Correction of the Sixth Moment Formula in GARCH(1,1)

1 Background

In the original GARCH(1,1) formulation by Bollerslev (1986), the existence and expressions of higher-order moments were derived under certain constraints on the parameters. However, the formula for the standardized sixth moment (Γ_6) is often misreported in secondary sources. Here, we provide the corrected version, derived both from the original theorem and from symbolic computation in `Mathematica`.

2 Correct Formulas

Let $x_t = \sigma_t Z_t$ with $Z_t \sim \mathcal{N}(0,1)$ i.i.d., and $\sigma_t^2 = \alpha_0 + \alpha x_{t-1}^2 + \beta \sigma_{t-1}^2$ where $\alpha = \alpha_1,\ \beta = \beta_1$. Then:

$$\mathbb{E}[x_t^2] = \frac{\alpha_0}{1-\alpha-\beta}, \tag{1}$$

$$\Gamma_4 = \frac{\mathbb{E}[x_t^4]}{(\mathbb{E}[x_t^2])^2} = 3 + \frac{6\alpha^2}{1-3\alpha^2-2\alpha\beta-\beta^2}, \tag{2}$$

$$\begin{aligned}\Gamma_6 &= \frac{\mathbb{E}[x_t^6]}{(\mathbb{E}[x_t^2])^3} \\ &= \frac{15(1-\alpha-\beta)^2\Big(1+2\beta+8\beta^2+7\beta^3+\alpha(2+16\beta+21\beta^2)+\alpha^2(24+35\beta)+21\alpha^3\Big)}{\big(1-3\alpha^2-2\alpha\beta-\beta^2\big)\big(1-15\alpha^3-9\alpha^2\beta-3\alpha\beta^2-\beta^3\big)}.\end{aligned} \tag{3}$$

3 Existence Conditions

The following inequalities are necessary for the existence of moments:

$$\alpha \geq 0, \quad \beta \geq 0, \quad \alpha+\beta < 1, \tag{4}$$

$$3\alpha^2 + 2\alpha\beta + \beta^2 < 1 \quad \text{(existence of } \Gamma_4\text{)}, \tag{5}$$

$$15\alpha^3 + 9\alpha^2\beta + 3\alpha\beta^2 + \beta^3 < 1 \quad \text{(existence of } \Gamma_6\text{)}. \tag{6}$$

4 Autocovariance of Squared Returns

For the autocovariance of squared returns at lag $n \geq 1$:

$$\gamma_n = \mathrm{Cov}(x_t^2, x_{t+n}^2) = \left(\mathbb{E}[x_t^2]\right)^2 \left(2 + \frac{6\alpha^2}{1 - 3\alpha^2 - 2\alpha\beta - \beta^2}\right) (\alpha + \beta)^n. \tag{7}$$

5 Remarks

- The corrected formula for Γ_6 ensures consistency with Bollerslev (1986) and with symbolic derivations in `Mathematica`.
- The correction also affects the autocovariance expression: the factor is $(\mathbb{E}[x^2])^2$ rather than $\mathbb{E}[x^2]$.
- Numerical experiments in the companion paper must be regenerated using this corrected expression.
- Suggested citation: T. Bollerslev, "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31, 307–327 (1986).