Spectral Expansion

Seminar Notes

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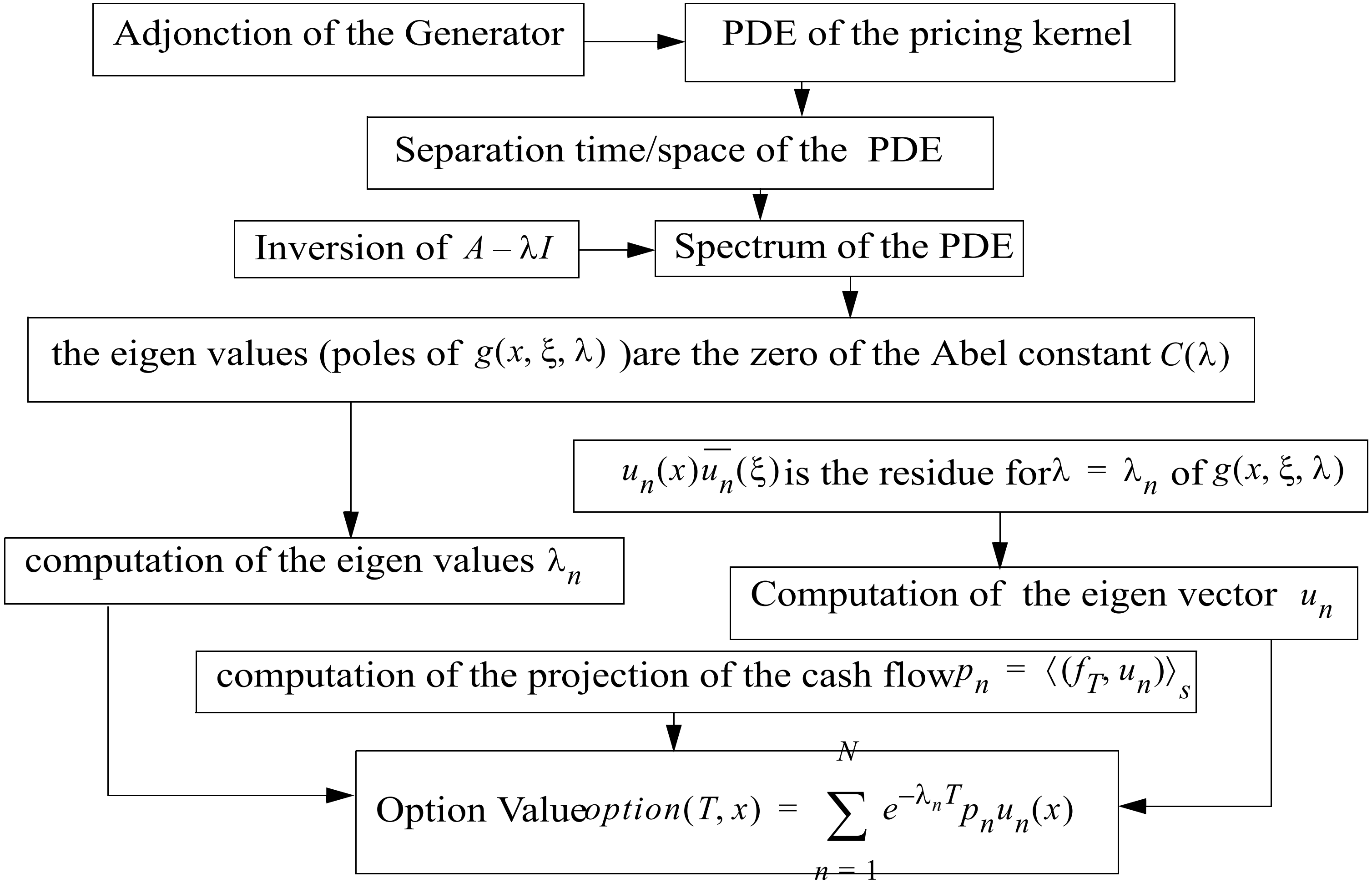
Plan

- Inversion of Operators
- Adjonction of Operators
- The Pricing Kernel => Adjoint PDE (Computation)
- Auto Adjonction of the 2nd order PDE (Computation)
- Spectrum of an Operator
- Definition of Green's Function
- Construction of the Green's function (Computation)
- Eigen Functions and Green's Function (lambda changed Pb) (Computation)
- Construction of the Green's function (lambda changed Pb) (Computation)
- Simple Exemple (Computation)

Principe of the Method



Inversion of Operators

- Operator defined with a domain => every injective operator has an inverse (Au=0 => u=0)

- A closable <=> $u_n \to 0$ then $Au_n \to 0$ or no limit, A closed <=> $u_n \to u$ then $Au_n \to Au$

- A has bounded inverse <=> Bounded away from 0: $\|Ax\| > C\|x\|$

- Type of operators

  - Type I : Bounded inverse
  - Type II : Unbounded inverse
  - Type III : no inverse

- $\frac{d}{dx}$ on $L_2(0, 1)$ with $u(0) = 0$ is closable, its closure is an extension , the boundary survives state I

- $\frac{d}{dx}$ on $L_2(0, 1)$ with $u'(0) = 0$ is closable, the boundary disappears, state III

Adjonction of Operators

- A bounded on the whole H (hilbert) , <Au,v> is a linear function of u, Riesz representation -> $\exists g, \langle Au, v\rangle = \langle u, g\rangle$ for all v
- A unbounded, A defined on dense subspace of H, for some v, $\exists g, \langle Au, v\rangle = \langle u, g\rangle$ -> (v,g)is an admissible pair written as (v,A*v), D_{A^*} is a subspace
- A is symmetric iff $\langle Au, v\rangle = \langle u, Av\rangle$ for all u,v of D_A
- A is self adjoint if $D_{A^*} = D_A$ and A is symmetric, every bounded symmetric is self adjoint
- A^* is closed and D_{A^*} is dense so closure of A closable on a dens e domain is A^{**}
- Exemple of symmetric operator non self adjoint: $-i\frac{d}{dx}$ with $\begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases}$ because the boundary condition disapears at the closure

The Pricing Kernel => Adjoint PDE

- $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ => $u(t,x) = E_x[f(X_t)] = f(x) + E_x\left[\int_0^t Af(X_s)ds\right]$ (dynkin formula)

- where $Af = b(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma(x)\sigma(x)^*\frac{\partial^2}{\partial x \partial x}f$ is the generator defined by $Af = \lim_{t \to 0}\frac{E_x[f(X_t)] - f(x)}{t}$

- Existence of a density: $P[X_t \in A, (X_0 = x)] = \int_A p(t,x,y)dy$ such $u(t,x) = \int p(t,x,y)f(y)$

- $\frac{\partial u}{\partial t} = E[Af(X_t)] = \int p(t,x,y)Af(y) = \frac{\partial}{\partial t}\int p(t,x,y)f(y)$ so $\int p(t,x,y)Af(y) = \int \frac{\partial}{\partial t}p(t,x,y)f(y)$ and

$$\int f(y)\left\{A^*p(t,x,y) - \frac{\partial}{\partial t}p(t,x,y)\right\} = 0$$

- => $\frac{\partial}{\partial t}p(t,x,y) = A^*p(t,x,y)$

Auto Adjonction of the 2nd order PDE

- $Lu = \frac{1}{s}(-(pu')' + qu)$ with $\begin{cases} u(a)\cos\alpha - u'(a)\sin\alpha = 0 \\ u(b)\cos\beta - u'(b)\sin\beta = 0 \end{cases}$

- $\begin{cases} u \in D_L \\ v \in D_L \end{cases}$ then $\int_a^b s(\bar{v}Lu - uL\bar{v}) = \int_a^b (\bar{v}(pu')' - u(p\bar{v}')') = \int_a^b (pw)'$ where $w = \bar{v}u' - u\bar{v}'$ wronskian

- $w(a) = w(b) = 0$ => Auto Adjoint

- Abel Lemma : $w(u_1, u_2, x) = \frac{C}{p(x)}$ $\forall x$ but $C = C(u_1, u_2)$

  - corrolary if there is x_0 such w(x_0)=0 , then w(x) =0 for all x

Spectrum of an Operator

- λ such $A - \lambda I$ has a bounded inverse is the resolvent set of A : the complement to C is the spectrum

  - $A - \lambda I$ in a state III (no inverse) -> point spectrum (eigen value/eigen vector)
  - $A - \lambda I$ in a state II (unbounded inverse) -> continuous spectrum
  - Range[$A - \lambda I$] < H -> compression spectrum (eigen value of A^*)

- A symmetric then $\langle Au, u \rangle$ is real so the approximate spectrum (eigen+continuous) is real and all eigen vectors are orthogonal

- A self adjoint then **point spectrum =compression spectrum**

Definition of Green's Function

- $Lu \equiv a_2(x)u'' + a_1(x)u' + a_0(x)u = f(x)$ and $\begin{matrix} B_1 u \equiv \alpha_{11}u(a) + \alpha_{12}u'(a) + \beta_{11}u(b) + \beta_{12}u'(b) \\ B_2 u \equiv \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) \end{matrix}$

- green function

  - $g(x, \xi)$: $Lg = 0$ for $a < x, \xi < b$ and $x \neq \xi$

  - $\begin{cases} B_1 g = 0 \\ B_2 g = 0 \end{cases}$ and $\left.\dfrac{\partial g}{\partial x}\right|_{x = \xi^+} - \left.\dfrac{\partial g}{\partial x}\right|_{x = \xi^-} = \dfrac{1}{a_2(\xi)}$

- equivalent to saying that $Lg = \delta(x - \xi)$ for $a < x, \xi < b$ and $\begin{cases} B_1 g = 0 \\ B_2 g = 0 \end{cases}$

Construction of the Green's function

- Hypothesis

  - Unmixed boundary conditions
    $$B_1 u \equiv \alpha_{11} u(a) + \alpha_{12} u'(a) = 0$$
    $$B_2 u \equiv \beta_{21} u(b) + \beta_{22} u'(b) = 0$$

  - The completely homogeneous system has only the trivial solution

- let u_1 such $Lu_1 = 0$ and $\begin{matrix} u_1(a) = \alpha_{12} \\ u_1'(a) = -\alpha_{11} \end{matrix}$ and u_2 such $Lu_2 = 0$ and $\begin{matrix} u_2(b) = \beta_{12} \\ u_2'(b) = -\beta_{11} \end{matrix}$

- g is such $g(x, \xi) = Au_1 \quad a < x < \xi$ and $g(x, \xi) = Bu_1 \quad \xi < x < b$ and we have $\begin{matrix} Au_1(\xi) - Bu_2(\xi) = 0 \\ -Au_1(\xi) + Bu_2(\xi) = \dfrac{1}{a_2(\xi)} \end{matrix}$

- the solution is : $A = \dfrac{u_2(\xi)}{a_2(\xi) W(u_1, u_2, \xi)} \qquad B = \dfrac{u_1(\xi)}{1 a_2(\xi) W(u_1, u_2, \xi)}$

- g can be represente $g(x, \xi) = \dfrac{u_1(min(x, \xi)) u_2(max(x, \xi))}{a_2(x) W(u_1, u_2, \xi)}$

Eigen Functions and Green's Function (lambda changed Pb)

- $-(pu')' + qu - \lambda su = 0$ and $\begin{array}{l} u(a)\cos(\alpha) - u'(a)\sin(\alpha) = 0 \\ u(b)\cos(\beta) + u'(b)\sin(\beta) = 0 \end{array}$ for $a < x < b$ -> green's $g(x, \xi, \lambda)$

- decomposition: $g(x, \xi, \lambda) = \sum_n g_n(\xi, \lambda) u_n(x)$ with $g_n(\xi, \lambda) = \langle g, u_n \rangle = \int_a^b s(x) g(x, \xi, \lambda) \bar{u}_n(x) dx$

- $-(pg')' + qg - \lambda sg = \delta(x - \xi)$ multiply by $\bar{u}$ and integrate : $\langle Lg, u_n \rangle - \lambda \langle g, u_n \rangle = \bar{u}(\xi)$

- Auto-adjonction of L : $\lambda_n \langle g, u_n \rangle - \lambda \langle g, u_n \rangle = \bar{u}(\xi)$ and $\langle g, u_n \rangle = \dfrac{\bar{u}(\xi)}{(\lambda_n - \lambda)}$

- so $g(x, \xi, \lambda) = \sum_n \dfrac{u_n(x)\overline{u_n}(\xi)}{\lambda_n - \lambda}$ <=> $\dfrac{1}{2i\pi} \int_{C_\infty} g(x, \xi, \lambda) d\lambda = -\sum_n u_n(x)\overline{u_n}(\xi)$ <=> $\dfrac{\delta(x - \xi)}{s(x)} = \sum_n u_n(x)\overline{u_n}(\xi)$

- so $u_n(x)\bar{u}(\xi)$ is the residue for $\lambda = \lambda_n$ of $g(x, \xi, \lambda)$

- generalisable to continuous spectrum : $\frac{1}{2i\pi}\int_{C_\infty} g(x, \xi, \lambda)\,d\lambda = -\sum_n u_n(x)\overline{u_n}(\xi) - \int u_\lambda(x)\overline{u_\lambda}(\xi)\,d\lambda$

Construction of the Green's function (lambda changed Pb)

- Hypothesis

  - Unmixed boundary conditions
  $$B_1 u \equiv \cos(\alpha)u(a) - \sin(\alpha)u'(a) = 0$$
  $$B_2 u \equiv \cos(\beta)u(b) - \sin(\beta)u'(b) = 0$$

  - The completely homogeneous system has only the trivial solution

- let u_1 such $Lu_1 = 0$ and $\begin{matrix} u_1(a) = \sin(\alpha) \\ u_1'(a) = \cos(\alpha) \end{matrix}$ and u_2 such $Lu_2 = 0$ and $\begin{matrix} u_2(b) = \sin(\beta) \\ u_2'(b) = \cos(\beta) \end{matrix}$

- g is such $\begin{cases} g(x, \xi, \lambda) = Au_1 \\ g(x, \xi, \lambda) = Bu_1 \end{cases}$ for $\begin{cases} a < x < \xi \\ \xi < x < b \end{cases}$ we have $\begin{matrix} Au_1(\xi) - Bu_2(\xi) = 0 \\ -Au_1(\xi) + Bu_2(\xi) = \dfrac{1}{-p(\xi)} \end{matrix}$

- the solution is : $A = \dfrac{-u_2(\xi)}{p(\xi)W(u_1, u_2, \xi)}$ $\quad B = \dfrac{-u_1(\xi)}{p(\xi)W(u_1, u_2, \xi)}$

- g can be represente $g(x, \xi, \lambda) = \dfrac{-u_1(min(x, \xi))u_2(max(x, \xi))}{p(\xi)W(u_1, u_2, \xi)} = \dfrac{-u_1(min(x, \xi))u_2(max(x, \xi))}{C(\lambda)}$

- if we define $k_n = \frac{u_1(x, \lambda_n)}{u_2(x, \lambda_n)}$ then $g(x, \xi, \lambda) = -k_n \frac{u_2(min(x, \xi))u_2(max(x, \xi))}{C(\lambda)} = -k_n \frac{u_2(x)u_2(\xi)}{C(\lambda)}$

- the eigen values (poles of $g(x, \xi, \lambda)$)are the zero of the Abel Constant $C(\lambda)$

- now we take the residue at λ_n to get $-u_n(x)\overline{u_n}(\xi) = -k_n \frac{u_2(x)u_2(\xi)}{C'(\lambda_n)}$

- we deduce $u_n(x) = \pm \frac{u_2(x)}{\sqrt{C'(\lambda_n)}}$

Simple Exemple

- We consider $\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial x^2} = 0$ for $0 < x < 1$ and $F(x, 0) = f(x)$ Plus $\begin{cases} F(0, t) = 0 \\ F(1, t)\cos(\beta) + F'(1, t)\sin(\beta) = 0 \end{cases}$

- we try solutions like $F(x, t) = u(x)T(t)$ we get : $\frac{1}{T}\frac{dT}{dt} = \frac{1}{u}\frac{d^2u}{dx^2} = -\lambda$

- the general solution will be therefore $F(x, t) = \int T_\lambda(t)u_\lambda(x)d\lambda$ where $d\lambda$ is a measure on C and u_λ solution of $-u'' - \lambda u = 0$ for $0 < x < 1$ and $u(x, 0) = f(x)$ Plus

$$\begin{cases} u(0, t) = 0 \\ u(1, t)\cos(\beta) + u'(1, t)\sin(\beta) = 0 \end{cases}$$

- the green function $-g'' - \lambda g = \delta(x - \xi)$ with $\begin{cases} g(0, t) = 0 \\ g(1, t)\cos(\beta) + g'(1, t)\sin(\beta) = 0 \end{cases}$

- solution of the homogeneous system:

$$v(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \qquad z(x,\lambda) = \sin(\beta)\cos(\sqrt{\lambda}(x-1)) - \frac{\cos(\beta)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}(x-1))$$

- we calculate the wronskian

$$C(\lambda) = -\sin(\beta)\cos(\sqrt{\lambda}) - \frac{\cos(\beta)}{\sqrt{\lambda}\sin(\sqrt{\lambda})}$$

- when $0 \le \beta < \frac{3\pi}{4}$ all eigenvalues are positive , we call them r_n

- the normalized eigenfunction are therefore $u_n = \frac{1}{N_n}\sin(\sqrt{r_n}x)$ with

$$N_n^2 = \frac{1}{2} - \frac{\sin(2\sqrt{r_n})}{4\sqrt{r_n}}$$