# Stochastic Volatility

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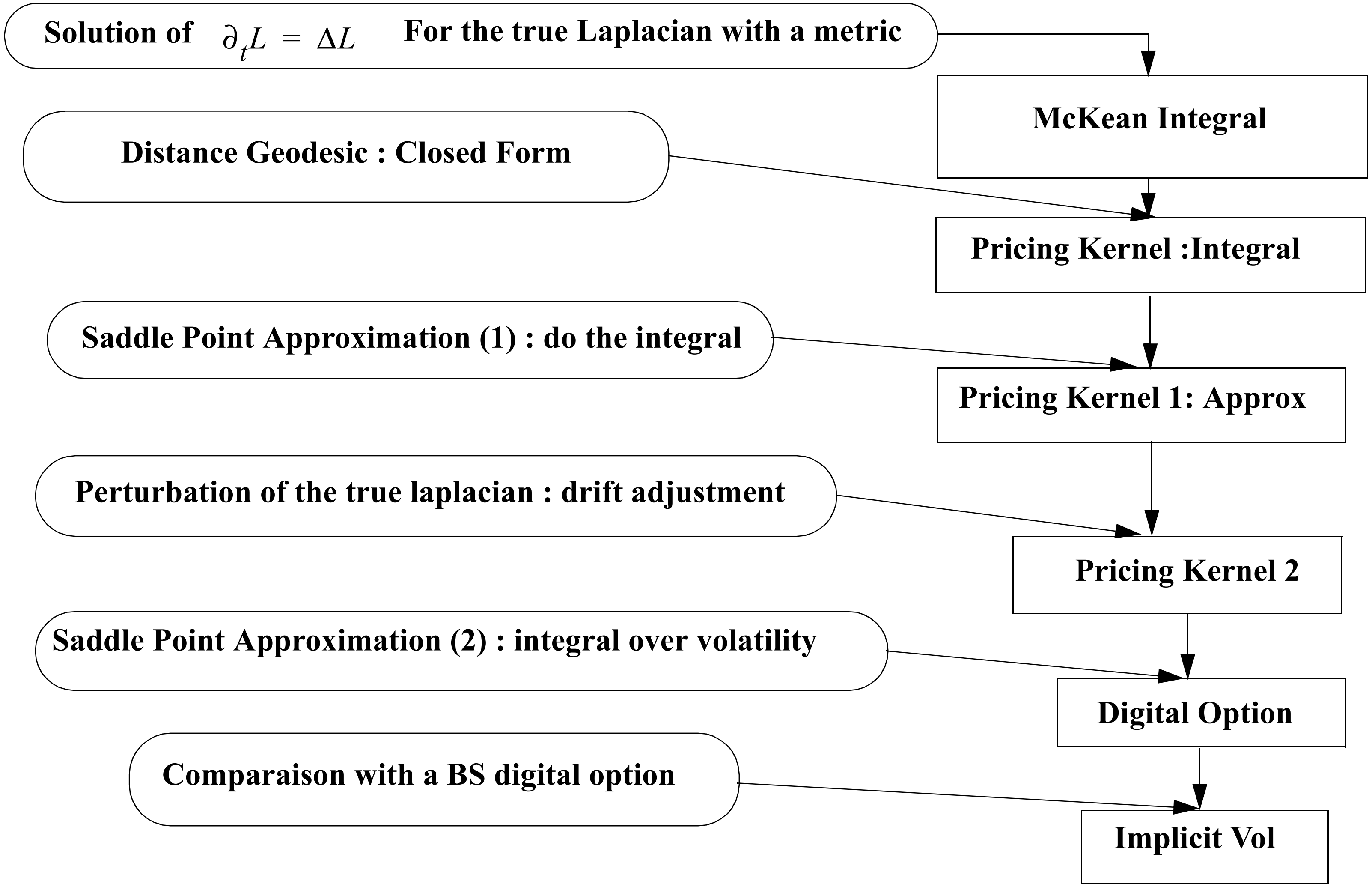
# Strategy of the Resolution



# We want to solve:

- $$\begin{aligned} dF_t &= \sigma_t b(F_t) dW_t \\ d\sigma_t &= v\sigma_t dZ_t \\ E[dW_t dZ_t] &= \rho dt \end{aligned}$$

- we do the change of variables :

- $$\begin{aligned} s &= (T-t)/T \\ y &= \sigma/v \end{aligned}$$

- The green problem we want to solve is therfore:

$$\frac{\partial}{\partial s}K(s,x,y) = \frac{1}{2}y^2\left(b(x)^2\frac{\partial^2}{\partial x^2} + 2\rho b(x)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right)K(s,x,y)$$
$$K(0,x,y) = \delta(x-x_0)\delta(y-y_0)$$

# Distance (without knowing it)

- Black and Sholes world with a deterministic volatility $\sigma(t)$:
  - in the moneyness & maturity coordinates

$$x_\tau = Log\left[\frac{S_{T-t}}{k}\right] \qquad \tau = T-t$$

  - $dX_t = \sigma(t)dW_t$ is associated with a kernel

$$\partial_\tau G(x_0, x, \tau) = \sigma(t)^2 \partial_{x,x} G(x_0, x, \tau)$$

  - any security (here a call) can be repriced by

$$Call = \int (e^x - 1) G(x_0, t_0, x, \tau) dx$$

  - it is well known that

$$G(x_0, x, \tau) = \frac{e^{-\frac{(x-x_0)^2}{2\Sigma_\tau^2}}}{\sqrt{2\pi\Sigma_\tau^2}} \qquad \Sigma_\tau = \int_0^\tau \sigma(s)^2 ds$$

# Distance (we almost see it)

- We can do it differently

  - we change of state variable :

$$y = \frac{x}{\sqrt{\int_0^\tau \sigma(s)^2 ds}}$$

  - the new equation reads now

$$\partial_\tau G(y, \tau) = \frac{\partial}{\partial y}\frac{\partial}{\partial y} G_2(y, \tau)$$

  - it is a standard diffusion

# Distance (local vol)

- Local Volatility context

$$\partial_\tau G(x_0, x, \tau) = \sigma(x)^2 \partial_{x,x} G(x_0, x, \tau)$$

- we change of state variable :

$$y = \int_0^x \frac{dz}{\sigma(z)} \qquad \frac{dy}{dx} = \frac{1}{\sigma(x)}$$

- the new equation reads now $G_2(y, \tau) = G_1(x(y), \tau)$

$$\frac{\partial}{\partial x} G_1(x, \tau) = \frac{dy}{dx}\frac{\partial}{\partial y} G_2(y, \tau) \qquad \frac{\partial}{\partial x}\frac{\partial}{\partial x} G_1(x, \tau) = \frac{\partial}{\partial x}\left(\frac{dy}{dx}\right)\frac{\partial}{\partial y} G_2(y, \tau) + \left(\frac{dy}{dx}\right)^2 \frac{\partial}{\partial y}\frac{\partial}{\partial y} G_2(y, \tau)$$

$$\partial_\tau G_2(y, \tau) = \sigma'(x)\frac{\partial}{\partial y} G_2(y, \tau) + \frac{\partial}{\partial y}\frac{\partial}{\partial y} G_2(y, \tau)$$

- by Feyman Kac it is a standard diffusion with a drift

- For very short times the behaviour is given by $\partial_\tau G_2(y, \tau) \approx \frac{\partial}{\partial y}\frac{\partial}{\partial y} G_2(y, \tau)$

# A Large Deviation Result(Varadhan)

- Standard diffusion

$$Prob[x < x_\tau < x + dx] \equiv p(x_0, x, \tau) = \left(\frac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-|x - x_0|^2}{2\tau}}$$

- So $\underset{\tau \to 0}{Lim}\{2t \cdot Log[p(x_0, x, \tau)]\} = -|x - x_0|^2$

- This is of course still true for multidimensional state

- Standard diffusion with a drift : $\underset{\tau \to 0}{Lim}\{2t \cdot Log[p(x_0, x, \tau)]\} = -|x - x_0|^2$ (Varadhan 1967)

- by normalisation we get

$$p(x_0, x, \tau) = \left(\frac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-|x - x_0|^2}{2\tau}} (1 + O(\tau))$$

# Synthesis

- For Determinist Volatility $\partial_\tau G(x_0, x, \tau) = \sigma(t)^2 \partial_{x,x} G(x_0, x, \tau)$

  - $d(x, x_0) = \dfrac{|x - x_0|}{\sqrt{\int_0^\tau \sigma(s)^2 ds}}$ $\qquad p(x_0, x, \tau) = \left(\dfrac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-d(x, x_0)^2}{2\tau}}$

- For Local Volatility $\partial_\tau G(x_0, x, \tau) = \sigma(x)^2 \partial_{x,x} G(x_0, x, \tau)$

  - $d(x, x_0) = \left|\int_{x_0}^{x} \dfrac{dz}{\sigma(z)}\right|$ $\qquad p(x_0, x, \tau) = \left(\dfrac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-d(x, x_0)^2}{2\tau}} (1 + O(\tau))$

- For Stochastic Volatility $\partial_\tau G(x_0, x, \tau) = \sum_{\mu,\nu} H_{\mu,\nu} \dfrac{\partial^2}{\partial x_\mu \partial x_\nu} G(x_0, x, \tau)$

  - $d(x, x_0) = \mathrm{GeodesicDistance}(x_0, x)$ $\qquad p(x_0, x, \tau) = \left(\dfrac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-d(x, x_0)^2}{2\tau}} (1 + O(\tau))$

# The geodesic distance

- For a metric $ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$ (Summation convention)
  - Geodesic is a path $x^i(t)$ such $\int_0^t \sqrt{g_{\mu\nu} dx^\mu(t) dx^\nu(t)}$ is locally extremal
  - Geodesic given by the solution of

$$\frac{d^2 x^i(t)}{dt^2} + \Gamma^i_{jk} \frac{dx^i(t)}{dt} \frac{dx^i(t)}{dt} \qquad \Gamma^i_{jk} = g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})$$

  - So for every (x,y) close enough, there is only one geodesic that goes from x to y.the distance is : $d(x, y) = \int_0^t \sqrt{g_{\mu\nu} dx^\mu(t) dx^\nu(t)}$ where $x(t)$ is a geodesic that goes from x to y.

- Exemple : for a poincare plan , $ds^2 = y^2(dx^2 + dy^2)$ we find that

$$d(x_0, y_0, x, y) = ArcCosh\left(1 + \frac{(x - x_0)^2 + (y - y_0)^2}{2yy_0}\right)$$

# Tensors and Conventions

**Function** **Vector Basis**

**Vector** $V = v^{\mu}\frac{\partial}{\partial x^{\mu}}$

**Form** $\omega = \omega_{\mu} dx^{\mu}$

**Duality** $\langle dx^{\mu}, \frac{\partial}{\partial x^{\nu}} \rangle = \delta^{\mu}_{\nu}$

**Metric Tensor** $g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$

**Duality biform-bivecteur** $g_{\mu\nu} g^{\nu\rho} = \delta^{\rho}_{\mu}$

**Transform a vector into a form** $g_{\mu\nu} v^{\mu} = a_{\nu}$

**Matrix and Vectors** $M^{\alpha}_{\beta} A^{\beta} = B^{\alpha}$

**Derivation** $A_{,\mu} \equiv \frac{\partial A}{\partial x^{\mu}}$

# A technique to extend our vocabulary of geodesics

- Pull-back : $\varphi$ : application $E \to F$ (diffeomorphism) such that the metric of E , $g^E_{\mu\nu}$ is isometrically diffeomorphic to $g^F_{\mu\nu}$ the metric of F
  - If $d(x, y)$ is a geodesic distance for F, then $d(\varphi((x'), \varphi(y')))$ is the geodesic distance for E

- Exemple : for the poincare plan : if b(x) is any positive function , $-1 < \rho < 1$, and $\varphi$ defined by : $\varphi(x, y) = \left(\frac{1}{\sqrt{1-\rho^2}}\left(\int_0^x \frac{du}{b(u)} - \rho y\right), y\right)$
  - then the metric is defined by the pull-back : $g^E(\vec{X}, \vec{Y}) = g^E(J_\varphi(\vec{X}), J_\varphi(\vec{Y}))$ where

$$J_\varphi(x, y) = \left(\frac{\frac{x}{b(x)} - \rho y}{\sqrt{1-\rho^2}}, y\right)$$

is the jacobian matrix. the Metric on E is

then: $$ds^2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{dx^2 - 2\rho dx dy}{y^2 b(x)^2} + \frac{dy^2}{y^2}\right)$$

  - the distance on E is then : $d(x_0, y_0, x, y) = ArcCosh\left(1 + \left(\left(\frac{1}{1-\rho^2}\left(\int_{x_0}^x \frac{du}{b(u)} - \rho(y - y_0)\right)\right)^2 + (y-y_0)^2\right) / (2yy_0)\right)$

# Generalization of Laplacian

- A differentiable  manifold equiped with a metric
  - There is locally a systeme of coordinate where the metric is the euclidian metric. This is the normal cooordinate (geodesic based) .
  - There is a standard laplacian expressed in the normal coordinates .
  - How to compute it in the current coordinates ?

- We need an intrinsic definition of the laplacian (invariant by a change of coordinate)
  - In an euclidian space $(\Delta(f), h) = \int_M \Delta(f) \cdot h dv = \sum_i \int_M \frac{\partial^2 f}{\partial x_i^2} \cdot h dv = -\sum_i \int_M \frac{\partial f}{\partial x_i} \cdot \frac{\partial h}{\partial x_i} dv = -(\nabla(f), \nabla(h))$

    for f and g  that are 0 at infinity
  - In a space with a metric g

$$-(\nabla(f), \nabla(h)) = -\int_M \nabla(f) \cdot \nabla(h) \sqrt{g} dv = -\int_M g^{\mu\nu} \left(\frac{\partial f}{\partial x^\mu}\right) \left(\frac{\partial h}{\partial x^\nu}\right) \sqrt{g} dv = \int_M \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \left( g^{\mu\nu} \left(\frac{\partial f}{\partial x^\mu}\right) \sqrt{g} \right) h \sqrt{g} dv$$

  - So the invariant definition is : $\Delta(f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \left( g^{\mu\nu} \left(\frac{\partial f}{\partial x^\mu}\right) \sqrt{g} \right)$

# Delta function business

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<u>Small Breviaire of delta fonctions</u>

$$\delta(ax) \equiv \frac{\delta(x)}{a}$$

If it exists $x_0$ such $\begin{cases} f(x_0) = 0 \\ f'(x_0) \neq 0 \end{cases}$ then $\delta(f(x)) \equiv \dfrac{\delta(x-x_0)}{f'(x_0)}$

If it exists $x_0$ such $\begin{cases} f(x_0) = 0 \\ f'(x_0) = 0 \quad f''(x_0) \neq 0 \end{cases}$ then $\delta(f(x)) \equiv \dfrac{\delta(x-x_0)}{2|f''(x_0)|}$

For multidim delta $\delta(f(x)) \equiv \dfrac{\delta(x-x_0)}{det(\nabla f)|_{x_0}}$

# Invariance of the Initial condition

- $\delta(x)$ is not an invariant, because:
- but $\delta(f(\mathrm{GeodesicDistance}(x((t), y(t)))))$ is of course an invariant
- so for the poincaré plan, $\delta(Cosh(d(x)) - 1)$ is an invariant.

$$\delta(Coshd(x) - 1) = \delta\left(\frac{(x-x_0)^2 + (y-y_0)^2}{2yy_0}\right) = \delta\left(\frac{(x-x_0)^2}{2yy_0}\right) = y_0^2\delta(x-x_0)$$

- So we we solve $\begin{cases} \frac{\partial}{\partial\tau}K(\tau, x, y) = \frac{1}{2}\Delta K(\tau, x, y) \\ K(0, x, y) = \delta(x)\delta(y) \end{cases}$

- the solution of $\begin{cases} \frac{\partial}{\partial\tau}G(\tau, x, y) = \frac{1}{2}\Delta G(\tau, x, y) \\ G(0, x, y) = y_0^2\delta(x) \end{cases}$ is obtained by:

  - $G(\tau, x, y) = y_0^2 K(\tau, x, y)$

# A Result from McKean (1970)

- In the Poincare Plan ( $ds^2 = y^2(dx^2 + dy^2)$ ), the invariant laplacian (laplace Beltrami operator) is :

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

- The Solution of the Green Problem : $\begin{cases} \frac{\partial}{\partial \tau}K(\tau, x, y) = \frac{1}{2}\Delta K(\tau, x, y) \\ K(0, x, y) = \delta(x)\delta(y) \end{cases}$ is :

$$K(\tau, x, y) = \frac{e^{-\tau/8}\sqrt{2}}{(2\pi\tau)^{3/2}y^2}\int_{d(0, 0, x, y)}^{\infty} \frac{ue^{-u^2/(2\tau)}}{\sqrt{\cosh(u) - \cosh(d(0, 0, x, y))}}du$$

- it is a closed form !
- it depends of the coordinates through the distance function.

# Pull Back

- When we change of coordinate, K is a Kernel, therfore we adjust it with a jacobian:

- through the pull-back $K^E(\tau, x, y) = det(J_\varphi)_{x_0, y_0}(K^F(\tau, \varphi(x, y))$ because of the way delta function of the initial condition transforms through the pull back

- $$\begin{cases} \frac{\partial}{\partial\tau}K(\tau, x, y) = \frac{1}{2}\Delta K(\tau, x, y) \\ K(0, x, y) = \delta(x - x_0)\delta(y - y_0) \end{cases}$$ transforms into

- $$\begin{cases} \frac{\partial}{\partial\tau}K_\varphi(\tau, x, y) = \frac{1}{2}\Delta K_\varphi(\tau, x, y) \\ K_\varphi(0, x, y) = det(\nabla\varphi)_{x_0, y_0}\delta(x - x_0)\delta(y - y_0) \end{cases}$$

- because $\Delta$ is the invariant beltrami operator and

$$\int g(\varphi_x(x, y), \varphi_y(x, y))\delta(\varphi_x(x, y))\delta(\varphi_y(x, y))dxdy = \int g(x, y)\delta(x)\delta(y)\frac{dxdy}{det(\nabla\varphi)_{0, 0}}$$

# We Solved the wrong problem !

- The diffusion that we want to solve: $\frac{\partial}{\partial\tau}K(\tau, x, y) = g^{\mu\nu}\partial_\mu\partial_\nu K((\tau, x, y))$

- The diffusion that we know how to solve: $\frac{\partial}{\partial\tau}K(\tau, x, y) = \Delta K((\tau, x, y))$

- In the case of $g^{\mu\nu}\partial_\mu\partial_\nu K((\tau, x, y)) = \frac{1}{2}y^2\left(b(x)^2\frac{\partial^2}{\partial x^2} + 2\rho b(x)\frac{\partial\ \partial}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\right)$ we have

$$g^{\mu\nu}\partial_\mu\partial_\nu - \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}\left(g^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\right)\sqrt{g}\right) = -\frac{y^2}{\sqrt{1-\rho^2}}b(x)b'(x)\frac{\partial}{\partial x}$$

- But we have the same leading derivatives. the difference is just a drift term

# Perturbation Techniques (1)

- To Solve the small misfit

  - We want to compute the solution of $\frac{dU}{ds} = LU \qquad U(0) = 1$ formally it is $U(s) = e^{sL}$

  - $L = L_0 + H$

  - At the second order we have

$$U(s) = \left\{ I + sH + \frac{s^2}{2}(L_0 H - H L_0 + H^2) + O(s^3) \right\} e^{sL_0}$$

# Perturbation Techniques (2)

- Computation of integrals: saddle point approximation

  - We want to compute $I = \int_0^{\infty} f(u) e^{-\frac{\varphi(u)}{\varepsilon}} du$ when $\varepsilon$ small

  - we have :

$$I \approx \sqrt{\frac{2\pi\varepsilon}{\varphi''(u_0)}} e^{-\frac{\varphi(u_0)}{\varepsilon}} \left\{ f(u_0) + \varepsilon \left[ \frac{f''(u_0)}{2\varphi''(u_0)} - \frac{f(u_0)\varphi^{(4)}(u_0) + 4(f'(u_0)\varphi^{(3)}(u_0))}{8\varphi''(u_0)^2} + \frac{5f(u_0)(\varphi^{(3)}(u_0))^2}{24\varphi''(u_0)^3} \right] \right\}$$

# Geodesics and Eikonal Equations(1)

- When $\tau \to 0$ The pricing kernel , according to Varadhan is

$$p(x_0, x, \tau) \to \int \left(\frac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-d(x, y, x_0, y_0)^2}{2\tau}} dy \qquad d(x, y, x_0, y_0) = \text{GeodesicDistance}(x, y, x_0, y_0)$$

- When the underlying is assumed Lognormal $p(x_0, x, \tau) = (x_0, x, \tau)\left(\frac{1}{\sqrt{2\pi\tau}}\right) e^{\frac{-(x - x_0)^2}{2\sigma^2\tau}}$

- When $\tau \to 0$ by using the saddle point approximation, if we assume $y \to d(x, y, x_0, y_0)$ differentiable, then it exist a point where $d(x, y, x_0, y_0)$ is minimum and has null derivative, so $\tau Log[p(x_0, x, \tau)] \approx \tau Log\left[ e^{\frac{-\underset{y}{Min}\{d(x, y, x_0, y_0)\}^2}{2\tau}} \right]$.Therefore when $\tau \to 0$ the implicit vol is equal to $\sigma \approx \dfrac{x - x_0}{\underset{y}{Min}\{d(x, y, x_0, y_0)\}}$

# Geodesics and Eikonal Equations(2)

- $ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu}$ defines a geodesic distance $d(x_0, x)$

- The variation of the geodesic distance is given by

$$g^{\mu\nu} \frac{\partial^2}{\partial x^{\mu} \partial x^{\nu}} d(x_0, x) = 1$$

# Reading Advice

- Book on Riemannian Geometry: one of the most readable and comprehensive is:
  - Riemannian Geometry and geometric analysis by Jurgen Jost, Universitext, Springer third edition : cost around 30 euros

# Project Schedule

- Short term
  - Check the Standard first order SABR formula
  - Get a second order SABR formula
  - Get a first and second order SABR with mean reverting Formula
- LongTerm
  - Term Structure with SABR Volatility, HW type or maybe BGM