General conditions for a symmetric matrix to be a correlation matrix

3 mai 2006

Table of contents

1 First method : Cholesky's decomposition **1**

2 Second method : derivatives of the determinant **1**

2.1 The theorem 1

2.2 The relations for a 4×4 matrix 2

1 First method : Cholesky's decomposition

There is a bijection between the ρ's, the correlation matrices, and the unique lower triangular matrix L with strictly positive diagonal terms, such as $\rho = LL^\top$. The unique explicit conditions on L are, for all $i \in [\![1, n]\!]$:

$$\left\{ \begin{array}{ccc} \sum_{l=1}^{i} l_{ij}^2 & = & 1 \\ l_{ii} & > & 0 \end{array} \right.$$

.

This decomposition is easily understood as the projection of correlated brownians on independant ones :

$$\left\{ \begin{array}{ccl} W_1 & = & l_{11}\tilde{W}_1 \\ W_2 & = & l_{21}\tilde{W}_1 + l_{22}\tilde{W}_2 \\ & \dots & \\ W_n & = & l_{n1}\tilde{W}_1 + \cdots + l_{nn}\tilde{W}_n \end{array} \right. ,$$

and $\rho = LL^\top$.

2 Second method : derivatives of the determinant

2.1 The theorem

Theorem. *Let ρ be a symmetric matrix in dimension n, with characteristic polynomial P. Then ρ is definite positive if and only if for all $k \in [\![0, n]\!]$*

$$(-1)^{n-k} P^{(k)}(0) > 0.$$

Démonstration. We only treat the even-n case.

If for all $k \in [\![0, n]\!]$ $(-1)^{n-k} P^{(k)}(0) > 0$, then for $x = -a$ negative (with Taylor's expansion, exact for a polynomial)

$$P(x) = \sum_{k=0}^{n} \frac{P^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{(-1)^k P^{(k)}(0)}{k!} a^k > 0,$$

so there is no root in $\mathbb{R}^-$, so all eigenvalues of ρ are positive, so it is positive definite.

Conversely, if all eigenvalues are strictly positive, we can suppose with a density argument that they are distinct. Let us call them $0 < a_1 < \cdots < a_n$. As P is 0 at a_1 and a_2, with Rolle's theorem P' has a root on $]a_1, a_2[$, and the same for $]a_2, a_3[, \ldots,]a_{n-1}, a_n[$. So we have found $n-1$ distinct roots for P', which has at most $n-1$ roots, so they are all of the roots : so P' has no root on $\mathbb{R}^-$; as it tends to $-\infty$ in $-\infty$, we have $P'(0) < 0$. The reader will easily reproduce the proof above to show that $P''(0) > 0$, $P'''(0) < 0$ and so on.

□

2.2 The relations for a 4×4 matrix

Imagine we have a 4×4 correlation matrix

$$\rho = \begin{bmatrix} 1 & \rho_{21} & \rho_{31} & \rho_{41} \\ \rho_{21} & 1 & \rho_{32} & \rho_{42} \\ \rho_{31} & \rho_{32} & 1 & \rho_{43} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 \end{bmatrix}.$$

Then the necessary and sufficient conditions from the theorem above are

$$\begin{aligned} &\rho_{21}^2 + \rho_{31}^2 + \rho_{32}^2 + \rho_{41}^2 + \rho_{42}^2 + \rho_{43}^2 + 2\rho_{31}\rho_{32}\rho_{41}\rho_{42} + 2\rho_{21}\rho_{32}\rho_{41}\rho_{43} + 2\rho_{21}\rho_{31}\rho_{42}\rho_{43} \\ &\qquad < 1 + 2\rho_{21}\rho_{31}\rho_{32} + \rho_{21}\rho_{41}\rho_{42} + \rho_{31}\rho_{41}\rho_{43} + \rho_{32}\rho_{42}\rho_{43} + \rho_{32}^2\rho_{41}^2 + \rho_{31}^2\rho_{42}^2 + \rho_{21}^2\rho_{43}^2 \end{aligned}$$

$$\rho_{21}^2 + \rho_{31}^2 + \rho_{32}^2 + \rho_{41}^2 + \rho_{42}^2 + \rho_{43}^2 < 2 + \rho_{21}\rho_{31}\rho_{32} + \rho_{21}\rho_{41}\rho_{42} + \rho_{31}\rho_{41}\rho_{43} + \rho_{32}\rho_{42}\rho_{43}$$

$$\rho_{21}^2 + \rho_{31}^2 + \rho_{32}^2 + \rho_{41}^2 + \rho_{42}^2 + \rho_{43}^2 < 6$$