Copulas

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Generic Copula

- Copula=Distribution sur $[0, 1]^n$ such that the marginals are uniform

- definition of distributions:

$$F(X_1, X_2, ..., X_n) = Prob[x_1 \le X_1, x_2 \le X_2, ..., x_n \le X_n]$$

- "Sklar Theorem" : modification of the marginals:

$$F(X_1, X_2, ..., X_n) = C(G_1((X_1), G_2(X_2), ..., G_n(X_n)))$$

- X_i are variables in the "marginal space"
- $G_i(X_i) \in [0, 1]$ is in the "probability space" ,
- G_i is a cummulative distribution: it is a bijective mapping from the marginal space to the probability space

Usual Copulas

- density: $pdf(X_1, X_2, ..., X_n) = \left(\frac{\partial}{\partial X_1} \cdot \frac{\partial}{\partial X_2} \cdot \ldots \cdot \frac{\partial}{\partial X_n}\right) F(X_1, X_2, ..., X_n)$

- Gaussian density: $pdf(X_1, X_2, ..., X_n) = \frac{1}{(2\pi)^{n/2}\sqrt{det(\Sigma)}} e^{-\frac{1}{2}\left((X-\mu)^{\mathrm{T}} \cdot \frac{1}{\Sigma} \cdot (X-\mu)\right)}$

- Student density $pdf(X_1, X_2, ..., X_n) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{n/2}\sqrt{det(\Sigma)}}\left(1 + \frac{(X-\mu)^{\mathrm{T}} \cdot \frac{1}{\Sigma} \cdot (X-\mu)}{\nu}\right)^{-\left(\frac{\nu+d}{2}\right)}$

- Elliptical Copula: the characteristic function $\varphi_{X-\mu}(t) = \phi(t^{\mathrm{T}}\Sigma t)$ (like gaussian, student,)

- Archimedian copula: generated by a decreasing function $\varphi(x)$ from $[0, 1]$ to $[0, \infty]$

  - $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$

  - exemple the Gumbel : $\varphi(u) = (-Log(u))^{\theta}$ or the Clayton $\varphi(u) = (u^{-\theta} - 1)/\theta$

Tail Dependence

- Upper tail dependence

$$\lim_{u \to 1} Prob[Y > F_Y^{-1}(u) | (X > F_X^{-1}(u))] = \lambda_{up}$$

- with a copula $\lambda_{up} = \lim_{u \to 1} \frac{(1 - 2u + C(u, u))}{1 - u}$

  - exemple Gumbel : $\lambda_{up} = 2 - 2^{\frac{1}{\theta}}$

  - Gaussian copula : $\lambda_{up} = 0$

- Lower tail dependence with a copula $\lambda_{down} = \lim_{u \to 1} \frac{C(u, u)}{u}$

  - Student bivariate : $\lambda_{up} = \lambda_{down} = \overline{t_{\nu + 1}}(\sqrt{\nu + 1}\sqrt{1 - R_{12}} / \sqrt{1 + R_{12}})$

Pricing with Copula (bivariate)

- $Option = \int h(X_1, X_2) \underbrace{\frac{\partial}{\partial X_1} \cdot \frac{\partial}{\partial X_2} F(X_1, X_2)}_{\textbf{density}} dX_1 dX_2$

  payoff

- we separate the copula from the marginals

$\gamma_i(X_i) = p_i$
pricing marginals

$\eta(Y) = p$
copula marginals

$$Option = \int h(X_1, X_2) \frac{\partial}{\partial X_1} \cdot \frac{\partial}{\partial X_2} C(\gamma_1(X_1), \gamma_2(X_2)) dX_1 dX_2$$

pricing marginal space

$$= \int h(X_1, X_2) \partial_1 \partial_2 C(\gamma_1(X_1), \gamma_2(X_2)) \gamma_1'(X_1) \gamma_2'(X_2) dX_1 dX_2$$

$$= \int h(\gamma_1^{-1}(p_1), \gamma_2^{-1}(p_2)) \partial_1 \partial_2 C(p_1, p_2) dp_1 dp_2$$

probability space

$$= \int h(\gamma_1^{-1} \bullet \eta(p_1), \gamma_2^{-1} \bullet \eta(p_2)) \partial_1 \partial_2 C(\eta(Y_1), \eta(Y_2)) \eta'(Y_1) \eta'(Y_1) dY_1 dY_2$$

$$= \int h(\gamma_1^{-1} \bullet \eta(Y_1), \gamma_2^{-1} \bullet \eta(Y_2)) \underbrace{\frac{\partial}{\partial X_1} \cdot \frac{\partial}{\partial X_2} C(\eta(Y_1), \eta(Y_2))}_{\textbf{total density associated with the copula}} dY_1 dY_2$$

copula marginal space

pricing quantile

copula marginals

Additional simplification for gaussian copulas:

$$= \int h\left(\gamma_1^{-1} \bullet N(Y_1), \gamma_2^{-1} N(Y_2)\right) \frac{e^{-\frac{1}{2(1-\rho^2)}(Y_1^2 + Y_2^2 - 2\rho Y_1 Y_2)}}{2\pi\sqrt{1-\rho^2}} dY_1 dY_2$$

$$= \int h\left(\gamma_1^{-1} \bullet N(Y_1), \gamma_2^{-1} N(Y_2)\right) \frac{e^{-\frac{1}{2}\left(\left(Y_1\sqrt{1-\rho^2} + \rho Y_2\right)^2 + Y_2^2\right)}}{2\pi\sqrt{1-\rho^2}} dY_1 dY_2$$

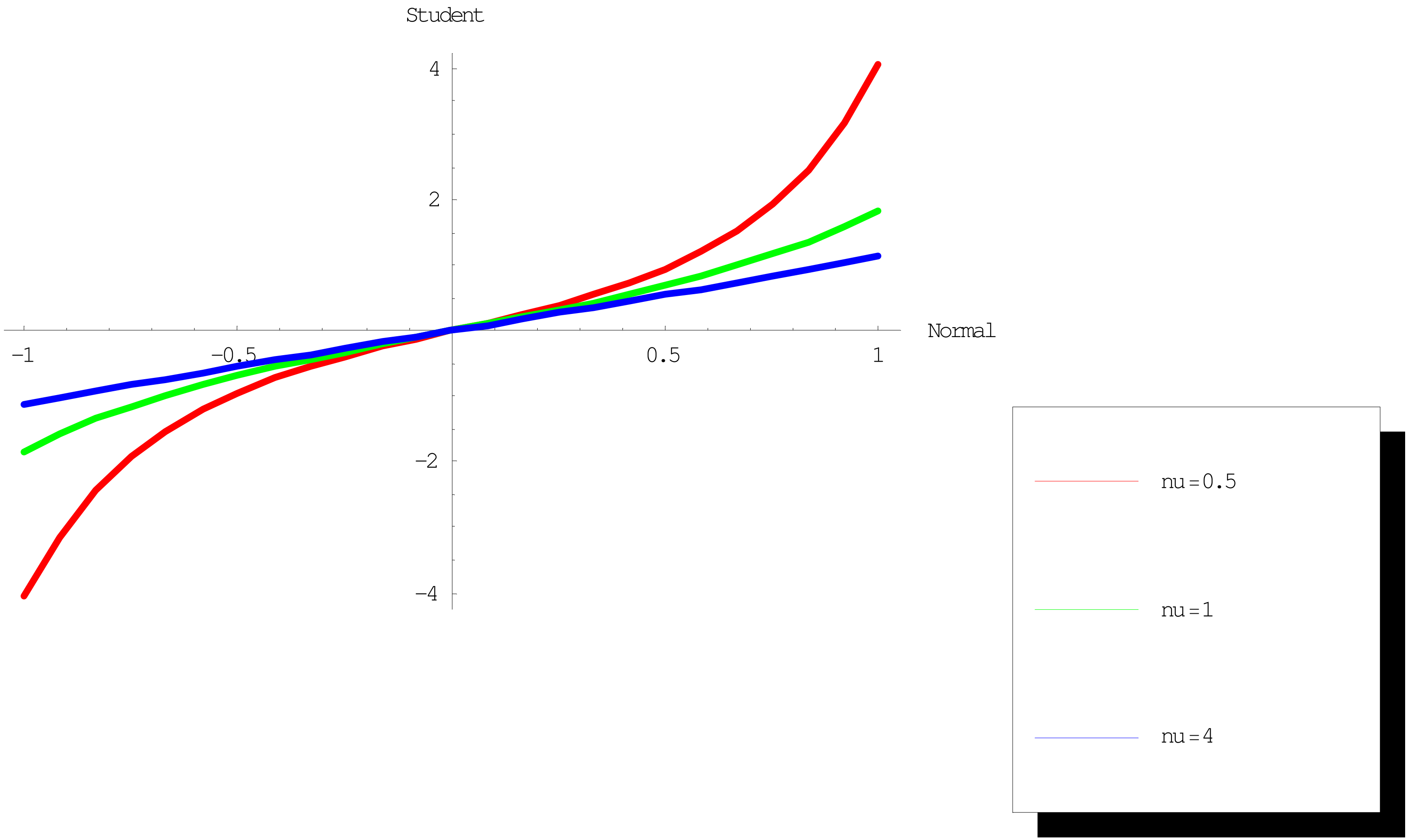
$$= \int h\left(\gamma_1^{-1} \bullet N\left(\frac{Y_1' - \rho Y_2}{\sqrt{1-\rho^2}}\right), \gamma_2^{-1} N(Y_2)\right) \frac{e^{-\frac{1}{2}(Y_1'^2 + Y_2^2)}}{2\pi} dY_1' dY_2$$

$$Y_1\sqrt{1-\rho^2} + \rho Y_2 = Y_1'$$

correlated marginal **non correlated bivariate density**

Gaussianity of a marginal

- We plot $\psi_\nu^{-1} \bullet N(x)$



Exemple of elliptic copulas

Gaussian Copula

Student Copula

associated density

$$\frac{e^{-\frac{1}{2(1-\rho^2)}(X^2+Y^2-2\rho XY)}}{2\pi\sqrt{1-\rho^2}}$$

$$\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)\sqrt{1-\rho^2}}\left(1+\frac{(X^2+Y^2-2\rho XY)}{\nu(1-\rho^2)}\right)^{-\left(\frac{\nu+2}{2}\right)}$$

Copula marginal

$$N[X]$$

$$\frac{1}{2}\left(1+Sign(X)Beta_R\left(\frac{\nu}{X^2+\nu},1,\frac{\nu}{2},\frac{1}{2}\right)\right)$$

Copula marginal density

$$\frac{e^{-\frac{1}{2}X^2}}{\sqrt{2\pi}}$$

$$\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1+\frac{X^2}{\nu}\right)^{-\left(\frac{\nu+2}{2}\right)}$$

simplification of the ratios

$$\frac{e^{-\frac{1}{2(1-\rho^2)}(X^2+Y^2-2\rho XY)}}{e^{-\frac{1}{2}X^2}\,e^{-\frac{1}{2}Y^2}} = e^{-\frac{1}{2}((X\sqrt{1-\rho^2}+\rho Y)^2+Y^2)}$$

none

Other way to get copulas

- Genest and Rivet Transformation : if $C(u, v)$ is a copule and γ is a one to one function of [0,1] to [0,1], then

$$C_{\gamma}(u, v) = \gamma^{-1}(C(\gamma(u), \gamma(v)))$$

- The Franck copule defined by :

$$C(u, v) = -\frac{1}{\theta}Log\left(1 - \frac{(1 - e^{-\theta u})(1 - e^{-\theta v})}{1 - e^{-\theta}}\right)$$

- can be compounded by $\gamma(x) = x^{1/\alpha}$ with $\alpha > 1$

The Symmetric Markov Algebra of copulas

- Product of 2 copulas :

$$(x, y) \to (C_1 \cdot C_2)(x, y) = \int_0^1 \partial_2 C_1(x, s) \partial_1 C_2(s, x) ds$$

- adjoint: $(x, y) \to (C^{\mathrm{T}})(x, y) = C(y, x)$

- The unit element is the product copula $C^{\mathrm{e}}(x, y) = xy$

- The null element is $C^{min}(x, y) = min(x, y)$

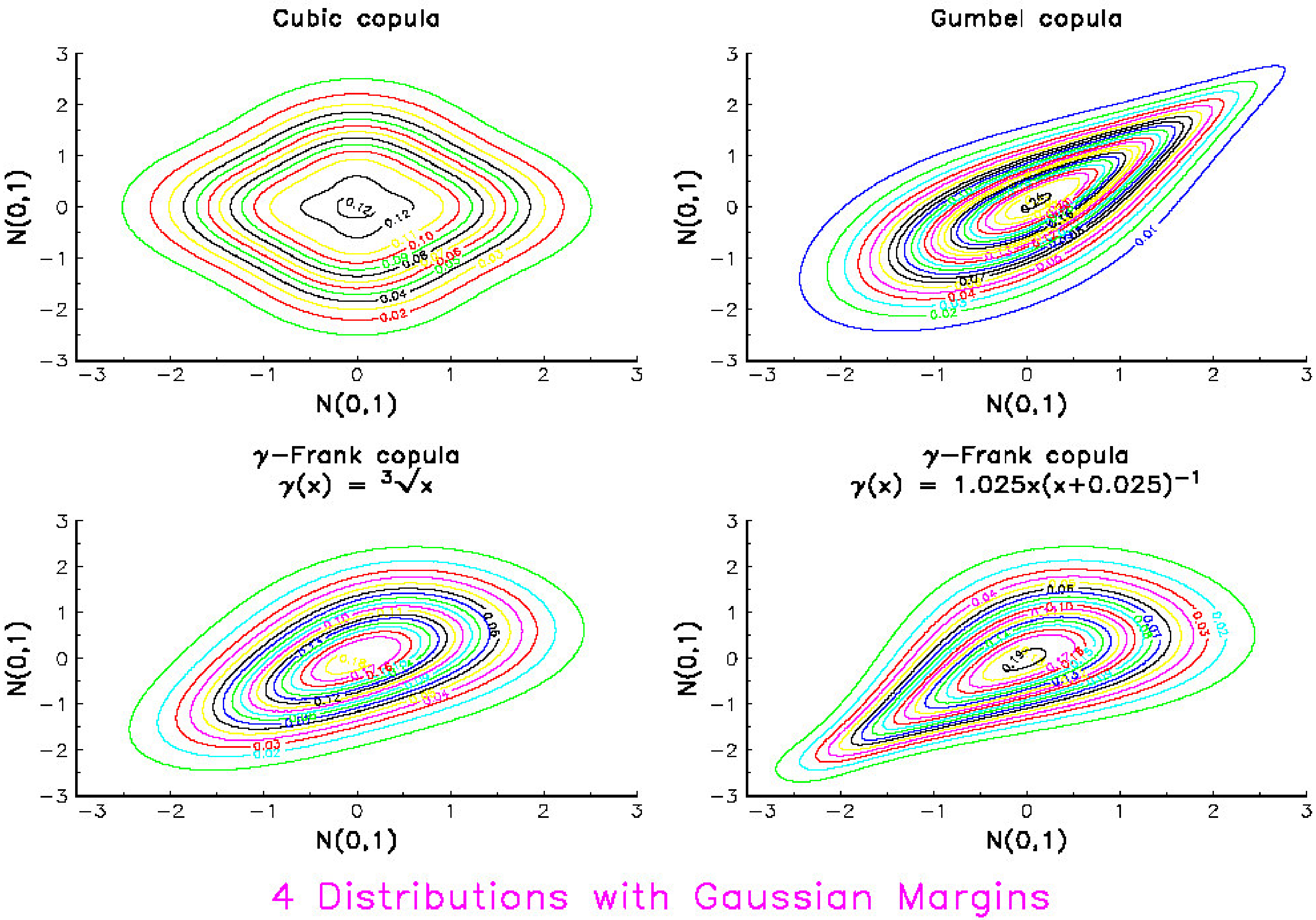
- For all copula C we have $C^{min} \le C \le C^{max}$ with the stochastic order

- with the gaussian copula we have $(C^{\max} = C_{-1}) < (C^{\mathrm{e}} = C_0) < (C^{\min} = C_1)$

- Any element that posses a right or lft inverse is extreme

  - exemple $C^{\max} \cdot C^{\max} = C^{\min}$

Exemples of equiprobability curves



Pricing formulas for

- Double Digital bivariate : payoff = 1 if $S_1 \le K_1$ and $S_2 \le K_2$ is :

$$OptFwdVal = C^{Q}(F_1(K_1), F_2(K_2))$$

- Digital Spreadoption: payoff =1 if $S_2 - S_1 > K$ is

:
$$OptFwdVal = \int_{-\infty}^{\infty} (\partial_1 C^{Q})(F_1(x), F_2(x+K))dx$$

- Spreadoption: payoff = $(S_2 - S_1 - K)^{+}$ is

:
$$OptFwdVal = S_2 - S_1 - K + \int_{-\infty}^{\infty}\int_{-x}^{K} f_1(x)(\partial_1 C^{Q})(F_1(x), F_2(x+y))dxdy$$

- Basketoption: payoff = $(S_2 + S_1 - K)^{+}$ is

:
$$OptFwdVal = S_2 + S_1 - K + \int_{-\infty}^{\infty}\int_{x}^{K} f_1(x)(\partial_1 C^{Q})(F_1(x), F_2(y-x))dxdy$$

More Option Formulas

- Max Digital option: payoff =1 if $Max(S_2, S_1) > 1$ is

: $$OptFwdVal = \int_{-\infty}^{K} C^{Q}(F_1(x), F_2(x))dx$$

- BestOf option 1 : payoff = $Max((K_1 - S_1)^+, (K_2 - S_2)^+)$ is

: $$OptFwdVal = Put(K_2) + \int_{-\infty}^{K_1}\int_{-\infty}^{\infty} f_2(x)(\partial_2 C^{Q})(F_1(Max(x + y - K_2, x)), F_2(x))dxdy$$

- BestOf option 2 : payoff = $Max((K_1 - S_1)^+, (S_2 - K_2)^+)$ is

: $$OptFwdVal = Put(K_2) + \int_{-\infty}^{K_1}\int_{-\infty}^{\infty} f_2(x)(\partial_2 C^{Q})(F_1(Max(x + y - K_2, x)), F_2(x))dxdy$$

Stochastic stability of copulas and time dependent copulas

- We look at the following model:

$$dX = (\mu(X,t) - \lambda(X,t))dt + \Sigma(X,t)dW \qquad dW \cdot dW = \rho dt$$

- Multivariate constant coefficients => Girsanov Do not change the copula

- Constant girsanov is not sufficient:

$$\begin{aligned} dX_1 &= \alpha_1(A_1 - X_1)dt + \Sigma_1 dW_1 \\ dX_2 &= \alpha_2(A_2 - X_2)dt + \Sigma_2 dW_2 \end{aligned} \qquad dW_1 \cdot dW_2 = \rho dt$$

  - Correlation of $\langle X_1(t), X_2(t) \rangle$ depends on time:

$$\rho(t) = \rho \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1 + \alpha_2} \frac{(1 - e^{-(\alpha_1 + \alpha_2)t})}{\sqrt{1 - e^{-\alpha_1 t}}\sqrt{1 - e^{-\alpha_2 t}}}$$

  - Conditional copula, pseudo-copula and others