Some simple remarks about diffusions with jumps

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1 A strange example : Merton's model

Let us consider the simplest jump-based model we know : Merton's model[1]. The process $(X_t)_{t\geq 0}$ is given by

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t(\lambda)} Y_i,$$

where :
- W is a standard brownian motion ;
- $N(\lambda)$ is a Poisson Process with intensity λ ;
- the Y_i are iid random variables, with density $\nu_0(x) = \frac{e^{-\frac{(x-\mu)^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}}$.

So $(X_t)_{t\geq 0}$ is a Levy process with Levy measure $\nu(x) = \lambda\nu_0(x)$.
An integro-differential equation followed by p, the probability density of $(X_t)_{t\geq 0}$, is

$$\partial_t p = \frac{1}{2}\partial_{xx} p - \gamma\partial_x p + \int_{\mathbb{R}} (p(x+y) - p(x))\,\nu(y)\mathrm{d}y.$$

The interesting point with this model is that we know an explicit formula[2] for p :

$$\boxed{p_t(x) = e^{-\lambda t} \sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{k!} \frac{e^{-\frac{(x-\gamma t-k\mu)^2}{2(\sigma^2 t+k\delta^2)}}}{\sqrt{2\pi(\sigma^2 t + k\delta^2)}}}. \tag{1}$$

[1] Merton, R. C., *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics, 3 (1976), 125-44.

[2] Ekaterina Voltchkova, *Équations intégro-différentielles d'évolution : méthodes numériques et applications à la finance*, thèse de l'Ecole Polytechnique.

This formula implies the following small-time asymptotics, where we need to distinguish the cases $\delta \neq 0$ and $\delta = 0$.

1.1 First case : continuous distribution of the jumps

Thanks to formula (1), we have

$$\boxed{p_t(x) \underset{t\to 0^+}{\sim} \lambda t \frac{e^{-\frac{(x-\mu)^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}} \quad \text{if } x \neq 0}$$

These asymptotics imply the following comments :
- we see that the integral of the asymptotics of p does not tend to 1 as t tends to 0 ; this is not a contradiction with our results, as it is generally forbidden to swap the integral and an equivalent ; a solution is to keep the formula for $x \neq 0$ and to put a Dirac of intensity the difference to 1 at $x = 0$ (ie intensity $1 - \lambda t$) ;
- the formula above expresses the following idea : for small times, $x \neq 0$ can be joined uniquely thanks to one jump, as the diffusion « cannot be fast enough ».

1.2 Second case : discrete distribution of the jumps

Here, $\delta = 0$, that is to say : we just have jumps of size μ. Then, a short discussion shows that

$$\boxed{p_t(x) \underset{t\to 0^+}{\sim} \frac{(\lambda t)^k}{k!} \frac{e^{-\frac{(x-k\mu)^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}}$$

where k is the shortest integer such as $|x - k\mu|$ is minimum.

Remark. Note that $\frac{(\lambda t)^k}{k!}$ is exactly the probability of doing k jumps in the time $[0, t]$. This remark will be useful in the following : everything happens as if, to reach x in a small time, one does the necessary number of jumps to get as near as possible, and then a classical diffusion happens.

1.3 Graphics for both cases

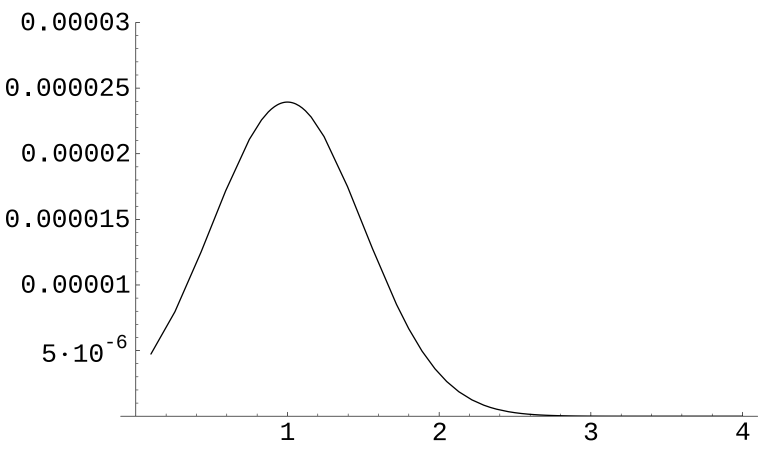

FIG. 1 – For a continuous distribution of jumps, the resulting asymptotics are a gaussian around the most probabilistic jump

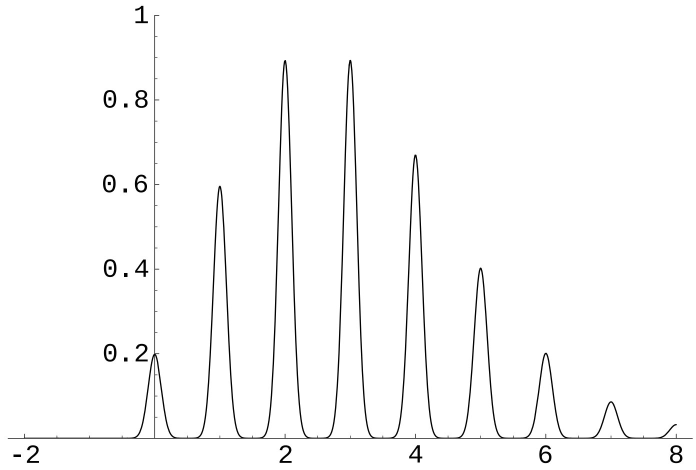

FIG. 2 – For a discrete distribution of jumps, the resulting asymptotics are gaussians around the nearest point accessible with jumps

2 Attempt for a satisfactory general formula

2.1 The conjectures

The moral of the example before, for small times, is the following :

- **for a continuous distribution of jumps, one has to reach x by one jump ;**
- **for a discrete distribution of jumps, one needs to go to x by jumps, minimising the geodesic distance of diffusion parts.**

So an analogy with the euclidian case suggests the following conjecture, where we are in a riemannian space with distance d, and the jumps are a compound Poisson process : $J_t = \sum_{i=0}^{N_t(\lambda)} Y_i$, with N a Poisson process with intensity λ, and the Y_i iid random variables of the riemannian space, with measure ν_0 (with Diracs eventually). So its Levy measure is $\nu(\mathrm{d}x) = \lambda\nu_0(\mathrm{d}x)$.

Conjecture 1 : jumps with a space-covering distribution (subcase of conjecture 2 actually). If the measure ν is a density measure, then

$$P(x,y,t) \underset{t\to 0^+}{\sim} \nu(\overrightarrow{xy})t.$$

Remark. With such a formula, as seen before, this is not a distribution as the integral is not equal to 1. So we need to keep a dirac of intensity $1-\lambda t$ at x.

Conjecture 2 : discrete distribution of jumps.

Suppose that there are a finite number of k jumps, each one with probability $p_1, \ldots, p_k$. Suppose that there is a unique way of jumps and geodesics from x to y such as $d_1 + \cdots + d_{n+1} := d$ is minimum (this generalizes the notion of geodesic in presence of jumps). Then, if d is sufficiently small,

$$P(x,y,t) \underset{t\to 0^+}{\sim} \frac{1}{(2\pi t)^{\frac{l}{2}}} e^{-\frac{d^2}{2t}} \prod_{i=1}^{n} p_{j_i} \frac{(\lambda t)^n}{n!}.$$

The condition « d sufficiently small » makes it possible to consider terms such as the Jacobi Field equal to 1. This condition is reasonable if the set of possible jumps is big enough.

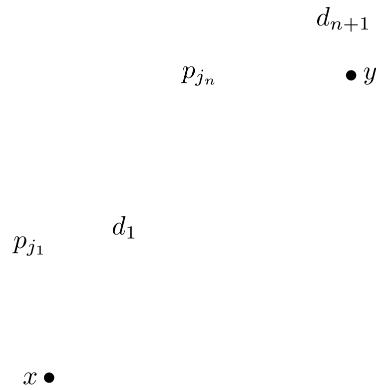

FIG. 3 – A geodesic with jumps

Remark. For other possible cases, such as a dense distribution of jumps, on rational numbers for example, or a distribution covering only a part of the space with positive Lebesgue measure, I don't have a precise idea of a possible formula.

2.2 Simple proof of conjecture 1

For a diffusion with jumps, from an initial point O, the evolution integro-differential equation has the following integro-differential form :

$$\begin{cases} \partial_t p &= \frac{1}{2}\Delta p + fp + \int_{\mathbb{R}^l} \left(p(x+y,t) - p(x,t)\right) \nu(-y) \mathrm{d}y \\ p(x,0) &= \delta_O(x) \end{cases}.$$

Here, we consider that for every y $\nu(y) \neq 0$. Contrary to the purely differential case, for $x \neq O$ and $t \to 0^+$, we don't get $0 = 0$ but $\partial_t p = \nu(x)$, so

$$P(O,x,t) \underset{t\to 0^+}{\sim} \nu(\overrightarrow{Ox})t.$$