Analytical Risks and Natural Sensitivities

by Olivier Croissant

What we are Looking for

- Fast Risk and Hedge Ratios Computation
- Comprehensive Risk Factor Handling
- Liquidity Risk Factor Handling
- Non Linear Instruments
- Auditability of the Risks
- Engineering of Hedge Ratios

How we will Achieve these Goals

- Define and compute sensitivites --> natural sensitivities
- Define standardized sensitivities --> Mappings
- Define non linear risk measure --> cumulants
- Define liquidity risk with cumulants and sensitvities
- Generalized hedge ratios : geometrization of the hedge ratios

Sensitivities

Bond_Example.nb

```
X2 = Bond[sell, 25 mm, usd, TT["15-May-01"], coupon → 0.065]

Seq[Inst[Fix[-574315., 0.353425, Currency[usd], 0]],
 Inst[Fix[-1.625×10^6, 1.35342, Currency[usd], 0]],
 Inst[Fix[-1.625×10^6, 2.35342, Currency[usd], 0]],
 Inst[Fix[-2.6625×10^7, 3.35342, Currency[usd], 0]]]

sens1 = NS1[X2]

200211. S1[Index[Rate[0.353425, Currency[usd]]]] +
 2.16437×10^6 S1[Index[Rate[1.35342, Currency[usd]]]] +
 3.73901×10^6 S1[Index[Rate[2.35342, Currency[usd]]]] +
 8.62917×10^7 S1[Index[Rate[3.35342, Currency[usd]]]]

MP[sens1]

69144.4 S1S[Index[Rate[0.25, Currency[usd]]], 4, 1] +
 131066. S1S[Index[Rate[0.5, Currency[usd]]], 5, 1] +
 1.00136×10^6 S1S[Index[Rate[1., Currency[usd]]], 6, 1] +
 3.95325×10^6 S1S[Index[Rate[2., Currency[usd]]], 7, 1] +
 6.42484×10^7 S1S[Index[Rate[3., Currency[usd]]], 8, 1] +
 2.29921×10^7 S1S[Index[Rate[4., Currency[usd]]], 9, 1]
```

Natural Sensitivities

Standardized Sensitivities

A More Complexe Example

FXOption.nb

```
In[12]:= inst =
    FxOption[buy, call, 100 mm, usd, TT["30-Mar-98"], currencybase → dem, strike → 1.5]

In[21]:= ns1 = NS1a[inst]

Out[21]= 1.56921 × 10^6 S1[Index[ExchangeRate[Currency[dem], Currency[usd]]]] -
    49669.7 S1[Index[Rate[0, 0.0657534, Currency[dem]]]] +
    2.40549 × 10^7 S1[Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0] +
    4.48115 × 10^7 S1[Volatility1[
      Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0.0657534, 0.],
     0]
```

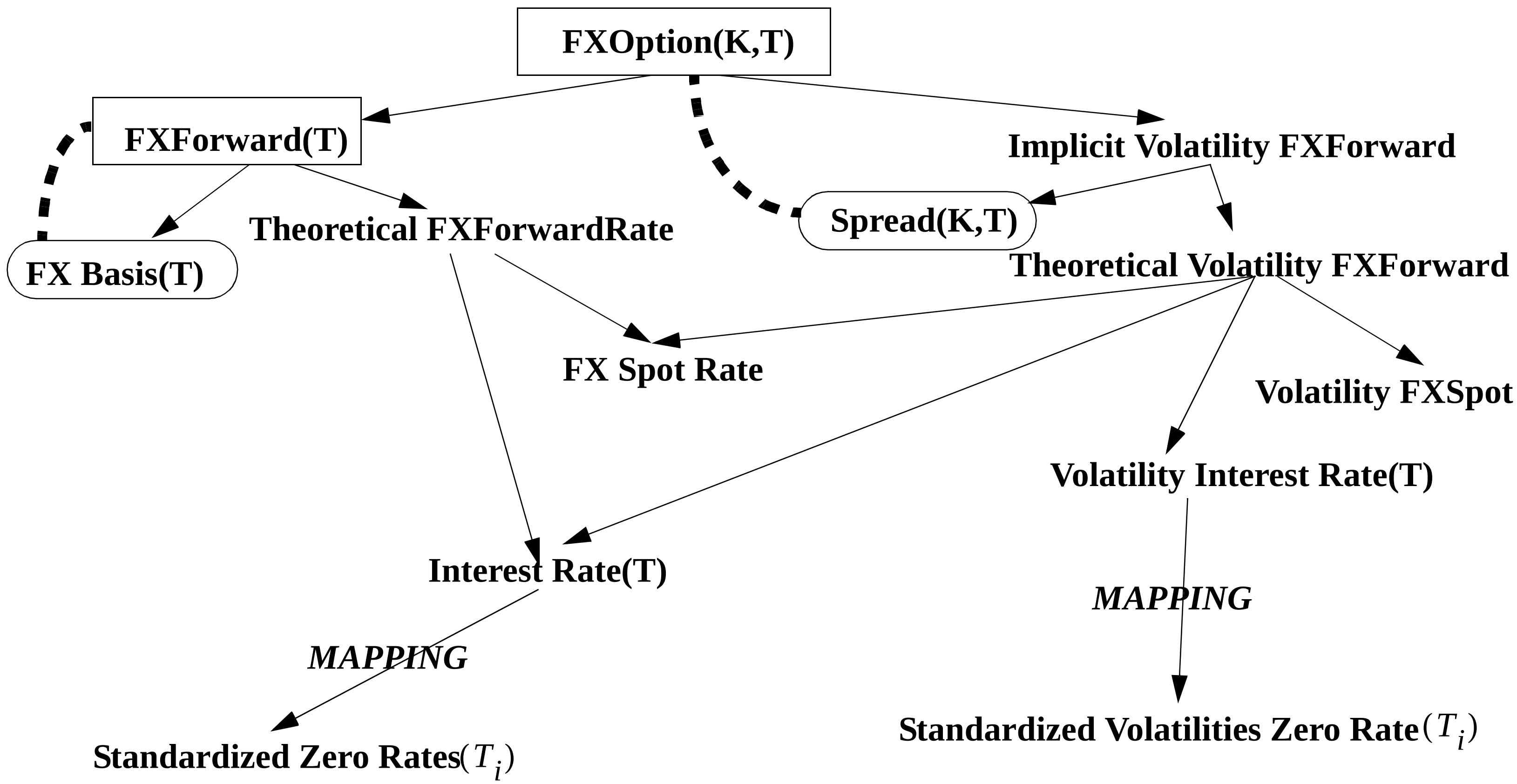
Top level Natural Sensitivities

```
In[15]:= MP[ns1]

Out[15]= -4.47051 × 10^7
     S1S[Index[ExchangeRate[Currency[dem], Currency[usd]]], 2] + 2.61083 × 10^6
     S1S[Volatility[Index[ExchangeRate[Currency[dem], Currency[usd]]]], 2] -
    1.48702 × 10^6 S1S[Index[Rate[0.0194444, Currency[dem]]], 2, 2] +
    1.45159 × 10^6 S1S[Index[Rate[0.0194444, Currency[usd]]], 2, 1] -
    745993. S1S[Index[Rate[0.0833333, Currency[dem]]], 3, 2] +
    732435. S1S[Index[Rate[0.0833333, Currency[usd]]], 3, 1] +
    354.266 S1S[Volatility[Index[Rate[0.00277778, Currency[dem]]]], 1, 2] -
    516.842 S1S[Volatility[Index[Rate[0.00277778, Currency[usd]]]], 1, 1] +
    12.6911 S1S[Volatility[Index[Rate[1, Currency[dem]]]], 2, 2] -
    16.3948 S1S[Volatility[Index[Rate[1, Currency[usd]]]], 2, 1]
```

Standardized Sensitivities

Hierachy of Natural Market Descriptors



- Node = Pricing Function = F(X1,X2,X3,...Xn)
- MAPPING = Interpolation function =Discretisation

Natural Sensitivities (First Idea)

- Compounding of Pricing Function -> Modularity of the differentials

$$Z = f(y_1, y_2) \qquad y_1 = g(x_1) \qquad y_2 = h(x_2)$$

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr] & \\
y_1 \arrow[d] & & y_2 \arrow[d] \\
x_1 & & x_2
\end{tikzcd}$$

- the Chain Rule of First Derivatives : we do not need the full space of risk factors

$$dZ = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2 \qquad dy_1 = \frac{dg}{dx_1} dx_1 \qquad dy_2 = \frac{dh}{dx_2} dx_2$$

First Derivatives of h

$$dZ = \frac{\partial f}{\partial y_1}\frac{dg}{dx_1} dx_1 + \frac{\partial f}{\partial y_2}\frac{dh}{dx_2} dx_2 = \begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \end{bmatrix} \begin{bmatrix} \frac{dg}{dx_1} dx_1 & \frac{dh}{dx_2} dx_2 \end{bmatrix}$$

First Derivatives of f

First Derivatives of g

Modularity of the Second Derivatives

- First differential $dZ : Z(world + d(world)) - Z(world)$ as a linear function of d(world)

- Second differential $d^2Z : Z(world + d(world)) - Z(world) - dZ$ as a quadratic function of d(world)

- if we have $Z = f(y_1, y_2, ..., y_n)$ and $y_i = y_i(x_1, x_2, ..., x_p)$

- --> $\frac{\partial^2}{\partial x_j \partial x_k} Z = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2}{\partial y_l \partial y_m} Z \cdot \frac{\partial y_l}{\partial x_j} \cdot \frac{\partial y_m}{\partial x_k} + \sum_{l=1}^{n} \frac{\partial}{\partial y_l} Z \cdot \frac{\partial^2}{\partial x_j \partial x_k} y_l$

- another way to say that is : $d^2Z = \sum_{i,j=1}^{n} \frac{\partial^2 Z}{\partial y_i \partial y_j} dy_i \otimes dy_j + \sum_{i=1}^{n} \frac{\partial Z}{\partial y_i} d^2 y_i$

Quadratic Forms

Sensitivities (order 1 and 2) of Z

Modularity of the Natural Sensitivities

- By Including the first order :

$$[dZ, d^2Z] \longleftarrow \text{Output of the node}$$

$$= \left[\sum_{i=1}^{n} \frac{\partial Z}{\partial y_i} d^2 y_i, \sum_{i,j=1}^{n} \frac{\partial^2 Z}{\partial y_i \partial y_j} dy_i \otimes dy_j + \sum_{i=1}^{n} \frac{\partial Z}{\partial y_i} d^2 y_i \right]$$

$$[dy_i, d^2 y_i] \longleftarrow \text{Input of the node}$$

- We compute a relative linear form and a relative quadratic form , **not a delta vector nor a gamma matrix**

- The difference is the same than between a sparse vector description and a full vector and between a sparse matrix description and a full matrix.

- Naturally every thing extends to higher order (but less interesting)

Example of a Natural Second Order Sensitivity

- Top Level Senstivities

FXOption.nb *

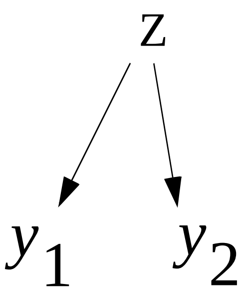
```
inst = FxOption[buy, call, 100 mm, usd, TT["30-Mar-98"], currencybase → dem, strike → 1.5]

In[22]:= NS2a[inst]

Out[22]= 3.46703 × 10^7 S1[Index[ExchangeRate[Currency[dem], Currency[usd]]], 0]
          S1[Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0] +
         3.94459 × 10^8 S1[Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0]^2 -
         103181. S1[Index[ExchangeRate[Currency[dem], Currency[usd]]], 0]
          S1[Index[Rate[0.0657534, Currency[dem]]], 0] -
         3.16338 × 10^6 S1[Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0]
          S1[Index[Rate[0.0657534, Currency[dem]]], 0] +
         4707.22 S1[Index[Rate[0.0657534, Currency[dem]]], 0]^2 +
         6.45868 × 10^7 S1[Index[ExchangeRate[Currency[dem], Currency[usd]]], 0] S1[Volatilityl[
            Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0.0657534, 0.], 0] +
         7.64823 × 10^7
          S1[Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0] S1[Volatilityl[
            Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0.0657534, 0.], 0] -
         5.89302 × 10^6 S1[Index[Rate[0.0657534, Currency[dem]]], 0] S1[Volatilityl[
            Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0.0657534, 0.], 0] +
         4.3603 × 10^8 S1[Volatilityl[Index[
              ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0.0657534, 0.], 0]^2 +
         2.40549 × 10^7 S2[Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0] -
         49669.7 S2[Index[Rate[0.0657534, Currency[dem]]], 0] - 4.48115 × 10^7 S2[Volatilityl[
            Index[ExchangeRate[0.0657534, Currency[usd], Currency[dem]]], 0.0657534, 0.],
           0]
```

Mapping (Introduction)

- The last step in the chain of chain computations of natural senstivities
- Necessary step to do risk and generalized hedge ratio computation



Natural factor

$$Z = f(y_1, y_2)$$

Standardized Factors

$$dZ = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2$$

$$d^2 Z = \frac{\partial^2 f}{(\partial y_1)^2}(dy_1 \otimes dy_1) + \frac{\partial^2 f}{(\partial y_2)^2}(dy_2 \otimes dy_2) + 2\frac{\partial^2 f}{\partial y_1 \partial y_2}(dy_1 \otimes dy_2)$$

$$+ \cancel{\frac{\partial f}{\partial y_1}(d^2 y_1) + \frac{\partial f}{\partial y_2}(d^2 y_2)}$$

$= 0$ By definition

Mapping Exemple from an Interpolation

- Assume that the value of a portfolio PV depends on r_{18m}

- Develop it at the second order $PV \approx PV_0 + \delta(r_{18m} - r_{18m,\,0}) + \frac{1}{2}\gamma(r_{18m} - r_{18m,\,0})^2$

- Assume we have an interpolation function $r_{18m} = \frac{r_1 + r_2}{2}$

- We replace to get :$PV \approx PV_0 + \delta\left(\frac{(r_1 - r_{1,\,0}) + (r_2 - r_{2,\,0})}{2}\right) + \frac{\gamma}{2}\left(\frac{(r_1 - r_{1,\,0}) + (r_2 - r_{2,\,0})}{2}\right)^2$

- It gives birth to the mappings : $PV \approx PV_0 + \Delta \cdot \begin{bmatrix} dr_1 \\ dr_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} dr_1 & dr_2 \end{bmatrix}\Gamma\begin{bmatrix} dr_1 \\ dr_2 \end{bmatrix}$

$$\delta \longrightarrow \Delta = \begin{bmatrix} \delta/2 \\ \delta/2 \end{bmatrix} \qquad \gamma \longrightarrow \Gamma = \begin{bmatrix} \gamma/2 & \gamma/2 \\ \gamma/2 & \gamma/2 \end{bmatrix}$$

- We say this mapping **derives** from the interpolation

Do all Mappings Derive from an Interpolation?

- The present value $PV = C_{18m}e^{-r_{18m}t} = C_{18m}B_{18m}$ ==> $(dPV = C_{18m}dB_{18m}$
- A Mapping should give birth to another decomposition $dPV = C_1 dB_1 + C_2 dB_2$
- and therefore we have :$dB_{18m} = (C_1 / C_{18m})dB_1 + (C_2 / C_{18m})dB_2$
- From $dB_t = -tB_t dr_t$ we get $dr_{18m} = \frac{C_1 B_1}{C_{18m}(3/2)B_{18m}}dr_1 + \frac{C_2 2B_2}{C_{18m}(3/2)B_{18m}}dr_2$
- Does this derive from an interpolation function $r_{18m} = f(r_1, r_2)$?
- Poincare Lemma tells us : <=>$\frac{\partial}{\partial r_2}\left[\frac{C_1 B_1}{C_{18m}(3/2)B_{18m}}\right] = \frac{\partial}{\partial r_1}\left[\frac{C_2 2B_2}{C_{18m}(3/2)B_{18m}}\right]$
- If it derives , ==> **Implicit Interpolation Function** . it does it for RiskMetrics.

Why should a Mapping Derives from an Interpolation?

- **All Sensitivities Mapping** : To have a consistent method to derive the mapping of sensitivities of any order.

- **Arbitrage** : having an interpolation inconsistent with a first order mapping lead to the following risk arbitrage : Any position with a convexity depending on the risk factor whose mapping is wrong will be mispriced.

- **Inconsistent Risk Calculation** : by having a risk computed for a position at a different market point from its actual Mark to Market , we do an inexact computation.The Risk mesures how the PV varies when the risk factors move. We therefore must have the correct linear relationships for it

Risk Metrics Mapping

- It determine the mapping by enforcing the followings:

$$Var[C_1 + C_2] = \sigma_1^2 Z_1^2 C_1^2 + 2\rho\sigma_1\sigma_2 Z_1 C_1 Z_2 C_2 + \sigma_2^2 Z_2^2 C_2^2$$

$$Var[C_t] = \sigma_t^2 Z_t^2 C_t^2$$

- And the preservation of the present value which is by no way a needed characteristic of an equivalent risk representation

- An ideal variance property would have been :

$$Var[P + C_1 + C_2] = \sigma_1^2 Z_1^2 C_1^2 + 2\rho\sigma_1\sigma_2 Z_1 C_1 Z_2 C_2 + \sigma_2^2 Z_2^2 C_2^2 + Var[P] + 2Cov[P, C_1 + C_2]$$

$$Var[P + C_t] = \sigma_t^2 Z_t^2 C_t^2 + Var[P] + 2Cov[P, C_t]$$

we can try to gess that

But we have no ways to determine that !

- The implicit volatility determined by a choosen mapping is by no way arbitrageable which is not the case of the implicit interpolation. Neither It gives a non consistency argument for the risk computation, which is not the case with the implicit interpolation.

Risk Measures for Non Linear Instruments

- Being able to get the exact sensitivities, we approximate the variation of the MtM by the quadratic polynomial : $PV \approx PV_0 + \Delta \cdot \begin{bmatrix} dx_1 \\ \dots \\ dx_n \end{bmatrix} + \frac{1}{2}\begin{bmatrix} dx_1 & \dots & dx_n \end{bmatrix} \Gamma \begin{bmatrix} dx_1 \\ \dots \\ dx_n \end{bmatrix}$

- We assume a jointly normal approximation of the risk factors $Cov[dx_i, dx_j] = \Sigma_{i,j} dt$

- The Risk Metrics Risk measure is just : $Var[PV_t - PV_0] = ({}^t\Delta\Sigma\Delta)t = Trace[t\Sigma(\Delta \otimes \Delta)]$

- If we take into account the gammas : $Var[PV_t - PV_0] = Trace\left[t\left(\Sigma(\Delta \otimes \Delta) + \frac{1}{2}(\Gamma\Sigma)^2\right)\right]$

- skewness and kurtosis :

$$Skew[PV_t - PV_0] = E[(PV_t - PV_0)^3] = Trace[3\Sigma\Gamma\Sigma(\Delta \otimes \Delta) + (\Gamma\Sigma)^3]$$

$$Kurtosis[PV_t - PV_0] = E[(PV_t - PV_0)^4 - 3(Var[PV_t - PV_0])^2] = Trace[12\Sigma(\Gamma\Sigma)^2(\Delta \otimes \Delta) + (3(\Gamma\Sigma))^4]$$

More about Cumulants

- The carateristic function $\Phi_X(t) = E[e^{itX}]$ is a very interesting object

- It is the moment generating function : $\mu_k[X] = E[(X)^k] = (-i)^k \frac{d^k}{dt}(\Phi_X(t)|_{t=0})$

- It simply multiply over independent variable :

$$\text{let} \qquad \Psi_X(t) = Log[\Phi_X(t)]$$

$$X \text{ and } Y \text{ independant} \Leftrightarrow \Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t) \Leftrightarrow \Psi_{X+Y}(t) = \Psi_X(t) + \Psi_Y(t)$$

- we define the cumulants as $Cum_k(X) = (-i)^k \frac{d^k}{dt}(\Psi_X(t)|_{t=0})$

- they cumul over independant variables :$Cum_k(X+Y) = Cum_k(X) + Cum_k(Y)$

More about Cumulants (2)

- The first cumulants are :

$$Cum_1(X) = E[X]$$

$$Cum_2(X) = E[X^2] - (E[X])^2 = Var[X]$$

$$Cum_3(X) = E[(X - E[X])^3] = Skewness[X]$$

$$Cum_4(X) = E[(X - E[X])^4 - 3(E[(X - E[X])^2])^2] = kurtosis[X]$$

- They measure how far from the normal distribution X is : $Cum_k[X] = 0 \quad \forall k > 2$ is equivalent to X Normal

- They caracterise a distribution :

Knowing the cumulants <=> Knowing the distribution

More about Cumulants (3)

- You can modify the set of cumulants by the superoperator

$$:g(x) = \exp\left[\sum_{i=1}^{\infty} \frac{\varepsilon_i(-D)^i}{i!}\right] h(x)$$ where h(x) and g(x) are densities

Then $$Cumulant_k[g] = Cumulant_k[h] + \varepsilon_k$$

- We can find the distribution associated with a set of cumulants c_2, c_3, c_4

- We can find option values in case where the market is non normal :

$$Option = B_T \int (x-K)^+ g(x)$$

Then by plugging the formula and integrating by parts we correct the option price

Computing Long Maturity Risks

- Using Lognormal Variables for the Risk Factors :

$$\frac{dX_t}{X_t} = a(t)dt + \sigma dW_t \qquad Cov\left[\frac{dX_t}{X_t}, \frac{dY_t}{Y_t}\right] = \sigma_{X,Y}dt$$

=> we just handle Log[X] instead of X

- Using Mean Reverting Variables for the Risk Factors :

$$dX_t = (a(t) - X_t)dt + \sigma dW_t \qquad \frac{dX_t}{X_t} = (a(t) - X_t)dt + \sigma dW_t \qquad \frac{dX_{i,j}(t)}{X_{i,j}(t)} = B_i(a_{i,j}(t) - X_i)dt + \overrightarrow{\sigma_{i,j}}\overrightarrow{dW_t}$$

=> we just replace the dependency in the horizon of risks

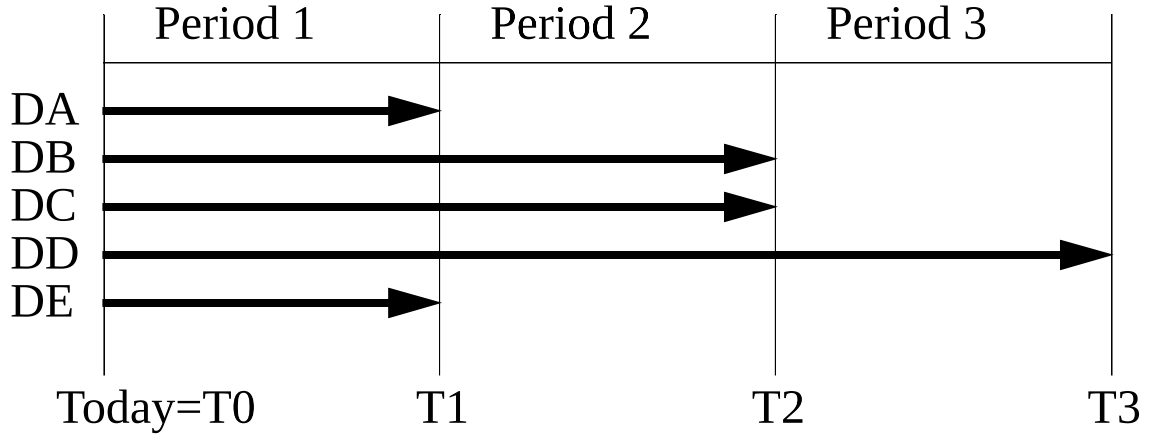
$\sigma_t = \sqrt{t}\left\|\vec{\sigma_i}\right\|$ by $\sigma_t = \sqrt{\frac{1 - e^{-2B_i t}}{2B_i}}\left\|\vec{\sigma_i}\right\|$ for independant brownians

-> principal component analysis to identify the blocs of factors i with the same $\boldsymbol{B_i}$

Handling of Liquidity Risks

- Thanks to cumulants we define a distribution of the P&L associated with a complete unwinding of the portfolio, knowing the unwinding period for each deals :

- No interest rates (the futur values have to be corrected by the zero coupon price)



- PL(3)-PL(0)={PL(3)-PL(2)} + {PL(2)-PL(1)} + {PL(1)-PL(0)}

- Markovian Markets : $Var[PL_3 - PL_0] = Var[PL_3 - PL_2] + Var[PL_2 - PL_1] + Var[PL_1 - PL_0]$

- More generally : $Cu_n[PL_3 - PL_0] = Cu_n[PL_3 - PL_2] + Cu_n[PL_2 - PL_1] + Cu_n[PL_1 - PL_0]$

Computation of Forward cumulants

- Portfolio :

Period 1	Period 2
$Variance = C_1$	Δ, Γ $Variance = C_2$

Today=T0 T1 T2

- Computation of the Variance for Period 1 + Period 2

$$Tr\left[(\Delta \otimes \Delta)(C_1 + C_2) + \frac{1}{2}(\Gamma(C_1 + C_2))^2\right] \neq Tr\left[(\Delta \otimes \Delta)C_1 + \frac{1}{2}(\Gamma C_1)^2\right] + Tr\left[(\Delta \otimes \Delta)C_2 + \frac{1}{2}(\Gamma C_2)^2\right]$$

Variance associated with $\Delta(t_1)$ unknown

Variance associated with $\Delta(t_1)$ known

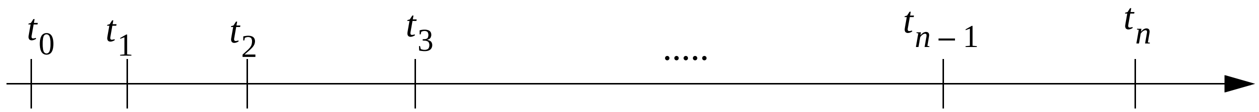
- But $\Delta(t_1)$ is not known due to Γ effect , so :

$$Var(Period_2) = Tr\left[(\Delta \otimes \Delta)(C_1 + C_2) + \frac{1}{2}(\Gamma(C_1 + C_2))^2\right] - Tr\left[(\Delta \otimes \Delta)C_1 + \frac{1}{2}(\Gamma C_1)^2\right]$$

- And the same for other cumulants : $Cu_n(Period_q) = Cu_n\left(\sum_{i \le q} Period_i\right) - Cu_n\left(\sum_{i \le q-1} Period_i\right)$

Equivalent Unwinding Periods

- Lets assume that the deals are sorted and regrouped by unwinding periods. Let be the number of classes be n



- Let be the pth forward cumulant associated with the period k : $[t_{k-1}, t_k]$ be $Cu_{p,k}$. There are normalized to 1 unit of time

- The total p cumulant of the total P&L is therefore $Cu_p = \sum_{k=1,n} Cu_{p,k}(t_k - t_{k-1})$

- The Equivalent unwinding period associated with the order p is

$$T_p = \frac{\sum_{k=1,n} Cu_{p,k}(t_k - t_{k-1})}{Cu_{p,1}}$$

Modularity of Unwinding Periods

- Let assume that every subportfolio k has an unwinding period of order p : $T_{p,k}$ with a normalized p cumulant associated with the period 1 (instantaneous cumulant) of the subportfolio k be

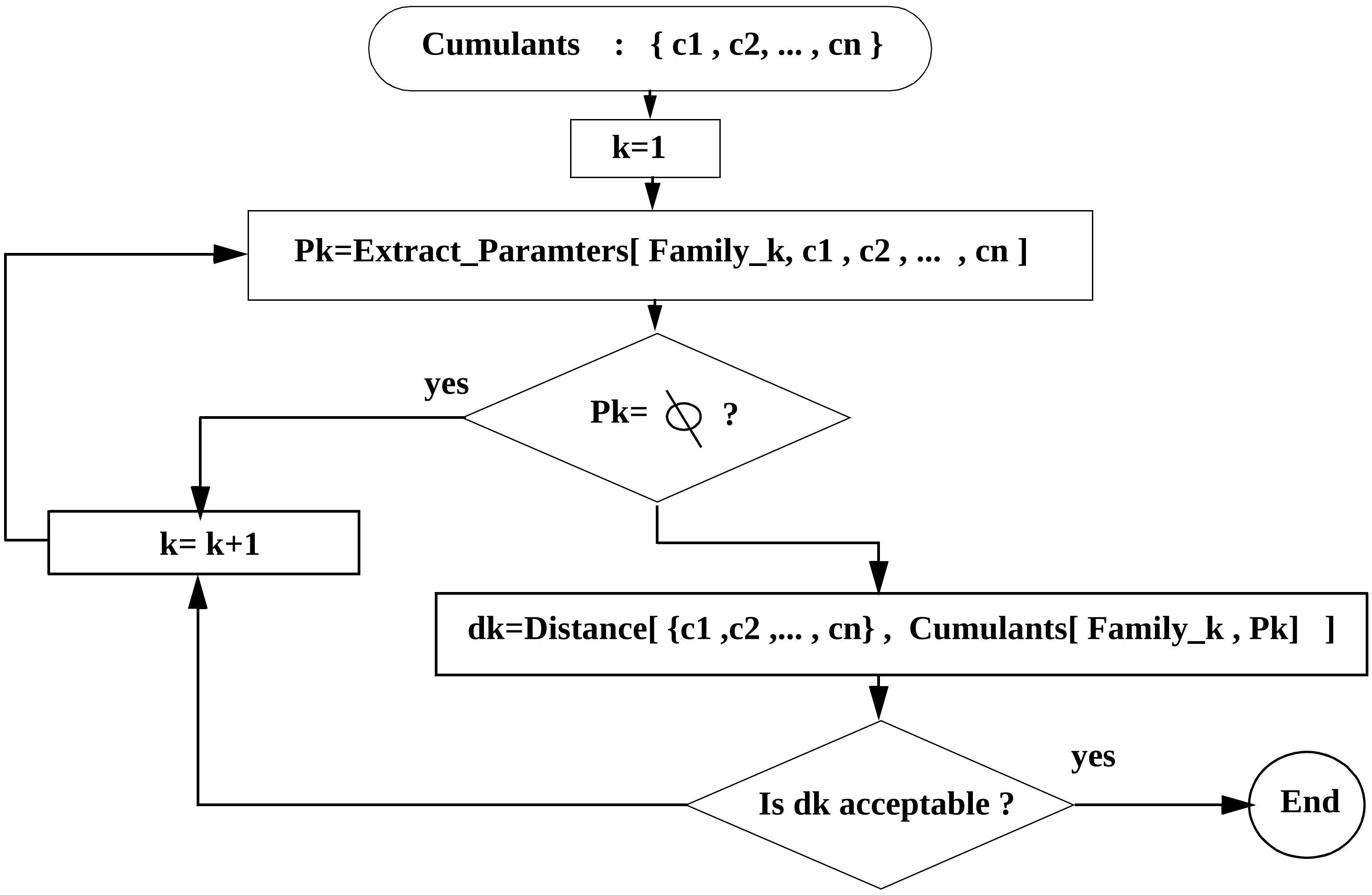
$$Cu_{p,1,k}$$

- The unwinding period associated with the total portfolio of order p is therefore :

$$T_p = \frac{\sum_{k=1,n} Cu_{p,1,k} T_{p,k}}{Cu_{p,1}}$$

- Because of central limit theorem, we observe that this compounding tend to make $T_p \to 0$ for $p > 2$ Which is an example of ergodicity property

Quadratic Risk Calculation: Fitting the Cumulants



Quadratic Risks : The Different Families of Distributions

- Extract_ Parameters [Cumulants]

- Compute_Cumulants [Parameters]

Distance [cumulants-1 , cumulants-2]

$$d[a, b] = \sum_{k=2}^{N} \alpha_k (a_k - b_k)^2 \longrightarrow$$ scalar product : easy optimization

- Example of families :

$$Z = \frac{1}{2} g X^2 + dX + hY^2 - \frac{1}{2} g - \frac{1}{2} h$$

X and Y are independant normal variables

$$Z = \frac{1}{2} g X^2 + dX + cY - \frac{1}{2} g$$

$$Z = \gamma + \delta \log\left[\frac{Y}{1-Y}\right] \quad Z = \gamma + \delta Sinh^{-1}[Y] \quad Z = \gamma + \delta \log[Y]$$ Johnson Family of Distribution

$$\frac{1}{p}\frac{dp}{dx} = -\frac{a+x}{c_0 + c_1 x + c_2 x^2}$$

Pearson Family of Distribution

Example of a Fitting

- let s study the following family of distribution : $U = \frac{1}{2}aX^2 + bX + cY$ with X and Y independant Normalized Normal variables

- The cumulants are $\begin{cases} c_2 = \frac{1}{2}a^2 + b^2 + c^2 \\ c_3 = a^3 + 3b^2 a \\ c_4 = 3a^4 + 12a^2 b^2 \end{cases}$

  => $a^4 - 4ac_3 + c_4 = 0$ that we solve to get a then

$$b = \sqrt{\frac{c_3 - a^3}{3a}} \qquad c = \sqrt{c_2 - \frac{a^2}{2} - b^2}$$

- but the 4th degree equation has a solution only if $c_4 < 3(c_3)^{4/3}$

Exemple of a Fitting : Suite

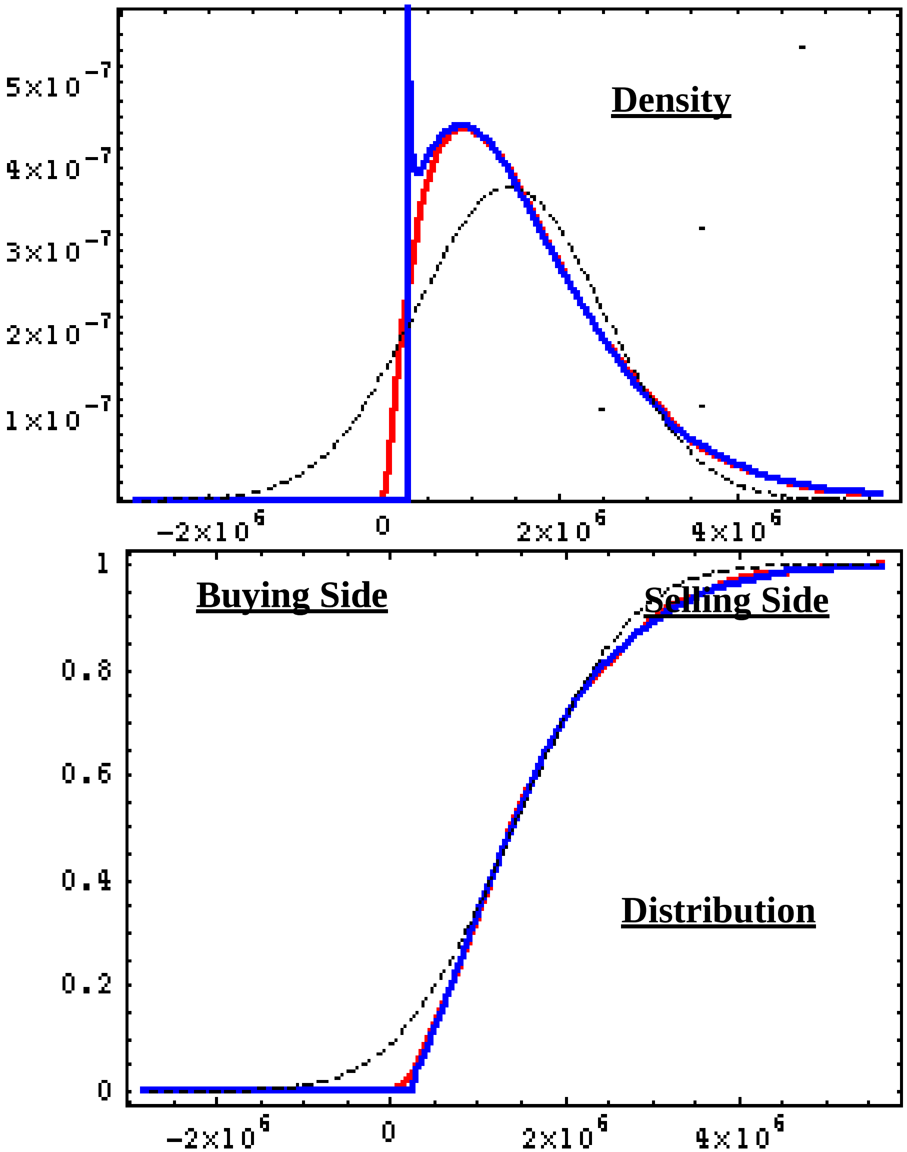
- Computation of the Distribution :

$$\alpha = \begin{cases} \displaystyle\int_{-\frac{b^2}{2a}}^{\infty} \frac{\phi'\left(\frac{R-y}{c}\right)}{c} \{\phi(x_2(y)) - \phi(x_1(y))\} dy & (g > 0) \\ \displaystyle 1 - \int_{-\infty}^{-\frac{b^2}{2a}} \frac{\phi'\left(\frac{R-y}{c}\right)}{c} \{\phi(x_2(y)) - \phi(x_1(y))\} dy & (g < 0) \end{cases}$$

- with the convention :

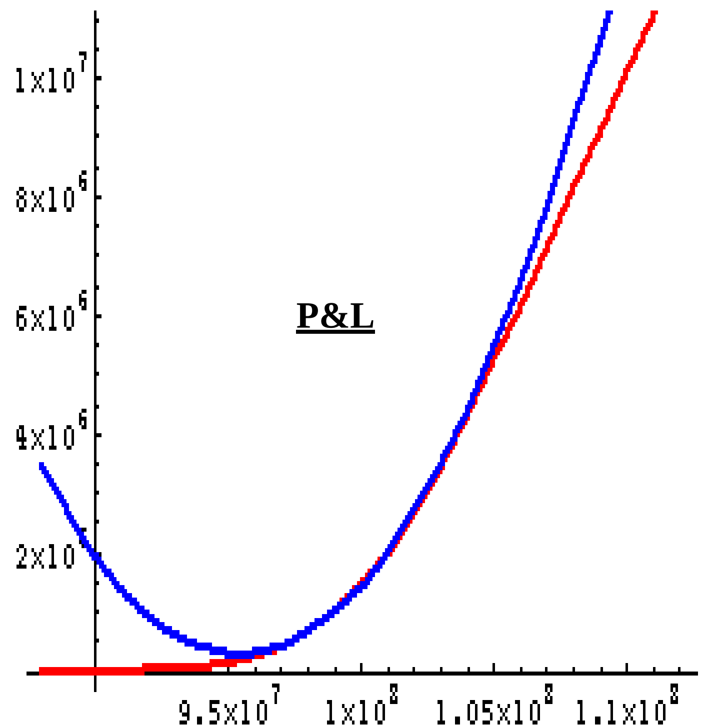
$$\begin{matrix} x_2(y) \\ x_1(y) \end{matrix} = \frac{-b}{a} \pm \frac{\sqrt{b^2 + 2ay}}{a} \qquad \phi'(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

Exemples of Quadratic Risks



Density of a Call at the money, 2 weeks from expiration on a market at 20% volatility per year

True Density of the call
Quadratic Approximation
Normal Approximation



Exact Results

- For the 2 weeks option at the money

	Risks	level	Real Risk	Quadratic	relat. err.	Linear	relat. err.	Improv./Lin.
Expected Value : 1.417524 Standard deviation : 1.06258 (in $ millions)	Buyer	5%	0.988	1.056	-3.29%	1.747	60.1%	94.7%
		1%	1.2721	1.1415	-11.4%	2.472	94.3%	89.1%
	Seller	5%	2.223	2.274	2.25%	1.747	-21.4%	89.2%
		1%	3.401	3.472	4.79%	2.472	-27.3	81.6%

- A “Normal Option” (6 month)

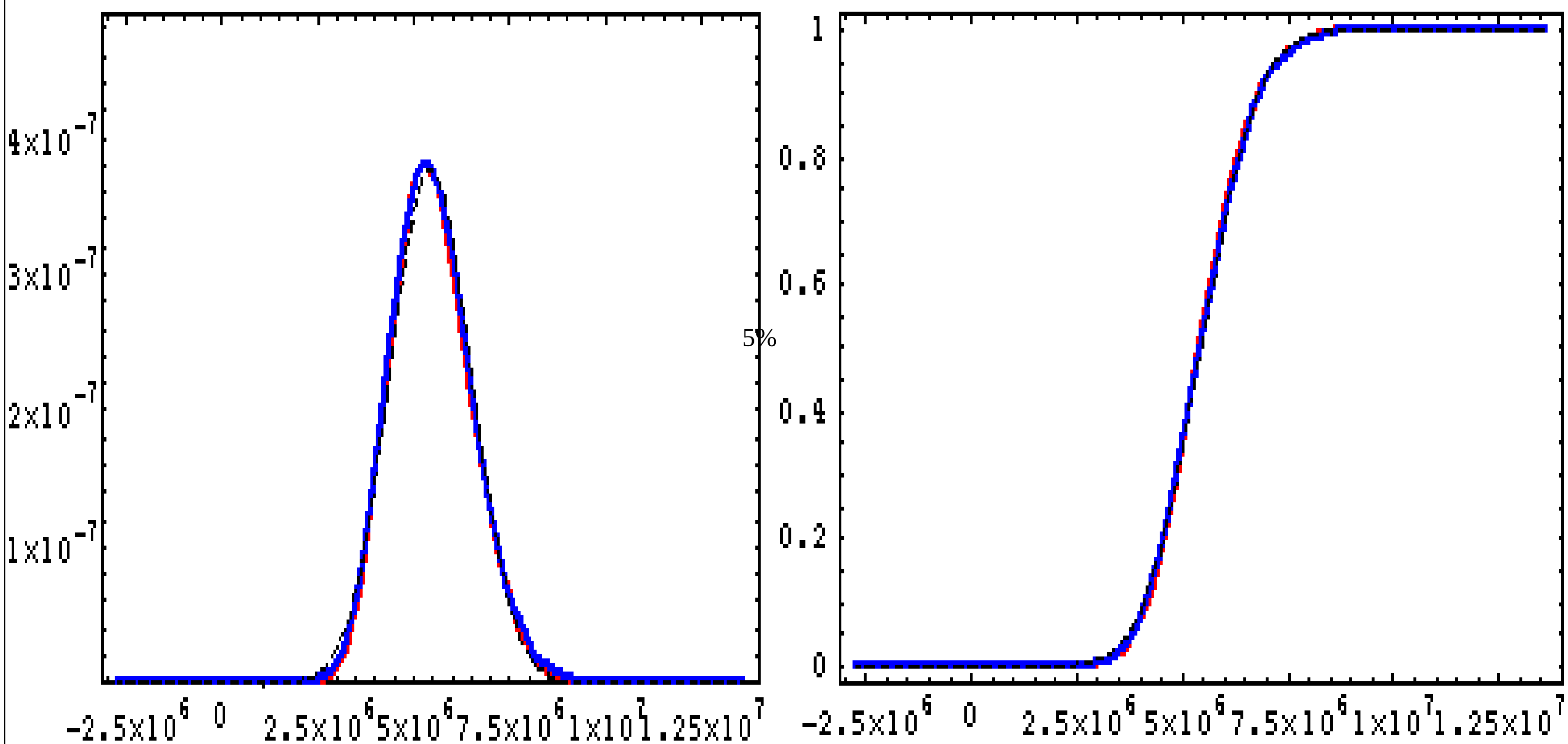
	Risks	level	Real Risk	Quadratic	relat. err.	Linear	relat. err.	Improv./Lin.
Expected Value :5.4455 Standard deviation : 1.0588 (in $ millions)	Buyer	5%	1.5427	1.5828	2.53%	1.7417	12.9%	79.8%
		1%	2.0922	2.1484	2.62%	2.4632	17.7%	84.8%
	Seller	5%	2.279	2.279	0.00%	1.7417	-92.4%	100%
		1%	2.279	2.279	0.00%	2.4632	-89.2%	100%

- A Nasty Option (6 days)

	Risks	level	Real Risk	Quadratic	relat. err.	Linear	relat. err.	Improv./Lin.
Expected Value : 0.7974 Standard deviation : 1.1551 (in $ millions)	Buyer	5%	0.7590	0.7675	1.11%	1.8999	150%	99.3%
		1%	0.9379	0.8287	-13.2%	2.6871	186%	93.8%
	Seller	5%	2.5358	2.7376	7.37%	1.8999	-25.1%	68.3%
		1%	3.8609	4.5038	14.3%	2.6871	-30.4%	45.2%

A “Normal Option”

<ins>*Call at the money, 6 months from expiration on a market at 20% volatility per year*</ins>



Expected Value :5.4455
Standard deviation : 1.0588

(in $ millions)

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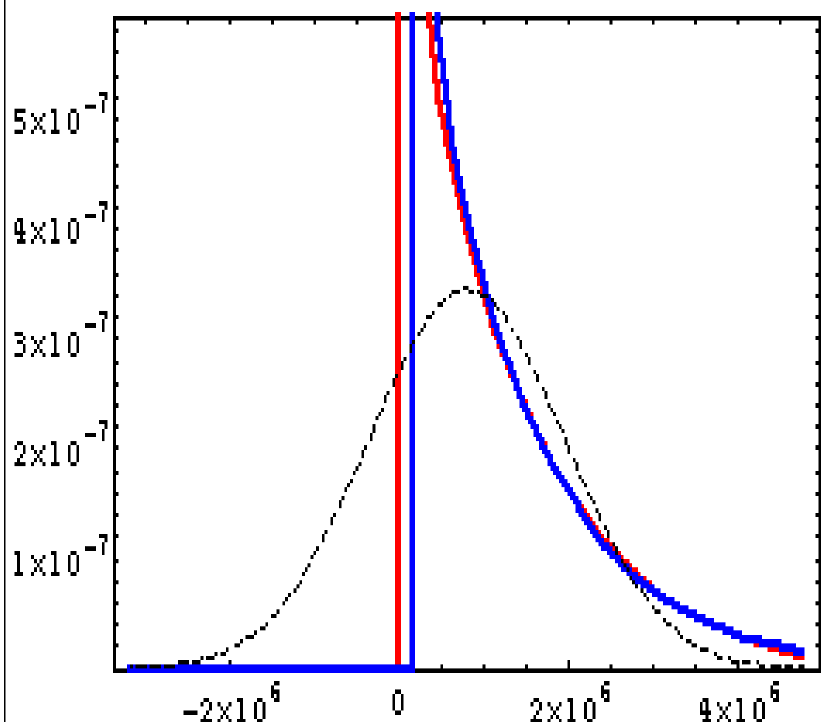
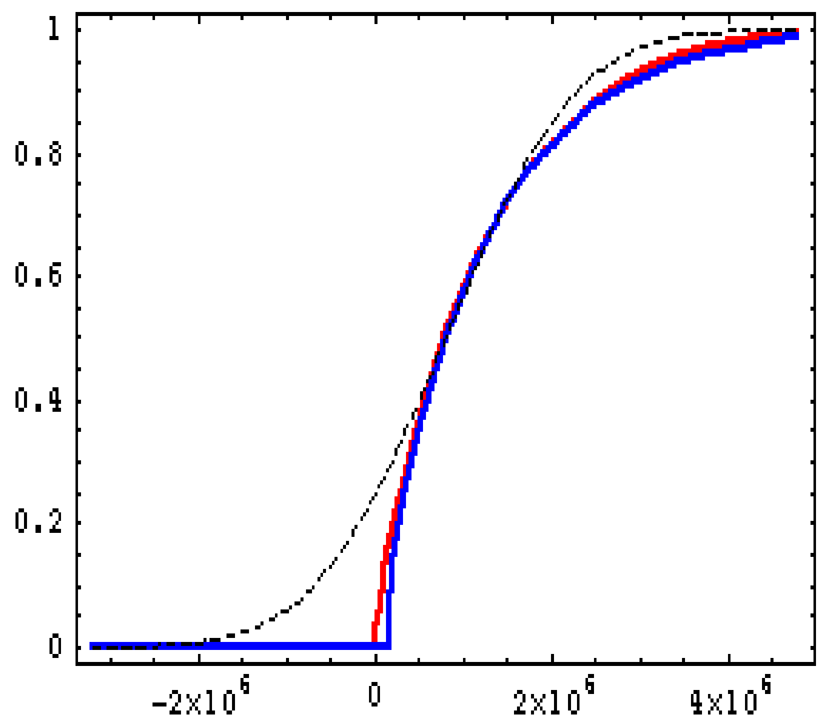
A Nasty Option

Call at the money, 6 days from expiration on a market at 20% volatility per year





Expected Value : 0.7974
Standard deviation : 1.1551

(in $ millions)

Risks	level	Real Risk	Quadratic	relat. err.	Linear	relat. err.	Improv./Lin.
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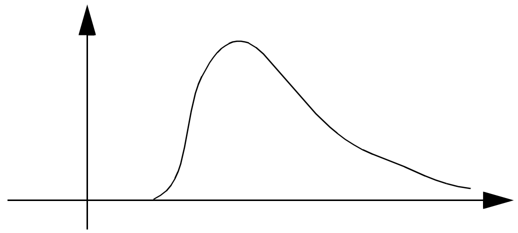
Difficulties associated with Risk Calculations

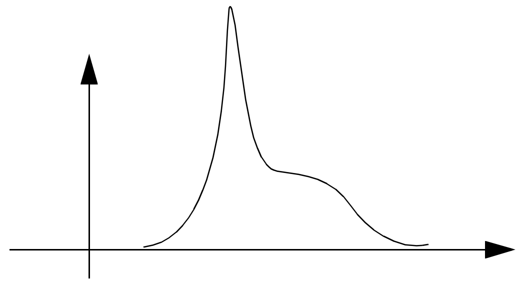
Skewness and Kurtosis

Risk of buying an option
#
Risk of selling an option



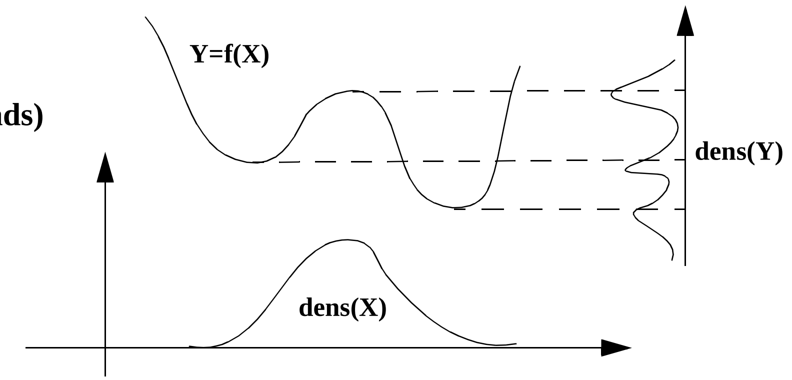
Spikes in the the density

Hedged portfolio :
high gamma without
corresponding delta
(->Infinite density)



Multimodality

Non local effects
(pile of option spreads)



Simple Hedge Ratio Computation

$$\boxed{\textbf{A Bond Option}} \longrightarrow \boxed{\textbf{Black \& Sholes (Fut[Bond], other param)}}$$

$$\textbf{Hedge Ratio (type I)} = \frac{\partial \text{Black (Fut(Bond) , other param)}}{\partial \text{ Fut (Bond)}}$$

Hedge Ratios using the First Order Differentials

$$\Delta_{Option} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \qquad \Delta_{Futur} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

Geometry determined by Σ

$$X \cdot Y = {}^{t}X\Sigma Y$$

$$\|X\|^2 = {}^{t}X\Sigma X$$

Hedge ratio (type II) = $\dfrac{\Delta_{Option} \cdot \Delta_{Futur}}{\|\Delta_{Futur}\|^2}$

If n=1, $\Delta_{Option} \; // \; \Delta_{Futur}$

=> Hedge ratio (type II)=Hedge ratio (type I)

Hedge Ratio as a Distance Minimizer

$$D_{Option} = \begin{bmatrix} \frac{\partial}{\partial X_i} Option \\ \frac{\partial^n}{\partial X_{i_1} \partial X_{i_2} \ldots \partial X_{i_n}} Option \\ \ldots \\ Option(X_k + a_k) \\ Option(h_k(X)) \\ \frac{\partial^n}{\partial X_{i_1} \partial X_{i_2} \ldots \partial X_{i_n}} Option(h_k(X)) \end{bmatrix}$$

Risk Descriptor:

<u>One constraint : being linear</u>

Risk Function : $D_{Option} \longrightarrow \left\| D_{Option} \right\| \in R^{+}$

Hedge Ratio (type III) = $\{h_i\}$ **such Minimum** $\left\| D_{Option} - \sum_{j=1}^{p} h_j D_{hedge(j)} \right\|$

If p=1 , Risk descriptor = Δ **, Risk function = Var[.]**

=> Hedge ratio (type III)=Hedge ratio (type II)

Conclusion

Second Order principal component analysis

- We have the universal formula : $Var[PV_t - PV_0] = Trace\left[t\left(C(\Delta \otimes \Delta) + \frac{1}{2}(\Gamma C)^2\right)\right]$

- If we set in the basis where C=1 and gamma is diagonal, this formula can decompose into :

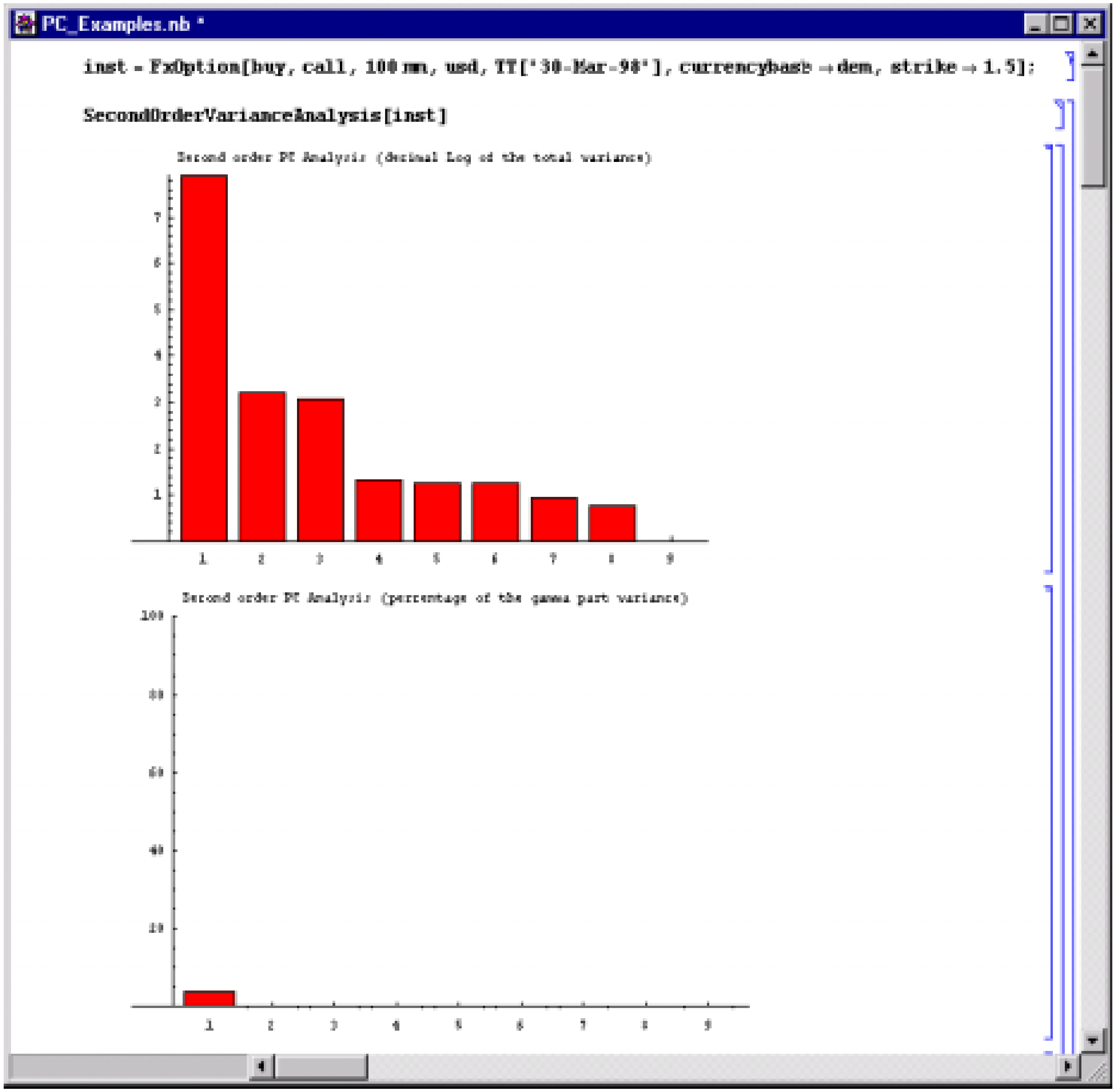
$$Var[PV_t - PV_0] = t\sum_{i=1}^{n}\left(\delta''^2_i + \frac{1}{2}\gamma''^2_i\right)$$

- Therefore we can analyse the origin of the variance, Sort the dimensions by importances

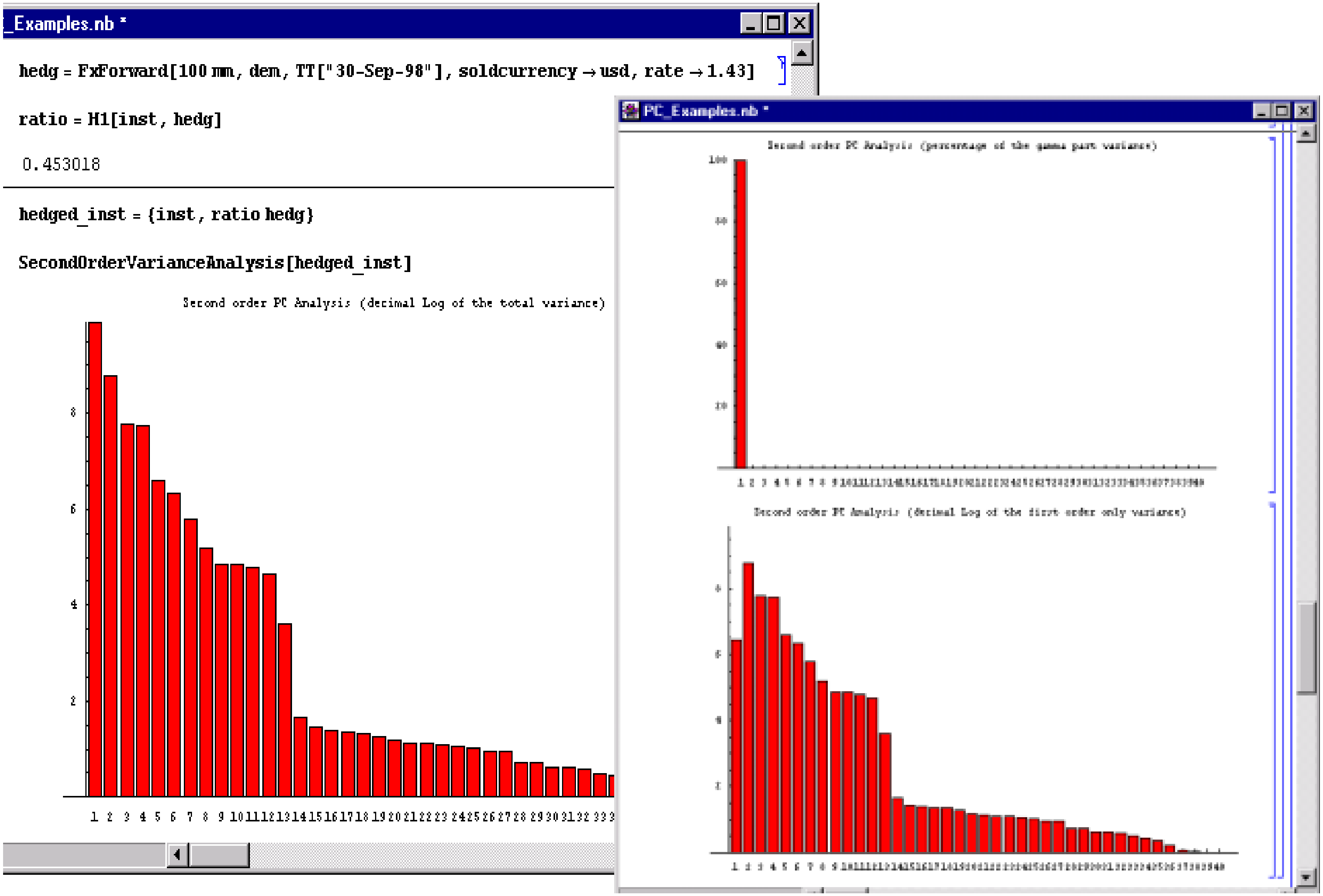
- Other Cumulants are also diagonal, allowing us to refine our discretisation depending on the severity of the nonlinearity :

- There is no third order principal component analysis because we cannot diagonalize the third order tensors $\frac{\partial^3 PV}{\partial x_i \partial x_j \partial x_k}$

Exemple of Second order PCA



PC Analysis : Strong Gamma



Reminder of Linear Agebra

- Matrix Multiplication

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ x + 5y \end{bmatrix}$$

- Norm of a Vector

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \|x\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

- Quadratic Form

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad {}^t xCx = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + 3y^2 + 5z^2 + 4xy - 2xz + 4yz$$

- Exemple of Quadratic Form

Covariance matrix, Correlation matrix, Gamma matrix

positive when ${}^t xCx \geq 0$ Diagonal when $C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reminder of linear Algebra (2)

- Scalar product

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad x \cdot y = 1 \times 3 + 2 \times 5$$

(projection)

- Tensorial product

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad x \otimes y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 \times 3 & 1 \times 5 \\ 2 \times 3 & 2 \times 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$$

- Trace

$$Trace \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix} = 3 + 10 = 13$$

(non commutative integration)

- Link beween the precedings :

$$Trace[x \otimes y] = x \cdot y \qquad Trace[(x \otimes y)C] = Trace[C(x \otimes y)] = {}^{t}xCy$$

Reminder of Differential Analysis

- Derivative

$$f(x) \qquad \frac{df}{dx} = Lim\left(\frac{f(x+dx)-f(x)}{dx}\right) \quad \text{when} \quad dx \to 0$$

- Partial Differential

$$f(x,y) \qquad \frac{\partial f}{\partial x} = Lim\left(\frac{f(x+dx,y)-f(x,y)}{dx}\right) \quad \text{when} \quad dx \to 0$$

- Total Diffential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Differential Analysis (2)

- Chain rule for the derivative

$$f(x) = g(h(x)) = g \bullet h(x) \qquad x \xrightarrow{h} y \xrightarrow{g} f(x) \qquad \frac{df}{dx} = \frac{dg}{dy} \times \frac{dh}{dx} \qquad df = \frac{df}{dy} \times dy$$

Notation Abuse

- Chain Rule for the differential

$$f(x_1, x_2) = g(h_1(x_1, x_2), h_2(x_1, x_2))$$

$$\begin{tikzcd}
x_1 \arrow[r, dashed, "h_1"] \arrow[ddr, dashed] & y_1 \arrow[dr, dashed, "g"] & \\
 & & f(x_1, x_2) \\
x_2 \arrow[r, dashed, "h_2"'] \arrow[uur, dashed] & y_2 \arrow[ur, dashed] &
\end{tikzcd}$$

$$\begin{cases} df = \dfrac{\partial g}{\partial h_1} dh_1 + \dfrac{\partial g}{\partial h_2} dh_2 \\ dh_1 = \dfrac{\partial h_1}{\partial x_1} dx_1 + \dfrac{\partial h_1}{\partial x_2} dx_2 \\ dh_2 = \dfrac{\partial h_2}{\partial x_1} dx_1 + \dfrac{\partial h_2}{\partial x_2} dx_2 \end{cases}$$

$$df = \left(\frac{\partial g}{\partial h_1} \frac{\partial h_1}{\partial x_1} + \frac{\partial g}{\partial h_2} \frac{\partial h_2}{\partial x_1} \right) dx_1 + \left(\frac{\partial g}{\partial h_1} \frac{\partial h_1}{\partial x_2} + \frac{\partial g}{\partial h_2} \frac{\partial h_2}{\partial x_2} \right) dx_2$$

another way to write it : $df = \dfrac{\overrightarrow{\partial g}}{\partial h} \cdot \overrightarrow{dh} = {}^t\dfrac{\partial f}{\partial y} \left[\dfrac{\partial y}{\partial x} \right] dx$

Notation Abuse

Jacobian (another quadratic form)

Simultaneous Diagonalization of C and Γ

- $P = \{V_1, V_2, ..., V_n\}$ is an eigen basis for C then $\begin{cases} \Delta' = P^{-1}\Delta \\ C' = P^* C P \\ \Gamma' = P^* \Gamma P \end{cases}$ because we can check :

  - $\Delta'^* C' \Delta' = \Delta^* C \Delta$ and $\Delta'^* \Gamma' \Delta' = \Delta^* \Gamma \Delta$ form all vector Δ

- Now $C' = \begin{bmatrix} \lambda_1 & ... & 0 \\ .. & ... & .. \\ 0 & ... & \lambda_n \end{bmatrix} = \Lambda^2$ because C is definite positive, therfore we can define a new basis in wich $C'' = 1$, and we will have : $\Delta'' = \Lambda\Delta'$, $\Gamma'' = \Lambda^{-1}\Gamma'\Lambda^1$ in order to have :

  - $\Delta''^* \Delta'' = \Delta'^* C' \Delta'$ and $\Delta''^* \Gamma'' \Delta'' = \Delta'^* \Gamma' \Delta'$ again

- We observe than Γ'' is still symetric because : $\Gamma''^* = \Lambda^{-1}\Gamma'^*\Lambda^1 = \Lambda^{-1}(P^*\Gamma P)^*\Lambda^1 = \Gamma''$

- Therefore Γ'' is diagonalizable and its eigen basis is orthonormal