A Few Considerations on Equity Tree Techniques (part 1)

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Risk Neutral Valuation

- Risk neutral Valuation of a european option

$$V_t = E_t[e^{-r(T-t)}V_T]$$

- Conditional Expectation rule

$$V_t = E_t[e^{-r(T_1-t)}E_{T_1}[V_T]]$$

$$= \int_Z e^{-r(T_1-t)}E_{T_1}[V_T, (S_{T_1} = Z)]p(S_t, (S_{T_1} = Z))dZ$$

$$= \int_Z e^{-r(T_1-t)}E_{T_1}[V_T, (S_{T_1} = Z)]\frac{e^{-\frac{(S_t-Z)^2}{2\sigma^2\Delta t}}}{\sqrt{2\pi\Delta t}\sigma}dZ = \int_Z e^{-r(T_1-t)}E_{T_1}[V_T, (S_{T_1} = Z)]\varphi_t(Z)dZ$$

Risk Neutral Valuation for an American Option

- Discretization of the preceding formula for european options

$$V_t = \int_Z e^{-r\Delta t} E_{t+\Delta t}[V_t, (S_{t+\Delta t} = Z)] \varphi_t(Z) dZ$$

- Introduction of the exercise prices X_t for bermudian options

$$V_t = Max\left[X_t, \int_Z e^{-r\Delta t} E_{t+\Delta t}[V_t, (S_{t+\Delta t} = Z)] \varphi_t(Z) dZ\right]$$

- We can show that the preceding rule converge toward american option prices when $\Delta t \rightarrow 0$

Approximate Computation of an Integral

- We want to compute

$$\int_a^b f(X)dX$$

- The newton-Cotes Rule
  - N equally spaced point x_i, we find the polynomial of order N that fit these points,
  - then we compute : $\int_a^b f(X)dX \approx \sum_{i=0}^{N} \frac{p_i}{i+1}(a^{i+1} - (b)^{i+1})$.

- The Simpson Rule : N=3
  - $\int_a^b f(X)dX = \frac{1}{3}(b-a)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$
  - Exact up to order 3
  - In general for odd order N, we get an exact computation for polynomial of order N

Gauss Integration Rules

- Can we do better ?
- Yes !
- With N terms, we can get exact calculation for polynomial up to order 2N-1 ! if we sample f(X) at optimized points
- Gauss Theorem : $\int_{-\infty}^{\infty} f(X)w(X)dX \approx \sum_{i=1}^{N} w_i f(X_i)$ exact for polynomial up to 2N-1 if:

  - X_i are the zeros of special polynomials $P_n(X)$ called the orthogonal polynomials associated with w(X)

  - $w_j = \dfrac{-a_{N+1,N+1}\left(\int_{-\infty}^{\infty} P_N^2(X)w(X)dX\right)}{a_{N,V}P'_N(X_j)P_{N+1}(X_j)}$ for $1 \le j \le N$

Example of Gauss Polynomial

- For $w(X) = \dfrac{e^{-\frac{(S_t - Z)^2}{2\sigma^2 \Delta t}}}{\sqrt{2\pi\Delta t}\,\sigma}$

- N=2 :

  - $w_1 = 1/2 \qquad X_1 = S_t - \sigma\sqrt{\Delta t}$
    $w_2 = 1/2 \qquad X_2 = S_t + \sigma\sqrt{\Delta t}$

- N=3

  - $w_1 = 1/6 \qquad X_1 = S_t - \sigma\sqrt{3\Delta t}$
    $w_2 = 2/3 \qquad X_2 = S_t$
    $w_3 = 1/3 \qquad X_3 = S_t + \sigma\sqrt{3\Delta t}$

Main Property of Gauss-Hermite Integration

- The moments are matched up to the order of 2N-1

Extension of the preceding ideas to non derivable functions.

- Allow for non derivable function to be repriced exactly (Non derivable at Z=0).

- N=3, we reprice exactly first and second moments :

$$
\begin{aligned}
w_1 &= 1/\pi & \qquad X_1 &= S_t - \sigma\frac{\sqrt{2\pi\Delta t}}{2} \\
w_2 &= 1 - 2/\pi & \qquad X_2 &= S_t \\
w_3 &= 1/\pi & \qquad X_3 &= S_t + \sigma\frac{\sqrt{2\pi\Delta t}}{2}
\end{aligned}
$$

- Comparaison of a representative set of 324 options : (source : Omberg 88)

Table 1:

Tree	Maximum error	Mean error
Binomial Cox	22.4 cts	3.9 cts
Binomial Gauss Hermite	22.3 cts	3.6 cts
Trinomial Gauss-Hermite	24.4 cts	2.9 cts
Trinomial Sharpened	4.5 cts	1.0 cts