Volatility Surfaces and

Local Volatility

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Plan

- 1) Volatility : Analysis and Synthesis
  - Gamma: Sensitivity to bad volatility
  - The static and dynamic replication of volatility
  - A dynamic replication of variance localized in strike
- 2) Implicit Diffusion
  - Backward and forward equations
  - Backward and forward transition probability
  - Local volatility
- 3) Stochastic Volatility Modeling
  - Static fitting of the smile by imcomplete beta function
  - Hull and White approach
  - Heston Approach
- Conclusion

Phenomenology

- Standard Model :

$$\frac{dS_t}{S_t} = (r_t - \delta_t)dt + \sigma_t dW_t$$

- Call Pricing :

$$Call(K, T) = B_T(F_T N(d_1) - KN(d_2)) \qquad d_1 = \frac{Log\left[\frac{F_T}{K}\right] + \frac{1}{2}V_T^{\ 2}}{V_T} \qquad d_1 = d_2 - \frac{V_T}{2} \qquad V_T = \sqrt{\int_0^T \sigma_s^2 ds}$$

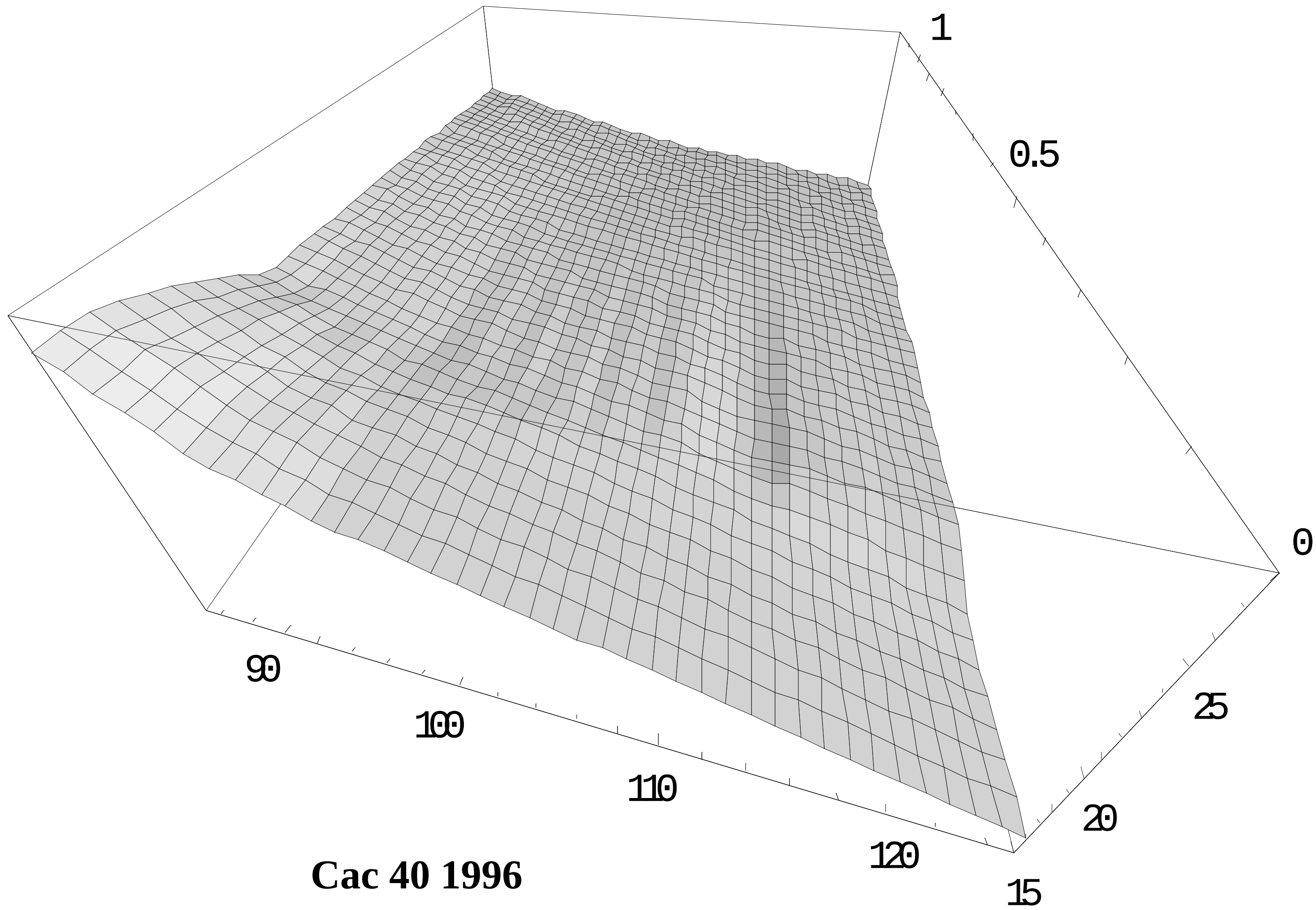
- Market Prices

$$Call(K, T) = B_T(F_T N(d_1) - KN(d_2)) \qquad V_T = V(K, T)$$

<u>implicit volatility = wrong number in the wrong model to get right price</u>

<u>Volatility Surfaces = Market Prices</u>

A Real Volatility Surface



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P&L of a Hedged Call

- Variation of the instantaneous P&L

$$d(\text{P\&L}) = d(C - \Delta S) = \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{\partial C}{\partial t}dt$$

- But the realized volatility

$$\sigma_{\text{realized}}\sqrt{dt} = \frac{|dS|}{S}$$

- The link between θ and Γ in a black and sholes formula is : (r=0, the S is the forward price)

$$\Gamma = \frac{\varphi(d_1)}{S\sigma\sqrt{T}} \qquad \theta = \frac{S\sigma\varphi(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2) = \frac{1}{2}\sigma^2 S^2 \Gamma$$

- We rewrite the instantaneous P&L

$$d(\text{P\&L}) = \frac{1}{2}\Gamma S^2(\sigma^2_{\text{realized}} - \sigma^2_{implicit})dt$$

Average P&L

- If $\varphi(\sigma, S, t)$ is the true joint probability density of the stock with the volatility, and $\Gamma_0 = \Gamma(\sigma_0)$ with σ_0 being the specified volatility

$$Avg[\text{P\&L}] = Avg[\text{P\&L,t=0}] + \frac{1}{2}\int\int E[\Gamma_0 S^2(\sigma^2 - \sigma_0^2)\big|S]\varphi\, dS\, dt$$

- If the vol process is deterministic:

$$Avg[\text{P\&L}] = Avg[\text{P\&L,t=0}] + \frac{1}{2}\int\int S^2(\sigma^2 - \sigma_0^2)E[\Gamma_0\big|S]\varphi\, dS\, dt$$

- If we look at european options and the vol is stochastic

$$Avg[\text{P\&L}] = Avg[\text{P\&L,t=0}] + \frac{1}{2}\int\int (\Gamma_0 S^2(E[\sigma^2\big|S] - \sigma_0^2))\varphi\, dS\, dt$$

Quizz

$$Avg[\text{P\&L}] = Avg[\text{P\&L},t=0] + \frac{1}{2}\int\int E[\Gamma_0 S^2(\sigma^2 - \sigma_0^2) \big| S]\varphi dSdt$$

- Buy an european option at 20% vol
- realized historical vol is 25%
- Have you made money ?

Not Necessarily !

- High vol with low gamma, low vol with high gamma --> you loose !

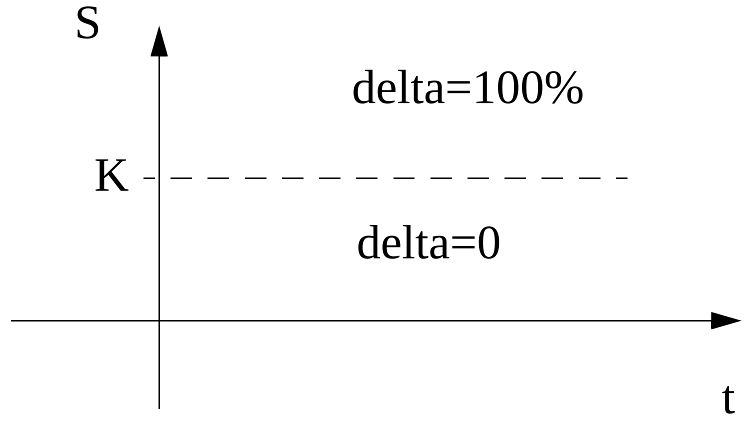
An Extreme Case

- Put $\sigma_0 = 0$ in the formula ,this implies that $\Gamma_0 = \delta(S-K)$ and you get the Tanaka formula

$$Avg[\text{P\&L}] = (S_0 - K)^+ + \frac{1}{2}\int K^2\sigma^2(K,t)\varphi(K,t)dt$$

- This is just the intrinsic value of the option plus the average price of the volatility effect.

- The hedging strategy is in this case the stop loss start gain strategy :



Completude of the Market

- Any terminal Payoff $f(F_T)$ can be replicated with calls , puts , and cash :

$$f(F) = \int_0^{\infty} \delta(F-K) f(K) dK = \int_0^{\kappa} \delta(F-K) f(K) dK + \int_{\kappa}^{\infty} \delta(F-K) f(K) dK$$

- by integrating by part twice we get the static replication formula :

$$\boxed{f(F) = \underbrace{f(\kappa) + f'(\kappa)[(F-\kappa)^+ - (\kappa - F)^+]}_{\text{conversion}} + \underbrace{\int_0^{\kappa} (K-F)^+ f''(K) dK}_{\text{put}} + \underbrace{\int_{\kappa}^{\infty} (F-K)^+ f''(K) dK}_{\text{call}}}$$

Static Replication of the Future Variance

- Let consider a forward contract F_T at maturity whose value today is : F_0 .

$$Vol^2{}_{[0,\,T]} = Var\left[Log\left[\frac{F_T}{F_0}\right]\right]$$

- If the payoff is $f(F) = Log\left[\frac{F}{F_0}\right]$. we apply the static replication formula , $f''(F) = -\frac{1}{K^2}$

$$H = E\left[Log\left[\frac{F_T}{F_0}\right]\right] = -\int_0^{F_0}\frac{1}{K^2}P_0(K,\,T)dK - \int_{F_0}^{\infty}\frac{1}{K^2}C_0(K,\,T)dK$$

- If the payoff is $\left(Log\left[\frac{F}{F_0}\right] - H\right)^2$, we reapply the formula to get :

$$Vol^2{}_{[0,\,T]} = \int_0^{F_0e^H}\frac{2}{K^2}\left(1-\left(Log\left[\frac{K}{F_0}\right]+H\right)\right)P_0(K,\,T)dK + \int_{F_0e^H}^{\infty}\frac{2}{K^2}\left(1-\left(Log\left[\frac{K}{F_0}\right]+H\right)\right)C_0(K,\,T)dK$$

- Which statically hedge any claim on volatility

Dynamic Hedging

- Applying Ito to the forward value of a claim $e^{r(T_1 - t)}V(t, \sigma_h)$:

$$V(T_1) = V(T)e^{r(T_1 - T)} + \int_T^{T_1} e^{r(T_1 - t)}\frac{\partial V}{\partial F}dF_t + \int_T^{T_1} e^{r(T_1 - t)}\left(-rV + \frac{\partial V}{\partial t} + F^2\sigma_t^2\frac{\partial^2 V}{\partial F^2}\right)dt$$

- where the volatility σ_h is just a parameter of th pricing function V , a priori different fro m the real volatility σ_t.

- By definition, V solve $-rV + \frac{\partial V}{\partial t} = -\frac{\sigma_h^2 F^2}{2}\frac{\partial^2 V}{\partial F^2}$ with terminal value $V(F, T_1, \sigma_h) = f(F)$

- So we get

$$\boxed{f(F_{T_1}) + \underbrace{\int_T^{T_1} e^{r(T_1 - t)}F^2\frac{\partial^2 V}{\partial F^2}(\sigma_h^2 - \sigma_t^2)dt}_{\text{error due to bad volatility}} = e^{r(T_1 - T)}V(F_T) + \int_T^{T_1} e^{r(T_1 - t)}\frac{\partial V}{\partial F}dF_t}$$

- So applying a hedging strategy, we miss the terminal value by an amount that we know

Stop Losses, Start Gain Strategy again

- Suppose that $\sigma_h = 0$ in the preceding formula:

$$\int_T^{T_1} \frac{F_t^2}{2} f''(F_t)\sigma_t dt = f(F_{T_1}) - f(F_T) + \int_T^{T_1} f'(F_t) dF_t$$

- We are going to apply it to 3 cases :

Table 1:

Description	$f''(F_t)$	payoff at T_1
Variance over a future period	$\frac{2}{F_t^2}$	$\int_T^{T_1} \sigma_t^2 dt$
Future Corridor variance	$\frac{2}{F_t^2} 1_{F_t \in [\kappa - \Delta\kappa, \kappa + \Delta\kappa]}$	$\int_T^{T_1} \sigma_t^2 1_{F_t \in [\kappa - \Delta\kappa, \kappa + \Delta\kappa]} dt$
Future Variance along a strike	$\frac{2}{F_t^2} \delta(F_t - \kappa)$	$\int_T^{T_1} \sigma_t^2 \delta(F_t - \kappa) dt$

Contract paying a future variance

- let consider the payoff function $f(F) = 2\left(Log\left[\frac{\kappa}{F}\right] + \frac{F}{\kappa} - 1\right)$ we see that the first derivative is : $f'(F) = 2\left(\frac{1}{\kappa} - \frac{1}{F}\right)$ and the second derivative is $f''(F) = \frac{2}{F^2}$ so by applying the preceding formula we get :

$$\int_T^{T_1} \sigma_t^2 dt = f(F_{T_1}) - f(F_T) - 2\int_T^{T_1} \left(\frac{1}{\kappa} - \frac{1}{F_t}\right) dF_t$$

- The initial cost is given by static replication :

$$f\left(F_{T_1}\right) - f(F_T) = \int_0^{\kappa} \frac{1}{K^2} P_0(K, T_1) dK + \int_{\kappa}^{\infty} \frac{1}{K^2} C_0(K, T_1) dK - e^{-r(T_1 - T)}\left(\int_0^{\kappa} \frac{1}{K^2} P_0(K, T) dK + \int_{\kappa}^{\infty} \frac{1}{K^2} C_0(K, T) dK\right)$$

- So the investor is assumed to start a dynamic strategy at T up to T_1

Contract paying a future Corridor Variance

- let consider the payoff function $f(F) = 2\left(Log\left[\frac{\kappa}{F^*}\right] + F\left(\frac{1}{\kappa} - \frac{1}{F^*}\right)\right)$ where $F^* = Max[\kappa - \Delta\kappa, Min[F, \kappa + \Delta\kappa]]$ represent F Floored and Capped. we see that the first derivative is : $f'(F) = 2\left(\frac{1}{\kappa} - \frac{1}{F^*}\right)$ and the second derivative is $f''(F) = \frac{2}{F_t^2} 1_{F_t \in [\kappa - \Delta(\kappa, \kappa + \Delta\kappa)]}$ so by applying the preceding formula we get :

$$\int_T^{T_1} \sigma_t^2 1_{F_t \in [\kappa - \Delta\kappa, \kappa + \Delta\kappa]} dt = f(F_{T_1}) - f(F_T) - 2\int_T^{T_1} \left(\frac{1}{\kappa} - \frac{1}{F_t^*}\right) dF_t$$

- The initial cost is given by static replication :

$$f\left(F_{T_1}\right) - f(F_T) = \int_{\kappa - \Delta\kappa}^{\kappa} \frac{1}{K^2} P_0(K, T_1) dK + \int_{\kappa}^{\kappa + \Delta\kappa} \frac{1}{K^2} C_0(K, T_1) dK - e^{-r(T_1 - T)} \left(\int_{\kappa - \Delta\kappa}^{\kappa} \frac{1}{K^2} P_0(K, T) dK + \int_{\kappa}^{\kappa + \Delta\kappa} \frac{1}{K^2} C_0(K, T) dK\right)$$

- So the investor is assumed to start a dynamic strategy at T up to T_1.

- The integrand doesn't vary outside the window for F => no trading (Semi-Static).

Contract paying a Future Variance Localized in Strike

- The game is to make $\Delta\kappa$ --> 0 in the preceding formula. In order to get a non zero value, we multiply the notional by $\frac{1}{\Delta\kappa}$.

- The delta strategy is simply $\frac{e^{-r(T_1 - t)}}{\kappa^2} Sgn[F_t, \kappa]$

- The initial cost is the ratioed calendar spread $\frac{1}{\kappa^2}[V_0(\kappa, T_1) - e^{-r(T_1 - T)} V_0(\kappa, T)]$ where $V_0(\kappa, T) = P_0(\kappa, T) + C_0(\kappa, T)$ is the straddle struck at κ and maturing at T.

- => The replication $:\int_T^{T_1} \sigma_t^2 \delta(F_t - \kappa) dt = \frac{1}{\kappa^2}\left[V_0(\kappa, T_1) - e^{-r(T_1 - T)} V_0(\kappa, T)\right] - \int_T^{T_1} \frac{e^{-r(T_1 - t)}}{\kappa^2} Sgn[F_t - \kappa] dF_t$

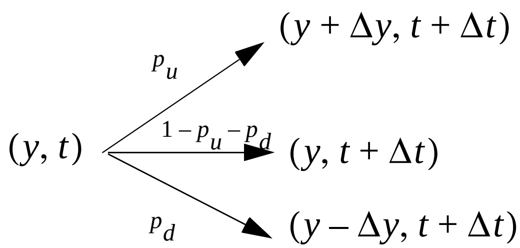
- Which is just an avatar of the Tanaka-Meyer formula :

  $|G_t - \kappa| = |G_0 - \kappa| + \int_0^t Sgn[G_s - \kappa] dM_s + 2\Lambda_t$ where $G_t = f(F_t)$ and $\Lambda_t = \int_0^t \delta(G_s - \kappa) d\langle M\rangle_s^2$ is the local time. This local time can be synthetized by straddles and a st.loss/st.gain strate.

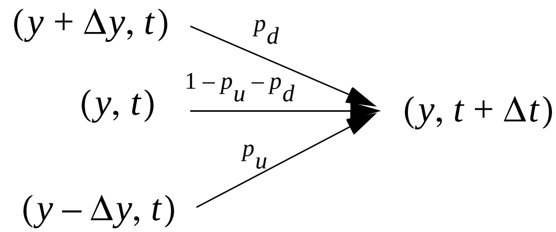
Forward and Backward Equations

- Let Assume a Diffusion $dy_t = \mu dt + \sigma dW_t$ is represented by a trinomial process:

backward framework



forward framework



- Matching of the two First moments

$$P_u \Delta y + (1 - p_u - p_d)0 + p_d(-\Delta y) = \mu \Delta t$$

$$P_u(\Delta y - \mu_1)^2 + (1 - p_u - p_d)(0 - \mu_1)^2 + p_d((-\Delta y) - \mu_1)^2 = \sigma^2 \Delta t$$

- Conservation of probability : p(x,s,y,t) is the transition probability

Forward $p(x, s, y, t + \Delta t) = p_d p(x, s, y + \Delta y, t) + (1 - p_u - p_d) p(x, s, y, t) + p_u p(x, s, y - \Delta y, t)$

=> Forward Equation (Fokker Planck) $\left(\frac{\partial}{\partial t} + (r - \delta) S \frac{\partial}{\partial S} + \frac{1}{2} \sigma^2_{S,t} S^2 \frac{\partial^2}{\partial S^2}\right) p(t, S, t_1, S_1) = 0$

Backward $p(y, t, x, s) = p_d p(y + \Delta y, t + \Delta t, x, s) + (1 - p_u - p_d) p(y, t + \Delta t, x, s) + p_u p(y - \Delta y, t + \Delta t, x, s)$

=> Backward Equation $\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial S_1}(r - \delta) S_1 - \frac{1}{2} \frac{\partial^2}{\partial S_1^2} \sigma^2_{S_1, t_1} S_1^2\right) p(t, S, t_1, S_1) = 0$

The Forward Transition Probability (FTP)

- As any markov process, the security prices have a forward transition probability :

$$\boxed{C_{K,T}(t,S) = e^{-r(t_1-t)}\int_0^{\infty} p(t,S,t_1,S_1)C_{K,T}(t_1,S_1)dS_1}$$

- This FTP follows a backward equation, a Chapman-Kolmogorov equation and a for-

$$\left(\frac{\partial}{\partial t} + (r-\delta)S\frac{\partial}{\partial S} + \frac{1}{2}\sigma_{S,t}^2 S^2\frac{\partial^2}{\partial S^2}\right)p(t,S,t_1,S_1) = 0 \qquad with \qquad p(t,S,t,S_1) = \delta(S-S_1)$$

ward equation

$$p(t,S,t_1,S_1) = \int_0^{\infty} p(t,S,t_2,S_2)p(t_2,S_2,t_1,S_1)dS_1 \qquad with \qquad t \le t_2 \le t_1$$

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial S_1}(r-\delta)S_1 - \frac{1}{2}\frac{\partial^2}{\partial S_1^2}\sigma_{S_1,t_1}^2 S_1^2\right)p(t,S,t_1,S_1) = 0$$

- Differentiating twice the definition, we get :

$$\boxed{p(t,S,T,K) = e^{r(T-t)}\frac{\partial^2}{\partial K^2}C_{K,T}(t,S)}$$

- **<u>So the knowledge of the call prices is equivalent to the knowledge of the diffusion</u>**

The Backward Transition Probability (BTP)

- The completude of the market can be expressed as :

$$\boxed{C_{K,T}(t,S) = e^{-\delta(T-T_1)}\int_0^{\infty}\Phi(K,T,K_1,T_1)C_{K_1,T_1}(t,S)dK_1}$$

- where $\Phi(K,T,K_1,T_1)$ is called the Backward Transition Probability ->Synthetic diffusion

- This BTP obeys a forward equation, a Chapman Kolmogoroff equation and a backward

$$\left(\frac{\partial}{\partial T} + (r-\delta)K\frac{\partial}{\partial K} - \frac{1}{2}\sigma^2_{K,T}K^2\frac{\partial^2}{\partial K^2}\right)\Phi(K,T,K_1,T_1) = 0 \qquad with \qquad \Phi(K,T,K_1,T) = \delta(K-K_1)$$

equation

$$\Phi(K,T,K_1,T_1) = \int_0^{\infty}\Phi(K,T,K_2,T_2)\Phi(K_2,T_2,K_1,T_1)dK_2 \qquad with \qquad T \le T_2 \le T_1$$

$$\left(\frac{\partial}{\partial T_1} + \frac{\partial}{\partial K_1}(r-\delta)K_1 + \frac{1}{2}\frac{\partial^2}{\partial K_1^2}\sigma^2_{K_1,T_1}K_1^2\right)\Phi(K,T,K_1,T_1) = 0$$

- And we can compute it from the call prices by differentiating twice its definition :

$$\boxed{\Phi(K,T,K_1,T_1) = e^{\delta(T-t)}\frac{\partial^2}{\partial S^2}C_{K,T}(t,S)}$$

Black and Sholes Equation and Local Volatility

- Backward Equation : From $C_{K,T}(S,t)$ by a classical arbitrage relationship,

$$\left(\frac{\partial}{\partial t} + (r-\delta)S\frac{\partial}{\partial S} + \frac{1}{2}\sigma_{S,t}^2 S^2 \frac{\partial^2}{\partial S^2} - r\right) C_{K,T}(S,t) = 0$$

- By using $C_{K,T}(t,S) = e^{-\delta(T-T_1)}\int_0^{\infty}\Phi(K,T,K_1,T_1)C_{K_1,T_1}(t,S)dK_1$, and the forward equation for Φ, we derive a new equation : The Forward Equation

$$\left(\frac{\partial}{\partial T} + (r-\delta)K\frac{\partial}{\partial K} - \frac{1}{2}\sigma_{K,T}^2 K^2 \frac{\partial^2}{\partial K^2} + \delta\right) C_{K,T}(S,t) = 0$$

- We could had derived it by just looking at $C_{K,T}(t,S) = e^{-r(T-t)}\int_0^{\infty} p(t,S,T,S_1)(S_1-K)^+ dS_1$

- It gives us a way to compute the volatility σ:

$$\boxed{\sigma_{K,T}^2 = 2\frac{\frac{\partial C_{K,T}}{\partial T} + (r-\delta)K\frac{\partial C_{K,T}}{\partial K} + \delta C_{K,T}}{K^2\frac{\partial^2 C_{K,T}}{\partial K^2}}}$$

Link Between Implied Volatility and Local Volatility

- From the Preceding formula we deduce that

$$\sigma_{K,T}^2 = \frac{2\frac{\partial\Sigma_{K,T}}{\partial T} + \frac{\Sigma_{K,T}}{T} + 2(r_T-\delta_T)K\frac{\partial\Sigma_{K,T}}{\partial K}}{K^2\left(\frac{\partial^2\Sigma_{K,T}}{\partial K^2} + \frac{1}{TK^2\Sigma_{K,T}} + \frac{d_+}{\Sigma_{K,T}K\sqrt{T}}\frac{\partial\Sigma_{K,T}}{\partial K} + \frac{d_+d_-}{\Sigma_{K,T}}\left(\frac{\partial\Sigma_{K,T}}{\partial K}\right)^2\right)}$$ where $d_\pm = \frac{Log\left[\frac{S}{K}\right] + (r_T-\delta_T)T}{\Sigma_{K,T}\sqrt{T}} \pm \Sigma_{K,T}\sqrt{T}$ and

$\Sigma_{K,T}$ is the implicit volatility .

- This formula allow us to compute $\sigma_{K,T}$ from $\Sigma_{K,T}$. The other way around ($\Sigma_{K,T}$ from $\sigma_{K,T}$) need a tree based model to be solved.

- In an arbitrage free theory $\sigma_{K,T}^2 > 0$.

- A possible parametrization of the local volatility (Brown, Randall 1999) is :

$$\sigma_{K,T} = \sigma_{atm}(T) + \sigma_{skew}(T)Tanh\left(\gamma_{skew}(T)Log\left[\frac{K}{S_0e^{(r_T-\delta_T)T}}\right] - \theta_{skew}(T)\right) + \sigma_{smile}(T)\left(1 - Sech\left(\gamma_{smile}(T)Log\left[\frac{K}{S_0e^{(r_T-\delta_T)T}}\right] - \theta_{smile}(T)\right)\right)$$

where $\sigma_i(T) = \frac{\beta_{i,1} + \beta_{i,2}T}{1 + \beta_{i,3}T}$ and we minimize : $\sum w_{1,k}\left(P_k(x) - \frac{(B_k + A_k)}{2}\right)^2 + w_{2,k}((B_k - P_k(x))^+ + (P_k(x) - A_k)^+)$

Parametrization of the Underlying Security Distribution

- Generalized beta distribution of the second kind (GB2) is a 4 parameters distribution :

$$\varphi_{a,b,p,q}(x) = \frac{|a|x^{ap-1}}{b^{ap}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q}B(p,q)} \qquad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

- The moments are $E[X^n] = \dfrac{b^n B\left(p+\frac{n}{a}, q-\frac{n}{a}\right)}{B(p,q)}$

- We can center this family around F (the forward price of X at maturity T) :

$$b = \lambda F \qquad \lambda = \frac{B(p,q)}{B\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}$$

- we eliminate one parameter but we now have an arbitrage free family of distribution with respect to the forward.

Generalized Black and Sholes

- We have $Call(K, T) = B_T \int_{-\infty}^{\infty} (x - K)^+ \varphi_{a, F, p, q}(x) dx$ and $Put(K, T) = B_T \int_{-\infty}^{\infty} (K - x)^+ \varphi_{a, F, p, q}(x) dx$

- Then These integrals are computable :

$$Call(K, T) = B_T\left(FI_z\left(q - \frac{1}{a}, p + \frac{1}{a}\right) - KI_z(q, p)\right) \qquad Put(K, T) = B_T\left(FI_{1-z}\left(p + \frac{1}{a}, q - \frac{1}{a}\right) - KI_{1-z}(q, p)\right)$$

$$z = \frac{(\lambda F / K)^a}{1 + (\lambda F / K)^a}$$

- where $0 \le I_z(p, q) \le 1$ is the incomplete beta function

- Greeks are easily computable :

$$\Delta_{Call} = I_z\left(q - \frac{1}{a}, p + \frac{1}{a}\right) \qquad \Delta_{Put} = I_{1-z}\left(p + \frac{1}{a}, q - \frac{1}{a}\right)$$

$$\Gamma_{Call} = \Gamma_{Put} = \frac{a(\lambda F / K)^{aq-1}\lambda}{FB(p, q)[1 + (\lambda F / K)^a]^{p+q}}$$

Fitting the Generalized BS to the Market

- We determine a,p,q by Minimizing

$$\sum_i \left\{ w_{C,\,i}[Call^{market}(K_i, T) - Call_{a,\,p,\,q}(K_i, T)] + w_{P,\,i}[Put^{market}(K_i, T) - Put_{a,\,p,\,q}(K_i, T)] \right\}$$

The best parameters for the fit to the FTSE data

Maturity	*a*	*p*	*q*
Sep-98	16.68819	0.511387	4.144175
Dec-98	22.78168	0.248835	0.816654
Mar-99	46.38553	0.09594	0.255468
Jun-99	48.61783	0.076328	0.231991
Sep-99	48.28537	0.066373	0.262421
Dec-99	49.1467	0.058059	0.305228
Mar-00	50.33071	0.051503	0.3619
Jun-00	49.34221	0.048309	0.37601

Stochastic Volatility - Hull and White Model

- Model : $\frac{dS}{S} = (r-\delta)dt + \sigma dW \qquad \frac{d\sigma^2}{\sigma^2} = \alpha dt + \gamma dW_1 \qquad E[dWdW_1] = 0$

- $E[S_t] = E[E[S_t|\sigma]] = E\left[(r-\delta)t - \frac{1}{2}\int_0^t \sigma_s^2 ds\right]$, $Var[S_t] = E[(S_t - E[S_t])^2] = E[E[(S_t - E[S_t])^2|\sigma]] = E\left[\int_0^t \sigma_s^2 ds\right]$

- So options are easily priced : $Call = E\left[BS\left(\int_0^t \sigma_s^2 ds\right)\right]$

- Taylor : $BS(V) = BS(E[V]) + \frac{\partial}{\partial V}BS(V - E[V]) + \frac{1}{2}\frac{\partial^2}{\partial V^2}BS(V - E[V])^2 + \frac{1}{6}\frac{\partial^3}{\partial V^3}BS(V - E[V])^3 + \ldots$

- So $Call = BS(E[V]) + \frac{1}{2}\frac{\partial^2}{\partial V^2}BS \cdot Var[V] + \frac{1}{6}\frac{\partial^3}{\partial V^3}BS \cdot Skew[V] + \ldots$ -> smile but very little skew

- in the case $\alpha = 0$, If we call $k = \gamma^2 T$: $Call = BS(\sigma_0) + \frac{1}{2}\underbrace{\frac{S\sqrt{T}\varphi(d_1)(d_1 d_2 - 1)}{4\sigma_0^3}}_{\frac{\partial^2}{\partial V^2}BS}\left(\underbrace{\frac{2\sigma_0^4(e^k - k - 1)}{k^2}}_{Var[V]} - \sigma_0^4\right) + \ldots$

Stochastic Volatility - Heston Model

- Model: $\frac{dS}{S} = (r-\delta)dt + \sqrt{v}dW \qquad dv = \kappa(\theta - v)dt + \sqrt{v}\sigma dW_1 \qquad E[dWdW_1] = \rho dt$

- one hedging instrument, two independent randomness => one market price of risk

- Market price of risk should be the same for all options

- $\dfrac{\frac{\partial U}{\partial t} + Generator[U]}{\frac{\partial U}{\partial v}} = MarketPrice[v]$ is independent of U

- Solution of the SDE :

$$\frac{\partial U}{\partial t} + \underbrace{\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS + \frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + (r-\delta)S\frac{\partial U}{\partial S} + \kappa(\theta - v)\frac{\partial U}{\partial v}}_{\text{bidimensional generator}} - rU = \lambda(S, v, t)\frac{\partial U}{\partial v}$$

$$\frac{1}{2}\sum_{i,j}\frac{\partial^2 U}{\partial S_i \partial S_j}\Sigma_{ij} + \sum_i \mu_i S_i$$

Heston Model - Path Breaking Resolution

- Formula for a european call (K,T) : $C(S, v, t) = SP_1(S, v, t) - KB(t, T)P_2(S, v, t)$

- we find $\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} + (r + u_j v)\frac{\partial P_j}{\partial x} + (a_j - b_j v)\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0$ (we note $x = Log[S]$) with

  $u_1 = -u_2 = \frac{1}{2}$ $\quad a_1 = a_2 = \kappa\theta$ $\quad b_1 = \kappa + \lambda - \rho\sigma$ $\quad b_2 = \kappa + \lambda$ and $P_j(x, v, T) = 1_{x \geq Log[K]}$

- P_j is the conditional probability that $x_T \geq Log[K]$ (risk neutral probabilities) in the case j

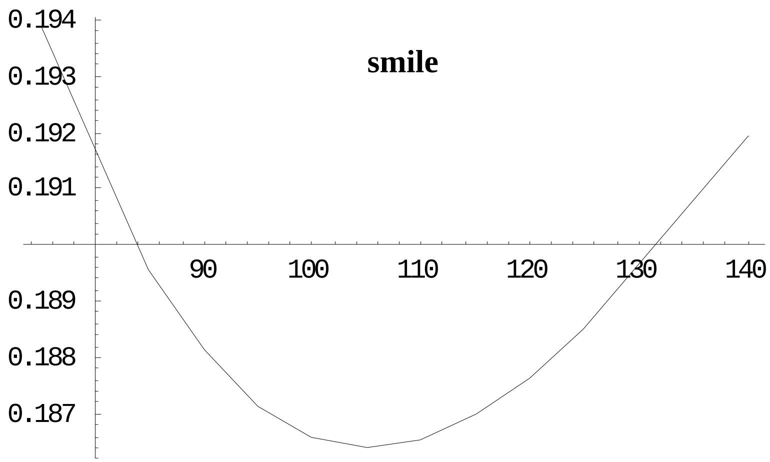
- The characteristic function of the P_j is computable and we have

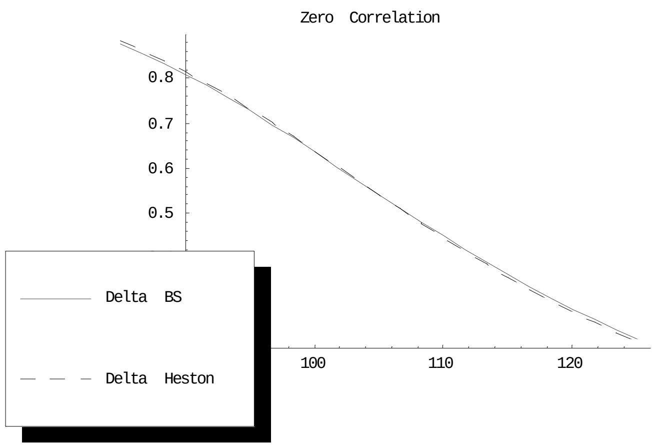
  : $P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re[e^{-i\varphi Log[K] + C[T-t, \varphi] + D[T-t, \varphi]v + i\varphi x}]d\varphi$ with

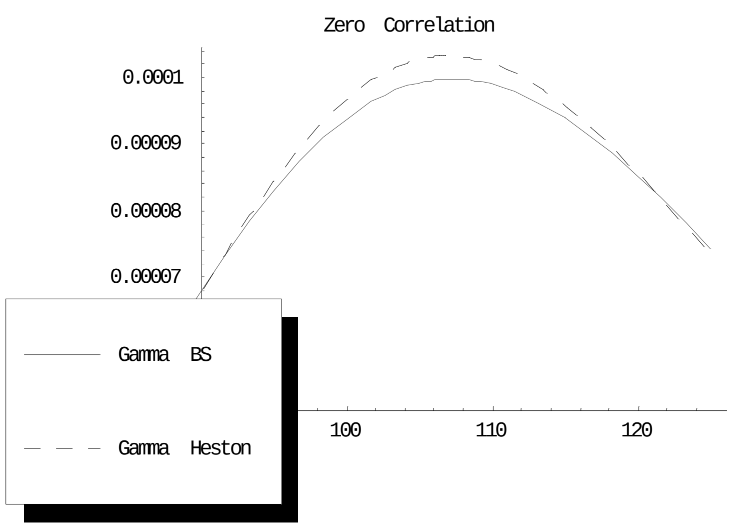
$$C[\tau, \varphi] = \left(i\tau r\varphi + \frac{a_j}{\sigma^2}\left((b_j - i\rho\sigma\varphi + d)\tau - 2Log\left[\frac{1 - ge^{d\tau}}{1-g}\right]\right)\right) \qquad D[\tau, \varphi] = \frac{b_j - i\rho\sigma\varphi + d}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right]$$

$$g = \frac{b_j - i\rho\sigma\varphi + d}{b_j - i\rho\sigma\varphi - d} \qquad d = \sqrt{(i\rho\sigma\varphi - b_j)^2 - \sigma^2(2iu_j\varphi - \varphi^2)}$$

Heston Model- Zero Correlation



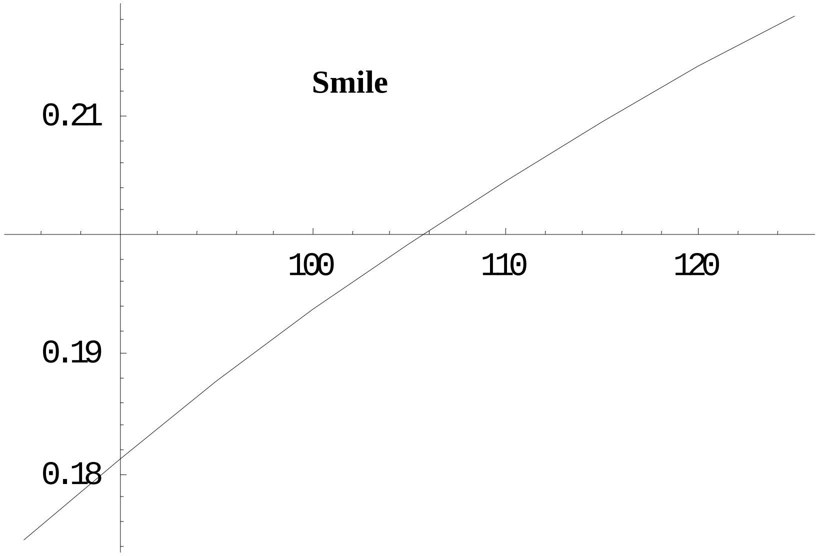


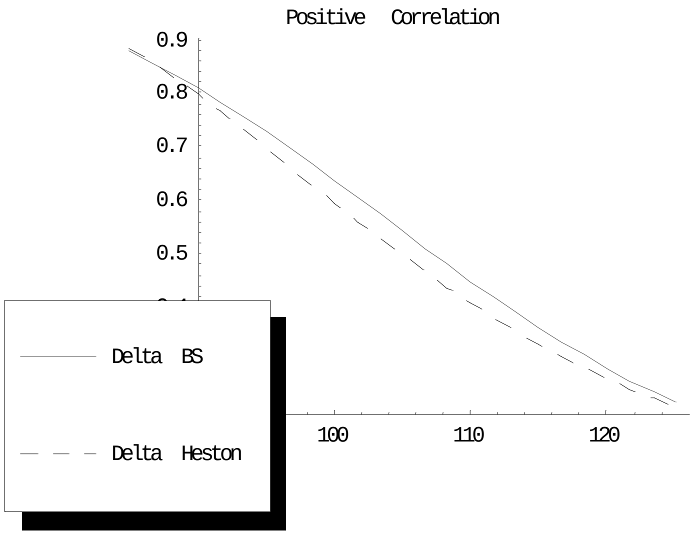


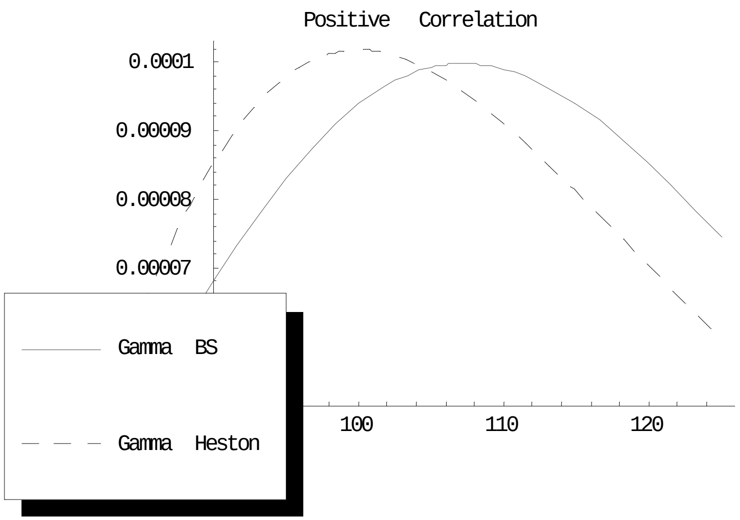
Ex: Currencies

Lesson: Stochastic vol=> higher sensitivity to bad vol

Heston Model - Positive Correlation

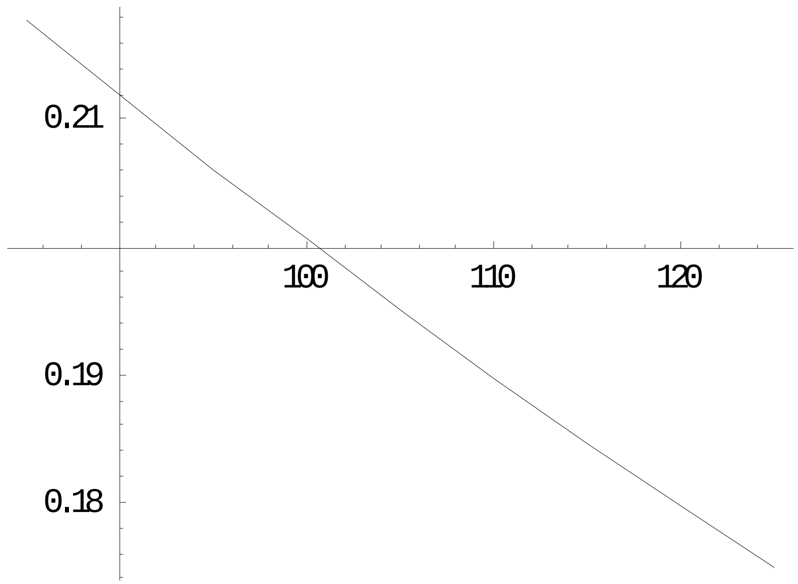


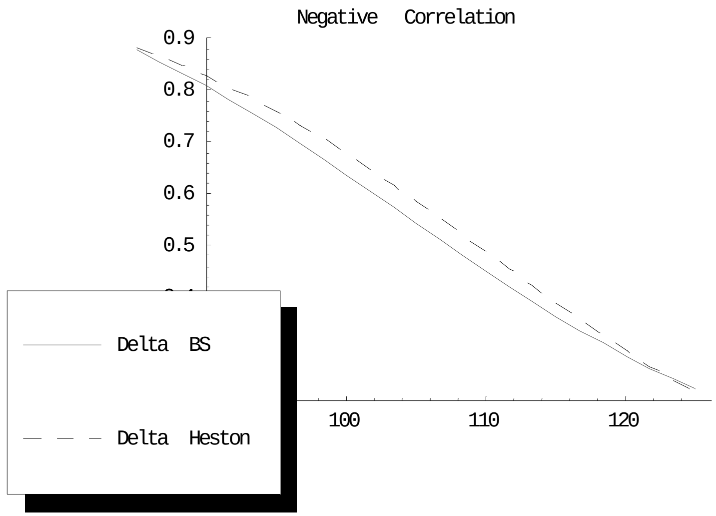


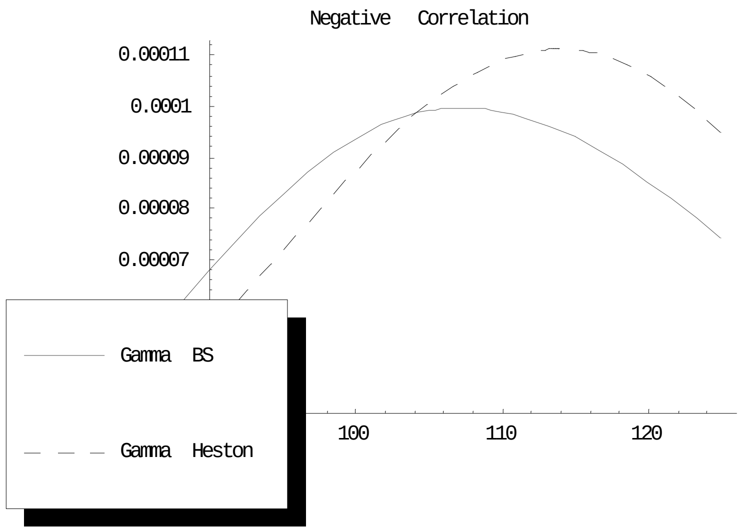


Ex : Commodities

Heston Model - Negative Correlation



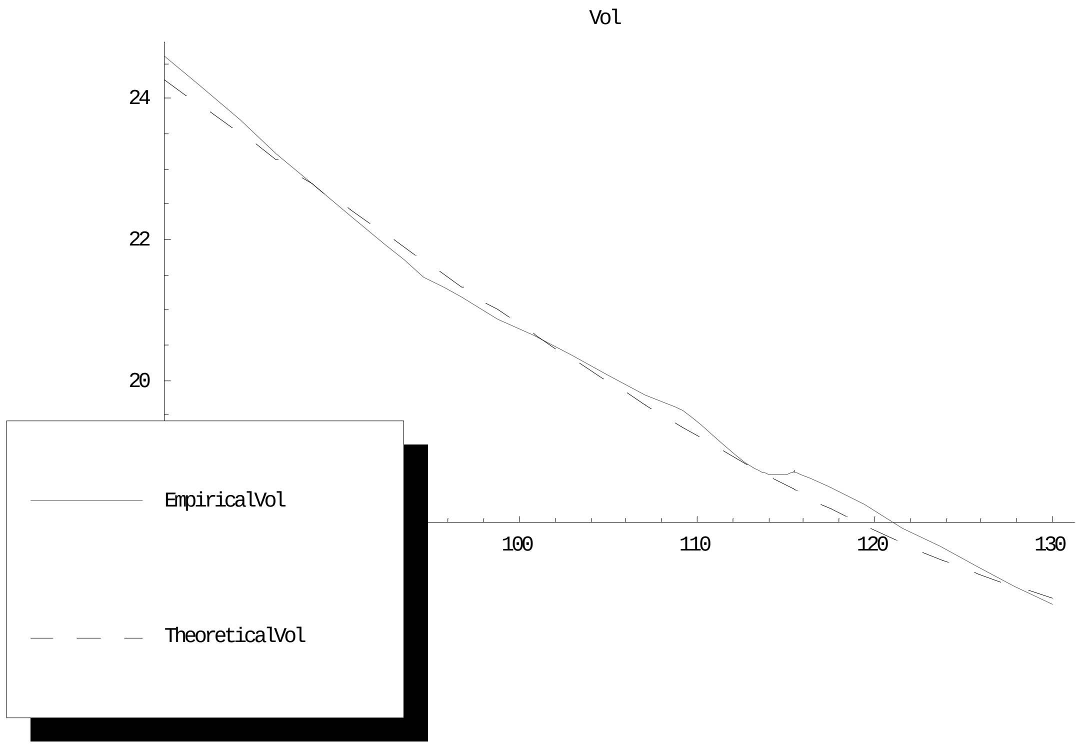




Ex : Equity

Lesson : on the Equity Markets, BS -> underhedges

Heston Model - Fitting the Equity Market



Fit of the smile at 1 year

Final Words

- Existence of volatility surface may modeled many ways
- Either we model a process for the underlying with a stochastic volatility , then compute the distribution at maturity , either we model directly the distribution
- There are other ways like the Derman implicit tree, but it is less used in practice because a parametrization is always more powerful to compare between different markets .
- There is also an aritrage-free approach of the smile, but it does not seem to be accepted in practice
- All the methods we have seen, apply to equity, currency and fixed income markets, but are more developed in the equity derivative business because of the simpler underlyings.

Reading Advices

- The cheaper documents are the Goldman Sachs documents downloadable from the web
  - 1994 Derman and Kani : The volatility smile and its implied tree
  - 1994 Derman and Kani : Static option replication.
  - 1996 Derman kani, Kamal : Trading and hedging local volatility
  - 1999 Derman, Kamal,.: More than you ever wanted to know about volatility swaps
- Two Risk publications are excellent :
  - 2000 Brockhaus, Farkas, Ferraris,.. :Equity Derivatives and Market Risk Models
  - 1998 Jarrow : Volatility (The more advanced texts , because its is just reprints)
- An excellent book very accessible and down to earth:
  - 2000 Rebonato :Volatility and correlation
- Always good :
  - 1998 Wilmott : Derivatives
  - 2000 Wilmott : Paul Wilmott on Quantitative Finance