MultiFractals In Finance

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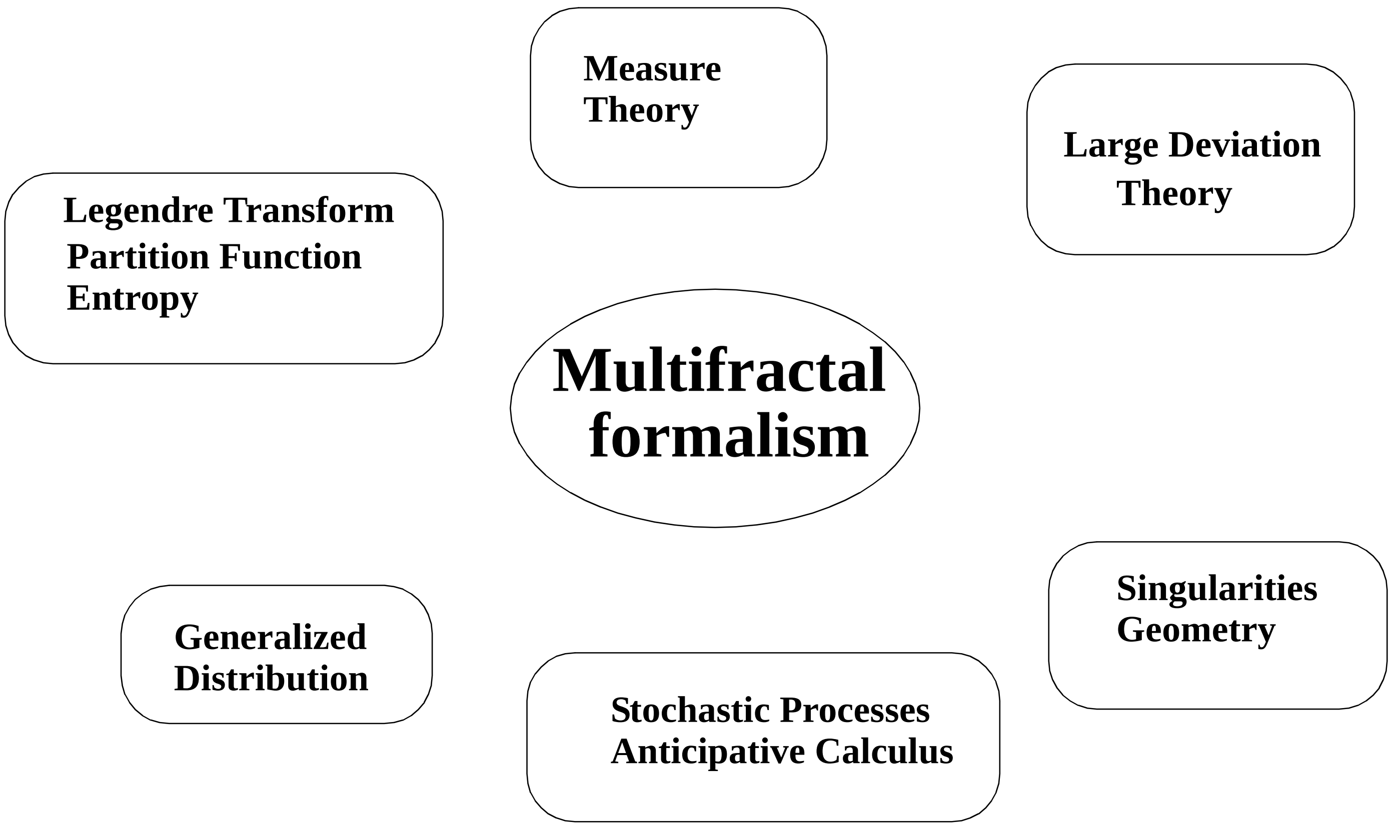
Plan

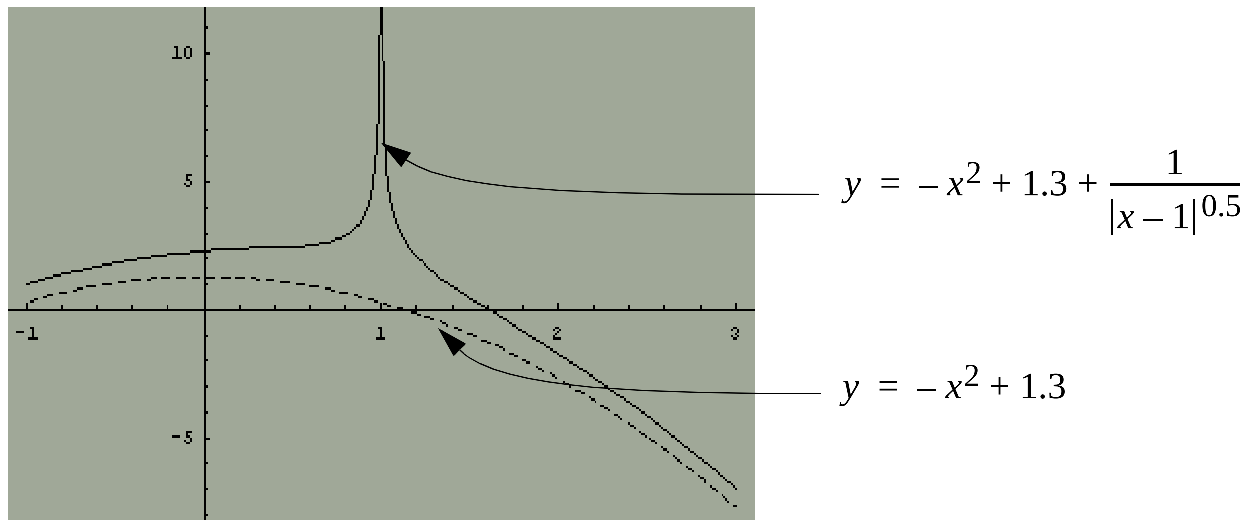
- Models for univariate factors
- Calibration of the multifractal brownian motion
- Multivariate multifractal brownian motion
- Expansion of the multifractal brownian motion around the brownian motion
- Optimization of a portfolio of multifractal assets

The Multifractal formalism



The Singularity Spectrum : Holder dimension

- For any singularity :



- Taylor series for singularities : It exists an exponent α and a polynomial P such that $F(x) - P(x) - (x - x_i)^{\alpha}$ is order strictly more than α :

$$F(x) = P(x) + (x - x_i)^{\alpha} + o\left(\frac{1}{x^{-\alpha}}\right)$$

- α is the Holder dimension (exponent) of the singularity

Singularity Spectrum : Fractal Dimension

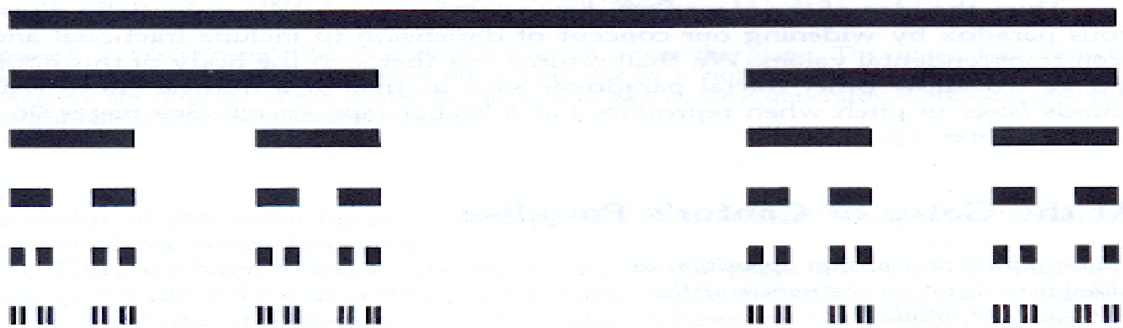
- Box Counting:
  - if $n(\delta)$ is the number of hypercubes to cover the set S (in a multidimensional vectorial space).
  - If S is a segment of line: the number of hypercube is equal to $n(\delta) = \frac{L}{\delta}$
  - If S is a Cube of dimension K: the number of hypercube is : $n(\delta) = \left(\frac{L}{\delta}\right)^K$
  - So in general we have

$$K = Lim_{\delta \to 0} \frac{Log[n(\delta)]}{Log[\delta]}$$

  - This the definition of the fractal dimension of a Set
  - The convergence does not have to be regular.

Singularity spectrum : Exemple of Fractals

- Cantor Set : Dimension $= \frac{Log[2]}{Log[3]} \approx 0.63093$



Stage 0

Stage 1

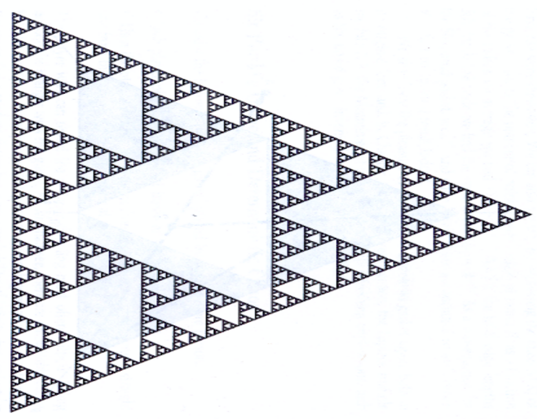
Stage 2

Stage 3

Stage 4

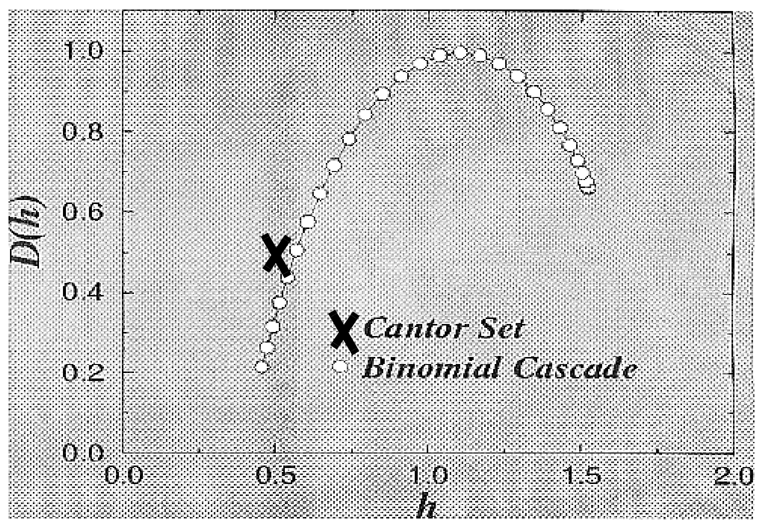
Stage 5

- Sierpinsky Gasket (wining game of Sir Pinsky): Dimension $= \frac{Log[3]}{Log[2]} \approx 1.584963$



Singularity Spectrum : Monofractal and Multifractal measures

- Instead of function we will look at distributions and measures.
- Equivalent going from a density to the measure associated with the density
- The Set of Singularities S of F(x) (as a density of the measure μ) can be partitioned with the holder dimension.
  - one holder dimension -> monofractal. D(h) is the fractal dimension of S
  - more than one dimension -> multifractal.
- Singularity spectrum



Multifractality of the financial time series

- Finite shift operator $\delta_l X(t) \equiv X(t+l) - X(t)$

- A process is scale invariant if for a stationary process $M(q,l) \equiv E[|\delta_l X(t)|^q] = C_q l^{\zeta_q}$

- A scale invariant process is :

  - monofractal if ζ_q is linear in q, ex : a self similar process is such $\alpha^{-H} X(\alpha t)$ has the same distribution than $X(t)$. ==> Then we have $M(q,l) = M(q,L)\left(\frac{l}{L}\right)^{qH}$

  - multifractal if ζ_q is non linear in q. ex : Castaing processes are given by mixture of self similar transformation : $P_l(\delta X) = \int G_{l,L}(u) e^{-u} P_L(e^{-u}\delta X) du$. The self similar processes are associated with kernel $G_{l,L}(u) = \delta\left(u - H Log\left[\frac{l}{L}\right]\right)$.

- For a Castaing process ==> $M(q,l) = M(q,L)\left(\frac{l}{L}\right)^{F[-iq]}$ $\quad F[k] = \dfrac{Log[\hat{G}_{l,L}[k]]}{Log\left[\frac{l}{L}\right]}$

Fourier transform

Multiplicative Cascades

- Definition : this a Castaing Process such that : W_i are independant identically distributed

$$\delta_{l_n} X(t) = \left(\prod_{i=1}^{n} W_i\right) \delta_L X(t) \qquad l_n = 2^{-n} L$$

- The magnitude is defined as the process $\omega(t, l) \equiv \frac{1}{2} \log[|\delta_l X(t)|^2]$

- The cascade equation becomes $\omega(t, l_{n+1}) = \omega(t, l_n) + Log[W_{n+1}]$

- If $\log[W_i]$ is normal $N[\mu, \lambda^2]$, $\omega(t, l_n) \sim N[\mu, \lambda^2]^{*n} \omega(t, L)$, if also $\omega(t, L) \sim N[\mu, \lambda^2]$, then $\omega(t, l_n) \sim N[n\mu, n\lambda^2]$

- For the process $X(t)$ to converge with a finite variance, we must choose $E[\omega_{\Delta t, k}] = -Var[\omega_{\Delta t, k}]$ because $E[e^Y] = e^{E[Y] + Var[Y]}$ for Y normal, this give the relationship $\mu(k) = -\lambda^2(k) = -\lambda^2 Log\left[\frac{T}{\Delta t}\right]$ for the multifractal brownian motion

The (standard) multifractal brownian motion

- three parameters : σ^2 : variance, λ : intermittency and T : integral scale

- Constructive definition :

$$X(t) = X(0) + \sum_{i=1}^{n} \varepsilon_{\Delta t, k} e^{\omega_{\Delta t, k}}$$

  - $\varepsilon_{\Delta t, k}$ is normal with standard deviation equal to $\sigma\Delta t$

  - all the $\omega_{\Delta t, k}$ are jointly normal with

$$Cov[\omega_{\Delta t, k}, \omega_{\Delta t, l}] = \lambda^2 Log\left[\frac{T}{(|k-l|+1)\Delta t}\right]$$

- Analytic definition

$$X(t) - X(s) = \sigma \int_{-\infty}^{\infty} \frac{e^{-\frac{H^2}{2\lambda^2}}}{\sqrt{2\pi}\lambda} \theta_{t-s, H}[u] dW_u dH \qquad \theta_r[u] = \begin{cases} 0 & -(\infty < u \le 0) \\ u^{H-\frac{1}{2}} & 0 < u \le T \\ u^{H-\frac{1}{2}} - (u-T)^{H-\frac{1}{2}} & T < u < \infty \end{cases}$$

Statistics of the multifractal brownian motion

- Moments :

$$M(2p, l) = K_{2p}\left(\frac{l}{T}\right)^{p - 2p(p-1)\lambda^2}$$

- when $l \gg T$, $M(2p, l) \approx C_p l^p$ which the brownian signature

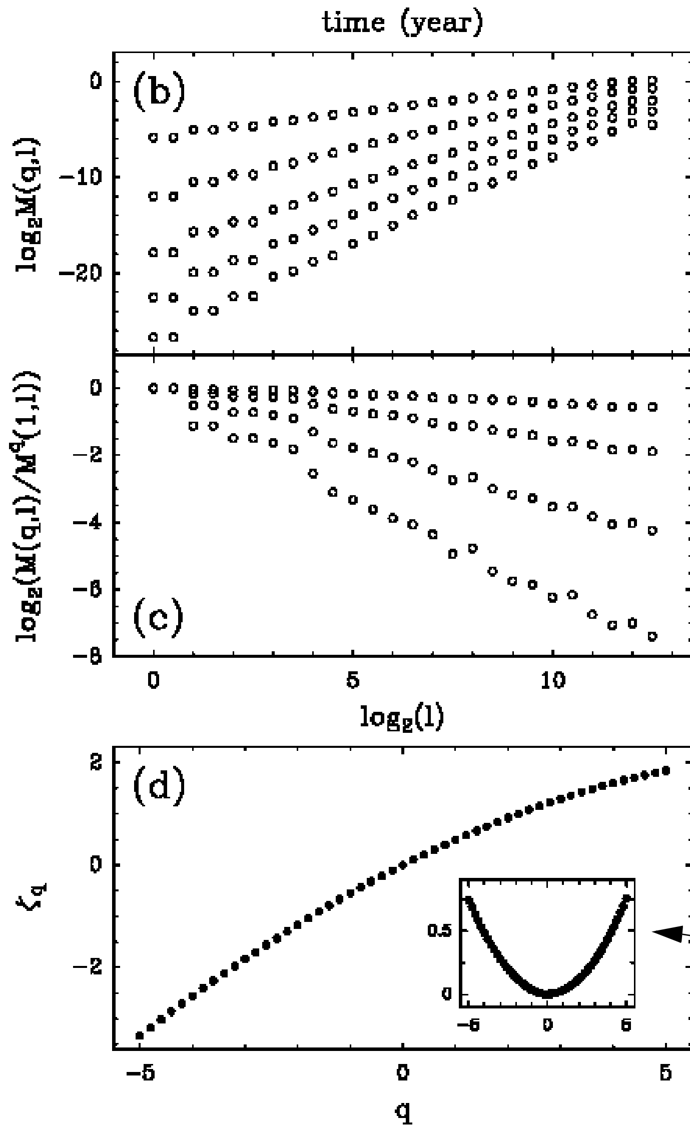
- Scale correlations :

$$C_p(l, \tau) \equiv E[(X_{\Delta t}(l+\tau) - X_{\Delta t}(l))^p (X_{\Delta t}(\tau))^p] = Q_p\left(\frac{\tau}{T}\right)^{2\zeta_p}\left(\frac{l}{T}\right)^{-\lambda^2 p^2}$$

- Magnitude correlation function

$$C_\omega(l, \tau) = Lim_{p \to 0}\left(\frac{C_p(l, \tau) - M(p, \tau)}{p^2}\right) = Cov[Log[\delta_l X(t+\tau)], Log[\delta_l X(t)]] \approx -\lambda^2 Log\left[\frac{l}{T}\right]$$

S&P 500 over 1988-1999

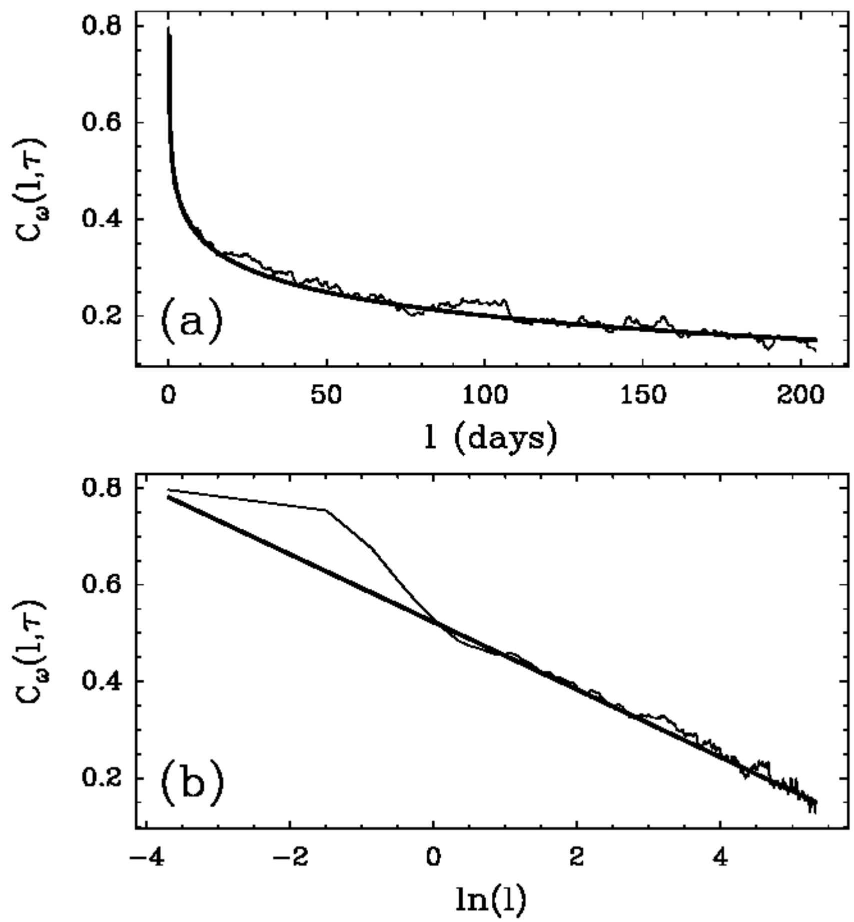


Log of moments M(q,l)
for q=1,2,3,4,5

ζ_q spectrum

non linear part showing
the multifractality

S&P 500

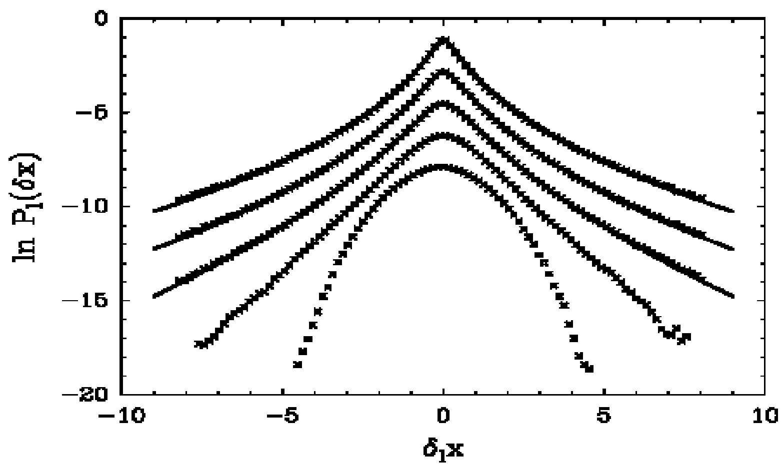


Magnitude correlation function

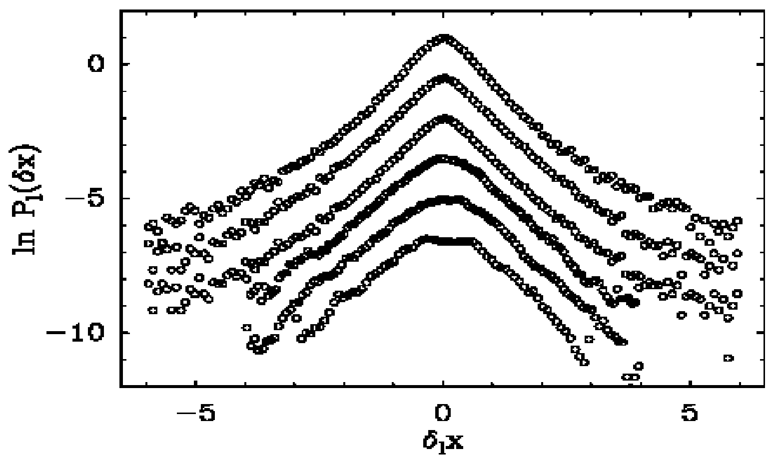
the small scale window is due to
the smoothing window ncessary
to to estimate the local magnitude

$$C_\omega(l,\tau) = Lim_{p\to 0}\left(\frac{E[(X_{\Delta t}(l+\tau) - X_{\Delta t}(l))^p (X_{\Delta t}(\tau))^p] - M(p,\tau)}{p^2}\right) \sim -\lambda^2 Log\left[\frac{l}{T}\right]$$

pdfs across scales for MBM and S&P500



Multifractal brownian motion



S&P 500

scales={10mn,40mn,160mn,1day,1week,1month}

Estimations for different Markets

Series	Size	λ^2	T
Future S&P500	7.10^4	0.025	3 years
Future JY/USD	7.10^4	0.02	6 months
Future Nikkei	7.10^4	0.02	6 months
Future FTSE100	7.10^4	0.02	1 year
S&P500 index	6.10^3	0.024	3 years
French index	6.10^3	0.029	2 years
Italian index	6.10^3	0.029	2 years
Canadian index	6.10^3	0.024	3 years
German index	6.10^3	0.027	3 years
UK index	6.10^3	0.026	6 years
hong-kong index	6.10^3	0.05	3 years

Multivariate Multifractal Formalism

- $X = \{X_1, X_1, ..., X_N\}$ is a multivariate process that exhibits a cascade equation: $\{\delta_l X_i(t)\}_{1 \le i \le N} \sim W_{i,\, l/L} \delta_L X_i(t)$ where W is a Log infinitly divisible stochastic vector.

- Then the pdf satisfies $P_l(\delta X) = \int du^N G_{l/L}(u) e^{-\sum u_i} P_T(e^{-u} \cdot \delta X)$

- Then the moments verify $M(q_1, q_2, ..., q_N, l) = E[|\delta_l X_1|^{q_1} ... |\delta_l X_N|^{q_N}] = K_{q_1, .., q_N} l^{\zeta_{q_1, ..., q_N}}$ where $\zeta_{q_1, .., q_N} = \hat{G}(-iq_1, ..., -iq_N)$.

Multivariate Multifractal Brownian Motion

- $X(t) = Lim_{\Delta t \to 0} \sum_{k=1}^{t/\Delta t} \varepsilon_{\Delta t, k} \cdot e^{\omega_{\Delta t, k}}$

first calibrating equation

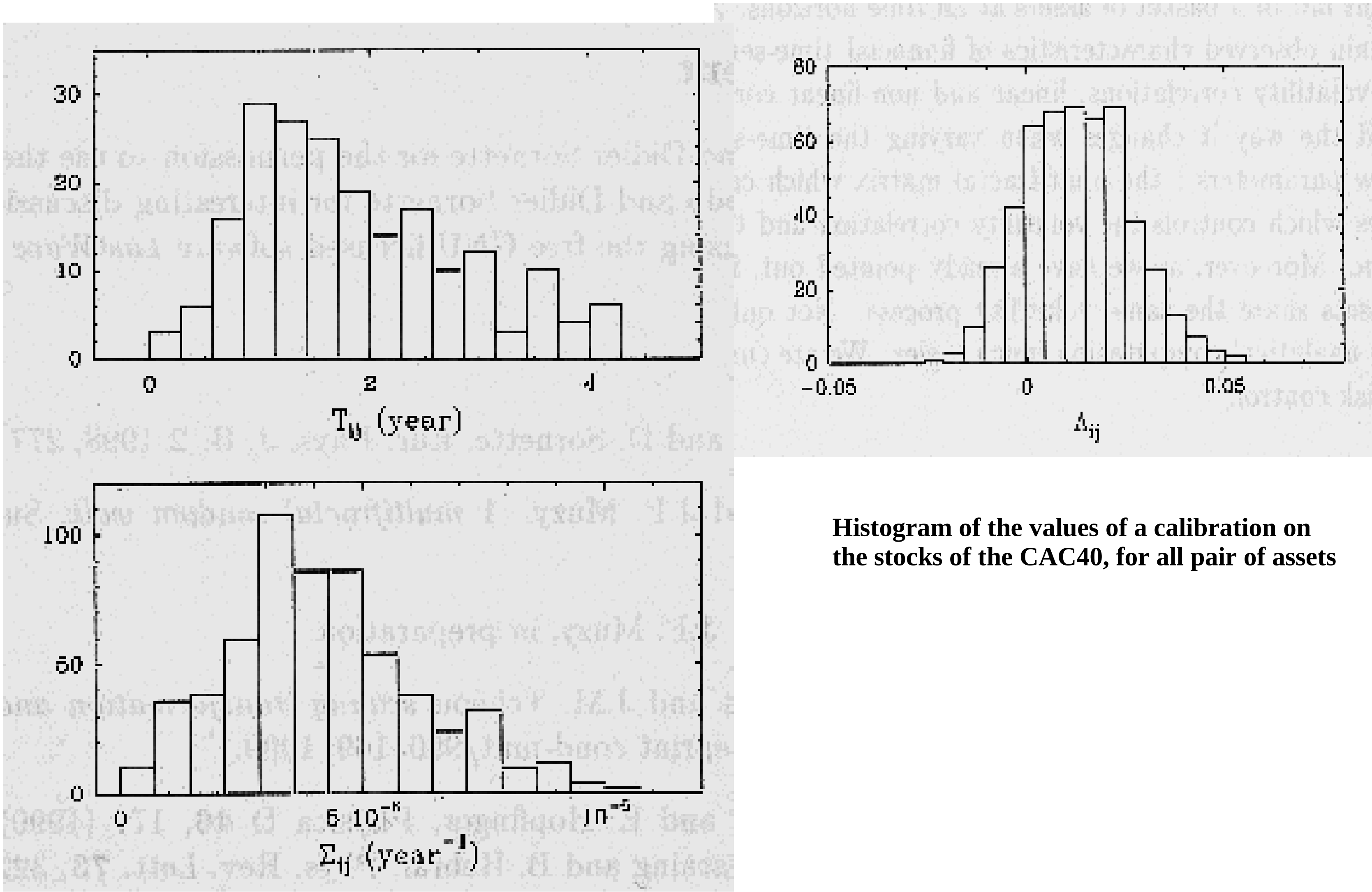
- $Cov[\varepsilon_{\Delta t, k, i}(t), \varepsilon_{\Delta t, k, j}(t+\tau)] = \delta(\tau)\Sigma_{i,j}\Delta t$ and $Cov[\omega_{\Delta t, k, i}, \omega_{\Delta t, l, j}] = \Lambda_{i,j} Log\left[\frac{T_{i,j}}{(|k-l|+1)\Delta t}\right]$ To have a convergence we also state $E[\omega_{\Delta t, k, i}] = -Var[\omega_{\Delta t, k, i}]$

- In the case where Σ is diagonal or $\omega_i = \omega_j$, we have : $\zeta_{q_1, q_2, \ldots, q_N} = \sum_i \zeta_{q_i} - \sum_{i<j} \Lambda_{i,j} q_i q_j$ .In general we have

  $Cov[Log[\delta_l X_i(t)], Log[\delta_l X_i(t+\tau)]] = K(-\Lambda_{i,j} Log[\tau] + C)$

- Calibration consistency $\begin{cases} Cov[X_i(l), X_j(l)] = \Sigma_{i,j} e^{\frac{1}{2}(\Lambda_{i,i} + \Lambda_{j,j} + 2\Lambda_{i,j})} l \\ \dfrac{E[|X_i(l)|^q |X_j(l)|^q]}{E[|X_i(l)|^q |X_j(l)|^q]} \sim l^{-\Lambda_{i,j} q^2} \end{cases}$ **Second calibrating equation**

Multivariate Multifractal Calibration



Histogram of the values of a calibration on the stocks of the CAC40, for all pair of assets

Cumulants of the MBM

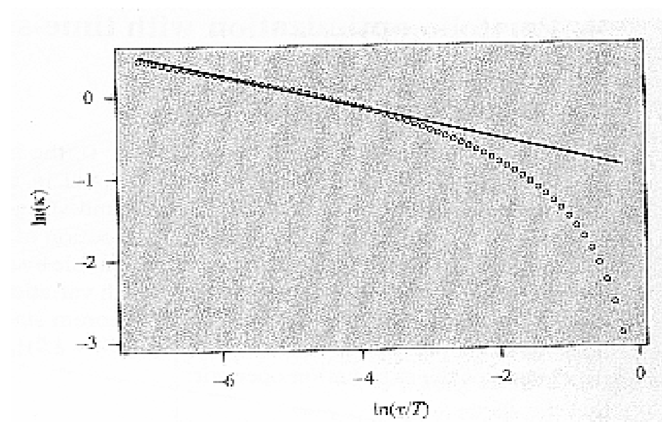
$$c_1 = l\mu$$
$$c_2 = \sigma^2\left(\frac{l}{T}\right)$$
$$c_3 = 0$$
$$c_4 = 3\sigma^4\left(\frac{l}{T}\right)^2 - 4\lambda^2\left(1 - \left(\frac{l}{T}\right)^{4\lambda^2}\right)$$
$$c_5 = 0$$
......

$$Y = e^x - 1$$
=>

$$C_1 = l\left(\mu + \frac{1}{2}\sigma^2\right)$$
$$C_2 = \sigma^2\left(\frac{l}{L}\right)$$
$$C_3 = l^2\left(3\sigma^4 - 18\sigma^4\lambda^2 Log\left[\frac{l}{T}\right]\right)$$
$$C_4 = 16\sigma^6 l^3 - 12\sigma^4\lambda^2 l^2 Log\left[\frac{l}{T}\right]$$
...

=> $\kappa = \dfrac{C_4}{C_2^2} = 16\sigma^2 l - 12\lambda^2 Log[l]$

=> **Long Range Correlations (S&P500 over 30 years)**



Portfolio Optimization : Market Portfolio

- we want to maximize $\int u(\delta S) P_l(\delta S) d\delta S$

- If we take $u(W) = -e^{-aW}$,we have to minimize $\int e^{-a\delta S} P_l(\delta S) d\delta S = \hat{P}(ia) = Exp\left[-aC_1 + \sum_{i=2}^{\infty} \frac{(-a)^i}{i!} C_i\right]$

- In the absence of correlations, using cumulants, for the first orders :

Lagrange

$$-\alpha \sum w_i - a \sum w_i l \mu_i + \frac{a^2}{2} \sum w_i^2 l \sigma_i^2 - \frac{a^4}{24} \sum w_i^4 12 \sigma_i^4 \lambda_i^2 l^2 Log\left[\frac{l}{T_i}\right]$$

- If no fractality, we find the markowitz solution : if $\sigma_i = \sigma$, $w_j^0 = \frac{1}{N} + \frac{1}{a\sigma^2}(\mu_j - \langle\mu\rangle)$,

- If fractality, $w_i \approx w_i^0\left(1 + 2a^2\sigma_i^2 w_i^0 \lambda_i^2 l Log\left[\frac{l}{T_i}\right]\right)$, Asset ,with $T_i \approx el$ will be the most depleted

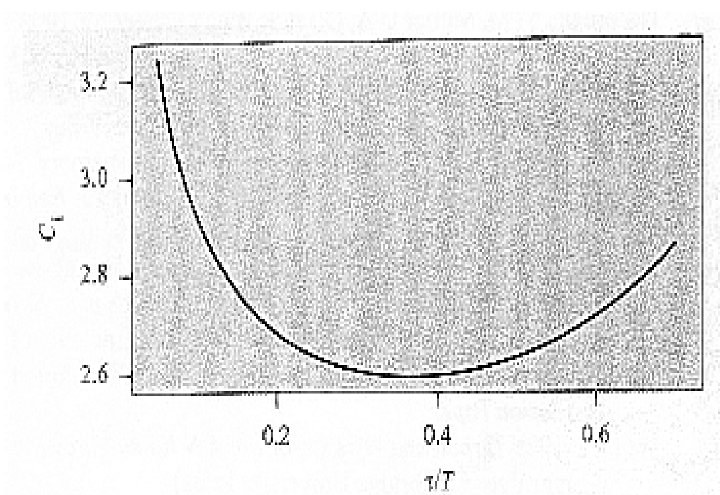
Portfolio Optimization : Efficient frontiers

- We are looking at multiperiod problems with rebalancing assumption (we maintain the composition of the portfolio).

- For the C1-C2 efficient frontier, there is no dependence on the Δt of rebalancing (Tobin result)

- For the C1-C4 efficient frontier, we have a dependence on Δt.

  - For a given risk ,there is better return for small Δt than for large ones
  - There is a worst horizon , for which the return is minimum



CAC40 Stocks

Conclusion

- Multi fractal processes adress the problem of parametrizing the change of the distribution between short horizon (1 minute) and long horizon (20 years) with all the intermediate stages.

- The model coming from turbulence physics is tractable, seems robust and corrobored by the financial data.

- It allows to address the problem of optimizing rebalancing horizons in asset management .

- Extension of the model to multivariate cases and with skewness (levy processes at integral time) is tractable