Options With Mean Reversion (One Dimension)

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1 Integration of the one dimensional mean reverting motion

The equation to integrate is :

$$dr = (b - ar)dt + \sigma dw_t$$

we perform the change of variable :

$$r = r_1 + r_2$$

such that :

$$\begin{cases} dr_1 = (b - ar_1)dt \\ dr_2 = -ar_2dt + dw_t \end{cases}$$

the first equation can be integrated in

$$r_1 = \frac{b}{a} + Ke^{-at}$$

in the second one we perform the change of variable :

$$r_2 = e^{-at}x$$

we can deduce that :

$$dr_2 = -ar_2dt + e^{-at}dx$$

therefore :

$$dx = e^{at}\sigma dW_t$$

we do here a change of time :

$$f(t)dW_t = d\overline{W}_{\int_0^t f^2(s)ds}$$

that gives us:

$$dx = d\overline{W}_{\frac{e^{2at}-1}{2a}\sigma^2}$$

therefore

$$x = \overline{W}_{\frac{e^{2at}-1}{2a}\sigma^2}$$

and :

the second one can be integrated into :

$$r_2 = e^{-at}\left(\overline{W}_{\frac{e^{2at}-1}{2a}\sigma^2}\right)$$

the result is therefore :

$$r = \frac{b}{a} + Ke^{-at} + e^{-at}\left(\overline{W}_{\frac{e^{2at}-1}{2a}\sigma^2}\right)$$

wher K is determined such that

$$r(t_0) = r_0$$

let ‘s call

$$\frac{b}{a} = r_\infty$$

we then can write :

$$r = r_\infty + (r_0 - r_\infty)e^{-at} + e^{-at}\left(\overline{W}_{\frac{e^{2at}-1}{2a}\sigma^2}\right)$$

which is easy to simulate :

$$r_{n+1} - r_n = (r_\infty - r_n)(1 - e^{-a(t_{n+1} - t_n)}) + N\left[\sigma\sqrt{\frac{1 - e^{-2a(t_{n+1} - t_n)}}{2a}}, 0\right]$$

2 Digital Option Pricing

So the distirbution of the forward at time t is given by :

$$F = r_\infty + (r_0 - r_\infty)e^{-at}$$

and the variance of this forward is given by :

$$C_2 = \frac{1 - e^{-2at}}{2a}\sigma^2$$

So the formula for a digital option that pays \$1 if the underlying is above K is

$$Call = \int_K^\infty \frac{e^{-\frac{(x-F)^2}{2C_2}}}{\sqrt{2\pi C_2}}dx = 1 - \Phi\left(\frac{K - F}{\sqrt{C_2}}\right)$$