Spectral Sheaf, Grothendieck Topos and Geometric Morphisms (with diagrams)

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1 The Spectral Sheaf in Quantum Mechanics

Let $\mathcal{A}$ be a von Neumann algebra (for example $B(\mathcal{H})$, the bounded operators on a Hilbert space). We define the category of *contexts* $\mathcal{V}(\mathcal{A})$:

- objects: maximal abelian subalgebras $V \subseteq \mathcal{A}$,
- morphisms: inclusions $i_{V'V} : V' \hookrightarrow V$.

For each V, we associate its Gelfand spectrum $\Sigma(V)$ (compact space of characters $V \to \mathbb{C}$).

We define the *spectral sheaf* (more precisely a presheaf)

$$\Sigma : \mathcal{V}(\mathcal{A})^{op} \longrightarrow \mathbf{Set}, \qquad \Sigma(V) = \mathrm{Spec}(V), \quad \Sigma(i_{V'V})(\lambda) = \lambda|_{V'}.$$

It lives in the topos of presheaves

$$\widehat{\mathcal{V}(\mathcal{A})} := \mathbf{Set}^{\mathcal{V}(\mathcal{A})^{op}}.$$

In topos logic, the spectral sheaf plays the role of a "generalized state space": it has no global section (Kochen–Specker theorem), only local sections.

2 Geometric Morphisms

A *geometric morphism* between topoi $\mathcal{E} \to \mathcal{F}$ is given by an adjunction

$$f^* \dashv f_* \quad \text{with} \quad f^* : \mathcal{F} \to \mathcal{E} \text{ left exact}, \; f_* : \mathcal{E} \to \mathcal{F},$$

and the natural isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(f^*X, Y) \cong \mathrm{Hom}_{\mathcal{F}}(X, f_*Y).$$

This generalizes continuous maps $f : X \to Y$ which induce $f : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

3 Sieves and Grothendieck Topos

Let $\mathcal{C}$ be small and $c \in \mathrm{Ob}(\mathcal{C})$. A *sieve* S on c is a set of arrows $d \to c$ closed under precomposition. A Grothendieck topology J associates to each c a collection of *covering* sieves. A presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ is a sheaf if every compatible family indexed by a covering sieve glues to a unique section. A *Grothendieck topos* is the category of sheaves on a site $(\mathcal{C}, J)$.

4 Concrete Example: Restriction of Contexts and Transport of the Spectral Presheaf

Let $\mathcal{W} \subseteq \mathcal{V}(\mathcal{A})$ be a subcategory of contexts (e.g., those experimentally accessible), and $i : \mathcal{W} \hookrightarrow \mathcal{V}(\mathcal{A})$ the inclusion. Then i induces a geometric morphism between presheaf topoi:

$$i : \widehat{\mathcal{W}} \longrightarrow \widehat{\mathcal{V}(\mathcal{A})}.$$

The inverse image functor i^* acts by *restriction* on objects, in particular

$$i^*(\Sigma) = \Sigma|_{\mathcal{W}}.$$

Diagrams

(1) Inclusion of sites.

$$\mathcal{W} \xhookrightarrow{\quad i \quad} \mathcal{V}(\mathcal{A})$$

(2) Induced geometric morphism (adjunction).

$$\begin{tikzcd} \widehat{\mathcal{W}} \arrow[r, bend left, "i^*"] & \widehat{\mathcal{V}(\mathcal{A})} \arrow[l, bend left, "i_*"] \end{tikzcd} \qquad \text{with } i^* \dashv i_*.$$

(3) Action on the spectral presheaf.

$$\begin{tikzcd} \Sigma \in \widehat{\mathcal{V}(\mathcal{A})} \arrow[d, mapsto, "i^*"] \\ i^*\Sigma = \Sigma|_{\mathcal{W}} \in \widehat{\mathcal{W}} \end{tikzcd}$$

Physical Interpretation

Restricting contexts amounts to limiting the available observables; the geometric morphism ensures the coherence of the internal logic during this transition, and the spectral presheaf is transported naturally by i^*.

Summary. The *spectral sheaf* encodes the contextual state via Gelfand spectra. *Geometric morphisms* are the appropriate arrows between topoi (adjunction $i^* \dashv i_*$). *Sieves* form the basis for the notion of a sheaf on a site. In the quantum example, the inclusion i restricts Σ to $\Sigma|_{\mathcal{W}}$.