From Black and Scholes to BGM

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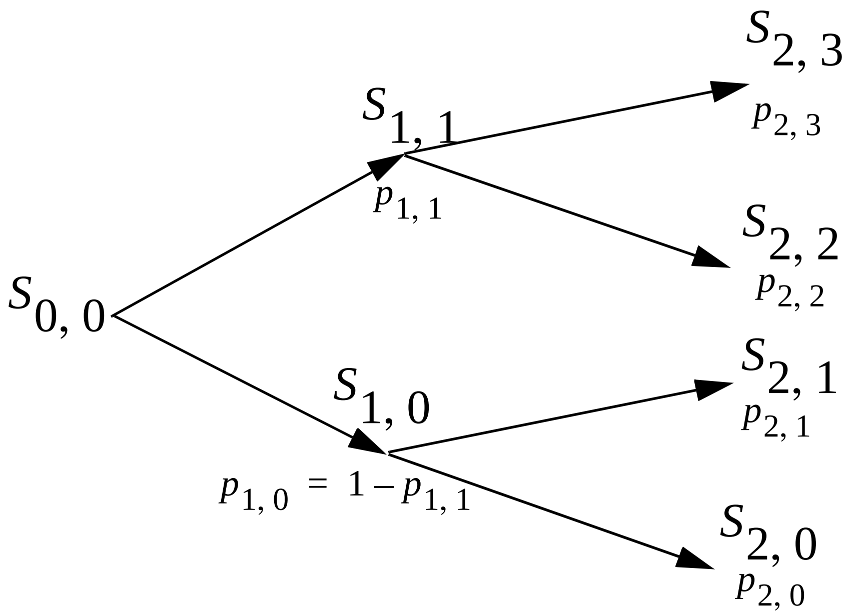
Messages

- A Black and Scholes formula is linked to the lognormality of the the future ratio of two assets
- Change of Numeraire => Change Of measures => Change of Drift=>derivative pricing
- Existence of an instantaneous interest rate is not needed for building an arbitrage free model (BGM). ====> Everything is simpler (more tractable)
- Using BGM, smiles from the cap market can be used to compute smiles for swaptions

Plan

- Revision of the Change of Measure in Ito Processes
- The Change of Numeraire Formula
- Black and Scholes as 1,2,3
- BGM
- Forward LIBOR Measure Foward swap measure
- Example of Calibration of a BGM Model
- Smiles Transfert
- Conclusion

The Change of Measure



A Process

=

A State Tree

+

Transition Probabilities

Changing of Measure

$\equiv$

Changing of probabilities

$$PV = E_P[S_T] = \sum p_{T,i} S_{T,i}$$

$$PV = E_P[S_T] = E_Q\left[\frac{dP}{dQ} S_T\right]$$

The 3 Equivalent Frameworks

Tree with Transition Probability

Change of probabilities

$$E_P[S_T] = \sum p_{T,i} S_{T,i}$$

$$\boxed{\frac{dP}{dQ}(S_{T,i}) = \frac{p_{T,i}}{Q_{T,i}}}$$

Expectations Computation

Change of Measure

$$E_P[S_T] = E_Q\left[\frac{dP}{dQ} S_T\right]$$

Ito Process

Change of Brownian

$$dS_t = \mu dt + \sigma dW_t$$

$$d\overline{S}_t = \mu dt + \sigma d\overline{W}_t$$

$$E[\overline{S}_t] = E\left[\frac{dP}{dQ} S_t\right]$$

$$\boxed{\begin{aligned} \overline{W}_t &= W_t + \int_0^t \gamma_s ds \\ \frac{dQ}{dP}(T) &= \exp\left[-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right] \end{aligned}}$$

Rule of Derivative Pricing

Set of Asset prices processes S_i **That defines the filtration** $\mathcal{F}_t$ **in the natural measure P**

We assume that we can build an arbitrage free system :

Therefore it exists a mesure Q such that all the prices are martingales

$$S_i(t) = E_Q[S_i(s) | \mathcal{F}_t] \qquad \textbf{for all s>t}$$

let be the associated change of measure process $\frac{dQ}{dP}(t) = \beta_t$ **with** $\beta_0 = 1$

we have in fact : $\beta_0 S_i(0) = E_P[\beta_t S_i(t) | \mathcal{F}_0] = E_P[\beta_s S_i(s) | \mathcal{F}_0]$

which means $\beta_t S_i(t) = E_P[\beta_s S_i(s) | \mathcal{F}_t]$

So any claim X at time T its price is :

$$\boxed{S_i(t) = \beta_t^{-1} E_P[\beta_T X(T) | \mathcal{F}_t]}$$

assuming the market is complete

<u>**Derivative Pricing Formula**</u>

Change of Numeraire

If we succeded finding a martingale measure P for a set of assets X_i
then at any t the price is $V_t = E_P[X|t]$

let be β_t **a previsible process**

Then it exists another measure, Q_β **under which** βX_i **are martingales, then at any time we have :**

$$\beta_t^{-1} V_t = E_{Q_\beta}[\beta_T^{-1} X_T | t] \qquad \textbf{with} \qquad \frac{dP}{dQ_\beta} = \beta$$

So we have $V_t = \beta_t E_{Q_\beta}[\beta_T^{-1} X_T | t] = \gamma_t E_{Q_\gamma}[\gamma_T^{-1} X_T | t] = \ldots$

The change of numeraire formula is therefore :

$$\gamma_t E_{Q_\gamma}[\gamma_T^{-1} X_T | t] = \beta_t E_{Q_\beta}[\beta_T^{-1} X_T | t]$$

$$\textbf{with} \qquad \frac{dQ_\gamma}{dQ_\beta} = \frac{\beta}{\gamma}$$

Black and Scholes as 1,2,3 (part 1)

$$Call = B_0 E_Q[B_T^{-1}(S_T - K)^+] = B_0 E_Q[B_T^{-1}(S_T - K)1_A]$$ **where** $A = \{S_T \geq K\}$

$$= B_0 E_Q[B_T^{-1} S_T 1_A] - K B_0 E_Q[B_T^{-1} 1_A]$$

change of numeraire

Numeraire=$B_t = S_t$

$B_T = S_T$

$Q = Q_Z$

assets= $\left\{1, \frac{P(t,T)}{S_t}\right\}$ Z_t

$$B_0 E_Q[B_T^{-1} S_T 1_A] = S_0 E_{Q_Z}[1_A] = S_0 Q_Z[A]$$

Numeraire=$B_t = P(t, T)$

$B_T = P(T, T) = 1$

$Q = Q_F$

assets= $\left\{1, \frac{S_t}{P(t,T)}\right\}$ F_t

$$K B_0 E_Q[B_T^{-1} 1_A] = K P(0, T) E_{Q_F}[1_A] = K P(0, T) Q_F[A]$$

Hypotheses

$F_t = \frac{S_t}{P(t,T)}$ **verifies** $\frac{dF_t}{F_t} = \sigma_t dW_t$ **with** σ_t **determinisitic**

F is a martingale under Q_F

W_t **Brownian Under the measure** Q_F

$Z_t = 1/F_t$

Ito => $\frac{dZ_t}{Z_t} = -\sigma_t dW_t + \sigma_t^2 dt$

Z is a martingale under Q_Z

Girsanov => $\frac{dZ_t}{Z_t} = \sigma_t dU_t$

U_t **Brownian Under the measure** Q_Z

Black and Scholes as 1,2,3 (part 2)

Lemma 1 When $\frac{dX_t}{X_t} = \sigma_t dW_t$ **(Ito)** Then $Log[X_t] = \int_0^t \sigma_s dW_s + Log[X_0] - \frac{1}{2}\int_0^t \sigma_s^2 ds$

Lemma 2 When $Y = \int_0^t \sigma_s dW_s + H$ Then $Y \sim Norm\left[H, \sqrt{\int_0^t \sigma_s^2 ds}\right]$

H deterministic **(Markov)**

Lemma 3 When $X \sim Norm[m, s]$ Then $Prob[X > 0] = N\left[\frac{m}{s}\right]$

Theorem

Put together : when $\frac{dX_t}{X_t} = \sigma_t dW_t$ Then $Prob[Log[X_t] > 0] = N\left[\frac{Log[X_0] - \frac{1}{2}\int_0^t \sigma_s^2 ds}{\sqrt{\int_0^t \sigma_s^2 ds}}\right]$

σ_t Deterministic

Black and Scholes as 1,2,3 (part 3)

$$Q_F[A] = Prob_{Q_F}[S_T > K] = Prob_{Q_F}[F_T > K] = Prob_{Q_F}\left[Log\left[\frac{F_T}{K}\right] > 0\right]$$

$$Q_Z[A] = Prob_{Q_Z}[S_T > K] = Prob_{Q_Z}\left[Z_T < \frac{1}{K}\right] = Prob_{Q_Z}[Log[KZ_T] < 0]$$

$\frac{d(F_t/K)}{(F_t/K)} = \sigma_t dW_t$ **implies that** $Prob\left[Log\left[\frac{F_t}{K}\right] > 0\right] \sim Norm\left[\left(Log\left[\frac{F_0}{K}\right] - \frac{1}{2}\int_0^t \sigma_s^2 ds\right) / \sqrt{\int_0^t \sigma_s^2 ds}\right]$

$\frac{d(KZ_t)}{(KZ_t)} = \sigma_t dU_t$ **implies that** $Prob[Log[KZ_t] > 0] \sim Norm\left[\left(Log[KZ_0] - \frac{1}{2}\int_0^t \sigma_s^2 ds\right) / \left(\sqrt{\int_0^t \sigma_s^2 ds}\right)\right]$

So $\begin{cases} Q_F[A] = N[d_2] \\ Q_Z[A] = N[d_1] \end{cases}$

And the value of the call is : $= B_0(F_0 N[d_1] - KN[d_2])$

(The only hypothesis is : $\frac{S_t}{P(t, T)}$ **has a deterministic volatility)**

The general Black and Scholes Formula

$$Call = B_p(F\Phi(d_1) - K\Phi[d_2])$$

$$d_1 = \left(Log\left[\frac{F}{K}\right] + \frac{V^2}{2}\right) / V$$

$$d_2 = d_1 - V$$

- Applicability:
  - Option on Bonds with Coupons and continuous yield
  - Quanto Option
  - term structure of volatility for the forward
  - Reset time and payment time different (structured options)
  - Approximation of asian option
  - Stochastic interest rate
  - Stochastic yield