Liquidity Risk (Part II)

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1 Introduction

In this paper we will add two other point of view to the three one presented in the part I

2 The Cumulant Approach

Let's assume that the interest rates are equal to zero.

Let F(t) be the value of the portfolio at any time t in the futur. Of course, the portfolio is assumed to be self-financing, that means no cash-in, no cash-out. The unwinding takes a finite time to happen , therefore, F(t) will be constant after a certain period of time. Let's call this value F_{∞}. To define a capital at risk , we are interested in the distribution of the following stochastic variable :

$$V = F_{\infty} - F_{t_0}$$

Now let's make the obvious decomposition :

$$V = X_1 + X_2 + X_3$$

$$\text{where}\begin{cases} X_1 = F_{t_1} - F_{t_0} \\ X_2 = F_{t_2} - F_{t_1} \\ X_1 = F_{t_3} - F_{t_2} \equiv F_{\infty} - F_{t_2} \end{cases}$$

This decomposition allows us to reduce the problem with multiple unwinding periods into a set of problems with unique unwinding period. Now we need two more ingredients to make the potage:

- The Markov property for market prices .This is checked inside our Ito process assumption . In this case, it translates into :X_1 ,X_2 and X_3 are all independent.
- The assumption that all distributions will be caracterized by the set of cumulants.This assumption, not obvious in the general probabilist framework is always accepted for the practical cases we are dealing with. We will use the cumulants rather than the moments because only the first ones sum over independent variables. (I think that their name come from this very particular property)

Let's call the nth cumulant of X : $c_n(X)$.

Therefore we have :

Formula 1

$$c_n(V) = c_n(X_1) + c_n(X_2) + c_n(X_3)$$

3 The path Integral approach

To Localize (i.e. to reveal the underlying density structure) the general formula we need to have to introduce densities over the η space. That means for every strategy

$$T \to \eta(T)$$

we have density $dp[\eta]$. To make sense of it , we can assume that there is only a finite number of them , but the formulation will be valid for a continuous infinity number of them. At this point we will use the path integrals known in physics as Feynman integrals. The principle of the substitution if the following :

$$f(a_1, a_2, ..., a_n) = \int \delta(a_1 - x_1)\delta(a_2 - x_2)...\delta(a_n - x_n) f(x_1, x_2, ..., x_n) dx_1 dx_2 ... dx_n$$

that we write as : $f[a] = \int \mathcal{D}x\,\delta(a-x) f(x)$ in which $\delta(a-x)$ is the density.

It is then natural to make n-> infinity in our formula and we get a path integral

Therefore we rewrite the total variance as :

Formula 4

$$W = \int \mathcal{D}a\,\mathcal{D}b \int_t \langle dP(a) | dP(b) \rangle dt$$

The localization of our formalism is then obvious.

delta function are not very natural, and we never know exactly in advance the liquidation strategy, therefore, let assume that we have an uncertainty on the quantity of asset to liquidate every day. let's assume that this uncertainty is expressed by a density associated with every trading. Then The expectation of the total variance is still given by the same formula:

$$E[W] = \int \mathcal{D}a\,\mathcal{D}b \int_t \langle dP(a) | dP(b) \rangle dt$$

but the density is not singular anymore.

let assume that the density is expressed by:

$$dP(a, j)_t = \frac{e^{-\int (a(s) - z(j, s))^2 \frac{ds}{2\gamma^2}} \delta\left(\int a(s)ds - \mu(j)\right)}{N} \sigma_{j, \beta}$$

Where N is a normalisation factor. Then we compute the capital at risk as :

$$\frac{1}{N}\int \mathcal{D}a\,\mathcal{D}b \int_t e^{-\int ((a(s) - z(j, s))^2 + (b(s) - z(j, s))^2) \frac{ds}{2\gamma^2}} \delta\left(\int a(s)ds - \mu(j)\right) \delta\left(\int b(s)ds - \mu(k)\right) \sigma_{j, \beta}\sigma_{j, \beta} dt$$

But we know that

$$\int e^{-ay^2 + by} dy = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

therefore the normalized density is :

$$\int dx_1 dx_2 \ldots dx_n \frac{e^{-\frac{(x_1 - a_1)^2 + .. + (x_n - a_n)^2}{2\gamma^2}}}{\gamma^{n-1}(2\pi)^{(n-1)/2}} \delta(x_1 + \ldots + x_n - \mu) = 1$$

and then

$$\int \mathcal{D}a\, \mathcal{D}b \int_t \langle dP(a) | dP(b) \rangle dt = \sum_{j,\beta} \sigma_{j,\beta} \sigma_{j,\beta}$$

which is unchanged with respect to case with no density. This is normal because the scalar product was constant. The real power of this method will appear when we will be handling non linear markets or markets with a dependency of the salar product of deltas on the paths followed by the liquidation strategy (See Liquidity RiskPart II). We put this derivation here for completness purpose only.

4 Annexes