Causal Fermion Systems and Computational Speed Limits

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Abstract

This note explores the formulation of Causal Fermion Systems (CFS) from the perspective of computational limits and the maximum speed of information transfer. Starting from the framework proposed by Finster, we reinterpret variational principles and the role of fermionic operators in terms of computational resources and time complexity. The finiteness of the speed of light is thus understood not only as a geometric constraint of spacetime but also as the manifestation of a bounded computational capacity. Possible links with oscillatory representations of fermionic states are discussed, accompanied by graphical illustrations. This approach suggests a bridge between the causal structures of physics and computation theory, opening the way towards an emergent description of spacetime where causality equates to constraints on algorithmic processes.

1 Causal Fermion Systems, Global Coherence and Complexity

1.1 Global Coherence of Causality

In the framework of *causal fermion systems* (CFS), a spacetime point is described by a finite-rank self-adjoint operator $x \in \mathrm{L}(\mathcal{H})$. The causality between two points x, y is defined from the spectrum of the operator

$$A_{xy} \;=\; P(x,y)\,P(y,x),$$

where P is the fermionic projector.

Definition 1.1 (Local Causal Relation)**.** Let x, y be two points of the CFS universe. We say that:

- x and y are *spacelike* if all eigenvalues of A_{xy} have the same modulus,
- x and y are *timelike* if all eigenvalues are real and not all of the same modulus,
- x and y are *lightlike* otherwise.

The transition from a local causality to a coherent global structure requires additional conditions.

Axiom 1.1 (Global Causal Coherence)**.** The causal relation $\prec$ defined by timelike pairs satisfies:

1. **Acyclicity**: there is no sequence $x_1 \prec x_2 \prec \cdots \prec x_n = x_1$,
2. **Transitivity**: if $x \prec y$ and $y \prec z$, then $x \prec z$,
3. **Local Spectral Compatibility**: for any triplet (x, y, z), the discrete holonomy

$$H_{xyz} := A_{xy}\, A_{yz}\, A_{zx}$$

satisfies $\|H_{xyz}\| \le \varepsilon$ for a small ε (control of short loops).

These conditions guarantee that the set $(M, \prec)$ is a causal partial order (DAG), analogous to the causal structure of a Lorentzian manifold.

Objective

To move from a *bivariate* criterion (on x, y) to a *global partial order* $(M, \preceq)$ (acyclic, transitive, "asymptotically Lorentzian").

"Actionable" Propositions

(A) Emergent Time Function via Spectral Asymmetry. Define a causal height $\tau(x)$ by integrating a spectral asymmetry "future vs past" of A_{xy}:

$$\tau(x) \;:=\; \int_M F\Big(\mathrm{Spec}(A_{xy})\Big)\, d\rho(y),$$

where F weights differently the "timelike before/after" configurations. If τ is *strictly increasing* along "timelike" edges, then

$$x \to y \;\Rightarrow\; \tau(x) < \tau(y),$$

which excludes cycles. *Practical test*: compute τ on numerical minimizers of the action; verify that a topological sort exists $\Rightarrow$ DAG.

(B) Local–Global Coherence (type *sheaf gluing*). Construct a presheaf $\mathcal{C}(U)$ of the induced causal relations on every $U \subset M$. Require:

1. **Compatibility on cover** U_i (agreement of $\mathcal{C}(U_i \cap U_j)$),
2. **Local acyclicity $\Rightarrow$ global acyclicity** (gluing theorem).

Actionable: add to the CFS action a penalty term for causal 2-cycles/3-cycles detected on triples (x, y, z):

$$\mathcal{S}_{\mathrm{loop}}[\rho] \;=\; \lambda \sum_{(x,y,z)} \Phi\Big(A_{xy}, A_{yz}, A_{zx}\Big),$$

where Φ is zero if the orientation induced by the spectra is acyclic, positive otherwise.

(C) "Causal Curvature" via Discrete Holonomy. Define the holonomy

$$H_{xyz} := A_{xy}\, A_{yz}\, A_{zx}.$$

Impose $\|H_{xyz}\|$ small (or a quasi-real positive spectrum) on short cycles. This forces an effective geometry close to a Lorentzian manifold and suppresses unphysical microscopic time loops.

1.2 Effective Dynamics and Collapse

Although the original formulation of CFS is essentially deterministic, quantum phenomenology requires an effective mechanism for state reduction.

Hypothesis 1.1 (Microscopic Mixing)**.** The universal measure ρ is not a unique minimizer, but a convex mixture of quasi-degenerate minimizers. The fermionic correlations decompose into quasi-orthogonal components

$$\rho \;\simeq\; \sum_i p_i\, \rho_i, \qquad p_i \geq 0, \quad \sum_i p_i = 1.$$

Proposition 1.1 (Effective Decoherence)**.** *Under the previous hypothesis, interaction with external degrees of freedom suppresses interference between the ρ_i. The observer sees an effective state*

$$\rho_{\text{obs}} \;\simeq\; \rho_j \quad \textit{with probability } p_j.$$

This reproduces the Born rule as a typical frequency law.

Furthermore, a nonlinear master equation may appear at finite resolution:

$$\dot{\Gamma}_t \;=\; \mathcal{L}[\Gamma_t] \;+\; \epsilon\,\mathcal{N}[\Gamma_t],$$

where $\mathcal{L}$ is an effective unitary dynamics and $\mathcal{N}$ is a weak contraction resulting from CFS *coarse-graining*. The parameter ϵ is fixed by the spectral cutoff scale.

1.3 Complexity Bound and Speed of Light

The speed of light c emerges as the limit for the propagation of informational complexity in the CFS network.

Definition 1.2 (Connection Complexity)**.** For two points x, y, we define the connection complexity $\mathrm{Comp}(x \to y)$ as the minimal cost of causal chains connecting x to y, measured by the spectral decay of A_{xy} and the depth of the chain.

Theorem 1.1 (Fermionic Lieb–Robinson Bound)**.** *There exist constants $C, \xi > 0$ and a speed v_* such that, for any pair of local observables O_X, O_Y,*

$$\|[O_X(t), O_Y]\| \;\leq\; C\, \exp\Big(-(\mathrm{dist}(X,Y) - v_* t)/\xi\Big).$$

We identify $c := v_*$ as the maximum speed of information propagation, interpreted as the bound on the computational complexity of the CFS fabric.

Thus, the finiteness of the speed of light is interpreted as the physical expression of a finite information processing capacity by the discrete substrate of the universe.

1.4 From the Euler–Lagrange Equation to Finite Propagation Speed

Within the causal action principle framework, the Euler–Lagrange equations are written as

$$\nabla_u\Big(\int_{\mathcal{F}} \mathcal{L}(x,y)\,d\rho(y) - \tfrac{\nu}{2}\Big) = 0, \qquad \forall\, u \in \mathfrak{J},\ \forall\, x \in M,$$

where $\mathcal{L}$ is the causal Lagrangian, ρ the universal measure, and ∇_u the derivative with respect to a jet u. These equations, although formulated in an integral and non-local manner, admit an effective structure analogous to hyperbolic equations.

Linearization. Consider an infinitesimal deformation of the universal measure, parameterized by v. Differentiation of the EL equations leads to the *linearized field equations*

$$\langle u,\, \Delta v\rangle(x) = 0, \qquad \forall\, u \in \mathfrak{J},\ \forall\, x \in M,$$

where Δ is a linear operator obtained from the second derivatives of $\mathcal{L}$. These equations describe the dynamics of perturbations around a reference state.

Lenticular Regions and Surface Layer Integrals. To establish a *hyperbolic character*, Finster and collaborators introduce *lenticular regions*, domains admitting a foliation by surface layers. In these domains, surface integrals are defined which play the role of conserved energies. These functionals allow controlling the norm of a solution in a region from the initial data on a previous layer.

Green's Operators and Causal Support. Using energy estimates, *retarded* and *advanced* Green's operators for the operator Δ are constructed. It is thus obtained that the solution at a point x depends only on data located in the causal past (resp. future) of x, defined by the structure induced by the A_{xy}. This establishes that linearized perturbations possess a *causal support*, meaning they propagate at *finite speed*.

Identification with the Speed of Light. In the continuous limit, where the CFS approximates a Lorentzian manifold, the propagation speed thus defined is naturally identified with the speed of light c. The relativistic causality bound is therefore recovered, but obtained *without postulate*, as a direct consequence of the Euler–Lagrange equations.

Remark 1.1. This proof differs conceptually from a Lieb–Robinson type inequality (used in quantum lattice systems), as it does not rely on a local Hamiltonian commutator, but on the variational analysis of the EL equations, linearization, and the construction of energy estimates in a non-smooth integral framework.

1.5 Fermionic Operators and Local Oscillators

In a causal fermion system, each spacetime point $x \in M$ is represented by a finite-rank self-adjoint operator $F(x)$ acting on the Hilbert space $\mathcal{H}$. These operators encode the fermionic structure through the projector of occupied states (the Dirac sea), via the correlation kernel

$$P(x,y) \;=\; \pi_x\, P\, \pi_y,$$

where π_x is the projection associated with the point x.

Link with Fermionic Oscillators. In second quantization, fermionic degrees of freedom can be realized as a set of local oscillators, i.e., creation and annihilation operators

$$\{a_x, a_y^\dagger\} = \delta_{xy}, \qquad \{a_x, a_y\} = 0,$$

satisfying the canonical anticommutation relations (CAR). We can therefore interpret each operator $F(x)$ as the effective data of a local fermionic mode (binary oscillator: occupied or empty).

Oscillator Network and Propagation. The global dynamics is equivalent to that of a *network of fermionic oscillators* coupled by the correlations $P(x, y)$. In such a structure:

- information propagation occurs along the edges of the network (analogous to local couplings),
- the CAR relations impose a bound on the propagation speed of perturbations, of the same nature as the Lieb–Robinson bound in spin systems.

Finster's Analysis. Rather than proving this bound via a Lieb–Robinson type inequality, Finster proceeds by starting from the Euler–Lagrange equations of the variational principle. The linearization of the EL equations leads to effective hyperbolic-type equations for the perturbations, with:

1. *energy estimates* defined by surface layer integrals,
2. the construction of advanced and retarded Green's operators,
3. the existence of a *causal support* for the solutions.

Thus, the propagation speed is finite and, in the continuous limit, is identified with the speed of light c.

Informational Reading. A complementary interpretation can then be proposed:

- the fermionic operators $F(x)$ form a register of local oscillators (fermionic quantum bits),
- the correlations $P(x, y)$ define the fundamental quantum circuit of the universe,
- the bound c appears as the *limit speed of information transmission* in this oscillator network,
- or, as the *intrinsic computational complexity bound* of the fabric of the universe.

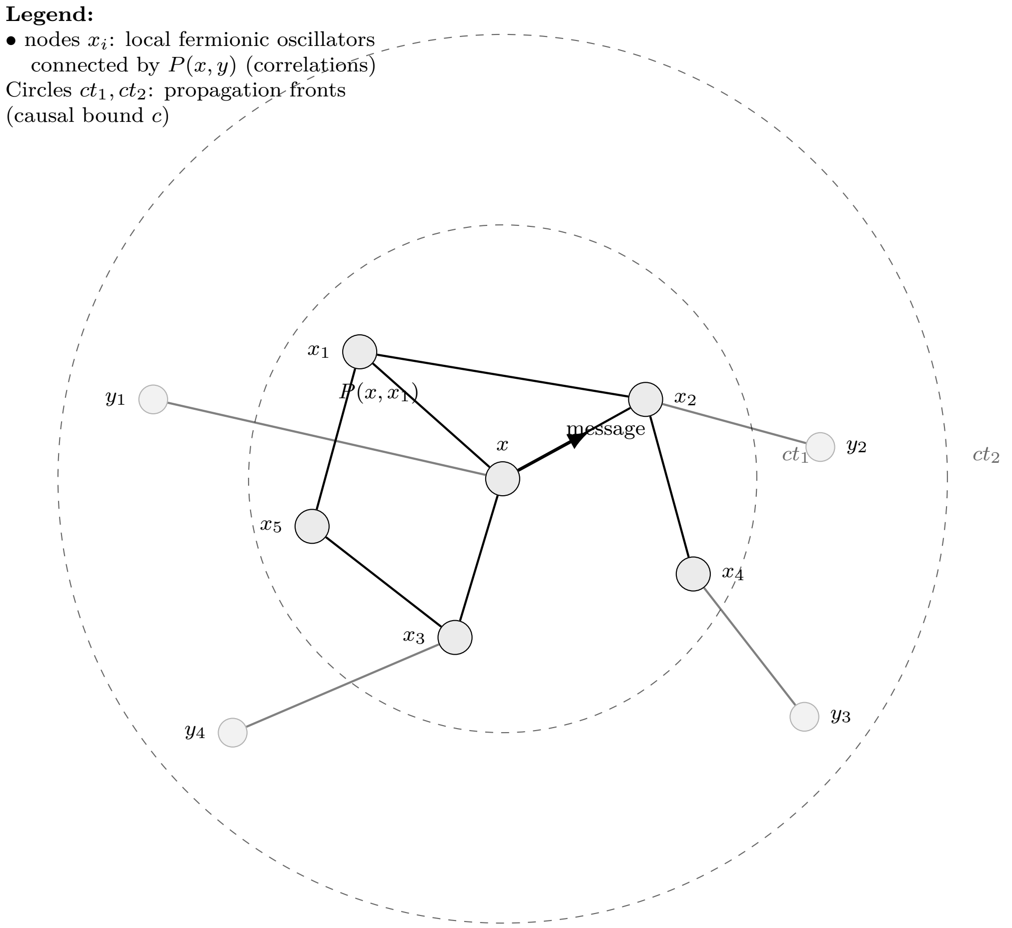

Figure 1: Network of local fermionic oscillators (nodes) coupled by $P(x, y)$. The dashed circles of radius ct illustrate the causal propagation bound: information from x only reaches nodes beyond ct_1 at time $t_2 > t_1$.

References

[1] F. Finster, *Causal Fermion Systems: Discrete Space-Times, Causation and Finite Propagation Speed*, arXiv:1812.00238 [math-ph], 2018. `https://arxiv.org/abs/1812.00238`.

[2] C. Dappiaggi and F. Finster, *Linearized Fields for Causal Variational Principles: Existence Theory and Causal Structure*, Methods and Applications of Analysis, vol. 27, no. 3, pp. 241–291, 2020. doi:10.4310/MAA.2020.v27.n3.a1, also available at arXiv:1808.03177 [math-ph].