Cumulants

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1 Cumulants

Generally we can easily compute the moments μ_n of the P&L of our portfolio The only problem is that past the 3rd order, central momentum are not summable.The correct quantities to use are the cumulants.The cumulants c_n are defined by the following equation;

$$e^{\sum_{k>0} c_k \frac{(it)^k}{k!}} = \sum_{k \geq 0} \mu_k \frac{(it)^k}{k!}$$

The cumulants are in fact as their names tell cumulative for independent variables.

A general expression for computing them from the momentum is

$$c_r = r! \sum_{m=1}^{r} \sum_{\substack{\pi_1 + \pi_2 + \ldots + \pi_m = \rho \\ p_1\pi_1 + p_2\pi_2 + \ldots + p_m\pi_m = r}} \left(\frac{\mu_{p_1}}{p_1!}\right)^{\pi_1} \left(\frac{\mu_{p_2}}{p_2!}\right)^{\pi_2} \ldots \left(\frac{\mu_{p_m}}{p_m!}\right)^{\pi_m} \frac{(-1)^{\rho-1}(\rho-1)!}{\pi_1!\pi_2!\ldots.\pi_m!}$$

the preceding summation is to be understood over all possible non negative π_i p_j and ρ is an intermediate result function of the particular choice of π_i and p_j

having the cumulants it is easy to come back the moments by inversing these relationships.

2 Computation of the 4 first cumulants (one factor)

When a portfolio is made of purchases of options, the variance could be important but the risk very limited.To appreciate this a new measure is needed.This will be given by the computation of the third momentum of the P&L

lets begin with the one factor model; x is a normal centered variable with a standard deviation of σ

$$y = f(x) = \frac{1}{2}\gamma x^2 + \delta x + c$$

we take for the definition of variance:

$$var[y] = E[(y - E[y])^2] = E[y^2] - (E[y])^2$$

skewness:

$$skew[y] = E[(y - E[y])^3] = E[y^3] + 2(E[y])^3 - 3E[y^2]E[y]$$

kurtosis

$$c_4 = E[(y - E[y])^4] - 3(E[(y - E[y])^2])^2$$

$$= E[y^4] - 4E[y^3]E[y] - 3(E[y^2])^2 + 12E[y^2](E[y])^2 - 6E[y]^4$$

then we compute the momentum of y using the formula given in annex:

$$E[x^n] = \frac{(n-1)!}{2^{\frac{n}{2}-1}\left(\frac{n}{2}-1\right)!}\sigma^n$$

applying:

$$E[y] = \frac{1}{2}\gamma\sigma^2 + c$$

$$E[y^2] = (\delta^2 + \gamma c)\sigma^2 + c^2 + \frac{3}{4}\gamma^2\sigma^4$$

$$E[y^3] = \frac{15}{8}\gamma^3\sigma^6 + \frac{9}{4}(\gamma^2 c + 2\delta^2\gamma)\sigma^4 + 3\left(\delta^2 c + \frac{c^2\gamma}{2}\right)\sigma^2 + c^3$$

$$E[y^4] = \frac{105}{16}\gamma^4\sigma^8 + \left(\frac{45}{2}\delta^2\gamma^2 + \frac{15}{2}\gamma^3 c\right)\sigma^6 + \left(3\delta^4 + 18\delta^2\gamma c + \frac{9}{2}\gamma^2 c^2\right)\sigma^4$$
$$+(6\delta^2 c^2 + 2\gamma c^3)\sigma^2 + c^4$$

So deduce the cumulants:

$$\boxed{\begin{aligned} var[y] &= \delta^2\sigma^2 + \frac{1}{2}\gamma^2\sigma^4 \\ skew[y] &= 3\delta^2\gamma\sigma^4 + \gamma^3\sigma^6 \\ kurtosis[y] &= 12\delta^2\gamma^2\sigma^6 + 3\gamma^4\sigma^8 \end{aligned}}$$

3 Computation of cumulants (n factors)

We use the preceding paragraph, knowing that in this case Δ is a vector Γ and Σ are matrices.

3.1 Simultaneous diagonalization of Γ and Σ^{-1}

It is known that symmetric quadratic form like Γ or Σ^{-1} can be diagonalized but generally not in the same basis.What does the job is the fact that Σ^{-1} is a definite positive quadratic form.This means that Σ^{-1} has only positive eigen values and no null eigen values.It is the case because eigen values are the volatilities of the independent factors in which the actual factors can decomposed through the diagonalization process.

More specifically: It is known that such a quadratic form could be decomposed using its quadratic form: let’s assume that we have diagonalized Σ^{-1} :

$$\Sigma^{-1} = P^{*} D_{\Sigma} P$$

because Σ has only strictly positive eigen values, D_{Σ} is composed only of a diagonal of strictly positive numbers: the variances of the independent factors that the matrix P allow us to built we can built a matrix with in its diagonal the square roots of these volatilities and the properties of this matrix D_V will be that;

$$D_{\Sigma} = D_V D_V$$

So by defining Y as the product of D_V by P we have extracted a “square root” of the matrix Σ^{-1} **in the following sense:**

$$\Sigma^{-1} = P^{*} D_V D_V P = Y^{*} Y \qquad Y = D_V P$$

The fact that Y is invertible allow us to state that:

$$(Y^{-1})^{*} \Sigma^{-1} Y^{-1} = I$$

Now if we consider the quadratic form Γ in this new basis, it is easy to show that its matrix is of the form:

$$\Gamma_V = (Y^{-1})^{*} \Gamma Y^{-1}$$

this matrix, because it is still a gamma matrix is still symmetric.You can also show it by taking the transpose of the preceding definition.So we can diagonalize this new quadratic form by finding a matrix U which is going to be assumed orthogonal that means that the new basis W will be deduced from the preceding by orthogonal symmetries and rotations:

$$\Gamma_V = U^{*} \Gamma_w U \qquad U^{*} U = U U^{*} = I$$

We can check that in this new basis W, Σ is still diagonal. The matrix that allow us to pass from the initial basis to W is equal to:

$$T = Y^{-1} U^{*}$$

we check that:

$$T^*\Sigma^{-1}T = U^* (Y^{-1})*\Sigma^{-1}Y^{-1}U = U^* U = I$$
$$T^*\Gamma T = U (Y^{-1})*\Gamma Y^{-1}U^* = UU^*\Gamma_w UU^* = \Gamma_w$$

Having proved that the matrix exists, we need to find a way to compute it easily:

$$\Gamma\Sigma = (Y)^*\Gamma_V YY^{-1}(Y^{-1})^* = Y^*\Gamma_V(Y^{-1})^* = Y^*U^*\Gamma_w U(Y^{-1})^*$$

and because u is unitary we have $U^* = U^{-1}$ and:

$$\Gamma\Sigma = P^{-1}\Gamma_w P \qquad P = U(Y^{-1})^* = T^{-1}$$

so to get P we just need to diagonalize $\Gamma\Sigma$ as an operator which is not in general symmetric but that is diagonalizable.

3.2 Computation of invariants

We then deduce that:

$$Trace[\Gamma\Sigma\Gamma\Sigma] = Trace[P^{-1}\Gamma_w PP^{-1}\Gamma_w P] = Trace[\Gamma_w\Gamma_w]$$

we could be interested in the way deltas are transforming; let state first that:

:

$$\frac{\partial}{\partial\tilde{x}_i} = \sum_k T_{i,k}\frac{\partial}{\partial x_k}$$
$$\frac{\partial^2}{\partial\tilde{x}_i\partial\tilde{x}_j} = \sum_{k,l} T_{i,k}T_{j,l}\frac{\partial^2}{\partial x_k\partial x_l}$$

where x_k are the new set of factors that diagonalize both Σ^{-1} and Γ.

this imply that the relation ships between the factors are:

$$\left(T_{i,k} = \frac{\partial x_i}{\partial \tilde{x}_k}\right) \Leftrightarrow \left(x_i = \sum_k T_{i,k}\tilde{x}_k\right) \Leftrightarrow (x = T\tilde{x})$$

So we have the following set of coherent matrix identities:

$$X = TX_W$$

$$\Delta_W = T^* \Delta$$

$$\Gamma_w = T^* \Gamma T$$

So if we know that

$$\Sigma = TT^*$$

This imply that:

$$Trace[(\Delta \otimes \Delta)\Sigma] = Trace[T^*(\Delta \otimes \Delta)T] = Trace[(T^*\Delta) \otimes (T^*\Delta)]$$
$$= Trace[\Delta_W \otimes \Delta_W]$$

more generally:

$$Trace[(\Delta_W \otimes \Delta_W)(\Gamma_W)^n] = Trace[(T^*\Delta \otimes T^*\Delta)(T^*\Gamma T)^n]$$
$$= Trace[T^*(\Delta \otimes \Delta)T(T^*\Gamma T)^n] = Trace[(\Delta \otimes \Delta)T(T^*\Gamma T)^n T^*]$$
$$= Trace[(\Delta \otimes \Delta)\Sigma(\Gamma\Sigma)^n]$$

and

$$Trace[(\Delta_W \otimes \Delta_W)^p(\Gamma_W)^n] = Trace[(T^*\Delta \otimes T^*\Delta)^p(T^*\Gamma T)^n]$$
$$= Trace[(T^*(\Delta \otimes \Delta)T)^p(T^*\Gamma T)^n] = Trace[((\Delta \otimes \Delta)TT^*)^p(\Gamma TT^*)^n]$$
$$= Trace[((\Delta \otimes \Delta)\Sigma)^p(\Gamma\Sigma)^n]$$

and we have also

$$Trace[(\Gamma_W)^n] = Trace[(T^*\Gamma T)^n]$$
$$= Trace[(\Gamma T T^*)^n] = Trace[(\Gamma\Sigma)^n]$$

3.3 Expression of cumulants in invariant terms

We know that C is symmetric with all eigen value strictly positive, so it is possible to find a basis (a linear combination of the factors) such that in this basis C is reduced to the identity matrix.Then in this new basis the gamma matrix Γ is symmetric (theorem of schwarz), so it could be diagonalized in a basis that is orthonormal under the preceding scalar product, so we can find a basis where both C and Γ are diagonal.let ‘s call this new basis B2, it is known that it could be orthonormal with respect to the scalar product defined by B1 that mean that keep norm of vectors invariant when going from the basis B2 to the basis B1

The value of the cumulants in the diagonalized base B2 is:

$$var\left[\sum_i y_i\right] = \sum_i\left(\delta_i^2 + \frac{1}{2}\gamma_{i,i}^2\right)$$

$$skew\left[\sum_i y_i\right] = \sum_i(3\delta_i^2\gamma_{i,i} + \gamma_{i,i}^3)$$

$$kurtosis\left[\sum_i y_i\right] = \sum_i(3\gamma_{i,i}^4 + 12\delta_i^2\gamma_{i,i}^2)$$

now

$$\sum_i \delta_i^2 = Trace[\Delta_W \otimes \Delta_W] \qquad \sum_i \gamma_{i,i}^2 = Trace[\Gamma_W^2]$$

$$\sum_i \delta_i^2\gamma_{i,i} = Trace[(\Delta_W \otimes \Delta_W)\Gamma_W] \qquad \sum_i \gamma_{i,i}^3 = Trace[\Gamma_W^3]$$

$$\sum_i \delta_i^2\gamma_{i,i}2 = Trace[(\Delta_W \otimes \Delta_W)\Gamma_W\Gamma_W] \qquad \sum_i \gamma_{i,i}^4 = Trace[\Gamma_W^4]$$

So if we use the formula of the preceding paragraphs

$$\boxed{\begin{aligned} var\left[\sum_i y_i\right] &= Trace\left[(\Delta\otimes\Delta)\Sigma + \frac{1}{2}(\Gamma\Sigma)^2\right] \\ skew\left[\sum_i y_i\right] &= Trace[3(\Delta\otimes\Delta)\Sigma\Gamma\Sigma + (\Gamma\Sigma)^3] \\ kurtosis\left[\sum_i y_i\right] &= Trace[12(\Delta\otimes\Delta)\Sigma(\Gamma\Sigma)^2 + 3(\Gamma\Sigma)^4] \end{aligned}}$$

If we apply the same technics than for the skew we find the intrinsic expression is:

4 5th and 6th cumulants

Using the preceding methods we get:

$$E[y^5]$$

$$= \frac{945}{32}\gamma^5\sigma^{10} + \left(\frac{525}{4}\delta^2\gamma^3 + \frac{525}{16}c\gamma^4\right)\sigma^8 + \left(\frac{75}{2}\delta^4\gamma + \frac{225}{2}c\delta^2\gamma^2 + \frac{75}{4}c^2\gamma^3\right)\sigma^6$$

$$+ \left(15c\delta^4 + 45c^2\delta^2\gamma + \frac{15}{2}c^3\gamma^2\right)\sigma^4 + \left(10c^3\delta^2 + \frac{5}{2}c^4\gamma\right)\sigma^2 + c^5$$

$$E[y^6]$$

$$= \frac{10395}{64}\gamma^6\sigma^{12} + \left(\frac{14175}{16}\delta^4\gamma^2 + \frac{2835}{16}c\gamma^5\right)\sigma^{10} + \frac{1575}{2}\left(\frac{1}{2}\delta^4\gamma^2 + c\delta^2\gamma^3 + \frac{1}{8}c^2\gamma^4\right)\sigma^8$$

$$+ \left(15\delta^6 + 225c\delta^4\gamma + \frac{675}{2}c^2\delta^2\gamma^2 + \frac{75}{2}c^3\gamma^3\right)\sigma^6$$

$$+ \left(45c^2\delta^4 + 90c^3\delta^2\gamma + \frac{45}{4}c^4\gamma^2\right)\sigma^4 + (15c^4\delta^2 + 3c^5\gamma)\sigma^2 + c^6$$

and the expression of the fifth and sixth cumulant are therefore:

$$c_5[y] = 60\delta^2\gamma^3\sigma^8 + 12\gamma^5\sigma^{10}$$
$$c_6[y] = 360\delta^2\gamma^4\sigma^{10} + 60\gamma^6\sigma^{12}$$

and the cumulated cumulant for the entire portfolio is therefore:

$$c_5[y] = Trace[60(\Delta \otimes \Delta)\Sigma(\Gamma\Sigma)^3 + 12(\Gamma\Sigma)^5]$$
$$c_6[y] = Trace[360(\Delta \otimes \Delta)\Sigma(\Gamma\Sigma)^4 + 60(\Gamma\Sigma)^6]$$

more generally it can be schown that

$$\boxed{c_n[y] = Trace\left[\frac{n!}{2}(\Delta \otimes \Delta)\Sigma(\Gamma\Sigma)^{n-1} + \frac{(n-1)!}{2}(\Gamma\Sigma)^n\right]}$$

5 Generalisation

If we allow for a non nul mean of the underlying normal variables,

$$Q = Z^*AZ + b^*Z + d$$

and the variable Z follow a normality $N(\mu, \Sigma)$, then the moment generating function is :

$$M_Q(t) = \frac{e^{td - \frac{1}{2}\mu^*\Sigma^{-1}\mu + \frac{1}{2}(\Sigma^{-1/2}\mu + t\Sigma^{1/2}b)^*(I - 2t\Sigma^{1/2}A^*\Sigma^{1/2})(\Sigma^{-1/2}\mu + t\Sigma^{1/2}b)}}{\left|I - 2tA^*\Sigma\right|^{1/2}}$$

and the cumulants are r>= 2:

$$K_r = 2^{r-1}r!\left\{\frac{1}{r}Tr(A^*\Sigma)^r + \frac{1}{4}b^*(\Sigma A^*)^{r-2}\Sigma b + \mu^*(A^*\Sigma)^{r-1}A^*\mu + b^*(\Sigma A^*)^{r-1}\mu\right\}$$

and

$$K_1 = Tr(A^*\Sigma) + \mu^*A^*\mu + b^*\mu + d$$

5.1 Integrals needed to do the computations

Generally we have the following integrals

$$\int_{-\infty}^{\infty} e^{-px^2 + 2qx}dx = \sqrt{\frac{\pi}{p}}e^{\frac{q^2}{p}}$$

$$\int_{-\infty}^{\infty} xe^{-px^2 + 2qx}dx = \frac{q}{p}\sqrt{\frac{\pi}{p}}e^{\frac{q^2}{p}}$$

$$\int_{-\infty}^{\infty} x^2e^{-px^2 + 2qx}dx = \frac{1}{2p}\sqrt{\frac{\pi}{p}}\left(1 + 2\frac{q^2}{p}\right)e^{\frac{q^2}{p}}$$

$$\int_{-\infty}^{\infty} x^3e^{-px^2 + 2qx}dx = 6e^{\frac{q^2}{p}}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^3\left(\frac{1}{6} + \frac{p}{4q^2}\right)$$

$$\int_{-\infty}^{\infty} x^4e^{-px^2 + 2qx}dx = 24e^{\frac{q^2}{p}}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^4\left(\frac{1}{24} + \frac{1}{2}\left(\frac{p}{4q^2}\right) + \frac{1}{2}\left(\frac{p}{4q^2}\right)^2\right)$$

and in general:

$$\int_{-\infty}^{\infty} x^n e^{-px^2+2qx}dx = n!e^{q^{2/p}}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^n\left[\sum_{k=0}^{E\left(\frac{n}{2}\right)}\frac{1}{(n-2k)!k!}\left(\frac{p}{4q^2}\right)^k\right]$$

The reader interested will look into Gradshteyn and Ryzhic[1],in particular

$$\int_{-\infty}^{\infty} e^{-px^2}dx = \sqrt{\frac{\pi}{p}} \qquad \int_{-\infty}^{\infty} xe^{-px^2}dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-px^2}dx = \frac{1}{2p}\sqrt{\frac{\pi}{p}} \qquad \int_{-\infty}^{\infty} x^3 e^{-px^2+2qx}dx = 0$$

$$\int_{-\infty}^{\infty} x^4 e^{-px^2}dx = \frac{3}{4p^2}\sqrt{\frac{\pi}{p}}$$

More generally

$$\int_{-\infty}^{\infty} x^n e^{-px^2}dx = \frac{(n-1)!}{2^{n-1}\left(\frac{n}{2}-1\right)!}\frac{1}{p^{n/2}}\sqrt{\frac{\pi}{p}} \qquad \textit{when}\ \text{n even} \qquad 0\ \ \text{n odd}$$

and the moments of the bivariate density are:

$$\int_{-\infty}^{\infty} x^n dx \int_{-\infty}^{\infty} y^m dy \frac{e^{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x}{\sigma_1}\right)^2 - \frac{2\rho xy}{\sigma_1\sigma_2} + \left(\frac{y}{\sigma_2}\right)^2\right\}}}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}$$

$$= \frac{m!}{2^{n+m}}\sigma_1^n\sigma_2^m \sum_{l=0}^{E\left[\frac{m}{2}\right]} \rho^{m-2l}(1-\rho^2)^l \frac{(n+m-2l)!}{\left(\frac{n+m-2l}{2}\right)!(m-2l)!l!}$$

which is valid only for n+m even. If n+m is odd then the corresponding bivariate moment is equal to 0