UPPER BOUND FOR AMERICAN PRICES

P. BOURGADE

Abstract. This note gives a synthesis of :

- the duality approach for upper bounds in American options, detailed in [1];
- its insertion in the so-called Longstaff-Schwartz algorithm to get American prices ([2]).

After a theoretical overview, we emphasize on the computational possibility of combining both ideas.

In the following, we consider an asset $(X_t)_{0\leq t\leq T}$ supposed to be markovian, an American option on $(X_t)_{0\leq t\leq T}$ with payoff h. Let $B_t = e^{\int_0^t r_s \mathrm{d}s}$, with r_s the instantaneous risk-free rate of return.

Let $\mathcal{F}_t$ be the completed σ-algebra generated by $(X_s)_{0\leq s\leq t}$. For $0 \leq t \leq T$ consider $\mathcal{T}_t$ the set of stopping times with respect to $(\mathcal{F}_t, 0 \leq t \leq T)$, and with values in $[t, T]$.

As $(X_t)_{0\leq t\leq T}$ is markovian, the price of the option at time t, conditionally of not having been exercised at this time, can be written $g(t, X_t)$, with g depending on the underlying model for $(X_t)_{0\leq t\leq T}$ and h. Then the classical theory of arbitrage freeness for American options states that

$$g(t, X_t) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}\left(\frac{h(X_\tau)B_t}{B_\tau} \mid \mathcal{F}_t\right).$$

1. Stopping time versus supermartingale

1.1. Basic idea.

It is well known that $(\frac{g(t,X_t)}{B_t}, 0 \leq t \leq T)$ is a $\mathcal{F}_t$-supermartingale: for $t < t'$

$$\begin{aligned}
\mathbb{E}\left(\frac{g(t', X_{t'})}{B_{t'}} \mid \mathcal{F}_t\right) &= \mathbb{E}\left(\sup_{\tau\in\mathcal{T}_{t'}} \mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} \mid \mathcal{F}_{t'}\right) \mid \mathcal{F}_t\right)\\
&= \mathbb{E}\left(\sup_{\tau\in\mathcal{T}_{t'}} \mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} \mid \mathcal{F}_t\right) \mid \mathcal{F}_t\right) = \sup_{\tau\in\mathcal{T}_{t'}} \mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} \mid \mathcal{F}_t\right)\\
&\leq \sup_{\tau\in\mathcal{T}_t} \mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} \mid \mathcal{F}_t\right) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} \mid \mathcal{F}_t\right) = \frac{g(t, X_t)}{B_t},
\end{aligned}$$

where the inequality results from $\mathcal{T}_{t'} \subset \mathcal{T}_t$. Therefore, $\left(\frac{g(t,X_t)}{B_t}, 0 \leq t \leq T\right)$ is a $\mathcal{F}_t$-supermartingale upper-bounding $\left(\frac{h(X_t)}{B_t}, 0 \leq t \leq T\right)$, so if we denote

Date: 26 September 2007.

$\Pi_{t\to T}$ the set of the $\mathcal{F}_t$-supermartingales,

$$\frac{g(t,X_t)}{B_t} = \sup_{s\in[t,T]}\left(\frac{h(X_s)}{B_s} - \frac{g(s,X_s)}{B_s}\right) + \frac{g(t,X_t)}{B_t}$$
$$\geq \inf_{\pi\in\Pi_{t\to T}}\left(\sup_{s\in[t,T]}\left(\frac{h(X_s)}{B_s} - \pi_s\right) + \pi_t\right).$$

Moreover, $\left(\frac{g(s,X_s)}{B_s}, t\leq s\leq T\right)$ is the unique (up to a version) $\mathcal{F}_t$-supermartingale for which this infimum is obtained : if $(\pi_s, t\leq s\leq T)$ is any $\mathcal{F}_t$ supermartingale,

$$\sup_{s\in[t,T]}\left(\frac{h(X_s)}{B_s} - \frac{g(s,X_s)}{B_s}\right) + \frac{g(t,X_t)}{B_t} = \frac{g(t,X_t)}{B_t} = \sup_{\tau\in\mathcal{T}_t}\mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} - \pi_\tau + \pi_\tau \mid \mathcal{F}_t\right)$$
$$\leq \sup_{\tau\in\mathcal{T}_t}\mathbb{E}\left(\frac{h(X_\tau)}{B_\tau} - \pi_\tau \mid \mathcal{F}_t\right) + \pi_t \leq \mathbb{E}\left(\sup_{s\in[t,T]}\left(\frac{h(X_s)}{B_s} - \pi_s\right) \mid \mathcal{F}_t\right) + \pi_t.$$

1.2. **Upper bound.** Consequently, for any supermartingale $(\pi_s, t\leq s\leq T)$,

$$\pi_t + \mathbb{E}\left(\sup_{s\in[t,T]}\left(\frac{h(X_s)}{B_s} - \pi_s\right) \mid \mathcal{F}_t\right)$$

is an upper bound for the discounted price $\frac{g(t,X_t)}{B_t}$. Pompously, this can be summarized by :

The price of a markovian American option is bounded by the price of a lookback between the payoff and any supermartingale.

Imagine now we have an approximation $\tilde{g}$ for the option price g. Then via N iterations of a trajectory $(X_s^{(k)}, t\leq s\leq T)$ $(1\leq k\leq N)$, a very probable upper bound is obtained by

$$\boxed{\overline{g}(t,X_t) = \tilde{g}(t,X_t) + \frac{B_t}{N}\sum_{k=1}^N \sup_{s\in[t,T]}\left(\frac{h(X_s^{(k)})}{B_s} - \frac{\tilde{g}(s,X_s^{(k)})}{B_s}\right).}$$

This upper bound is *probable* for two reasons :

- the quality of the Monte Carlo and its convergence in $O\left(\frac{1}{\sqrt{N}}\right)$;
- $\left(\frac{\tilde{g}(s,X_s)}{B_s}, t\leq s\leq T\right)$ is generally not a supermartingale. This difficulty can be overcome by replacing $\tilde{g}$ by its supermartingale part $\tilde{\tilde{g}}$:

$$\tilde{\tilde{g}}(t_{k+1}, X_{t_{k+1}}) - \tilde{\tilde{g}}(t_k, X_{t_k})$$
$$:= \tilde{g}(t_{k+1}, X_{t_{k+1}}) - \tilde{g}(t_k, X_{t_k}) - \mathbb{E}\left(\tilde{g}(t_{k+1}, X_{t_{k+1}}) - \tilde{g}(t_k, X_{t_k}) \mid \mathcal{F}_{t_k}\right)$$

for the Bermudean options, and the following analogue for American options can easily be shown for $(X_t, t \geq 0)$ an Itô process with drift $\mu(t, X_t)$ and volatility $\sigma(t, X_t)$ $(t \leq s \leq T)$:

$$\tilde{\tilde{g}}(s, X_s) := \tilde{g}(s, X_s) - \int_{u=t}^{s} \left(\partial_u \tilde{g}(u, X_u) + \mu(u, X_u)\partial_x \tilde{g}(u, X_u) + \frac{\sigma(u, X_u)^2}{2} \partial_{xx} \tilde{g}(u, X_u) \right) \mathrm{d}u.$$

However, our opinion herein is that, from a computational point of view, such a correction is unnecessary and could give a worse result: generally the distance between $\tilde{\tilde{g}}$ and g is bigger than that between $\tilde{g}$ and g.

2. Implementation for an upper bound.

2.1. Getting the *frontière d'exercice*. From the Longstaff Schwarz or Andersen algorithm we get an approximation $\tilde{g}$ of g, and therefore the $\tilde{f}(t)$ implicitely defined by $g(t, \tilde{f}(t)) = h(\tilde{f}(t))$.

2.2. Results. ??

3. Summary of the method.

(1) With the Longstaff Schwartz or Andersen algorithm (first Monte Carlo, N_1 simulations), we get an approximation $\tilde{g}$ for g, lower bound of g, beginning from $\tilde{g}(x, T)$ till $\tilde{g}(x, 0)$. A lower bound for the price is therefore $\tilde{g}(0, X_0)$.

(2) We then consider the stopping time τ consisting in stopping as soon as $X_t > \tilde{f}(t)$, with $\tilde{f}$ the approximate *frontière d'exercice*. With a second Monte Carlo (N_2 simulations) we get the following lower and upper bounds :

$$\begin{cases} \underline{g}_0 &= \frac{1}{N_2} \sum_{k=1}^{N_2} \frac{f(\tau_k)}{B_{\tau_k}} \\ \overline{g}_0 &= \underline{g}_0 + \frac{1}{N_2} \sum_{k=1}^{N_2} \sup_{s \in [0,T]} \left(\frac{h(X_s^{(k)})}{B_s} - \frac{\tilde{g}(s, X_s^{(k)})}{B_s} \right) \end{cases}$$

Then $\underline{g}_0$ is the lower bound for the price of the American option, and $\overline{g}_0$ is an upper bound.

References

[1] M. B. Haugh, L. Kogan, Pricing American options : a duality approach.
[2] F. A. Longstaff, E. S. Schwartz, Valuing American options by simulation : a simple least-squares approach.

ENST, 46 rue Barrault, 75634 Paris Cedex 13. Université Paris 6, LPMA, 175, rue du Chevaleret F-75013 Paris.
E-mail address: bourgade@enst.fr