From Black and Scholes to BGM

by Olivier Croissant

Messages

- A Black and Scholes formula is linked to the lognormality of the the future ratio of two assets

- Change of Numeraire => Change Of measures => Change of Drift=>derivative pricing

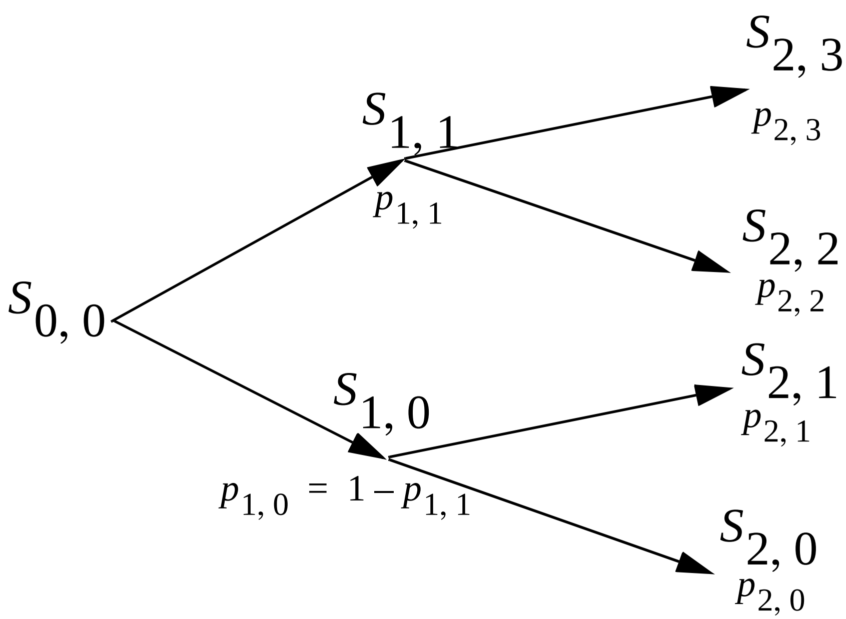
- Existence of an instantaneous interest rate is not needed for building an arbitrage free model (BGM). ====> Everything is simpler (more tractable)

- Using BGM, smiles from the cap market can be used to compute smiles for swaptions

Plan

- Revision of the Change of Measure in Ito Processes
- The Change of Numeraire Formula
- Black and Scholes as 1,2,3
- BGM
- Forward LIBOR Measure Foward swap measure
- Example of Calibration of a BGM Model
- Smiles Transfert
- Conclusion

The Change of Measure



A Process

=

A State Tree

+

Transition Probabilities

Changing of Measure

$\equiv$

Changing of probabilities

$$PV = E_P[S_T] = \sum p_{T,i} S_{T,i}$$

$$PV = E_P[S_T] = E_Q\left[\frac{dP}{dQ} S_T\right]$$

The 3 Equivalent Frameworks

Tree with Transition Probability

Change of probabilities

$$E_P[S_T] = \sum p_{T,i} S_{T,i}$$

$$\boxed{\frac{dP}{dQ}(S_{T,i}) = \frac{p_{T,i}}{Q_{T,i}}}$$

Expectations Computation

Change of Measure

$$E_P[S_T] = E_Q\left[\frac{dP}{dQ} S_T\right]$$

$$\boxed{\begin{aligned} \overline{W}_t &= W_t + \int_0^t \gamma_s ds \\ \frac{dQ}{dP}(T) &= \exp\left[-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right] \end{aligned}}$$

Ito Process

Change of Brownian

$$dS_t = \mu dt + \sigma dW_t$$

$$d\overline{S}_t = \mu dt + \sigma d\overline{W}_t$$

$$E[\overline{S}_t] = E\left[\frac{dP}{dQ} S_t\right]$$

Rule of Derivative Pricing

Set of Asset prices processes S_i **That defines the filtration** $\mathcal{F}_t$ **in the natural measure P**

We assume that we can build an arbitrage free system :

Therefore it exists a mesure Q such that all the prices are martingales

$$S_i(t) = E_Q[S_i(s) | \mathcal{F}_t] \qquad \textbf{for all s>t}$$

let be the associated change of measure process $\frac{dQ}{dP}(t) = \beta_t$ **with** $\beta_0 = 1$

we have in fact : $\beta_0 S_i(0) = E_P[\beta_t S_i(t) | \mathcal{F}_0] = E_P[\beta_s S_i(s) | \mathcal{F}_0]$

which means $\beta_t S_i(t) = E_P[\beta_s S_i(s) | \mathcal{F}_t]$

So any claim X at time T its price is :

assuming the market is complete

$$\boxed{S_i(t) = \beta_t^{-1} E_P[\beta_T X(T) | \mathcal{F}_t]}$$

Derivative Pricing Formula

Change of Numeraire

If we succeded finding a martingale measure P for a set of assets X_i
then at any t the price is $V_t = E_P[X|t]$

let be β_t **a previsible process**

Then it exists another measure, Q_β **under which** βX_i **are martingales, then at any time we have :**

$$\beta_t^{-1} V_t = E_{Q_\beta}[\beta_T^{-1} X_T | t] \qquad \textbf{with} \qquad \frac{dP}{dQ_\beta} = \beta$$

So we have $V_t = \beta_t E_{Q_\beta}[\beta_T^{-1} X_T | t] = \gamma_t E_{Q_\gamma}[\gamma_T^{-1} X_T | t] = \ldots$

The change of numeraire formula is therefore :

$$\gamma_t E_{Q_\gamma}[\gamma_T^{-1} X_T | t] = \beta_t E_{Q_\beta}[\beta_T^{-1} X_T | t]$$

$$\textbf{with} \qquad \frac{dQ_\gamma}{dQ_\beta} = \frac{\beta}{\gamma}$$

Black and Scholes as 1,2,3 (part 1)

$$Call = B_0 E_Q[B_T^{-1}(S_T - K)^+] = B_0 E_Q[B_T^{-1}(S_T - K)1_A]$$ **where** $A = \{S_T \geq K\}$

$$= B_0 E_Q[B_T^{-1} S_T 1_A] - K B_0 E_Q[B_T^{-1} 1_A]$$

change of numeraire

Numeraire=$B_t = S_t$

$B_T = S_T$

$Q = Q_Z$

assets= $\left\{1, \dfrac{P(t,T)}{S_t}\right\}$ Z_t

Numeraire=$B_t = P(t, T)$

$B_T = P(T, T) = 1$

$Q = Q_F$

assets= $\left\{1, \dfrac{S_t}{P(t,T)}\right\}$ F_t

$$B_0 E_Q[B_T^{-1} S_T 1_A] = S_0 E_{Q_Z}[1_A] = S_0 Q_Z[A]$$

$$K B_0 E_Q[B_T^{-1} 1_A] = KP(0, T) E_{Q_F}[1_A] = KP(0, T) Q_F[A]$$

Hypotheses

$F_t = \dfrac{S_t}{P(t,T)}$ **verifies** $\dfrac{dF_t}{F_t} = \sigma_t dW_t$ **with** σ_t **determinisitic**

F is a martingale under Q_F

W_t **Brownian Under the measure** Q_F

$Z_t = 1/F_t$

Ito => $\dfrac{dZ_t}{Z_t} = -\sigma_t dW_t + \sigma_t^2 dt$

Girsanov => $\dfrac{dZ_t}{Z_t} = \sigma_t dU_t$

Z is a martingale under Q_Z

U_t **Brownian Under the measure** Q_Z

Black and Scholes as 1,2,3 (part 2)

Lemma 1 When $\frac{dX_t}{X_t} = \sigma_t dW_t$ **(Ito)** Then $Log[X_t] = \int_0^t \sigma_s dW_s + Log[X_0] - \frac{1}{2}\int_0^t \sigma_s^2 ds$

Lemma 2 When $Y = \int_0^t \sigma_s dW_s + H$ Then $Y \sim Norm\left[H, \sqrt{\int_0^t \sigma_s^2 ds}\right]$

H deterministic **(Markov)**

Lemma 3 When $X \sim Norm[m, s]$ Then $Prob[X > 0] = N\left[\frac{m}{s}\right]$

Theorem

Put together : when $\frac{dX_t}{X_t} = \sigma_t dW_t$ Then $Prob[Log[X_t] > 0] = N\left[\frac{Log[X_0] - \frac{1}{2}\int_0^t \sigma_s^2 ds}{\sqrt{\int_0^t \sigma_s^2 ds}}\right]$

σ_t Deterministic

Black and Scholes as 1,2,3 (part 3)

$$Q_F[A] = Prob_{Q_F}[S_T > K] = Prob_{Q_F}[F_T > K] = Prob_{Q_F}\left[Log\left[\frac{F_T}{K}\right] > 0\right]$$

$$Q_Z[A] = Prob_{Q_Z}[S_T > K] = Prob_{Q_Z}\left[Z_T < \frac{1}{K}\right] = Prob_{Q_Z}[Log[KZ_T] < 0]$$

$\frac{d(F_t/K)}{(F_t/K)} = \sigma_t dW_t$ **implies that** $Prob\left[Log\left[\frac{F_t}{K}\right] > 0\right] \sim Norm\left[\left(Log\left[\frac{F_0}{K}\right] - \frac{1}{2}\int_0^t \sigma_s^2 ds\right) \Big/ \sqrt{\int_0^t \sigma_s^2 ds}\right]$

$\frac{d(KZ_t)}{(KZ_t)} = \sigma_t dU_t$ **implies that** $Prob[Log[KZ_t] > 0] \sim Norm\left[\left(Log[KZ_0] - \frac{1}{2}\int_0^t \sigma_s^2 ds\right) \Big/ \left(\sqrt{\int_0^t \sigma_s^2 ds}\right)\right]$

So $\begin{cases} Q_F[A] = N[d_2] \\ Q_Z[A] = N[d_1] \end{cases}$

And the value of the call is : $= B_0(F_0 N[d_1] - KN[d_2])$

(The only hypothesis is : $\frac{S_t}{P(t, T)}$ **has a deterministic volatility)**

The Classical No-Arbitrage Condition

1) Assume the existence of an instantaneous continuously compounded interest rate

1') Equivalently assume the existence of a "continuous money market bond" : $B(t) = \exp\left(\int_0^t r(s)\,ds\right)$

2) Assume the exisitence of an equivalent measure called risk neutral measure under which all relative prices P/B are martingales for any asset price P(t)

2') Equivalently assume the existence of a semi-martingale $\xi > 0$ **such that :**

(a) ξB **is a martingale**

(b) ξP **is a martingale for any asset P**

Inadequacies of the Traditional No-Arbitrage Condition

we do not need assumption 1) or 1')

-The Libor Market (Forward, Swap, Cap, Floor).
rates are simple , quaterly or semi annual

-The Swap Market (Swap, Swaption)

or much simply

-A Margrabe Framework with 2 assets B_1 **and** B_2

and an option paying $Max[0, B_1(T) - B_2(T)]$

assume that B_1 / B_2 **has a deterministic volatility**

=> we can hedge it with a long position in B_1 **and a short position in** B_2

we do not need a position on r(t) or B(t)

A more flexible No-Arbitrage condition

let $B_i, i \in I$ **a set of semi-martingales**

B_i **is a (locally) arbitrage free price system iff**

there is a positive semimartingale $\xi(t)$ **such that**

$\xi(0) = 1$ **and** ξB_i **are (local) martingales** $i \in I$

(under the original measure P)

$\xi(t)$ **is called the state price density or deflator**

$\xi(t, \omega)dP(\omega)$ **is the price a time 0 of a security whose payoff at time t is 0 except in the state** ω **where it is 1**

The HJM Models

-In the classical framework ,

$P(t, T)$ **is a bond maturing at T**

we define the forward rate as : $f(t, T) = -\frac{d}{dt}Log(P(t, T))$ $r_t \equiv f(t, t)$

we assume the existence of n independant brownian W_i

$$df(t, T) = \underset{\alpha(t, T, \omega)}{(\dots.)}dt + \sum_{i = 1, n} \sigma_i(t, T, \omega)dW_i(t, \omega)$$

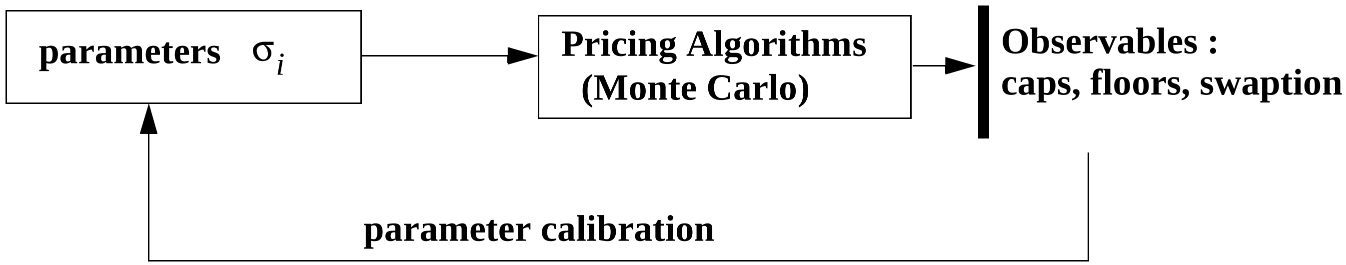
Assumptions
- **No-Arbitrage between bonds P(t,T)**
- **Numeraire = Money Market Account :** $B(t) = e^{\int_0^t r_s dS}$

In the "risk-neutral" measure : $\alpha(t) = \sum_{i = 1, n} \sigma_i(t, T)\int_t^T \sigma_i(t, s)dS$

Tradition : Measure associated with the MMA Numeraire

Problems with HJM pricing models

- Only one " real " factor models are tractable



- The Complexity of the calibration makes the parameters unstables

- There is an Industry Standard Black and Sholes formula

If we try to build a model with lognormal forwards : $\dfrac{P(T_i)}{P(T_{i-1})}$

it explodes !

The BGM Revelation

Let assume a Tenor Structure : $\{T_1, T_2, T_3, ..., T_N\}$

Let define the forward LIBOR by : $(1 + (T_i - T_{i-1})L_i(t)) = \dfrac{P(t, T_i)}{P(t, T_{i-1})}$

Does it exist an arbitrage free model in which the volatility of L_i **are all simultaneously deterministic ?**

The Answer is Yes !

(Musiela and Rutkovski 1995 for the Forward measure,Jamishidian 1997 for the spot LIBOR measure)

Forward Measures and Forward Rates

Let's consider a forward rate r_{t_1, t_2}

If we take as a numeraire the bond maturing at t_2 **:** $P(t, t_2)$ **, be Q the measure associated (called the forward measure associated with** t_2 **)**

For any security f(t) : $$\frac{f(0)}{P(0, t_2)} = E_Q\left[\frac{f(t)}{P(t, t_2)}\right]$$

It is True for $$f(t) = P(t, t_1) - P(t, t_2)$$

So we have : $$\frac{P(0, t_1) - P(0, t_2)}{P(0, t_2)} = E_Q\left[\frac{P(t, t_1) - P(t, t_2)}{P(t, t_2)}\right]$$

but we have $$\frac{P(t, t_1) - P(t, t_2)}{P(t, t_2)} \equiv 1 + (t_2 - t_1)R_{t_1, t_2}(t)$$

"forward rate"

In Conclusion $$R_{t_1, t_2}(0) = E_Q[R_{t_1, t_2}(t)]$$

The usual forward rate is the expectation of the stochastic forward rate under the forward measure Q

Swap Measures and Swap Rates

Let's define a day-coupon set : $$A_{n,N}(t) = \sum_{i=n}^{N} (t_i - t_{i-1}) P(t, t_i)$$

Let's take as a numeraire a day-coupon set $A_{n,N}(t)$

For any security f(t) : $$\frac{f(0)}{A_{n,N}(t)} = E_Q\left[\frac{f(t)}{A_{n,N}(t)}\right]$$

It is True for $f(t) = P(t, t_n) - P(t, t_{N+1})$

So we have : $$\frac{P(0, t_n) - P(0, t_{N+1})}{A_{n,N}(0)} = E_Q\left[\frac{P(t, t_n) - P(t, t_{N+1})}{A_{n,N}(t)}\right]$$

but we have $$\frac{P(t, t_n) - P(t, t_{N+1})}{A_{n,N}(t)} \equiv S_{n,N}(t)$$ **"forward swap rate"**

In Conclusion $$S_{n,N}(0) = E_Q[S_{n,N}(t)]$$

The usual forward swap rate is the expectation of the stochastic forward swap rate under the forward swap measure Q

Spot Libor Measure

We want to define an object equivalent to the money market account : $B(t) = e^{\int_0^t r_s dS}$

~1

$$B^*(t) = \frac{B_{m(t)}(t)}{B_{T_1}(0)} \prod_{j = 1,\, m(t)-1} (1 + (T_{j+1} - T_j)L_j(T_j))$$

First period Interest rate **Interest rate compounding up to Tm(t)**

So if $T_{m(t)-1} \le t \le T_{m(t)}$ **Then** $B^*(t) = P(t, T_{m(t)})$

therefore $$\frac{f(t_{m(t)-1})}{P(t_{m(t)-1}, t_{m(t)})} = E_Q\left[\frac{f(t_{m(t)})}{P(t_{m(t)}, t_{m(t)})}\right]$$

This implies that $$f(t_{m(t)-1}) = P(t_{m(t)-1}, t_{m(t)})E_Q[f(t_{m(t)})]$$

In the spot libor measure, we discount expected values one accrual period at a time

Arbitrage-free relationships in a BGM Model

Let assume any volatility vectors : $\lambda_i(t, \omega) \in R^k$

There is a unique set of processes $L_i(t)$ **Solution of the no-arbitrage condition :**

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=m(t)}^{n-1} \underbrace{\frac{\delta_j L_j(t)[\lambda_i(t) \cdot \lambda_j(t)]}{1 + \delta_j L_j(t)}}_{\alpha} dt + \lambda_i(t) dW(t)$$

1) Even if $\lambda_i(t)$ **is not stochastic, the drift** α **of the libor rates is stochastic**

2) When $T_{j+1} - T_j \to 0$ **the drift tends toward** $\sigma_i(t, T)\int_t^T \sigma_i(t, s) ds$

where $\sigma_i(t, T) = \int_t^T \lambda_i(s, \omega) ds$

BGM with the spot libor measure -> HJM neutral risk measure

Simple Example of Calibration of BGM (1)

let assume n forwards rates defined by : $(1+\delta_i L_i(t)) = \dfrac{P(t,T_i)}{P(t,T_{i-1})}$

he SDE for the forwards is is therefore $\dfrac{dL_i(t)}{L_i(t)} = \sum_{j=m(t)}^{n-1} \dfrac{\delta_j L_j(t)[\lambda_i \cdot \lambda_j]}{1+\delta_j L_j} dt + [\lambda_i \cdot dW(t)]$

For every caplet i , changing for the forward measure -> $\dfrac{dL_i(t)}{L_i(t)} = [\lambda_i \cdot \overline{dW_i(t)}]$

Pricing of a caplet -> Black and Scholes ($\|\lambda_i\|$)

For every couple of forwards (1,2)

$$S_{1,2}(t) = \frac{1 - \dfrac{1}{1+\delta L_1}}{\dfrac{\delta}{(1+\delta L_1)(1+\delta L_2)} + \dfrac{\delta}{1+\delta L_1}} = \frac{(1+\delta L_1)(1+\delta L_2) - 1}{2\delta + \delta^2 L_2} \approx \frac{L_1 + L_2}{2}$$

applying Ito gives : $\dfrac{dS_{1,2}(t)}{S_{1,2}(t)} = (\ldots)dt + \dfrac{\lambda_1 L_1 + \lambda_2 L_2}{L_1 + L_2} \cdot dW(t)$

Changing for the forward swap measure -> $\dfrac{dS_{1,2}(t)}{S_{1,2}(t)} = \left[\dfrac{\lambda_1 L_1 + \lambda_2 L_2}{L_1 + L_2} \cdot \overline{dW_{1,2}(t)}\right]$

Pricing of the swaption -> Black and Scholes ($\|\lambda_1 + \lambda_2\|$)

Simple Example of Calibration of BGM (2)

the knowledge of $\|\lambda_1 + \lambda_2\|$ **,** $\|\lambda_1\|$ **and** $\|\lambda_2\|$ **is equivalent to the knowledge of** λ_1 **and** λ_2 **up to a rotation**

Cap volatility -> Norms

Swaption volatilities -> Angles

The problem is a linear algebra problem :

- determine a set of vectors such that :

$$\begin{cases} \|\lambda_i\| = \sigma_i \\ \dfrac{\lambda_i \cdot \lambda_j}{\|\lambda_i\|\|\lambda_j\|} = \rho_{i,j} \end{cases}$$

let be given the n caplet volatilities σ_i **with their correlation** $\rho_{i,j}$

which is obvious if dim[dW] >=N

In order to get a more robust model we try to have a much lower dim[dW]

=> Least Square Minimisation Algorithms **(-> Rebonato 1998)**

Computing a complex pricing in a BGM model

1

In the spot Libor measure

First we simulate the independant martingales $\overrightarrow{dW}$

2

Then we get the libor structure L_i **by applying the ArbitrageFree SDE**

$$\frac{\Delta L_i(t)}{L_i(t)} = \sum_{j = \lfloor t \rfloor}^{n-1} \frac{\delta_j L_j(t)[\lambda_i \cdot \lambda_j]}{1 + \delta_j L_j} \Delta t + [\lambda_i \cdot \Delta W(t)]$$

up to the time T_n

3

Then we Compute the Cashflow to be paid by the security at time T_n
in function of the $L_i(t)$

4

Then we multiply by the discount factor in the spot libor measure :

$$D_n(0) = \frac{1}{B_1(0)} \prod_{j=1}^{n-1} \frac{1}{(1 + \delta_j L_j(0))}$$

More Sophisticated Calibration : Smiles for Caps

- Let assume that the smile is represented by a CEV process :

$$\frac{dF_i(t)}{F_i(t)^{\alpha}} = (\ldots)dt + \zeta_i \cdot dW_t$$

- Then if we create the variable $Q_i = \frac{1}{1-\alpha} F_i^{1-\alpha}$ then we compute its drift and assume it constant.

- We deduce the price of a call :

$Call = P(0, T_{i+1}) L(T_{i+1} - T_i)[F_i(0)(1 - \chi^2(a, b+2, c)) - K\chi^2(c, b, a)]$ with the conventions:

$$a = \frac{K^{2(1-\alpha)}}{(1-\alpha)^2 \sigma_i^2 T_i} \qquad b = \frac{1}{1-\alpha} \qquad c = \frac{F_i(0)^{2(1-\alpha)}}{(1-\alpha)^2 \sigma_i^2 T_i} \qquad \sigma_i = \frac{1}{T_i} \sum_{j=1,i} \|\zeta_{i-j}\|^2 T_j$$

- $\chi^2(x, l, n)$ is the cumulative distr. func. of $U_1^2 + U_2^2 + U_3^2 + \ldots + U_{n-1}^2 + (U_n + l)^2$ where U_i are independent standard normal variables

More Sophisticated Calibration : Smiles For Swaptions

- We can describe the smiles the same way :

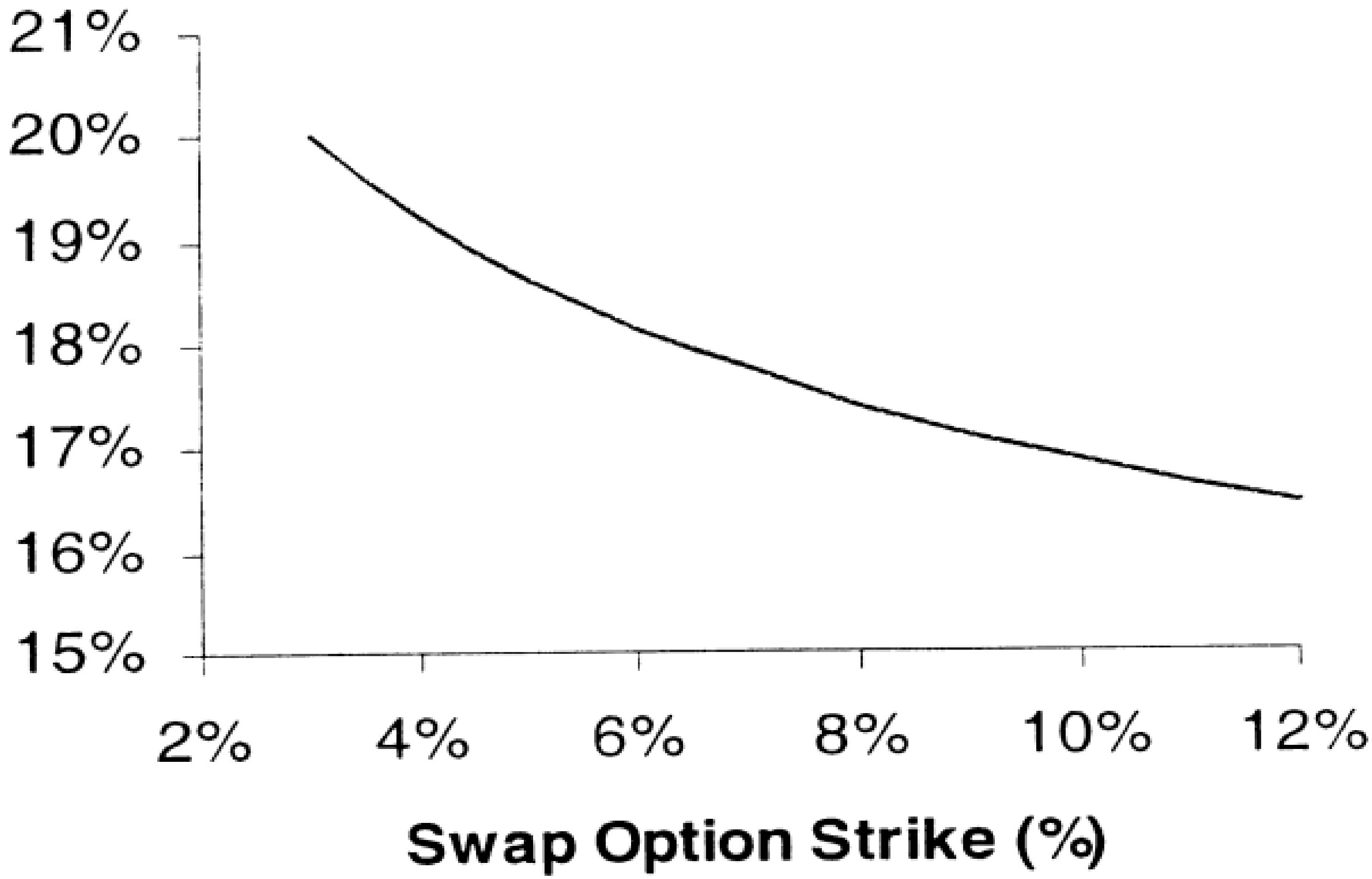
$$\frac{dS_{n,N}(t)}{S_{n,N}(t)^{\beta}} = (\ldots)dt + \eta_{n,N}(t)\cdot dW_t$$

- We define $\Theta^2{}_{n,N} = \frac{1}{T_n}\int_0^{T_n}\left\|\eta_{n,N}(t)\right\|^2 ds$

- Using $S_{n,N(t)} = \dfrac{\prod_{j=n,N}(1+(T_{j+1}-T_j)F_j(t))-1}{\sum_{i=n,N}(T_{i+1}-T_i)\prod_{j=i+1,N}(1+(T_{j+1}-T_j)F_j(t))}$, Hull and White (1999) show an analytical formula linking $\Theta^2{}_{n,N}$ and σ_i and and the other parameters. The formula is a first approximation but sufficient to describe the smile transfert.

Exemple of Smiles

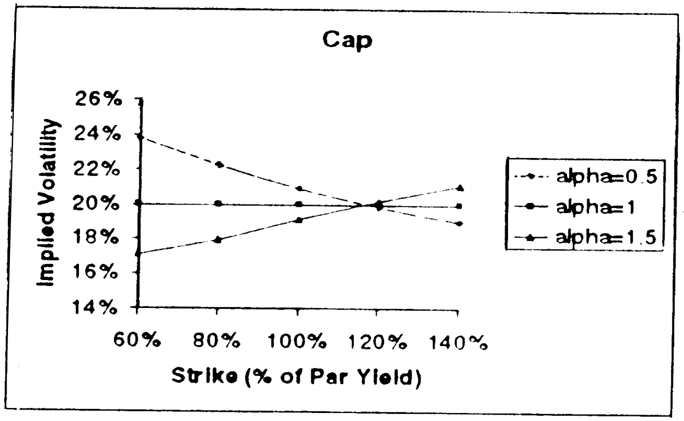
- alpha=1.25 , 5X5 Swaption

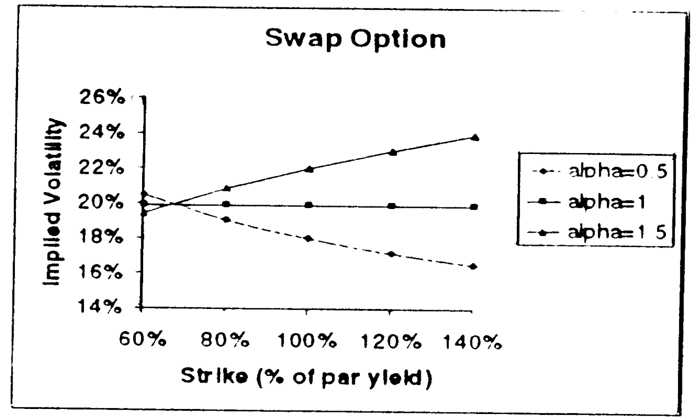


Exemple of Smiles Transfert

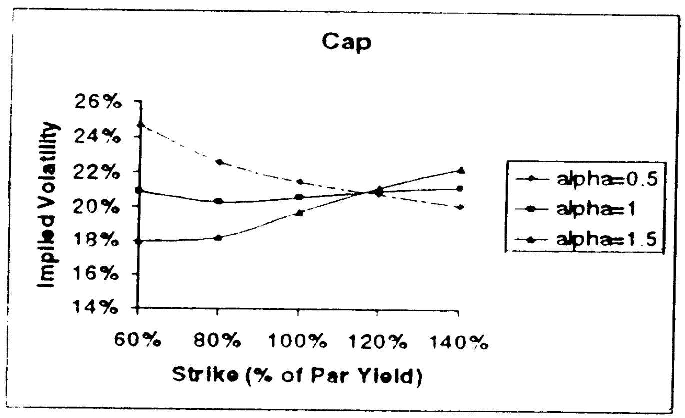
- Transfert of the smile from the cap market to the swaption market

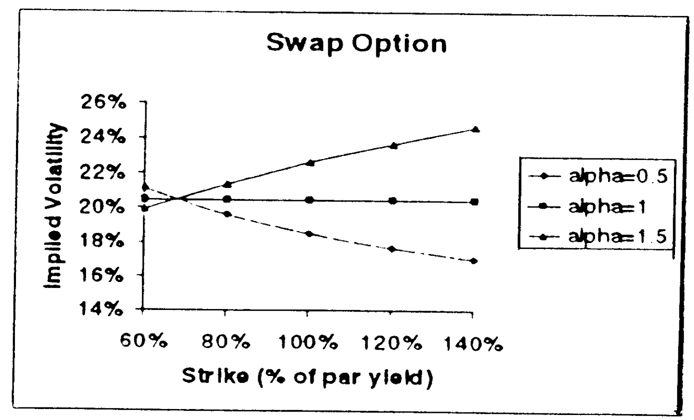
Flat Volatility Structure





Humped Volatility Structure





Conclusion

- BGM is a very flexible framework that allows a calibration to the caps and swaption simultaneously with handling of smiles

- Black and Scholes Formula is not linked to an instantaneous interest rate . Instead, it comes from the lognormality of the ratio of the underlying vehicule with another hedging asset (ex: a bond)

- Every pricing results from applying the most convenient numeraire and therefore from using the most convenient measure to compute the expectations.

Reading Advices

- Changes of Numeraire, Changes of Probability measure and option pricing by H.Geman, N El Karoui and J.C. Rochet (1995): J. Appl. Probability 32, 443-458 : a very good article that presents the technique with a few applications.

- LIBOR and swap market models and measures, by F. Jamshidian (1997) : Finance and Stochastics ,1, 293-330: the fondamental article that presents the BGM ideas and the market models . It requires to spend some time to rederive the equations, by far the most powerful presentation that I know.

- Forward Rate Volatilities, Swap rate volatilities and the implementation of the LIBOR Market Model (1999) by J. Hull and A. White : working paper : a very practical and comprehensive exposition of the market models, no general picture or theoretical framework as usual by these authors.

- Martingale Methods in Financial Modelling (1998) Springer, by M. Musiela and M. Rutkowski : the only book up to now that presents the numeraire technique and the BGM ideas; it is a sophisticated book that requires attention.