CORRECTIONS TO HAGAN

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Abstract. This note presents a summary of all existing results about the first-order term in the expansion of the implied volatility (SABR model).

The SABR model is defined via the SDE

$$\left\{\begin{array}{rcl} \mathrm{d}S_t & = & \sigma_t S_t^{\beta} \mathrm{d}W_t^1 \\ \mathrm{d}\sigma_t & = & \nu\sigma_t \mathrm{d}W_t^2 \end{array}\right. ,$$

with $\mathrm{d} < W^1, W^2 >_t = \rho \mathrm{d}t, \rho \in (-1,1)$, $S_0 = s$, $\sigma_0 = \alpha$.

Let K be the strike and $x = \ln(s/K)$. The smile associated to the SABR model admits an expansion with respect to maturity τ :

$$\sigma_{imp}(x,\tau) = \sigma_0(x) + O(\tau).$$

After calculations, here are the results for $\sigma_0(x)$ from Hagan, Henry-Labordre and Berestycki ($0 < \beta < 1$).

- Hagan :

$$\sigma_0(x) = \frac{\zeta}{z} \frac{\nu x}{\ln\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right)}$$

- Henry-Labordre and Berstycki :

$$\sigma_0(x) = \frac{\nu x}{\ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)}$$

Notations : $z = \frac{\nu}{\alpha}\frac{s^{1-\beta}-K^{1-\beta}}{1-\beta}$, and $\zeta = \frac{\nu}{\alpha}\frac{S-K}{(SK)^{\frac{\beta}{2}}}$.

Note that Hagan's formula is inconsistant with the log normal model ($\beta \to 1$), whereas Henry-Labordre and Berstycki's one is consistant :

$$\sigma_0^{(\beta=1)}(x) = \frac{\nu x}{\ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)}$$

with $z = \frac{\nu x}{\alpha}$.

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