Connecting Martingale Optimal Transport, Reinforcement Learning, and Dynamic Hedging

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Abstract

This document explores the connection between Martingale Optimal Transport (MOT) problems, reinforcement learning frameworks, and dynamic hedging strategies. Specifically, we analyze how the actor-critic reinforcement learning framework provides a natural solution method for dynamic MOT problems and relate this to the Schrödinger Bridge Problem (SBP) and entropically regularized optimal transport.

1 Martingale Optimal Transport (MOT)

MOT involves finding the coupling $\pi(x, y)$ that minimizes a cost function $c(x, y)$ under martingale constraints:

$$\inf_{\pi \in \Pi(\mu_0, \mu_T)} \int c(x, y)\, d\pi(x, y), \tag{1}$$

where $\Pi(\mu_0, \mu_T)$ is the set of couplings between μ_0 and μ_T such that the martingale property holds:

$$\mathbb{E}[Y \mid X] = X. \tag{2}$$

The dynamic formulation of MOT introduces time evolution via measures ρ_t and velocity fields v_t constrained by the continuity equation:

$$\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0. \tag{3}$$

To make this problem computationally tractable, we use entropic regularization, which modifies the objective:

$$\inf_{\pi \in \Pi(\mu_0, \mu_T)} \int c(x, y)\, d\pi(x, y) + \frac{1}{\epsilon} H(\pi \| \pi_0), \tag{4}$$

where $H(\pi \| \pi_0)$ is the Kullback-Leibler (KL) divergence relative to a prior measure π_0.

2 Reinforcement Learning Framework

Dynamic hedging can be framed as a Markov Decision Process (MDP) with the following components:

- **States (s_t)**: Represent market conditions and portfolio states.

- **Actions (a_t)**: Trading decisions to hedge risk.
- **Rewards (r_t)**: Cashflows and penalties based on transaction costs and risk aversion.

The value function $V(s_t)$, which captures the risk-adjusted expected return, satisfies the Bellman equation:

$$V(s_t) = \sup_{a_t} \mathbb{E}\left[r_t + \gamma V(s_{t+1}) \mid s_t, a_t\right], \tag{5}$$

where γ is a discount factor.

For risk-averse reinforcement learning, we use a risk-sensitive value function:

$$V(s_t) = -\frac{1}{\lambda} \log \mathbb{E}\left[\exp\left(-\lambda \sum_{i=t}^{T} r(s_i, a_i)\right)\right], \tag{6}$$

where λ controls the degree of risk aversion.

3 Actor-Critic Framework for MOT

The actor-critic method provides a solution framework for MOT problems by splitting optimization into two parts:

- **Actor**: Parameterizes the policy $\pi_\theta(a \mid s)$ and updates it to maximize the value function.
- **Critic**: Estimates the value function $V(s)$ or the action-value function $Q(s, a)$ to guide the Actor.

The policy gradient is computed as:

$$\nabla_\theta J(\pi_\theta) = \mathbb{E}_{s\sim\pi, a\sim\pi}\left[\nabla_\theta \log \pi_\theta(a \mid s) Q(s, a)\right]. \tag{7}$$

For MOT, the actor parameterizes the control variables h_t and q, while the critic estimates the cost-to-go function, providing feedback for policy improvement.

4 Connection to Schrödinger Bridge Problem

The Schrödinger Bridge Problem (SBP) seeks the most likely stochastic process Q connecting two distributions μ_0 and μ_T, minimizing the relative entropy with respect to a reference process R:

$$\inf_{Q \in \mathcal{P}(\mu_0, \mu_T)} H(Q \| R). \tag{8}$$

The entropic regularization in OT is equivalent to solving SBP dynamically, where the reference measure R corresponds to the prior π_0 in OT.

Using the Pythagorean theorem for relative entropy, the solution can be decomposed as:

$$H(P \| R) = H(P \| Q^*) + H(Q^* \| R), \tag{9}$$

where Q^* is the intermediate calibration measure satisfying the marginal constraints.

5 Unified Framework

By introducing a calibration measure Q^* (or π^*), we unify MOT, SBP, and reinforcement learning under the same inequality:

$$H(\text{Solution}\|\text{Reference}) = H(\text{Solution}\|\text{Calibration}) + H(\text{Calibration}\|\text{Reference}). \quad (10)$$

This decomposition highlights the dual role of regularization: smoothing the solution and introducing a bias toward the reference measure.

6 Conclusion

The connection between MOT, reinforcement learning, and SBP provides a powerful framework for solving dynamic problems in finance and beyond. The actor-critic method offers a computationally efficient way to tackle MOT problems, while the equivalence to SBP unifies the treatment of entropic regularization across dynamic and static settings.