V(JS)-V(S))] + \sigma^2 S \frac{\partial V}{\partial S}}{\lambda S E[(J-1)^2] + \sigma^2 S}$ and We get an equation :

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}\left(\mu - \frac{\sigma^2}{d}(\mu + \lambda\varepsilon - r)\right) - rV + \lambda E\left[(V(SJ) - V(S))\left(1 - \frac{J-1}{d}(\mu + \lambda\varepsilon - r)\right)\right] = 0$$

- Intregro-differential (because of E[]) to solve with fourier methods.

Simulation of a JD Process for a Brownian Motion

- Let assume $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (J-1)dq_t$ then $S_t = S_0 e^{\int_0^t \left\{\left(\mu - \frac{\sigma^2}{2}\right)ds + \sigma dW_s + Log(J_s)dq_s\right\}}$

- Therefore $S_t \sim S_0 e^{N\left[\left(\mu - \frac{\sigma^2}{2}\right)t,\ \sigma^2 t\right] + NP[\lambda, \varepsilon, \delta^2, t]}$ where $NP[\lambda, \varepsilon, \delta^2, t]$ means the value of a poisson process of parameter λ at time t and a jump which is normal with parameters ε and δ^2

- To Simulate $NP[\lambda, \varepsilon, \delta^2, t]$ we first conditionate by the number of jumps n and simulate the conditional variable: $NP[n, \varepsilon, \delta^2] = NP[\lambda, \varepsilon, \delta^2, t]\big|n_{\lambda, t} \sim N[n\varepsilon, n\delta^2]$

- So $S_t \sim S_0 e^{N\left[\left(\mu - \frac{\sigma^2}{2}\right)t + n_{\lambda, t},\ \varepsilon,\ \sigma^2 t + n_{\lambda, t}\delta^2\right]}$ and to simulate S_t, we first simulate $n_{\lambda, t}$ then we simulate the exponential of a normal law

- The simulation of the counting process $n_{\lambda, t}$ uses the density $p(n) = e^{-\lambda(1+\varepsilon)T}\left(\frac{(\lambda(1+\varepsilon)T)^n}{n!}\right)$

Calibration (Standard)

- Let assume $\left(dS_t = \left(\alpha - \lambda\varepsilon - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t + Log[J]dq_t\right) \Leftrightarrow \left(\frac{d\left(e^{S_t}\right)}{e^{S_t}} = (\alpha - \lambda\mu_0)dt + \sigma dW_t + (J-1)dq_t\right)$ where the jumps have the distribution $Log[J] \sim N[\varepsilon, \delta^2]$

- The density : $p(x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\Phi(x, \mu + n\varepsilon, \sigma^2 + n\delta^2)$ of $x = Log\left[\frac{S_t}{S_{t-1}}\right]$ leads us to the following cumulants

$$: \begin{bmatrix} C_1 = \lambda\varepsilon \\ C_2 = \sigma^2 + \lambda(\varepsilon^2 + \delta^2) \\ C_3 = \lambda\varepsilon(\varepsilon^2 + 3\delta^2) \\ C_4 = \lambda(\varepsilon^4 + 6\varepsilon^2\nu^2 + 3\delta^4) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \left(x^4 - \frac{2C_3}{C_1}x^2 + \frac{3}{2}\frac{C_4}{C_1}x - \frac{C_3^2}{2C_1^2} = 0\right) \to \varepsilon = \text{real root x / } (\lambda > 0) \\ \lambda = \frac{C_1}{x} \\ \delta^2 = \frac{C_3 - x^2 C_1}{3C_1} \\ \sigma^2 = C_2 - \frac{C_1}{x}\left(x^2 + \frac{C_3 - x^2 C_1}{3C_1}\right) \end{bmatrix}$$

- You get the cumulants from $C_1 = M_1 \quad C_2 = M_2 - (M_1)^2$
$C_3 = M_3 - 3M_1M_2 + 2(M_1)^3 \quad C_4 = M_4 - 4M_3M_1 - 3M_2^2 + 12M_2(M_1)^2 - 6(M_1)^4$

Entropy : Reminder

- N repetitions of an experiment with K possible outcomes -> $\sum_{1 \le i \le K} N_i = N$
- The frequency : $p_i = \frac{N_i}{N}$
- The number of ways to do that : $W = \frac{N!}{N_1!N_2!N_3!....!N_{k-1}!N_k!}$
- Two independent systems : $W_{tot} = W_1 \times W_2$ -> to have an additive theory we consider:

  $-Log[W] = -NLog[N] + \sum_{i=1}^{K} N_i Log[N_i] = N\left(\sum_{i=1}^{K} p_i Log[p_i]\right)$ by using the stirling formula : $Log(x!) \sim xLog(x) - x$

- Definition : relative richness of a system = Entropy (Shannon) $H \equiv \frac{Log[W]}{N} = \sum p_i Log[p_i]$

Brownian Likelihood Maximisation with Indirect Estimation

- The Joint density : $f(t_0, s_0, t_1, s_1, t_2, s_2, ..., t_n, s_n) = f(t_0, s_0)\prod_{k=1}^{n} f(t_k, s_k | t_{k-1}, s_{k-1})$ Conditional density

- An Approximate distribution : Bernouilli law for a short term Poisson law

$$s_{k+1} = s_k + \left(\mu_k(\beta) - \lambda\varepsilon - \frac{(\sigma_k(\beta))^2}{2}\right)\Delta_k + \sigma_k(\beta)\sqrt{\Delta_k}a_k + c_k(\varepsilon + \delta b_k)$$

$$f(t_{k+1}, s_{k+1} | t_k, s_k) = (1-\lambda\Delta_k)N(s_k + \mu_k(\beta)\Delta_k, (\sigma_k(\beta))^2\Delta_k, s_{k+1}) + \lambda\Delta_k N(s_k + \mu_k(\beta)\Delta_k + \varepsilon, (\sigma_k(\beta))^2\Delta_k + \delta^2, s_{k+1})$$

- lets define $\Theta_I(\Theta)$ as the solution for the Maximization of $\sum_{p=1}^{m}\sum_{k=1}^{n} Log[f(t_k, S_{p,k}(\Theta) | t_{k-1}, S_{p,k-1}(\Theta), \Theta_I)]$

  with respect to $\Theta_I = (\lambda(\Theta), \varepsilon(\Theta), \delta(\Theta), \beta(\Theta))$ where $S_{p,k}(\Theta)$ is generated as the value at time t_k of the p-th sample generated by the true jump-diffusion process and where β can be multi dimensional (i.e.the drift and the instantaneous volatility include a term structure)

- Find Θ such it Minimizes a distance $\|\Theta - \Theta_I(\Theta)\|$ provided by $\|\Theta - \Theta_I(\Theta)\|^2 = (\Theta - \Theta_I(\Theta))^T\Omega(\Theta - \Theta_I(\Theta))$

Black and Sholes with Mean Meverting Process

- Usual Assumption : $\left(\frac{dS_t}{S_t} = (r-y)dt + \sigma dW_t\right) \Leftrightarrow \left(d(Log(S_t)) = \left(r - y - \frac{\sigma^2}{2}\right)dt + \sigma dW_t\right)$

- Mean reverting : $\left(\frac{dS_t}{S_t} = \left(b(l_\infty - Log(S_t)) + \frac{\sigma^2}{2}\right)dt + \sigma dW_t\right) \Leftrightarrow (d(Log(S_t)) = b(l_\infty - Log(S_t))dt + \sigma dW_t)$

- Let assume that S is an index , on which we can have derivatives, then

$$S_t = S_0\left(\frac{S_\infty}{S_0}\right)^{1-e^{-Bt}} e^{\sqrt{\frac{1-e^{-2Bt}}{2B}}\sigma\xi_t}$$ where ξ_t is Normal(0,1)

- In this case the Black and Sholes formula holds : $Call = e^{-rT}(F_{T,0}N(d_1) - KN(d_2))$ with ,

$$\begin{cases} d_1 = \dfrac{Log\left[\dfrac{F_T}{K}\right] + \dfrac{1}{2}(\bar{\sigma}^2)}{\bar{\sigma}\sqrt{T}} \\ d_2 = d_1 - \bar{\sigma} \end{cases}$$ and $\bar{\sigma} = \sqrt{\frac{1-e^{-2bT}}{2b}}\sigma$ with the log-expected forward being :

$$F_T = S_0\left(\frac{S_\infty}{S_0}\right)^{(1-e^{-BT})}$$

Simulation of a JD for a Mean Reverting Process

- Let assume $dS_t = b(S_\infty - S_t)dt + \sigma dW_t + Log[J]dq_t$

  let's split s into : $S = S_1 + S_2$ such that : $\begin{array}{l} dS_1 = b(S_\infty - S_1)dt \\ S_2 = -bS_2 dt + \sigma dW + Log[J]dq \end{array}$

- We solve : $S_1 = S_\infty + K\, e^{-bt}$ and if $S_2 = e^{-bt}x$ then $dS_2 = -bS_2 dt + e^{-bt}dx$ and $dx = e^{bt}(\sigma dW + Log[J]dq)$

- We can solve $x = \int_0^t e^{bt}\sigma dW_s + \int_0^t e^{bt}Log[J]dq_s$ but

- The following holds : $e^{bt}\sigma N[0, dt] \sim N[0, e^{2bt}\sigma^2 dt]$ and $e^{bt}NP[\lambda dt, \varepsilon, \delta^2] \sim NP[\lambda dt, e^{bt}\varepsilon, e^{2bt}\delta^2]$

- $\sum_k N[0, v^2{}_k] \sim N\left[0, \sum_k v^2{}_k\right]$ and by conditioning we have $NP[\lambda dt, e^{bt}\varepsilon, e^{2bt}\delta^2] \sim N[n_{\lambda,\, dt}e^{bt}\varepsilon, n_{\lambda,\, dt}e^{2bt}\delta^2]$

- Therefore $x \sim N\left[0, \frac{e^{2bt} - 1}{2b}\sigma^2\right] + \int_0^t NP[\lambda dt, e^{bt}\varepsilon, e^{2bt}\delta^2]$ and

$$S \sim N\left[S_\infty + (S_0 - S_\infty)\, e^{-bt}, \frac{1 - e^{-2bt}}{2b}\sigma^2\right] + e^{-bt}\int_0^t N[n_{\lambda,\, dt}e^{bt}\varepsilon, n_{\lambda,\, dt}e^{2bt}\delta^2]$$

Mean Reverting Likelihood Maximisation with Indirect Estimation

- The Joint density : $f(t_0, s_0, t_1, s_1, t_2, s_2, \ldots, t_n, s_n) = f(t_0, s_0)\prod_{k=1}^{n} f(t_k, s_k | t_{k-1}, s_{k-1})$ Conditional density

- An Approximate distribution : Bernouilli law for a short term Poisson law

$$s_{k+1} = s_k + b(s_\infty - s_k)\Delta_k + \sigma_k(\beta)\sqrt{\Delta_k}a_k + c_k(\varepsilon + \delta b_k)$$

$$f(t_{k+1}, s_{k+1} | t_k, s_k) = (1 - \lambda\Delta_k)N(s_k + b(s_\infty - s_k)\Delta_k, (\sigma_k(\beta))^2\Delta_k, s_{k+1}) + \lambda\Delta_k N(s_k + b(s_\infty - s_k)\Delta_k + \varepsilon, (\sigma_k(\beta))^2\Delta_k + \delta^2, s_{k+1})$$

- lets define $\Theta_I(\Theta)$ as the solution for the Maximization of $\sum_{p=1}^{m}\sum_{k=1}^{n} Log[f(t_k, S_{p,k}(\Theta) | t_{k-1}, S_{p,k-1}(\Theta), \Theta_I)]$

  with respect to $\Theta_I = (\lambda(\Theta), \varepsilon(\Theta), \delta(\Theta), \beta(\Theta))$ where $S_{p,k}(\Theta)$ is generated as the value at time t_k of the p-th sample generated by the true jump-diffusion process and where β can be multi dimensional (i.e.the mean reverting coefficient and the instantaneous volatility include a term structure)

- Find Θ such it Minimizes a distance $\|\Theta - \Theta_I(\Theta)\|$ provided by $\|\Theta - \Theta_I(\Theta)\|^2 = (\Theta - \Theta_I(\Theta))^T\Omega(\Theta - \Theta_I(\Theta))$

Multi Dimensional Processes

- Let assume we have the following processes

$$dS_{i,t} = \left(\alpha_i - \sum_k \lambda_k \varepsilon_{i,k} - \frac{1}{2}\sum_j \sigma_{i,j}^2\right)dt + \sum_j \sigma_{i,j} dW_{j,t} + \sum_k Log[J_{i,k}]dq_{k,t} \qquad \begin{cases} 1 \le i \le n \\ 1 \le j \le p \\ 1 \le k \le l \end{cases}$$

$$dQ_{i,t} = \beta_i(Q_{i,\infty} - Q_{i,t})dt + \sum_j \sigma_{i,j} dW_{j,t} + \sum_k Log[J_{i,k}]dq_{k,t} \qquad \begin{cases} 1 \le i \le m \\ 1 \le j \le p \\ 1 \le k \le l \end{cases}$$

- Independent L jumps and P brownian motions dynamise N Asset processes and M mean reverting indexes

- As the sensitivity of the process i to the brownian j is given by the instantaneous vola-tility $\sigma_{i,j}$, the sensitivity of the processes i to the jumps k is given by the couple (mean,variance) $(\varepsilon_{i,k}, \delta_{i,k}^2)$

- Every processes can be the final process or the logarithm of the final process

MultiDimensional Approximation

- The Joint density :

$$f(t_0, s_0, q_0, t_1, s_1, q_1, t_2, s_2, q_2, \ldots, t_n, s_n, q_n) = f(t_0, s_0, q_0) \prod_{k=1}^{n} f(t_k, s_k, q_k | t_{k-1}, s_{k-1}, q_{k-1})$$

Conditional density

- An Approximate distribution : Bernouilli law for a short term Poisson law

$$S_{i,t+1} - S_{i,t} = \left(\alpha_i - \sum_k \lambda_k \varepsilon_{i,k} - \frac{1}{2}\sum_j \sigma_{i,j}^2\right)\Delta_t + \sum_j \sigma_{i,j} a_{j,t} \sqrt{\Delta_t} + \sum_k b_{k,t}(\varepsilon_{i,k} + \delta_{i,k} c_{k,t})$$

$$Q_{i,t+1} - Q_{i,t} = \beta_i (Q_{i,\infty} - Q_{i,t})\Delta_t + \sum_j \sigma'_{i,j} a_{j,t} \sqrt{\Delta_t} + \sum_k b_{k,t}(\varepsilon'_{i,k} + \delta'_{i,k} c_{k,t})$$

- Every time steps we do P normal drawing for the $a_{j,t}$, L bernouilli (0 or 1) drawing for the $b_{k,t}$ and L normal drawing for the $c_{k,t}$.

- We have left the size of the jumps uncorrelated with the other brownians. We could have correlated it -> more complex formulas.

Generalized MultiDimensional Normal Density

- Let assume that the multivariate gaussian density id given by

$$\varphi(M, C, X) = \frac{1}{\sqrt{(2\pi)^n Det[C]}} e^{-\frac{1}{2}(X-M)^{\mathrm{T}} C^{-1} (X-M)}$$

- Where M is the vector mean, C is the covariance Matrix, and X is the multidimension point at which we want to compute the density. It is the density of BX+M if X is made of independent normal variables and B the “square root ot C” as a symetric matrix

- If Dim[X]=n and Y=BX with Dim[Y]=p, with p#n, we still define a generalised density, with $C = B^{\mathrm{T}} B$

- If Rank[C]<p, (Null Eigenvalue..), We can decompose the Y space in a direct sum: $Y = Y_1 + Y_2$ such that C restricted to the Y_1 is definite positive and C restricted to the Y_2 is =0. In this case the density is written as : $\varphi(M, C, \{Y_1 + Y_2\}) = \varphi(M_1, C_{11}, Y_1)\delta(B_2(Y_1 + Y_2))$ where C_{11} and B_2 are the corresponding restriction of C and B

MultiDimensional Approximated Density

- Then the density is given by:

$$f(t+1, S_{1,t+1}, \dots, S_{n,t+1}, Q_{1,t+1}, \dots, Q_{m,t+1} \,|\, t, S_{1,t}, \dots, S_{n,t}, Q_{1,t}, \dots, Q_{m,t})$$

$$= \sum_{\nu_k \in \{0,1\}} \prod_{1 \le k \le l} (1-\lambda_k \Delta_t)^{\nu_k} (\lambda_k \Delta_t)^{1-\nu_k} \varphi\left(M_{\nu_1, \nu_2, \dots, \nu_l}, C_{\nu_1, \nu_2, \dots, \nu_l}, X\right)$$

- where

$$\begin{cases} M_{\nu_1, \nu_2, \dots, \nu_l, S, i} = \left(\alpha_i - \sum_k \lambda_k \varepsilon_{i,k} - \frac{1}{2} \sum_j \sigma_{i,j}^2\right) \Delta_t + \sum_k \nu_k \varepsilon_{i,k} + S_{i,t} \\ M_{\nu_1, \nu_2, \dots, \nu_l, Q, i} = \beta_i (Q_{i,\infty} - Q_{i,t}) \Delta_t + \sum_k \nu_k \varepsilon'_{i,k} + Q_{i,t} \\ C_{\nu_1, \nu_2, \dots, \nu_l, S, W, i, j} = \sigma_{i,j} \sqrt{\Delta_t} \qquad C_{\nu_1, \nu_2, \dots, \nu_l, S, J, i, k} = \nu_k \delta_{i,k} \\ C_{\nu_1, \nu_2, \dots, \nu_l, Q, W, i, j} = \sigma'_{i,j} \sqrt{\Delta_t} \qquad C_{\nu_1, \nu_2, \dots, \nu_l, Q, J, i, k} = \nu_k \delta'_{i,k} \end{cases}$$

Reflected Brownian Motion

- density of $Ref\left|W_t\right|_0\right|$:

$$p[x,E] = \frac{1}{2\sqrt{2\pi}}\left(\int_E\left(\exp\left[-\frac{(x-y)^2}{2t}\right]dy + \exp\left[-\frac{(x+y)^2}{2t}\right]dy\right)\right)$$

- density of $Ref\left|W_t\right|_k\right|$

$$p[x,E] = \frac{1}{2\sqrt{2\pi}}\left(\int_E\left(\exp\left[-\frac{(x-y)^2}{2t}\right]dy + \exp\left[-\frac{(x+y-2k)^2}{2t}\right]dy\right)\right)$$

- density of $Ref\left|W_t\right|_{k_1}^{k_2}\right|$

$$p[x,E] = \frac{1}{2\sqrt{2\pi}}\left(\sum_{-\infty<n<\infty}\int_E\left(\exp\left[-\frac{(x-y-2n(k_2-k_1))^2}{2t}\right]dy + \exp\left[-\frac{(x+y-2k_1+2n(k_2-k_1))^2}{2t}\right]dy\right)\right)$$

Two Jumps Processes

- Two jumps : the conditional density is changed like

$$(1-\lambda_1 f_1(t_k)\Delta_k) f(t_{k+1}, s_{k+1} | t_k, s_k) =$$

$$(1-\lambda_2 f_2(t_k)\Delta_k)\left\{(1-\lambda_1 f_1(t_k)\Delta_k)\varphi[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma, s_\infty, 0, 0] + \lambda_1 f_1(t_k)\Delta_k \varphi\left[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma, s_\infty, l_1, \delta^2{}_1\right]\right\} +$$

$$\lambda_2 f_2(t_k)\Delta_k\left\{(1-\lambda_1 f_1(t_k)\Delta_k)\varphi\left[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma, s_\infty, l_2, \delta^2{}_2\right] + \lambda_1 f_1(t_k)\Delta_k \varphi\left[s_k, s_{k+1}, \Delta_k, k_1, k_2, \sigma, s_\infty, l_1 + l_2, \delta^2{}_1 + \delta^2{}_2\right]\right\}$$

- And the Set of parameters on which we do the optimization is now : $\theta = \{\lambda_1, \lambda_2, \sigma\}$

- the structure functions $f_1(t)$ and $f_2(t)$ and the parameters $\{l_1, \delta^2{}_1, l_2, \delta^2{}_2\}$ will have to be determined exogeneously by studying their statistics