Computational Complexity and Spatiotemporal Emergence: A Geometric Approach

Olivier Croissant

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Abstract

We propose an approach where computational complexity is viewed as a fundamental ingredient for the emergence of spatiotemporal geometry. By integrating perspectives from information theory, geometric dynamical systems, and quantum theories of spacetime (notably the works of Finster and Miranda), we lay the foundations for a universal computational structure where complexity becomes a dynamic, topological, and asymptotic invariant.

1 Introduction

The emergence of spacetime is now considered not as an axiom, but as a consequence of a more fundamental dynamics, potentially informational or computational. With this in mind, we propose to define a geometric notion of computational complexity that plays a role analogous to that of action in physical theories.

2 Turing Machines and the Two-Dimensional Cantor Set

Eva Miranda and her collaborators have demonstrated a bijective correspondence between the configurations of a Turing machine and the points of a subset of the two-dimensional Cantor set. The two-dimensional Cantor set, defined as a product of Cantor sets $\mathcal{C} \times \mathcal{C}$, is a totally discontinuous but uncountable topological space, which can contain sufficient structural richness to encode a complete computational dynamics.

In this framework, each configuration of a Turing machine (state, position of the read/write head, tape content) can be injectively mapped to a point in the two-dimensional Cantor set using continuous encoding functions. Conversely, the dynamics of computation is represented as a continuous (or locally affine) transformation on this space. Thus, computation is no longer seen as a discrete sequence of states, but as a continuous trajectory in an abstract topological space.

This result allows reformulating Turing computations within a geometric framework, using tools from topological dynamics, Bratteli–Vershik systems, or singular foliations. This approach paves the way for simulating computational behaviors via geometric dynamical systems.

3 Complexity as Symplectic Volume

Following the work of Eva Miranda and others, we propose to model a computational substrate using a Poisson manifold $(\mathcal{M}, \pi)$ or symplectic manifold $(\mathcal{M}, \omega)$, equipped with a Hamiltonian flow with singularities. The local computational complexity is then modeled by a function $C : \mathcal{M} \to \mathbb{R}^+$ such that:

$$C(x) = \int_{\Lambda_x} \omega, \tag{1}$$

where Λ_x is the leaf of the Hamiltonian foliation passing through x.

4 Universality and Geometric Models of Computation

Certain geometric dynamical systems (singular foliations, focus-focus type bifurcations) are Turing-universal. The complexity associated with these structures is therefore a *universal complexity*, potentially surpassing the limits of Turing machines or even standard quantum computers. We adopt here a perspective inspired by "Topological Kleene Field Theories" (TKFT), where computation is inherent to the topology of the substrate.

5 Asymptotic Index and Effective Dimension

We propose to characterize the dimension of spacetime as an asymptotic value extracted from a topological or spectral index. For example, starting from a sequence of computational graphs $\{G_N\}$ with a Dirac-type operator D_N, one can define:

$$\mathrm{Ind}(D_N) \sim c \cdot N^{d_{\mathrm{eff}}/k}, \tag{2}$$

and deduce the effective dimension d_{eff} in the limit $N \to \infty$. This is consistent with spectral approaches à la Connes or Seeley–DeWitt type expansions.

6 Towards an Informational Metric

The distance between two regions can be reconstructed from mutual information or the covariance of observables. For example:

$$ds^2(A, B) = \lim_{\kappa \to \infty} \frac{\log I(A, B)}{\kappa}, \tag{3}$$

where $I(A, B)$ is the mutual information between regions A and B, and κ is the complexity bound.

7 Conclusion and Perspectives

This approach allows laying computational foundations for spacetime, where complexity, dimension, and the metric are asymptotic invariants arising from a geometric dynamics. The continuation of this program consists of: (1) formalizing TKFTs, (2) constructing a spectral triple on computational substrates, (3) extracting physical observables via quantization à la Miranda.

References

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