Papier original : Bollerslev

Notations (suivant le papier)

- $z_t \sim \mathcal{N}(0,1)$, $\varepsilon_t = \sigma_t z_t$, $\sigma_t^2 = \omega + \alpha\,\varepsilon_{t-1}^2 + \beta\,\sigma_{t-1}^2$.
- $a_m := \mathbb{E}[z^{2m}] = (2m-1)!!$ (donc $a_0 = 1,\ a_1 = 1,\ a_2 = 3,\ a_3 = 15$).
- $\mu(\alpha,\beta,m) := \mathbb{E}\big[(\beta+\alpha z^2)^m\big] = \sum_{j=0}^{m} \binom{m}{j} \beta^{m-j}\alpha^j a_j$.
  - En particulier
    $\mu_1 := \mu(\alpha,\beta,1) = \alpha+\beta$
    $\mu_2 := \mu(\alpha,\beta,2) = \beta^2 + 2\alpha\beta + 3\alpha^2$
    $\mu_3 := \mu(\alpha,\beta,3) = \beta^3 + 3\alpha\beta^2 + 9\alpha^2\beta + 15\alpha^3$.

Formule récursive de Bollerslev (Th. 2)

Pour tout $m \geq 1$,

$$\boxed{\mathbb{E}[\varepsilon_t^{2m}] = \frac{a_m \sum_{n=0}^{m-1} \binom{m}{n}\, \omega^{\,m-n}\, \mu(\alpha,\beta,n)\, \mathbb{E}[\varepsilon_t^{2n}]}{1-\mu(\alpha,\beta,m)}\ ;}$$

avec la convention $\mathbb{E}[\varepsilon^0] = 1$ et $\mu(\alpha,\beta,0) = 1$.

In[∘]:=
```
(*double factorial (2m-1)!!=a_m*)a[m_] := Factorial2[2 m - 1]
```

In[∘]:=
```
(*mu(alpha,beta,m)*)
mu[α_, β_, m_] := Sum[Binomial[m, j] β^(m - j) α^j a[j], {j, 0, m}]
```

In[∘]:=
```
(*recursive definition of even moments E[ε^(2m)]*)
Clear[M]
M[0, α_, β_, ω_] := 1 (*convention E[ε^0]=1*)
M[m_, α_, β_, ω_] := a[m] *
  Sum[Binomial[m, n] ω^(m - n) mu[α, β, n] × M[n, α, β, ω], {n, 0, m - 1}] / (1 - mu[α, β, m])
```

In[∘]:=
```
(*Examples*)
M[1, α, β, ω]  (*E[ε^2]*)
```

Out[∘]= $\dfrac{\omega}{1-\alpha-\beta}$

In[∘]:=
```
FullSimplify[M[2, α, β, ω] / (M[1, α, β, ω]^2)]
```

Out[∘]= $3 - \dfrac{6\,\alpha^2}{-1 + 3\,\alpha^2 + 2\,\alpha\,\beta + \beta^2}$

In[∘]:= `Simplify[M[3, α, β, ω] ] (*E[ε^6]*)`

Out[∘]= $$-\frac{15\times\left(1+21\,\alpha^3+2\,\beta+8\,\beta^2+7\,\beta^3+\alpha^2\,(24+35\,\beta)+\alpha\,\left(2+16\,\beta+21\,\beta^2\right)\right)\omega^3}{(-1+\alpha+\beta)\times\left(-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2\right)\times\left(-1+15\,\alpha^3+9\,\alpha^2\,\beta+3\,\alpha\,\beta^2+\beta^3\right)}$$

In[∘]:= `(*Normalized ratio E[ε^6]/(E[ε^2])^3*)`
`Simplify[M[3, α, β, ω] / (M[1, α, β, ω]^3)]`

Out[∘]= $$\frac{15\,(-1+\alpha+\beta)^2\left(1+21\,\alpha^3+2\,\beta+8\,\beta^2+7\,\beta^3+\alpha^2\,(24+35\,\beta)+\alpha\,\left(2+16\,\beta+21\,\beta^2\right)\right)}{\left(-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2\right)\times\left(-1+15\,\alpha^3+9\,\alpha^2\,\beta+3\,\alpha\,\beta^2+\beta^3\right)}$$

Formule de ChatGPT

In[∘]:= $$\text{FullSimplify}\left[15/(1-\mu 3)\times\left(1+\frac{3\,\mu 1}{1-\mu 1}+\frac{9\times(1+\mu 1)\,\mu 2}{(1-\mu 1)\times(1-\mu 2)}\right)\ /.\ \{\mu 1\to\alpha+\beta,\ \mu 2\to\ \beta\text{^}2+2\,\alpha\,\beta+3\,\alpha\text{^}2,\ \mu 3\to\beta\text{^}3+3\,\alpha\,\beta\text{^}2\ +\ 9\,\alpha\text{^}2\,\beta\ +15\,\alpha\text{^}3\}\right]$$

Out[∘]= $$-\frac{15\times\left(1+\alpha\,(2+3\,\alpha\,(8+7\,\alpha))+2\,\beta+\alpha\,(16+35\,\alpha)\,\beta+(8+21\,\alpha)\,\beta^2+7\,\beta^3\right)}{(-1+\alpha+\beta)\times\left(-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2\right)\times\left(-1+15\,\alpha^3+9\,\alpha^2\,\beta+3\,\alpha\,\beta^2+\beta^3\right)}$$

In[∘]:= $$\text{FullSimplify}\left[\frac{15\,\omega\text{^}3}{1-\mu 3}\left(1+\frac{3\,\mu 1}{1-\mu 1}+\frac{9\times(1+\mu 1)\,\mu 2}{(1-\mu 1)\times(1-\mu 2)}\right)\Big/\ (\text{M}[1,\alpha,\beta,\omega])\text{^}3\ /.\ \{\mu 1\to\alpha+\beta,\ \mu 2\to\ \beta\text{^}2+2\,\alpha\,\beta+3\,\alpha\text{^}2,\ \mu 3\to\beta\text{^}3+3\,\alpha\,\beta\text{^}2\ +\ 9\,\alpha\text{^}2\,\beta\ +15\,\alpha\text{^}3\}\right]$$

Out[∘]= $$\frac{15\,(-1+\alpha+\beta)^2\left(1+\alpha\,(2+3\,\alpha\,(8+7\,\alpha))+2\,\beta+\alpha\,(16+35\,\alpha)\,\beta+(8+21\,\alpha)\,\beta^2+7\,\beta^3\right)}{\left(-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2\right)\times\left(-1+15\,\alpha^3+9\,\alpha^2\,\beta+3\,\alpha\,\beta^2+\beta^3\right)}$$

In[∘]:= $$\text{FullSimplify}\left[\left(1+21\,\alpha^3+2\,\beta+8\,\beta^2+7\,\beta^3+\alpha^2\,(24+35\,\beta)+\alpha\,\left(2+16\,\beta+21\,\beta^2\right)\right)-\left(1+\alpha\,(2+3\,\alpha\,(8+7\,\alpha))+2\,\beta+\alpha\,(16+35\,\alpha)\,\beta+(8+21\,\alpha)\,\beta^2+7\,\beta^3\right)\right]$$

Out[∘]= 0

Formule de Luke

In[∘]:= $$\text{FullSimplify}\left[\ (-1+\alpha+\beta)\times\left(1+\frac{3\,(\alpha+\beta)}{1-\alpha-\beta}+3\left(\frac{1+2\,\frac{(\alpha+\beta)}{1-\alpha-\beta}\left(3\,\alpha^2+2\,\alpha\,\beta+\beta^2\right)}{\left(1-3\,\alpha^2-2\,\alpha\,\beta-\beta^2\right)}\right)\right)\right]$$

Out[∘]= $$-1+4\,\alpha+4\,\beta+\frac{3\times(1+\alpha+\beta)}{-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2}$$

In[∘]:= $$\text{FullSimplify}\left[\left(1+21\,\alpha^3+2\,\beta+8\,\beta^2+7\,\beta^3+\alpha^2\,(24+35\,\beta)+\alpha\,\left(2+16\,\beta+21\,\beta^2\right)\right)\Big/\left(-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2\right)\right]$$

Out[∘]= $$8+7\,\alpha+7\,\beta+\frac{9\times(1+\alpha+\beta)}{-1+3\,\alpha^2+2\,\alpha\,\beta+\beta^2}$$

In[∘]:=

Formule de Mohamed

Out[∘]= de Formule Mohamed

In[∘]:=
```
AA = Simplify[Together[
  ExpandAll[ 15 × (1 - α - β) / (3 × (1 + (3 (α+β))/(1-α-β) + 3 ((1+2 (α+β)/(1-α-β) (3 α^2+2 α β+β^2))/(1-3 α^2-2 α β-β^2))) - 15 α^3 - 9 α^2 β - 3 α β^2 - β^3) -
    (15 (-1 + α + β)^2 (1 + 21 α^3 + 2 β + 8 β^2 + 7 β^3 + α^2 (24 + 35 β) + α (2 + 16 β + 21 β^2))) / ((-1 + 3 α^2 + 2 α β + β^2) × (-1 + 15 α^3 + 9 α^2 β + 3 α β^2 + β^3)) ]]]
```

Out[∘]= $-\left(\left(15\,(-1+\alpha+\beta)^2\,\left(945\,\alpha^9+9\,\alpha^8\,(15+413\,\beta)+6\,\alpha^7\,\left(-240+66\,\beta+1099\,\beta^2\right)+6\,\alpha^6\,\left(-141-514\,\beta+91\,\beta^2+1197\,\beta^3\right)+6\,\alpha^5\,\left(-50-454\,\beta-595\,\beta^2+80\,\beta^3+896\,\beta^4\right)+\alpha^4\,\left(231-1140\,\beta-3810\,\beta^2-2550\,\beta^3+288\,\beta^4+2884\,\beta^5\right)+2\,\alpha^3\,\left(-99+150\,\beta-672\,\beta^2-1572\,\beta^3-626\,\beta^4+62\,\beta^5+567\,\beta^6\right)+(1+\beta)^2\,\left(-11+\beta-80\,\beta^2+93\,\beta^3-83\,\beta^4+5\,\beta^5-13\,\beta^6+7\,\beta^7\right)+\alpha^2\,\left(-279-330\,\beta+238\,\beta^2-920\,\beta^3-1642\,\beta^4-424\,\beta^5+38\,\beta^6+322\,\beta^7\right)+\alpha\,\left(-21-178\,\beta-198\,\beta^2+92\,\beta^3-340\,\beta^4-516\,\beta^5-98\,\beta^6+8\,\beta^7+63\,\beta^8\right)\right)\right)\Big/\left(\left(1+45\,\alpha^5+57\,\alpha^4\,\beta-\beta^2-\beta^3+\beta^5+3\,\alpha^3\,\left(-5+14\,\beta^2\right)+\alpha\,\beta\,\left(-2-3\,\beta+5\,\beta^3\right)+3\,\alpha^2\,\left(-1-3\,\beta+6\,\beta^3\right)\right)\times\left(-12+45\,\alpha^6+3\,\beta+3\,\beta^2-11\,\beta^3-\beta^4-\beta^5+\beta^6+3\,\alpha^5\,(-15+34\,\beta)+3\,\alpha^4\,\left(-5-19\,\beta+33\,\beta^2\right)+3\,\alpha^3\,\left(-7-8\,\beta-14\,\beta^2+20\,\beta^3\right)+\alpha^2\,\left(9-51\,\beta-12\,\beta^2-18\,\beta^3+23\,\beta^4\right)+\alpha\,\left(3+6\,\beta-33\,\beta^2-4\,\beta^3-5\,\beta^4+6\,\beta^5\right)\right)\right)\right)$

In[∘]:=
```
Series[AA, {α, 0, 1}, {β, 0, 1}]
```

Out[∘]= $\left(-\frac{55}{4}-\frac{35\,\beta}{16}+O[\beta]^2\right)+\left(-\frac{35}{16}-\frac{5775\,\beta}{32}+O[\beta]^2\right)\alpha+O[\alpha]^2$