Liquidity Risk (Part I): liquidation risk for linear markets

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1 General Introduction

The first thing to do when whe want to handle a subject like liquidity risk, is to define the object. Liquidity risk manifests when, we find it impossible to trade a certain quantity of assets at the price quoted by the market, either because the market is totally illiquid, or because the trade is so large that it is impossible to find the appropriate conterparty in the market. We are faced with two possiblities, either we accept to pospone the execution of the whole trade by trading a small percentage of the total trade every day, either we accept a reaction of the market in front of the proposed trade, and the price will be affected.

This article is the Part I of a paper on liquidity risk made of three parts:

- Part I deals with the first possibility (we spread the deal over the time) in the case of linear markets.
- Part II also deals with the first possibility but with non linear markets.
- Part III deals with the second possibility and the modeling of a reactive market. This general framework allows the modeling of strategies that spread over time and includes a reaction of the market.

1.1 Daily Earnings At Risk versus Capital At Risk

We want to examine the notion of VaR in a market where unwinding periods are different for different types of position. Therefore, it is important to distinguish between two different preoccupations.

1) The daily mark to market practice leads to define a VaR as the daily earnings at risk, measuring the risk associated with a dynamic management in mind. Usually, this

practice is associated with the implicit assumption that the unwinding period for all markets is either one business day or the time horizon taken for the VaR calculation. What is essential here is to understand that behind this quick unwinding assumption, the time horizon is a parameter of the risk calculation.

2) Different is the VaR with no time horizon. In this case, we compute the risk exactly as if we wanted to liquidate the whole portfolio and unwind all positions (we may want to get out of the business). This VaR measure is more adequate for measuring the risk undertaken by stock holders of a financial institution. For example this risk measure would have spotted the risk associated with LTCM positions, better than a VaR with a time horizon, because it would have deaded us to ask about correlative and systemic effects. Of course, the correct model for price behaviour in a liquidative behaviour has to be used.

This kind of VaR is called Capital at Risk

The purpose of this paper is to show how to compute such VaR measure and to examine the main properties associated with it. Part I only deal with the linear markets and linear VaR measures. In a forthcoming part II we will be dealing with non linear markets and all the new features that are associated with them. We present three different points of view to compute the liquidity risk. All of them bring a piece of light on the same computation. We emphasize on the unity behind these apparent differences in formulation. To conclude this first part we present a stochastic generalisation where we allow the liquidation strategy to depend on the market evolution.

2 Assumptions for Part I

We assume that the market follows an Ito process with a deterministic volatility. All dimensions of the market will be denoted by the index α :

$$dS_\alpha = \mu_\alpha dt + \sum_\beta \sigma_{\alpha,\beta} dW_{\beta,t}$$

where the processes dW_β are independent brownian motions.This hypothesis is acceptable for short term risks. We may think that the shorter the term of the analysis, the more true this statement will be. In fact when we get to very short term, the effects coming from the non-continuity of the market prices begin to be important. So we will assume a medium term risk : typicaly between one day and one month: the usual period of time necessary to unwind a position ,which is the central topic of this paper.

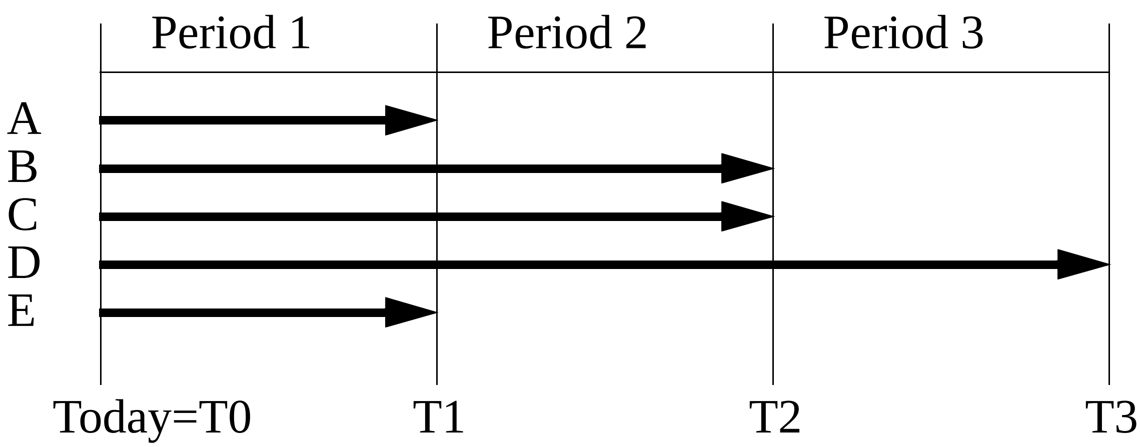
We will use the following convention : $\sigma_{\alpha,\beta} \equiv \frac{\partial S_\alpha}{\partial W_\beta}$

This convention emphasizes on the fact that the instantaneous variance effectively behaves like the derivatives of the value with respect to the borwnian motion, but also that these derivatives will be considered to be the real and true “deltas” of the porfolio. In such context, it is also useful to remember that as such deltas are completly linear with respect to partitioning the portfolios and subportfolios.

We assume in most of the article that the interest rates are equal to zero. Which is sensible for very short term considerations and usual portfolios. We examin in Paragraph 9 how to modifiy our calculations to take into account inerest rates

3 A Simple Example

We first look at a very simple example : Let's assume that the portfolio to liquidate is made of 5 positions A,B,C,D and E.



Let's assume that we know that in average,

- to liquidate a deal of type A or E we need one period of time of length period-1
- to liquidate a deal of type B or C, we need an additional period of time : period-1 plus Period-2.
- to liquidate a deal of type D we need an other additional period of time : period-1 plus period-2 plus period-3.

We also assume that the position will be liquidated in block, at the end of the sus-mentioned period. In fact these periods may be defined that way (more on this to come)

Let F(t) be the value of the portfolio at any time t in the futur. Of course, the portfolio is assumed to be self-financing, that means no cash-in, no cash-out. The unwinding takes a finite time to happen , therefore, F(t) will be constant after a certain period of time. Let's call this value F_∞ . To define a capital at risk , we are interested in the distribution of the following stochastic variable :

$$V = F_\infty - F_{t_0}$$

Now let's make the obvious decomposition :

$$V = X_1 + X_2 + X_3$$

where
$$\begin{cases} X_1 = F_{t_1} - F_{t_0} \\ X_2 = F_{t_2} - F_{t_1} \\ X_1 = F_{t_3} - F_{t_2} \equiv F_\infty - F_{t_2} \end{cases}$$

This decomposition allows us to reduce the problem with multiple unwinding periods into a set of problems with unique unwinding period. Now we need two more ingredients to make the potage:

- The Markov property for market prices .This is checked inside our Ito process assumption . In this case, it translates into : X_1 , X_2 and X_3 are all independent.
- The assumption that all distributions will be caracterized by the set of cumulants.This assumption, not obvious in the general probabilist framework is always accepted for the practical cases we are dealing with. We will use the cumulants rather than the moments because only the first ones sum over independent variables. (I think that their name come from this very particular property)

Let's call the expectation of X : $\mu(X)$ and the variance of X $\sigma^2(X)$.

we know that because of the independence assumption, we have :

$$\mu(V) = \mu(X_1) + \mu(X_2) + \mu(X_3)$$

$$\sigma^2(V) = \sigma^2(X_1) + \sigma^2(X_2) + \sigma^2(X_3)$$

We can show that this is sufficient to caraterize the statistical properties ov V, because the distribution of V is gaussian (see the demonstration in part II) , assuming that we can compute the expectation and variances for each of the X_i.

We use the following notations:

$$\delta_{i,j} = E\left[\int_{[t_{i-1}, t_i]} dS_j\right] = \int_{[t_{i-1}, t_i]} \mu_j dt$$

$$C_{i,j_1,j_2} = E\left[\left(\int_{[t_{i-1}, t_i]} dS_{j_1} - \delta_{i,j_1}\right)\left(\int_{[t_{i-1}, t_i]} dS_{j_2} - \delta_{i,j_2}\right)\right] = \int_{[t_{i-1}, t_i]} \sum_\beta \sigma_{j_1,\beta}\sigma_{j_2,\beta} dt$$

These local cumulants are the only ones that are non zero.

If we follow our preceeding prescription, the only non-zero cumulants for the variable V are :

$$\mu(V) = \mu(X_1) + \mu(X_2) + \mu(X_3)$$
$$= (\delta_{1,A} + \delta_{1,B} + \delta_{1,C} + \delta_{1,D} + \delta_{1,E}) + (\delta_{2,B} + \delta_{2,C} + \delta_{2,D}) + \delta_{3,D}$$

and

$$\sigma^2(V) = \sigma^2(X_1) + \sigma^2(X_2) + \sigma^2(X_3)$$
$$= \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} C_1 \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}^t + \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix} C_2 \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}^t + \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} C_3 \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}^t$$
$$= \sum_{\substack{i \in \{A, B, C, D, E\} \\ j \in \{A, B, C, D, E\}}} C_{1,i,j} + \sum_{\substack{i \in \{B, C, D\} \\ j \in \{B, C, D\}}} C_{2,i,j} + C_{3,D,D}$$

In the case of Capital at Risk calculation, the expectation does not matter because it is the best estimation of the profit that would have been made in the market. By arbitrage, this profit should be a short term interest rate on the capital . The interesting part is represented by $\sigma^2(V)$ which represents the variance associated with the Capital at Risk measure.

3.1 Portfolio Unwinding Period

Using the preceeding example, we can define another Variance, which is the instantaneous variance that we would have computed to define a RiskMetrics-like VaR.

$$\mathcal{V}(V, 0)dt = E[(dS_\alpha - \mu_\alpha dt)(dS_\alpha - \mu_\alpha dt)] = \sum_\beta \sigma_{j_1,\beta}\sigma_{j_2,\beta} dt$$

Therefore the Instantaneous variance is equal to

$$\mathcal{V}(V) = \sum_{\beta} \sigma_{j_1, \beta} \sigma_{j_2, \beta}$$

We define the unwinding period associated with the portfolio as

$$\mathcal{T}(V) = \frac{\sigma^2(V)}{\mathcal{V}(V)}$$

To finish the computation, let's make another assumption

Let's assume that $\sigma_{j, \beta}(t) = \sigma_{j, \beta}$ is constant over $[t_0, t_3]$.

Then the portfolio unwinding period is equal to :

$$\mathcal{T}(V) = \frac{\left(\sum_{\substack{i \in \{A, B, C, D, E\} \\ j \in \{A, B, C, D, E\}}} \sum_{\beta} \sigma_{i, \beta} \sigma_{j, \beta} \right)(t_1 - t_0) + \left(\sum_{\substack{i \in \{B, C, D\} \\ j \in \{B, C, D\}}} \sum_{\beta} \sigma_{i, \beta} \sigma_{j, \beta} \right)(t_2 - t_1) + \left(\sum_{\beta} \sigma_{D, \beta} \sigma_{D, \beta} \right)(t_3 - t_2)}{\left(\sum_{\substack{i \in \{A, B, C, D, E\} \\ j \in \{A, B, C, D, E\}}} \sum_{\beta} \sigma_{i, \beta} \sigma_{j, \beta} \right)}$$

4 Total variance and portfolio unwinding period

The preceeding example leads us to make the following definitions :

1. For a position P whose instantaneous variance is $\mathcal{V}$ and whose unwinding period is T, we define the **Total Variance** W as

$$W = \mathcal{V} \times T$$

2. For a portfolio $\mathcal{P}$ whose instantaneous variance is $\mathcal{V}$ and whose Total variance W computed as chown before $(\sigma^2(V))$, we define the **Unwinding Period** as

$$T = \frac{W}{\mathcal{V}}$$

These two rules allow us to associate a hierarchy of total variances or equivalently a hierarchy of unwinding periods to a hierarchy of portfolios . These heriarchies possess an important associativity property which is the modularity of our computation:

> To compute the unwinding period of a portfolio, it does not matter whether you do it directly from the basic deals or from intermediate portfolios for which you will have computed intermediate unwinding periods

5 Point of view 1 : Period Variances Computation

The preceding example allows us to define the first way to compute the liquidity risk associated with a portfolio. Let's summarize it by the following ::

$$W[P] = \sigma^2(X_1) + \sigma^2(X_2) + \sigma^2(X_3)$$

Point of view 1

$$\mathcal{V}[P] = \sigma^2(X_1)$$

5.0.1 Example 1 (Point of view 1)

Let's consider a bond portfolio with two positions :

- Position A has a future equivalent of 10 contracts and an unwinding period of 3 days.
- Position B has a future equivalent of 30 contracts and an unwinding period of 1 day.

Therefore we have two periods of respective length of 1 day and 2 days

We assume that the standard deviation with 1 contract is $ 1000 for 1 day. Therefore the variance for one day is 1000,000.

The total variance is equal to :

$$W = 40000^2 + 10000^2 \times 2 = 1700M$$

while the instantaneous variance is equal to

$$\mathcal{V} = 40000^2$$

therefore the unwinding period associated with the portfolio is equal to :

$$T = \frac{1700M}{40000^2} = 1.0625 \text{ days}$$

5.0.2 Example 2 (Point of view 1)

Let’s reverse the positions : consider a bond portfolio with two positions :

- Position A has a future equivalent of 30 contracts and an unwinding period of 3 days.
- Position B has a future equivalent of 10 contracts and an unwinding period of 1 day

The total variance is equal to :

$$W[P] = 40000^2 + 30000^2 \times 2 = 3400M$$

while the instantaneous variance is equal to

$$\mathcal{V}[P] = 40000^2$$

therefore the unwinding period associated with the portfolio is equal to :

$$T[P] = \frac{3400M}{40000^2} = 2.125 \text{ days}$$

which is midway between 1 day and 3 days. This shows a decreasing importance of the long unwinding periods in the portfolio which is due to the square root law of the standard deviation evolution with time.

5.0.3 Example 3 (Point of view 1)

Let’s consider a portfolio made of two assets A and B with volatility of respectively 20% and 25%. The correlation between them is 0.7.

- The first position of \$10M in asset A has an unwinding perid of 3 days,
- The second position of \$5M in asset B has an unwinding period of 1 day.

During the first period of time, the variance of the portfolio is

$$Var[X_1] = \begin{bmatrix} 10M & 5M \end{bmatrix} \begin{bmatrix} 0.20^2 & 0.7 \times 0.2 \times 0.25 \\ 0.7 \times 0.2 \times 0.25 & 0.25^2 \end{bmatrix} \begin{bmatrix} 10M \\ 5M \end{bmatrix} \times \frac{1}{365} = 2.483 \times 10^{10}$$

During the second period of time the variance of the portfolio is : \

$$Var[X_1] = \begin{bmatrix}10M & 0\end{bmatrix}\begin{bmatrix}0.20^2 & 0.7\times 0.2\times 0.25 \\ 0.7\times 0.2\times 0.25 & 0.25^2\end{bmatrix}\begin{bmatrix}10M \\ 0\end{bmatrix}\times\frac{2}{365} = 2.192\times 10^{10}$$

Therfore the total variance is :

$$W[P] = 2.483\times 10^{10} + 2.192\times 10^{10} = 4.6745\times 10^{10}$$

and the equivalent unwinding period is :

$$T[P] = \frac{4.6745\times 10^{10}}{2.483\times 10^{10}} = 1.88 \text{ days}$$

6 Point of View 2 : A Delta Density Point of View

Let's assume that we have a continuum (for the discretized case , we use generalized functions or distributions) of deal whose unwinding period are between T and $T + dT$. Let's call the total value of associated deals $dP(T)$.

and let's define

$$\langle dP(T_1)|dP(T_2)\rangle = \sum_{\alpha}\left[\frac{\partial}{\partial W_\alpha}dP(T_1)\right]\left[\frac{\partial}{\partial W_\alpha}dP(T_2)\right]$$

we show in annexe that the total variance and the unwinding period can be put in the following forms :

$$W[P] = \int_{(T_1, T_2)} Min[T_1, T_2]\langle dP(T_1)|dP(T_2)\rangle$$

Point of view 2

$$\mathcal{V}[P] = \int_{(T_1, T_2)} \langle dP(T_1)|dP(T_2)\rangle$$

where the scalar product is nothing but the scalar product between the densities of deltas along the liquidation period .The associativity property derives from the fact that the portfolio enters into the formula as a density. This is shown in annexe.

An interesting interpretation of the preceeding formula is that the total variance is just the expectation of the operator Min(x,y) under the bidimensional measure $\langle dP(T_1) | dP(T_2) \rangle$.

We can also say that the unwinding period is just a weighted average of min(x,y) where x and y are the unwinding periods associated with the deals. The bidimensionality comes from the fact that we need to consider variances as quadratic forms.

Another benefit of the preceeding formula is that it stays valid when we introduce a covariance matrix. The covariance structure is entirely confined into the scalar product.This result comes from the fonctorial nature of the formula where the fonctor goes from the category of euclidian vector spaces (vector spaces with a scalar product and their morphisms) to the category of measured spaces.Therefore the formula remains valid when the covariance structure changes.

6.1 Example 1 (point of view 2)

Let consider a bond portfolio with two positions :

- Position A has a future equivalent of 10 contracts and an unwinding period of 3 days.
- Position B has a future equivalent of 30 contracts and an unwinding period of 1 day

There are not sufficient data to compute the total variance of the portfolio, but we can compute the unwinding period of the portfolio,The delta density is equal to :

$$\frac{dP}{dT}(T) = -10 \times \delta(T-3) - 30 \times \delta(T-1)$$

The negative sign , which has no influence on the rest of the computation, is here to remind us that the position should decrease with time, and therefore for a positive delta the density should be negative . We have to compute :

$$T[P] = \frac{\displaystyle\int_{(T_1, T_2)} Min[T_1, T_2]\left(\frac{dP}{dT}(T_1)\frac{dP}{dT}(T_2)\right) dT_1 dT_2}{\left(\displaystyle\int \frac{dP}{dT}(T) dT\right)^2}$$

In this case the integrals to perform is made only of terms like :

$$\int_{(T_1, T_2)} Min[T_1, T_2](a\delta(T_1 - T_a) \times b\delta(T_2 - T_b))dT_1 dT_2 = ab(Min[T_a, T_b])$$

$$\int_{(T_1, T_2)} (a\delta(T_1 - T_a) \times b\delta(T_2 - T_b))dT_1 dT_2 = ab$$

therefore we find :

$$T[P] = \frac{30^2 \times 1 + 10 \times 30 \times 1 + 30 \times 10 \times 1 + 10^2 \times 3}{30^2 + 10 \times 30 + 30 \times 10 + 10^2} = \frac{1700}{1600} = 1.0625$$

which is consistent with point of view 1.

6.2 Example 3 (point of view 2)

Let’s consider a portfolio made of two assets A and B with volatility of respectively 20% and 25%. The correlation between them is 0.7.

- The first position of \$10M in asset A has an unwinding perid of 3 days,
- The second position of \$5M in asset B has an unwinding period of 1 day.

$$\frac{dP}{dT}(T) = -\begin{pmatrix} 10M \times \delta(T-3) \\ 5M \times \delta(T-1) \end{pmatrix}$$

we have to compute :

$$T[P] = \frac{\displaystyle\int_{(T_1, T_2)} Min[T_1, T_2]\left(\frac{dP}{dT}(T_1)\begin{bmatrix} 0.20^2 & 0.7 \times 0.2 \times 0.25 \\ 0.7 \times 0.2 \times 0.25 & 0.25^2 \end{bmatrix} \times \frac{dP}{dT}(T_2)\right)dT_1 dT_2}{\displaystyle\int \frac{dP}{dT}(T_1) \times \begin{bmatrix} 0.20^2 & 0.7 \times 0.2 \times 0.25 \\ 0.7 \times 0.2 \times 0.25 & 0.25^2 \end{bmatrix} \times \frac{dP}{dT}(T_2)dT_1 dT_2}$$

but

$$\frac{dP}{dT}(T_1) \times \begin{bmatrix} 0.20^2 & 0.7 \times 0.2 \times 0.25 \\ 0.7 \times 0.2 \times 0.25 & 0.25^2 \end{bmatrix} \times \frac{dP}{dT}(T_2)$$

$$= 4\delta(T_1 - 3)\delta(T_2 - 3) + 1.75\delta(T_1 - 1)\delta(T_2 - 3) + 1.75\delta(T_1 - 3)\delta(T_2 - 1) + 1.5625\delta(T_1 - 1)\delta(T_2 - 1)$$

so using the preceeding identities we find :

$$T[P] = \frac{1.70625 \times 10^{13}}{9.0625 \times 10^{12}} = 1.88$$

which is consistent with the calculation done using point of view 1

In order to do these calculations we have used the presentation of the formula with a covariance matrix that describes the geometry of the delta density space. We could have used pure deltas with respect to a basis where the covariance matrix was equal to identity, in this case we would have had to use much simpler formula like in the example 1 and 2 .

6.3 Example 4 (point of view 2)

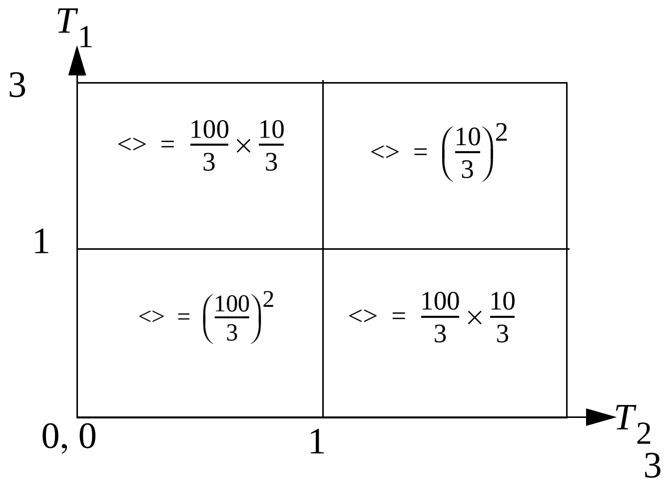
Let's consider a bond portfolio with two positions :

- Position A has a future equivalent of 10 contracts and its unwinding period is 3 days if we trade it in one block but we want to trade it all along the 3 days .
- Position B has a future equivalent of 30 contracts and its unwinding period is 1 day if we trade it in one block but we want to trade it regurlarly all along the day

The difference with the preceeding case is that we assume that we unwind the portfolio regularly . We will interpret that by assuming that the positions decrease linearly to 0 at the given unwinding date. therefore we have :

$$\frac{dP}{dT}(T) = -10 \times \left(\frac{1}{3}\right)_{0 \le T \le 3} - 30 \times (1)_{0 \le T \le 1} = -\left(\frac{100}{3}\right)_{0 \le T \le 1} - \left(\frac{10}{3}\right)_{1 < T \le 3}$$

For this piecewise constant function we partition the Plan $\{T_1, T_2\}$:into 4 regions



We also use : $\int_a^b\left[\int_a^b Min[x, y]dy\right]dx = \frac{2a^3}{3} - a^2b + \frac{b^3}{3}$

therefore:

$$T[P] = \frac{\left(\frac{100}{3}\right)^2\times\left(\frac{1}{3}\right) + 2\times\frac{100}{3}\times\frac{10}{3}\times\left(\frac{1}{2}\times 2\right) + 10^2\times\left(\frac{2}{3} - 3 + 9\right)}{40^2} = \frac{666.667}{1600} = 0.41667 \text{ days}$$

the result shows a drastic reduction of the unwinding period due to the intraday trading.There is no miracle here, we are simply comparing two different ways of computing the unwinding period. We are computing an unwinding period measure that assumes we get rid of the position at the end of the unwinding period, and we using it to measure a strategy that unwinds regurlarly and smoothly the position up to the unwinding date.

7 Point of View 3 : A Liquidation Positions Point of View

Let’s assume now that we have a liquidation strategy for each asset:

During every period $[t_{i-1}, t_i]$ and for every asset j we will liquidate a certain quantitity and keep the quantity $\eta_{i,j}$. We do not allow trading (that means buying back the securities for speculation purposes). Only liquidation is allowed, therefore the following equations hold:

$$\eta_{i,j} \geq \eta_{i+1,j} \qquad \eta_{0,j} = 1 \qquad \eta_{T,j} = 0$$

Then the preceeding prescription for the computation of the total variance extends to :

$$\sigma^2(V) = \sum_{i,j_1,j_2} \eta_{i,j_1} C_{i,j_1,j_2} \eta_{i,j_2}$$

Where i represents the ith period of time.

From which we can deduce the risk measures :

$$W(V) = \sum_{i,j_1,j_2} \eta_{i,j_1} \left(\int_{[t_{i-1},t_i]} \sum_{\beta} \sigma_{j_1,\beta} \sigma_{j_2,\beta} dt \right) \eta_{i,j_2}$$

$$\mathcal{U}(V) = \sum_{j_1,j_2,\beta} \sigma_{j_1,\beta} \sigma_{j_2,\beta} \qquad \text{(in t=0)}$$

If we continuize the time (dimension i) we get :

$$W(P) = \int_t \left(\sum_{j_1,j_2,\beta} \eta_{t,j_1} \sigma_{j_1,\beta} \sigma_{j_2,\beta} \eta_{t,j_2} \right) dt$$

Point of view 3

$$\mathcal{U}[P] = \sum_{j_1,j_2,\beta} \sigma_{j_1,\beta} \sigma_{j_2,\beta}$$

this expression for the liquidation risk is equivalent to Point of view 2 where we only take the density of deltas into account. It is very instructive to make the contact directly with Point of view 2. It is described in annexe

7.1 Example 4 (point of view 3)

Let’s consider our simple bond portfolio with two positions :

- Position A has a future equivalent of 10 contracts and an unwinding period of 3 days.
- Position B has a future equivalent of 30 contracts and an unwinding period of 1 day

As before we assume that the standard deviation of 1 contract is $ 1000 for 1 day and the unwinding occurs at the end of the unwinding period.

Therefore the position is

$$\eta[T] = 10 \times \left(1 - \frac{T}{3}\right)_{0 \le T \le 3} + 30 \times (1 - T)_{0 \le T \le 1} = \left(40 - \frac{100T}{3}\right)_{0 \le T \le 1} + \left(\frac{20}{3} - 10\frac{(T-1)}{3}\right)_{1 < T \le 3}$$

The total variance is equal to :

$$W(P) = \int_t (\eta[T]\sigma^2\eta[T])dt$$

therefore

$$W(P) = \int_{[0,1]} \left(40 - \frac{100T}{3}\right)^2 \sigma^2 dt + \int_{[1,3]} \left(\frac{20}{3} - 10\frac{(T-1)}{3}\right)^2 \sigma^2 dt$$

we get

$$W(P) = \frac{17200\sigma^2}{27} + \frac{800\sigma^2}{27} = 666.667\sigma^2$$

which is consistent with Point of view 2. We notice that in this case, the computation is much simpler using Point of view 3 than using Point of view 2. This true for this example, not true in general.

8 Point of View 4 : Stochastic Strategies Point of View

It is possible to extend the preceeding ideas to stochastic liquidations. That means that the $\eta_{t, j}$ depends now on the market. In annexe we show that under these new assumptions, the expression for the total variance becomes :

$$W[P] = E\left[\int \sum_{i, j} \eta_i \eta_j d[S_i, S_j] dt\right]$$

Stochastic generalization

where the integrator (the differential term) is the density of quadratic covariance.

As before, the computation of the total variance enables us to define the unwinding period by dividing it by the intantaneous variance.

8.1 Example 5 (point of view 4)

Let consider a bond portfolio with two positions :

- Position A has a future equivalent of 10 contracts and its unwinding period is 3 days if we trade it in one block, but we want to get rid of it in a much smarter way, taking into advantage of the market moves.
- Position B has a future equivalent of 30 contracts and its unwinding period is 1 day if we trade it in one block, but as for the preceding position we want to trade it in a smart way.

Unlike the preceeding handlings, in this case we will get rid of the position when the security price is rising, not when it is decreasing.

we assume that the strategy is given by the rule :

- Sell a percentage of the position if the security price is above a limit. let's call the respective limits : l_a and l_b and the respective percentage per unit of time :p_a and p_b.

The differential equations for η_i is :

$$d\eta_{i, t} = -\eta_{i, 0} p_i 1_{(S_{i, s} > l_i) \cap (\eta_{i, t} < 1)}$$

The solution of this equation is :

$$\eta_{i, t} = \eta_{i, 0} Max\left[1 - p_i \int_0^t 1_{(S_{i, s} > l_i)} ds, 0\right]$$

which is stochastic. Therefore we have to compute

$$W[P] = (\eta_{i,0}\sigma_i)^2 \int E\left[\left(Max\left[1 - p_i \int_0^t 1_{(S_{i,s} > l_i)} ds, 0\right]\right)^2\right] dt$$

Let's define H such

$$W[P] = (\eta_{i,0}\sigma_i)^2 \int H(t) dt$$

we can express H(t) as:

$$H(t) = E\left[\left(1 - p_i \int_0^t 1_{(S_{i,s} > l_i)} ds\right)^2 1_{\int_0^t 1_{(S_{i,s} > l_i)} ds < \frac{1}{p_i}}\right]$$

which is equal to :

$$H(t) = H_1(t) - 2p_i H_2(t) + p_i^2 H_3(t)$$

where

$$H_1(t) = Prob\left[\int_0^t 1_{(S_{i,s} > l_i)} ds < \frac{1}{p_i}\right]$$

$$H_2(t) = E\left[\left(\int_0^t 1_{(S_{i,s} > l_i)} ds\right) 1_{\int_0^t 1_{(S_{i,s} > l_i)} ds < \frac{1}{p_i}}\right]$$

$$H_3(t) = E\left[\left(\int_0^t 1_{(S_{i,s} > l_i)} ds\right)^2 1_{\int_0^t 1_{(S_{i,s} > l_i)} ds < \frac{1}{p_i}}\right]$$

We can make the link with classical probability problems by introducing $L_i(t, x)$ the local time of the process i,

then we can rewrite : $\int_0^t 1_{(S_{i,s} > l_i)} ds = \int_{l_i}^{\infty} L_i(t, x) dx$

these integrals can be computed using Monte Carlo methods.

One particular case is tractable : If $l_i = S_{i,0}$ and $\mu_i = 0$ then we can introduce the arc sin law by defining :

$$X_t = \int_0^t 1_{(W_s > 0)} ds$$

where W is a standard brownian

Then the law of X is : $Prob[X_t < y] = \frac{2}{\pi} ArcSin\left[\sqrt{\frac{y}{t}}\right]$

and the integrals can be computed analytically:

$$H_1\left(\frac{t}{\sigma_i^2}\right) = Prob\left[X_t < \frac{1}{p_i}\right] = \frac{2}{\pi} ArcSin\left[\sqrt{\frac{1}{p_i t}}\right]$$

$$H_2\left(\frac{t}{\sigma_i^2}\right) = E\left[X_t 1_{X_t < \frac{1}{p_i}}\right] = \int_0^{\frac{1}{p_i}} \frac{2x}{\pi t \sqrt{1 - \frac{x^2}{t^2}}} dx = \frac{2}{\pi t}\left(t^2 + \frac{1 - p_i^2 t^2}{p_i^2 \sqrt{1 - \frac{1}{p_i^2 t^2}}}\right)$$

$$H_3\left(\frac{t}{\sigma_i^2}\right) = E\left[X_t^2 1_{X_t < \frac{1}{p_i}}\right] = \int_0^{\frac{1}{p_i}} \frac{2x^2}{\pi t \sqrt{1 - \frac{x^2}{t^2}}} dx = \frac{1 - p_i^2 t^2 - p_i^3 t^2 \sqrt{t^2 - \frac{1}{p_i^2}} ArcTan\left[\frac{p\sqrt{t^2 - \frac{1}{p_i^2}}}{1 - p_i^2 t^2}\right]}{p^3\left(\pi t \sqrt{1 - \frac{1}{p_i^2 t^2}}\right)}$$

9 Taking into account Interest rates

We assume that interest rates are deterministic. That means that when we write an equation like ,

$$dX_t = \mu dt + \sigma dW_t$$

we may think that it is expressed in a currency that we need to converted.The exchange rate is precisely the discount ratio that will transform our zero interest rate equation into a deterministic interest rate equation. Let's call r_t the continuous zero coupon interest rate . The present value equation should now look like :

$$d\bar{X}_t = \left(\mu e^{-r_t t}\right) dt + \left(\sigma e^{-r_t t}\right) dW_t$$

This gives us a hint about how to include the interest rates into our point of views:

we just have to weight all the instantaneous variances σ^2 with a factor of $e^{-2r_t t}$ as long as they are associated with a position that continues a t time t.

A precise demonstration of it is done in the annexes in the case of the stochastic strategies.

10 Conclusion

In this paper, we described a way to compound the risk associated with the unwindind of different positions with different unwinding periods. The associativity and modularity of the method give us a convenient tool because of . The main hypothesis is the linearity of the evolution equations. This is an acceptable hypothesis for portfolios of bonds and stocks because we are only concerned with a very short term evolution.

11 Annexes

Mathematical demonstrations of results used in the paper

11.1 Formula for the point of view 2

Let's formalize the construction we have done to compute the unwinding period of a portfolio from the unwinding periods of the underlying deals

Let's assume a portfolio $P = \bigcup_i P_i$ made of sub-portfolios

with no common positions,that means that $\{P_i\}$ is a partition of P.We also assume that $T[P_i] < T[P_{i+1}]$ only for convenience of the presentation of the computations.

Let's call T the unwinding period associated with P and T_i the unwinding period associated with P_i . We always assume the linearity of the processes with respect to a set of brownian motions W_α . Therefore, The Sensitivity of the portfolio with respect the these brownians is additive:

$$S[P]_\alpha = \frac{\partial P}{\partial W_\alpha} = \sum_i \frac{\partial P_i}{\partial W_\alpha} = \sum_i S[P_i]_\alpha$$

The Instantaneous variance associated with every portfolio depends only on this sensitivity. It can be defined as the square of a norm on this sensitivity:

$$\mathcal{V}(P) = S[P]^*S[P] = \sum_{i,j} S[P_i]^*S[P_J]$$

therefore we can write the reccurence property as :

$$\mathcal{V}(P)T[P] = \sum_i (T[P_i] - T[P_{i-1}])\,\mathcal{V}\left(\bigcup_{j \geq i} P_j\right)$$

$$\mathcal{V}(P) = S[P]^*S[P]$$

$$S[P] = \sum_i S[P_i]$$

which also allows us to write :

$$T[P] = \frac{\sum_i (T[P_i] - T[P_{i-1}])\left(\sum_{j \geq i} S[P_j]\right)^*\left(\sum_{k \geq i} S[P_k]\right)}{\left(\sum_j S[P_j]\right)^*\left(\sum_k S[P_k]\right)}$$

To generalize the preceeding equation, let's assume that we have a continuum (or not if we use generalized functions or distributions) of deals whose unwinding period are between T and $T + dT$. let's call this number $dP(T)$.

Let's also introduce the scalar product

$$\langle dP(T) | dP(T') \rangle = S[dP(T)]^*S[dP(T')]$$

Then we can write the preceeding equation as :

$$T[P] = \frac{\int_T \left(\int_{(T_1 > T) \cap (T_2 > T)} \langle dP(T_1) | dP(T_2) \rangle \right) dT}{\int_{(T_1,\, T_2)} \langle dP(T_1) | dP(T_2) \rangle}$$

We then can integrate once this integral into

$$T[P] = \frac{\int_{(T_1,\, T_2)} Min[T_1, T_2] \langle dP(T_1) | dP(T_2) \rangle}{\int_{(T_1,\, T_2)} \langle dP(T_1) | dP(T_2) \rangle}$$

The advantage of the above equation, besides its obvious simplicity and generality, is that we do not need to implicitly assume any order in the components of the portfolio.

To prove the modularity and therefore the consistency of our formula, let's assume that

$$dP(T) = \int_x (dp(T, x)) dx$$

then we can rewrite the total unwinding period :

$$T[P] = \frac{\int_{(T_1,\, T_2,\, x,\, y)} Min[T_1, T_2] \langle dp(T_1, x) | dp(T_2, y) \rangle}{\int_{(T_1,\, T_2,\, x,\, y)} \langle dp(T_1, x) | dp(T_2, y) \rangle}$$

It is obvious that this implies

$$W[P] = \int_{(T_1, T_2)} Min[T_1, T_2]\langle dP(T_1)|dP(T_2)\rangle$$

$$\mathcal{V}[P] = \int_{(T_1, T_2)} \langle dP(T_1)|dP(T_2)\rangle$$

11.2 Direct demonstration of the equivalence between point of view 2 and point of view 3

We can check at this point that we can still make contact between point of view 2 and point of view 3 by making the following substitutions :

$$\left.\int_t \left(\sum_{j_1, j_2, \beta} \eta_{t, j_1} \sigma_{j_1, \beta} \sigma_{j_2, \beta} \eta_{t, j_2} \right) dt \right|_{\eta = \Theta(T(P_j) - T)} = \sum_{j_1, j_2, \beta} Min\left[T\left(P_{j_1}\right), T\left(P_{j_2}\right)\right] \sigma_{j_1, \beta} \sigma_{j_2, \beta}$$

because $\int \Theta(A - T)\Theta(B - T) dT = Min[A, B]$

and

$$\left.\int_{(T_1, T_2)} Min[T_1, T_2]\langle dP(T_1)|dP(T_2)\rangle \right|_{dP(T) = \sum_j \delta(T(P_j) - T)} = \sum_{j_1, j_2, \beta} Min\left[T\left(P_{j_1}\right), T\left(P_{j_2}\right)\right] \sigma_{j_1, \beta} \sigma_{j_2, \beta}$$

because $\int_{(T_1, T_2)} \delta(A - T)\delta(B - T) Min[T_1, T_2] dT_1 dT_2 = Min[A, B]$

This shows that both points of view are equivalent.

11.3 Case of a Stochastic Strategy

It is simple to see that from the point of view of semimartingale theory, we have the equality

$$dV_t = \eta_t \cdot dS_t = \sum_i \eta_{i,t} dS_{i,t}$$

and from this formula, if we apply the formula (theorem 29 p.68 of [Protter])

$$\left[\int H dX, \int K dY\right] = \int HK d[X, Y]$$

So we can write

$$[V_t, V_t] = \sum_{i,j} \left[\int dS_i, \int dS_j\right] = \sum_{i,j} \int \eta_i \eta_j d[S_i, S_j] = \int \sum_{i,j} \eta_i \eta_j d[S_i, S_j]$$

But the definition of the quadratic variation is :

$$[V, V] = V^2 - 2\int V dV$$

Therefore, If V is a martingale, (we have to work in the martingale measure), we can set the prices such that $V_0 = 0$ and :

$$Var[V] = E[V^2] - (E[V])^2 = E[V^2] - V_0^2 = E[[V, V]]$$

Then if we remember that :

$$[dS_i, dS_j] = \left(\sum_\alpha \frac{\partial S_i}{\partial W_\alpha} \frac{\partial S_j}{\partial W_\alpha}\right) dt = (S[S_i] \cdot S[S_j]) dt$$

We get the preceeding formula for the total variance.

$$W = E\left[\int \sum_{i,j} \eta_i \eta_j (S[S_i] \cdot S[S_j]) dt\right]$$

This derivation is more powerful than the preceeding ones because it can be used even if the unwinding processes are correlated with the market prices. The only constraint here is that positions should be predictable with respect to the filtration generated by the asset prices. Also this formula is valid for non-continuous market .

$$W = E\left[\int \sum_{i,j} \eta_i \eta_j \frac{d[S_i, S_j]}{dt} dt\right]$$

This formula gives back the formula of point of view 3 when the η are deterministic, and therefore includes all other point of view as well.

11.3.1 Handling of interest rates

We want to include interest rates in our calculation . Let’s first look at the discretized case : the value of the liquidation portfolio is the value of the sum of future cash flows induced by the liquidation:

$$V = \sum_{i=0}^{n} S_i (\eta_{i+1} - \eta_i) e^{-r_i t_i}$$

then we can rearrange :

$$V = -S_0 \eta_0 e^{-r_0 t_0} + \sum_{i=0}^{n-1} \eta_i \left(S_i e^{-r_i t_i} - S_{i+1} e^{-r_{i+1} t_{i+1}}\right) + S_n \eta_n e^{-r_n t_n}$$

The preceeding equations have to be interpreted in a vectorial way: η_i and S_i are vectors.

In the preceeding equation, by hypothesis we have: $S_0 \eta_0 e^{-r_0 t_0} = V_0$, $\eta_n = 0$,

Therefore we can write the continuous version of this equation:

$$dV_t = \eta_t d\left[S_t e^{-r_t t}\right]$$

As long as we assume that $S_t e^{-r_t t}$ is a vector of martingales, the rest of the argument applies and we find :

$$W = E\left[\int \sum_{i,j} \eta_i \eta_j \frac{d[S_i, S_j]}{dt} e^{-2 r_t t} dt\right]$$

Which validates the remark made in paragraph 9.

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