Mean Reversion Calibration with Maximum Likelihood

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1 Mean reversion with a time dependent volatility

The process is

$$f(t+dt,T)-f(t,T) = s_T(h_T - f(t,T))dt + g_T(T-t)dW^T_t$$

with

$$g_T(x) = a_T f(b_T, x)$$

where b_T is a parameter and f is any two variables function

The likelihood for a maturity is :

$$L = \sum_{0 \le n \le N-1} Log\left[\varphi_{[f_{n+1}|f_n]}\right]$$

and we have

$$\varphi_{[f_{n+1}|f_n]} = \frac{e^{\frac{-(f_{n+1}-f_n-s_T(h_T-f_n)\Delta t_n)^2}{2[g_T(T-t_n)]^2\Delta t_n}}}{g_T(T-t_n)\Delta t_n\sqrt{2\pi}}$$

where b_T is a parameter and f is any two variables function

The equations that we have to solve for maximizing the likelihood are :

$$\frac{\partial L}{\partial \sigma} = \frac{\partial L}{\partial s} = \frac{\partial L}{\partial h} = \frac{\partial L}{\partial b_T}$$

The solution of these equations can be written as :

$$a_T(b_T) = \sqrt{\frac{1}{N}\sum_{0 \le n \le N-1} \frac{I_n(f_{n+1}-f_n-s_T(b_T)(h_T(b_T)-f_n)\Delta t_n)^2}{\Delta t_n}}$$

$$s_T(b_T) = \frac{\sum_{0 \le n \le N-1} I_n(h_T(b_T)-f_n)(f_{n+1}-f_n)}{\sum_{0 \le n \le N-1} I_n\Delta t_n(h_T(b_T)-f_n)^2} \qquad I_n = \frac{1}{(f(b_T, T-t_n))^2}$$

$$h_T(b_T) = \frac{\left(\sum_{0 \le n \le N-1} I_n\Delta t_n f_n\right)\left(\sum_{0 \le n \le N-1} I_n f_n(f_{n+1}-f_n)\right) - \left(\sum_{0 \le n \le N-1} I_n(f_{n+1}-f_n)\right)\left(\sum_{0 \le n \le N-1} I_n\Delta t_n f_n^2\right)}{\left(\sum_{0 \le n \le N-1} I_n\Delta t_n\right)\left(\sum_{0 \le n \le N-1} I_n f_n(f_{n+1}-f_n)\right) - \left(\sum_{0 \le n \le N-1} I_n(f_{n+1}-f_n)\right)\left(\sum_{0 \le n \le N-1} I_n\Delta t_n f_n\right)}$$

If the parameter b_T is multidimensional, let's be it k, we have to solve k simultaneous equations that are exactly given by the preceding equation where we interpret the derivative $\frac{\partial f}{\partial b_T}(b_T, T-t_n)$ as a vector.

2 Mean reversion with time dependent coefficients

So

$$Log\left[\varphi_{[f_{n+1}|f_n]}\right] = \frac{-(f_{n+1}-f_n-s_TS_n(hH_n-f_n))^2}{2(v_TV_n)^2}-\frac{1}{2}Log[2\pi]-Log[vV_n]$$

we therefore derive:

$$L = -\frac{1}{2v_T^2}\sum_{0\le n\le N-1}\frac{-(f_{n+1}-f_n-s_TS_n(h_TH_n-f_n))^2}{(V_n)^2}-Log(v_T)N-\sum_{0\le n\le N-1}Log(V_n)$$

we have to solve

$$\begin{cases}\dfrac{\partial L}{\partial v_T} = \dfrac{1}{v_T^3}\displaystyle\sum_{0\le n\le N-1}\frac{-(f_{n+1}-f_n-s_TS_n(h_TH_n-f_n))^2}{V_n^2}-\frac{1}{v_T}N = 0\\[2ex] \dfrac{\partial L}{\partial s_T} = \dfrac{1}{v_T^2}\displaystyle\sum_{0\le n\le N-1}\frac{S_n(h_TH_n-f_n)(f_{n+1}-f_n-s_TS_n(h_TH_n-f_n))}{V_n^2} = 0\\[2ex] \dfrac{\partial L}{\partial h_T} = \dfrac{1}{v_T^2}\displaystyle\sum_{0\le n\le N-1}\frac{s_TS_nH_n\left(f_{n+1}-f_n-s_TS_n(h_TH_n-f_n)\right)}{V_n^2} = 0\end{cases}$$

which is equivalent to

$$\begin{cases} v_T = \sqrt{\sum_{0 \le n \le N-1} \frac{(f_{n+1} - f_n - s_T S_n (h_T H_n - f_n))^2}{N V_n^2}} \\ \sum_{0 \le n \le N-1} \frac{S_n f_n \left(f_{n+1} - f_n - s_T S_n (h_T H_n - f_n)\right)}{V_n^2} = 0 \\ \sum_{0 \le n \le N-1} \frac{S_n H_n \left(f_{n+1} - f_n - s_T S_n (h_T H_n - f_n)\right)}{V_n^2} = 0 \end{cases}$$

we define the statisitics

$$F_{sfd} = \sum_{0 \le n \le N-1} \frac{S_n f_n (f_{n+1} - f_n)}{V_n^2} \qquad F_{sshf} = \sum_{0 \le n \le N-1} \frac{S_n^2 H_n f_n}{V_n^2}$$

$$F_{ssff} = \sum_{0 \le n \le N-1} \frac{S_n^2 f_n^2}{V_n^2} \qquad F_{sshh} = \sum_{0 \le n \le N-1} \frac{S_n^2 H_n^2}{V_n^2}$$

$$F_{shd} = \sum_{0 \le n \le N-1} \frac{S_n H_n (f_{n+1} - f_n)}{V_n^2}$$

and we have to solve

$$\begin{gathered} F_{sfd} - s_T h_T F_{sshf} + s_T F_{ssff} = 0 \\ F_{shd} - s_T h_T F_{sshh} + s_T F_{sshf} \end{gathered}$$

The solution of this system is:

$$s_T = \frac{F_{shd} F_{ssff} - F_{sfd} F_{sshf}}{F_{shd} F_{sshf} - F_{sfd} F_{sshh}}$$

$$h_T = \frac{F_{sfd} F_{sshh} - F_{shd} F_{sshf}}{F_{sshf}^2 - F_{ssff} F_{sshh}}$$

$$v_T = \sqrt{\sum_{0 \le n \le N-1} \frac{(f_{n+1} - f_n - s_T S_n (h_T H_n - f_n))^2}{N V_n^2}}$$