Holographic Entanglement, Quantum Gravity and Fundamental Computational Complexity

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Abstract

This review explores recent developments linking holographic entanglement entropy, the emergence of Einstein's equations, and the role of a fundamental computational complexity. We present a unifying vision where spacetime locality emerges from an intrinsic limit on computing power, with implications for quantum gravity and cosmology. The approach integrates recent advances in string theory, loop quantum gravity, and quantum information theory.

Contents

1 Introduction

The holographic principle, embodied by the AdS/CFT correspondence, postulates a deep equivalence between a gravitational theory in an anti-de Sitter spacetime (AdS) and a conformal field theory (CFT) on its boundary. This review examines how entanglement entropy serves as a bridge between these descriptions, and how a fundamental limit of computational complexity might explain the emergence of spatial locality. We extend the standard formalism by integrating topoi and higher categories to include the observer in the physical description.

2 Evolution of Holographic Entropy Formulas

2.1 Ryu-Takayanagi Formula (RT)

For a CFT in a static state, the entanglement entropy S_A of a subregion A is given by:

$$S_A = \frac{\mathscr{A}(\gamma_A)}{4G_N} \tag{1}$$

where γ_A is the minimal surface anchored to the boundary ∂A, and $\mathscr{A}$ denotes the area.

2.2 Hubeny-Rangamani-Takayanagi Formula (HRT)

For dynamical states, the generalization uses extremal surfaces:

$$S_A = \frac{\mathscr{A}(\gamma_A^{\text{ext}})}{4G_N} \tag{2}$$

with γ_A^{ext} being a spacelike extremal surface satisfying:

$$\delta\mathscr{A} = 0, \quad K = 0 \tag{3}$$

where K is the mean extrinsic curvature.

2.3 Quantum Extremal Surfaces (QES)

Quantum corrections introduce:

$$S_A = \min_X \ \text{ext} \left[\frac{\mathscr{A}(X)}{4G_N} + S_{\text{bulk}}(\Sigma_X) \right] \tag{4}$$

where $S_{\text{bulk}}(\Sigma_X)$ captures quantum effects in the region Σ_X. This formulation resolves the information paradox via *entropy islands*:

$$\mathcal{I} = \Sigma_X \cap \text{black hole interior} \tag{5}$$

3 Gravitational and Thermodynamic Emergence

3.1 Modular Hamiltonian and First Law

The emergence of Einstein's equations stems from:

$$T\,\delta S = \delta E, \quad E_A = -\log(\rho_A) \tag{6}$$

where E_A is the modular Hamiltonian. This relation implies locally:

Theorem 1 (Gravitational Emergence (Faulkner et al.)). *The condition of holographic thermodynamic equilibrium implies:*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{7}$$

via the variational derivation:

$$\delta S_{EE} = \int_{\partial\mathcal{M}} \varepsilon^{\mu\nu}\delta g_{\mu\nu}\sqrt{h}d^{d-1}x \tag{8}$$

3.2 Quantum Corrections to Field Equations

Corrections to S_A induce modifications to Einstein's equations:

$$G_{\mu\nu} + \kappa\mathcal{K}_{\mu\nu} = 8\pi G_N T_{\mu\nu}^{(\text{eff})} \tag{9}$$

where $\mathcal{K}_{\mu\nu}$ is a higher-order tensor:

$$\mathcal{K}_{\mu\nu} = \nabla_\mu\nabla_\nu K - g_{\mu\nu}\Box K + \cdots \tag{10}$$

and κ is the complexity parameter related to the computational limit.

4 Computational Complexity and Emergent Locality

4.1 Locality Paradox

The *Quantum Error Correction* (QEC) mechanism explains apparent locality by a non-local encoding:

$$|\psi_{\text{bulk}}\rangle = \mathcal{U}_{\text{QEC}}\,|\psi_{\text{CFT}}\rangle\,, \quad \mathcal{U}_{\text{QEC}} \in \mathrm{U}(e^{\kappa}) \tag{11}$$

but the selection of degrees of freedom to encode a local region presents a logical circularity.

4.2 Finite Complexity Postulate

Postulate 1. *There exists a fundamental limit κ on the number of degrees of freedom that can be coherently entangled to form an elementary spatial region:*

$$\kappa = \dim \mathscr{H}_{local} \leq e^{S_{BH}} \tag{12}$$

with $S_{BH} = A/4G_N$ the Bekenstein-Hawking entropy.

4.3 Emergence of the Metric

Spatial distance emerges as:

$$ds^2 = \lim_{\kappa\to\infty} \frac{\log \mathscr{I}(A,B)}{\kappa} \eta_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

where $\mathscr{I}(A,B) = S_A + S_B - S_{A\cup B}$ is the mutual information.

5 Cosmological Implications and Tests

5.1 Signature in the Cosmic Microwave Background

Corrections induce characteristic non-Gaussianities:

$$f_{\text{NL}}^{(\text{eq})} \sim \kappa^{-1}\left(\frac{H}{M_{\text{Pl}}}\right)^2, \quad f_{\text{NL}}^{(\text{orth})} \sim 10^{-2}\kappa^{-1} \tag{14}$$

detectable by CMB-S4 and LiteBIRD observatories.

5.2 Corrections to Gravitational Waves

Gravitational wave dispersion shows a deviation:

$$\omega^2 = c^2k^2\left(1 + \alpha\frac{\kappa\ell_{\text{Pl}}^2k^2}{1+\beta\kappa\ell_{\text{Pl}}^2k^2}\right) \tag{15}$$

with $\ell_{\text{Pl}} = \sqrt{G\hbar/c^3}$, detectable by LISA and Einstein Telescope.

5.3 Fundamental Limit and Holographic Universe

The theoretical estimate gives:

$$\kappa \sim \frac{\mathscr{A}_{\text{horizon}}}{4G_N} \approx 10^{122} \tag{16}$$

in agreement with the Bekenstein bound for the observable Universe.

6 Conclusion and Perspectives

This review has demonstrated how:

- Holographic entanglement entropy links quantum information and geometry
- Finite computational complexity resolves the locality paradox
- Quantum corrections to Einstein's equations offer testable signatures

Future perspectives:

1. Formalization in ∞-topoi to include the observer:

$$\mathscr{C}_{\mathrm{QG}} = \int [\mathcal{D}g][\mathcal{D}\phi] e^{iS_{\mathrm{grav}}} \Rightarrow \mathrm{Obj}(\infty\text{-Topos}) \tag{17}$$

2. Generalization of the index theorem for dimensional emergence:

$$\mathrm{ind}(D) = \frac{1}{(4\pi)^{d/2}} \int_M \mathrm{tr}\left(a_{d/2}\right) \implies \dim \mathcal{M} = 4 \tag{18}$$

3. Search for signatures in gravitational observatory data

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