Correlation Problems in Stochastic Volatility Situations

by Olivier Croissant

Stochastic volatility pricing

- Processes

$$\begin{cases} dS_1 = \mu_1(S_1, S_2, .., S_n, t)dt + \sigma_1(S_1, S_2, .., S_n, t)dW_1 \\ dS_2 = \mu_2((S_1, S_2, .., S_n)c, t)dt + \sigma_2(S_1, S_2, .., S_n, t)dW_2 \\ \ldots \\ dS_n = \mu_n(S_1, S_2, .., S_n, t)dt + \sigma_n(S_1, S_2, .., S_n, t)dW_1 \end{cases}$$

- with

$$\langle dW_i, dW_j \rangle = \rho(S_1, S_2, .., S_n, t)dt$$

- By Feynman Kac: Differential equation for the option that pays at T :

$f(S_1(T), S_2(T), .., S_n(T)) - K$ if positive :

$$\begin{cases} \frac{\partial}{\partial T}P = \frac{1}{2}\left(\sum_{i,j} \rho_{i,j}\sigma_i\sigma_j\frac{\partial^2 p}{\partial S_i \partial S_j}\right) + \sum_{i,j} \mu_i \frac{\partial p}{\partial S_i} - rp \\ p(S_1, S_2, .., S_n, 0) = (f(S_1(T), S_2(T), .., S_n(T)) - K)^+ \end{cases}$$

A Second Order Differential Operator (dream)

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$$P = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

$$\Big\downarrow\, ? \qquad x = x(y)$$

$$P = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$$

A Second Order Differential Operator (Reality :Dif Geo)

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$$P = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

y **geodesic coordinates**

$$\partial_i \to \nabla_i = \partial_i + \mathcal{A}_i$$

$$P = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} + Q$$

$$P = \sum_{i=1}^{n} \nabla_i^2 + Q$$

Generalization of Laplacian

- A differentiable manifold equiped with a metric (the density measure is $\sqrt{g}dv$)
  - There is locally a systeme of coordinate where the metric is the euclidian metric. This is the normal cooordinates (geodesic based) .
  - There is a standard laplacian expressed in the normal coordinates .
  - How to compute it in the current coordinates ?

- We need an intrinsic definition of the laplacian (invariant by a change of coordinate)
  - In an euclidian space $(\Delta(f), h) = \int_M \Delta(f) \cdot h dv = \sum_i \int_M \frac{\partial^2 f}{\partial x_i^2} \cdot h dv = -\sum_i \int_M \frac{\partial f}{\partial x_i} \cdot \frac{\partial h}{\partial x_i} dv = -(\nabla(f), \nabla(h))$

    for f and g that are 0 at infinity
  - In a space with a metric g

integration by part !

$$-(\nabla(f), \nabla(h)) = -\int_M \nabla(f) \cdot \nabla(h) \sqrt{g} dv = -\int_M g^{ij}\left(\frac{\partial f}{\partial x^i}\right)\left(\frac{\partial h}{\partial x^j}\right)\sqrt{g} dv = \int_M \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(g^{ij}\left(\frac{\partial f}{\partial x^j}\right)\sqrt{g}\right) h \sqrt{g} dv$$

  - So the invariant definition is : $\Delta(f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(g^{ij}\left(\frac{\partial f}{\partial x^j}\right)\sqrt{g}\right)$

The Differential Geometry Solution

- Transition Probability from (t, S) to (t', S'):

Determinant of the metric — Synge world function — Scalar curvature — Higher order terms

$$p(t, S, t', S') = \frac{\sqrt{g(S)}}{(4\pi(t'-t))^{n/2}}\sqrt{\Delta(S, S')}\,P(S, S')\,e^{-\frac{\sigma(S, S')}{2(t'-t)}}\left(1 + (t'-t)\left(\frac{1}{6}R + Q\right) + \ldots\right)$$

Parralel transport of the connection

Van Vleck Morette determinant — Mass term

- Local volatility of f :

$$\sigma_f(t, f) = \frac{\int\left(\sum_{i, j}\rho_{i, j}\sigma_i\sigma_j\frac{\partial^2 f}{\partial S_i\partial S_j}(S)\right)\delta(f(S)-f)\,p(t, S, t', S')\,dS'}{\int p(t, S, t', S')\,dS'}$$

- Implicit volatility of the option:

$$\sigma_{BS}(t, S, t', S') = \frac{Log\left(\frac{K}{f_0}\right)}{\int_{f_0}^{K}\frac{df}{\sigma_f(f)}}\left(1 + T\frac{\sigma_f^2(f)}{24}\left(2\frac{\sigma_f''(f)}{\sigma_f(f)} - \left(\frac{\sigma_f'(f)}{\sigma_f(f)}\right)^2 + \frac{1}{f^2} + 12\frac{\frac{\partial}{\partial t}\sigma_f(f)}{\sigma_f^3(f)}\right) + \ldots\right)$$

Integrating on the State Variables

- $\sigma_f(t,f) = \dfrac{\int\left(\sum_{i,j}\rho_{i,j}\sigma_i\sigma_j\frac{\partial^2 f}{\partial S_i \partial S_j}(S)\right)\delta(f(S)-f)p(t,S,t',S')dS'}{\int p(t,S,t',S')dS'}$

- we have to perform $\displaystyle\int_{(f(y,S)=f)} \psi(y)e^{\frac{\varphi(y)}{\varepsilon}}dy$

- Steepest descent algorithms $\displaystyle\int_{(f(y,S)=f)} \psi(y)e^{\frac{\varphi(y)}{\varepsilon}}dy = e^{\frac{\varphi(y^*)}{\varepsilon}}\left(\sum_k a_k(\psi,\varphi,y^*)\varepsilon^k\right)$ where y^* is the minimum of $\varphi(y)$ constraint by $f(y,S) = f$ (Lagrange minimizing shema)

The Laplace Development

$$\int_{-\infty}^{\infty} f(u)\, e^{-\phi(u)/\epsilon} =$$

$$\frac{\sqrt{2\pi\epsilon}}{\sqrt{\phi''[u0]}}\, e^{-\frac{\phi[u0]}{\epsilon}}$$

$$\biggl(f[u0] + \epsilon \left(\frac{f''[u0]}{2\phi''[u0]} - \frac{f'[u0]\,\phi^{(3)}[u0]}{2\phi''[u0]^2} + \frac{5 f[u0]\,\phi^{(3)}[u0]^2}{24\phi''[u0]^3} - \frac{f[u0]\,\phi^{(4)}[u0]}{8\phi''[u0]^2}\right) +$$

$$\epsilon^2 \biggl(-\frac{5 f^{(3)}[u0]\,\phi^{(3)}[u0]}{12\phi''[u0]^3} + \frac{35 f''[u0]\,\phi^{(3)}[u0]^2}{48\phi''[u0]^4} - \frac{35 f'[u0]\,\phi^{(3)}[u0]^3}{48\phi''[u0]^5} + \frac{f^{(4)}[u0]}{8\phi''[u0]^2} - \frac{5 f''[u0]\,\phi^{(4)}[u0]}{16\phi''[u0]^3} +$$

$$\frac{35 f'[u0]\,\phi^{(3)}[u0]\,\phi^{(4)}[u0]}{48\phi''[u0]^4} - \frac{35 f[u0]\,\phi^{(3)}[u0]^2\,\phi^{(4)}[u0]}{64\phi''[u0]^5} + \frac{35 f[u0]\,\phi^{(4)}[u0]^2}{384\phi''[u0]^4} - \frac{f'[u0]\,\phi^{(5)}[u0]}{8\phi''[u0]^3} +$$

$$\frac{7 f[u0]\,\phi^{(3)}[u0]\,\phi^{(5)}[u0]}{48\phi''[u0]^4} - \frac{f[u0]\,\phi^{(6)}[u0]}{48\phi''[u0]^3}\biggr)\biggr)$$

$$\int_{0}^{\infty} f(u)\, e^{-\phi(u)/\epsilon} =$$

$$\frac{1}{12\,\phi''[u0]^3}$$

$$\Biggl(e^{-\frac{2\phi[u0]+u0^2\,\phi''[u0]}{2\epsilon}} \Biggl(2 f[u0] \Biggl(3\, e^{\frac{u0^2\,\phi''[u0]}{2\epsilon}} \sqrt{2\pi} \Biggl(1 + \mathrm{Erf}\Biggl[\frac{u0\sqrt{\frac{\phi''[u0]}{\epsilon}}}{\sqrt{2}}\Biggr]\Biggr) \sqrt{\epsilon\,\phi''[u0]^5} - 2\epsilon\,\phi''[u0]\,\phi^{(3)}[u0] - u0^2\,\phi''[u0]^2\,\phi^{(3)}[u0]\Biggr) +$$

$$f'[u0] \Biggl(6\, u0\, \epsilon\, \phi''[u0]\,\phi^{(3)}[u0] - 3\, e^{\frac{u0^2\,\phi''[u0]}{2\epsilon}} \sqrt{2\pi} \Biggl(1 + \mathrm{Erf}\Biggl[\frac{u0\sqrt{\frac{\phi''[u0]}{\epsilon}}}{\sqrt{2}}\Biggr]\Biggr) \sqrt{\epsilon^3\,\phi''[u0]}\;\phi^{(3)}[u0] + 2\,\phi''[u0]^2 \left(6\epsilon + u0^3\,\phi^{(3)}[u0]\right)\Biggr) +$$

$$f''[u0] \Biggl(-8\,\epsilon^2\,\phi^{(3)}[u0] + \epsilon\,\phi''[u0] \Biggl(3\, e^{\frac{u0^2\,\phi''[u0]}{2\epsilon}} \sqrt{2\pi} \Biggl(1 + \mathrm{Erf}\Biggl[\frac{u0\sqrt{\frac{\phi''[u0]}{\epsilon}}}{\sqrt{2}}\Biggr]\Biggr) \sqrt{\epsilon\,\phi''[u0]} - 4\, u0^2\,\phi^{(3)}[u0]\Biggr) - \phi''[u0]^2 \left(6\, u0\, \epsilon + u0^4\,\phi^{(3)}[u0]\right)\Biggr)\Biggr)\Biggr)$$

Metric, Connections and Mass Term

- let be a differential operator (Einstein convention!)

$$D = g^{ij}\partial_i\partial_j + b^i\partial_i \qquad g^{ij} = \rho_{i,j}\sigma_i\sigma_j \quad \text{: covariance matrix}$$

- the metric tensor is the inverse :

$$g_{ij} = [g^{ij}]^{-1} \qquad g_{ij} = \frac{[\rho_{i,j}]^{-1}}{\sigma_i\sigma_j} \qquad g = Det[g_{ij}] \text{ : Determinant of the metric}$$

- (Theorem) there is only one connection $\mathcal{A}_i$ and one mass term Q such

$$D = g^{-1/2}\underbrace{(\partial_i + \mathcal{A}_i)}_{\textbf{Covariant Derivative :}\nabla_i}g^{1/2}g^{ij}(\partial_i + \mathcal{A}_i) + Q$$

- We can compute them by

$$\mathcal{A}^i = \frac{1}{2}(b^i - g^{-1/2}\partial_j(g^{1/2}g^{ij})) \qquad \mathcal{A}_i = g_{ij}\mathcal{A}^j$$

$$Q = g^{ij}(\mathcal{A}_i\mathcal{A}_j - b_j\mathcal{A}_i - \partial_j\mathcal{A}_i)$$

Gauge Transformation and Girsanov Theorem

- The connection is a true form: by change of coordinate, $x'(x)$

$$\mathcal{A}_i dx^i = \mathcal{A}'_j dx'^j = \frac{\partial x'^j}{\partial x^i}\mathcal{A}'_j dx^i \qquad \text{so} \qquad \mathcal{A}_i = \frac{\partial x'^j}{\partial x^i}\mathcal{A}'_j$$

- But b^i is not a vector , because of the Ito Lemma, unless we use stratanovich

- if we multiply the transition probability by $e^{\chi} = e^{\int_0^t \Lambda(s)ds}$ then $p' = e^{\chi}p$ satisfy

$$D = g^{-1/2}(\partial_i + \mathcal{A}'_i)g^{1/2}g^{ij}(\partial_i + \mathcal{A}'_i) + Q'$$

  - where $\mathcal{A}'_i = \mathcal{A}_i - \partial_i\chi$ and $Q' = Q - \partial_t\chi = Q - \Lambda$: Gauge transformation
  - IF $\mathcal{A}_i dx^i$ is exact <=> there is ϕ such $d\phi = \mathcal{A}_i dx^i$ we can eliminate the connection by a gauge transformation <=>There is a change a measure that do the trick

The connection is exact <=> A Girsanov Transformation can cancel the connection

The Synge world function

- Its is the square of the geodesique distance

$$\sigma(S, S') = s^2 = \underset{x}{Min}\left\{\oint_x g^{ij}\partial_i x(s)\partial_j x(s)ds\right\}$$

- For the black and sholes equation it is just

$$\sigma(S, S') = {}^tS(\Sigma^{-1})S \qquad \Sigma \text{ is the covariance matrix}$$

- For the SABR normalized variables :

$$\sigma(x, y, x', y') = \mathrm{Cosh}^{-1}\left(1 + \frac{(x - x')^2 + (y - y')^2}{2yy'}\right)$$

- Solution of a non linear equation :

$$g^{ij}\partial_i\sigma\partial_j\sigma = 2\sigma$$

- Will play a more important role (as the action) in a symplectic version of the pricing

The VanVleck Morrette Determinant

parralel propagator

- Determinant of the operator $\Delta(S, S') = det(\Delta^i_j(S, S')) = det(-g^i_k(S, S')\sigma^k_j(S, S'))$ where $\sigma^k_j(S, S') = g^{kl}\sigma_{lj}(S, S') \equiv g^{kl}\nabla_l\sigma_j \equiv g^{kl}(\partial_i + \mathcal{A}_i)\partial_j\sigma$. The VVMD governs the congru-
  True 2-tensor True vector
  ence of geodesics. $\Delta > 1$ means focusing and $\Delta < 1$ means defocusing. It is the inverse of the density of illumination on the space of solid angle

  - . $\sqrt{g(S, S')} = \dfrac{1}{\Delta(S, S')}$ for Rieman coordinates

- For the Black and Sholes world: $\Delta(S, S') = det(\Sigma^{-1}) = \dfrac{1}{det(\Sigma)}$ Σ is the covariance matrix

- For the SABR normalized variables : $\Delta(S, S') = \dfrac{\sigma(S, S')}{Sinh(\sigma(S, S'))}$

- we can show that $\Delta(S, S') = \dfrac{det(\sigma_{lj}(S, S'))}{\sqrt{g(S)g(S')}}$ and the Differential Equation $\nabla_i(\Delta\partial_i\sigma) = 4\Delta$

Parralel Transport of the Connection

- Definition :

$$\mathcal{P}(S,S') = e^{\oint_{C(S,S')} \mathcal{A}_i dS^i} \qquad C(S,S') \quad \text{geodesic going from } S \text{ to } S'$$

- When the connection derive from a potential : $\mathcal{A}_i = \frac{\partial \chi}{\partial S^i}$ then $\oint_{C(S,S')} \mathcal{A}_i dS^i = \chi(S') - \chi(S)$

- For Black and Scholes world: $\oint_{C(S,S')} \mathcal{A}_i dS^i = \int_{f_0}^{F} \frac{-df}{2f} = \left(-\frac{1}{2}\right) Log\left(\frac{F}{f_0}\right)$

- For SABR $l = \sqrt{(x'-x)^2 + (y'-y)^2}$, $r = \sqrt{\left(\frac{x'-x}{2}\right)^2 + \left(\frac{y'-y}{2(x'-x)}\right)^2 + \frac{y'^2+y^2}{2}}$ and x, y reduced variables

$$\oint_{C(S,S')} \mathcal{A}_i dS^i = \frac{-\beta r^2}{2(1-\beta)\sqrt{1-\rho^2}}\left(H\left(a,\rho,\frac{x'-l}{r}\right) - H\left(a,\rho,\frac{x-l}{r}\right)\right) \quad \text{where} \quad H(a,\rho,x) = \int_0^x \frac{(1-y^2)dy}{a + y\sqrt{1-\rho^2} + \rho\sqrt{1-y^2}}$$

- (see details in PBourgade[1])

Scalar Curvature

- Rieman curvature a 4th order tensor defined by :

$$R(u, v)w = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]})w = R^i_{jkl} u^j v^k w^l$$

- The scalar curvature

$$R = \langle R(e_i, e_j)e_j, e_i \rangle = g^{kl} R^i_{ikl}$$

- Black and Scholes world: $R = 0$
- SABR world : $R = -1$
- Delta Model : $R = -\delta y^{2\delta - 2}$ (see details in PBourgade[1])

Mass Term

- True scalar computed from the connection $Q = g^{ij}(\mathcal{A}_i\mathcal{A}_j - b_j\mathcal{A}_i - \partial_j\mathcal{A}_i)$

- Name comes from Field Theory : Klein Gordon Equation for a massive particle like a Spin 0 Meson : $\Box\Psi + m\Psi = 0$

- For the Black and scholes world: $Q = \frac{-\sigma_0^2}{8}$

- For the SABR World : $Q = \frac{\alpha^2}{4}\left(C\partial^2{}_f C - \frac{\partial_f C}{2\sqrt{1-\rho^2}}\right)$

The Local Vol paradigm

- Given a stochastic vol model, we can use the Dupire formula to define the local vol:

$$\sigma_{local}(K, T) = \frac{\partial_T Call(K, T) + rK\partial_K Call(K, T)}{K^2 \partial^2{}_K Call(K, T)}$$

- Derman and Kani showed that (see also Beresticky[1])

$$\sigma^2{}_{local}(K, T) = \frac{\int p(S_0, t_0, S, T)\sigma_f^2(S_0, t_0, S, T)dS}{\int p(S_0, t_0, S, T)dS}$$

- So we just have to solve $\begin{cases} \partial_T Call(K, T) + rK\partial_K Call(K, T) = \sigma_{local}(K, T)K^2\partial^2{}_K Call(K, T) \\ Call(K, 0) = (f_0 - K)^+ \end{cases}$

The Local Vol Solution

- Equation: $df = C(f)dW$

  - Geometrization : $\mathcal{A}_f = -\frac{1}{2}\partial_f(Log(C(f)))$ and $Q = \frac{C^2(f)}{4}\left(\frac{\partial_f^2 C(f)}{C(f)} - \frac{1}{2}\left(\frac{\partial_f C(f)}{C(f)}\right)^2\right)$

  - We compute : $\Delta(f, f') = \frac{1}{C(f)^2}$ and $\mathcal{P}(f, f') = \sqrt{\frac{C(f)}{C(f')}}$

- We perform the integration for local vol $C(f)$ and BS $C(f) = \sigma f$:

  - (Local Vol) $Call = (f-K)^+ + \sqrt{\frac{TC(K)C(f)}{8\pi}}(H_1(\omega) + Q(K)TH_2(\omega))$

  - (BS) $Call = (f-K)^+ + \sqrt{\frac{TKf\sigma^2}{8\pi}}(H_1(\bar{\omega}) + Q(K)TH_2(\bar{\omega}))$

  - where $\omega = \frac{1}{\sqrt{2T}}\int_f^K \frac{dz}{C(z)}$ and $\bar{\omega} = \frac{Log\left(\frac{f}{K}\right)}{\sqrt{2T}\sigma}$ with

$$H_2(\omega) = \frac{2}{3}\left(e^{-\omega^2}(1-2\omega^2) - 2\sqrt{\pi\omega^6}\left(N\left(\sqrt{2\omega^2}\right)-1\right)\right)$$ and $$H_1(\omega) = 2\left(e^{-\omega^2} + \sqrt{\pi\omega^2}\left(N\left(\sqrt{2\omega^2}\right)-1\right)\right)$$

- by identification we get $\sigma_{BS} = \sqrt{\frac{C(K)C(f)}{Kf}}\frac{H_1(\omega)}{H_1(\bar{\omega})}\left(1 + Q(K)T\frac{H_2(\omega)}{H_1(\omega)}\right) + \frac{\sigma_{BS}^2}{8}\frac{H_2(\bar{\omega})}{H_1(\bar{\omega})}$

- By expanding, iteratively, we get:

- Order 0 : $\sigma_{BS} = \frac{Log\left(\frac{f}{K}\right)}{\int_f^K \frac{dz}{C(z)}}$

- Order 1 : $(\sigma f)_{BS} = \frac{Log\left(\frac{f}{K}\right)}{\int_f^K \frac{dz}{C(z)}}\left(1 + \frac{C^2(f_{avg})T}{24}\left(2\frac{C''(f_{avg})}{C(f_{avg})} - \left(\frac{C'(f_{avg})}{C(f_{avg})}\right)^2 + \frac{1}{f_{avg}{}^2}\right)\right)$

- where the best f_{avg} can be shown to be : $f_{avg} = \sqrt{\frac{K^2 - f^2}{Log\left(\frac{K}{f}\right)}}$

  - (See PBourgade[1])

Known Formula

- SABR Option formula :
  - Hagan Version (-> Hagan[1])
  - Analytical Complex Black and Scholes Version (-> OC)
- Delta Model Option Formula (->Bourgade[1])
- BiSABR SpreadOption Formula (->OC)
- Lambda Model Option Formula (->Labordere[1])
- Stochastic Vol BGM Swaption Formula (-> Labordere[2])
- Local Vol Basket Option (->Avellaneda[1])
- Stochastic Vol Basket Option (->OC)

BiSABR

- $$\begin{cases} dS_1 = \alpha_1 S_1^{\beta_1} dW_1 \\ d\alpha_1 = \nu_1 \alpha_1 dW_2 \\ dS_2 = \alpha_2 S_2^{\beta_2} dW_3 \\ d\alpha_2 = \nu_2 \alpha_2 dW_4 \end{cases}$$ with $$\begin{cases} dW_1 \cdot dW_2 = \rho_1 dt \\ dW_3 \cdot dW_4 = \rho_2 dt \end{cases}$$ and $$\begin{cases} dW_1 \cdot dW_3 = \rho_s dt \\ dW_2 \cdot dW_4 = \rho_v dt \end{cases} \quad \begin{cases} dW_1 \cdot dW_4 = \rho_{c12} dt \\ dW_2 \cdot dW_3 = \rho_{c21} dt \end{cases}$$

- Modified formula for a option on a spread that can be positive or negative: $\begin{matrix} dF = \alpha C(F) dW \\ d\alpha = \nu\alpha dW' + \mu\alpha dt \end{matrix}$ => $Call = (f - K)^{+} + F_1 \alpha \sqrt{B_0 B_z} \sqrt{I_0}\, G\left(\sqrt{-\kappa}, \frac{|x|}{\sqrt{2}}, \sqrt{T}\right)$

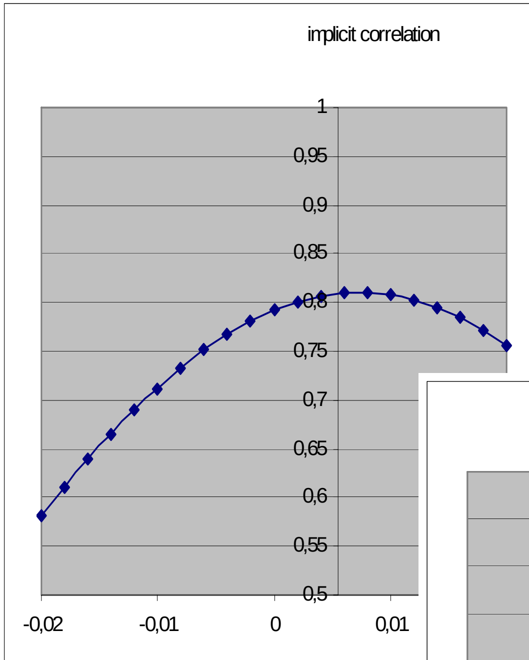
    - Notation of Hagan[1], $F_1 = e^{\left(\frac{1}{4}\alpha b_1 \nu\rho + \frac{\mu}{2}\right) z^2}$ drift of the vol of spread

    - where $G(a, b, x) = \int_0^x e^{-a^2 s^2 - \frac{b^2}{s^2}} ds$ and $I_0 = \sqrt{1 - 2\nu\rho z + \nu^2 z^2}$ $\quad I_1 = \frac{z\nu - \rho}{I_0}$ $\quad I_2 = \frac{1 - \rho^2}{I_0^3}$

$$\kappa = \frac{12\mu + \left(-3b_1^2\alpha^2 + 2b_2\alpha^2 - 4z^2\mu^2 - I_1^2\nu^2 + 2I_0 I_2 \nu^2 + 6\alpha b_1 \rho\nu - 16 z\mu\nu\rho\right)}{8I_0}$$

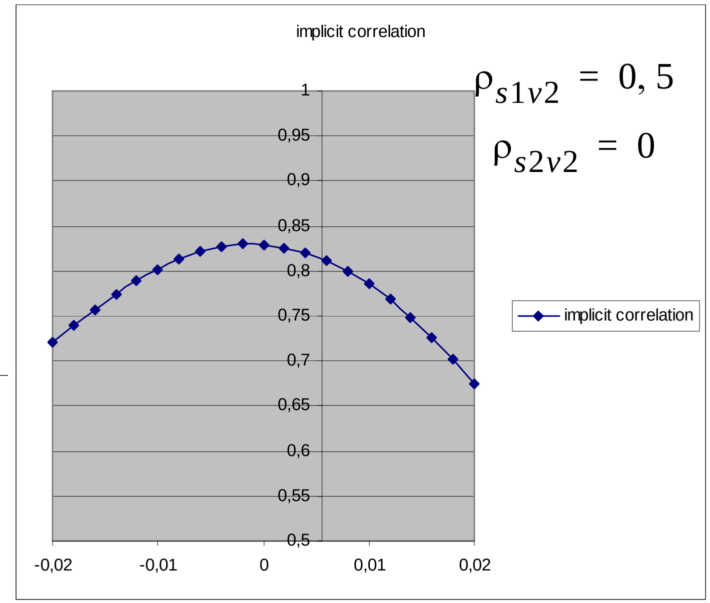
BiSABR : Phenomenology

- $\rho_s = 0,8 \qquad \rho_v = 0,5$



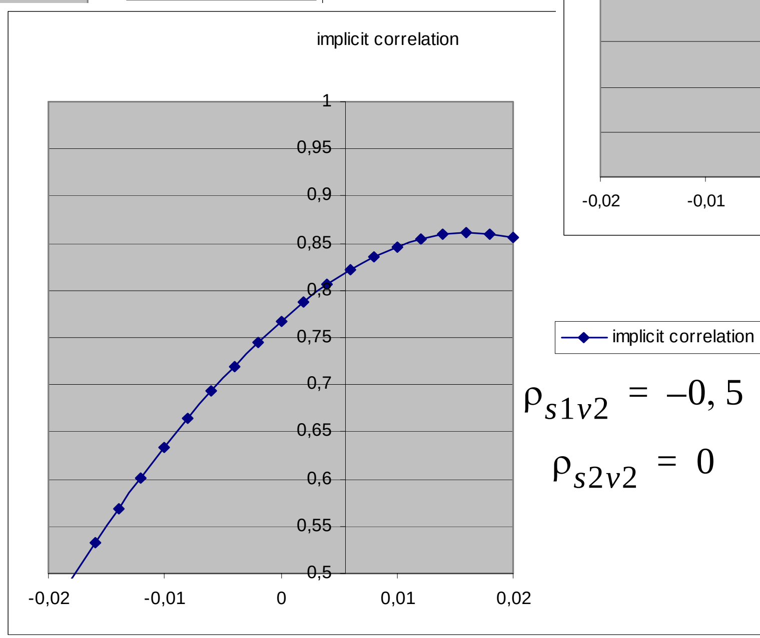
$\rho_{s1v2} = 0$

$\rho_{s2v2} = 0$



$\rho_{s1v2} = 0,5$

$\rho_{s2v2} = 0$

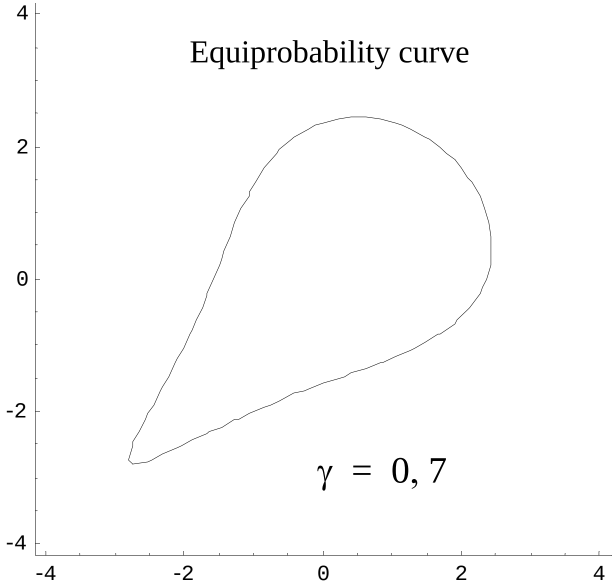


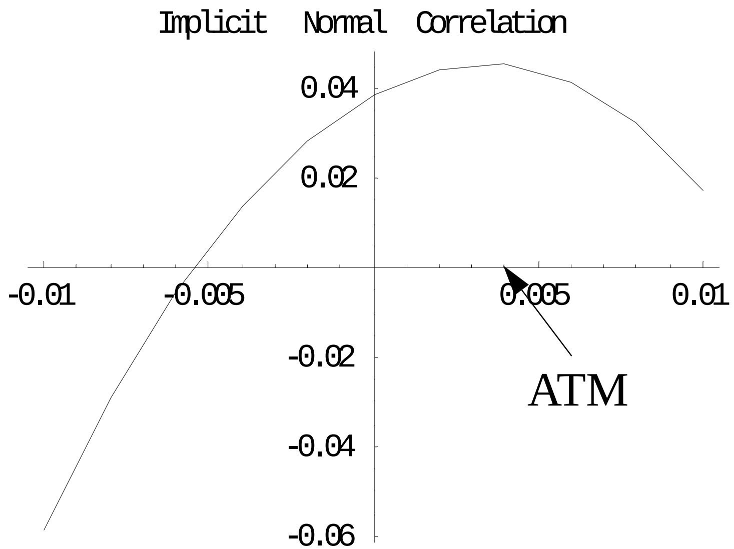
$\rho_{s1v2} = -0,5$

$\rho_{s2v2} = 0$

SpreadOption with copula and SABR

- Franck Gamma changed copula (Genest transformation with a power function)



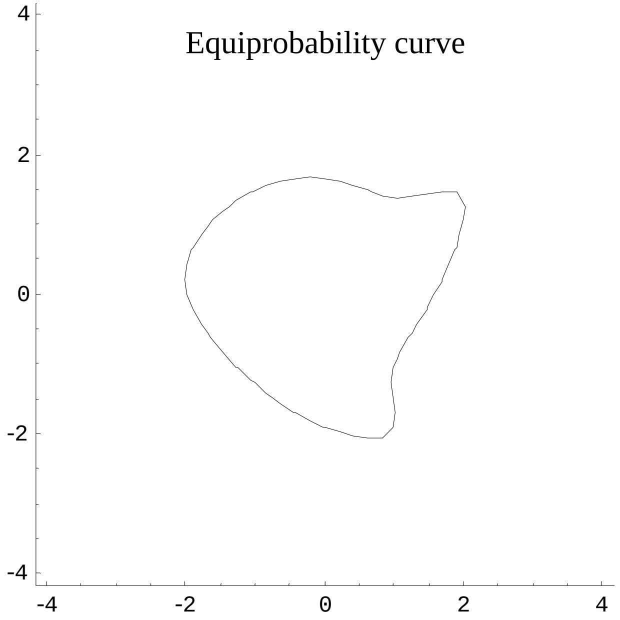


- Generally speaking, any **symmetric copula** gives a **correlation smile slope ~ 0** , in particular, Student, Gumbel, and all popular copulas.Only for student we can control the correlation level.

NonSymmetric Copula

- very little is known for non symmetric copulas (non exchangeable): Khoudraji(1995), Genest(1998): $C_{\kappa,\lambda}(u,v) = C_1(u^{1-\kappa}, v^{1-\lambda})C_2(u^{\kappa}, v^{\lambda})$

- Density of non Symmetric Product of Gaussian



$\rho_1 = 0,9$

$\rho_2 = -0,9$

$\kappa = 0,2$

$\lambda = 0,8$

- Lack of control on the correlation, unnaturality of of the implied density and the implied correlation.

Option on a Porfolio of Hedge Funds(N-SABR)

- In the Forward measure of maturity T

$$dF_i = \alpha C_i(F_i)dW_i \qquad 1 \le i \le n$$

$$d\alpha = \nu\alpha dW_{n+1}$$

$$dW_i dW_j = \rho_{ij}dt \qquad 1 \le i, j \le n+1$$

- We use the cholesky matrix: L_{ij} $\qquad LL^* = [\rho]$

$$x_i = \nu \sum_{k=1}^{n} L_{ik} \int_{F_{k,0}}^{F_k} \frac{dz_k}{C_k(z_k)} + \alpha \sum_{k=1}^{n} \rho_{i,n+1} L_{ki} \qquad 1 \le i \le n$$

- We define natural variables:

$$x_{n+1} = \alpha \sqrt{1 - \sum_{(i,j)}^{n} \rho_{i,n+1}\rho_{j,n+1}\rho_{i,j}}$$

- The Metric is therefore:

$$ds^2 = \frac{2}{\nu^2}\left(1 - \sum_{(i,j)}^{n} \rho_{i,n+1}\rho_{j,n+1}\rho_{i,j}\right)\left(\sum_{i=1}^{n+1} dx_i^2\right) / x_{n+1}^2$$

The Generalized hyperbolic Space H_n

- The metric : $g_{ij} = \frac{2}{\nu^2}\left(1 - \sum_{(l,k)}^{n} \rho_{l,n+1}\rho_{k,n+1}\rho_{l,k}\right)\delta_{ij}/x_{n+1}^2$ $\qquad \sqrt{g} = \frac{2^{(n+1)/2}}{\nu\alpha^2\sqrt{det([\rho])}\prod_{i=1}^{n} C[F_{i,0}]}$

- The Connection: $\mathcal{A}_i = \frac{-1}{C_i[F_i]}\sum_{j=1}^{n}\rho^{ij}\partial_j C_j[F_j]$,

- The Synge Function: $\sigma(x,x') = \frac{2}{\nu^2}\left(1 - \sum_{(i,j)}^{n}\rho_{i,n+1}\rho_{j,n+1}\rho_{i,j}\right)Cosh^{-1}\left(1 + \frac{\sum_{i=1}^{n+1}(x_i - x_i')^2}{2x_{n+1}x'_{n+1}}\right)$

- The Van Vleck Morette Determinant: $\Delta(x,x') = \frac{\sigma(x,x')}{Sinh(\sigma(x,x'))}$

- The parralel transport of the connection

$$Log(P(x,x')) = \frac{1}{\nu}\sum_{j=1}^{n}\left(-\frac{\partial_j C_j(F_{j,0})}{2}\right)\left(\nu\sum_{i=1}^{n}\rho_{ij}(x_i - x_i') + \rho_{n+1,j}(x_{n+1} - x_{n+1}')\right)$$

Toward a Perturbative Approach of Codependence

- Take 2 N-state Variables . And a bivariate distribution describing its joint distribution.

  It is given by a N*N matrix $P = \begin{bmatrix} a_{11} & .. & a_{1n} \\ .. & .. & .. \\ a_{n1} & .. & a_{nn} \end{bmatrix}$ $a_{ij} \geq 0$ $\sum_{i,j=1}^{n} a_{ij} = 1$ if we knows

  the marginals: $\sum_{i=1}^{n} a_{ij} = b_j$ $\sum_{j=1}^{n} a_{ij} = c_i$, the codependence is described by

  $n^2 - 1 - 2(n-1) = (n-1)^2$ freedom degres

- Fixing the the correlation = 1 dim

- Spreadoption smile slope = 1 dim

- $$\rho_{imp}(K) = \underbrace{\rho_{ATM}}_{\text{Correlation}} + (K - ATM) \times \underbrace{\omega_{ATM}}_{\text{Smile Slope}} + (K - ATM)^2 \times \underbrace{\varphi_{ATM}}_{\text{Smile Curvature}} + ..$$

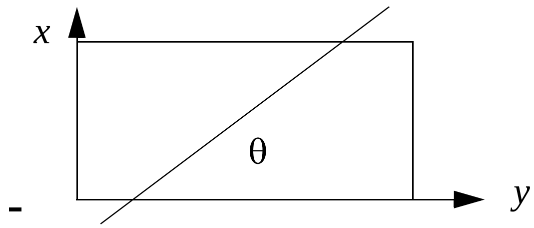
SpreadOption and Joint Distribution

- In general Knowing the spreadoption smile does not imply knowing the joint Distribution: It is like knowing n supplementary conditions when you have (n-1)*(n-1) unkowns

- For a class of stochastic volatility process, it is the case. If the smile determines the parameters, the joint distribution is defined.

$$\begin{aligned} dF_1 &= \alpha_1 F_1^{\beta_1} dW_1 \\ d\alpha_1 &= \nu_1 \alpha_1 dW_2 \end{aligned} \qquad \begin{aligned} dF_2 &= \alpha_2 F_2^{\beta_2} dW_3 \\ d\alpha_2 &= \nu_2 \alpha_2 dW_4 \end{aligned} \qquad E[(F_1 - F_2 - K)^+] \quad => \quad E[(Cos(\theta)F_1 - Sin(\theta)F_2 - K)^+]$$

The Iterative Proportional Fitting Procedure(IPFP) :



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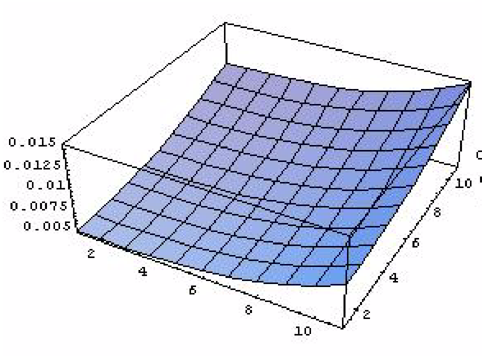
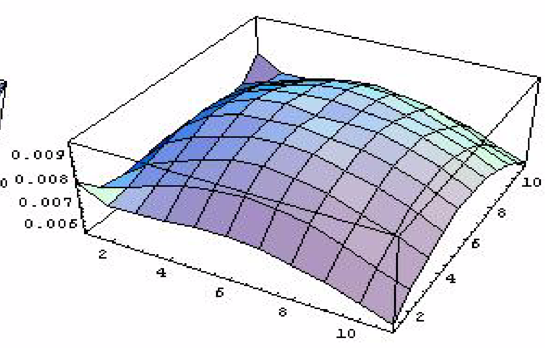
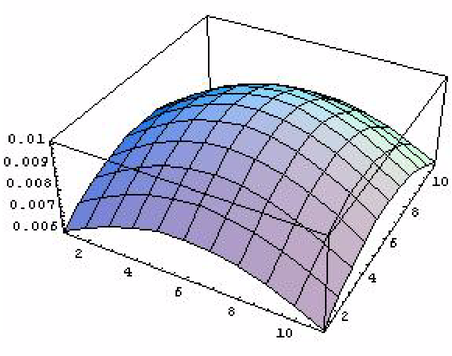


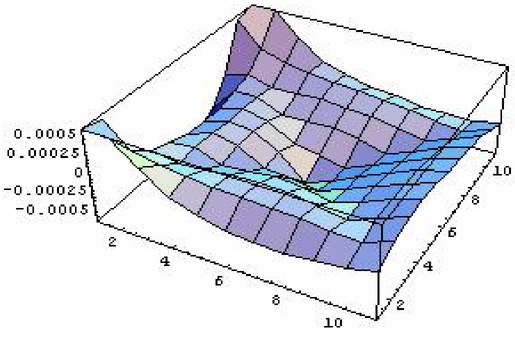
Initial Disitribution



After 20 iterations



Goal



Conclusion

- The Stochastic Volatility systems and particularly N-SABR allow to represent codependency that goes beyond simple linear correlations, starting a perturbative approach of a class of codependency.

- From the point of view of the stochastic volatility systems, The most important effect after the correlation has been overlooked in the classical symmetrical copula based approach. It can be identified as an implied correlation slope

- Future Development:
  - Finish the N-SABR Formula
  - Include Jump processes . Certain Results on Heat Kernel Coefficients have been identified as possible candidates to implement integral operators

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