BiSABR SPREADOPTION

$dF_1 = \alpha_1 C_1[F_1]\, dW_1;$

$d\alpha_1 = \nu_1 \alpha_1\, dW_{v1};$

$dW_1\, dW_{v1} = \rho_1\, dt;$

$dF_2 = \alpha_2 C_2[F_2]\, dW_2;$

$d\alpha_2 = \nu_2 \alpha_2\, dW_{v2};$

$dW_2\, dW_{v2} = \rho_2\, dt;$

plus the links

$dW_1\, dW_2 = \rho_s\, dt;\ dW_1\, dW_{v2} = \rho_{c12}\, dt;\ dW_2\, dW_{v1} = \rho_{c21}\, dt;\ dW_{v1}\, dW_{v2} = \rho_v\, dt;$

the pricing equation is :

$$-\partial_t H == \frac{1}{2}\epsilon^2 \alpha_1^2 C_1[f_1]^2 \frac{\partial^2 H}{\partial f_1^2} + \frac{1}{2}\epsilon^2 \alpha_2^2 C_2[f_2]^2 \frac{\partial^2 H}{\partial f_2^2} + \frac{1}{2}\epsilon^2 \nu_1 \alpha_1 \frac{\partial^2 H}{\partial \alpha_1^2} + \frac{1}{2}\epsilon^2 \nu_2[\alpha_2] \frac{\partial^2 H}{\partial \alpha_2^2} + \epsilon^2 \rho_1 \alpha_1 \nu_1 \alpha_1 C_1[f_1] \frac{\partial^2 H}{\partial \alpha_1 \partial f_1} + \epsilon^2 \rho_2 \alpha_2 \nu_2 \alpha_2 C_2[f_2] \frac{\partial^2 H}{\partial \alpha_2 \partial f_2} + \epsilon^2 \rho_{c12} \nu_2 \alpha_2 \alpha_1 C_1[f_1] \frac{\partial^2 H}{\partial \alpha_2 \partial f_1} + \epsilon^2 \rho_{c21} \nu_1 \alpha_1 \alpha_2 C_2[f_2] \frac{\partial^2 H}{\partial \alpha_1 \partial f_2} + \epsilon^2 \rho_s \alpha_1 C_1[F_1]\, \alpha_2 C_2[F_2] \frac{\partial^2 H}{\partial f_1 \partial f_2} + \epsilon^2 \rho_v \nu_1 \alpha_1 \nu_2 \alpha_2 \frac{\partial^2 H}{\partial \alpha_1 \partial \alpha_2}$$

with the boundary condition

$H[0] = \max[f_1 - f_2 - K, 0]$

the probability density is defined by :

$$\text{prob}\{F_1 < f_1[T] < F_1 + dF_1,\ F_2 < f_2[T] < F_2 + dF_2,\ A_1 < \alpha_1(T) < A_1 + dA_1,\ A_2 < \alpha_2(T) < A_2 + dA_2 \mid f_1[t] = f_1,\ f_2[t] = f_2,\ \alpha_1[t] = \alpha_1,\ \alpha_2[t] = \alpha_2\} = p[t, f_1, \alpha_1, f_2, \alpha_2, T, F_1, F_2, A_1, A_2]$$

The forward kolmogoroff equation of this density is :

$$p_T = \frac{1}{2}\epsilon^2 A_1^2 \left[C_1[F_1]^2 p\right]_{F_1 F_1} + \frac{1}{2}\epsilon^2 A_2^2 \left[C_2[F_2]^2 p\right]_{F_2 F_2} + \frac{1}{2}\epsilon^2 \nu_1^2 \left[A_1^2 p\right]_{A_1 A_1} + \frac{1}{2}\epsilon^2 \nu_2^2 \left[A_2^2 p\right]_{A_2 A_2} + \epsilon^2 \rho_1 \nu_1 \left[A_1^2 C_1[F_1]\, p\right]_{A_1 F_1} + \epsilon^2 \rho_2 \nu_2 \left[A_2^2 C_2[F_2]\, p\right]_{A_2 F_2} + \epsilon^2 \rho_s A_1 A_2 [C_1[F_1]\, C_2[F_2]\, p]_{F_1 F_2} + \epsilon^2 \rho_v \nu_1 \nu_2 [A_1 A_2 p]_{A_1 A_2} + \epsilon^2 \rho_{c12} A_1 \nu_2 [C_1[F_1]\, A_2 p]_{F_1 A_2} + \epsilon^2 \rho_{c21} A_2 \nu_1 [C_2[F_2]\, A_1 p]_{F_2 A_1}$$

with $p = \delta[F_1 - f_1] \times \delta[F_2 - f_2] \times \delta[A_1 - \alpha_1] \times \delta[A_2 - \alpha_2]$ at $T = t$

let $V[t, f_1, f_2, \alpha_1, \alpha_2]$ be the value of the spreadoption at date t strike K and maturity t_{tex}

we have :

$$V[t, f_1, f_2, \alpha_1, \alpha_2] = E[(F_1[t_{tex}] - F_2[t_{tex}] - K)^+ \mid F_1[t] = f_1, F_2[t] = f_2, A_1[t] = \alpha_1, A_2[t] = \alpha_2]$$

$$= \int_0^\infty dA_1 \int_0^\infty dA_2 \int_{F_2+K}^\infty dF_1 \int_0^\infty dF_2\ (F_1 - F_2 - K)\ p[t, f_1, \alpha_1, f_2, \alpha_2, T, F_1, F_2, A_1, A_2]$$

Change of variable :

$S = F_1 - F_2;$

$B = F_2;$

so :

$$\frac{\partial H}{\partial F_1} = \frac{\partial H}{\partial S};$$

$$\frac{\partial H}{\partial F_2} = \frac{\partial H}{\partial B} - \frac{\partial H}{\partial S};$$

$$\frac{\partial^2 H}{\partial F_1^2} = \frac{\partial^2 H}{\partial S^2};$$

$$\frac{\partial^2 H}{\partial F_1 \partial F_2} = \frac{\partial^2 H}{\partial B \partial S} - \frac{\partial^2 H}{\partial S^2};$$

$$\frac{\partial^2 H}{\partial F_2^2} = \frac{\partial^2 H}{\partial B^2} - 2\frac{\partial^2 H}{\partial B \partial S} + \frac{\partial^2 H}{\partial S^2}$$

so the Kolmogoroff forward equation rewrites :

$$p_T = \frac{1}{2}\epsilon^2 A_1^2 \left[C_1[S+B]^2 p\right]_{SS} + \frac{1}{2}\epsilon^2 A_2^2 \left(\left[C_2[B]^2 p\right]_{BB} - 2\left[C_2[B]^2 p\right]_{BS} + 2\left[C_2[B]^2 p\right]_{SS}\right) + \frac{1}{2}\epsilon^2 \nu_1^2 \left[A_1^2 p\right]_{A_1 A_1} + \frac{1}{2}\epsilon^2 \nu_2^2 \left[A_2^2 p\right]_{A_2 A_2} + \epsilon^2 \rho_1 \nu_1 \left[A_1^2 C_1[S+B]\ p\right]_{A_1 S} + \epsilon^2 \rho_2 \nu_2 \left(\left[A_2^2 C_2[B]\ p\right]_{A_2 B} - \left[A_2^2 C_2[B]\ p\right]_{A_2 S}\right) + \epsilon^2 \rho_S A_1 A_2 \left([C_1[S+B]\ C_2[B]\ p]_{SB} - [C_1[S+B]\ C_2[B]\ p]_{SS}\right) + \epsilon^2 \rho_V \nu_1 \nu_2 [A_1 A_2 p]_{A_1 A_2} + \epsilon^2 \rho_{c12} A_1 \nu_2 [C_1[S+B]\ A_2 p]_{SA_2} + \epsilon^2 \rho_{c21} A_2 \nu_1 [C_2[B]\ A_1 p]_{BA_1}$$

and we have :

$$V[t, f_1, f_2, \alpha_1, \alpha_2] = \int_0^\infty dA_1 \int_0^\infty dA_2 \int_K^\infty dS \int_0^\infty dB\ (S - K)\ p[t, f_1, \alpha_1, f_2, \alpha_2, T, S+B, B, A_1, A_2]$$

so by replacing and taking into account the integration over A variables giving 0 results :

$$= (f_1 - f_2 - K)^+ + \int_0^{t_{tex}} dT \int_0^\infty dA_1 \int_0^\infty dA_2 \int_K^\infty dS \int_0^\infty dB\ (S-K)^+ \left\{\frac{1}{2}\epsilon^2 A_1^2 \left[C_1[S+B]^2 p\right]_{SS} + \frac{1}{2}\epsilon^2 A_2^2 \left(\left[C_2[B]^2 p\right]_{BB} - 2\left[C_2[B]^2 p\right]_{BS} + \left[C_2[B]^2 p\right]_{SS}\right) + \epsilon^2 \rho_S A_1 A_2 \left([C_1[S+B]\ C_2[B]\ p]_{SB} - [C_1[S+B]\ C_2[B]\ p]_{SS}\right)\right\}$$

by regrouping :

$$= (f_1 - f_2 - K)^+ + \int_0^{t_{tex}} dT \int_0^\infty dA_1 \int_0^\infty dA_2 \int_K^\infty dS \int_0^\infty dB\, (S-K)^+$$
$$\left\{\left[\left(\frac{1}{2}\epsilon^2 A_1^2 C_1[S+B]^2 + \frac{1}{2}\epsilon^2 A_2^2 C_2[B]^2 - \epsilon^2 \rho_s A_1 A_2 C_1[S+B] C_2[B]\right) p\right]_{SS} + \frac{1}{2}\epsilon^2 A_2^2 \left[C_2[B]^2 p\right]_{BB} + \left[\left(\epsilon^2 \rho_s A_1 A_2 C_1[S+B] C_2[B] - \epsilon^2 A_2^2 C_2[B]^2\right) p\right]_{SB}\right\}$$

Now we integrate by parts with S and taking into account $C_2[0] = 0$

$$\int_0^\infty dB \int_0^\infty dS\, (S-K)^+ \left[\left(\frac{1}{2}\epsilon^2 A_1^2 C_1[S+B]^2 + \frac{1}{2}\epsilon^2 A_2^2 C_2[B]^2 - \epsilon^2 \rho_s A_1 A_2 C_1[S+B] C_2[B]\right) p\right]_{SS}$$
$$= \int_0^\infty dB \left(\frac{1}{2}\epsilon^2 A_1^2 C_1[K+B]^2 + \frac{1}{2}\epsilon^2 A_2^2 C_2[B]^2 - \epsilon^2 \rho_s A_1 A_2 C_1[K+B] C_2[B]\right) p_{S=K}$$

$$\int_0^\infty dB \int_K^\infty dS\, (S-K) \left[\left(\epsilon^2 \rho_s A_1 A_2 C_1[S+B] C_2[B] - \epsilon^2 A_2^2 C_2[B]^2\right) p\right]_{SB} =$$
$$\int_0^\infty dB \int_K^\infty dS \left[\left(\epsilon^2 \rho_s A_1 A_2 C_1[S+B] C_2[B] - \epsilon^2 A_2^2 C_2[B]^2\right) p\right]_B =$$
$$\int_K^\infty dS \left[\left(\epsilon^2 \rho_s A_1 A_2 C_1[S+B] C_2[0] - \epsilon^2 A_2^2 C_2[0]^2\right) p\right] = 0$$

we also have :

$$\int_0^\infty dS\, (S-K)^+ \int_0^\infty dB \frac{1}{2}\epsilon^2 A_2^2 \left[C_2[B]^2 p\right]_{BB} = 0$$

so in summary we have :

$$V[t, f_1, f_2, \alpha_1, \alpha_2] = (f_1 - f_2 - K)^+ + \int_0^{t_{tex}} dT \int_0^\infty dA_1 \int_0^\infty dA_2 \int_0^\infty dB \left(\frac{1}{2}\epsilon^2 A_1^2 C_1[K+B]^2 + \frac{1}{2}\epsilon^2 A_2^2 C_2[B]^2 - \epsilon^2 \rho_s A_1 A_2 C_1[K+B] C_2[B]\right) p_{S=K}$$

let ' s define

$$P[t, f_1, f_2, \alpha_1, \alpha_2] = \int_0^\infty dA_1 \int_0^\infty dA_2 \int_0^\infty dB \left(\frac{1}{2}\epsilon^2 A_1^2 C_1[K+B]^2 + \frac{1}{2}\epsilon^2 A_2^2 C_2[B]^2 - \epsilon^2 \rho_s A_1 A_2 C_1[K+B] C_2[B]\right) p_{S=K}$$

we know that the density satisfy the backward kolmogoroff equation :

$$-\partial_t p == \frac{1}{2}\epsilon^2 \alpha_1^2 C_1[f_1]^2 \frac{\partial^2 p}{\partial f_1^2} + \frac{1}{2}\epsilon^2 \alpha_2^2 C_2[f_2]^2 \frac{\partial^2 p}{\partial f_2^2} + \frac{1}{2}\epsilon^2 \nu_1^2 \alpha_1^2 \frac{\partial^2 p}{\partial \alpha_1^2} +$$
$$\frac{1}{2}\epsilon^2 \nu_2^2 \alpha_2^2 \frac{\partial^2 p}{\partial \alpha_2^2} + \epsilon^2 \rho_1 \nu_1 \alpha_1^2 C_1[f_1] \frac{\partial^2 p}{\partial \alpha_1 \partial f_1} + \epsilon^2 \rho_2 \nu_2 \alpha_2^2 C_2[f_2] \frac{\partial^2 p}{\partial \alpha_2 \partial f_2} +$$
$$\epsilon^2 \rho_{c12} \nu_2 \alpha_2 \alpha_1 C_1[f_1] \frac{\partial^2 p}{\partial \alpha_2 \partial f_1} + \epsilon^2 \rho_{c21} \nu_1 \alpha_1 \alpha_2 C_2[f_2] \frac{\partial^2 p}{\partial \alpha_1 \partial f_2} +$$
$$\epsilon^2 \rho_s \alpha_1 C_1[F_1] \alpha_2 C_2[F_2] \frac{\partial^2 p}{\partial f_1 \partial f_2} + \epsilon^2 \rho_v \nu_1 \alpha_1 \nu_2 \alpha_2 \frac{\partial^2 p}{\partial \alpha_1 \partial \alpha_2}$$

with $p = \delta[S-(f_1-f_2)] \times \delta[B-f_2] \times \delta[A_1-\alpha_1] \times \delta[A_2-\alpha_2]$ **at** $T = t$

so we deduce that P satisfy :

$$-\partial_t P == \frac{1}{2}\epsilon^2 \alpha_1^2 C_1[f_1]^2 \frac{\partial^2 P}{\partial f_1^2} + \frac{1}{2}\epsilon^2 \alpha_2^2 C_2[f_2]^2 \frac{\partial^2 P}{\partial f_2^2} + \frac{1}{2}\epsilon^2 \nu_1^2 \alpha_1^2 \frac{\partial^2 P}{\partial \alpha_1^2} + \frac{1}{2}\epsilon^2 \nu_2^2 \alpha_2^2 \frac{\partial^2 P}{\partial \alpha_2^2} + \epsilon^2 \rho_1 \alpha_1 \nu_1 \alpha_1 C_1[f_1] \frac{\partial^2 P}{\partial \alpha_1 \partial f_1} + \epsilon^2 \rho_2 \alpha_2 \nu_2 \alpha_2 C_2[f_2] \frac{\partial^2 P}{\partial \alpha_2 \partial f_2} + \epsilon^2 \rho_{c12} \nu_2 \alpha_2 \alpha_1 C_1[f_1] \frac{\partial^2 P}{\partial \alpha_2 \partial f_1} + \epsilon^2 \rho_{c21} \nu_1 \alpha_1 \alpha_2 C_2[f_2] \frac{\partial^2 P}{\partial \alpha_1 \partial f_2} + \epsilon^2 \rho_s \alpha_1 C_1[F_1] \alpha_2 C_2[F_2] \frac{\partial^2 P}{\partial f_1 \partial f_2} + \epsilon^2 \rho_v \nu_1 \alpha_1 \nu_2 \alpha_2 \frac{\partial^2 P}{\partial \alpha_1 \partial \alpha_2}$$

with $P[t] =$

$$\int_0^\infty dA_1 \int_0^\infty dA_2 \int_0^\infty dB \left(\frac{1}{2}\epsilon^2 A_1^2 C_1[K+B]^2 + \frac{1}{2}\epsilon^2 A_2^2 C_2[B]^2 - \epsilon^2 \rho_s A_1 A_2 C_1[K+B] C_2[B]\right) \delta[S-K] \times \delta[B-f_2] \times \delta[A_1-\alpha_1] \times \delta[A_2-\alpha_2] = \left(\frac{1}{2}\epsilon^2 \alpha_1^2 C_1[K+f_2]^2 + \frac{1}{2}\epsilon^2 \alpha_2^2 C_2[f_2]^2 - \epsilon^2 \rho_s \alpha_1 \alpha_2 C_1[K+f_2] C_2[f_2]\right) \delta[S-K]$$

So in summary we have to solve :

$$-\partial_t P == \frac{1}{2}\epsilon^2 \alpha_1^2 C_1[f_1]^2 \frac{\partial^2 P}{\partial f_1^2} + \frac{1}{2}\epsilon^2 \alpha_2^2 C_2[f_2]^2 \frac{\partial^2 P}{\partial f_2^2} + \frac{1}{2}\epsilon^2 \nu_1^2 \alpha_1^2 \frac{\partial^2 P}{\partial \alpha_1^2} + \frac{1}{2}\epsilon^2 \nu_2^2 \alpha_2^2 \frac{\partial^2 P}{\partial \alpha_2^2} + \epsilon^2 \rho_1 \nu_1 \alpha_1^2 C_1[f_1] \frac{\partial^2 P}{\partial \alpha_1 \partial f_1} \; \epsilon^2 \rho_2 \nu_2 \alpha_2^2 C_2[f_2] \frac{\partial^2 P}{\partial \alpha_2 \partial f_2} + \epsilon^2 \rho_{c12} \nu_2 \alpha_2 \alpha_1 C_1[f_1] \frac{\partial^2 P}{\partial \alpha_2 \partial f_1} + \epsilon^2 \rho_{c21} \nu_1 \alpha_1 \alpha_2 C_2[f_2] \frac{\partial^2 P}{\partial \alpha_1 \partial f_2} + \epsilon^2 \rho_s \alpha_1 C_1[f_1] \alpha_2 C_2[f_2] \frac{\partial^2 P}{\partial f_1 \partial f_2} + \epsilon^2 \rho_v \nu_1 \alpha_1 \nu_2 \alpha_2 \frac{\partial^2 P}{\partial \alpha_1 \partial \alpha_2}$$

with $P[0] = \left(\frac{1}{2}\epsilon^2 \alpha_1^2 C_1[K+f_2]^2 + \frac{1}{2}\epsilon^2 \alpha_2^2 C_2[f_2]^2 - \epsilon^2 \rho_s \alpha_1 \alpha_2 C_1[K+f_2] C_2[f_2]\right) \delta[f_1 - f_2 - K]$

Then the value of the call spreadoption is :

$$V[t_{tex}, f_1, f_2, \alpha_1, \alpha_2] = (f_1 - f_2 - K)^+ + \int_0^{t_{tex}} dT P[t, f_1, f_2, \alpha_1, \alpha_2]$$

that we can also write in the spread variables as :

$$-\partial_t P == \frac{1}{2}\,\epsilon^2\,\alpha_1{}^2\,C_1[f_1]^2\left(\frac{\partial^2 P}{\partial s^2}\right)+$$
$$\frac{1}{2}\,\epsilon^2\,\alpha_2{}^2\,C_2[f_2]^2\left(\frac{\partial^2 P}{\partial b^2}+\frac{\partial^2 P}{\partial s^2}-2\,\frac{\partial^2 P}{\partial b\,\partial s}\right)+\frac{1}{2}\,\epsilon^2\,\nu_1{}^2\,\alpha_1{}^2\,\frac{\partial^2 P}{\partial \alpha_1{}^2}+\frac{1}{2}\,\epsilon^2\,\nu_2{}^2\,\alpha_2{}^2\,\frac{\partial^2 P}{\partial \alpha_2{}^2}+$$
$$\epsilon^2\,\rho_1\,\nu_1\,\alpha_1{}^2\,C_1[f_1]\,\frac{\partial^2 P}{\partial \alpha_1\,\partial s}+\epsilon^2\,\rho_2\,\nu_2\,\alpha_2{}^2\,C_2[f_2]\left(\frac{\partial^2 P}{\partial \alpha_2\,\partial b}-\frac{\partial^2 P}{\partial \alpha_2\,\partial s}\right)+$$
$$\epsilon^2\,\rho_{c12}\,\nu_2\,\alpha_2\,\alpha_1\,C_1[f_1]\,\frac{\partial^2 P}{\partial \alpha_2\,\partial s}+\epsilon^2\,\rho_{c21}\,\nu_1\,\alpha_1\,\alpha_2\,C_2[f_2]\left(\frac{\partial^2 P}{\partial \alpha_1\,\partial b}-\frac{\partial^2 P}{\partial \alpha_1\,\partial s}\right)+$$
$$\epsilon^2\,\rho_s\,\alpha_1\,C_1[f_1]\,\alpha_2\,C_2[f_2]\left(\frac{\partial^2 P}{\partial s\,\partial b}-\frac{\partial^2 P}{\partial s^2}\right)+\epsilon^2\,\rho_v\,\nu_1\,\alpha_1\,\nu_2\,\alpha_2\,\frac{\partial^2 P}{\partial \alpha_1\,\partial \alpha_2}$$

with $P[0]=\left(\frac{1}{2}\,\epsilon^2\,\alpha_1{}^2\,C_1[K+b]^2+\frac{1}{2}\,\epsilon^2\,\alpha_2{}^2\,C_2[b]^2-\epsilon^2\,\rho_s\,\alpha_1\,\alpha_2\,C_1[K+b]\,C_2[b]\right)\delta[s-K]$

Then the value of the call spreadoption is :

$$V[t_{tex},\,f_1,\,f_2,\,\alpha_1,\,\alpha_2]=(f_1-f_2-K)^+ +\int_0^{t_{tex}} dT\,P[t,\,s+b,\,b,\,\alpha_1,\,\alpha_2]$$

that we reorganize as :

$$-\partial_t P == \frac{1}{2}\,\epsilon^2\left(\alpha_1{}^2\,C_1[s+b]^2+\alpha_2{}^2\,C_2[b]^2-2\,\rho_s\,\alpha_1\,C_1[s+b]\,\alpha_2\,C_2[b]\right)\frac{\partial^2 P}{\partial s^2}+$$
$$\frac{1}{2}\,\epsilon^2\,\alpha_2{}^2\,C_2[b]^2\,\frac{\partial^2 P}{\partial b^2}+$$
$$\epsilon^2\,\alpha_2\,C_2[b]\,\left(\rho_s\,\alpha_1\,C_1[s+b]-\alpha_2\,C_2[b]\right)\frac{\partial^2 P}{\partial b\,\partial s}+$$
$$\epsilon^2\,\nu_1\,\alpha_1\,\left(\rho_1\,\alpha_1\,C_1[s+b]-\rho_{c21}\,\alpha_2\,C_2[b]\right)\frac{\partial^2 P}{\partial \alpha_1\,\partial s}+$$
$$\epsilon^2\,\nu_2\,\alpha_2\,\left(\rho_{c12}\,\alpha_1\,C_1[s+b]-\rho_2\,\alpha_2\,C_2[b]\right)\frac{\partial^2 P}{\partial \alpha_2\,\partial s}+\epsilon^2\,\rho_{c21}\,\nu_1\,\alpha_1\,\alpha_2\,C_2[b]\left(\frac{\partial^2 P}{\partial \alpha_1\,\partial b}\right)+$$
$$\epsilon^2\,\rho_2\,\alpha_2\,\nu_2\,\alpha_2\,C_2[b]\left(\frac{\partial^2 P}{\partial \alpha_2\,\partial b}\right)+$$
$$\frac{1}{2}\,\epsilon^2\,\nu_1{}^2\,\alpha_1{}^2\,\frac{\partial^2 P}{\partial \alpha_1{}^2}+\frac{1}{2}\,\epsilon^2\,\nu_2{}^2\,\alpha_2{}^2\,\frac{\partial^2 P}{\partial \alpha_2{}^2}+\epsilon^2\,\rho_v\,\nu_1\,\alpha_1\,\nu_2\,\alpha_2\,\frac{\partial^2 P}{\partial \alpha_1\,\partial \alpha_2}$$

with $P[0]=\left(\frac{1}{2}\,\epsilon^2\,\alpha_1{}^2\,C_1[K+b]^2+\frac{1}{2}\,\epsilon^2\,\alpha_2{}^2\,C_2[b]^2-\epsilon^2\,\rho_s\,\alpha_1\,\alpha_2\,C_1[K+b]\,C_2[b]\right)\delta[s-K]$

Then the value of the call spreadoption is :

$$V[t_{tex},\,f_1,\,f_2,\,\alpha_1,\,\alpha_2]=(f_1-f_2-K)^+ +\int_0^{t_{tex}} dT\,P[t,\,s+b,\,b,\,\alpha_1,\,\alpha_2]$$

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* Start of the singular
perturbation simplification process *
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** ** ** ** ** ** *

Try to get closer to a SABR like equation

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(* We want to get closer to the
  following equation following stochastic process:
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$$ds = \alpha\, U[s]\ dW_1;$$

$$d\alpha = \mu\ dt + \nu\ \alpha\ dW_2;$$

$$dW_1 dW_2 = \rho\ dt;$$

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*)
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where U[F] is never 0 , that means U[F] keep the same sign

and $\int_K^f \frac{1}{U[s]} \, ds$ is defined so $\sqrt{B[\alpha\, z\, \epsilon] \,/\, B[0]}$ is also defined

In order to Keep the spread with the same sign,
we will shift the spread axis and the strike,
so the type of functions U[s] will be :

$$U[s] = (s + A)^{\beta}$$

First Change of Variable : purpose : to introduce the volatility of the spread as a variable

$$s_1 = s$$

$$b_1 = b$$

$$\alpha_{11} = \sqrt{\alpha_1^2\, C_1[s+b]^2 + \alpha_2^2\, C_2[b]^2 - 2\,\rho_s\, \alpha_1\, C_1[s+b]\, \alpha_2\, C_2[b]}$$

$$\alpha_{22} = \alpha_2$$

$$\frac{\partial \alpha_{11}}{\partial s} = \frac{\alpha_1\, (\alpha_1\, C_1[s+b] - \rho_s\, \alpha_2\, C_2[b])\, C_1'[s+b]}{\sqrt{\alpha_1^2\, C_1[s+b]^2 + \alpha_2^2\, C_2[b]^2 - 2\,\rho_s\, \alpha_1\, C_1[s+b]\, \alpha_2\, C_2[b]}} = \frac{Q_s}{Q}$$

$$\frac{\partial \alpha_{11}}{\partial b} = \frac{\alpha_1\, (\alpha_1\, C_1[s+b] - \rho_s\, \alpha_2\, C_2[b])\, C_1'[s+b] + \alpha_2\, (\alpha_2\, C_2[b] - \rho_s\, \alpha_1\, C_1[s+b])\, C_2'[b]}{\sqrt{\alpha_1^2\, C_1[s+b]^2 + \alpha_2^2\, C_2[b]^2 - 2\,\rho_s\, \alpha_1\, C_1[s+b]\, \alpha_2\, C_2[b]}} = \frac{Q_b}{Q}$$

$$\frac{\partial \alpha_{11}}{\partial \alpha_1} = \frac{C_1[s+b]\, (\alpha_1\, C_1[s+b] - \rho_s\, \alpha_2\, C_2[b])}{\sqrt{\alpha_1^2\, C_1[s+b]^2 + \alpha_2^2\, C_2[b]^2 - 2\,\rho_s\, \alpha_1\, C_1[s+b]\, \alpha_2\, C_2[b]}} = \frac{Q_1}{Q}$$

$$\frac{\partial \alpha_{11}}{\partial \alpha_2} = \frac{C_2[b]\, (\alpha_2\, C_2[b] - \rho_s\, \alpha_1\, C_1[s+b])}{\sqrt{\alpha_1^2\, C_1[s+b]^2 + \alpha_2^2\, C_2[b]^2 - 2\,\rho_s\, \alpha_1\, C_1[s+b]\, \alpha_2\, C_2[b]}} = \frac{Q_2}{Q}$$

or to find a change of variable that normalize in one shot the quadratic form

We have :

$$\frac{\partial P}{\partial s} = \frac{\partial P}{\partial s_1}\frac{\partial s_1}{\partial s} + \frac{\partial P}{\partial b_1}\frac{\partial b_1}{\partial s} + \frac{\partial P}{\partial \alpha_{11}}\frac{\partial \alpha_{11}}{\partial s} + \frac{\partial P}{\partial \alpha_{22}}\frac{\partial \alpha_{22}}{\partial s} =$$

$$\frac{\partial P}{\partial s_1} + \frac{\alpha_1\,(\alpha_1\,C_1[s+b] - \rho_s\,\alpha_2\,C_2[b])\,C_1'[s+b]}{\sqrt{\alpha_1^2\,C_1[s+b]^2 + \alpha_2^2\,C_2[b]^2 - 2\,\rho_s\,\alpha_1\,C_1[s+b]\,\alpha_2\,C_2[b]}}\frac{\partial P}{\partial \alpha_{11}} = \frac{\partial P}{\partial s_1} + \frac{Q_s}{Q}\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial P}{\partial b} = \frac{\partial P}{\partial s_1}\frac{\partial s_1}{\partial b} + \frac{\partial P}{\partial b_1}\frac{\partial b_1}{\partial b} + \frac{\partial P}{\partial \alpha_{11}}\frac{\partial \alpha_{11}}{\partial b} + \frac{\partial P}{\partial \alpha_{22}}\frac{\partial \alpha_{22}}{\partial b} =$$

$$\frac{\partial P}{\partial b_1} + \frac{\alpha_1\,(\alpha_1\,C_1[s+b] - \rho_s\,\alpha_2\,C_2[b])\,C_1'[s+b] + \alpha_2\,(\alpha_2\,C_2[b] - \rho_s\,\alpha_1\,C_1[s+b])\,C_2'[b]}{\sqrt{\alpha_1^2\,C_1[s+b]^2 + \alpha_2^2\,C_2[b]^2 - 2\,\rho_s\,\alpha_1\,C_1[s+b]\,\alpha_2\,C_2[b]}}$$

$$\frac{\partial P}{\partial \alpha_{11}} = \frac{\partial P}{\partial b_1} + \frac{Q_b}{Q}\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial P}{\partial \alpha_1} = \frac{\partial P}{\partial s_1}\frac{\partial s_1}{\partial \alpha_1} + \frac{\partial P}{\partial b_1}\frac{\partial b_1}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_{11}}\frac{\partial \alpha_{11}}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_{22}}\frac{\partial \alpha_{22}}{\partial \alpha_1} =$$

$$\frac{2\,C_1[s+b]\,(\alpha_1\,C_1[s+b] - \rho_s\,\alpha_2\,C_2[b])}{2\,\sqrt{\alpha_1^2\,C_1[s+b]^2 + \alpha_2^2\,C_2[b]^2 - 2\,\rho_s\,\alpha_1\,C_1[s+b]\,\alpha_2\,C_2[b]}}\frac{\partial P}{\partial \alpha_{11}} = \frac{Q_1}{Q}\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial P}{\partial \alpha_2} = \frac{\partial P}{\partial s_1}\frac{\partial s_1}{\partial \alpha_2} + \frac{\partial P}{\partial b_1}\frac{\partial b_1}{\partial \alpha_2} + \frac{\partial P}{\partial \alpha_{11}}\frac{\partial \alpha_{11}}{\partial \alpha_2} + \frac{\partial P}{\partial \alpha_{22}}\frac{\partial \alpha_{22}}{\partial \alpha_2} =$$

$$\frac{2\,C_2[b]\,(\alpha_2\,C_2[b] - \rho_s\,\alpha_1\,C_1[s+b])}{2\,\sqrt{\alpha_1^2\,C_1[s+b]^2 + \alpha_2^2\,C_2[b]^2 - 2\,\rho_s\,\alpha_1\,C_1[s+b]\,\alpha_2\,C_2[b]}}\frac{\partial P}{\partial \alpha_{11}} + \frac{\partial P}{\partial \alpha_{22}} =$$

$$\frac{Q_2}{Q}\frac{\partial P}{\partial \alpha_{11}} + \frac{\partial P}{\partial \alpha_{22}}$$

where $Q_s = Q\,\frac{\partial Q}{\partial s}$ and so on . and also $Q_s = \frac{1}{2}\frac{\partial\left(Q^2\right)}{\partial s}$

therefore

$$\frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right) = \frac{\partial Q_s}{\partial s}\Big/ Q - Q_s\,\frac{\partial Q}{\partial s}\Big/ Q^2 = \frac{\partial Q_s}{\partial s}\Big/ Q - \frac{Q_s^{\,2}}{Q^3} = \frac{Q^2\,\frac{\partial Q_s}{\partial s} - Q_s^{\,2}}{Q^3}$$

the same for the others :

$$\frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right) = \frac{\partial Q_b}{\partial s}\Big/ Q - Q_b\,\frac{\partial Q}{\partial s}\Big/ Q^2 = \frac{Q^2\,\frac{\partial Q_b}{\partial s} - Q_s\,Q_b}{Q^3}$$

In the case of the SABR Moels : we have :

$C_1[s+b] = (F_1)^{\beta_1}$;
$C_2[b] = (F_2)^{\beta_2}$;
$C_1'[b+s] = \beta_1\,(F_1)^{\beta_1-1}$;
$C_2'[b] = \beta_2\,(F_2)^{\beta_2-1}$;
$C_1''[b+s] = (\beta_1-1)\,\beta_1\,(F_1)^{\beta_1-2}$;
$C_2''[b] = (\beta_2-1)\,\beta_2\,(F_2)^{\beta_2-2}$;
Null

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Clear[α, β, ρ, ν]
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so we define

$$Q = \sqrt{\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\;\alpha_2\, C_2[b]}$$

$$\sqrt{\alpha_1^2\, C_1[b+s]^2 - 2\,\alpha_1\,\alpha_2\,\rho_{spread}\, C_1[b+s]\; C_2[b] + \alpha_2^2\, C_2[b]^2}$$

$$Q_1 = 2\, C_1[s+b]\;(\alpha_1\, C_1[s+b] - \rho_s\,\alpha_2\, C_2[b])$$

$$Q_2 = 2\, C_2[b]\;(\alpha_2\, C_2[b] - \rho_s\,\alpha_1\, C_1[s+b])$$

$$Q_s = \alpha_1^2\, C_1[b+s]\; C_1{}'[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[b+s]$$

$$Q_s = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\;\alpha_2\, C_2[b],\; s\right]}{2}\right]$$

$$\alpha_1^2\, C_1[b+s]\; C_1{}'[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[b+s]$$

$$Q_{F1} = \alpha_1^2\, C_1[F1]\; C_1{}'[F1] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[F1]$$

$$Q_{F1} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[F1]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[F1]\;\alpha_2\, C_2[b],\; F1\right]}{2}\right]$$

$$\alpha_1^2\, C_1[F1]\; C_1{}'[F1] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[F1]$$

$$Q_b = \alpha_1^2\, C_1[b+s]\; C_1{}'[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_1[b+s]\; C_2{}'[b] + \alpha_2^2\, C_2[b]\; C_2{}'[b]$$

$$Q_b = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\;\alpha_2\, C_2[b],\; b\right]}{2}\right]$$

$$\alpha_1^2\, C_1[b+s]\; C_1{}'[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_1[b+s]\; C_2{}'[b] + \alpha_2^2\, C_2[b]\; C_2{}'[b]$$

$$Q_{F2} = -\alpha_1\,\alpha_2\,\rho_{spread}\, C_1[F1]\; C_2{}'[b] + \alpha_2^2\, C_2[b]\; C_2{}'[b]$$

$$Q_{F2} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[F1]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[F1]\;\alpha_2\, C_2[b],\; b\right]}{2}\right]$$

$$-\alpha_1\,\alpha_2\,\rho_{spread}\, C_1[F1]\; C_2{}'[b] + \alpha_2^2\, C_2[b]\; C_2{}'[b]$$

$$Q_{\alpha_1} = \alpha_1\, C_1[b+s]^2 - \alpha_2\,\rho_{spread}\, C_1[b+s]\; C_2[b]$$

$$Q_{\alpha_1} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\;\alpha_2\, C_2[b],\; \alpha_1\right]}{2}\right]$$

$$\alpha_1\, C_1[b+s]^2 - \alpha_2\,\rho_{spread}\, C_1[b+s]\; C_2[b]$$

$$Q_{\alpha_2} = -\alpha_1\,\rho_{spread}\, C_1[b+s]\; C_2[b] + \alpha_2\, C_2[b]^2$$

$$Q_{\alpha_2} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\,\alpha_2\, C_2[b],\ \alpha_2\right]}{2}\right]$$

$$-\alpha_1\,\rho_{spread}\, C_1[b+s]\; C_2[b] + \alpha_2\, C_2[b]^2$$

$$(*\ Q_{ss} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial s^2}\ *)$$

$$Q_{ss} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\,\alpha_2\, C_2[b],\ s,\ s\right]}{2}\right]$$

$$\alpha_1^2\, C_1{}'[b+s]^2 + \alpha_1^2\, C_1[b+s]\; C_1{}''[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}''[b+s]$$

$$\frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right) = \frac{Q^2\, Q_{ss} - Q_s{}^2}{Q^3}$$

$$(*\ Q_{sb} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial s \partial b}\ *)$$

$$Q_{sb} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\,\alpha_2\, C_2[b],\ s,\ b\right]}{2}\right]$$

$$\alpha_1^2\, C_1{}'[b+s]^2 - \alpha_1\,\alpha_2\,\rho_{spread}\, C_1{}'[b+s]\; C_2{}'[b] + \alpha_1^2\, C_1[b+s]\; C_1{}''[b+s] - \alpha_1\,\alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}''[b+s]$$

$$\frac{\partial}{\partial b}\left(\frac{Q_s}{Q}\right) = \frac{Q^2\, Q_{sb} - Q_s\, Q_b}{Q^3}$$

$$(*\ Q_{s\alpha_1} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial s \partial \alpha_1}\ *)$$

$$Q_{s\alpha_1} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\,\alpha_2\, C_2[b],\ s,\ \alpha_1\right]}{2}\right]$$

$$2\,\alpha_1\, C_1[b+s]\; C_1{}'[b+s] - \alpha_2\,\rho_{spread}\, C_2[b]\; C_1{}'[b+s]$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{Q_s}{Q}\right) = \frac{Q^2\, Q_{s\alpha_1} - Q_s\, Q_{\alpha_1}}{Q^3}$$

$$(*\ Q_{s\alpha_2} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial s \partial \alpha_2}\ *)$$

$$Q_{s\alpha_2} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\,\alpha_2\, C_2[b],\ s,\ \alpha_2\right]}{2}\right]$$

$$-\alpha_1\,\rho_{spread}\, C_2[b]\; C_1{}'[b+s]$$

$$\frac{\partial}{\partial \alpha_2}\left(\frac{Q_s}{Q}\right) = \frac{Q^2\, Q_{s\alpha_2} - Q_s\, Q_{\alpha_2}}{Q^3}$$

$$(*\ Q_{bb} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial b^2}\ *)\, Q_{bb} = \text{Simplify}\left[\text{Distribute}\left[\frac{D\left[\alpha_1{}^2\, C_1[s+b]^2 + \alpha_2{}^2\, C_2[b]^2 - 2\,\rho_{spread}\,\alpha_1\, C_1[s+b]\,\alpha_2\, C_2[b],\ b,\ b\right]}{2}\right]\right]$$

$$\alpha_1^2\left(C_1{}'[b+s]^2 + C_1[b+s]\; C_1{}''[b+s]\right) - \alpha_1\,\alpha_2\,\rho_{spread}\,\left(2\, C_1{}'[b+s]\; C_2{}'[b] + C_2[b]\; C_1{}''[b+s] + C_1[b+s]\; C_2{}''[b]\right) + \alpha_2^2\left(C_2{}'[b]^2 + C_2[b]\; C_2{}''[b]\right)$$

$$\frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right) = \frac{Q^2 Q_{bb} - Q_b{}^2}{Q^3}$$

$(*\ Q_{b\alpha_1} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial b \partial \alpha_1}\ *)$

$$Q_{b\alpha_1} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2 C_1[s+b]^2 + \alpha_2{}^2 C_2[b]^2 - 2\rho_{spread}\,\alpha_1 C_1[s+b]\,\alpha_2 C_2[b],\ b,\ \alpha_1\right]}{2}\right]$$

$2\,\alpha_1\,C_1[b+s]\,C_1'[b+s] - \alpha_2\,\rho_{spread}\,C_2[b]\,C_1'[b+s] - \alpha_2\,\rho_{spread}\,C_1[b+s]\,C_2'[b]$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{Q_b}{Q}\right) = \frac{Q^2 Q_{b\alpha_1} - Q_b\, Q_{\alpha_1}}{Q^3}$$

$(*\ Q_{b\alpha_2} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial b \partial \alpha_2}\ *)$

$$Q_{b\alpha_2} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2 C_1[s+b]^2 + \alpha_2{}^2 C_2[b]^2 - 2\rho_{spread}\,\alpha_1 C_1[s+b]\,\alpha_2 C_2[b],\ b,\ \alpha_2\right]}{2}\right]$$

$-\alpha_1\,\rho_{spread}\,C_2[b]\,C_1'[b+s] - \alpha_1\,\rho_{spread}\,C_1[b+s]\,C_2'[b] + 2\,\alpha_2\,C_2[b]\,C_2'[b]$

$$\frac{\partial}{\partial \alpha_2}\left(\frac{Q_b}{Q}\right) = \frac{Q^2 Q_{b\alpha_2} - Q_b\, Q_{\alpha_2}}{Q^3}$$

$$\frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right) = \frac{Q^2 Q_{bs} - Q_s\, Q_b}{Q^3}$$

$(*\ \ Q_{\alpha_1\alpha_1} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial \alpha_1{}^2}\ *)$

$$Q_{\alpha_1\,\alpha_1} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2 C_1[s+b]^2 + \alpha_2{}^2 C_2[b]^2 - 2\rho_{spread}\,\alpha_1 C_1[s+b]\,\alpha_2 C_2[b],\ \alpha_1,\ \alpha_1\right]}{2}\right]$$

$C_1[b+s]^2$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{Q_{\alpha_1}}{Q}\right) = \frac{Q^2 Q_{\alpha_1\,\alpha_1} - Q_{\alpha_1}{}^2}{Q^3}$$

$(*\ Q_{\alpha_2\alpha_2} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial \alpha_2{}^2}\ *)$

$$Q_{\alpha_2\,\alpha_2} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2 C_1[s+b]^2 + \alpha_2{}^2 C_2[b]^2 - 2\rho_{spread}\,\alpha_1 C_1[s+b]\,\alpha_2 C_2[b],\ \alpha_2,\ \alpha_2\right]}{2}\right]$$

$C_2[b]^2$

$$\frac{\partial}{\partial \alpha_2}\left(\frac{Q_{\alpha_2}}{Q}\right) = \frac{Q^2 Q_{\alpha_2\,\alpha_2} - Q_{\alpha_2}{}^2}{Q^3}$$

$(*\ Q_{\alpha_1\alpha_2} = \frac{1}{2}\ \frac{\partial^2 (Q^2)}{\partial \alpha_1 \partial \alpha_2}\ *)$

$$Q_{\alpha_1\,\alpha_2} = \text{Distribute}\left[\frac{D\left[\alpha_1{}^2 C_1[s+b]^2 + \alpha_2{}^2 C_2[b]^2 - 2\rho_{spread}\,\alpha_1 C_1[s+b]\,\alpha_2 C_2[b],\ \alpha_1,\ \alpha_2\right]}{2}\right]$$

$-\rho_{spread}\,C_1[b+s]\,C_2[b]$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{Q_{\alpha_2}}{Q}\right) = \frac{Q^2 Q_{\alpha_1\,\alpha_2} - Q_{\alpha_2}\, Q_{\alpha_1}}{Q^3}$$

Summary using tangent vectors

$$=>\ \frac{\partial}{\partial s}=\frac{\partial}{\partial s_1}+\frac{Q_s}{Q}\frac{\partial}{\partial \alpha_{11}}$$

$$=>\ \frac{\partial}{\partial b}=\frac{\partial}{\partial b_1}+\frac{Q_b}{Q}\frac{\partial}{\partial \alpha_{11}}$$

$$=>\ \frac{\partial}{\partial \alpha_1}=\frac{Q_1}{Q}\frac{\partial}{\partial \alpha_{11}}$$

$$=>\ \frac{\partial}{\partial \alpha_2}=\frac{Q_2}{Q}\frac{\partial}{\partial \alpha_{11}}+\frac{\partial}{\partial \alpha_{22}}$$

So the equation rewrites as :

$$-\partial_t P == \frac{1}{2} A_{s_1{}^2}\frac{\partial^2 P}{\partial s_1{}^2}+A_{\alpha_{11}\,s_1}\frac{\partial^2 P}{\partial \alpha_{11}\,\partial s_1}+A_{\alpha_{11}{}^2}\frac{\partial^2 P}{\partial \alpha_{11}{}^2}+A_{\alpha_{11}}\frac{\partial P}{\partial \alpha_{11}}$$

with P[0] =

$$\left(\frac{1}{2}\epsilon^2\alpha_1{}^2 C_1[K+b]^2+\frac{1}{2}\epsilon^2\alpha_2{}^2 C_2[b]^2-\epsilon^2\rho_s\,\alpha_1\,\alpha_2\,C_1[K+b]\,C_2[b]\right)\delta[s-K]$$

Then the value of the call spreadoption is :

$$V[t_{tex}, f_1, f_2, \alpha_1, \alpha_2] = (f_1-f_2-K)^+ + \int_0^{t_{tex}} dT\,P[t, s+b, b, \alpha_1, \alpha_2]$$

$$\frac{\partial^2 P}{\partial s^2}=\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial s_1}+\frac{\partial P}{\partial \alpha_{11}}\frac{Q_s}{Q}\right)=$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial s_1}\right)+\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial \alpha_{11}}\right)\frac{Q_s}{Q}+\frac{\partial P}{\partial \alpha_{11}}\frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right)=$$

$$\frac{\frac{\partial}{\partial s_1}\partial P}{\partial s_1}+\frac{Q_s}{Q}\frac{\frac{\partial}{\partial \alpha_{11}}\partial P}{\partial s_1}+\frac{Q_s}{Q}\left(\frac{\frac{\partial}{\partial s_1}\partial P}{\partial \alpha_{11}}+\frac{Q_s}{Q}\frac{\frac{\partial}{\partial \alpha_{11}}\partial P}{\partial \alpha_{11}}\right)+\frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}=$$

$$\frac{\partial^2 P}{\partial s_1{}^2}+2\frac{Q_s}{Q}\frac{\partial^2 P}{\partial s_1\,\partial \alpha_{11}}+\left(\frac{Q_s}{Q}\right)^2\frac{\partial^2 P}{\partial \alpha_{11}{}^2}+\left(\frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right)\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b^2}=\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial b_1}+\frac{Q_b}{Q}\frac{\partial P}{\partial \alpha_{11}}\right)=$$

$$\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial b_1}\right)+\frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}+\frac{Q_b}{Q}\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_{11}}\right)=$$

$$\frac{\partial^2 P}{\partial b_1{}^2}+\frac{Q_b}{Q}\frac{\partial^2 P}{\partial b_1\,\partial \alpha_{11}}+\frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}+\frac{Q_b}{Q}\left(\frac{\partial^2 P}{\partial b_1\,\partial \alpha_{11}}+\frac{Q_b}{Q}\frac{\partial^2 P}{\partial \alpha_{11}{}^2}\right)=$$

$$\frac{\partial^2 P}{\partial b_1{}^2}+2\left(\frac{Q_b}{Q}\right)\frac{\partial^2 P}{\partial b_1\,\partial \alpha_{11}}+\left(\frac{Q_b}{Q}\right)^2\frac{\partial^2 P}{\partial \alpha_{11}{}^2}+\left(\frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right)\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial^2 P}{\partial \alpha_1{}^2}=\frac{Q_1}{Q}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_{11}}\right)+\frac{\partial}{\partial \alpha_1}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}=\left(\frac{Q_1}{Q}\right)^2\frac{\partial^2 P}{\partial \alpha_{11}{}^2}+\frac{\partial}{\partial \alpha_1}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial^2 P}{\partial \alpha_2^2} = \frac{\partial}{\partial \alpha_2}\left(\frac{\partial P}{\partial \alpha_{11}}\frac{Q_2}{Q}\right) + \frac{\partial P}{\partial \alpha_{22}} = \frac{Q_2}{Q}\frac{\partial}{\partial \alpha_2}\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{\partial}{\partial \alpha_2}\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\frac{Q_2}{Q}\left(\frac{Q_2}{Q}\frac{\partial}{\partial \alpha_{11}} + \frac{\partial}{\partial \alpha_{22}}\right)\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \left(\frac{Q_2}{Q}\frac{\partial}{\partial \alpha_{11}} + \frac{\partial}{\partial \alpha_{22}}\right)\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\left(\frac{Q_2}{Q}\right)^2\frac{\partial^2 P}{\partial \alpha_{11}^2} + 2\frac{Q_2}{Q}\frac{\partial^2 P}{\partial \alpha_{11}\,\partial \alpha_{22}} + \frac{\partial^2 P}{\partial \alpha_{22}^2} + \frac{\partial}{\partial \alpha_2}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial b}\right) = \frac{\partial}{\partial s}\left(\frac{\partial P}{\partial b_1} + \frac{\partial P}{\partial \alpha_{11}}\frac{Q_b}{Q}\right) =$$
$$\left(\frac{\partial}{\partial s_1} + \frac{Q_s}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial b_1}\right) + \frac{Q_b}{Q}\left(\frac{\partial}{\partial s_1} + \frac{Q_s}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} =$$
$$\frac{\partial^2 P}{\partial s_1\,\partial b_1} + \frac{Q_s}{Q}\left(\frac{\partial^2 P}{\partial \alpha_{11}\,\partial b_1}\right) + \frac{Q_b}{Q}\frac{\partial^2 P}{\partial s_1\,\partial \alpha_{11}} + \frac{Q_b}{Q}\frac{Q_s}{Q}\frac{\partial^2 P}{\partial \alpha_{11}^2} + \frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial \alpha_1}\right) = \frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\frac{\partial P}{\partial \alpha_{11}}\right) =$$
$$\frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_1}{Q}\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial \alpha_{11}}\right) = \frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_1}{Q}\left(\frac{\partial}{\partial s_1} + \frac{Q_s}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{11}}\right) =$$
$$\frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_1}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial \alpha_{11}}\right) + \frac{Q_1}{Q}\left(\frac{Q_s}{Q}\right)\left(\frac{\partial^2 P}{\partial \alpha_{11}^2}\right)$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial \alpha_2}\right) = \frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\frac{\partial P}{\partial \alpha_{11}} + \frac{\partial P}{\partial \alpha_{22}}\right) = \frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_2}{Q}\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \frac{\partial}{\partial s}\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_2}{Q}\left(\frac{\partial}{\partial s_1} + \frac{Q_s}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \left(\frac{\partial}{\partial s_1} + \frac{Q_s}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\left(\frac{\partial^2 P}{\partial s_1\,\partial \alpha_{22}}\right) + \frac{Q_2}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial \alpha_{11}}\right) + \frac{Q_s}{Q}\left(\frac{\partial^2 P}{\partial \alpha_{11}\,\partial \alpha_{22}}\right) + \frac{Q_2}{Q}\frac{Q_s}{Q}\left(\frac{\partial^2 P}{\partial \alpha_{11}^2}\right) + \frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_1}\right) = \frac{\partial}{\partial b}\left(\frac{Q_1}{Q}\frac{\partial P}{\partial \alpha_{11}}\right) =$$
$$\frac{\partial}{\partial b}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_1}{Q}\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_{11}}\right) = \frac{\partial}{\partial b}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_1}{Q}\left(\frac{\partial}{\partial b_1} + \frac{Q_b}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{11}}\right) =$$
$$\frac{\partial}{\partial b}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_1}{Q}\left(\frac{\partial^2 P}{\partial b_1\,\partial \alpha_{11}}\right) + \frac{Q_1}{Q}\left(\frac{Q_b}{Q}\right)\left(\frac{\partial^2 P}{\partial \alpha_{11}^2}\right)$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial b}\right) = \frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial b_1} + \frac{Q_b}{Q}\frac{\partial P}{\partial \alpha_{11}}\right) = \frac{Q_1}{Q}\frac{\partial P}{\partial b_1\,\partial \alpha_{11}} + \left(\frac{Q_b}{Q}\right)\frac{Q_1}{Q}\frac{\partial^2 P}{\partial \alpha_{11}^2} + \frac{\partial}{\partial \alpha_1}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_2}\right) = \frac{\partial}{\partial b}\left(\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{\partial P}{\partial \alpha_{22}}\right) = \frac{\partial}{\partial b}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_2}{Q}\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\frac{\partial}{\partial b}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}} + \frac{Q_2}{Q}\left(\frac{\partial}{\partial b_1} + \frac{Q_b}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{11}}\right) + \left(\frac{\partial}{\partial b_1} + \frac{Q_b}{Q}\frac{\partial}{\partial \alpha_{11}}\right)\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\left(\frac{\partial^2 P}{\partial b_1\,\partial \alpha_{22}}\right) + \frac{Q_2}{Q}\left(\frac{\partial^2 P}{\partial b_1\,\partial \alpha_{11}}\right) + \frac{Q_2}{Q}\frac{Q_b}{Q}\left(\frac{\partial^2 P}{\partial \alpha_{11}^2}\right) + \frac{Q_b}{Q}\left(\frac{\partial^2 P}{\partial \alpha_{11}\,\partial \alpha_{22}}\right) + \frac{\partial}{\partial b}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial \alpha_{11}}$$

$$\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial\alpha_2}\right) = \frac{\partial}{\partial\alpha_1}\left(\frac{Q_2}{Q}\frac{\partial P}{\partial\alpha_{11}} + \frac{\partial P}{\partial\alpha_{22}}\right) = \frac{\partial}{\partial\alpha_1}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}} + \frac{Q_2}{Q}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial\alpha_{11}}\right) + \frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial\alpha_{22}}\right) =$$

$$\frac{Q_2}{Q}\left(\frac{Q_1}{Q}\right)\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \left(\frac{Q_1}{Q}\right)\left(\frac{\partial^2 P}{\partial\alpha_{11}\,\partial\alpha_{22}}\right) + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

In Summary :

$$\frac{\partial^2 P}{\partial s^2} = \frac{\partial^2 P}{\partial s_1{}^2} + \left(\frac{Q_s}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + 2\,\frac{Q_s}{Q}\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}} + \frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b^2} = \frac{\partial^2 P}{\partial b_1{}^2} + \left(\frac{Q_b}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + 2\,\frac{Q_b}{Q}\frac{\partial^2 P}{\partial b_1\,\partial\alpha_{11}} + \frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial\alpha_2{}^2} = \frac{\partial^2 P}{\partial\alpha_{22}{}^2} + \left(\frac{Q_2}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + 2\,\frac{Q_2}{Q}\frac{\partial^2 P}{\partial\alpha_{11}\,\partial\alpha_{22}} + \frac{\partial}{\partial\alpha_2}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial\alpha_1{}^2} = \left(\frac{Q_1}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial s\,\partial b} = \frac{\partial^2 P}{\partial s_1\,\partial b_1} + \frac{Q_b}{Q}\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}} + \frac{Q_b\,Q_s}{Q^2}\frac{\partial^2 P}{\partial\alpha_{11}{}^2} + \frac{Q_s}{Q}\left(\frac{\partial^2 P}{\partial\alpha_{11}\,\partial b_1}\right) + \frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial s\,\partial\alpha_2} = \frac{\partial^2 P}{\partial s_1\,\partial\alpha_{22}} + \frac{Q_2}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}}\right) + \frac{Q_2\,Q_s}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{Q_s}{Q}\left(\frac{\partial^2 P}{\partial\alpha_{11}\,\partial\alpha_{22}}\right) + \frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b\,\partial\alpha_2} = \frac{\partial^2 P}{\partial b_1\,\partial\alpha_{22}} + \frac{Q_2}{Q}\left(\frac{\partial^2 P}{\partial b_1\,\partial\alpha_{11}}\right) + \frac{Q_2\,Q_b}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{Q_b}{Q}\left(\frac{\partial^2 P}{\partial\alpha_{11}\,\partial\alpha_{22}}\right) + \frac{\partial}{\partial b}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial s\,\partial\alpha_1} = \frac{Q_1}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}}\right) + \frac{Q_1\,Q_s}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b\,\partial\alpha_1} = \frac{Q_1}{Q}\left(\frac{\partial^2 P}{\partial b_1\,\partial\alpha_{11}}\right) + \frac{Q_1\,Q_b}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial\alpha_1\,\partial\alpha_2} = \frac{Q_1}{Q}\left(\frac{\partial^2 P}{\partial\alpha_{11}\,\partial\alpha_{22}}\right) + \frac{Q_2\,Q_1}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

If we remove the unused derivatives :

$$\frac{\partial^2 P}{\partial s^2} = \frac{\partial^2 P}{\partial s_1{}^2} + \left(\frac{Q_s}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + 2\,\frac{Q_s}{Q}\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}} + \frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b^2} = \left(\frac{Q_b}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + \frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial\alpha_1{}^2} = \left(\frac{Q_1}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial\alpha_2{}^2} = \left(\frac{Q_2}{Q}\right)^2 \frac{\partial^2 P}{\partial\alpha_{11}{}^2} + \frac{\partial}{\partial\alpha_2}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial s\,\partial b} = \frac{Q_b\,Q_s}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{Q_b}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}}\right) + \frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial s\,\partial\alpha_1} = \frac{Q_1 Q_s}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{Q_1}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}}\right) + \frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial s\,\partial\alpha_2} = \frac{Q_2 Q_s}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{Q_2}{Q}\left(\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{11}}\right) + \frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b\,\partial\alpha_2} = \frac{Q_2 Q_b}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{\partial}{\partial b}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial b\,\partial\alpha_1} = \frac{Q_1 Q_b}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_b}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$\frac{\partial^2 P}{\partial\alpha_1\,\partial\alpha_2} = \frac{Q_2 Q_1}{Q^2}\left(\frac{\partial^2 P}{\partial\alpha_{11}{}^2}\right) + \frac{\partial}{\partial\alpha_1}\left(\frac{Q_2}{Q}\right)\frac{\partial P}{\partial\alpha_{11}}$$

$$-\partial_t P == \frac{1}{2}\epsilon^2\left(Q^2\right)\frac{\partial^2 P}{\partial s^2} + \frac{1}{2}\epsilon^2\alpha_2{}^2 C_2[b]^2\frac{\partial^2 P}{\partial b^2} +$$
$$\epsilon^2\alpha_2 C_2[b]\,(\rho_s\alpha_1 C_1[s+b] - \alpha_2 C_2[b])\frac{\partial^2 P}{\partial b\,\partial s} +$$
$$\epsilon^2\nu_1\alpha_1\,(\rho_1\alpha_1 C_1[s+b] - \rho_{c21}\alpha_2 C_2[b])\frac{\partial^2 P}{\partial\alpha_1\,\partial s} +$$
$$\epsilon^2\nu_2\alpha_2\,(\rho_{c12}\alpha_1 C_1[s+b] - \rho_2\alpha_2 C_2[b])\frac{\partial^2 P}{\partial\alpha_2\,\partial s} +$$
$$\epsilon^2\rho_{c21}\nu_1\alpha_1\alpha_2 C_2[b]\left(\frac{\partial^2 P}{\partial\alpha_1\,\partial b}\right) + \epsilon^2\rho_2\alpha_2\nu_2\alpha_2 C_2[b]\left(\frac{\partial^2 P}{\partial\alpha_2\,\partial b}\right) +$$
$$\frac{1}{2}\epsilon^2\nu_1{}^2\alpha_1{}^2\frac{\partial^2 P}{\partial\alpha_1{}^2} + \frac{1}{2}\epsilon^2\nu_2{}^2\alpha_2{}^2\frac{\partial^2 P}{\partial\alpha_2{}^2} + \epsilon^2\rho_v\nu_1\alpha_1\nu_2\alpha_2\frac{\partial^2 P}{\partial\alpha_1\,\partial\alpha_2}$$

can be rewrited as :

that we rewrite as :

$$-\partial_t P == \frac{1}{2}A_{s_1{}^2}\frac{\partial^2 P}{\partial s_1{}^2} + A_{\alpha_{11}\,s_1}\frac{\partial^2 P}{\partial\alpha_{11}\,\partial s_1} + A_{\alpha_{11}{}^2}\frac{\partial^2 P}{\partial\alpha_{11}{}^2} + A_{\alpha_{11}}\frac{\partial P}{\partial\alpha_{11}}\times$$

with

$$A_{s_1{}^2} = \frac{1}{2}\epsilon^2\left(Q^2\right)$$

$$A_{\alpha_{11}\,s_1} = Q\,Q_s + \alpha_2 C_2[b]\,(\rho_s\alpha_1 C_1[s+b] - \alpha_2 C_2[b])\frac{Q_b}{Q} +$$
$$\nu_1\alpha_1\,(\rho_1\alpha_1 C_1[s+b] - \rho_{c21}\alpha_2 C_2[b])\frac{Q_1}{Q} + \nu_2\alpha_2\,(\rho_{c12}\alpha_1 C_1[s+b] - \rho_2\alpha_2 C_2[b])\frac{Q_2}{Q}$$

$$A_{\alpha_{11}^2} = \frac{1}{2}\left(\left(\frac{Q_s}{Q}\right)^2 Q^2\right) + \frac{1}{2}\ \alpha_2^2\, C_2[b]^2 \left(\frac{Q_b}{Q}\right)^2 +$$
$$\frac{1}{2}\ \nu_1^2\, \alpha_1^2 \left(\frac{Q_1}{Q}\right)^2 + \frac{1}{2}\ \nu_2^2\, \alpha_2^2 \left(\frac{Q_2}{Q}\right)^2 + \rho_v\, \nu_1\, \alpha_1\, \nu_2\, \alpha_2\, \frac{Q_2\, Q_1}{Q^2} +$$
$$\alpha_2\, C_2[b]\ (\rho_s\, \alpha_1\, C_1[s+b] - \alpha_2\, C_2[b])\ \frac{Q_b\, Q_s}{Q^2} +$$
$$\nu_1\, \alpha_1\ (\rho_1\, \alpha_1\, C_1[s+b] - \rho_{c21}\, \alpha_2\, C_2[b])\ \frac{Q_1\, Q_s}{Q^2} +$$
$$\nu_2\, \alpha_2\ (\rho_{c12}\, \alpha_1\, C_1[s+b] - \rho_2\, \alpha_2\, C_2[b])\ \frac{Q_2\, Q_s}{Q^2} +$$
$$\rho_{c21}\, \nu_1\, \alpha_1\, \alpha_2\, C_2[b]\ \frac{Q_1\, Q_b}{Q^2} + \rho_2\, \alpha_2\, \nu_2\, \alpha_2\, C_2[b]\ \frac{Q_2\, Q_b}{Q^2}$$

$$\text{Simplify}\Big[\frac{1}{2}\ \nu_1^2\, \alpha_1^2\ (Q_1)^2 + \frac{1}{2}\ \nu_2^2\, \alpha_2^2\ (Q_2)^2 + \rho_v\, \nu_1\, \alpha_1\, \nu_2\, \alpha_2\, Q_2\, Q_1 +$$
$$\frac{1}{2}\ \alpha_2^2\, C_2[b]^2\ (Q_b)^2 + \alpha_2\, C_2[b]\ (\rho_s\, \alpha_1\, C_1[s+b] - \alpha_2\, C_2[b])\ Q_b\, Q_s +$$
$$\nu_1\, \alpha_1\ (\rho_1\, \alpha_1\, C_1[s+b] - \rho_{c21}\, \alpha_2\, C_2[b])\ Q_1\, Q_s +$$
$$\nu_2\, \alpha_2\ (\rho_{c12}\, \alpha_1\, C_1[s+b] - \rho_2\, \alpha_2\, C_2[b])\ Q_2\, Q_s +$$
$$\rho_{c21}\, \nu_1\, \alpha_1\, \alpha_2\, C_2[b]\ Q_1\, Q_b + \rho_2\, \alpha_2\, \nu_2\, \alpha_2\, C_2[b]\ Q_2\, Q_b\Big]$$

$$\frac{1}{2}\ \Big(Q_1^2\, \alpha_1^2\, \nu_1^2 + 2\, Q_1\, \alpha_1\, \nu_1\ (Q_2\, \alpha_2\, \nu_2\, \rho_v + Q_b\, \alpha_2\, \rho_{c21}\, C_2[b] + Q_s\ (\alpha_1\, \rho_1\, C_1[b+s] - \alpha_2\, \rho_{c21}\, C_2[b])) +$$
$$\alpha_2\ \Big(Q_2^2\, \alpha_2\, \nu_2^2 + Q_b\, C_2[b]\ (Q_b\, \alpha_2\, C_2[b] + 2\, Q_s\ (\alpha_1\, \rho_s\, C_1[b+s] - \alpha_2\, C_2[b])) +$$
$$2\, Q_2\, \nu_2\ (Q_b\, \alpha_2\, \rho_2\, C_2[b] + Q_s\ (\alpha_1\, \rho_{c12}\, C_1[b+s] - \alpha_2\, \rho_2\, C_2[b]))\Big)\Big)$$

$$A_{\alpha_{11}} = \frac{1}{2}\ (Q^2)\ \frac{\partial}{\partial s}\left(\frac{Q_s}{Q}\right) + \frac{1}{2}\ \alpha_2^2\, C_2[b]^2\ \frac{\partial}{\partial b}\left(\frac{Q_b}{Q}\right) +$$
$$\alpha_2\, C_2[b]\ (\rho_s\, \alpha_1\, C_1[s+b] - \alpha_2\, C_2[b])\ \frac{\partial}{\partial s}\left(\frac{Q_b}{Q}\right) +$$
$$\nu_1\, \alpha_1\ (\rho_1\, \alpha_1\, C_1[s+b] - \rho_{c21}\, \alpha_2\, C_2[b])\ \frac{\partial}{\partial s}\left(\frac{Q_1}{Q}\right) +$$
$$\nu_2\, \alpha_2\ (\rho_{c12}\, \alpha_1\, C_1[s+b] - \rho_2\, \alpha_2\, C_2[b])\ \frac{\partial}{\partial s}\left(\frac{Q_2}{Q}\right) +$$
$$\rho_{c21}\, \nu_1\, \alpha_1\, \alpha_2\, C_2[b]\ \frac{\partial}{\partial \alpha_1}\left(\frac{Q_b}{Q}\right) + \rho_2\, \alpha_2\, \nu_2\, \alpha_2\, C_2[b]\ \frac{\partial}{\partial b}\left(\frac{Q_2}{Q}\right) + \frac{1}{2}\ \nu_1^2\, \alpha_1^2\ \frac{\partial}{\partial \alpha_1}\left(\frac{Q_1}{Q}\right) +$$
$$\frac{1}{2}\ \nu_2^2\, \alpha_2^2\ \frac{\partial}{\partial \alpha_2}\left(\frac{Q_2}{Q}\right) + \left(\rho_v\, \nu_1\, \alpha_1\, \nu_2\, \alpha_2\ \frac{\partial}{\partial \alpha_1}\left(\frac{Q_2}{Q}\right)\right)^2$$

To use the extended SABR Formula we just have to specify :

$$C[s] = \frac{\sqrt{\left(\alpha_1^2\, C_1[s+F_2]^2 +{}^2\, \alpha_2^2\, C_2[F_2]^2 - 2\, \rho_s\, \alpha_1\, \alpha_2\, C_1[s+F_2]\ C_2[F_2]\right)}}{\sqrt{\left(\alpha_1^2\, C_1[F_1]^2 +{}^2\, \alpha_2^2\, C_2[F_2]^2 - 2\, \rho_s\, \alpha_1\, \alpha_2\, C_1[F_1]\ C_2[F_2]\right)}}$$

$$C'[s] = \frac{Q_s[s, F_2]}{Q} = \frac{\alpha_1^2 C_1[F_2 + s]\, C_1'[F_2 + s] - \alpha_1 \alpha_2 \rho_{spread} C_2[F_2]\, C_1'[F_2 + s]}{\sqrt{\left(\alpha_1{}^2 C_1[F_1]^2 +^2 \alpha_2{}^2 C_2[F_2]^2 - 2 \rho_s \alpha_1 \alpha_2 C_1[F_1]\, C_2[F_2]\right)}};$$

$$C''[s] = \frac{\left(\alpha_1^2 C_1'[F_2 + s]^2 + \alpha_1^2 C_1[F_2 + s]\, C_1''[F_2 + s] - \alpha_1 \alpha_2 \rho_s C_2[F_2]\, C_1''[F_2 + s]\right)}{\sqrt{\left(\alpha_1{}^2 C_1[F_1]^2 +^2 \alpha_2{}^2 C_2[F_2]^2 - 2 \rho_s \alpha_1 \alpha_2 C_1[F_1]\, C_2[F_2]\right)}};$$

the vol of the spead is scaled by is scaled by

$$\sqrt{\left(\alpha_1{}^2 C_1[F_1]^2 +^2 \alpha_2{}^2 C_2[F_2]^2 - 2 \rho_s \alpha_1 \alpha_2 C_1[F_1]\, C_2[F_2]\right)}$$

Neglected Terms

the terms we neglect are :

$$\begin{aligned}
N = {} & \frac{1}{2} \alpha_2{}^2 C_2[b]^2 \left\{\frac{\partial^2 P}{\partial b_1{}^2} + 2 \frac{Q_b}{Q} \frac{\partial^2 P}{\partial b_1 \partial \alpha_{11}}\right\} + \\
& \alpha_2 C_2[b] \left(\rho_s \alpha_1 C_1[s + b] - \alpha_2 C_2[b]\right) \left\{\frac{\partial^2 P}{\partial s_1 \partial b_1} + \frac{Q_s}{Q} \left(\frac{\partial^2 P}{\partial \alpha_{11} \partial b_1}\right)\right\} + \\
& \nu_2 \alpha_2 \left(\rho_{c12} \alpha_1 C_1[s + b] - \rho_2 \alpha_2 C_2[b]\right) \left\{\frac{\partial^2 P}{\partial s_1 \partial \alpha_{22}} + \frac{Q_s}{Q} \left(\frac{\partial^2 P}{\partial \alpha_{11} \partial \alpha_{22}}\right)\right\} + \\
& \rho_{c21} \nu_1 \alpha_1 \alpha_2 C_2[b] \left\{\frac{Q_1}{Q} \left(\frac{\partial^2 P}{\partial b_1 \partial \alpha_{11}}\right)\right\} + \\
& \rho_2 \alpha_2 \nu_2 \alpha_2 C_2[b] \left\{\frac{\partial^2 P}{\partial b_1 \partial \alpha_{22}} + \frac{Q_2}{Q} \left(\frac{\partial^2 P}{\partial b_1 \partial \alpha_{11}}\right) + \frac{Q_b}{Q} \left(\frac{\partial^2 P}{\partial \alpha_{11} \partial \alpha_{22}}\right)\right\} + \\
& \frac{1}{2} \nu_2{}^2 \alpha_2{}^2 \left\{\frac{\partial^2 P}{\partial \alpha_{22}{}^2} + 2 \frac{Q_2}{Q} \frac{\partial^2 P}{\partial \alpha_{11} \partial \alpha_{22}}\right\} + \rho_v \nu_1 \alpha_1 \nu_2 \alpha_2 \left\{\frac{Q_1}{Q} \left(\frac{\partial^2 P}{\partial \alpha_{11} \partial \alpha_{22}}\right)\right\}
\end{aligned}$$

that we rewrite :

$$N = \left\{\frac{1}{2}\ \alpha_2{}^2\, C_2[b]^2\right\} \frac{\partial^2 P}{\partial b_1{}^2} + \left\{\frac{1}{2}\ \nu_2{}^2\, \alpha_2{}^2\right\} \frac{\partial^2 P}{\partial \alpha_{22}{}^2} +$$
$$\left\{\frac{1}{2}\, \alpha_2{}^2\, C_2[b]^2\, 2\, \frac{Q_b}{Q} + \alpha_2\, C_2[b]\ (\rho_s\, \alpha_1\, C_1[s+b] - \alpha_2\, C_2[b])\ \frac{Q_s}{Q} +\right.$$
$$\left. \rho_{c21}\, \nu_1\, \alpha_1\, \alpha_2\, C_2[b]\, \frac{Q_1}{Q} + \rho_2\, \alpha_2\, \nu_2\, \alpha_2\, C_2[b]\, \frac{Q_2}{Q}\right\} \frac{\partial^2 P}{\partial b_1\, \partial \alpha_{11}} +$$
$$\{\alpha_2\, C_2[b]\ (\rho_s\, \alpha_1\, C_1[s+b] - \alpha_2\, C_2[b])\}\ \frac{\partial^2 P}{\partial s_1\, \partial b_1} +$$
$$\{\nu_2\, \alpha_2\ (\rho_{c12}\, \alpha_1\, C_1[s+b] - \rho_2\, \alpha_2\, C_2[b])\}\ \frac{\partial^2 P}{\partial s_1\, \partial \alpha_{22}} +$$
$$\left\{\nu_2\, \alpha_2\ (\rho_{c12}\, \alpha_1\, C_1[s+b] - \rho_2\, \alpha_2\, C_2[b])\ \frac{Q_s}{Q} +\right.$$
$$\left. \rho_2\, \alpha_2\, \nu_2\, \alpha_2\, C_2[b]\, \frac{Q_b}{Q} + \frac{1}{2}\ \nu_2{}^2\, \alpha_2{}^2\, 2\, \frac{Q_2}{Q} + \rho_v\, \nu_1\, \alpha_1\, \nu_2\, \alpha_2\, \frac{Q_1}{Q}\right\} \frac{\partial^2 P}{\partial \alpha_{11}\, \partial \alpha_{22}} +$$
$$\{\rho_2\, \alpha_2\, \nu_2\, \alpha_2\, C_2[b]\}\ \frac{\partial^2 P}{\partial b_1\, \partial \alpha_{22}}$$

We need to compute inverse transformation

The inverse transformation can be written as :

Condition : $\alpha_{11}{}^2 - \left(1 - \rho_s{}^2\right)\, \alpha_{22}{}^2\, C_2[b_1]^2 > 0$

$s = s_1$

$b = b_1$

$$\alpha_1 = \frac{\rho_s\, \alpha_{22}\, C_2[b_1] + \sqrt{\alpha_{11}{}^2 - \left(1 - \rho_s{}^2\right)\, \alpha_{22}{}^2\, C_2[b_1]^2}}{C_1[s_1 + b_1]}$$

$\alpha_2 = \alpha_{22}$

so the following identity will be useful later

$$\sqrt{\alpha_{11}{}^2 - \left(1 - \rho_s{}^2\right)\, \alpha_{22}{}^2\, C_2[b_1]^2} = \alpha_1\, C_1[s_1 + b_1] - \rho_s\, \alpha_{22}\, C_2[b_1]$$

we need :

$$\frac{\partial \alpha_1}{\partial s_1} = -\frac{\left(\alpha_{22}\, \rho_s\, C_2[b_1] + \sqrt{\alpha_{11}^2 - \alpha_{22}^2\, \left(1 - \rho_s^2\right)\, C_2[b_1]^2}\right)\, C_1{}'[b_1 + s_1]}{C_1[b_1 + s_1]^2} = -\alpha_1\, \frac{C_1{}'[b + s]}{C_1[b + s]}$$

$$\frac{\partial \alpha_1}{\partial b_1} = -\frac{\left(\alpha_{22}\, \rho_s\, C_2[b_1] + \sqrt{\alpha_{11}^2 - \alpha_{22}^2\, \left(1 - \rho_s^2\right)\, C_2[b_1]^2}\right)\, C_1{}'[b_1 + s_1]}{C_1[b_1 + s_1]^2} +$$
$$\frac{\alpha_{22}\, \rho_s\, C_2{}'[b_1] - \frac{\alpha_{22}^2\, \left(1 - \rho_s^2\right)\, C_2[b_1]\, C_2{}'[b_1]}{\sqrt{\alpha_{11}^2 - \alpha_{22}^2\, \left(1 - \rho_s^2\right)\, C_2[b_1]^2}}}{C_1[b_1 + s_1]} =$$
$$\frac{-\alpha_1^2\, C_1[b + s]\, C_1{}'[b + s] - \alpha_2^2\, C_2[b]\, C_2{}'[b] + \alpha_1\, \alpha_2\, \rho_s\, \left(C_2[b]\, C_1{}'[b + s] + C_1[b + s]\, C_2{}'[b]\right)}{C_1[b + s]\, \left(\alpha_1\, C_1[b + s] - \alpha_2\, \rho_s\, C_2[b]\right)}$$

$$\frac{\partial \alpha_1}{\partial \alpha_{11}} = \frac{\alpha_{11}}{C_1[b_1+s_1]\ \sqrt{\alpha_{11}^2-\alpha_{22}^2\ \left(1-\rho_s^2\right)\ C_2[b_1]^2}} =$$

$$\frac{\sqrt{\alpha_1{}^2\ C_1[s+b]^2+\ \alpha_2{}^2\ C_2[b]^2-2\ \rho_s\ \alpha_1\ \alpha_2\ C_1[s+b]\ C_2[b]}}{C_1[b+s]\ \left(\alpha_1\ C_1[s+b]-\rho_s\ \alpha_{22}\ C_2[b]\right)}$$

$$\frac{\partial \alpha_1}{\partial \alpha_{22}} = \frac{\rho_s\ C_2[b_1]-\frac{\alpha_{22}\ \left(1-\rho_s^2\right)\ C_2[b_1]^2}{\sqrt{\alpha_{11}^2-\alpha_{22}^2\ \left(1-\rho_s^2\right)\ C_2[b_1]^2}}}{C_1[b_1+s_1]} =$$

$$\frac{\rho_s\ C_2[b]-\frac{\alpha_2\ \left(1-\rho_s^2\right)\ C_2[b]^2}{\left(\alpha_1\ C_1[s+b]-\rho_s\ \alpha_2\ C_2[b]\right)}}{C_1[b+s]} = \frac{C_2[b]\ \left(\alpha_1\ \rho_s\ C_1[b+s]-\alpha_2\ C_2[b]\right)}{C_1[b+s]\ \left(\alpha_1\ C_1[b+s]-\alpha_2\ \rho_s\ C_2[b]\right)}$$

We need the following second derivatives :

$$\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial b_1}\right) = \frac{1}{C_1[b+s]^2\ \left(\alpha_1\ C_1[b+s]-\alpha_2\ \rho_s\ C_2[b]\right)^2}$$
$$\Big(\alpha_1^3\ C_1[b+s]^2\ \left(C_1{}'[b+s]^2-C_1[b+s]\ C_1{}''[b+s]\right)+$$
$$\alpha_2^3\ \rho_s\ C_2[b]^2\ \left(-C_1{}'[b+s]\ C_2{}'[b]+C_1[b+s]\ C_2{}''[b]\right)+$$
$$\alpha_1^2\ \alpha_2\ \rho_s\ C_1[b+s]\ \Big(-2\ C_2[b]\ \left(C_1{}'[b+s]^2-C_1[b+s]\ C_1{}''[b+s]\right)+$$
$$C_1[b+s]\ \left(-C_1{}'[b+s]\ C_2{}'[b]+C_1[b+s]\ C_2{}''[b]\right)\Big)+\alpha_1\ \alpha_2^2\ \Big(\rho_s^2\ \Big(C_1[b+s]^2\ C_2{}'[b]^2+$$
$$C_2[b]^2\ \left(C_1{}'[b+s]^2-C_1[b+s]\ C_1{}''[b+s]\right)-C_1[b+s]^2\ C_2[b]\ C_2{}''[b]\Big)-$$
$$C_1[b+s]\ \left(C_1[b+s]\ C_2{}'[b]^2+C_2[b]\ \left(-2\ C_1{}'[b+s]\ C_2{}'[b]+C_1[b+s]\ C_2{}''[b]\right)\right)\Big)\Big)$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial b_1}\right) = -\Big(\Big(\alpha_1^2\ C_1[b+s]^2\ C_1{}'[b+s]-2\ \alpha_1\ \alpha_2\ \rho_s\ C_1[b+s]\ C_2[b]\ C_1{}'[b+s]+$$
$$\alpha_2^2\ C_2[b]\ \left(-C_1[b+s]\ C_2{}'[b]+\rho_s^2\ \left(C_2[b]\ C_1{}'[b+s]+C_1[b+s]\ C_2{}'[b]\right)\right)\Big)\Big/$$
$$\left(C_1[b+s]\ \left(\alpha_1\ C_1[b+s]-\alpha_2\ \rho_s\ C_2[b]\right)^2\right)\Big)$$

$$\frac{\partial}{\partial s}\left(\frac{\partial \alpha_1}{\partial b_1}\right) = \frac{1}{C_1[b+s]^2\ \left(\alpha_1\ C_1[b+s]-\alpha_2\ \rho_{ss}\ C_2[b]\right)^2}\ \Big(-\alpha_2^3\ \rho_{ss}\ C_2[b]^2\ C_1{}'[b+s]\ C_2{}'[b]+$$
$$\alpha_1^3\ C_1[b+s]^2\ \left(C_1{}'[b+s]^2-C_1[b+s]\ C_1{}''[b+s]\right)+\alpha_1^2\ \alpha_2\ \rho_{ss}\ C_1[b+s]$$
$$\left(-C_1[b+s]\ C_1{}'[b+s]\ C_2{}'[b]-2\ C_2[b]\ \left(C_1{}'[b+s]^2-C_1[b+s]\ C_1{}''[b+s]\right)\right)+\alpha_1\ \alpha_2^2$$
$$C_2[b]\ \left(2\ C_1[b+s]\ C_1{}'[b+s]\ C_2{}'[b]+\rho_{ss}^2\ C_2[b]\ \left(C_1{}'[b+s]^2-C_1[b+s]\ C_1{}''[b+s]\right)\right)\Big)$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial \alpha_{11}}\right) = \Big(-\alpha_1^3\ C_1[b+s]^3\ C_1{}'[b+s]+\alpha_2^2\ \alpha_{22}\ \rho_s\ C_2[b]^3\ C_1{}'[b+s]+$$
$$\alpha_1^2\ \rho_s\ C_1[b+s]^2\ \left(\alpha_{22}\ C_1[b+s]\ C_2{}'[b]+\alpha_2\ \left(3\ C_2[b]\ C_1{}'[b+s]-C_1[b+s]\ C_2{}'[b]\right)\right)+$$
$$\alpha_1\ \alpha_2\ C_1[b+s]\ C_2[b]\ \Big(\alpha_2\ \left(-2\ C_2[b]\ C_1{}'[b+s]+C_1[b+s]\ C_2{}'[b]\right)-$$
$$\alpha_{22}\ \rho_s^2\ \left(C_2[b]\ C_1{}'[b+s]+C_1[b+s]\ C_2{}'[b]\right)\Big)\Big)\Big/\ \Big(C_1[b+s]^2$$
$$\left(\alpha_1\ C_1[b+s]-\alpha_{22}\ \rho_s\ C_2[b]\right)^2\ \sqrt{\alpha_1^2\ C_1[b+s]^2-2\ \alpha_1\ \alpha_2\ \rho_s\ C_1[b+s]\ C_2[b]+\alpha_2^2\ C_2[b]^2}\Big)$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{11}}\right) =$$
$$\frac{C_2[b]\ \left(\alpha_1\ \left(\alpha_2-\alpha_{22}\right)\ \rho_s\ C_1[b+s]+\alpha_2\ \left(-\alpha_2+\alpha_{22}\ \rho_s^2\right)\ C_2[b]\right)}{\left(\alpha_1\ C_1[b+s]-\alpha_{22}\ \rho_s\ C_2[b]\right)^2\ \sqrt{\alpha_1^2\ C_1[b+s]^2-2\ \alpha_1\ \alpha_2\ \rho_s\ C_1[b+s]\ C_2[b]+\alpha_2^2\ C_2[b]^2}}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) = -\frac{C_2[b]\ \left(\alpha_1^2\ \rho_{ss}\ C_1[b+s]^2-2\ \alpha_1\ \alpha_2\ C_1[b+s]\ C_2[b]+\alpha_2^2\ \rho_{ss}\ C_2[b]^2\right)\ C_1{}'[b+s]}{C_1[b+s]^2\ \left(\alpha_1\ C_1[b+s]-\alpha_2\ \rho_{ss}\ C_2[b]\right)^2}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) = \left(\left(\alpha_1^2 \rho_s C_1[b+s]^2 - 2\alpha_1 \alpha_2 C_1[b+s] C_2[b] + \alpha_2^2 \rho_s C_2[b]^2\right)\left(-C_2[b] C_1'[b+s] + C_1[b+s] C_2'[b]\right)\right) \Big/ \left(C_1[b+s]^2 \left(\alpha_1 C_1[b+s] - \alpha_2 \rho_s C_2[b]\right)^2\right)$$

$$\frac{\partial}{\partial \alpha_2}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) = \frac{\alpha_1\left(-1+\rho_s^2\right) C_2[b]^2}{\left(\alpha_1 C_1[b+s] - \alpha_2 \rho_s C_2[b]\right)^2}$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) = -\frac{\alpha_2\left(-1+\rho_s^2\right) C_2[b]^2}{\left(\alpha_1 C_1[b+s] - \alpha_2 \rho_s C_2[b]\right)^2}$$

First Order

$$\frac{\partial P}{\partial s_1} = \frac{\partial P}{\partial s}\frac{\partial s}{\partial s_1} + \frac{\partial P}{\partial b}\frac{\partial b}{\partial s_1} + \frac{\partial P}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial s_1} + \frac{\partial P}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial s_1} = \frac{\partial P}{\partial s} + \frac{\partial P}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial s_1}$$

$$\frac{\partial P}{\partial b_1} = \frac{\partial P}{\partial b} + \frac{\partial \alpha_1}{\partial b_1}\frac{\partial P}{\partial \alpha_1}$$

$$\frac{\partial P}{\partial \alpha_{11}} = \frac{\partial P}{\partial s}\frac{\partial s}{\partial \alpha_{11}} + \frac{\partial P}{\partial b}\frac{\partial b}{\partial \alpha_{11}} + \frac{\partial P}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \alpha_{11}} + \frac{\partial P}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial \alpha_{11}} = \frac{\partial \alpha_1}{\partial \alpha_{11}}\frac{\partial P}{\partial \alpha_1}$$

$$\frac{\partial P}{\partial \alpha_{22}} = \frac{\partial P}{\partial s}\frac{\partial s}{\partial \alpha_{22}} + \frac{\partial P}{\partial b}\frac{\partial b}{\partial \alpha_{22}} + \frac{\partial P}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial \alpha_{22}} + \frac{\partial P}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial \alpha_{22}} = \frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial P}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_2}$$

Second Order

We actually need only :

$$\frac{\partial^2 P}{\partial b_1^2} =$$

$$\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial b} + \frac{\partial \alpha_1}{\partial b_1}\frac{\partial P}{\partial \alpha_1}\right) + \frac{\partial \alpha_1}{\partial b_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial b} + \frac{\partial \alpha_1}{\partial b_1}\frac{\partial P}{\partial \alpha_1}\right) =$$

$$\left(\frac{\partial^2 P}{\partial b^2}\right) + 2\frac{\partial \alpha_1}{\partial b_1}\left(\frac{\partial^2 P}{\partial \alpha_1 \partial b}\right) + \frac{\partial \alpha_1}{\partial b_1}\frac{\partial \alpha_1}{\partial b_1}\left(\frac{\partial^2 P}{\partial \alpha_1{}^2}\right) + \left(\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial b_1}\right) + \frac{\partial \alpha_1}{\partial b_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial b_1}\right)\right)\frac{\partial P}{\partial \alpha_1}$$

$$\frac{\partial^2 P}{\partial \alpha_{22}{}^2} =$$

$$\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_{22}}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$

$$\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial P}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_2}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial P}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_2}\right) =$$

$$\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial P}{\partial \alpha_1}\right) + \frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_2}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial P}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{\partial P}{\partial \alpha_2}\right) =$$

$$\left(\frac{\partial^2 P}{\partial \alpha_2{}^2}\right) + 2\frac{\partial \alpha_1}{\partial \alpha_{22}}\left(\frac{\partial^2 P}{\partial \alpha_2 \partial \alpha_1}\right) +$$

$$\frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial \alpha_1}{\partial \alpha_{22}}\left(\frac{\partial^2 P}{\partial \alpha_1{}^2}\right) + \left(\frac{\partial}{\partial \alpha_2}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) + \frac{\partial \alpha_1}{\partial \alpha_{22}}\frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right)\right)\frac{\partial P}{\partial \alpha_1}$$

$$\frac{\partial^2 P}{\partial b_1\,\partial\alpha_{11}} =$$

$$\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial\alpha_{11}}\right)+\frac{\partial\alpha_1}{\partial b_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial\alpha_{11}}\right)=$$

$$\frac{\partial}{\partial b}\left(\frac{\partial\alpha_1}{\partial\alpha_{11}}\frac{\partial P}{\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial b_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial\alpha_{11}}\frac{\partial P}{\partial\alpha_1}\right)=$$

$$\frac{\partial\alpha_1}{\partial\alpha_{11}}\left(\frac{\partial^2 P}{\partial b\,\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial b_1}\frac{\partial\alpha_1}{\partial\alpha_{11}}\left(\frac{\partial^2 P}{\partial\alpha_1^2}\right)+\left(\frac{\partial}{\partial b}\left(\frac{\partial\alpha_1}{\partial\alpha_{11}}\right)+\frac{\partial\alpha_1}{\partial b_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial\alpha_{11}}\right)\right)\frac{\partial P}{\partial\alpha_1}$$

$$\frac{\partial^2 P}{\partial s_1\,\partial b_1} =$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial b_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial b_1}\right)=$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial b}+\frac{\partial\alpha_1}{\partial b_1}\frac{\partial P}{\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial b}+\frac{\partial\alpha_1}{\partial b_1}\frac{\partial P}{\partial\alpha_1}\right)=$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial b}\right)+\frac{\partial}{\partial s}\left(\frac{\partial\alpha_1}{\partial b_1}\frac{\partial P}{\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial b}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial b_1}\frac{\partial P}{\partial\alpha_1}\right)=$$

$$\left(\frac{\partial^2 P}{\partial b\,\partial s}\right)+\frac{\partial\alpha_1}{\partial b_1}\left(\frac{\partial^2 P}{\partial s\,\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\left(\frac{\partial^2 P}{\partial\alpha_1\,\partial b}\right)+$$

$$\frac{\partial\alpha_1}{\partial s_1}\frac{\partial\alpha_1}{\partial b_1}\left(\frac{\partial^2 P}{\partial\alpha_1^2}\right)+\left(\frac{\partial}{\partial s}\left(\frac{\partial\alpha_1}{\partial b_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial b_1}\right)\right)\frac{\partial P}{\partial\alpha_1}$$

$$\frac{\partial^2 P}{\partial s_1\,\partial\alpha_{22}} =$$

$$\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial\alpha_{22}}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial\alpha_{22}}\right)=$$

$$\frac{\partial}{\partial s}\left(\frac{\partial\alpha_1}{\partial\alpha_{22}}\frac{\partial P}{\partial\alpha_1}+\frac{\partial P}{\partial\alpha_2}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial\alpha_{22}}\frac{\partial P}{\partial\alpha_1}+\frac{\partial P}{\partial\alpha_2}\right)=$$

$$\frac{\partial}{\partial s}\left(\frac{\partial\alpha_1}{\partial\alpha_{22}}\frac{\partial P}{\partial\alpha_1}\right)+\frac{\partial}{\partial s}\left(\frac{\partial P}{\partial\alpha_2}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial\alpha_{22}}\frac{\partial P}{\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial P}{\partial\alpha_2}\right)=$$

$$\left(\frac{\partial^2 P}{\partial s\,\partial\alpha_2}\right)+\frac{\partial\alpha_1}{\partial\alpha_{22}}\left(\frac{\partial^2 P}{\partial s\,\partial\alpha_1}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial\alpha_1}{\partial\alpha_{22}}\left(\frac{\partial^2 P}{\partial\alpha_1^2}\right)+$$

$$\frac{\partial\alpha_1}{\partial s_1}\left(\frac{\partial^2 P}{\partial\alpha_1\,\partial\alpha_2}\right)+\left(\frac{\partial}{\partial s}\left(\frac{\partial\alpha_1}{\partial\alpha_{22}}\right)+\frac{\partial\alpha_1}{\partial s_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\alpha_1}{\partial\alpha_{22}}\right)\right)\frac{\partial P}{\partial\alpha_1}$$

$$\frac{\partial^2 P}{\partial \alpha_{11}\, \partial \alpha_{22}} =$$
$$\frac{\partial \alpha_1}{\partial \alpha_{11}} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\frac{\partial \alpha_1}{\partial \alpha_{11}} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}} \frac{\partial P}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_2}\right) =$$
$$\frac{\partial \alpha_1}{\partial \alpha_{11}} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}} \frac{\partial P}{\partial \alpha_1}\right) + \frac{\partial \alpha_1}{\partial \alpha_{11}} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_2}\right) =$$
$$\frac{\partial \alpha_1}{\partial \alpha_{11}} \frac{\partial \alpha_1}{\partial \alpha_{22}}\left(\frac{\partial^2 P}{\partial {\alpha_1}^2}\right) + \frac{\partial \alpha_1}{\partial \alpha_{11}}\left(\frac{\partial^2 P}{\partial \alpha_1\, \partial \alpha_2}\right) + \frac{\partial \alpha_1}{\partial \alpha_{11}} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) \frac{\partial P}{\partial \alpha_1}$$

$$\frac{\partial^2 P}{\partial b_1\, \partial \alpha_{22}} =$$
$$\frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_{22}}\right) + \frac{\partial \alpha_1}{\partial b_1} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_{22}}\right) =$$
$$\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}} \frac{\partial P}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_2}\right) + \frac{\partial \alpha_1}{\partial b_1} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}} \frac{\partial P}{\partial \alpha_1} + \frac{\partial P}{\partial \alpha_2}\right) =$$
$$\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}} \frac{\partial P}{\partial \alpha_1}\right) + \frac{\partial}{\partial b}\left(\frac{\partial P}{\partial \alpha_2}\right) + \frac{\partial \alpha_1}{\partial b_1} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial P}{\partial \alpha_2}\right) + \frac{\partial \alpha_1}{\partial b_1} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}} \frac{\partial P}{\partial \alpha_1}\right) =$$
$$\left(\frac{\partial^2 P}{\partial b\, \partial \alpha_2}\right) + \frac{\partial \alpha_1}{\partial \alpha_{22}}\left(\frac{\partial^2 P}{\partial b\, \partial \alpha_1}\right) + \frac{\partial \alpha_1}{\partial b_1}\left(\frac{\partial^2 P}{\partial \alpha_1\, \partial \alpha_2}\right) +$$
$$\frac{\partial \alpha_1}{\partial b_1} \frac{\partial \alpha_1}{\partial \alpha_{22}}\left(\frac{\partial^2 P}{\partial {\alpha_1}^2}\right) + \left(\frac{\partial}{\partial b}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right) + \frac{\partial \alpha_1}{\partial b_1} \frac{\partial}{\partial \alpha_1}\left(\frac{\partial \alpha_1}{\partial \alpha_{22}}\right)\right) \frac{\partial P}{\partial \alpha_1}$$

Generalized

BiSABR SPREADOPTION

$dF_1 = \alpha_1\, F_1{}^{\beta_1}\, dW_1;$

$d\alpha_1 = \nu_1\, \alpha_1\, dW_{v1};$

$dW_1\, dW_{v1} = \rho_1\, dt;$

$dF_2 = \alpha_2\, F_2{}^{\beta_2}\, dW_2;$

$d\alpha_2 = \nu_2\, \alpha_2\, dW_{v2};$

$dW_2\, dW_{v2} = \rho_2\, dt;$

plus the links

$dW_1\, dW_2 = \rho_s\, dt;\ dW_1\, dW_{v2} = \rho_{c12}\, dt;\ dW_2\, dW_{v1} = \rho_{c21}\, dt;\ dW_{v1}\, dW_{v2} = \rho_v\, dt;$

and the final payment is :

$\text{Payoff} = \max[a_1\, f_1 - a_2\, f_2 - K,\ 0]$

we do the change of process : $F_1{}^a = a_1\, F_1$ and $F_2{}^a = a_2\, F_2$

So the preceding equations change to :

$$dF_1{}^a = a_1{}^{1-\beta_1}\, \alpha_1\, \left(F_1{}^a\right)^{\beta_1} dW_1\ ;$$

$$d\alpha_1 = \nu_1\, \alpha_1\, dW_{v1};$$

$$dF_2{}^a = a_2{}^{1-\beta_2}\, \alpha_2\, \left(F_2{}^a\right)^{\beta_1} dW_2;$$

$$d\alpha_2 = \nu_2\, \alpha_2\, dW_{v2};$$

this equivalent to redefine α_1 and α_2 by : $\alpha_1{}^a = a_1{}^{1-\beta_1}\, \alpha_1$ and $\alpha_2{}^a = a_2{}^{1-\beta_2}\, \alpha_2$

with this new set of variables the problem is to price the following process :

$$dF_1{}^a = \alpha_1{}^a\, \left(F_1{}^a\right)^{\beta_1} dW_1\ ;$$

$$d\alpha_1{}^a = \nu_1\, \alpha_1{}^a\, dW_{v1};$$

$$dF_2{}^a = \alpha_2{}^a\, \left(F_2{}^a\right)^{\beta_1} dW_2;$$

$$d\alpha_2{}^a = \nu_2\, \alpha_2{}^a\, dW_{v2};$$

and the final payment is :

$\text{Payoff} = \max\left[f_1{}^a - f_2{}^a - K,\ 0\right]$