Eigen Values
and
Eigen Vectors
an introduction ....

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Plan

- Matrix of a linear operator , Eigen vector , eigen values, linear dependence, invertible matrix, determinant, characteristic polynomial, similitude, trigonalisation.
- Matrix of a quadratic operator, diagonalization of a symmetric matrix, orthogonality of the eigen vectors, diagonalization of a quadratic form, link with the linear operator case, square root of a positive definite form. Analytical calculus with matrices
- SVD decomposition, spectral decomposition
- Simultaneous diagonalization of two quadratic forms, case of two linear operators
- Application of the preceding to Multidimensional Monte Carlo
- Application of the preceding ideas to the perturbation of a definite positive quadratic form, stress testing of the correlations
- Application of the preceding ideas to variance analysis, principal component analysis and portfolio principal components

Reminder of Linear Agebra

- Matrix Multiplication

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ x + 5y \end{bmatrix}$$

- Norm of a Vector

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \|x\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

- Quadratic Form

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad {}^t xCx = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + 3y^2 + 5z^2 + 4xy - 2xz + 4yz$$

- Exemple of Quadratic Form

Covariance matrix, Correlation matrix, Gamma matrix

positive when ${}^t xCx \geq 0$ Diagonal when $C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reminder of linear Algebra (2)

- Scalar product

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad x \cdot y = 1 \times 3 + 2 \times 5$$

(projection)

- Tensorial product

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad x \otimes y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 \times 3 & 1 \times 5 \\ 2 \times 3 & 2 \times 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$$

- Trace

$$Trace \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix} = 3 + 10 = 13$$

(non commutative integration)

- Link beween the precedings :

$$Trace[x \otimes y] = x \cdot y \qquad Trace[(x \otimes y)C] = Trace[C(x \otimes y)] = {}^{t}xCy$$

Eigen Systems

- Given a matrix M , any couple (λ, x) such $Mx = \lambda x$ with λ a real number and x a non zero vector is called an eigen system with λ the eigen value and x the eigen vector.

- Any matrix of dimension n has a maximum of n eigen values (and a maxium of n inpendendent eigen vectors!). could be less (in R). If 0 is an eigen vector then the matrix is not inversible.

- A way to find all eigen values is to solve :

$$Det[M - \lambda I] = Det\begin{bmatrix} m_{11}-\lambda & m_{12} & m_{13} & .. & m_{1n} \\ m_{21} & m_{22}-\lambda & m_{23} & .. & m_{2n} \\ .. & .. & .. & .. & .. \\ m_{n1} & m_{n2} & .. & .. & m_{nn}-\lambda \end{bmatrix} = Det[M] - \lambda Trace[M] + ... + (-\lambda)^n = P(\lambda) = 0$$

- The determinant is defined by :

$$Det\begin{bmatrix} m_{11} & m_{12} & m_{13} & .. & m_{1n} \\ m_{21} & m_{22} & m_{23} & .. & m_{2n} \\ .. & .. & .. & .. & .. \\ m_{n1} & m_{n2} & .. & .. & m_{nn} \end{bmatrix} = \sum_{i_k \in \{1, 2, 3, .., n\}} (-1)^{i_1 + i_2 + .. + i_n} m_{1, i_1} m_{2, i_2} .. m_{n, i_n} = Volume\left\{\begin{bmatrix} m_{11} \\ m_{12} \\ .. \\ m_{1n} \end{bmatrix}, \begin{bmatrix} m_{21} \\ m_{22} \\ .. \\ m_{2n} \end{bmatrix}, .., \begin{bmatrix} m_{n1} \\ m_{n2} \\ .. \\ m_{nn} \end{bmatrix}\right\}$$

- $Volume[V_1, V_2, ..., V_n] = 0$ is equivalent to say that $\{V_1, V_2, ..., V_n\}$ are not independent

Diagonalization of an operator

- If possible we can have a new basis such that

$$Y = PX \qquad f(Y) = f\left(\begin{bmatrix} Y_1 \\ .. \\ Y_n \end{bmatrix}\right) = \begin{bmatrix} \lambda_1 Y_1 \\ .. \\ \lambda_n Y_n \end{bmatrix} \qquad \overline{M} = P^{-1}MP$$

  - where $\overline{M}$ is diagonal.

- It is always possible to find two unitary invertible matrices Q and R such that

$$\overline{M} = QMR$$

  - where $\overline{M}$ is diagonal. this is called the SVD (singular value decomposition).In this case the diagonal elements of $\overline{M}$ are called the singular values of M

- This can be extended to $m \times n$ matrices and be used to define pseudo inverse of matrices

Symmetric Matrices

- M is symmetric if $m_{i,j} = m_{j,i}$. It is associated with a quadratic form :

$$X_i \to f(X) = \sum_{i,j} m_{i,j} X_i X_j = {}^tXMX = Trace[M(X \otimes X)] = Trace[(X \otimes X)M]$$

- We can diagonalize quadratic forms: find an orthonormal basis where

$$Y = PX \qquad f(Y) = \sum_i \lambda_i Y_j^2 \qquad \overline{M} = {}^tPMP$$

- Theorem : we can always diagonalize real symetric matrices

- The eigen vectors are orthogonal because $\mu\langle x|y\rangle = \langle x|My\rangle = \langle xM^*|y\rangle = \langle xM|y\rangle = \lambda\langle x|y\rangle$ implies that $(\mu - \lambda)\langle x|y\rangle = 0$. So the passage matrix is unitary with ${}^tP = P^{-1}$. Therefore diagonalizing M as an operator is equivalent as diagonalizing it as a quadratic form : We

$$(\overline{M} = {}^tPMP) \Leftrightarrow (\overline{M} = P^{-1}MP)$$

can do two ways :

  - Find an Eigen orthonormal basis in which $\overline{M}$ is diagonal
  - Find a basis in which $\overline{M}$ is the identity=> $M = {}^tQQ$ this is the “square root” of M

Exemple of diagonalization

- $M = \begin{bmatrix} 3 & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$ gives us the characteristic polynomial

  : $p(\lambda) = (3-\lambda)\left(\frac{3}{2}-\lambda\right) - 1 = \frac{7}{2} - \frac{9}{2}\lambda + \lambda^2 = \left(\lambda - \frac{9}{4}\right)^2 + \frac{-81+56}{16} = \left(\lambda - \frac{9}{4}\right)^2 - \frac{25}{16}$ so the eigen values are : $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = \frac{7}{2} \end{cases}$

- The eigen vectors are given by :

  - for $\lambda_1 = 1$: we solve $(3-1)x + y = 0$ so $V_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ we normalize by : $V_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \Big/ \left(\sqrt{\frac{5}{4}}\right)$

  - for $\lambda_2 = \frac{7}{2}$: we solve $\left(3 - \frac{7}{2}\right)x + y = 0$ so $V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ we normalize by : $V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Big/ (\sqrt{5})$

- So we can write the decomposition : $M = {}^t\begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{2} \end{bmatrix} \times \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

Projectors

- Idempotent linear application : $m[m[x]] = m^2 x = mx$
- It is caracterized by the invariant space : the set of x such $mx = x$
- Given an invariant space by a free orthonormal system : $\{e_i\}_{i \in I}$ (not always generator)

  we can write : $m = \sum_{i \in I} e_i \otimes e_i$

- For every symmetric operator we have: $M = \sum_{i \in I} \lambda_i e_i \otimes e_i$: spectral decomposition
- $(e_i \otimes e_i)(e_j \otimes e_j) = (e_i \otimes e_i)\delta_{ij}$ therefore if $\lambda_i > 0$, $M = \left(\sum_{i \in I} (\sqrt{\lambda_i} e_i \otimes e_i)\right)\left(\sum_{j \in I} (\sqrt{\lambda_j} e_j \otimes e_j)\right) = {}^t QQ$

Simultaneous Diagonalization of C and Γ

- $P = \{V_1, V_2, ..., V_n\}$ is an eigen basis for C then $\begin{cases} \Delta' = P^{-1}\Delta \\ C' = P^*CP \\ \Gamma' = P^*\Gamma P \end{cases}$ because we can check :

  - and Δ behave like a vector and C and Γ behaves like a quadratic form under an orthonormal basis change

- Now $C' = \begin{bmatrix} \lambda_1 & ... & 0 \\ .. & ... & .. \\ 0 & ... & \lambda_n \end{bmatrix} = \Lambda^2$ because C is definite positive, therfore we can define a new basis in which $C" = 1$. This new basis is defined by $Q^*C'Q = I$ so $Q = \Lambda^{-1}$.

- We observe than $\Gamma"$ is still symetric because it is still second derivatives / $X"$

- Therefore $\Gamma"$ is diagonalizable and its eigen basis is orthogonal , so can be made orthonormal and therefore is diagonal / C

Application to Multidimensional Monte Carlo

- discretization of $dX_i = \sum_j M_{i,j} dW_j$ with $[dW_j, dW_i] = C_{i,j} dt$
- Change of variable $dW_j = \sum_k R_{j,k} dZ_k$ with dZ independent :$[dZ_i, dZ_j] = \delta_{ij} dt$
- this implies: $[dW_j, dW_i] = \sum_{k_1, k_2} R_{j,k_1} R_{i,k_2} dt = C_{j,i} dt$
- Using matrices : =>$C = {}^t RR$
- Simulation of X : $\Delta X_i = \sum_{j,k} M_{i,j} R_{j,k} \eta_k \Delta t$ where η_k are normal independent of variance 1

Application to Correlation perturbation

- We are given a correlation matrix $\{\rho_{i,j}\}$ => Definite Positive=> all eigen values are >0. We want to perturbe $\rho_{i,j}$ while keeping the perturbated matrix still definite positive

- Let's call A the initial correlation matrix and let call $B_{2,3} = \begin{bmatrix} 0 & 0 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 \end{bmatrix}$

- $A + tB_{2,3}$ definite positive <=> All eigen values >0 => $Det[A + tB_{2,3}] > 0$

- $f(t) = Det[A + tB_{2,3}]$ is a quadratic polynomial in t :$Det\begin{bmatrix} a_{11} & a_{12} & a_{13} & .. \\ a_{21} & a_{22} & a_{23}+t & .. \\ a_{31} & a_{32}+t & a_{33} & .. \\ .. & .. & .. & .. \end{bmatrix} = a + bt + ct^2$

- We determine a,b,c by : $\begin{cases} f(0) = a \\ f(-1) = a - b + c \\ f(1) = a + b + c \end{cases} \Leftrightarrow \begin{cases} a = f(0) \\ b = (f(1) - f(-1))/2 \\ c = (f(1) + f(-1))/2 \end{cases}$ and f(x) by SVD computation

- We find the limit correlations ρ_{min} and ρ_{max} by solving : $a + b\rho + c\rho^2 = 0$

Application to Principal Component Analysis

- The evolution of the P&L of a portfolio at very short term is given by :

$$P(t) - p(0) = \sum_i \frac{\partial P}{\partial x_i}(x_i(t) - x_i(t)) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 P}{\partial x_i \partial x_j}(x_i(t) - x_i(0))(x_j(t) - x_j(0)) + \frac{\partial P}{\partial t}t$$

- by changing of risk factor basis (-> basis that diagonalize C and Γ) where the approxi-

$$P(t) - P(0) = \sum_u P_u(t) + \frac{\partial P}{\partial t}t$$

  mations P_u are independent (at short term).

- Therefore variance and other cumulants cumul over these independent variables:

$$Variance[P(t) - P(0)] = \sum_u Variance[P_u(t)] = \sum_u Trace\left[tC(\Delta_u \otimes \Delta_u) + \frac{t^2}{2}(\Gamma_u C)^2\right]$$

- The variance analysis can be done per dimension and per order (go to 2 when 1~0). the most important u will be called portfolio principal components .