Local time for the SABR model
Connection with the "complex" Black Scholes
And application to CMS and Spread Options

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Abstract: It is well known that the cost of a call and put option is equal to its intrinsic value plus the cost of a stop loss strategy. This stop loss strategy can be re-expressed in terms of the local time. It provides easily closed forms solution for model like Black Scholes [8] or [3]. This paper examines the theory of local time for stochastic volatility models and in particular the SABR model [5]. It gives an approximated formula for the local time in SABR and shows that this model can be valued using a Black Scholes formula but where all the terms are complex number. This formula turns out to be more robust for low and high strikes. This solves in particular the problem of valuing the whole smile in SABR as required in the replication method for CMS rate and spread option with gaussian copula where the whole smile is required.

1. Expected local time and vanilla option prices

The strength of the local time theory lies in its general framework. Let the forward asset F_t follow a stochastic volatility diffusion:

$$dF_t = \sigma(t, F_t) f(F_t) dW_t^F \qquad F_0 = f \tag{1.1}$$

$$\text{and } d\sigma(t, F_t) = a(t, \sigma)dt + b(t, \sigma)dW_t^\sigma \qquad \sigma_0 = \alpha \tag{1.2}$$

where W_t^F and W_t^σ are two standard Brownian motions potentially correlated $\left\langle dW_t^F, dW_t^\sigma \right\rangle = \rho_t dt$ and $f(F_t)$ is a mapping function (usually a CEV $f(F_t) = F_t^\beta$ or a displaced diffusion function $f(F_t) = F_t + \mu_t$. The Meyer Tanaka formula, extension of the Ito formula to convex payoff ([6] or [9]) provides an interesting framework to compute the price of a call with strike k given by ([3]):

$$(F_T - k)^+ = (f - k)^+ + \int_0^T 1_{F_u > k} dF_u + \frac{1}{2}\int_0^T 1_{F_u = K} \sigma^2(F_u, u) f(F_u) dF_u \tag{1.3}$$

The above equation summarizes that the payoff of a call option is the sum of its intrinsic and its time value represented by its continuous local time. But more than a new formulation for the value of a vanilla option, it shows that this value lies in the computation of the local time for any model. In particular, in Black Scholes, the local time is easy to compute and given by ([3]):

$$BS\_Call(f, k, \alpha, T) = (f - k)^+ + \frac{1}{2}\alpha^2 k^2 \int_0^T \frac{1}{\alpha k \sqrt{u}} \frac{e^{-\frac{1}{2}\left(\frac{Log(f/k)}{\alpha\sqrt{u}} - \frac{1}{2}\alpha\sqrt{u}\right)^2}}{\sqrt{2\pi}} du \tag{1.4}$$

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where $BS\_Call(f,k,\alpha,T)$ is the forward value of a Black Scholes call, with strike k, with volatility α, with maturity T and whose forward is worth f today. The above result can be derived either from a direct computation of the local time for geometric Brownian motion or by relating the vega to the local time (see appendix section 7.1).

2. Local time and probability density

General framework

Define the probability density $p(t,f,\alpha;T,F,A)$ by

$$p(t,f,\alpha;T,F,A)dFdA = \mathrm{Prob}\{F < F_T < F+dF, A < \sigma_T < A+dA | F_t = f, \sigma_t = \alpha\} \tag{2.1}$$

we then have the following connection between the local time and the probability density:

Proposition 2.1

The forward price of a call option $V(f,k,\alpha,T)$ with forward worth f today, with strike k, with volatility with boundary condition equal to α and with maturity T is related to the density probability as follows:

$$V(f,k,\alpha,\mathrm{T}) = (f-k)^+ + \frac{1}{2} f(k)^2 \int_0^T A^2 p(t,f,\alpha,\tau,k,A) dA d\tau \tag{2.2}$$

The integral part is precisely the local time part.

Proof: see 7.2. □

The equation (2.2) shows that any local time problem can be reformulated as a probability density problem.

3. Explicit computation of the local time for the SABR model

Computing the local time in stochastic volatility models is not an easy task in general. The above theory assumes either an explicit function for the local time or of the probability density. But this is far from being the case for general stochastic volatility models.

However, for specific form of the stochastic volatility models, we can find good approximation using perturbation theory. For instance in the case of the SABR model [5], we can find a good approximation on the local time part. A SABR mode assumes a constant volatility martingale process for the underlying with a lognormal martingale stochastic volatility:

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dW_t^F \qquad \text{with } F_0 = f \\ d\sigma_t &= \nu \sigma_t dW_t^{\sigma} \qquad \text{with } \sigma_0 = \alpha \end{aligned} \tag{3.1}$$

where the two Brownian motions are correlated by

$$\left\langle dW_t^F , \mathrm{dW}_t^{\sigma} \right\rangle = \rho dt \tag{3.2}$$

Using the results of [5], we can prove a first approximation of the local time as follows:

Proposition 3.1 The forward value of a call option is given in a SABR model by

$$SABR\_Call(f,k,T,\alpha,\beta,\rho,\nu) = (f-k)^+ + \frac{f-k}{2x\sqrt{2\pi}} e^{\theta} \int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du \tag{3.3}$$

with

$$\theta = \frac{1}{4}\rho\nu\alpha b_1 z^2 + Log\left(\frac{\alpha (fk)^{\frac{\beta}{2}} z}{f-k}\right) + Log\left(\frac{x}{z}\left(1-2\nu\rho z+\nu^2 z^2\right)^{\frac{1}{4}}\right) \tag{3.4}$$

$$\kappa = \frac{1}{8}\left(\frac{\alpha^2(\beta-2)\beta k^{2\beta}}{\left(k-\alpha(\beta-1)k^{\beta} z_0\right)^2} + \frac{6\alpha\beta k^{\beta}\nu\rho}{k-\alpha(\beta-1)k^{\beta} z_0} + \frac{\nu^2\left(2-3\rho^2+2\nu\rho z_0-\nu^2 z_0{}^2\right)}{1-2\nu\rho z_0+\nu^2 z_0{}^2}\right) \tag{3.5}$$

$$x = \frac{1}{\nu} Log\left(\frac{-\rho+\nu z+\sqrt{1-2\nu\rho z+\nu^2 z^2}}{1-\rho}\right) \tag{3.6}$$

$$z = \frac{f^{1-\beta} - k^{1-\beta}}{\alpha(1-\beta)} \qquad z_0 = \frac{f_0^{1-\beta} - k^{1-\beta}}{\alpha(1-\beta)} \tag{3.7}$$

$$b_1 = \frac{\beta}{\alpha(1-\beta)z_0 + k^{1-\beta}} \tag{3.8}$$

where f_0 is given by (FIX FIX FIX).

Proof: see 7.3. □

Compared to the final result of [5] (2.17a), the solution is more accurate as it does not involve the last approximation on the integral term $\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du$ as in (B.48c) where it is approximated as $\int_0^T \frac{e^{-\frac{x^2}{2u}}}{\sqrt{u}\left(1-\frac{2}{3}\kappa u\right)^{3/2}} du$. This approximation is clearly very off in general. We will see now how to efficiently compute this integral.

4. Fast computation of the SABR Stochastic integral and connection with Black Scholes

The SABR standard integral given by $\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du$ can be computed as follows:

$$\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2i\sqrt{\kappa}} \left[e^{i\sqrt{2}x\sqrt{\kappa}} \left(erf\left(\frac{x}{\sqrt{2T}} + i\sqrt{\kappa T} \right) - 1 \right) + e^{-i\sqrt{2}x\sqrt{\kappa}} \left(erf\left(\frac{-x}{\sqrt{2T}} + i\sqrt{\kappa T} \right) + 1 \right) \right] \tag{4.1}$$

with the additional formula for the complex error function

$$\begin{aligned} erf(x+iy) &= erf(x) + \frac{e^{-x^2}}{2\pi x}\left[1-\cos(2xy) + i\sin(2xy)\right] \\ &+ \frac{2}{\pi} e^{-x^2} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{4}n^2}}{n^2+4x^2} \left\{2x - 2x\cosh(ny)\cos(2xy) + n\sinh(ny)\sin(2xy) + i\left(2x\cosh(ny)\sin(2xy) + n\sinh(ny)\cos(2xy)\right)\right\} \end{aligned} \tag{4.2}$$

Proof: see 7.4. □

The last series converges very quickly (5 terms is in general enough for a good precision).

Connection with Black Scholes

Identifying the local time component both in Black Scholes and SABR, we can find a relationship between SABR and Black Scholes. Namely, we can prove that the SABR model can be seen as an extension of the Black Scholes formula in the complex filed as proven by proposition 3.2

Proposition 3.2

The SABR model is equivalent to a Black Scholes models with the following parameters (complex numbers potentially):

$$\tilde{f} = \frac{x\sqrt{-2\kappa}\, e^{\pm x\left(\sqrt{-\frac{\kappa}{2}}+\sqrt{-2\kappa}\right)-\theta}}{\left(e^{\pm x\sqrt{-2\kappa}} - 1\right)} x\sqrt{-2\kappa}\, e^{x\sqrt{-\frac{\kappa}{2}}-\theta} \tag{3.11}$$

$$\tilde{k} = \frac{x\sqrt{-2\kappa}\, e^{\pm x\sqrt{-\frac{\kappa}{2}}-\theta}}{\left(e^{\pm x\sqrt{-2\kappa}} - 1\right)} x\sqrt{-2\kappa}\, e^{x\sqrt{-\frac{\kappa}{2}}-\theta} \tag{3.12}$$

$$\tilde{\alpha} = \sqrt{-2\kappa} \tag{3.13}$$

$$SABR\_Call(f,k,T,\alpha,\beta,\rho,v) = BS\_Call\left(\tilde{f},\tilde{k},\tilde{\alpha},T\right) \tag{3.3}$$

Proof: see 7.5. □

Compared to the Hagan et al. formula, described in [5], the above computation of the SABR model requires to compute the Black Scholes formula with the complex error function $erf(x+iy)$ using the remark (4.1) and (4.2).

Quality of our approximation and numerical examples
TO BE COMPLETED (using Olivier graphics).

5. Application to CMS replication pricing

CMS can be replicated as explained in [3]. Namely, one

6. Spread option with Copula

TO BE COMPLETED

7. Conclusion

In this paper, we show that the local time formulation of the vanilla option problem enables us to relate the SABR model to the Black Scholes one. After providing a closed form approximation for the local time problem in SABR, we explain how to numerically compute the complex error function $erf(x+iy)$ using fast convergent series. This "complex Black Scholes" formula turns out to be more robust for low and high strike. This is very useful for CMS replication and spread option pricing using copula.

8. References

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9. Appendix

Proof 7.1: rather than computing the local time for a geometric Brownian motion, one can notice the following vega decomposition of a call option:

$$BS\_Call(f,k,\alpha,T)=\int_0^{\alpha}\frac{\partial}{\partial\sigma}BS\_Call(f,k,\sigma,T)d\sigma+(f-k)^+ \quad (7.1.1)$$

which can be re-expressed when plugging the vega formula for call option as

$$BS\_Call(f,k,\alpha,T)=\int_0^{\alpha}k\sqrt{T}N'(d_2(\sigma))d\sigma+(f-k)^+ \quad (7.1.2)$$

where $N'(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is the standard normal density and $d_2(\sigma)=\frac{\log(f/k)}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}$. The result (1.4) is then trivially obtained by doing a change of variable that interchanges time u and volatility σ:

$$u=\frac{\sigma^2T}{\alpha^2} \quad (7.1.3).\square$$

Proof 7.2: this proof adapts the one in [5] to more general stochastic volatility. We will stretch the difference with [5]. As a function of the forward variables T,F,A, the density p defined in (2.3) satisfies the forward Kolmogorov equation also referred to as the Fokker-Plank equation:

$$p_T=\frac{1}{2}A^2\left[f(F)^2p\right]_{FF}+\frac{1}{2}\left[b(T,A)^2p\right]_{AA}+\rho_t\left[Af(F)b(t,A)p\right]_{FA}+\left[a(t,A)p\right]_A \quad (7.2.1),$$

with the boundary condition given by $p=\delta(F-f)\delta(A-\alpha)$ (7.2.2).

Compared to [5], we have an additional term from the stochastic volatility diffusion. Integrating the forward value of a call option with the probability density leads to

$$V(f,k,\alpha,\mathrm{T})=E\left[(F_T-k)^+\middle|F_0=f,\sigma_0=\alpha\right]=\int_{A=0}^{\infty}\int_{F=K}^{\infty}(F-k)p(t,f,\alpha;T,F,A)dFdA \quad (7.2.3).$$

But using the fact that the density is given by the standard Chapman Kolmogorov equation:

$$p(t,f,\sigma;T,F,A)=\delta(F-f)\delta(A-\alpha)+\int_t^T P_T(t,f,\alpha;\tau,F,\mathrm{A})d\tau \quad (7.2.4),$$

we can rewrite (7.2.3) as

$$V(f,k,\alpha,\mathrm{T})=(f-k)^++\int_0^T\int_{A=0}^{\infty}\int_K^{\infty}(F-k)p_T(t,f,\alpha;\tau,F,A)dFdAd\tau \quad (7.2.5).$$

Using the forward Kolmogorov equation (7.2.1) and integrating with respect to the volatility knowing that all terms over all A yields zero, leads to

$$V(f,k,\alpha,\mathrm{T})=(f-k)^++\frac{1}{2}\int_0^T\int_{A=0}^{\infty}\int_{F=K}^{\infty}(F-k)A^2\left[f(F)^2p\right]_{FF}dFdAd\tau \quad (7.2.6).$$

This finally conducts to the result after two integration by parts with respect to F.$\square$

Proof 7.3: Result (B.47) of [5] noticing the following fact the expression given in [5] can be completely expressed in terms of the initial parameter since,

$$\theta = \log\left(\alpha \frac{z}{f-k}\sqrt{B(0)B(\alpha z)}\right) + Log\left(\frac{x}{z}\sqrt{I(vz)}\right) + \frac{1}{4}\rho v \alpha b_1 z^2$$
$$= \frac{1}{4}\rho v \alpha b_1 z^2 + Log\left(\frac{\alpha (fk)^{\frac{\beta}{2}} z}{f-k}\right) + Log\left(\frac{x}{z}\left(1 - 2v\rho z + v^2 z^2\right)^{\frac{1}{4}}\right) \tag{7.2.7}$$

$$\kappa = v^2\left(\frac{1}{4}I''(vz_0)I(vz_0) - \frac{1}{8}\left[I'(vz_0)\right]^2\right) + \alpha^2\left(\frac{b_2}{4} - \frac{3{b_1}^2}{8}\right) + \frac{3}{4}\rho v \alpha b_1$$
$$= \frac{1}{8}\left(\frac{\alpha^2(\beta-2)\beta k^{2\beta}}{\left(k - \alpha(\beta-1)k^{\beta} z_0\right)^2} + \frac{6\alpha\beta k^{\beta} v\rho}{k - \alpha(\beta-1)k^{\beta} z_0} + \frac{v^2\left(2 - 3\rho^2 + 2v\rho z_0 - v^2 {z_0}^2\right)}{1 - 2v\rho z_0 + v^2 {z_0}^2}\right) \tag{7.2.8}$$ □

Proof 7.4: in Abramowitz et al. [1] formula 7.4.33, we have

$$\int_0^t e^{-b^2x^2 - \frac{a^2}{x^2}} dx = \frac{\sqrt{\pi}}{4a}\left[e^{2ab}\left(erf\left(bx + \frac{a}{x}\right)\right) + e^{-2ab}\left(erf\left(bx - \frac{a}{x}\right)\right)\right] \tag{7.4.1}$$

But doing a change of variable $u = x^2$, we have

$$\int_0^t \frac{e^{-b^2u - \frac{a^2}{u}}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2b}\left[e^{2ab}\left(erf\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right) - 1\right) + e^{-2ab}\left(erf\left(\frac{-a}{\sqrt{t}} + b\sqrt{t}\right) + 1\right)\right] \tag{7.4.2}$$

By analyticity, with $b = i\bar{b}$ and dropping the bar, we get

$$\int_0^t \frac{e^{-\frac{a^2}{u} + b^2 u}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2ib}\left[e^{2iab}\left(erf\left(\frac{a}{\sqrt{t}} + ib\sqrt{t}\right) - 1\right) + e^{-2iab}\left(erf\left(\frac{-a}{\sqrt{t}} + ib\sqrt{t}\right) + 1\right)\right] \tag{7.4.3}$$,

which is exactly the formula (4.1). (4.2) is a standard expansion for the complex erf function given in Abramowitz et al. [1] in 7.1.29.□

Proof 7.5:
Immediate, when comparing the local time formulation for the Black Scholes and the SABR model given by (1.4), (3.3):

$$BS\_Call(f,k,\alpha,T) = (f-k)^+ + \frac{1}{2}\alpha^2 k^2 \int_0^T \frac{1}{\alpha k\sqrt{u}} \frac{e^{-\frac{1}{2}\left(\frac{Log(f/k)}{\alpha\sqrt{u}} - \frac{1}{2}\alpha\sqrt{u}\right)^2}}{\sqrt{2\pi}} du \tag{7.5.1}$$

$$SABR\_Call(f,k,T,\alpha,\beta,\rho,v) = (f-k)^+ + \frac{f-k}{2x\sqrt{2\pi}} e^{\theta} \int_0^T \frac{e^{-\frac{x^2}{2u} + \kappa u}}{\sqrt{u}} du \tag{7.5.2}$$ □