Mean Reversion Processes

(revised paper)

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1 RiskWatch implementation of Multi Dimensional Mean Reversion Process

Let assume we have N risk factors with mean reversion :

$$dS_i = B_i(S_{\infty,i} - S_i)dt + S_i\vec{A_i} \cdot \overrightarrow{dW}$$

where $\vec{A_i}$ and $\overrightarrow{dW}$ are vectorial.

This continuous process is positive, so we take the logarithm:

$$s_i = Log[S_i]$$

by aplying the Ito-Lemma we have :

$$ds_i = \left(\frac{B_i(S_{\infty,i} - S_i)}{S_i} - \frac{1}{2}\|A_i\|^2\right)dt + \vec{A_i} \cdot \overrightarrow{dW}$$

If we want to simulate it :

$$s_{n+1} = s_n + \left(\frac{B_i(S_{\infty,i} - S_i)}{S_i} - \frac{1}{2}\|A_i\|^2\right)(t_{n+1} - t_n) + \sum_k A_{i,k}\xi_{n,k}\sqrt{t_{n+1} - t_n}$$

then coming back to the "big S" dimension:

$$S_{n+1} = S_n e^{\left(\frac{B_i(S_{\infty,i} - S_i)}{S_i} - \frac{1}{2}\|A_i\|^2\right)(t_{n+1} - t_n) + \sum_k A_{i,k}\xi_{n,k}\sqrt{t_{n+1} - t_n}}$$

The problem with this formulation is that it can be used only with small time steps, as testified by the "HistoRisk Bug of RNB"

The troubles occurs when we have a big volatility with large time steps.

For example on the japanese markets we observe :

- daily volatility : 46%,
- daily mean reversion 2% .
- starting rate 0.01,
- long term rate : 0.01

with a time step of one year we see that we generate easily the following rate :

$$0.01 \times e^{-\frac{1}{2}0.46^2 \times 365 + 0.46 \times \sqrt{365}} = 1.1 \times 10^{-13}$$

which is almost zero, then at the next time steps the mean reversion will play to give :

$$1.1 \times 10^{-13} \times e^{\left(\frac{0.01}{1.1 \times 10^{-13}} \times 0.02 - \frac{1}{2}0.46^2\right) \times 365 + 0.46 \times \sqrt{365}} = 10^{181818\,000\,000}$$

this is why Riskwatch explodes !!!

This behavior is caracteristic of linear approximations outside their validity domain.

2 The Suggested Multi Dimensional mean reverting processes

Let Assume we have n Risk factors called s_i

$$ds_i = B_i(s_{\infty,i} - s_i)dt + \vec{A_i} \cdot \overrightarrow{dW_t}$$

This risk factors may be Log of interest rates, Log of volatilities, spreads or volatility spreads

this means that to compare with the preceding cases we need first to take the Log.

But The mean reverting feature is comparable with the Hull and White Process.It is in fact a multidimensional extension of the Hull and White Process for the short itme risk , without a numerical long term mean reverting limit depending on the time .

we can write this set of equation with a matrix form :

$$ds = B \cdot (s_\infty - s)dt + A \cdot dW$$

where B is a diagonal matrix

let’s split s into :

$$s = s_1 + s_2$$

such that :

$$ds_1 = B \cdot (s_\infty - s_1)dt$$
$$ds_2 = -B \cdot s_2 dt + A \cdot dW$$

we integrate easily s_1 by :

$$s_1 = s_\infty + K \cdot e^{-Bt}$$

where K is a diagonal matrix of constant.

In the second equation we do a change of variable

$$s_2 = e^{-Bt} \cdot x$$

then we have the following differential equation :

$$ds_2 = -B \cdot s_2 dt + e^{-Bt} \cdot dx$$

then we deduce that :

$$e^{-Bt} \cdot dx = A \cdot dW$$

which is equivalent to :

$$dx = e^{Bt}A \cdot dW$$

which really means that :

$$dx_{i,\,t} = \sum_k e^{B_i t} A_{i,\,k} dW_{k,\,t} = \sum_k A_{i,\,k} e^{B_i t} dW_{k,\,t}$$

Therefore we can write the final solution as :

$$x_{i,\,t} = \int_0^t \sum_k A_{i,\,k} e^{B_i s} dW_{k,\,s}$$

and :

$$s_{i,\,t} = s_\infty + K \cdot e^{-Bt} + e^{-Bt} \int_0^t \sum_k A_{i,\,k} e^{B_i s} dW_{k,\,s}$$

the constant K are determined by

$$s(0) = s_0$$

which give the final solution :

$$s_{i,\,t} = s_{\infty,\,i} + (s_{0,\,i} - s_{\infty,\,i}) e^{-B_i t} + e^{-B_i t} \int_0^t \sum_k A_{i,\,k} e^{B_i s} dW_{k,\,s}$$

We know that $= e^{-B_i t} \int_0^t \sum_k A_{i,\,k} e^{B_i s} dW_{k,\,s}$ is gaussian with a covariance equal to :

$$Cov[c_i, c_j] = \int_0^t \sum_k A_{i,k} A_{j,k} e^{(B_i + B_j)(s-t)} ds = \sum_k A_{i,k} A_{j,k} \frac{1 - e^{-(B_i + B_j)t}}{(B_i + B_j)}$$

We can therefore extract the square root of this matrix by a standard algorithm. such that :

$$\sum_k A_{i,k} A_{j,k} \frac{1 - e^{-(B_i + B_j)t}}{(B_i + B_j)} = \sum_k \overline{A_{i,k}(t)}\,\overline{A_{j,k}(t)}$$

and the result can be written as :

$$s_{i,t} = s_{\infty,i} + (s_{0,i} - s_{\infty,i}) e^{-B_i t} + \sum_k \overline{A_{i,k}(t)}\,\overline{W_{k,t}}$$

for simulation purposes we extract the time shifted form :

$$s_{i,t_{n+1}} = s_{\infty,i} + (s_{i,t_n} - s_{\infty,i}) e^{-B_i(t_{n+1} - t_n)} + \sum_k \overline{A_{i,k}(t)}\,\overline{W}_{k,t}$$

And using standard normal variables $\xi_{n,k}$:

$$s_{i,t_{n+1}} = s_{\infty,i} + \left(s_{i,t_n} - s_{\infty,i}\right) e^{-B_i(t_{n+1} - t_n)} + \sum_k \overline{A_{i,k}(t_{n+1} - t_n)}\,\xi_{n,k}$$

equation 1

if we want to compare with the riskwatch approach: let's introduce

$$S_i = e^{s_i}$$

we then have :

$$S_{i,t_{n+1}} = S_{i,t_n} \times \left(\frac{S_{\infty,i}}{S_{i,t_n}}\right)^{\left(1-e^{-B_i(t_{n+1}-t_n)}\right)} \times e^{\sum_k \overline{A_{i,k}(t_{n+1}-t_n)}\xi_{n,k}}$$

equation 2

The Preceding example : on the japanese markets we observe :

- daily volatility : 46%,
- daily mean reversion 2% .
- starting rate 0.01,
- long term rate : 0.01

By assuming that the eigenvalues of the matrix $\overline{A(t)}^* \overline{A(t)}$ will not be very different from the eigen value of : $A(0)^* A(0)$ which is the original covariance matrix.

with a time step of one year, with the same residuals we see that we generate the following rate :

$$0.01 \times e^{0.46} = 0.0997$$

which is 9% practically the upperbound for the one standard deviation quantile at infinity . Then at the next time steps the mean reversion will play to give :

$$0.0997 \times \left(\frac{0.01}{0.0997}\right)^{(1-e^{-0.02 \times 365})} \times e^{0.46} = 0.0998$$

which is still the upperbound of the one standard deviation quantile at infinity, a lower bound would have given us :

$$0.0997 \times \left(\frac{0.01}{0.0997}\right)^{(1 - e^{-0.02 \times 365})} \times e^{-0.46} = 0.0010$$

values with which we feel very confortable

Of course for very short period of time we remark that

$$\frac{1 - e^{-B_i(t_{n+1} - t_n)}}{B_i} \approx (t_{n+1} - t_n) \qquad \left(\frac{S_{\infty, i}}{S_{i, t_n}}\right)^{\left(1 - e^{-B_i(t_{n+1} - t_n)}\right)} \approx 1$$

and we find the classical lognormal behavior :

$$S_{i, t_{n+1}} \approx S_{i, t_n} \times e^{\sqrt{(t_{n+1} - t_n)} \sum_k A_{i, k} \xi_{n, k}}$$

3 Comparison of the RiskWatch Process with the Proposed Alternative

Both approach have the same type of parameters :

Mean reversion, Long term limit for every factor plus a covariance matrix .The differences are the following :

TABLE 1.

Aspects	RiskWatch	Suggested Approach
Time Steps	small	any size
Finite time calculation	approximated	exact
type of factors	only positive risk factors	only positive risk factors or risk factors that can be positive or negative

4 Calibration

To get the mean reverting level andthe mean reverting coefficient, we will minimize the same quadratic function than for the Actual Historisk . The difference , in case of a restriction to the positive factors is that we work at the logarithmic level instead of the raw risk factor level.

5 Conclusion : Suggested next setp

To solve the RNB Bug and hedge against similar bug reports from all other client, we suggest to simply swap the Riskwatch formula with preceding **equation 2**, and adjust the calibration module accordingly.

Then in a next release, we will use **equation 1** in order to mix the different type of variables.