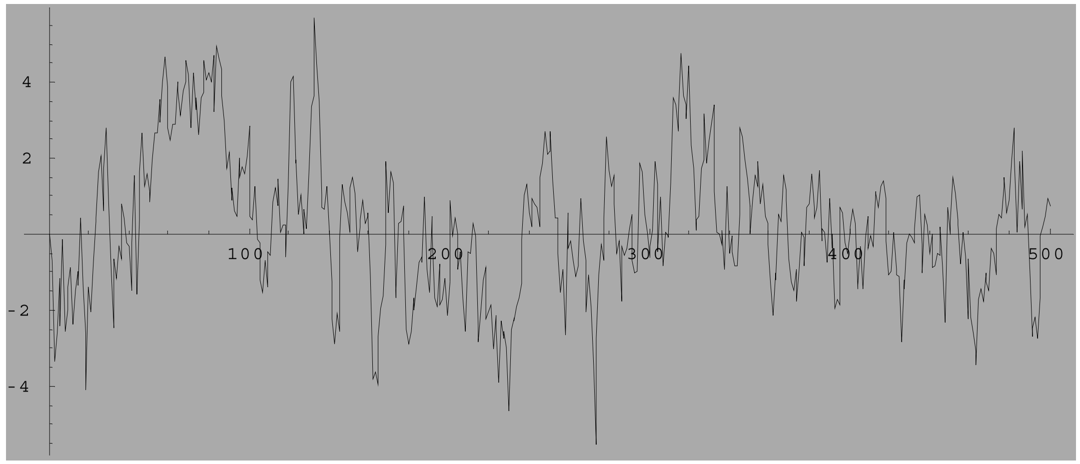
Mean Reversion Models

by Olivier Croissant

Mono Dimensional Normal Mean Reverting Motion



- Ito Equation :

$$dX_t = s(l - X_t)dt + \sigma dW_t$$

s mean reversion speed

l long term limit

σ standard deviation of the noise

Requirement for a Model

- Existence of continuous model :

  - We can discretize the model at any scale (it is not the case for a GARCH model)

- Simulation : Existence of an analytic (or quasi-analytic) formula for the computation of the finite time increment

- Calibration : Robust procedure

- Option Formula

- Existence Of a Bridge Formula

Checking the Standard Mean Reverting Model (Simulation)

- we can show that

$$X_t = l + (X_0 - l)e^{-at} + e^{-st}\left(\overline{W}_{\frac{e^{2st}-1}{2s}\sigma^2}\right)$$

- so we can simulate with any finite steps :

$$X_{n+1} - X_n = (l - X_n)\left(1 - e^{-s(t_{n+1} - t_n)}\right) + N\left[\sigma\sqrt{\frac{1 - e^{-2s(t_{n+1} - t_n)}}{2s}}, 0\right]$$

Checking the Standard Mean Reverting Model (Calibration)

- maximum likelihood formula
  - The residuals should be independant and

$$L(s, l, \sigma, n) = \frac{\left(X_{n+1} - X_n e^{-s(t_{n+1} - t_n)} - l\left(1 - e^{-s(t_{n+1} - t_n)}\right)\right)}{\sigma\sqrt{\frac{1 - e^{-2s(t_{n+1} - t_n)}}{2s}}} \sim N[1, 0]$$

  - we can solve $Max\left\{\prod_n L(s, l, \sigma, n)\right\}$
  - They are simple formula for the solution of the preceding problem if we linearize the problem, <=> small speed or small time steps
  - The difficult part is the computation of the couple (s,l). σ is computed by

$$\sigma = \sqrt{\frac{1}{N}\sum_{0 \le n \le N-1} \frac{\left(X_{n+1} - X_n e^{-s(t_{n+1} - t_n)} - l\left(1 - e^{-s(t_{n+1} - t_n)}\right)\right)^2}{\left(\frac{1 - e^{-2s(t_{n+1} - t_n)}}{2s}\right)}}$$

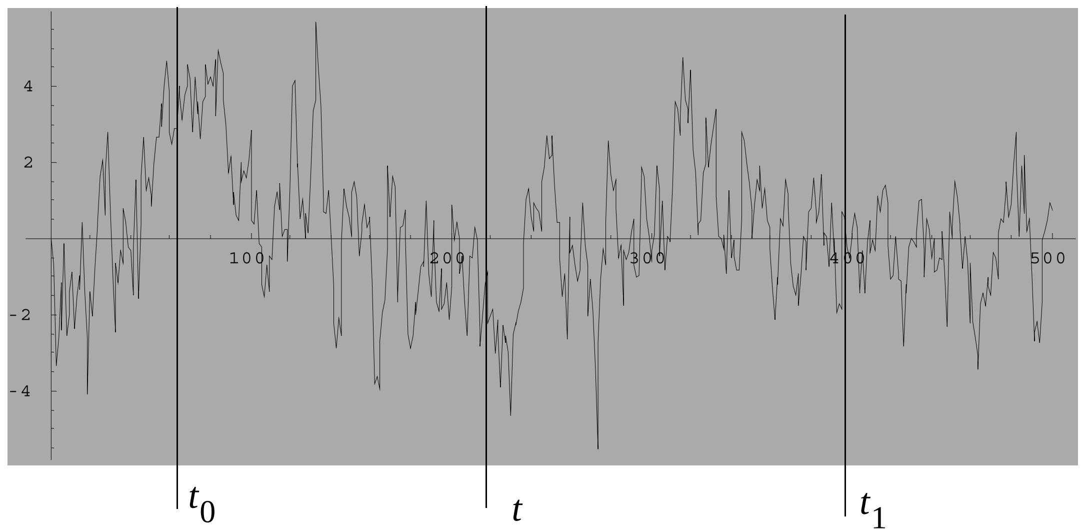
Checking the Standard Mean Reverting Model (Option)

- So the distirbution of the forward at time t is given by : $F = l + (X_0 - l)e^{-at}$
- and the variance of this forward is given by : $C_2 = \frac{1 - e^{-2st}}{2s}\sigma^2$
- So A digital Call is given by : $Call = \int_K^\infty \frac{e^{-\frac{(x-F)^2}{2C_2}}}{\sqrt{2\pi C_2}} dx = \Phi\left(\frac{F-K}{\sqrt{C_2}}\right)$
- And a regular call is given by

$$Call = \int_K^\infty \frac{(x-K)e^{-\frac{(x-F)^2}{2C_2}}}{\sqrt{2\pi C_2}} dx = \sqrt{\frac{C_2}{2\pi}} e^{-\frac{(F-K)^2}{2C_2}} + (F-K)\Phi\left(\frac{F-K}{\sqrt{C_2}}\right)$$

The Standard Mean Reverting Bridge

- Definition: Conditional on t_0 and t_1: the distribution in t is:



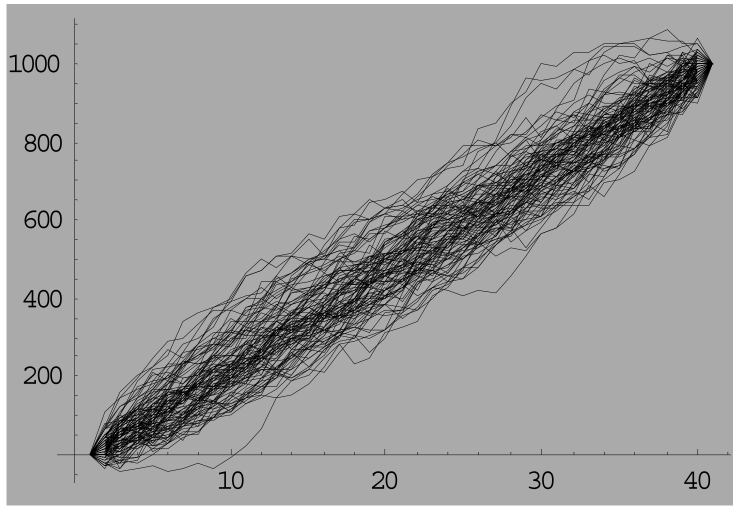
- The distribution in t is:

$$l + \left(\frac{(x-l)}{\sqrt{1+2s(t-t_0)}} + \frac{\sigma}{\sqrt{1+2s(t-t_0)}} \left(\left(\frac{\sqrt{1+2s(t_1-t_0)}(y-l)-(x-l)}{\sigma} \right) \frac{t-t_0}{t_1-t_0} + \frac{t_1-t}{t_1-t_0} W_{\frac{(t_1-t_0)(t-t_0)}{(t_1-t)}} \right) \right).$$

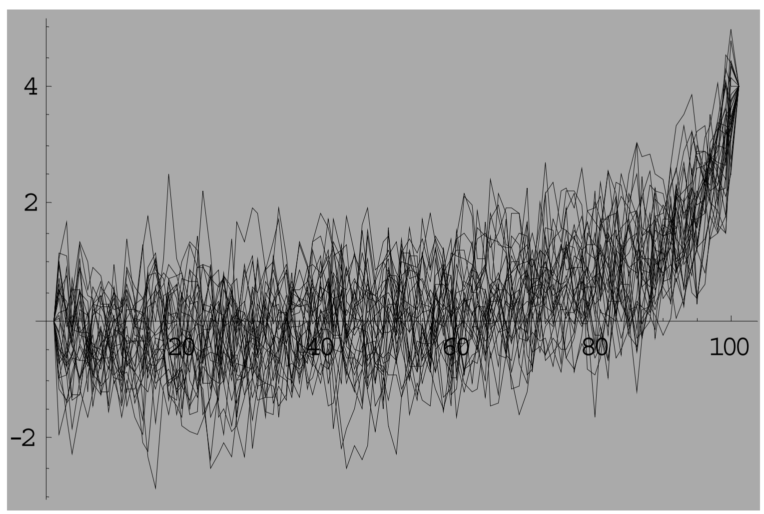
- Bridges to compute risk on path dependant portfolios, it gives a perturbative approcah

Standard Mean Reverting Bridge / Brownian Bridge

- Brownian Bridge :



- Mean Reverting Bridge :



Stationary Distribution

- Long term distribution of values (prices). May not exist. Loss of memory of the initial points

- Defined by : $X_t + dX_t \sim X_t$. But these are normal distribution , so the expectation and the variance should be constant and $\begin{pmatrix} Var[X_t + dX_t] = Var[X_t] \\ E[X_t + dX_t] = E[X_t] \end{pmatrix}$ implies that $\begin{pmatrix} Var[X_t] = \dfrac{\sigma^2}{s} \\ E[X_t] = l \end{pmatrix}$

- The Stationary distribution and the instataneous distribution can be used to calibrate the process :

$$\begin{cases} Var[X_t] = \dfrac{\sigma^2}{s} \\ E[X_t] = l \\ Var[dX_t] = \sigma^2 \end{cases}$$

Non Synchronicity of the MR process

- We could have deducted the stationary from the limit when $t \to \infty$ of the distribution

$$X_t = l + (X_0 - l)e^{-st} + e^{-st}\left(\overline{W}_{\frac{e^{2st}-1}{2s}\sigma^2}\right) \longrightarrow l + \overline{W}_{\frac{1}{2s}\sigma^2}$$

- we can put this formula as : $\boxed{X_t = l + (X_0 - l)e^{-st} + e^{-st}\int_0^t e^{sz} d(\sigma W_z)}$ showing

the time structure of the relationship between the process and the brownian motion

MultiDimensional Gaussian Mean Reverting Processes

- We observe in a time series $x_{t,j}$

  - The Instantaneous Distribution: $S_{i,j}$ and the Stationary Distribution: $L_{i,j}$

  - Example of estimators $\begin{cases} r_{i,j} = x_{i,j} - x_{i-1,j} \qquad S_{i,j} = \dfrac{1}{N-1}\left(\sum_i (r_{i,j} - \overline{r_{i,j}})(r_{i,k} - \overline{r_{i,k}})\right) \\ L_{i,j} = \dfrac{1}{N-1}\left(\sum_i (x_{i,j} - \overline{x_{i,j}})(x_{i,k} - \overline{x_{i,k}})\right) \end{cases}$

- We double diagonalize S and L to get a linear combination $y_{t,j} = \sum_k M_{j,k} x_{t,k}$ such :

  - In this new basis : $dy_j(t) = a_j(b_j - y_j(t))dt + \sigma_j dw_j(t)$

- In Summary $dx_j = \left(\alpha_j - \sum_k Q_{j,k} x_k\right)dt + \sum_k P_{j,k} dw_k$ with $P^{-1}QP$ diagonal

Definition of a (even non gaussian) Mean Reverting Process

- from $\begin{cases} X_t = l + (X_0 - l)e^{-st} + e^{-st}\int_0^t e^{sz} d(\sigma W_z) \\ \Leftrightarrow \quad dX_t = s(l - X_t)dt + \sigma dW_t \end{cases}$

- if y is a non gaussian stochastic process such that the integrals exist ,

$$\begin{cases} X_t = l + (X_0 - l)e^{-st} + e^{-st}\int_0^t e^{sz} d(y_z) \\ \Leftrightarrow \quad dX_t = s(l - X_t)dt + dy_t \end{cases}$$

- The best class on non gaussian markov processes is the levy processes

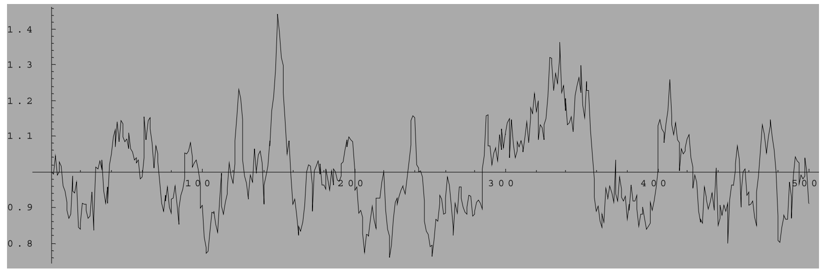
- y is called the Background driving Levy process associated with z

- z is the Ornstein-Uhlenbeck process associated with y

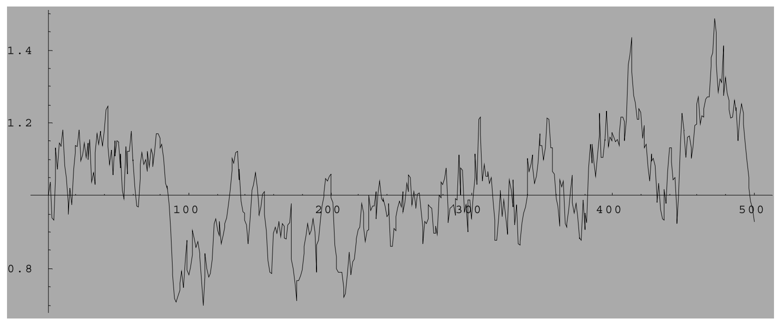
Positive Mean Reverting Processes

- $X(0) > 0$: Three practical cases :

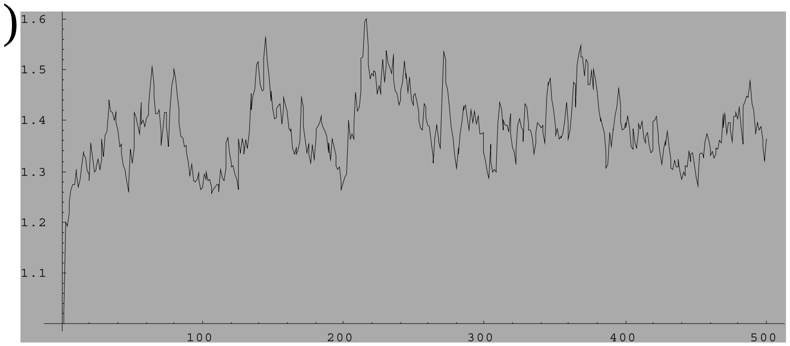
  - Lognormal noise



  - Square root noise



  - Positive Noise (here Abs[Brownian])



Lognormal Mean Reverting Processes

- $dX_t = s(l - X_t)dt + \sigma X_t dW_t$

- non explosion condition : $2s > \sigma^2$

- Stationary distribution : $\begin{cases} E[X_s] = l \\ E[X_s^2] = l^2 \dfrac{2s}{2s - \sigma^2} \end{cases}$

- The correlation between two BDLM create a covariance beetween the associated OU processes : $Cov[X_s, Y_s] = \dfrac{l_x l_y}{s_x + s_y - (Cov[dW_x, dW_y]/dt)}(Cov[dW_x, dW_y]/dt)$

Levy Process

- Continuous in probability, Cadlag, independent and stationary increment in probability

- $z(t)$ such that $z(0) = 0$ the cumulant function $C[\zeta \lozenge z(t)] \equiv Log[E[e^{i\zeta z(t)}]]$ verify :

$$\boxed{C[\zeta \lozenge z(t)] = i\mu_t\zeta - \frac{1}{2}\zeta^* C_t \zeta + \int_R \{e^{i\zeta x} - 1 - i\zeta\tau(x)\} U_t(dx)}$$

Levy-Khintchine representation

  - μ is a location parameter
  - C is a diffusion matrix
  - $U(dx)$ is the levy measure of the process (distribution of the jumps)
  - $\tau(x) \equiv 1_{|x| \le 1} x + 1_{|x| > 1} \frac{x}{|x|}$

- $\mu = C = 0$ and $Support[U] \subset R^+$ then it is a subordinator (positive increments)

- $C[\zeta \lozenge z(t)] = t(C[\zeta \lozenge z(1)])$

Self-Decomposability

- X is self decomposable <=> $X = \int_{-\infty}^{0} e^{t} dz(t)$ with $z(t)$ being a Levy process

- x_t is self decomposable => $x_t = e^{-\lambda t} x_0 + \int_0^t e^{-\lambda(t-s)} dz(\lambda s)$

  - the marginal distribution of x_t is independent of λ

- X is self decomposable <=> $\exists\{\phi_c, c \in [0, 1]\}$ such that $\phi(\zeta) = \phi(c\zeta)\phi_c(\zeta)\ \forall\zeta$

- X is self decomposable <=> X is limit $b_n^{-1}(x_1 + x_2 + .. + x_n) - a_n$ with $\{x_i\}_i$ satisfying the uniform asymptotic negligeability

- $X_{>0}$ is self decomposable <=> the BDLP is with ≥ 0 increments

The Integrated Process

- For every OU process $dX_t = \lambda(l - X_t)dt + dz(\lambda t)$ the integrated process is defeined

  by :$X^*(t) = \int_0^t X(s)ds$

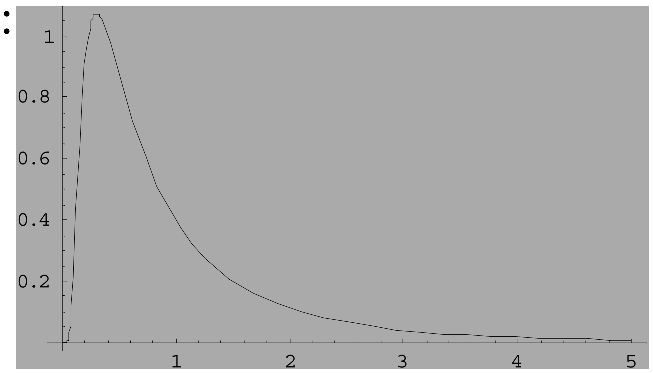
- $X^*(t) = \dfrac{Z(\lambda t) - X(t) + X(0)}{\lambda}$

- Therefore OU are great to model instaneous volatility $\sigma^2(t)$!

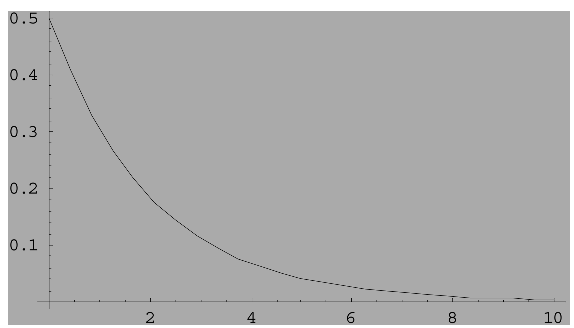
The Generalized Inverse Gaussian Distribution

- density : $f_{\lambda,\,\delta,\,\gamma}(x) \;=\; \dfrac{(\gamma/\delta)^{\lambda}}{2K_{\lambda}(\delta\gamma)} x^{\lambda-1} e^{-\frac{1}{2}\left(\frac{\delta^2}{x} + \gamma^2 x\right)}$, it is self-decomposable (1979)

- Exemple inverse gaussain $\lambda \;=\; -\frac{1}{2}$:



- Exemple Gamma $\delta \;=\; 0$



Behaviour of an increment with stochastic GIG volatility

- An increment with volatility σ is represented by a return equal to $\sigma\varepsilon$ where ε is N[0,1]

- If $\sigma \sim GIG[\lambda, \delta, \gamma]$ then $\sigma\varepsilon$ has a generalized hyperbolic distribution with den-

  sity $$\frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi}\gamma^{\lambda-\frac{1}{2}}K_{\lambda}(\delta\gamma)}\sqrt{\delta^2+x^2}^{\lambda-\frac{1}{2}}K_{\lambda-\frac{1}{2}}(\gamma\sqrt{\delta^2+x^2})$$

Simulation of OU Processes

- We assume that we know the BDLP $z(t)$

- Then $X(t) = e^{-\lambda t}\int_0^{\lambda t} e^s dz(s)$

- Result: $X(t) \sim \sum_{i=1}^{\infty} W^{-1}\left(\frac{a_i}{\lambda t}\right) e^{\lambda(r_i - t)}$ where

  - $W(x) = \int_x^{\infty} U(dx)$ and U is the levy measure of the process z

  - $a_1 < a_2 < a_3 < \ldots < a_n < ..$ arrival time of a poisson process with intensity 1

- In the case of GIG process z, W^{-1} has exponentials in it, and the formula converges very fast (2-5 terms of the series are sufficient)

The Asset Model

- Black-Scholes extended by Nelson(91) model :

$$dx(t) = \{\mu + \beta\sigma^2(t)\}dt + \sigma(t)dW(t) + \rho d\bar{z}(t)$$

where $\bar{z}(t) = z(t) - E[z(t)]$ Arbitrage freeness

levrage
(a fall in price -> increase in vol)

- If $\beta = 0$ the process is continuous, otherwise it has jumps
- If $\beta = \mu = \rho = 0$, it is a subordinated brownian process
- Aggregated returns $y_n = \int_{(n-1)\Delta}^{n\Delta} x(s)ds$ Aggragated volatility $\sigma^2{}_n = \int_{(n-1)\Delta}^{n\Delta} \sigma^2(s)ds$
- $y_n|\sigma^2{}_n \sim N(\mu\Delta + \beta\sigma^2{}_n, \sigma^2{}_n)$

Caracteristique Function of an average of the returns

- $C\left\{\zeta \lozenge \int_0^\infty f(s)dx(s)\right\} = \int_0^\infty \left(k\left[\int_0^\infty N(s,u)e^{-\lambda u}du + i\zeta\rho f(\lambda^{-1}s)\right] - i\zeta(\mu + \lambda\rho\xi)f(s)\right)ds$

  - where k is the cumulant function of the BDLP $k[\zeta] = C\{\zeta \lozenge z(1)\}$, and

  - $N(s,u) = \frac{1}{2}\zeta^2\{f^2(u)e^{-s} + f^2(\lambda^{-1}s+u)\} - i\zeta\beta\{f(u)e^{-s} + f(\lambda^{-1}s+u)\}$

  - $\xi = E[\sigma^2(t)]$

- This formula makes the link between observables and simple functional of the parameters of z

- Exemple of the IG

Kalman Filter

- Dynamic behaviour of a system : $x_{k+1} = F_k x_k + f_k + w_k$ where w_k is noise (centered gaussian)

- Observables : $y_k = H_k x_k + h_k + b_k$ where b_k is noise (centered gaussian)

- A Filter is defined by the $S_1, S_2, ..., S_n$ and the linear estimate is given by

$$\overset{\frown}{x}_n = \overline{x_n} + \sum_{i=0}^{n-1} S_i(y_i - \overline{y_i})$$

  - where the expectations are $\overline{x_{k+1}} = F_k \overline{x_k} + f_k$ and $\overline{y_k} = H_k \overline{x_k} + h_k$

- There is a best filter that minimize $Var[x_n - \overset{\frown}{x}_n]$

- it is also obtained by minimzing a likelihood function associated with b_k

Linearization of the problem

- $y_n^2 = \sigma^2_n + u_n$ observables with a noise : $u_n = \sigma^2_n(\varepsilon^2 - 1)$
- $\sigma^2_{n+1} = (1 - e^{-\lambda})\xi + e^{-\lambda}\sigma^2_n + v_n$ unobservable with noise $v_n = (1 - e^{-\lambda})\{\eta_n - \xi\}$
- $\hat{\sigma}^2_{n+1|n} = e^{-\lambda}\frac{p}{p+1}y^2_n + e^{-\lambda}\frac{1}{p+1}\hat{\sigma}^2_{n|n-1} + (1 - e^{-\lambda})\xi$ is a KF recursion
- where p is $p = Lim_{n \to \infty}\{p_{n+1|n}\}$ and $p_{n+1|n} = \frac{e^{-2\lambda}}{p_{n!n-1} + 1} + \frac{Var[v_n]}{Var[u_n]}$. p is the GARCH filter of the non conditional volatility
- The output of this KF is a likelihood that we can write $:L = -\frac{1}{2}\sum_{n=1}^{T} Log[p_{n|n-1}] - \frac{1}{2}\sum_{n=1}^{T}\frac{1}{p_{n|n-1}}\left(y_n^2 - \hat{\sigma}^2_{n|n-1}\right)$ that we minimize -> parameters