From Ito Lemma to HJM

by Olivier Croissant

Plan

- Sigma-algebra and Processes
- Ito Lemma
- Girsanov Theorem
- Arbitrage Free Models
- HJM

Name of the Game

- Playing with the drift to price derivatives
- Playing with the drift to define an arbitrage free model

Information and Processes

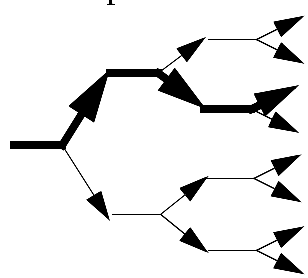
The Pb : Representing the evolution of the world and the incertitude attached to future events

<u>The Solution : <=> Representing the increase of available information</u>

Time 1	Time 2	Time 3	Time 4	
Attribute-t1	Attribute-t1 Attribute-t2	Attribute-t1 Attribute-t2 Attribute-t3	Attribute-t1 Attribute-t2 Attribute-t3 Attribute-t4	<u>Filtration</u>
Sigma algebra-1	Sigma algebra-2	Sigma algebra-3	Sigma algebra-4	
$\Sigma_1 \subset$	$\Sigma_2 \subset$	$\Sigma_3 \subset$	$\Sigma_4 \subset$	$(\ldots \subset \Sigma_\infty)$

sigma-algebra (n) = {the set of possible histories up to time n}

<u>Type of attribute</u> :Binary



a path <=> an event <=> an element of a sigma algebra(n)

(actually, a sigma algebra is a little bit more sophisticated, but we do not care.**)**

Dynamic and Probability Measures

The Pb : Representing evolution laws, or "a priori" knowledge about the future

<u>Solution : add a probability measure to every sigma-algebra</u>

Time 4

Attribute-t1
Attribute-t2
Attribute-t3
Attribute-t4

Sigma algebra-4

$$(att1, att2, att3, att4) \rightarrow dens_4(att1, att2, att3, att4)$$

$$\int dens_4(x1, x2, x3, x4)(dx1)(dx2)(dx3)(dx4) = 1$$

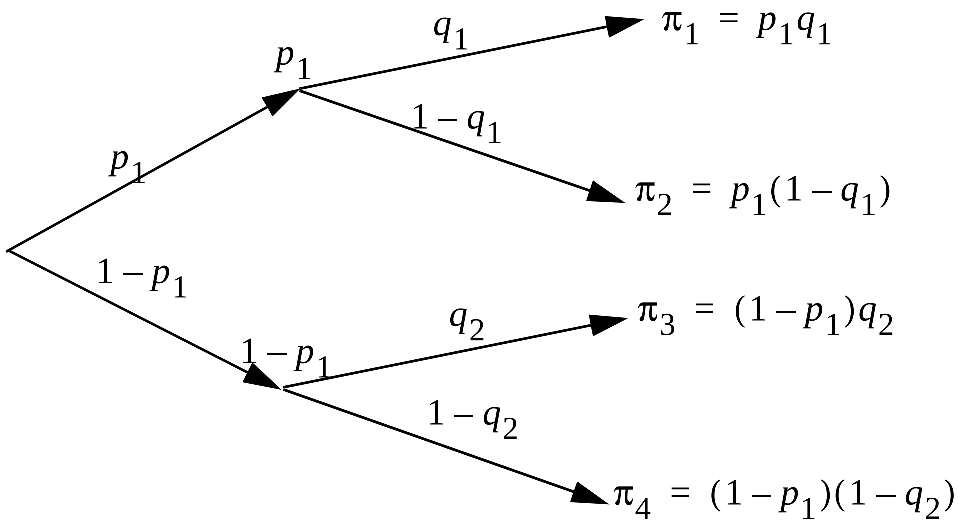
Compatibility condition : $\int dens_4(x1, x2, x3, x4)(dx4) = dens_3(x1, x2, x3)$

Filtration= Set [Sigma -algebra(t)] : describes the evolution of the knowledge about the system

At every Stochastic Process --> Filtration + Probability Measures (t)

Transition Probabilities in a Tree

n=2 steps



$$p_1 = \pi_1 + \pi_2$$

$$q_1 = \frac{\pi_1}{\pi_1 + \pi_2}$$

$$q_2 = \frac{\pi_3}{1 - (\pi_1 + \pi_2)}$$

---It is possible to generalize this to trees with n steps an p order transitions

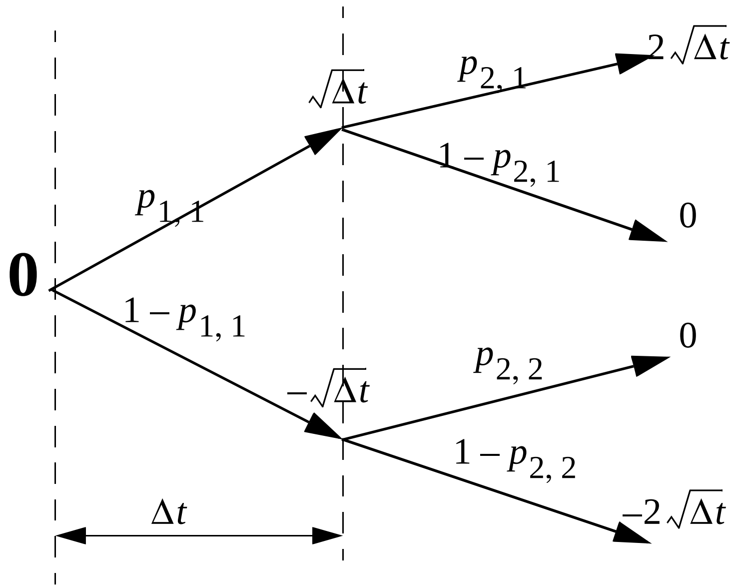
---Therefore, giving the terminal probability dens[x1,x2,...xn] is equivalent to specifiy the local transitions dens[xn | x1,x2,...xn-1], which was also obvious from the definition of the conditional densities...

---Every Process is Markov if the State Space is the Maximal one

Transition Probabilities Determine Everything

- Transition Probabilities define Evolution
- Expectations define Transition probabilities:
  - Given the Arrow Debreu Prices for the maximal markov process (<=> terminal probabilities) -> Transition probabilties
- Knowing How to compute Expectations <=> Knowing transition Probabilties <=> Knowing the processes
- An example of process :
  - Transition Probabilities are defined by $Prob[X(t+dt) \in A | (X(t) = x)] = \int_A \frac{e^{-\frac{(y-(x+\mu dt))^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dy$
  - Expectations are given by $E[f(X_T)] = \int_A f(y) \frac{e^{-\frac{(y-\mu T)^2}{2\sigma^2 T}}}{\sqrt{2\pi T}\sigma} dy$
  - We call it an Ito Process with Constant drift and vol : $dX_t = \mu dt + \sigma dW_t$

Transition Probabilities: Other Examples



$$p_{1,1} = p_{2,1} = p_{2,2} = \frac{1}{2}$$

=> Brownian Motion

$$K = Floor\left[\frac{1}{\sqrt{\Delta t}}\right]$$

$$p_{n,j} = \sum_{p}^{N_n} \delta_{j, \frac{n}{2} + pK} \frac{(\lambda \Delta t)^p}{p!} e^{-\lambda \Delta t}$$

=> Poisson Process

Discret Processes with Density

- Discret set of dates

- If x is a process, we may have a probability density $dens(x_1, x_2, x_3, x_4)$ with a normalisation condition $\int dens(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4 = 1$

- For a markov process : $dens(x_1, x_2, x_3, x_4) = dens(x_3, x_4)$

- $\int dens(x_1, x_2, x_3, x_4) dx_1 dx_2$ will be called the conditonal density to time 3

Conditional Expectations

Time 4

Attribute-t1
Attribute-t2
Attribute-t3
Attribute-t4

Sigma algebra-4

$$(att1, att2, att3, att4) \to dens_4(att1, att2, att3, att4)$$

$$\int dens_4(x1, x2, x3, x4)(dx1)(dx2)(dx3)(dx4) = 1$$

Compatibility condition : $\int dens_4(x1, x2, x3, x4)(dx4) = dens_3(x1, x2, x3)$

This creates a set of interesting identities : the conditional expectation chain rule :

if A is a function A(x1,x2,x3,x4), let's call E[A| 3] the integral $\dfrac{\int A(x1, x2, x3, x4) dens_4(x1, x2, x3, x4)(dx4)}{dens_3(x1, x2, x3)}$

Then we have : E[E[A| 3] | 2] = E[A | 2]

both equal to $\dfrac{\int \left\{ \dfrac{\int A(x1, x2, x3, x4) dens_4(x1, x2, x3, x4)(dx4)}{dens_3(x1, x2, x3)} \right\} dens_3(x1, x2, x3) dx3}{dens_2(x1, x2)}$

Predictable Process

Space = {Set of stochastic Process with the same filtration}

a{ F , P1} + b{ F ,P2} = { F , P3}

$L \in Real$: Vectorial Space
$L \in$ Any stochastic process : Ring

L { F, P} = {F , P4}

Def : Predictable processes are stochastic processes whose probability measures are delta functions :

$$dens_4(x1, x2, x3, x4) = \delta(x4 - f(x1, x2, x3))$$

$(x4 = x_{t = 4})$

<=> **The only x4 that matter is a function of (x1,x2,x3)**

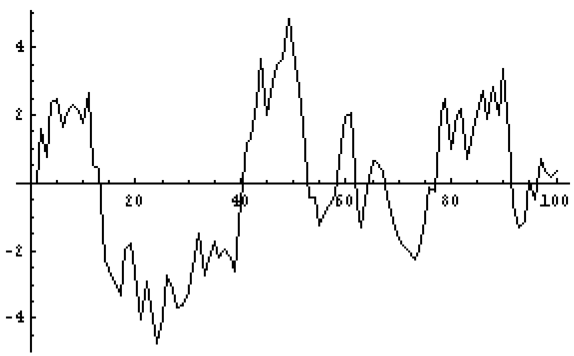
<=> **The value at time 4 is computable as soon as time 3**

The set of previsible processes (on the same filtration) is a commutative ring for + and *

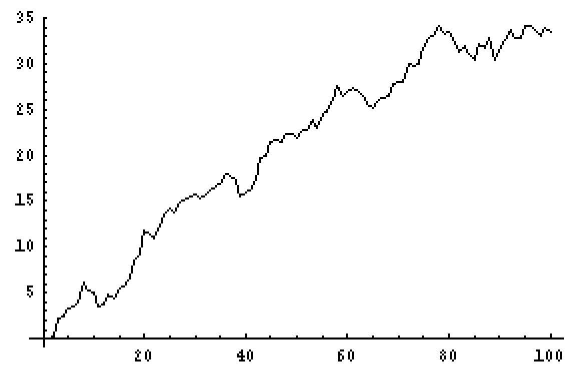
Martingale

S_t **is a martingale iff** $E_t\left[S_{t_1}\right] = S_t$ **For any time** t

=> $E_0[S_t] = S_0$



A Martingale



Not a Martingale

S_t **Not a Martingale =>** $\frac{1}{E_0[S_t]}S_t$ **and** $S_t - E_0[S_t]$

$S_t > 0$

are not always Martingales !

Ito Processes

- Ito processes

$$dx_t = \mu_t dt + \sigma_t dW_t \qquad <=> \qquad x_t - x_0 = \int_0^t \mu_s ds + t\int_0^t \sigma_t dW_t$$

- Ordinary functions

$$d(f(y(t))) = \frac{df}{dy} \times \frac{dy}{dt} dt = \frac{df}{dy} dy$$

- Stochastic function

$$d(f(x_t)) = \frac{df}{dy} dx_t \overset{?}{=} \frac{df}{dy} \mu_t dt + \frac{df}{dy} \sigma_t dW_t$$

Naive Differential Calculus on Ito Processes Does not Work !

- Take $f(W_t) = W_t^2$

- Do we have

$$W_t^2 \overset{?}{=} \int_0^t 2W_s dW_s \qquad <=> \qquad d(W_t^2) \overset{?}{=} 2W_t dW_t$$

- But the expectations are different:

$$E[W_t^2] = t \qquad E\left[\int_0^t 2W_s dW_s\right] = Lim\ 2\sum_{i=0}^{n-1} E\left[W\left(\frac{it}{n}\right)\left(W\left(\frac{(i+1)t}{n}\right) - W\left(\frac{it}{n}\right)\right)\right] = 0$$

What Went Wrong

- Taylor expansion

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 + \frac{1}{3!}f'''(W_t)(dW_t)^3 + \ldots$$

- Property of brownian motion

$$\int_0^t dW_s{}^2 = Lim\ \frac{t}{n}\sum_{i=1}^{n}\frac{\left(W\left(\frac{(i+1)t}{n}\right) - W\left(\frac{it}{n}\right)\right)^2}{t/n}$$

$$= Lim\ \frac{t}{n}\sum_{i=1}^{n} Z_i^2 \qquad \rightarrow t$$

- Conclusion

$$\int_0^T dW_t{}^2 \sim \int_0^T dt \qquad \int_0^T \frac{1}{2}f''(W_t)(dW_t)^2 \sim \int_0^T \frac{1}{2}f''(W_t)dt$$

Ito Lemma

- Mono Dimensional: if the initial process is : $dx_t = \mu dt + \sigma dW_t$,

$$df(x) = \left(\mu\frac{df}{dx} + \frac{\sigma^2}{2}\frac{d^2f}{dx^2}\right)dt + \sigma\frac{df}{dx}dW_t$$

- Multi Dimensional: if the initial process is : $dx_{i,t} = \mu_i dt + \sum_j \sigma_{i,j} dW_{j,t}$

$$df(x) = \left(\mu\frac{df}{dx} + \frac{1}{2}Trace[\sigma^* H\sigma]\right)dt + \frac{df}{dx}\sigma dW_t$$

- with vector notations and $H_{i,j} = \dfrac{d^2f}{dx_i dx_j}$ is the hessian (second derivatives)

Example of Application

- Two dimensions

$$\frac{dX_t}{X_t} = \mu dt + \sigma \cdot dW_t$$

$$\frac{dY_t}{Y_t} = \nu dt + \rho \cdot dW_t$$

  - then

$$\frac{d(X_tY_t)}{X_tY_t} = \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + (\sigma \cdot \rho)dt = X_t dY_t + Y_t dX_t + (\|\sigma\| \times \|\rho\| \times Corr[X_t, Y_t])dt$$

- A Quanto Forward Contract:
  - Commitment to buy the US Stock IBM at USD 100 and we agree now on the exchange rate : 1USD = 1.5 CAN
  - Difference between no correlation and correlation is a factor of

$K = e^{(\|\sigma\| \times \|\rho\| \times Corr[X_t, Y_t])\Delta t}$ so the price is $K \times Forward[X]Forward[Y]$

Martingale (2)

- An important property of the martingales is : $\int_0^t A_t dM_t$ is a martingale

    - Exemple the profit made on a market in an arbitrage-free model is just: $\int_0^t \Delta_t \cdot dX_t$

    It has to be a martingale

- To have this property we have to have more than simply the conservation of expectation, we have to have the conservation of conditional expectation: $E[M_{t+T}|t] = E[M_t]$.

- Exemple of a process keeping the expectation but definitively not a martingale: $x_{n+1} = 2x_n$ but $x_1 = \begin{cases} 1 & prob = \frac{1}{2} \\ -1 & prob = \frac{1}{2} \end{cases}$

- if x is an Ito process : $dx_t = \mu dt + \sigma dW_t$, $x_t - \int_0^t \mu dt$ and $x_t / \left\{ \int_0^t \mu dt \right\}$ are martingales but not $x_t - E_0[x_t]$ or $x_t / E_0[x_t]$

Integration

Ito : $d(W_t^2) = 2W_t dW_t + dt$ **Then we define :** $\int W_t dW_t = \frac{W_t^2}{2} - t$

$$d(W_t^n) = nW_t^{n-1}dW_t + \frac{n(n-1)}{2}W_t^{n-2}dt$$

$$\int_0^T W_t^{n-1}dW_t = \frac{(W_T)^n}{n} - \frac{(n-1)}{2}\int_0^T W_t^{n-2}dt$$

Compensator

It is possible to define another stochastic multiplication between a previsible process and a any martingale :

$\{P, B\} \to P \cdot B = \int PdB$ **(dot product)** **-> It is still a martingale**

$Q = \int PdB$ **->** $\int HdQ = \int HPdB$ **-> It is an associative product**

Radon -Nikodym

We want to change from a probability measure P to another Q .

Only if the probability measures are equivalent :

P~Q <=> $(Q_4(x1, x2, x3, x4) = 0) \Leftrightarrow (P_4(x1, x2, x3, x4) = 0)$

we define : $\dfrac{dQ}{dP}(4) = \dfrac{Q_4(x1, x2, x3, x4)}{P_4(x1, x2, x3, x4)}$ **It is called the Radon Nikodym derivative**

therefore we have :

$$E_Q[A] = \int A(x1, x2, x3, x4) Q_4(x1, x2, x3, x4)(dx4)(dx3)(dx2)(dx1)$$

$$= \int A(x1, x2, x3, x4) \frac{dQ}{dP}(4) P_4(x1, x2, x3, x4)(dx4)(dx3)(dx2)(dx1)$$

$$= E_P\left[\frac{dQ}{dP} A\right]$$ **Change of measure of an expectation**

Radon -Nikodym (2)

We want to change from a probability measure P to another Q .

Only if the probability measures are equivalent :

P~Q <=> $(Q_4(x1, x2, x3, x4) = 0) \Leftrightarrow (P_4(x1, x2, x3, x4) = 0)$

we define : $\frac{dQ}{dP}(4) = \frac{Q_4(x1, x2, x3, x4)}{P_4(x1, x2, x3, x4)}$

therefore we have :

$$E_Q[A|2] = \frac{\int A(x1, x2, x3, x4) Q_4(x1, x2, x3, x4)(dx4)(dx3)}{\int Q_4(x1, x2, x3, x4)(dx4)(dx3)}$$

$$= \frac{\int \frac{dQ}{dP}(4) A(x1, x2, x3, x4) P_4(x1, x2, x3, x4)(dx4)(dx3)}{\int P_4(x1, x2, x3, x4)(dx4)(dx3)} \times \frac{\int P_4(x1, x2, x3, x4)(dx4)(dx3)}{\int \frac{dQ}{dP}(4) P_4(x1, x2, x3, x4)(dx4)(dx3)} = \frac{E_P\left[\frac{dQ}{dP}A \middle| 2\right]}{E_P\left[\frac{dQ}{dP} \middle| 2\right]}$$

Conditional Expectation and Radon Nikodym	$E_Q[A\|n] = \frac{E_P\left[\frac{dQ}{dP}A \middle\| n\right]}{E_P\left[\frac{dQ}{dP} \middle\| n\right]}$

What We Want to do

- A change of measure on a tree <=> change of measure in the expectations
- Expectation can be defined as expectation of Ito processes
- What is the equivalent of a change of measure for the associated Ito process?
- We do matter because it will be the key to make arbitrage free theories

A Way to Recognize a Gaussian Variable

If X is a normal (μ, σ)

The caracteristic function of X is $$E[e^{\theta X}] = \int \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2} + \theta x}}{\sigma\sqrt{2\pi}} dx = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}$$

If a process X_t **verify** $$E[e^{\theta X_t}] = e^{\theta\mu t + \frac{1}{2}\theta^2\sigma^2 t}$$

Then there a brownian motion W such that it is an ito process given by

$$dX_t = \mu dt + \sigma dW_t$$

Change of Measure and Brownians

Given an Ito Process $dX_t = \mu dt + \sigma dW_t$ **and** γ **a real process (predictable)**

Let's create the change of measure $\frac{dQ}{dP}(T) = e^{-\gamma W_T - \frac{1}{2}\gamma^2 T}$

$$E_Q[e^{\theta W_T}] = E_P\left[\frac{dQ}{dP}(T)e^{\theta W_T}\right] = E\left[e^{-\gamma W_T - \frac{1}{2}\gamma^2 T + \theta W_T}\right]$$

$$= e^{-\frac{1}{2}\gamma^2 T} E[e^{(\theta - \gamma) W_T}] = e^{-\frac{1}{2}\gamma^2 T + \frac{1}{2}(\theta - \gamma)^2 T} = e^{-\theta\gamma T + \frac{1}{2}\theta^2 T}$$

It is the carateristic function of a brownian shifted by $-\gamma T$

Therefore shifting a brownian by $-\gamma T$ **is equivalent to do a change of measure** $\frac{dQ}{dP}(T) = e^{-\gamma W_T - \frac{1}{2}\gamma^2 T}$

(Girsanov theorem)

If W_t is a P-Brownian motion then $W_t - \gamma t$ is a Q-Brownian motion

C-M-G and Inverse C-M-G

- If W_t is a P-Brownian and γ_t is a previsible process
  - then it exist a measure Q [expectation point of view] such
  - 1) Q is equivalent to P
  - 2) $\frac{dQ}{dP}(T) = \exp\left[-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right]$
  - 3) $W_t + \int_0^t \gamma_s ds$ is a Q-Brownian
- If W_t is a P-Brownian and Q is measure equivalent to P[expectation point of view]
  - then there exists a previsible process γ_t such that $W_t + \int_0^t \gamma_s ds$ is a Q-Brownian
  - $\frac{dQ}{dP}(T) = \exp\left[-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right]$

Arbitrage Free Models

- Given N asset prices that are represented by processes $S_i(t)$
- Its today price is the expectation of the discounted price in the future: For a given measure P , it always exists a process B_t such that

  - $Price[S] = E_P\left[\frac{1}{B_t}S_t\right]$

- Therefore $\frac{1}{B_t}S_t$ is a martingale in the measure P
- Therefore it exists a positive process $\left(X_t = \frac{1}{B_t}\right)$ such that $X_t S_t$ is a P-martingale
- If We call this process a change of measure $\frac{dQ_S}{dP}(t) = X_t$, therefore it exists a measure Q_S equivalent to P such that $E_{Q_S}[S_t]$ is the price of the asset
- Harrison and Pliska showed that: No Arbitrage => $Q_S = Q$

Abitrage Free Price Processes

- Let's assume we have N prices process given by Ito processes:

$$dX_{i,t} = \mu_i dt + \sigma_i \cdot dW_t$$

- Arbitrage Freeness => It exist a change of measure such all prices are martingales

- <=> (Girsanov) It exist a new set of brownians $\bar{W}_t$ and a change of drift γ_t such that $d\bar{X}_{i,t} = (\mu_i - \gamma_t)dt + \sigma_i \cdot d\bar{W}_t$ are martingales and the prices are the expectation of this new processes

- Theorem : An Ito Process is a martingale iff its drift =0

- Definition : γ_t is the instantaneous spot rate : $\gamma_t \equiv r_t$

- Arbitrage Freeness => it exist a brownian such that $dX_{i,t} = r_t dt + \sigma_i \cdot dW_t$

The HJM Models

-In the classical framework ,

$P(t, T)$ **is a bond maturing at T**

we define the forward rate as : $f(t, T) = -\frac{d}{dt}Log(P(t, T))$

Therefore we have $P(t, T) = P(0, T)e^{\int_0^t f(s, T)ds}$

we assume the existence of n independant brownian W_i

$$df(t, T) = \underbrace{(....)}_{\alpha(t, T, \omega)}dt + \sum_{i = 1, n} \sigma_i(t, T, \omega)dW_i(t, \omega)$$

- It is a good model because, by modeling the forward rate we avoid all static arbitrages: (Compounding N rates against another)

Deriving the HJM Relationship

We Start From $$df(t, T) = \alpha(t, T, \omega)dt + \sum_{i=1,n} \sigma_i(t, T, \omega)dW_i(t, \omega)$$

We Apply Ito on $$P(t, T) = P(0, T)e^{\int_0^t f(s, T)ds}$$

We find : $$\frac{dP(t, T)}{P} = \left[-\int_t^T \alpha(t, v)dv + f(t, t) + \frac{1}{2}\left(\int_t^T \sigma(t, v)dv\right)^2\right]dt - \left(\int_t^T \sigma(t, v)dv\right) \cdot dW_t$$

Independent of T

No-Arbitrage between bonds P(t,T) =>

In the risk-neutral measure : $$\boxed{\alpha(t) = \sum_{i=1,n} \sigma_i(t, T)\int_t^T \sigma_i(t, s)ds}$$

Universality of HJM

- Link with short term models: $P(t,T) = E_{N,t}\left[e^{-\int_0^T r_s ds}\right]$ and $f(t,T) = -\frac{\partial}{\partial T}P(t,T)$

- Vasicek : $dr_t = (\theta - \alpha r_t) + \sigma dW_t$ we can show that if we take

  - $\begin{cases} df(t,T) = (..)dt + \sigma e^{-\alpha(t-T)}dW_t \\ f(0,T) = \frac{\theta}{\alpha} + e^{-\alpha T}\left(r_0 - \frac{\theta}{\alpha}\right) - \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha T}) \end{cases}$ we get the same model (HW -> Id.)

- CIR : $dr_t = (\theta_t - \alpha_t r_t) + \sigma_t \sqrt{r_t} dW_t$

  - $\begin{cases} df(t,T) = (..)dt + \sigma_t \sqrt{r_t} D(t,T) dW_t \\ f(0,T) = r_0 D(0,T) + \int_0^T \theta_s D(s,T) ds \end{cases}$ we get the same model $\begin{cases} D(t,T) = \frac{\partial}{\partial T}B(t,T) \\ \frac{\partial B}{\partial t} = \frac{\sigma_t^2}{2}B^2 + \alpha\rho_t B - 1 \qquad B(T,T) = 1 \end{cases}$

Conclusion

- Transition probabilities specify a stochastic process.
- We can adjust transition probabilities by multiplying by a change of measure coefficient
- This is equivalent to adjust the drift of the associated Ito processes
- If a model is able to be made arbitrage-free, You make it arbitrage-free by playing on the drift
- The only sensible interest rate framework that assumes the existence of an instantaneous forward rate is HJM. In this case, the drift is completly determined by the volatilities
- Only one factor HJM models are used in practice
- But HJM is the gate toward a more powerful framework : BGM -> (To be Continued)

Reading Advice

- Financial Calculus (An Introduction to Derivative Pricing): by M. Baxter and A. Rennie, Cambridge. It gives the most gentle and friendly introduction to Ito, Girsanov and the other guys that I ever seen.

- Arbitrage Theory in Continuous Time : by T. Bjork, Oxford. It goes beyond the preceding and gives more material while keeping a very pedagogic presentation

- I do not recomend the Hull and White book or the Wilmott's or the Musiela -Rutkowski books. They do not explain stochastic concepts

- Harrison and Pliska (1981) Martingales and Stochastic Integrals in the Theory of Continuous Trading (Stochastic Processes Applications,11,215-260

- Heath, Jarrow, Morton (1992) Bond Pricing and the Term Structure of Interest Rates: a new Methodology for contingent Claims Valuation (Econometrica Vol 60, #1, January,77-105