Jump Diffusion Processes

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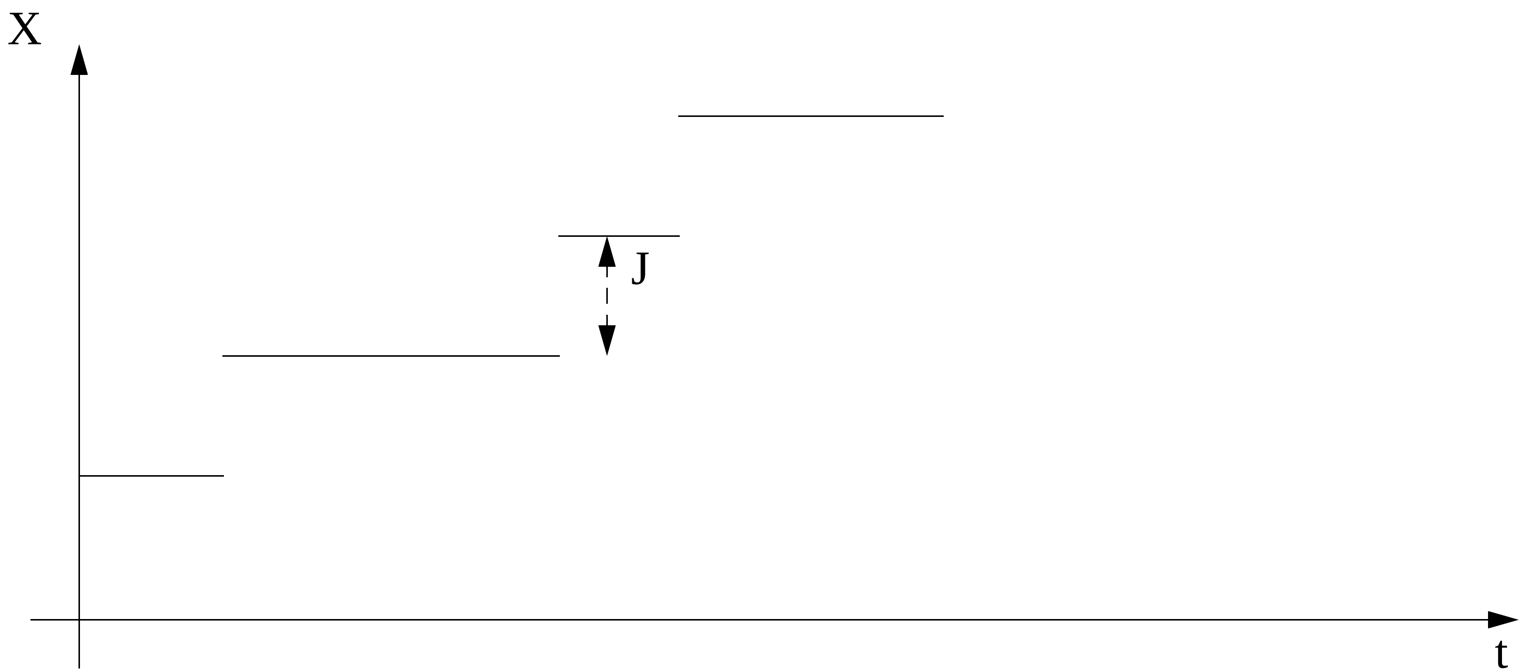
Plan

- Jump Processes : what is it ?
- Stochastic calculus with Jump diffusion processes : Ito lemma
- Calibration of JD Processes : Statistical Methods
- Calibration of JD Processes : Arbitrage Free Approach and The Merton Formula, CAPM, APT, Market Price of Risk,...
- Conclusion

Pure Jump Process (poisson process)

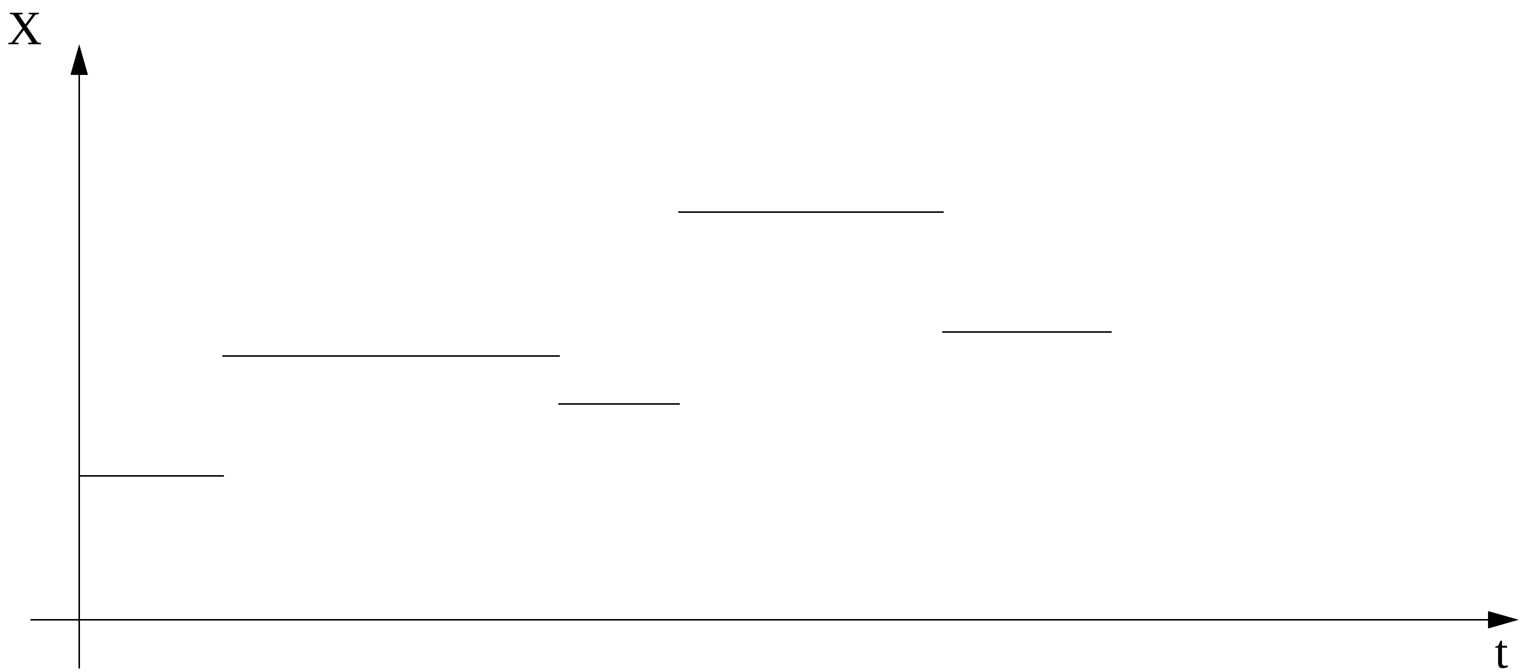
- Pure jumps of fixed size J with an intensity λ :



- $dX_t = Jdq_t$ <=> $X_t = JN_t$: N_t is the counting process : $p(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$

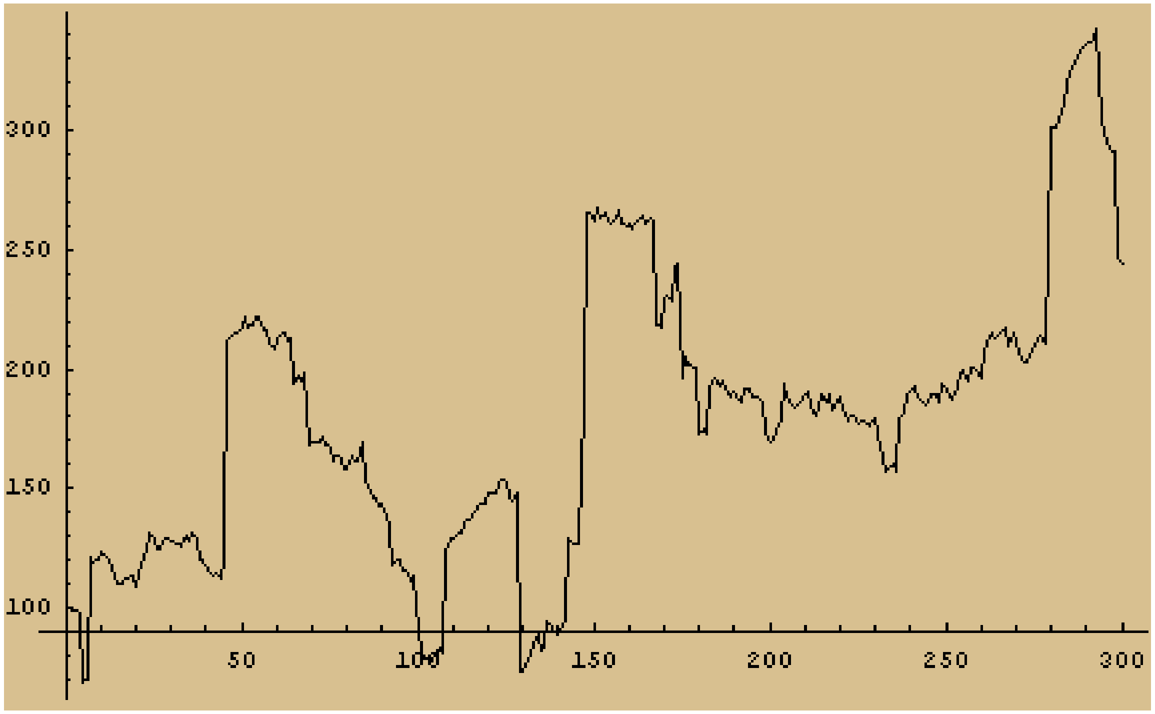
Pure Jump Process

- Pure jumps of Normal size (m,s) with an intensity λ :



- $dX_t = J_t dq_t$ <=> $X_t = \int_0^t J_s dq_s + X_0$ -> : $d(x,t) = (1 - e^{-\lambda t})\delta(x - x_0) + \sum_{k = 1, \infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\frac{e^{-\frac{(x - km)^2}{2ks^2}}}{s\sqrt{2\pi k}}$

Jump Diffusion Process



$$dX = \sigma dW + J dq$$

(m, s) λ

Ito Formula for Ito Processes

- Starting process

$$dX = \mu dt + \sigma dW$$

- image through f

$$f = F(X, t) \qquad df = \left(\frac{\partial F}{\partial X}\mu + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2\right)dt + \frac{\partial F}{\partial X}\sigma dW$$

Ito formula for Ito processes with jumps

- Starting process

$$dX = \mu dt + \sigma dW + J dq$$

- Image through f

$$f = F(X, t) \qquad df = \left(\frac{\partial F}{\partial X}\mu + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2\right)dt + \frac{\partial F}{\partial X}\sigma dW + (F(X^- + J) - F(X^-))dq$$

Ito Formula (integrated)

- The Formula (One Dimension)

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0 < s \le t} \{f(X_s) - (f(X_{s-}) - f'(X_{s-})\Delta X_s)\} + \frac{1}{2}\int_0^t f''(X_{s-})\sigma_s^2 ds$$

- The Formula (N Dimensions)

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0 < s \le t} \left\{ f(X_s) - \left(f(X_{s-}) - \sum_{1 \le j \le N} D_j f(X_{s-})\Delta X^j{}_s \right) \right\} + \frac{1}{2}\int_0^t \sum_{\substack{1 \le i \le N \\ 1 \le j \le N}} D_{ij} f(X_{s-})\rho_{i,j}\sigma_i\sigma_j ds$$

Exemple of application

- Let assume $dS_t = S_t \mu ds + S_t \sigma dW_t + S_t (J_t - 1) dq_t$ where $dq_t = \begin{cases} 0 \text{ with probability} (1-\lambda) dt \\ 1 \text{ with probability } \lambda dt \end{cases}$

- Let apply Ito to Log[S] , this is equivalent to

$$d(Log[S_t]) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t + (Log[S_t + (J_t - 1) S_t] - Log[S_t]) dq_t$$

- with that we simplify :

$$d(Log[S_t]) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t + Log[J_t] dq_t$$

Another Exemple

- Let assume $dS_t = S_t \mu dt + S_t \sigma dW_t + S_t (J_t - 1) dq_t$ where $dq_s = \begin{cases} 0 \text{ with probability} (1-\lambda)ds \\ 1 \text{ with probability } \lambda ds \end{cases}$

- Let apply Ito to f[S,t] , we have

$$df(S_t) = \left(f_x S_t \mu + f_t - \frac{1}{2} f_{xx} (S_t \sigma)^2\right) dt + f_x S_t \sigma dW_t + (f(S_{t-} J_t) - (f(S_{t-}))) dq_t$$

- which is equivalent to :

$$f(S_t) = f(S_0) + \int_0^t \left(\frac{\partial f}{\partial S} S_s \mu + \frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} S_s^2 \sigma^2\right) ds + \int_0^t \frac{\partial f}{\partial S} S_s \sigma dW_s + \int_0^t (f(S_{s-} J_s) - f(S_{s-})) dq_s$$

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Warning ! The process $\int_0^t \frac{\partial f}{\partial S} S_s \sigma dW_s$ is a martingale, but $\int_0^t (f(S_{s-} J_s) - f(S_{s-})) dq_s$ is not !!

Simulation of a JD Process for a Brownian Motion

- Let assume $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (J-1)dq_t$ then $S_t = S_0 e^{\int_0^t \left\{\left(\mu - \frac{\sigma^2}{2}\right)ds + \sigma dW_s + Log(J_s)dq_s\right\}}$

- Therefore $S_t \sim S_0 e^{N\left[\left(\mu - \frac{\sigma^2}{2}\right)t,\, \sigma^2 t\right] + NP[\lambda, \varepsilon, \delta^2, t]}$ where $NP[\lambda, \varepsilon, \delta^2, t]$ means the value of a poisson process of parameter λ at time t and a jump which is normal with parameters ε and δ^2

- To Simulate $NP[\lambda, \varepsilon, \delta^2, t]$ we first conditionate by the number of jumps n and simulate the conditional variable: $NP[n, \varepsilon, \delta^2] = NP[\lambda, \varepsilon, \delta^2, t]\big|n_{\lambda, t} \sim N[n\varepsilon, n\delta^2]$

- So $S_t \sim S_0 e^{N\left[\left(\mu - \frac{\sigma^2}{2}\right)t + n_{\lambda, t},\, \varepsilon,\, \sigma^2 t + n_{\lambda, t}\delta^2\right]}$ and to simulate S_t, we first simulate $n_{\lambda, t}$ then we simulate the exponential of a normal law

- The simulation of the counting process $n_{\lambda, t}$ uses the density $p(n) = e^{-\lambda(1+\varepsilon)T}\left(\frac{(\lambda(1+\varepsilon)T)^n}{n!}\right)$

Calibration (Standard)

- $\left(dS_t = \left(\alpha - \lambda\varepsilon - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t + Log[J]dq_t\right) \Leftrightarrow \left(\frac{d\left(e^{S_t}\right)}{e^{S_t}} = (\alpha - \lambda\mu_0)dt + \sigma dW_t + (J-1)dq_t\right)$ where $Log[J] \sim N[\varepsilon, \delta^2]$

  - => Density : $p(x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\Phi(x, \mu + n\varepsilon, \sigma^2 + n\delta^2)$ of $x = Log\left[\frac{S_t}{S_{t-1}}\right]$

- Cumulants (moments) = $C_1 = M_1 \quad C_2 = M_2 - (M_1)^2$
  $C_3 = M_3 - 3M_1M_2 + 2(M_1)^3 \quad C_4 = M_4 - 4M_3M_1 - 3M_2^2 + 12M_2(M_1)^2 - 6(M_1)^4$

  - => Cumulants : $\begin{bmatrix} C_1 = \lambda\varepsilon \\ C_2 = \sigma^2 + \lambda(\varepsilon^2 + \delta^2) \\ C_3 = \lambda\varepsilon(\varepsilon^2 + 3\delta^2) \\ C_4 = \lambda(\varepsilon^4 + 6\varepsilon^2\nu^2 + 3\delta^4) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \left(x^4 - \frac{2C_3}{C_1}x^2 + \frac{3}{2}\frac{C_4}{C_1}x - \frac{C_3^2}{2C_1^2} = 0\right) \to \varepsilon = \text{real root x / } (\lambda > 0) \\ \lambda = \frac{C_1}{x} \\ \delta^2 = \frac{C_3 - x^2C_1}{3C_1} \\ \sigma^2 = C_2 - \frac{C_1}{x}\left(x^2 + \frac{C_3 - x^2C_1}{3C_1}\right) \end{bmatrix}$

  - => Parameters (cumulants)

Maximum Likelihood Method

- In Theory
  - definition of a "probability" on the parameters: $Jdens_P(\{x_0, x_1, x_2, ..., x_n\}) = Dens(P)$
  - Most Likely P<=>Max[density]<=>{derivatives=0}<=>{derivatives[Log]=0}
  - In case of independent processes $\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}$

$$Jdens_P(\{x_0, x_1, x_2, ..., x_n\}) = \prod_{1 \le n \le T} Cdens_{P,n}(x_n | x_{n-1})$$

$$\frac{\partial}{\partial P} \sum_{1 \le n \le T} Log[Cdens_{P,n}(x_n | x_{n-1})] = 0$$

- In Practice a two steps process
  - simplification : $L[S_1] = L[S_2]$

$$\underbrace{\{x_0, x_1, x_2, ..., x_n\}}_{S_1} \Leftrightarrow \underbrace{\{x_0, y_1, y_2, ..., y_n\}}_{S_2} \qquad y_{n+1} = x_{n+1} - x_n - b_n(S_n - x_n)$$

  - computation of the Likelihood

$$L[S_2] = \sum_{1 \le n \le T} Log[dens(y_n | x_n)]$$

Calibration with the maximum likelihood estimator

- The process is

$$f(t+dt,T)-f(t,T) = s(h-f(t,T))dt + \sigma dW_t + J_t dq_t$$

  - the no jumps conditional density of the process is given by:

$$\varphi_0(f_{n+1},f_n,t) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(f_{n+1}-f_n-s(h-f_n))^2}{2\sigma^2}}$$

  - the conditional density with 1 jump of type 1 of the process is given by:

$$\varphi_1(f_{n+1},f_n,t) = \frac{1}{\sqrt{2\pi(\sigma^2+\rho^2)}}e^{-\frac{(f_{n+1}-f_n-s(h-f_n)-\varepsilon)^2}{2(\sigma^2+\rho^2)}}$$

  - The Log Likelihood is :

$$L = -\sum_n Log[(1-\lambda)\varphi_0(f_{n+1},f_n,t) + \lambda\varphi_1(f_{n+1},f_n,t)]$$

  - The critical point equation are :

$$\frac{\partial L}{\partial \sigma} = \frac{\partial L}{\partial s} = \frac{\partial L}{\partial h} = \frac{\partial L}{\partial \varepsilon} = \frac{\partial L}{\partial \rho} = \frac{\partial L}{\partial \lambda}$$

A more robust Calibration (pragmatic)

- Separation of the jumps from the diffusion (approximatif)

$$\Delta f_n = f_{n+1} - f_n \qquad \begin{cases} \Delta f^j_n = \Delta f_n 1_{|\Delta f_n| > l} \\ \Delta f^c_n = \Delta f_n 1_{|\Delta f_n| \le l} \end{cases}$$

- Calibration of the jumps : maximum of likelihood

  - We assume a normal distribution $N(0, \rho)$ the likelihood is

$$L = \sum_{Jump[n]} \frac{-(\Delta f^j_n)^2}{2(\rho)^2} - Log[\rho] \quad \text{and the solution is} \quad \rho = \sqrt{\sum_{Jump[n]} \frac{\left(\Delta f^j_n - \overline{\Delta f^j_n}\right)^2}{N_{jumps}}}$$

- Calibration of the continuous part: maximum of likelihood

  - geometrical brownian motion $\sigma = \sqrt{\sum_{NoJump[n]} \frac{\left(\Delta f^c_n - \overline{\Delta f^c_n}\right)}{N_{Nojumps}}}$

Jump Diffusion Formula For a Brownian Motion

- Let assume that the forward price is following :

$$F_{T,t} = F_{T,0} + \int_0^t (F_{T,s}\mu ds + F_{T,s}\sigma dW_s + F_{T,s}(J_s - 1)dq_s) \text{ where } dq_s = \begin{cases} 0 \text{ with probability}(1-\lambda)ds \\ 1 \text{ with probability } \lambda ds \end{cases}$$

- We also assume that the jump distribution conditional to it to appear is normal.

- We want to compute the value of an European call, maturity T, strike price K with $\varepsilon = E[Y-1]$ and δ^2 is the variance of the size of the jump conditional to this one to occur: $\delta^2 = E[(Y-1)^2 | \Delta S > 0] - \varepsilon^2$

- $Call = e^{-\lambda(1+\varepsilon)T} \sum_n \frac{(\lambda(1+\varepsilon)T)^n}{n!} \left[F_{T,0} e^{b_n T} N(d_{1,n}) - KN(d_{2,n})\right]$ where

  - $b_n = -\lambda + nLog[1+\varepsilon]$, $d_{1,n} = \dfrac{Log\left[\frac{F}{K}\right] + b_n T + \frac{1}{2}(\sigma^2 T + n\delta^2)}{\sqrt{\sigma^2 T + n\delta^2}}$ and $d_{2,n} = d_{1,n} - \sqrt{\sigma^2 T + n\delta^2}$

  (Merton Formula)

Jump Diffusion Model For a Brownian Explained

- Structure :

$$Call = \sum_n e^{-\lambda T}\frac{(\lambda T)^n}{n!}(e^{-\lambda\varepsilon T}(1+\varepsilon)^n)[Fe^{b_n T}N(d_{1,n}) - KN(d_{2,n})]$$

Prob(n Jumps)

$$E_N[Max\{F_T - X, 0\} \mid \text{n Jumps}]$$

- Drift coming from non zero expectation of the jumps : $b_n = nLog[1+\varepsilon]$ If no arbitrage

$$b_n = -\lambda + nLog[1+\varepsilon]$$

Drift of the risk neutral

- Volatility spread coming from the jumps : $d_{1,n} = \dfrac{Log\left[\frac{F}{K}\right] + b_n T + \frac{1}{2}(\sigma^2 T + n\delta^2)}{\sqrt{\sigma^2 T + n\delta^2}}$

Volatility spread

Jump Diffusion Model For a Brownian : Arbitrages

- Equilibrium between the spot price and the forward price :

$$(dS_t = S_t\mu dt + S_t\sigma dW_t + S_t(J_t - 1)dq_t) \Leftrightarrow (dF_{T,t} = F_{T,t}(\mu - r_t + y_t)dt + F_{T,t}\sigma dW_t + F_{T,t}(J_t - 1)dq_t)$$

because $F_{T,t} = S_t e^{\int_t^T (r_s - y_s)ds}$ by arbitrage

- Risk neutral equilibrium with the bond prices : $\left(\frac{dB_{T,t}}{B_{T,t}} = r_t dt\right) \Leftrightarrow \left(E_{NR}\left[\frac{dS_t}{S_t}\right] = r_t dt\right)$ implies

that :$\mu + \lambda E[J_t - 1] = r_t$ or with our preceding notation : $\mu = r_t - \lambda\varepsilon$

- Therefore the risk neutral equations are :

$$\begin{aligned} dS_t &= S_t(r_t - \lambda\varepsilon)dt + S_t\sigma dW_t + S_t(J_t - 1)dq_t \\ dF_{T,t} &= F_{T,t}(y_t - \lambda\varepsilon)dt + F_{T,t}\sigma dW_t + F_{T,t}(J_t - 1)dq_t \end{aligned}$$

Origin of the Jump Diffusion Formula and Generalisation

- If $dS_t = S_t\mu_t dt + S_t\sigma_t dW_t + S_t(J_t - 1)dq_t$ then $S_t = S_0 Exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right)ds\right] Exp\left[\int_0^t \sigma_s dW_s\right] Y_t$

  where the variable Y_t follows $Y_t = Y_{n_t} \equiv \prod_{i=1}^{n_t} Y_i$ with n_t is poisson distributed with a parameter

  equal to $\lambda_Y = \int_0^t \lambda_s ds$ and Y_j is a sequence of independent variable distributed like J

- Then the option formula looks like

$$Call = \sum_n e^{-\lambda_Y} \frac{(\lambda_Y)^n}{n!} E[S_0 Y_{n_t} N(d_1) - KN(d_2) | (n_t = n)]$$

$$\boxed{\text{Assumptions : } \mu_s, \sigma_s, \lambda_s \text{ deterministic}}$$

- with $d_1 = \left(Log\left[\frac{S_0}{K}\right] + \int_0^t \left(\mu_s + \frac{\sigma_s^2}{2}\right)ds\right) \Big/ \int_0^t \frac{\sigma_s^2}{2} ds$ and $d_2 = d_1 - \int_0^t \frac{\sigma_s^2}{2} ds$

Jump Diffusion : Hedging the Option

- Let the hedged portfolio : $\Pi = V(S,t) - \Delta S$ by applying Ito we get :

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS + (V[JS] - V[S] - \Delta(J-1)S)dq$$

- If we hedge only the diffusion, $\Delta = \frac{\partial V}{\partial S}$, we can adjust $E[d\Pi] = rdt$ we get the classical jump diffusion option formula (Merton 1976)

- We can try to find Δ to minimise the variance of $d\Pi$ and then equate the expectation of $d\Pi$ to the risk free rate . We find : $\Delta = \dfrac{\lambda E[(J-1)(V(JS) - V(S))] + \sigma^2 S \frac{\partial V}{\partial S}}{\lambda S E[(J-1)^2] + \sigma^2 S}$ and We get an equation :

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}\left(\mu - \frac{\sigma^2}{d}(\mu + \lambda\varepsilon - r)\right) - rV + \lambda E\left[(V(SJ) - V(S))\left(1 - \frac{J-1}{d}(\mu + \lambda\varepsilon - r)\right)\right] = 0$$

- Intregro-differential (because of E[]) to solve with fourier or laplace methods.

The Market Price of Risk (1)

- CAPM Type of reasoning (optimality of mean/variance criterium) =>

Excess of return over the risk free rate = Market price of Risk X (Risk)

$$E(S_i) - r = \underbrace{\frac{E(Market) - r}{\sigma(Market)}}_{\text{Market Price of risk}} \underbrace{\sigma(S_i)}_{\text{risk associated with } S_i}$$

- Instantaneous view , One Dimensional Risk :

$$\left(\frac{\partial}{\partial t} + \underbrace{\mu X \frac{\partial}{\partial \boldsymbol{X}} + \frac{1}{2}(X\sigma)^2 \frac{\partial^2}{\partial X^2}}_{\text{Generator of the process X}}\right) - r = \underbrace{\lambda}_{\text{Market price of normalized risk}} \underbrace{\frac{\partial}{\partial \boldsymbol{X}}}_{\text{Normalized Risk}}$$

Applied to any security

- True Arbitrage Free Approach => Market price of risk should be determined implicitly

Market Price of Risk (2)

- APT Reasoning : Extension to multidimensional risks based on arbitrage free ideas:

$$E(S_i) - r = \lambda_1 \frac{\partial S_i}{\partial I_1} + \lambda_1 \frac{\partial S_i}{\partial I_2} + \dots . + \lambda_k \frac{\partial S_i}{\partial I_k}$$

Sensitivity to the kth factor

kth price of risk

- Instantaneous Relationship valid for stochastic arbitrage-free theory

$$\left(\frac{\partial}{\partial t} + \underbrace{\sum_n X_n \mu_n \frac{\partial}{\partial \boldsymbol{X}_n} + \frac{1}{2} \sum_{n,\, m,\, l} X_n X_m \sigma_{n,\, l} \sigma_{m,\, l} \frac{\partial^2}{\partial X_n \partial X_m}}_{\text{Generator of the process X}}\right) - r = \lambda_1 \frac{\partial}{\partial I_1} + \lambda_1 \frac{\partial}{\partial I_2} + \dots . + \lambda_k \frac{\partial}{\partial I_k}$$

Sensitivity to the kth factor

kth price of risk

- Radical Arbitrage Free theory => all market prices are determined implicitly

- Mixed approach => Historical determination of the market prices of risks : $\lambda_k = E[I_k] - r$

Market Price of Risk (3)

- A market with jumps : $d(Log[X_t]) = \mu dt + \sigma dW_t + Log[J_t]dQ_t$

  - is associated with two sources of risks: diffusions and jumps

- It is natural to suppose that arbitragefreeness implies that

$$\left[\frac{\partial}{\partial t} + Generator[Log[X]] - r\right]S = \lambda_W \sigma S \frac{\partial}{\partial X} S + \lambda_Q E[S(JX) - S(X)]$$

- The Generator is defined by $S(X_t) - \int_0^t (Generator[X] \cdot S)dt$ is a martingale wich is an extension of the idea $Generator[X] \equiv \frac{d(E[X])}{dt}$ for jumps and stochastic variables.

- Therefore we will define the generator of a jump diffusion by :

$$Generator[X] \cdot S = \mu S \frac{\partial}{\partial \boldsymbol{X}} S + \frac{1}{2}(S\sigma)^2 \frac{\partial^2}{\partial X^2} S + \lambda E[S(JX) - S(X)]$$

Arbitrage Free Jump Diffusion

- If we apply the equation :

$$\frac{\partial}{\partial t}S + \mu S\frac{\partial}{\partial \boldsymbol{X}}S + \frac{1}{2}(S\sigma)^2\frac{\partial^2}{\partial X^2}S + \lambda E[S(JX) - S(X)] - rS - \left(\lambda_W\sigma S\frac{\partial}{\partial X}S + \lambda_Q E[S(JX) - S(X)]\right) = 0$$

- to the security X if it is traded : $\mu E[X] - rE[X] - (\lambda_W\sigma E[X] + (\lambda_q - \lambda)E[JX - X]) = 0$
- If we assume independence between J and X We get : $\mu - r - \lambda_W\sigma - (\lambda_q - \lambda)E[J - 1] = 0$
- Therefore, we need another market price to identify λ_q and λ_W
-

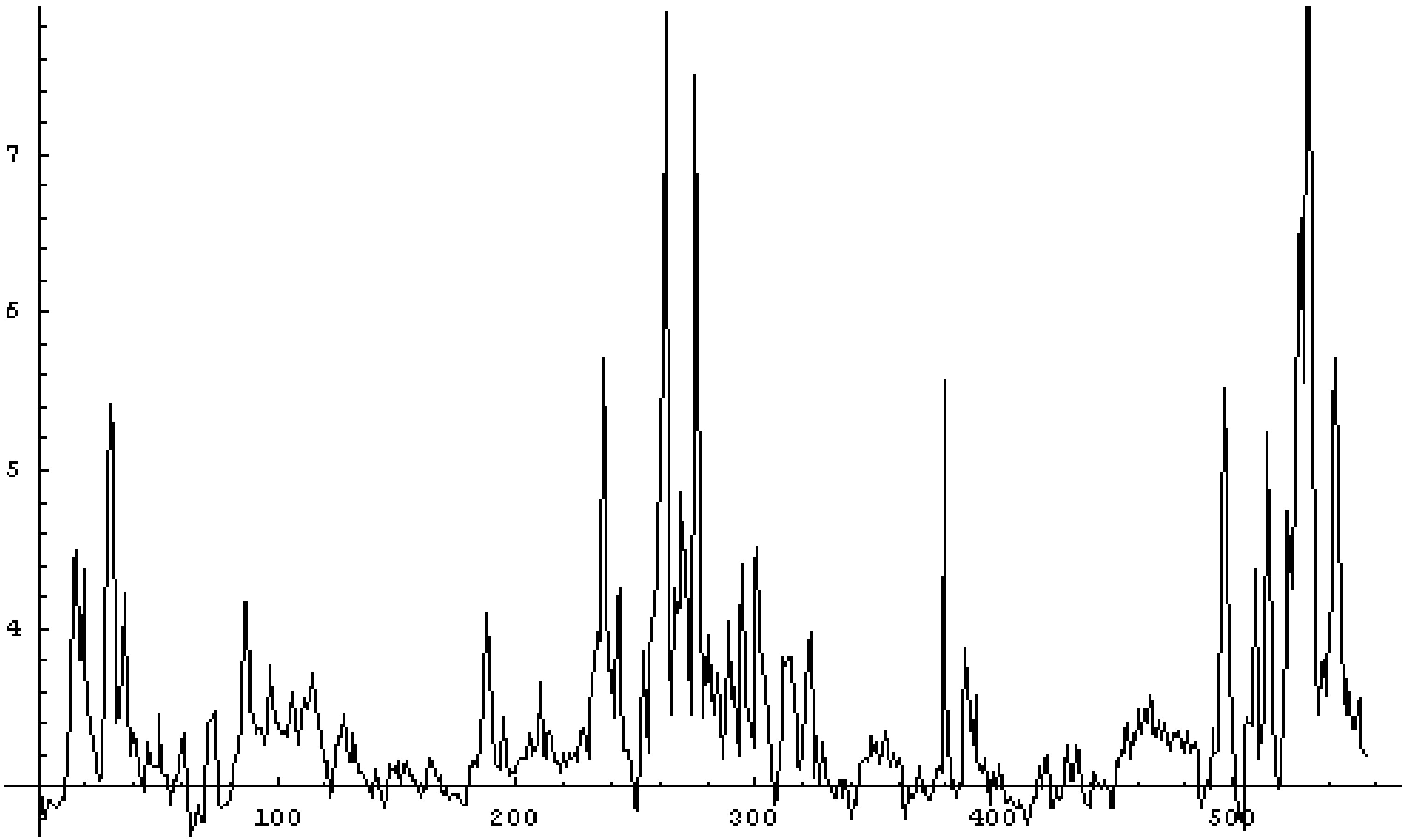
$\lambda_q = 0$ => Merton Model

$\lambda_q \neq 0$ => Arbitrage Free Model with an implicit λ_q

Jump Diffusion Models : List of Applications

- Equity Stocks with uncertain dividends -> Derivatives
- Equity Option Markets -> Risk of Crash -> Smile Modeling
- Emerging country FX option -> Risk of Devaluation
- Defaultable Security -> Derivatives
- Electricity Markets -> Forward Curve Simulation and Options

Application : Electricity Markets



- Log of Spot price Florida 97-98

Conclusion

- Jump Diffusion Models are tractable. They may be arbitrage free. But the calibration and the implicit parameters determination are delicate.

- A key concept is the market price of risk, it is a generalization of the CAPM anf APT idea

- All classical concepts have a natural generalization : Ito lemma, complete markets, ...

- A very hot application is the electricity markets

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