Rectification of Labordère's article, for n-SABR

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1 The model, the isometry

Let us consider the following 1-volatility (a) /n-assets ($F_1, \ldots, F_n$) simple process :

$$\left\{\begin{array}{ccll} \mathrm{d}F_k & = & aC_k(F_k)\ \mathrm{d}Z_k & k \in [\![1, n]\!] \\ \mathrm{d}a & = & \nu a\ \mathrm{d}Z_{n+1} & \\ < \mathrm{d}Z_i, \mathrm{d}Z_j > & = & \rho_{ij}\ \mathrm{d}t & (i,j) \in [\![1, n+1]\!]^2 \end{array}\right.$$

We can associate the following metric to this process :

$$\mathrm{d}s^2 = \frac{2}{\nu^2 a^2} \left(\sum_{i,j=1}^{n} \rho^{ij} \frac{\nu \mathrm{d}F_i}{C_i(F_i)} \frac{\nu \mathrm{d}F_j}{C_j(F_j)} + 2 \sum_{i=1}^{n} \rho^{ia} \frac{\nu \mathrm{d}F_i}{C_i(F_i)} \mathrm{d}a + \rho^{aa} \mathrm{d}a^2 \right),$$

that is to say the block matrix

$$G = \frac{2}{\nu^2 a^2} \left[\begin{array}{cc} \left(\rho^{ij} \frac{\nu}{C_i(F_i)} \frac{\nu}{C_j(F_j)}\right)_{1 \leq i,j \leq n} & \left(\rho^{ia} \frac{\nu}{C_i(F_i)}\right)_{1 \leq i \leq n} \\ \left(\rho^{aj} \frac{\nu}{C_j(F_j)}\right)_{1 \leq j \leq n} & \rho^{aa} \end{array} \right].$$

Let us introduce the following notations (where the index $n+1$ corresponds to the volatility a) :

- $\rho = \left[\begin{array}{cc} U & u \\ u^\top & \rho_{aa} \end{array} \right]$ is the $(n+1) \times (n+1)$ correlation matrix ;
- $\rho^{-1} = \left[\begin{array}{cc} V & v \\ v^\top & \rho^{aa} \end{array} \right]$ is the $(n+1) \times (n+1)$ inverse of the correlation matrix ;
- X is a $n \times n$ matrix ;
- Y is a $n \times n$ matrix ;
- Z is a $n \times n$ matrix.

We introduce the following change of variables :

$$\left\{\begin{array}{lcl} x_k & = & \frac{\sqrt{2}}{\nu}\left(\rho^{aa}-\sum_{i,j=1}^n \rho^{ia}\rho^{ja}z_{ij}\right)^{1/2}\left(\sum_{i=1}^n \nu X_{ki}\int_{F_i^0}^{F_i}\frac{\mathrm{d}F_i}{C_i(F_i)}+\left(\sum_{i=1}^n \rho^{ia}Y_{ik}\right)a\right), \quad k\in[\![1,n]\!] \\ x_{n+1} & = & \frac{\sqrt{2}}{\nu}\left(\rho^{aa}-\sum_{i,j=1}^n \rho^{ia}\rho^{ja}z_{ij}\right)\ a \end{array}\right.$$

This change of variables corresponds to a function $\Phi(F_1,\ldots,F_n,a)=(x_1,\ldots,x_n,x_{n+1})$, with (as a block matrix)

$$\begin{aligned}\nabla\Phi = \frac{\sqrt{2}}{\nu}\left(\rho^{aa}-\sum_{i,j=1}^n \rho^{ia}\rho^{ja}z_{ij}\right)^{1/2} & \left[\begin{array}{cc} \left(\nu\frac{X_{ij}}{C_j(F_j)}\right)_{1\le i,j\le n} & \left(\sum_{k=1}^n \rho^{ka}Y_{ki}\right)_{1\le i\le n} \\ 0 & \left(\rho^{aa}-\sum_{i,j=1}^n \rho^{ia}\rho^{ja}z_{ij}\right)^{1/2} \end{array}\right] \\ &= \frac{\sqrt{2}}{\nu}\left(\rho^{aa}-\sum_{i,j=1}^n \rho^{ia}\rho^{ja}z_{ij}\right)^{1/2}\left[\begin{array}{cc} A & T \\ 0 & c\end{array}\right].\end{aligned}$$

Consider the hyperbolic space $\mathbb{H}^{n+1}$, with metric

$$H=\frac{1}{x_{n+1}^2}I_{n+1}.$$

Then Φ is an isometry between $\mathbb{G}$ and $\mathbb{H}$ if and only if

$$\boxed{G=(\nabla\Phi)^\top H\nabla\Phi)},$$

that is to say

$$\left[\begin{array}{cc} \left(\rho^{ij}\frac{\nu}{C_i(F_i)}\frac{\nu}{C_j(F_j)}\right)_{1\le i,j\le n} & \left(\rho^{ia}\frac{\nu}{C_i(F_i)}\right)_{1\le i\le n} \\ \left(\rho^{aj}\frac{\nu}{C_j(F_j)}\right)_{1\le j\le n} & \rho^{aa}\end{array}\right]=\left[\begin{array}{cc} A^\top A & A^\top T \\ (A^\top T)^\top & T^\top T+c^2\end{array}\right]$$

Let us do the calculation for each of the three blocks of the matrix, in order to see what condition over X and Y makes Φ an isometry.

$$(A^\top A)_{ij}=\sum_{k=1}^n (A^\top)_{ik}A_{kj}=\sum_{k=1}^n A_{ki}A_{kj}=\frac{\nu^2}{C_i(F_i)C_j(F_j)}\sum_{k=1}^n X_{ki}X_{kj}=\frac{\nu^2}{C_i(F_i)C_j(F_j)}(X^\top X)_{ij}\overset{?}{=}G_{ij}$$

$$\begin{aligned} T^\top T+c^2 &= \sum_{i=1}^n\left(\sum_{k=1}^n \rho^{ka}X_{ki}\right)^2+c^2=\sum_{i,k,l=1}^n \rho^{ka}\rho^{la}Y_{ki}Y_{li}+c^2 \\ &= \sum_{k,l=1}^n \rho^{ka}\rho^{la}(YY^\top)_{kl}+\left(\rho^{aa}-\sum_{i,j=1}^n \rho^{ia}\rho^{ja}z_{ij}\right) \\ &= \rho^{aa}+\left(v^\top YY^\top v-v^\top Zv\right)\overset{?}{=}\rho^{aa}=G_{aa}\end{aligned}$$

$$\begin{aligned}(A^\top T)_{ia} &= \sum_{k=1}^n (A^\top)_{ik}T_{ka}=\sum_{k=1}^n A_{ki}T_{ka}=\sum_{k=1}^n \nu\frac{X_{ki}}{C_i(F_i)}\left(\sum_{l=1}^n \rho^{la}Y_{lk}\right) \\ &= \frac{\nu}{C_i(F_i)}\sum_{k,l=1}^n \rho^{la}Y_{lk}X_{ki}=\frac{\nu}{C_i(F_i)}\left(v^\top YX\right)_i\overset{?}{=}G_{ia}\end{aligned}$$

To sum up, Φ is an isometry if and only if X and Y are such as

$$\left\{ \begin{array}{ccc} X^\top X & = & V \\ v^\top Y Y^\top v & = & v^\top Z v \\ v^\top Y X & = & v^\top \end{array} \right. . \tag{1}$$

Consider X is given and write $x = Y^\top v$. Then

$$\left\{ \begin{array}{ccc} |x|^2 & = & v^\top Z v \\ x & = & {X^{-1}}^\top v \end{array} \right. .$$

This is possible if and only if $v^\top X^{-1} {X^{-1}}^\top v = v^\top Z v$, that is to say

$$\boxed{v^\top V^{-1} v = v^\top Z v}.$$

Let's go back in the calculation to choose the matrixes X, Y and Z :

– X is any matrix such as $X^\top X = U$, so for instance X is the Cholesky matrix associated to U ;
– $Z = V^{-1}$ is convenient ;
– $Y = X^{-1}$ is convenient.

Remark. For $n \geq 2$, the system (1) has many solutions, because of $3n^2$ unknown variables for $n^2 + n + 1$ equations. The intuitive reason is that the hyperbolic space $\mathbb{H}^{n+1}$ has isometries : all those of the subspace $\mathcal{R}^n$.

2 Van Vleck's determinant in hyperbolic geometry, Molchanov's expansion of the heat kernel

First of all, in our special case, we don't need to calculate the Van Vleck Morette determinant of the hyperbolic metric, because of the following formula by Molchanov[1] for a diffusion in the hyperbolic space $\mathbb{H}^{n+1}$:

$$\boxed{p(t,x,y) = \frac{1}{(2\pi t)^{\frac{n+1}{2}}} e^{-\frac{d^2}{2t}} \left(\frac{d}{\sinh d}\right)^{\frac{n}{2}}},$$

with $\cosh d = 1 + \frac{\sum_{i=1}^{n+1}(x_i - x_i^0)^2}{2x_{n+1}^0 x_{n+1}}$.

Let us calculate the Van Vleck determinant for the Hyperbolic metric (that is to say $\frac{1}{\sqrt{gg^0}} \det(\frac{\partial^2 (d^2/2)}{\partial x_i^0 \partial x_j})$). First, we find a relation between these determinants for $\mathbb{H}^{n+1}$ (d_{n+1}) and $\mathbb{H}^2$ (d_2).

Consider the initial point $(x_i^0)_{1 \leq i \leq n+1}$ and the final point $(x_j)_{1 \leq i \leq n+1}$. With no loss of generality, we can consider the axis of coordinates for (x_n^0, x_n) and (x_{n+1}^0, x_{n+1}) to be on the same plane as the geodesic, and the $(n-1)$ other coordinates to be perpendicular to this plane.

Consider $i \in [\![1, n-1]\!]$ or $j \in [\![1, n-1]\!]$. Then

$$\frac{\partial}{\partial x_i^0} \frac{\partial}{\partial x_j} \left(\frac{d^2}{2}\right) = \frac{\partial}{\partial x_i^0} \left(d \frac{\partial d}{\partial x_j}\right) = \frac{\partial d}{\partial x_i^0} \frac{\partial d}{\partial x_j} + d \frac{\partial^2 d}{\partial x_i^0 \partial x_j} = d \frac{\partial^2 d}{\partial x_i^0 \partial x_j},$$

because $\frac{\partial d}{\partial x_i^0} = 0$ or $\frac{\partial d}{\partial x_i^0} = 0$ because i or j is a coordinate relative to an axis orthogonal to the plane of the trajectory.

Moreover,

$$\frac{\partial}{\partial x_i^0} \frac{\partial}{\partial x_j} \cosh d = \frac{\partial}{\partial x_i^0} \left(\sinh d \frac{\partial d}{\partial x_j}\right) = \sinh d \frac{\partial^2 d}{\partial x_i^0 \partial x_j} + \cosh d \frac{\partial d}{\partial x_i^0} \frac{\partial d}{\partial x_j} = \sinh d \frac{\partial^2 d}{\partial x_i^0 \partial x_j},$$

[1] S. A. Molchanov, *Diffusion Processes and Riemannian Geometry.*

for the same reason as before. This is the reason why

$$\frac{\partial^2}{\partial x_i^0 x_j}\left(\frac{d^2}{2}\right) = \frac{d}{\sinh d}\frac{\partial^2}{\partial x_i^0 x_j}\cosh d = \frac{d}{\sinh d}\frac{\partial^2}{\partial x_i^0 x_j}\left(\frac{\sum_{k=1}^{n+1}(x_k - x_k^0)^2}{2x_{n+1}^0 x_{n+1}}\right) = \frac{d}{\sinh d}\frac{-1}{x_{n+1}x_{n+1}^0}\delta_i^j,$$

because $i \in [\![1, n-1]\!]$ or $j \in [\![1, n-1]\!]$. So, as a block matrix of sizes $n-1$ and 2, we have

$$d_{n+1} = \left(\frac{d}{\sinh d}\frac{-1}{x_{n+1}x_{n+1}^0}\right)^{n-1}\frac{\sqrt{g_2 g_2^0}}{\sqrt{g g^0}}d_2 = \left(-\frac{d}{\sinh d}\right)^{n-1}d_2.$$

A calculation of d_2 may be directly done using Mathematica...