Topos and Causal Fermion Systems: Undecidability, Decoherence, and the Emergence of Time

Olivier Croissant

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Contents

Abstract

We propose a unified framework combining topos theory and causal fermion systems (CFS) to analyze undecidability and decoherence as internal structures of the theory, rather than external limitations. The intuitionistic logic of topos localizes undecidability via slicing and local booleanization, while the CFS density ρ encodes fermionic correlations from which effective coherence parameters (time, metric, cosmological constants) emerge.

We introduce quantitative invariants (purity, entropy, interference, complexity, CFS signatures) and effective equations (influence functionals, GKSL master equations, weak constraints) to characterize decoherence phase transitions. These transitions are linked to topological accidents in CFS correlations and provide a new interpretation of the relationship between computational complexity and the emergence of time.

This perspective differs from Loop Quantum Gravity, which imposes granularity via spin networks, and string theory, where the landscape of vacua reflects massive undecidability. Here, undecidability is a structuring and generative principle, whose crossing delineates the boundary between quantum and classical regimes. We formulate falsifiable laws (L1–L3) and identify observational signatures (CMB anomalies, topological correlations in large-scale structure, gravitational waves detected by LIGO/Virgo and future missions), offering a clear experimental program to test the relevance of the topos–CFS approach.

1 Basic Intuition: Measurement as Creation

Quantum measurement is often presented as a destructive "collapse." Here, we adopt a different reading: measurement is an **act of creation**. Before observation, several outcomes remain possible (open truths). After measurement, a reality is fixed: the proposition becomes **decidable** (true/false) in the appropriate context.

Conceptual framework. It is like drawing a card from a deck: as long as one hasn't chosen, all cards are potential; at the moment of drawing, only one becomes real. Measurement does not "remove" anything: it *adds* a truth to the world.

Key point. This act of creation is **outside external time**: it assumes no absolute clock. It refines the "landscape of available truths," consistent with theories where the fundamental equation (such as the Wheeler–DeWitt equation) involves no global time [10, 17].

Perspective. This intuition motivates the proposed framework: combining the internal logic of topos theory and the causal structure of causal fermion systems (CFS) to address in a unified way three fundamental questions: (i) the undecidability inherent in quantum gravity theories, (ii) decoherence as an informational phase transition, and (iii) the emergence of time as an effective coherence parameter. We thus propose a conceptual alternative to dominant approaches (Loop Quantum Gravity and string theory), in which undecidability is not a flaw but a fundamental structure organizing the quantum-to-classical transition.

2 Topos: A Framework for Contextual Logic

The previous intuition—measurement as creation of information—requires a language capable of expressing both context-dependence and the absence of global truth. This is precisely what the notion of a **topos** provides. A topos is a generalization of the universe of sets: one can do mathematics "as usual," but the internal logic is no longer necessarily classical; it becomes **intuitionistic**. This allows modeling situations where certain propositions remain *open* (neither globally true nor false), until an act of measurement makes them decidable.

2.1 Elementary Definition

A **topos** is a "universe of discourse" where one can do mathematics as in the category of sets, but with an often **intuitionistic** logic (truth is not necessarily binary everywhere). Intuition:

- *Objects* representing collections depending on a **context** (point of view);
- *Arrows* (morphisms) translating changes of viewpoint;
- A **subobject classifier** Ω encoding internal "truth values" (Heyting rather than pure Boolean) [14, 1].

2.2 Measurement as a Topos Morphism

In this reading, a **measurement proposition** (e.g., "$\sigma_z = +1$") is a subobject U of the state object Σ. The act of measuring decomposes into two conceptual gestures:

1. **Slicing/Conditioning**: passage to the *slice* $\mathcal{E}/U$, which internalizes "viewing the world *under the condition* U";
2. **Localization/Local Booleanization**: choice of an internal topology j_U that makes U **decidable** in a subtopos $\mathrm{Sh}_{j_U}(\mathcal{E}/U)$.

This yields a **geometric morphism**

$$\mathcal{E} \longrightarrow \mathcal{E}_U^{\mathrm{meas}} := \mathrm{Sh}_{j_U}(\mathcal{E}/U),$$

where U becomes locally true/false (decidable), without manufacturing a "global truth" forbidden by the Kochen–Specker theorem [11].

Created logical information. The decidability gain is measured by

$$\Delta I(U) := -\log \mu(U),$$

where μ is the "internal Born measure" (a probability evaluation inside the topos). This is a **semantic gain** (refinement of truth), *independent* of a thermodynamic entropy cost [21].

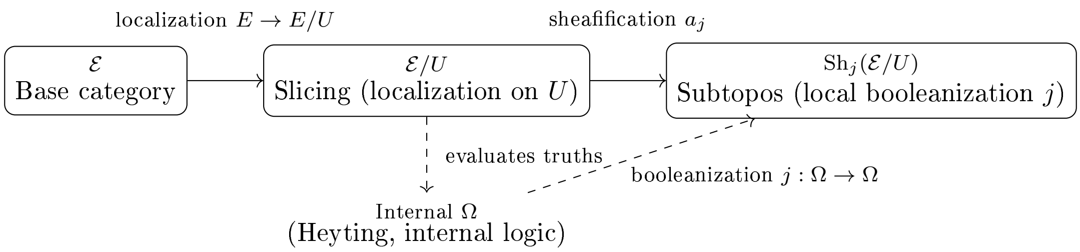

Figure 1: Creative measurement: slicing $E \to E/U$ followed by local booleanization j leading to the subtopos $\mathrm{Sh}_j(\mathcal{E}/U)$.

3 Time and Wheeler–DeWitt: From "Strong" to "Weak"

3.1 Two Sentences on Wheeler–DeWitt

The **Wheeler–DeWitt equation** is at the heart of canonical quantum gravity. It expresses the **gauge invariance of general relativity**: since the theory is invariant under time reparameterization, there is no longer a privileged global time [5, 20].

This equation is a **strong constraint**: it must be satisfied for every state Ψ, which can be thought of as a **wave function on the space of metrics**. That is, $\Psi[h_{ij}]$ describes the probability of a geometric configuration of the universe, and the equation

$$\hat{H}\Psi[h_{ij}] = 0$$

ensures the consistency of this wave function with the fundamental symmetry of general relativity. It thus imposes a constraint of **fundamental atemporality**: classical time disappears from the basic equations.

3.2 Strong and Weak Versions of the Wheeler–DeWitt Constraint

In the *strong* formulation, the fundamental constraint of canonical quantum gravity is written

$$\hat{H}\Psi = 0,$$

valid for every state Ψ in the space of metrics. This equality is understood *pointwise*, within the framework of classical (set-theoretic) logic.

In a *weak and contextual* reading, more suited to the language of topos theory, we do not require strict annihilation, but only its validity after localization in each context C. Formally, for every test function φ_C representative of the context, we impose

$$\int \varphi_C(x)\hat{H}(x)\Psi \, d\mu(x) = 0.$$

In other words, the internal proposition

$$[\hat{H}\Psi = 0] \in \Omega(E/U)$$

is not necessarily true in the ambient topos E, but it becomes *locally decidable* in the subtopos $\mathrm{Sh}_{j_C}(E/U)$ associated with context C. Thus, the Wheeler–DeWitt constraint is interpreted as a contextual coherence between viewpoints, rather than an absolute global equality [18, 6].

3.3 Development of Contexts and Heyting Algebra

The progressive development of harmoniously integrated contexts can be described by a **Heyting algebra**. This structure is not necessarily a global tree, except locally. In practice, there exist **logical rendezvous**: situations reachable by several different informational paths. It is this multiplicity of paths that largely explains the success of the **Feynman path integral** [8], where contributions from all possible histories leading to the same observable result are summed.

3.4 Where do CFS Appear?

In *Causal Fermion Systems* [7], geometry is encoded by a measure ρ on operators (causal and spectral relations are carried by "fermionic chains" A_{xy}). The weak constraint is then written as a **spectral stationarity**:

$$\int \phi_C(x)(\ell_\rho(x) - \lambda) \, d\rho(x) = 0, \quad \ell_\rho(x) := \int L(x,y) \, d\rho(y),$$

where L is a causal Lagrangian. This form is natural in a framework where time emerges from **correlations** rather than a clock.

4 Quantitative Invariants and Order Parameters

In the combined Topos–CFS framework, decoherence can be characterized operationally using quantitative invariants. These quantities serve as *order parameters*: they indicate the passage from an undecidable regime, rich in correlations, to a decidable regime where probabilities emerge [21, 19].

4.1 Purity and Reduced Entropy

We define the purity and von Neumann entropy of the reduced state ρ_S by:

$$\mathcal{P} = \mathrm{Tr}(\rho_S^2), \tag{1}$$

$$S = -\mathrm{Tr}(\rho_S \log \rho_S). \tag{2}$$

Decoherence leads to a decrease in purity and an increase in entropy:

$$\text{Decoherence} \uparrow \quad \Rightarrow \quad \mathcal{P} \downarrow,\ S \uparrow .$$

4.2 Suppression of Interference

In a pointer basis $\{|i\rangle\}$, we introduce the interference measure:

$$\mathcal{D} = \sum_{i \neq j} |\langle i| \rho_S |j\rangle|^2 .$$

This invariant decreases as decoherence progresses. It is interpreted as an *order parameter*: $\mathcal{D} \to 0$ corresponds to the extinction of interference.

4.3 Complexity and Computational Budget

Decoherence can also be analyzed in terms of complexity:

- *Operator complexity*: growth of out-of-time-order commutators (OTOC) [15].
- *Descriptive complexity*: cost $K(\text{state} \mid \text{model})$, in the manner of Kolmogorov or the minimum description length principle [13].

A decoherence threshold is introduced when $K > B$, where B is the observer's available information budget. In topos language, this corresponds to choosing a sheafification functor j, which generates internal probabilities.

4.4 CFS Signatures

In the CFS framework, the following signatures are particularly relevant:

- Correlator $C(x, y) = \mathrm{Tr} A_{xy}$ or norm $\|A_{xy}\|$.
- Decoherence lengths and times $\ell_{\text{dec}}, \tau_{\text{dec}}$ obtained from spatial or temporal decay of C.
- Spectral gaps in the operators A_{xy}.
- Profile of $\ell_\rho(x) = \int L(x, y)\, d\rho(y)$.

A *topological accident* manifests as a jump in the persistence of cycles in the graph G_τ built on $C(x,y)$ [4].

4.5 Homotopical Diagnostics and Sheaves

Decoherence can also be quantified via topological tools:

- **Sheaf cohomology**: a jump in $\dim H^1$ signals a gluing defect (regime transition) [2].
- **Gluing operator** $\mathcal{G}$: a drop in rank corresponds to the emergence of probabilistic "classical islands" within an undecidable background.

4.6 Loschmidt Echo and OTOC

Finally, the dynamics of decoherence is measured by:

$$\mathcal{L}(t) = \left|\langle\psi|\, e^{iHt}e^{-i(H+\delta H)t}\,|\psi\rangle\right|^2, \tag{3}$$

$$\mathrm{OTOC}(t) = \langle [W(t), V]^2\rangle. \tag{4}$$

The decay of $\mathcal{L}(t)$ and the growth of the OTOC are direct indicators of coherence loss and information scrambling [15].

4.7 From Weak Wheeler to CFS: Weakness as Correlation Effect

CFS framework. Let $\mathcal{H}$ be a complex Hilbert space, $M \subset \mathrm{L}(\mathcal{H})$ a set of self-adjoint operators of bounded rank, and ρ a (so-called universal) measure on M. The *causal Lagrangian* $L(x,y)$ depends on the *fermionic chains* $A_{xy} := P(x,y)P(y,x)$ (fermionic correlations between x and y). The causal action is

$$\mathcal{S}[\rho] = \iint_{M\times M} L(x,y)\, d\rho(x)\, d\rho(y),$$

and its Euler–Lagrange equations are written, abstractly,

$$\delta\mathcal{S}[\rho;\delta\rho] = 0 \quad \text{for all admissible variations } \delta\rho.$$

Admissible variations & correlation foliations. The variations $\delta\rho$ are not arbitrary: they must *preserve the microstructure* induced by the spectrum of the A_{xy} (causality, rank, spectral constraints). A useful model is to consider *push-forwards* by diffeomorphisms $F_\varepsilon = \mathrm{id} + \varepsilon X$ that are *tangent* to the correlation foliations: $\rho_\varepsilon := (F_\varepsilon)_*\rho$. Then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{S}[\rho_\varepsilon] = \int_M \langle \nabla\ell_\rho(x), X(x)\rangle\, d\rho(x) = 0,$$

where $\ell_\rho(x) := \int_M L(x,y)\, d\rho(y)$ and X ranges over a restricted space of fields (*variations compatible with correlations*).

Induced weak identity. By integration by parts (in the sense of ρ) in the effective geometry carried by the correlation foliations, we obtain a conservation law

$$\int_M \varphi_C(x)\mathcal{H}_{\mathrm{eff}}[\rho](x)\, d\rho(x) = 0 \quad \text{for every test } \varphi_C \in \mathcal{D}_{\mathrm{corr}}(C),$$

where $\mathcal{H}_{\mathrm{eff}}$ is the effective operator (Hamiltonian constraint in the emerging continuum) and $\mathcal{D}_{\mathrm{corr}}(C)$ denotes test functions *compatible with the correlation structure* in context C (i.e., constant along certain "forbidden" directions and sufficiently regular on the leaves). In other words, the fundamental constraint is only satisfied *after testing*:

$$\boxed{\forall C, \int \varphi_C \mathcal{H}_{\mathrm{eff}}\, d\rho = 0}$$

This is exactly the *weak/contextual version* of the Wheeler–DeWitt type [6].

Topos reading. The contexts C form a small category $\mathcal{C}$ (e.g., correlated subobjects, or adapted coverings). The subtopos $\mathrm{Sh}_{j_C}(E/U_C)$ encodes the internal logic where $[\mathcal{H}_{\mathrm{eff}} = 0]$ becomes *locally decidable*. Aggregation over contexts is expressed by a *coend*

$$\int^{C\in\mathcal{C}} [\mathcal{H}_{\mathrm{eff}} = 0]_C \in \mathcal{N} \subset \Omega,$$

where $\mathcal{N}$ is the "logical kernel" (potentially of measure one) of compatible but globally undecidable truths. The "weakness" of Wheeler is thus seen as the projection of CFS stationarity onto the variation directions allowed by fermionic correlations.

Conceptual consequence. If the correlations A_{xy} are strongly non-local, the space of admissible variations is *narrow*, and the EL-equations can only enforce a *weak constraint* (after testing). The "strong" rigidity $\mathcal{H}_{\mathrm{eff}} = 0$ pointwise only reappears in regimes where the measure ρ is sufficiently decorrelated (or localized) to make all test directions admissible.

5 Decoherence as a Correlational Phase Transition in the Topos+CFS Framework

5.1 Origin of Decoherence

Decoherence is generally understood as the loss of quantum interference due to coupling with an environment. In a *Topos + Causal Fermion Systems (CFS)* perspective, we propose a broader structural reading: decoherence is a **correlational phase transition**, where certain long-range entanglements or correlations become inaccessible to an observer with finite resources. This inaccessibility imposes a **local booleanization** (via a sheafification operator j in the topos), and manifests as an emergence of probabilities [21, 19].

Usual time then appears as an effective quantity, stabilized by the hierarchy of contexts where decoherence operates. Its natural ecosystem is general relativity equipped with a cosmological constant [17].

5.2 Taxonomy of Decoherence

We distinguish several orthogonal axes:

5.2.1 Origin of the Phenomenon

1. **Environmental**: information loss through coupling to external degrees of freedom (Feynman–Vernon, GKSL) [3].
2. **Complexity-induced**: descriptive cost K exceeding the observer's budget B, imposing a cut j.
3. **Contextual (topos)**: choice of a subtopos $\mathrm{Sh}_j(\mathcal{E})$ that makes certain propositions decidable [11].
4. **Gravitational**: horizons (black holes, de Sitter), metric fluctuations [12].
5. **Cosmological**: expansion, cosmological constant, era transitions [16].

5.2.2 By Scale

Local (qubits, laboratory), mesoscopic (condensed matter), cosmological (CMB, large-scale structure, GW).

5.2.3 By Correlation Topology (CFS)

- Fusion or separation of components of the support of ρ.
- Appearance/disappearance of robust cycles in the correlation graph G_τ (TDA) [4].
- Spectral crossings in the invariants of fermionic chains A_{xy}.

5.2.4 By Reversibility

Quasi-reversible (high Loschmidt echo) versus irreversible (loss behind an operational horizon).

5.3 Quantitative Invariants of Decoherence

5.3.1 Standard Measures

$$\mathcal{P} = \mathrm{Tr}(\rho_S^2), \quad S = -\mathrm{Tr}(\rho_S \log \rho_S), \tag{5}$$

$$\mathcal{D} = \sum_{i \neq j} |\langle i|\rho_S|j\rangle|^2. \tag{6}$$

Purity $\mathcal{P}$ decreases and entropy S increases under decoherence; interference $\mathcal{D}$ collapses.

5.3.2 Complexity

Decoherence threshold when $K > B$ (descriptive complexity K greater than budget B). Decoherence is then the effect of an information constraint [13].

5.3.3 CFS Signatures

- Correlator $C(x, y) = \mathrm{Tr}A_{xy}$: spatial/temporal decay of correlation $\ell_{\mathrm{dec}}, \tau_{\mathrm{dec}}$.
- Spectral crossings of A_{xy}.
- Energy functional $\ell_\rho(x) = \int L(x, y)\, d\rho(y)$: ridge bifurcations.

5.3.4 Topological Diagnostics (TDA/Sheaves)

- **TDA**: jumps in Betti numbers β_k in G_τ [4].
- **Sheaf cohomology**: increase in $\dim H^1$ = gluing defect [2].
- **Gluing operator** $\mathcal{G}$: rank loss = transition.

5.3.5 Echoes and Scrambling

$$\mathcal{L}(t) = \left|\langle\psi|e^{iHt}e^{-i(H+\delta H)t}|\psi\rangle\right|^2, \tag{7}$$

$$\mathrm{OTOC}(t) = \langle [W(t), V]^2 \rangle. \tag{8}$$

Decoherence is accompanied by a collapse of $\mathcal{L}$ and growth of OTOC (scrambling) [15].

5.4 Decoherence as a Phase Transition

We introduce a control parameter λ (environmental coupling, amplitude Λ, budget B). A decoherence accident is detected by an abrupt change in one of the above invariants. In CFS, it corresponds to a topological accident in the correlation graph or a crossing in the spectrum of A_{xy}.

5.5 Emergent Time and General Relativity

Usual time emerges as an effective coherence parameter: it stabilizes pointer trajectories resulting from local booleanization. General relativity is then seen as an effective theory valid in the decidable region. The cosmological constant acts as a background gravitational noise source (Gibbons–Hawking temperature [9]), ensuring a minimal decoherence rate at large scales.

5.6 Consequences

In the Topos+CFS framework, decoherence is a rich and structured phase transition, for which we can propose a taxonomy and quantitative invariants. It directly links the structure of microscopic correlations to the emergence of probabilities, time, and general relativity. Thus, topological accidents of ρ become the key to understanding the genesis of effective regimes in observable physics.

6 Effective Equations (Computational Interfaces)

6.1 Influence Functional (Feynman–Vernon)

The influence of the environment is integrated via noise and dissipation kernels, defining an effective rate γ. In the CFS framework, this rate can be expressed from the correlations A_{xy} or the measure ρ:

$$\gamma \sim \int \mathcal{K}[A_{xy}]\, d\rho(x)\, d\rho(y),$$

where $\mathcal{K}$ is a functional (to be calibrated) of spectra and gaps [3].

6.2 GKSL Master Equation

The evolution of the reduced state follows:

$$\dot{\rho}_S = -i[H_{\text{eff}}, \rho_S] + \sum_\alpha \Gamma_\alpha \left(L_\alpha \rho_S L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho_S\} \right).$$

The operators L_α represent pointer observables associated with correlation foliations, and the coefficients Γ_α are functionals of the A_{xy} [3].

6.3 Weak and Contextual Version (Topos)

For each context C,

$$\int \phi_C \mathcal{H}_{\text{eff}}[\rho]\, d\rho = 0,$$

and the internal probability of an event U is given by $\mu_C(U)$. Decoherence then corresponds to the choice of a j_C (local booleanization), leading to the loss of interference terms in μ_C [11].

7 Phase Diagram of Decoherence

We introduce a control parameter λ (environmental coupling, expansion rate, amplitude Λ, budget B). The observables $\{\mathcal{P}, \mathcal{D}, \ell_{\mathrm{dec}}, \tau_{\mathrm{dec}}, \dim H^1, \mathrm{persist}\}$ allow tracing the transition:

- non-analytic rupture or crossover,
- in CFS: coincidence with a topological accident of the graph G_τ or a crossing of the spectrum of A_{xy} [4].

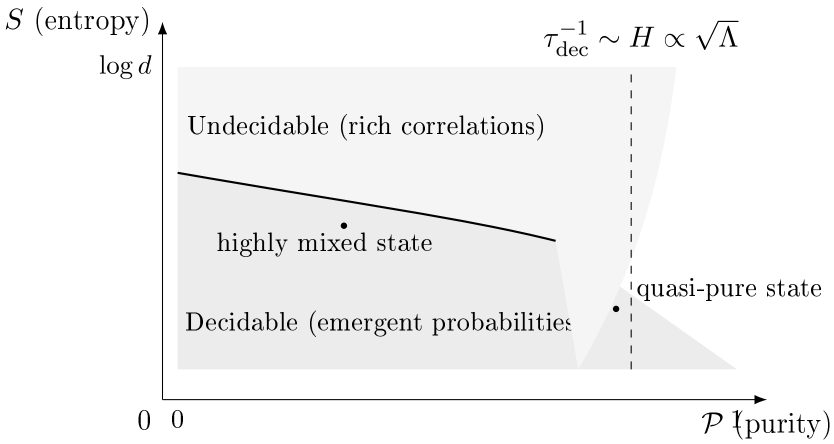

Figure 2: Phase diagram of decoherence.

8 Emergent Time and the Role of Λ

Usual time appears as an effective coherence parameter, stabilized by the hierarchy of cuts j. General relativity is interpreted as an effective theory in the decidable region (where weak constraints align into geodesics) [17]. The cosmological constant Λ acts as a background gravitational noise (horizon, Gibbons–Hawking temperature [9]), imposing a minimal rate τ_{dec}^{-1} at large scales.

9 Maps and Cosmological Observables

- **Lensing (weak/strong)**: an increase in $\Delta\mathrm{persist}(\beta_1)$ implies local anomalies in convergence κ (arcs, flux ratios).

- **Large-scale structure (LSS)**: persistent cycles $\leftrightarrow$ filaments/voids; decoherence $\leftrightarrow$ local gaussianization (kurtosis $\downarrow$, skewness $\to 0$).
- **CMB / CMB lensing**: gluing defects $\Rightarrow$ local excesses in $C_\ell^{\phi\phi}$.
- **Gravitational waves (GW)**: phase transitions $\Rightarrow$ bumps in $\Omega_{\rm GW}(f)$ and exchanges of quasi-normal modes.

10 Operational Pipeline

1. Construct G_τ on $C(x,y) = \mathrm{Tr}A_{xy}$ (or $\|A_{xy}\|$).
2. Apply topological analysis (TDA): barcodes, persistence $\beta_k(\tau)$ [4].
3. Study sheaf gluing constraints on a covering $\{C\}$; track $\dim H^1$ [2].
4. Compute invariants $\{\mathcal{P}, \mathcal{D}, \ell_{\rm dec}, \tau_{\rm dec}, \mathrm{OTOC}, \mathcal{L}(t)\}$.
5. Scan λ (or Λ, B) and detect ruptures.
6. Map results to cosmological observables $(\kappa, C_\ell, \Omega_{\rm GW})$ and compare with data.

11 Three Falsifiable Laws

L1 (Complexity threshold). There exists B^* such that if $K > B^*$, then $\mathcal{D} \to 0$ and $\dim H^1 \uparrow$ (complexity-induced decoherence) [13].

L2 (CFS accident $\Rightarrow$ signature). A robust persistence jump in G_τ precedes an observable variation (κ, $C_\ell^{\phi\phi}$, peak in $\Omega_{\rm GW}$) [4].

L3 (Λ-decoherence). At large scales, $\tau_{\rm dec}^{-1}$ admits a floor controlled by Λ, testable via large-scale lensing maps [9].

12 Internal Measure, Coherence, and Toy Example

12.1 Born and Internal Measure in the Topos

Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. The contexts V are its abelian subalgebras; their Gelfand spectra Σ_V form the *spectral presheaf* $\underline{\Sigma}$. To each projection $P \in \mathcal{N}$ we associate its outer daseinization $\delta^o_V(P) \subseteq \Sigma_V$, clopen in each context V, which defines a clopen subobject $\underline{\delta^o(P)} \subseteq \underline{\Sigma}$.

A state ρ induces an internal measure on clopens by

$$\mu_\rho(\underline{U}) = \mathrm{tr}(\rho E_{\underline{U}}),$$

where $E_{\underline{U}}$ is the projection corresponding to $\underline{U}$ (via the projection $\leftrightarrow$ clopen translation by daseinization). Restricted to a context V, μ_ρ coincides with the classical Born rule on Σ_V.

12.2 Coherence Axioms (Intuitionistic Valuation)

On the Heyting algebra of clopens $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$, an internal measure is a valuation μ such that: (i) $\mu(\bot) = 0$, $\mu(\top) = 1$; (ii) $U \leq V \Rightarrow \mu(U) \leq \mu(V)$; (iii) $\mu(U \vee V) + \mu(U \wedge V) = \mu(U) + \mu(V)$; (iv) $\mu(\bigvee_{i\in I} U_i) = \sup_{J \Subset I} \mu(\bigvee_{i\in J} U_i)$ for any directed family (U_i); (v) naturality under context restrictions. These conditions replace classical σ-additivity with directed continuity adapted to intuitionistic logic.

12.3 Link with Gleason and the 2D Case

For $\dim \mathcal{H} \geq 3$, any finitely additive measure on the lattice of projections (and regular) comes from a state (Gleason's theorem). Via daseinization $P \mapsto \underline{\delta^o(P)}$, we get $\mu_\rho(\underline{\delta^o(P)}) = \mathrm{tr}(\rho P)$, which satisfies the previous axioms. In dimension 2, we use a Busch–Gleason type extension (POVMs) or a regularity hypothesis to reconstruct ρ.

12.4 Toy Example: Qubit and Interaction Between Contexts

Let $\rho = \frac{1}{2}(\mathbf{1} + \vec{r} \cdot \vec{\sigma})$. The context $C_{\vec{n}}$ is the abelian algebra generated by $\sigma_{\vec{n}}$. For the proposition $P_+^{(\vec{n}_0)} = \frac{1}{2}(\mathbf{1} + \vec{n}_0 \cdot \vec{\sigma})$:

$$\mu_\rho(\underline{\delta^o(P_+^{(\vec{n}_0)})}) = \mathrm{tr}(\rho P_+^{(\vec{n}_0)}) = \frac{1 + \vec{r} \cdot \vec{n}_0}{2}.$$

Viewed in $C_{\vec{n}}$, daseinization provides a probability $\mu_\rho^{(\vec{n})} = \frac{1+\vec{r}\cdot\vec{n}}{2}$, which depends on the angle between $\vec{n}$ and $\vec{n}_0$. We can quantify the interaction between contexts by

$$\mathcal{I}(C_{\vec{n}}, C_{\vec{m}} | \underline{\delta^o(P_+^{(\vec{n}_0)})}) := \left| \log \frac{1 + \vec{r} \cdot \vec{m}}{1 + \vec{r} \cdot \vec{n}} \right|,$$

or via the Jensen–Shannon distance between the corresponding binary distributions. The calibration of $\Delta I(U) = -\log \mu_\rho(U)$ is then immediate in each context.

13 Positioning Relative to Existing Approaches

13.1 Loop Quantum Gravity (LQG)

The topos–CFS approach can be compared to Loop Quantum Gravity (LQG), developed notably by Rovelli [17, 22] and Ashtekar [23, 24]. LQG imposes a discretization of time

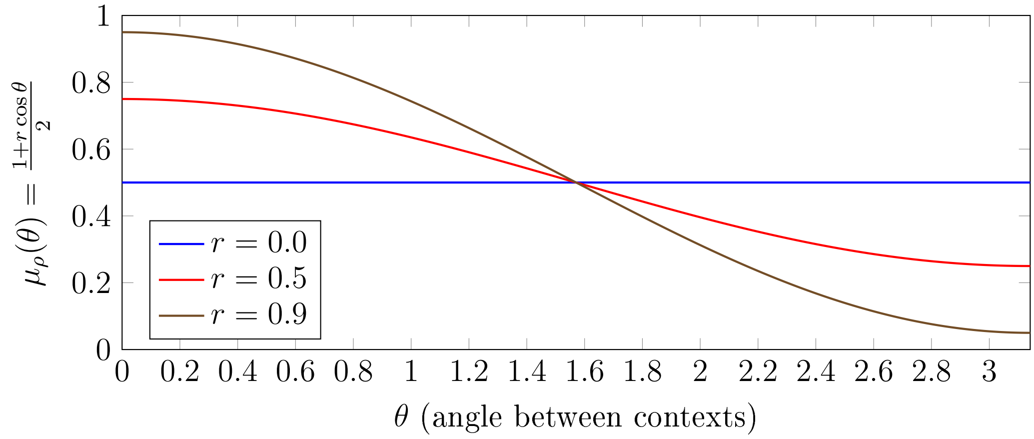

Figure 3: Qubit: probability $\mu_\rho(\theta)$ as a function of the angle between contexts.

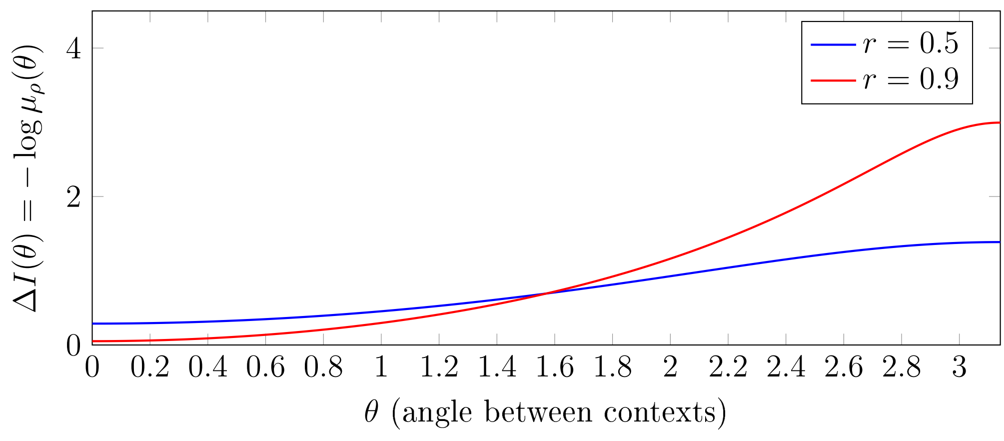

Figure 4: Qubit: information gain $\Delta I(\theta) = -\log \mu_\rho(\theta)$.

and space via geometric spin networks, whereas our weak constraint remains *atemporal*, rooted in contextual logic and the daseinization of observables.

A crucial point is that LQG confronts *global undecidability* (measurement paradox, difficulty extracting a unified dynamics), while the topos–CFS framework localizes this undecidability through slicing $\mathcal{E}/U$ and contextual booleanization. CFS signatures provide a potentially testable alternative to spin networks, opening the possibility of examining fermionic correlations in gravitational data (e.g., LIGO/Virgo [27]).

13.2 String Theory

String theory suffers from the proliferation of vacua (the *landscape*) [25, 26], interpreted here as *massive undecidability*. While strings aim for a global truth through compactifications, the topos–CFS approach considers undecidability as a fundamental structure: it belongs to the toposic kernel and is not a contingent defect.

The link between dark energy and the "decidable frontier" is a fruitful hypothesis: the decoherence threshold $\tau_{\text{dec}} \sim 1/\sqrt{\Lambda}$ would be directly related to the cosmological constant. Finally, the possibility that CFS produce topological signatures (absent in strings) opens the way to observational tests, for example through CMB anomalies (Planck [28], Simons Observatory [29]).

14 Conclusion

We have proposed a framework combining topos theory and causal fermion systems (CFS) to describe decoherence and undecidability as internal structures, rather than external limitations of the theory. The intuitionistic logic of topos localizes undecidability via slicing $\mathcal{E}/U$, while the CFS density ρ encodes fermionic correlations that shape observable signatures. In this perspective, usual time emerges as an effective coherence parameter, and general relativity is read as an effective theory in the decidable region.

This framework distinguishes itself from Loop Quantum Gravity (where granularity is imposed by spin networks) and string theory (where the landscape of vacua reflects massive undecidability). Here, undecidability is a fundamental and productive structure, organizing the quantum-to-classical transition through topological phase transitions. The decoherence threshold $\tau_{\text{dec}} \sim 1/\sqrt{\Lambda}$ directly links this dynamics to the cosmological constant, opening a path to relate dark energy and the emergence of time.

Finally, we have formulated falsifiable laws (L1–L3) and identified observable signatures: CMB anomalies, topological correlations in large-scale structure, and patterns in gravitational waves (LIGO/Virgo, future space missions). These avenues provide a clear experimental program, allowing us to test whether the topos–CFS structure constitutes a viable and predictive alternative to traditional approaches to quantum gravity.

A Intuitionistic Valuation & Internal Born Measure

A.1 Intuitionistic Valuation ⇒ Weak Coherence of WDW

Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and $\underline{\Sigma}$ the spectral presheaf on the category of abelian contexts $V \subset \mathcal{N}$. On the Heyting algebra of clopens $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$, a *valuation* is a map $\mu : \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \to [0,1]$ satisfying: (i) $\mu(\bot) = 0$, $\mu(\top) = 1$; (ii) $U \leq V \Rightarrow \mu(U) \leq \mu(V)$; (iii) modularity $\mu(U \vee V) + \mu(U \wedge V) = \mu(U) + \mu(V)$; (iv) *directed continuity* (Scott-continuity): for any directed family (U_i), $\mu(\bigvee_i U_i) = \sup_{J \Subset I} \mu(\bigvee_{i \in J} U_i)$; (v) naturality under context restrictions.

Lemma 1 (Modular linearity on clopens)**.** *For any finite family (U_k) of pairwise disjoint clopens, $\mu(\bigvee_k U_k) = \sum_k \mu(U_k)$.*

Proof. By induction using (iii) and disjointness ($U_i \wedge U_j = \bot$ for $i \neq j$). □

Consider the weak Wheeler–DeWitt (WDW) constraint written in a context C as

$$\int \phi_C(x) \widehat{H}(x) \Psi \, d\mu(x) = 0,$$

where the integral is internal integration with respect to the valuation μ. By representing ϕ_C and the spectral projectors of $\widehat{H}$ by clopens, modular linearity ensures finite additivity of contributions and Scott-continuity ensures stability under refinement of tests (directed families of clopens). Thus, the validity of the constraint in every context C is stable under refinement and gluing: this is *weak coherence.*

A.2 Internal Born Measure via Daseinization and Link with Gleason

To each projection $P \in \mathcal{N}$ associate its outer daseinization $V \mapsto \delta^o_V(P)$, clopen of Σ_V, which assembles a clopen subobject $\underline{\delta^o(P)} \subset \underline{\Sigma}$. Let ρ be a normal state on $\mathcal{N}$. Define the induced internal measure

$$\mu_\rho(\underline{U}) := \mathrm{tr}(\rho E_{\underline{U}}),$$

where $E_{\underline{U}}$ is the projection corresponding to $\underline{U}$ via the projection $\leftrightarrow$ clopen translation. Then for any projection P,

$$\mu_\rho(\underline{\delta^o(P)}) = \mathrm{tr}(\rho P),$$

that is, the Born rule, viewed through contexts. For $\dim \mathcal{H} \geq 3$, any finitely additive measure on the lattice of projections satisfying regularity conditions is given by a state ρ (Gleason's theorem). The above construction then transports Born to an internal valuation satisfying (i)–(v). In dimension 2, we use a Busch–Gleason type extension (POVMs) or a regularity hypothesis to reconstruct ρ.

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