Towards a Weak Wheeler–DeWitt Equation in the Framework of Causal Fermion Systems and Topoi of Viewpoints

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Abstract

We propose a reformulation of the Wheeler–DeWitt equation, not as a strong functional differential equation on the space of metrics, but as a *weak and contextual* constraint. This perspective relies both on the framework of *Causal Fermion Systems* (CFS), where geometry is replaced by a universal measure on a space of operators, and on the logic of *topoi*, where truth and physical states become intrinsically contextual. The weak Wheeler–DeWitt equation thus becomes a condition of spectral stationarity on measures, ensuring coherence between contexts.

1 Introduction

In the ADM formulation of general relativity, gravitation is interpreted as a *dynamic geometry*, and the geometrodynamic wave function $\Psi[h_{ij}]$ satisfies the Wheeler–DeWitt equation

$$\widehat{H}\,\Psi[h_{ij}] = 0, \tag{1}$$

which is a quantum Hamiltonian constraint. This equation is *strong*: it is defined as a functional differential equation on the infinite-dimensional space of metrics h_{ij}.

However, two difficulties arise:

- the rigorous definition of functional differential operators is problematic,
- the internal logic of quantum mechanics suggests that dynamic constraints are always *contextual*.

CFS and topos logic offer an alternative path: replace strong constraints with *weak* constraints, expressed as integral equalities on the spectral correlations defined by the universal measure ρ.

By *weak version*, we mean a condition obtained after coupling to test functions: instead of requiring $\widehat{H}\Psi = 0$ everywhere, we impose

$$\int \Psi^* \, \widehat{H} \, \Psi \, \mathcal{T} \;=\; 0 \qquad \text{for every allowed test observable } \mathcal{T}. \tag{2}$$

The space of test functions will be organized contextually below.

2 Topos of Contexts

Consider a category $\mathcal{C}$ of contexts:

- objects: $C \in \mathrm{Obj}(\mathcal{C})$, interpreted as *viewpoints* (local choice of spin basis S_x, or commutative subalgebras),

- morphisms: $u : C \to C'$, interpreted as changes of basis or spin connections $D_{x,y}$.

A *presheaf of states* is a contravariant functor

$$\mathsf{State} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Vect},$$

which associates to each context C a space of states $\mathsf{State}(C)$, and similarly for test observables.

$$\begin{array}{ccc} C & & \mathsf{State}(C) \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{State}(u)} \\ C' & & \mathsf{State}(C') \end{array}$$

The topos $\mathbf{Sets}^{\mathcal{C}^{op}}$ thus describes the multiplicity of viewpoints, with an internal intuitionistic logic (as in the spectral sheaf of quantum mechanics).

3 Causal Fermion Systems: Recap

In a CFS, one starts from a Hilbert space $(\mathcal{H}, \langle .|.\rangle)$ and a universal measure ρ on a space $\mathcal{F}$ of finite-rank self-adjoint operators. The spacetime is $M = \operatorname{supp} \rho$. For $x, y \in M$, the fermionic kernel $P(x, y)$ and the fermionic product $A_{xy} = P(x, y)P(y, x)$ encode spectral causality. The causal Lagrangian $L(x, y)$ defines the action

$$S[\rho] = \iint L(x, y)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y). \tag{3}$$

4 From Strong to Weak Wheeler–DeWitt

We associate to each context C a *contextual Hamiltonian density*, not as a differential operator, but as a linear form on test functions:

$$\mathcal{H}_\rho[C] : \mathsf{Test}(C) \longrightarrow \mathbb{R}, \qquad \mathcal{H}_\rho[C](\varphi) \;:=\; \int_M \varphi_C(x)\, \ell_\rho(x)\, \mathrm{d}\rho(x), \tag{4}$$

where

$$\ell_\rho(x) \;:=\; \int_M L(x, y)\, \mathrm{d}\rho(y) \tag{5}$$

is the *potential function* associated to the causal Lagrangian.[1] The *contextual weak Wheeler–DeWitt constraint* is then:

$$\boxed{\forall\, C \in \mathrm{Obj}(\mathcal{C}),\ \forall\, \varphi \in \mathsf{Test}(C), \qquad \mathcal{H}_\rho[C](\varphi) \;=\; \lambda \int_M \varphi_C(x)\, \mathrm{d}\rho(x)} \tag{6}$$

for a (contextually invariant) constant $\lambda \in \mathbb{R}$. In other words,

$$\int_M \varphi_C(x)\, \big(\ell_\rho(x) - \lambda\big)\, \mathrm{d}\rho(x) \;=\; 0 \qquad \forall\, (C, \varphi). \tag{7}$$

In ADM, $\widehat{H}\Psi = 0$ is a strong differential constraint. We propose the *weak* analogue in CFS:

$$\forall C \in \mathrm{Obj}(\mathcal{C}),\ \forall \varphi \in \mathsf{Test}(C), \quad \int_M \varphi_C(x)\, (\ell_\rho(x) - \lambda)\, \mathrm{d}\rho(x) = 0, \tag{8}$$

where

$$\ell_\rho(x) = \int_M L(x, y)\, \mathrm{d}\rho(y) \tag{9}$$

is the associated potential function. Here λ is a Lagrange multiplier related to global constraints (volume, normalization, etc.).

[1] Constraints (*e.g.* functional $T[\rho]$) can be incorporated via Lagrange multipliers in ℓ_ρ.

5 Toy Example: Discrete Mini-Superspace

Consider a CFS with spin dimension $n = 1$, with a discrete support $M = \{x_1, \dots, x_m\}$. The measure is $\rho = \sum_i \rho_i \delta_{x_i}$. The weak constraint (8) becomes

$$\sum_{i=1}^{m} \varphi_C(x_i)\,(\ell_\rho(x_i) - \lambda)\,\rho_i = 0 \quad \forall (C, \varphi). \tag{10}$$

In this case, minimization imposes that all $\ell_\rho(x_i)$ coincide (modulo λ), which is exactly the discrete stationarity of viewpoints.

6 Diagram of the Topos of Viewpoints

$$\begin{tikzcd}
& \mathsf{State}(C_1) \arrow[dl, "\mathsf{State}(u_{10})"'] & \\
\mathsf{State}(C_0) \arrow[rr, "\mathsf{State}(u_{20})"] & & \mathsf{State}(C_2) \arrow[ul, "\mathsf{State}(u_{21})"']
\end{tikzcd}$$

The morphisms ensure coherence between contexts, and the weak Wheeler–DeWitt constraint is a condition of *global stationarity* respecting these compatibilities.

7 Comparison with the Classical Wheeler–DeWitt

The usual Wheeler–DeWitt equation reads

$$\left(-16\pi G\, G_{ijkl}\, \frac{\delta^2}{\delta h_{ij}\, \delta h_{kl}} + \frac{\sqrt{h}}{16\pi G}\, ({}^{(3)}R - 2\Lambda) \right) \Psi[h_{ij}] = 0, \tag{11}$$

where G_{ijkl} is the DeWitt super-metric.

In the continuous limit of CFS:

- the potential function $\ell_\rho(x)$ corresponds to an effective Hamiltonian density obtained by expanding the causal Lagrangian $L(x, y)$,
- the weak integral

$$\int_M \varphi(x)(\ell_\rho(x) - \lambda)\, \mathrm{d}\rho(x) = 0$$

  reproduces the integrated ADM constraint against a test function φ,
- the role of the functional differential operators $\frac{\delta^2}{\delta h\, \delta h}$ is replaced by the spectral structure of the fermionic product A_{xy} and its eigenvalues.

Thus, the weak version in CFS is a *spectral analogue* of the Wheeler–DeWitt constraint: it encodes the dynamics of geometry not by functional derivatives on h_{ij}, but by coherence conditions on the measures ρ and the spectra A_{xy}, in the spirit of noncommutative geometry.

8 Explicit Example: Short-Distance Expansion and Curvature Term

We sketch a (heuristic) calculation showing how a scalar curvature term can emerge in $\ell_\rho(x)$ from the bi-local kernel.

Short-Distance Parametrix

Let y be in a small geodesic neighborhood of x, denote ξ^μ the normal coordinates at x, and $\sigma(x, y)$ the Synge world function. A parametrix (of Hadamard type) for a fermionic kernel in curved spacetime takes the schematic form

$$P(x, y) \simeq \frac{\Gamma(\xi)}{\sigma^2}\Big(a_0(x, y) + a_1(x, y)\,\sigma + \cdots\Big) + m\,\frac{\mathbf{1}}{\sigma}\Big(b_0(x, y) + \cdots\Big), \tag{12}$$

where $\Gamma(\xi)$ is linear in $\gamma \cdot \xi$, and a_k, b_k are related to the Seeley–DeWitt coefficients (*Hadamard*). The first corrections contain curvature, in particular $a_1(x, x) \propto R(x)$.

Fermionic Chain and Spectral Invariants

The fermionic product $A_{xy} := P(x, y)P(y, x)$ then admits an expansion in powers of ξ. After angular averaging over the sphere S^3 (in dimension 4), we use

$$\langle \xi^\mu \xi^\nu \rangle = \frac{\xi^2}{4}\,\eta^{\mu\nu}, \qquad \langle \xi^\mu \xi^\nu \xi^\alpha \xi^\beta \rangle = \frac{\xi^4}{24}\,(\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}). \tag{13}$$

The spectral invariants of A_{xy} (e.g., traces of powers) are then expressed as series in ξ whose local coefficients involve R, $R_{\mu\nu}$, etc.

Radial Integration and Cutoff

By introducing an ultraviolet cutoff ε (geodesic radius) and a local density $w(x)$ such that $\mathrm{d}\rho(y) \simeq w(x)\,\mathrm{d}^4\xi$, we obtain, for the potential function,

$$\ell_\rho(x) = \int L(x, y)\,\mathrm{d}\rho(y) \simeq c_0\,\varepsilon^{-4} + c_1\,\varepsilon^{-2}\,R(x) + c_2\,\log\!\left(\frac{1}{\varepsilon}\right)\,\big(\alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu}\big) + \cdots, \tag{14}$$

where c_0, c_1, c_2, α_i are constants (depending on the details of the parametrix and the precise definition of L). The term $\varepsilon^{-2}R$ is the analogue of the Einstein–Hilbert term (geometric potential).

Link with Wheeler–DeWitt

Inserting this expansion into the weak constraint (8) and grouping the divergences into a renormalization of constants (λ, effective cosmological constant, etc.), we obtain, at the dominant level,

$$\int_M \varphi_C(x)\,\sqrt{h(x)}\,\big(\alpha\,{}^{(3)}R(x) - 2\Lambda_{\text{eff}} + \cdots\big)\,\mathrm{d}^3x = 0, \tag{15}$$

which reproduces the *potential* part (scalar curvature and cosmological constant) of the Wheeler–DeWitt constraint, in a *weak* manner (integrated against contextual tests φ_C). The *kinetic* part (quadratic terms in canonical momenta via the DeWitt super-metric) generally appears in the continuum via a canonical analysis of the non-local dependence of $P(x, y)$ and the choice of foliation (beyond the scope of this sketch).

9 Physical Perspectives

This reformulation opens several paths:

(a) **Emergent Time.** In the strong version, external time disappears ("problem of time"). Here, time is replaced by coherence between contexts: the causal order and spectral correlations induce an emergent arrow of time.

(b) **Gauge Invariance.** Local transformations $U(x) \in U(p, q)$ on each fiber S_x translate into changes of viewpoint. The weak equation is naturally invariant if the tests φ are chosen covariantly, analogous to gauge-invariant observables.

(c) **Decoherence and Collapse.** The "discretization" of minimizing measures in CFS suggests a natural mechanism for wave function reduction. In the language of topoi, there is no global truth, only contextual truths — which accounts for the relative nature of measurement and collapse.

In summary, the weak Wheeler–DeWitt equation is not just a mathematical rewriting: it offers a conceptual interpretation of dynamics as *inter-context coherence*, where causality, time, and collapse emerge from the internal logic.

10 Conclusion

We have proposed a topos-theoretic reading of Wheeler–DeWitt, where the Hamiltonian constraint becomes a weak condition of spectral coherence between contexts, expressed directly in terms of the universal measure of a CFS. This suggests a natural bridge between quantum gravity, spectral causality, and intuitionistic logic.

References

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