Mean reverting bridge

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1 Brownian bridge

The different ways to define a brownian bridge are :

$$B_{x,y,l,t} = B_{x,x,l,t} + (y-x)\frac{t}{l}$$

where $B_{x,y,l,t}$ is the process whose value at time 0 is x , and whose value at time l is y

two ways to write the process $B_{x,x,l,t}$:

$$B_{x,x,l,t} \sim x + \frac{(l-t)}{l} W_{\frac{lt}{l-t}} \sim x + W_t - \frac{t}{l} W_l$$

The process can be defined by a differential equation :

$$dB_t = \frac{y - B_t}{l - t}dt + dWt$$

with $B_0 = x$

and we can therefore write :

$$B_t = x + (y - x)\frac{t}{l} + (l - t)\int_0^t \frac{dW_s}{l - s}$$

2 Mean Reverting Process

The mean reverting process described by the following equation :

$$dY_t = \alpha(Y_\infty - Y_t)dt + \sigma dW_t$$

has the following integrated solution:

$$Y_t \sim Y_\infty + (Y_0 - Y_\infty)e^{-\alpha t} + e^{-\alpha t}\sigma W_{\frac{e^{2\alpha t} - 1}{2\alpha}}$$

3 Mean Reverting Bridge

The brownian bridge can be understood at the limit (small boule around y tending toward 0) as Bconditionned by $B_l = y$

It means that we can partition the brownian path issued from $B_0 = 0$ into sets Ω_y of paths such that

$$p \in \Omega_{y,l} \Leftrightarrow (p_l = y)$$

the process described by $\Omega_{y,t}$ can be associated with the brownian bridge

and we have for any process Y:

$$E_t[Y] = \int_{(p \in P)} Y[p,t]d\mu_p = \int_y dy\left(\int_{(p \in \Omega_{y,l})} Y[p,t]d\mu_p\right) = \int_y dyE[Y|p_t = y]$$

now if we look at the following process :

$$C_{y,l,t} = C_\infty + (C_0 - C_\infty)e^{-\alpha t} + e^{-\alpha t}\sigma B_{0,y,l,\frac{e^{2\alpha t}-1}{2\alpha}}$$

The path are essentially the same but slightly distorded in value in a deterministic way and with a redefinition of the time through the function $g : t \to \frac{e^{2\alpha t}-1}{2\alpha}$ which is a bijection of the positive reals

The property of this process is that

$$\begin{aligned} C_{y,l,g^{-1}(l)} &= C_\infty + (C_0 - C_\infty)e^{-\alpha g^{-1}(l)} + e^{-\alpha g^{-1}(l)}\sigma B_{0,y,l,l} \\ &= C_\infty + (C_0 - C_\infty)e^{-\alpha g^{-1}(l)} + e^{-\alpha g^{-1}(l)}\sigma y \end{aligned}$$

so if we define

$$Y(y) = \frac{y - C_\infty - (C_0 - C_\infty)e^{-\alpha g^{-1}(l)}}{e^{-\alpha g^{-1}(l)}\sigma}$$

the following process

$$D_{y,\,l,\,t} = C_{\infty} + (C_0 - C_{\infty})e^{-\alpha g^{-1}(t)} + e^{-\alpha g^{-1}(t)}\sigma B_{0,\,Y(y),\,l,\,t}$$

will satisfy

$$D_{y,\,l,\,l} = y \qquad D_{y,\,l,\,0} = C_0$$

All these considerations lead us to define the mean reverting bridge as :

$$M_{x,\,y,\,l,\,\alpha,\,M_{\infty},\,\sigma,\,t} = M_{\infty} + \frac{(x - M_{\infty}) + \sigma B_{0,\,Y(y,\,l),\,l,\,t}}{\sqrt{1 + 2\alpha t}}$$

where

$$B_{0,\,y,\,l,\,t} \sim y\frac{t}{l} + \frac{(l-t)}{l}W_{\frac{lt}{l-t}} \sim y\frac{t}{l} + W_t - \frac{t}{l}W_l$$
$$\sim y\frac{t}{l} + (l-t)\int_0^t \frac{dW_s}{l-s}$$

and

$$Y(y,\,l) = \frac{\sqrt{1+2\alpha l}(y - M_{\infty}) - (x - M_{\infty})}{\sigma}$$