The Quadratic Approximation for Value at risk (around the mode)

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1 Value at Risk

An algorithm for risk evaluation that would have all the needed qualities preconised by the BIS and G30 recommendation is difficult to built.It should incorporate all type of market risks.But the drawback of such algorithm usually based on historical simulation or monte carlo methods, is their slowness. It is why, even if it is recommended to run such algorithm periodically (for example every month) another type of method is needed to have on a day to day basis an estimation of the market risk at all the necessary levels (Portfolio,desk,floor,bank).

The Normal approximation seems to have the required caracterisitics.But usually, it include attention to only delta positions.The handling of a portfolio of derivatives is therefore extremely naive because the gamma effects is not taken into account.

In this Article we show how to use the normal hypothesis to compute a value at risk that take into account the gamma effect.That mean that we will use a quadratic description of the P&L of the portfolio.This measure of risk is adequate to be used on a day to day basis for defining operational trading limits.

We assume that we are interested in the P&L of a portfolio in T days, This P&L will depends on a set of variables that we will call factors.For example, for each currency we will identify a finite set of interest rate factors and a few others to describe the equity risk and the exchange rates if any.To summarize, we need to arrive to a modelisation where the P&L in T days is a quadratic expression of this factors.:

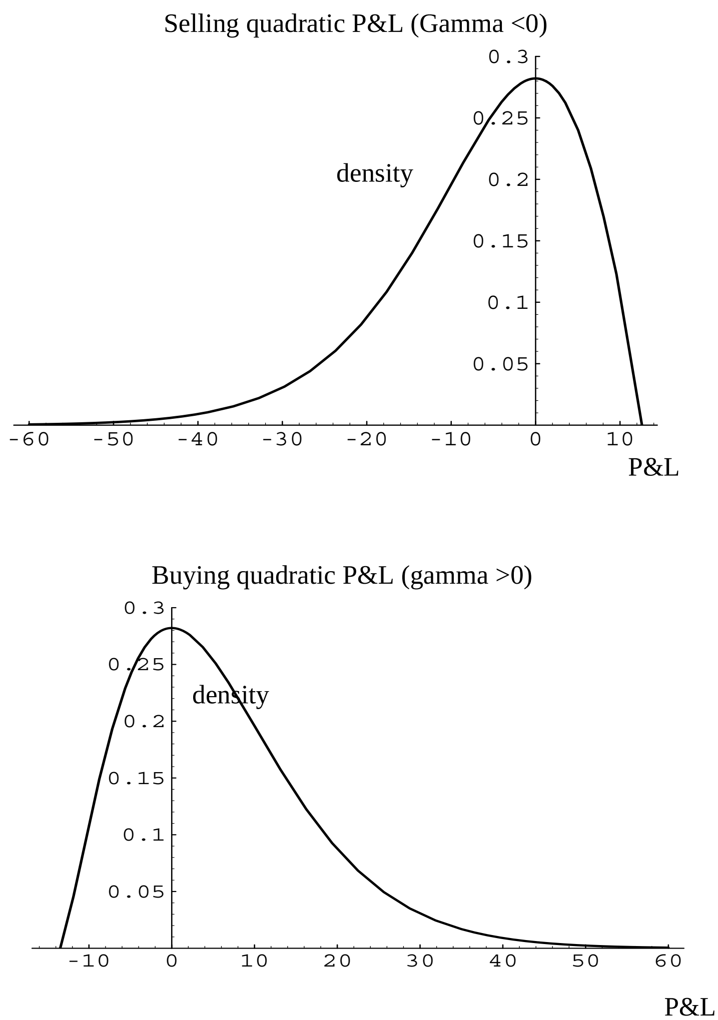
$$\text{P\&L} \quad = \sum_{i} \Delta_i S_i + \frac{1}{2} \sum_{i,\, j} \Gamma_{ij} S_i S_j$$

We need now to represent the dynamical behavior of these factors.Because we are interested in the evolution of the factor in a very short period of time, we will not build an arbitrage-free world, we will simply assume that all factors are jointly normal. The power of this hypothesis comes from the fact that any linear combination of factors are then also normal.Of course the introduction of a gamma risk introduce a distortion in this framework. Without it or neglecting it, the P&L would have been simply normal,and we would have talked of the normal approximation.With the gamma term, the P&L is a quadratic from of normal variables.

Let's define what will call the risk.Following the recent trend we fix a level of probability α and we want to determine the amount R such that the probability that at time T, the P&L will incur a loss greater than R is α.We will assume that a loss of 0 is equivalent to a P&L equal to its expected value.Because we want to be general, the interesting object is the function that associate to a level α, the corresponding risk R_α=R(α).But this function is completely determined by the distribution of the P&L.

If we assume that the P&L is normal then it is sufficient to determine the volatility,then using the inverse of the N function to determine R_α.

But if we want to take into account the gamma effect, the distribution could have a lot of different shape, and the volatility of the P&L is not sufficient to describe its statistical properties, we will need more informations about the distribution.The introduction of the quadratic term in the P&L, introduces a skew and a kurtosis in the distribution that reflects the fact that selling or buying and option, even at the right price doesn't bring you the same type of risk.(See fig 1)



Probability density of Y=(+/-) 2 X^2+10X when X is a standard normal variable

Fig 1

We need to compute the skew,the kurtosis and all the statistical properties that will allow us to deduce the correct shape of the distribution.Such a set of parameters exists; the momentums.But the problem of the momentums is they don’t behave friendly when we need to sum up the effect over 500 factors.It is why a different but equivalent set of parameters have to be chosen: The cumulants.The cumulant of order n is a polynomial of the momentums up to the order of n.They have a very nice property; as their names indicates it,they cumul over a set of independent variables (See annex 1).the problem is so to find a set of independent variables whose sum will give the P&L.

This could achieved by considering the Γ_{ij} and the Σ_{ij} matrices.It is well known that they can be diagonalized in the same basis of factor (See [4] Wilson or [3] Prasolov).In this basis of factors called Q, the P&L could be expressed as:

$$\mathrm{P\&L} = \sum_i \left[\tilde{\Delta}_i Q_i + \frac{1}{2} \tilde{\Gamma}_{ii} Q_i Q_i \right] = \sum_i \mathrm{p\&l}_i$$

where Δ_i is the delta and Γ_{ij} (Diagonal) expressed in the Q basis.

the Q_i being independent because the covariance matrix is diagonal,it follows that the

$\mathrm{p\&l}_i$ are also independent and the cumulant property could apply.That mean that the cumulants of the P&L will simply be the sum of the cumulants of the elementary $\mathrm{p\&l}_i$.

In appearance we need to diagonalize the gamma matrix and the covariance matrix to be able to compute those elementary cumulants then to sum up them to get the cumulant of the P&LBut this diagonalization is very problematic because when we want to do it in the 500 factor case with the incertitude associated with every correlation factor used to build the covariance matrix, we come up with unsolvable problems about the reliability of the diagonal factors, and with it, very little confidence in the further calculations that have to be performed to compute the final risk.It is just where an other important branch of mathematics brings its contribution: the theory of algebraic invariant.It it possible to show that we don’t need to diagonalize these matrices to compute the cumulants of the total P&L.Using only the Γ, and Σ matrices and the delta vector, it is possible to compute these cumulants with only simple operations like matrix multiplication, matrix sum,tensorial product and Traces (see Annex 2).These computation are further more robust to incertitude than everything related to diagonalization process.

lets assume that we know the cumulants of the desired distribution, how can we actually compute it? A first possibility is to use the definition of the characteristic function from which the cumulants are drawn and using a software package like Mathematica compute the risk at any level.if c_i with $i \in \{1, 2, ..., n\}$ are the known cumulants, then assuming the other cumulant being 0, the corresponding R(α) could computed as the inverse function α(R) using:

$$1 - 2\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{-Sin\left(tR + \sum\limits_{k \geq 0} c_{2k+1} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\right)}{t} e^{\sum\limits_{k > 0} c_{2k} \frac{(-1)^k t^{2k}}{(2k)!}} dt$$

But this can be rather uncertain due to the difficulty of the numerical integration of this very oscillating function. Another way is to use edgeworth expansionsThis technics has already successfully been used to price option on averaged index (See [6] Turnbull & Wakeman).Using the fact that the distribution is the distribution of a nice function of normal variables, we can develop it around the normal distribution.Actually, if we consider the diversification, it is a path that lead ineluctably to the normality.More precisely we can show that an approximation of the searched distribution F(x) could be written as:

$$F(x) = N\left(\frac{x-c_1}{\sqrt{c_2}}\right) - \left\{\sum_{l=3}^{\infty} z_l c_2^{-\left(\frac{l-1}{2}\right)} H_{l-1}\left(\frac{x-c_1}{\sqrt{c_2}}\right) \alpha\left(\frac{x-c_1}{\sqrt{c_2}}\right)\right\}$$

where $\omega(x)$ is the derivative of the N function $H_l(x)$ are the Hermite polynomials of order l and z_l are numbers that are a function of the cumulants(See Annex 3)

The risk function can therefore be computed by assuming that the first cumulant is equal to 0:

$$\boxed{\alpha = N\left(\frac{-R}{\sqrt{c_2}}\right) - \left\{\sum_{l=3}^{\infty} z_l H_{l-1}\left(\frac{-R}{\sqrt{c_2}}\right) \alpha\left(\frac{-R}{\sqrt{c_2}}\right) \Big/ \left(c_2^{\frac{(l-1)}{2}}\right)\right\}}$$

where we see how the Hermite polynomials terms correct the normal approximation.

Using the formulas given in annex, we can compute for example the first correction to the normality introduced by the gamma effect:

$$\alpha = N\left(\frac{-R}{\sqrt{c_2}}\right) - \frac{c_3}{6c_2^{3/2}}\left(\left(\frac{R}{\sqrt{c_2}}\right)^2 - 1\right)\frac{e^{-\frac{R^2}{2c_2}}}{\sqrt{2\pi}}$$

The preceding result can be extended to polynomials of any degree for the P&L, and can be used to study the probabilistic characteristics of complex non linear positions like portfolio of mortgaged based instruments.(See [7])

2 Annexes

2.1 **Annex 1:**Definition of cumulants

It is known that a distribution is characterized by its momentum μ_k that are defined by:

$\mu_k[X] = E[(X - E[X])^k]$ and if we define the characteristic function by:

$\Phi_X(t) = E[e^{itX}]$ we can show that the k-momentum is simply

the k-derivative in 0 of the characteristic function.It is known that if X and Y are independent variables, then the first 3 momentums are additive.But this is not true for order greater or equal than 4. The trick is to take the logarithm of the characteristic function, let's call it Ψ.

A very powerful result is that if X and Y are independent then the characteristic function of the sum X+Y is the product of characteristic function of X and Y.This imply that the Ψ function is additive, and so are its derivative in 0:

$$X \text{ and } Y \text{ independant} \Leftrightarrow \Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t) \Leftrightarrow \Psi_{X+Y}(t) = \Psi_X(t) + \Psi_Y(t)$$

the derivative of Ψ in 0 are called the cumulants.They can be computed in function of the momentums.(See [5] Kendall's) The first cumulants for a centralized variable are:

$$c_3 = \mu_3 \qquad c_4 = \mu_4 - 3\mu_2^2$$

2.2 **Annex 2:** The cumulants of a quadratic form of normal variables

Let's remind the definition of the tensorial product of two vector X and Y:It is the matrix M such that: $M_{ij} = X_i Y_j$ and this matrix is noted: $X \otimes Y$.

The other important operation is the trace of a matrix.It is simply the sum of its diagonal elements so: $Trace[M] = \sum_i M_{ii}$.

The trace has a lot of very nice properties.First of all, being equal to the sum of the eigen values of the matrix, it is independent of the basis in which it is computed.An other properties is that the trace of the product of two matrix is independent of the order in which the product is done.

We can show that in the context of VAR, the first cumulants of the P&L are the followings:

$$c_2 = Trace\left[(\Delta \otimes \Delta)\Sigma + \frac{1}{2}(\Gamma\Sigma)^2\right]$$

$$c_3 = Trace[3(\Delta \otimes \Delta)\Sigma\Gamma\Sigma + (\Gamma\Sigma)^3]$$

$$c_4 = Trace[12(\Delta \otimes \Delta)\Sigma(\Gamma\Sigma)^2 + 3(\Gamma\Sigma)^4]$$

Using this definition we make the link with the classical value at risk calculation (normal approximation gamma=0)

$$\Delta^*\Sigma\Delta = \sum_{i,j} \Delta_i \Sigma_{i,j} \Delta_j = Trace[(\Delta \otimes \Delta)\Sigma]$$

2.3 **Annex 3:** The Hermite polynomials and the exponential coefficients

The hermite polynomials are defined by

$$\omega(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \qquad \frac{d^r\omega}{dx^r}(x) = (-1)^r H_r(x)\omega(x)$$

so the first polynomials are:

$$\boxed{\begin{aligned} H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \\ H_5(x) &= x^5 - 10x^3 + 15x \\ H_6(x) &= x^6 - 15x^4 + 45x^2 - 15 \end{aligned}}$$

The exponential coefficients z are defined by

$$Exp\left\{\sum_{k=3}^{\infty} \frac{c_k y^k}{k!}\right\} = 1 + \sum_{l=3}^{\infty} z_l y^l$$

So for a normalized centralized distribution the first coefficients are:

$$\boxed{\begin{aligned} z_3 &= \frac{c_3}{6} \\ z_4 &= \frac{c_4}{24} \\ z_5 &= \frac{c_5}{120} \\ z_6 &= \frac{1}{720}(c_6 + 10c_3^2) \end{aligned}}$$

3 References

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[7]: to be published