Multivariate Mean Reversion Calibration

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Let's Asume we want to model a multivariate time serie, with real factors (positve or negative, they can obtained by taking the log of an always positive time serie).

Let's also assume that the multivariate time serie is of dimension N. We can compute the short term covariance matrix S and a long term covariance matrix L .Defined by :

$$r_{i,j} = x_{i,j} - x_{i-1,j} \qquad S_{i,j} = \frac{1}{N-1}\left(\sum_i (r_{i,j} - \overline{r_{i,j}})(r_{i,k} - \overline{r_{i,k}})\right)$$

$$L_{i,j} = \frac{1}{N-1}\left(\sum_i (x_{i,j} - \overline{x_{i,j}})(x_{i,k} - \overline{x_{i,k}})\right)$$

Then because S is definite sysmetric, we can find a basis in which the short term matrix is equal to the identity. We may have to regularize S in order to keep it well conditionned.

In this Basis, L is still a symetric matrix that we diagonalize. Therefore we have a new basis in which L is diagonal and S is still diagonal because it is the scalar product. This double diagonalization works because when we apply a linear transfomation to $r_{i,j}$, it is equivalent to apply the same transformation to $x_{i,j}$.

So we have shown the existence of an invertible matrix M such that :

$$ {}^{t}MSM = \begin{bmatrix} 1 & ... & 0 \\ ... & ... & ... \\ 0 & ... & 1 \end{bmatrix} $$

and

$$ {}^{t}MLM = \begin{vmatrix} v_1 & ... & 0 \\ ... & ... & ... \\ 0 & ... & v_N \end{vmatrix} $$

In this new basis, the short term dynamic is diagonal, and the independance between the dimension is robust because the long term correlation matrix is equal to 1.So we make the assumption that the dynamic is completely diagonalized and is of the form:

$$ dy_j(t) = a_j(b_j - y_j(t))dt + \sigma_j dw_j(t) $$

where we calibrate the reversion speed as

$$ a_j = \frac{1}{2v_j} $$

we then determine the long term mean reversion . The long term mean reversion can safly be estimated by

$$ b_j = \frac{1}{\sum_i 1} \sum_i y_j(t_i) $$

and the volatility parameters will be computed as : $\sigma_j = 1$ because S isequal to the identity matrix in this basis . then by inverse reconstruction, we get the equation of the initial risk factors x_j : (we have $\vec{y} = M\vec{x}$

$$dx = M^{-1}a(b - My)dt + M^{-1}dw$$

which is equivalent to :

$$dx_j = \left(\alpha_j - \sum_k Q_{j,k}x_k\right)dt + \sum_k P_{j,k}dw_k$$

where Q and P are two matrix such that it exist a vector a and a matrix M such that

$$Q = M^{-1}D_aM \qquad P = M^{-1}$$

which is equivalent to the statement that $P^{-1}QP$ should be diagonal or that

$$[P^{-1}QP, D] = 0$$

for every diagonal matrix